RWTH BUSINESS SCHOOL

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Outline

- Matrices
- Matrix Manipulation
- Gaussian Elimination
- Vectors
- Determinants

Outline

The matrix **A** is said to have **order** $m \times n$. The mn numbers that constitute **A** are called its **elements** or **entries**. In particular, a_{ij} denotes the element in the ith row and the jth column. For brevity, the $m \times n$ matrix is often expressed as $(a_{ij})_{m \times n}$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Matrices – Problem Set 1

Construct the 4×3 matrix $\mathbf{A} = (a_{ij})_{4\times 3}$ with $a_{ij} = 2i - j$.

Matrices – Problem Set 1 (Solution)

Construct the 4×3 matrix $\mathbf{A} = (a_{ij})_{4\times3}$ with $a_{ij} = 2i - j$.

Solution: The matrix $\mathbf{A}$ has $4 \cdot 3 = 12$ entries. Because $a_{ij} = 2i - j$, it follows that $a_{11} = 2 \cdot 1 - 1 = 1$, $a_{12} = 2 \cdot 1 - 2 = 0$, $a_{13} = 2 \cdot 1 - 3 = -1$, and so on. The complete matrix is

$$\mathbf{A} = \begin{pmatrix} 2\cdot1-1 & 2\cdot1-2 & 2\cdot1-3 \\ 2\cdot2-1 & 2\cdot2-2 & 2\cdot2-3 \\ 2\cdot3-1 & 2\cdot3-2 & 2\cdot3-3 \\ 2\cdot4-1 & 2\cdot4-2 & 2\cdot4-3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \\ 7 & 6 & 5 \end{pmatrix}$$

Matrices

In general, if $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n}$ are two matrices of the same order, we define the **sum** of $\mathbf{A}$ and $\mathbf{B}$ as the $m \times n$ matrix $(a_{ij} + b_{ij})_{m\times n}$. Thus,

$$\mathbf{A} + \mathbf{B} = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$$

So we add two matrices of the same order by adding their corresponding entries.

If α is a real number, we define $\alpha\mathbf{A}$ by

$$\alpha\mathbf{A} = \alpha(a_{ij})_{m\times n} = (\alpha a_{ij})_{m\times n}$$

Thus, to multiply a matrix by a scalar, multiply *each* entry in the matrix by that scalar. Returning to the chain of stores, the matrix equation $\mathbf{B} = 2\mathbf{A}$ would mean that all the entries in $\mathbf{B}$ are twice the corresponding elements in $\mathbf{A}$—that is, the sales revenue for each commodity in each of the outlets has exactly doubled from one month to the next. (Of course, this is a rather unlikely event.) Equivalently, $2\mathbf{A} = \mathbf{A} + \mathbf{A}$.

Problem Set 2

Compute $\mathbf{A}+\mathbf{B}$, $3\mathbf{A}$, and $(-\frac{1}{2})\mathbf{B}$, if $\mathbf{A}=\begin{pmatrix}1 & 2 & 0\\ 4 & -3 & -1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}0 & 1 & 2\\ 1 & 0 & 2\end{pmatrix}$.

Problem Set 2

Compute $\mathbf{A}+\mathbf{B}$, $3\mathbf{A}$, and $(-\frac{1}{2})\mathbf{B}$, if $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & -3 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$.

Solution:

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 1 & 3 & 2 \\ 5 & -3 & 1 \end{pmatrix}, \quad 3\mathbf{A} = \begin{pmatrix} 3 & 6 & 0 \\ 12 & -9 & -3 \end{pmatrix}, \quad (-\tfrac{1}{2})\mathbf{B} = \begin{pmatrix} 0 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & 0 & -1 \end{pmatrix}$$

Matrices – Some Basic Rules

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$\mathbf{A} + \mathbf{0} = \mathbf{A}$$

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$$

$$(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$$

$$\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$$

Matrix Multiplication

Suppose that $\mathbf{A} = (a_{ij})_{m\times n}$ and that $\mathbf{B} = (b_{ij})_{n\times p}$. Then the product $\mathbf{C} = \mathbf{AB}$ is the $m \times p$ matrix $\mathbf{C} = (c_{ij})_{m\times p}$, whose element in the ith row and the jth column is the inner product

$$c_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} + \cdots + a_{in}b_{nj}$$

of the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$.

Matrix Multiplication

Note that to get c_{ij} we multiply each component a_{ir} in the ith row of $\mathbf{A}$ by the corresponding component b_{rj} in the jth column of $\mathbf{B}$, then add all the products. One way of visualizing matrix multiplication is this:

$$\begin{pmatrix} a_{11} & \dots & a_{1k} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \boxed{a_{i1} \quad \dots \quad a_{ik} \quad \dots \quad a_{in}} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mk} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \dots & b_{kj} & \dots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \dots & b_{nj} & \dots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1j} & \dots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \dots & \boxed{c_{ij}} & \dots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \dots & c_{mj} & \dots & c_{mp} \end{pmatrix}$$

It bears repeating that the matrix product $\mathbf{AB}$ is defined only if the number of columns in $\mathbf{A}$ is equal to the number of rows in $\mathbf{B}$. Also, if $\mathbf{A}$ and $\mathbf{B}$ are two matrices, then $\mathbf{AB}$ might be defined, even if $\mathbf{BA}$ is not. For instance, if $\mathbf{A}$ is 6×3 and $\mathbf{B}$ is 3×5, then $\mathbf{AB}$ is defined (and is 6×5), whereas $\mathbf{BA}$ is not defined.

Matrices – Problem Set 3

Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & -1 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$. Compute the matrix product $\mathbf{AB}$. Is the product $\mathbf{BA}$ defined?

Matrices – Problem Set 3

Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & -1 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$. Compute the matrix product $\mathbf{AB}$. Is the product $\mathbf{BA}$ defined?

$\mathbf{A}$ is 3×3 and $\mathbf{B}$ is 3×2, so $\mathbf{AB}$ is a 3×2 matrix:

$$\mathbf{AB} = \begin{pmatrix} 0 & 1 & 2 \\ \boxed{2 \quad 3 \quad 1} \\ 4 & -1 & 6 \end{pmatrix} \left(\boxed{\begin{matrix} 3 \\ 1 \\ -1 \end{matrix}} \begin{matrix} 2 \\ 0 \\ 1 \end{matrix} \right) = \begin{pmatrix} -1 & 2 \\ \boxed{8} & 5 \\ 5 & 14 \end{pmatrix}$$

The matrix product $\mathbf{BA}$ is not defined because the number of columns in $\mathbf{B}$ $(= 2)$ is not equal to the number of rows in $\mathbf{A}$ $(= 3)$.

Matrices – Problem Set 4

Initially, three firms A, B, and C (numbered 1, 2, and 3) share the market for a certain commodity. Firm A has 20% of the market, B has 60%, and C has 20%. In the course of the next year, the following changes occur:

$$\begin{cases} A \text{ keeps } 85\% \text{ of its customers, while losing } 5\% \text{ to } B \text{ and } 10\% \text{ to } C \\ B \text{ keeps } 55\% \text{ of its customers, while losing } 10\% \text{ to } A \text{ and } 35\% \text{ to } C \\ C \text{ keeps } 85\% \text{ of its customers, while losing } 10\% \text{ to } A \text{ and } 5\% \text{ to } B \end{cases}$$

Matrices – Problem Set 4

We can represent market shares of the three firms by means of a *market share vector*, defined as a column vector $\mathbf{s}$ whose components are all nonnegative and sum to 1. Define the matrix $\mathbf{T}$ and the initial market share vector $\mathbf{s}$ by

$$\mathbf{T} = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \quad \text{and} \quad \mathbf{s} = \begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix}$$

Notice that t_{ij} is the percentage of j's customers who become i's customers in the next period. So $\mathbf{T}$ is called the *transition matrix*.

Compute the vector $\mathbf{Ts}$, show that it is also a market share vector, and give an interpretation. What is the interpretation of $\mathbf{T}(\mathbf{Ts})$, $\mathbf{T}(\mathbf{T}(\mathbf{Ts}))$, ...?

Matrices – Problem Set 4 (Solution)

$$\mathbf{Ts} = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.35 \\ 0.40 \end{pmatrix}$$

Because $0.25 + 0.35 + 0.40 = 1$, the product $\mathbf{Ts}$ is also a market share vector. The first entry in $\mathbf{Ts}$ is obtained from the calculation

$$0.85 \cdot 0.2 + 0.10 \cdot 0.6 + 0.10 \cdot 0.2 = 0.25$$

Here $0.85 \cdot 0.2$ is A's share of the market that it retains after 1 year, $0.10 \cdot 0.6$ is the share A gains from B, and $0.10 \cdot 0.2$ is the share A gains from C. The sum is therefore A's total share of the market after 1 year. The other entries in $\mathbf{Ts}$ can be interpreted similarly, so $\mathbf{Ts}$ must be the new market share vector after 1 year. Then $\mathbf{T}(\mathbf{Ts})$ is the market share vector after one more year—that is, after 2 years, and so on

Matrices – Systems of Equation

The definition of matrix multiplication was introduced in order to allow systems of equations to be manipulated. Indeed, it turns out that we can write linear systems of equations very compactly by means of matrix multiplication. For instance, consider the system

$$3x_1 + 4x_2 = 5$$
$$7x_1 - 2x_2 = 2$$

Now define $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Then we see that

$$\mathbf{Ax} = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 4x_2 \\ 7x_1 - 2x_2 \end{pmatrix}$$

So the original system is equivalent to the matrix equation

$$\mathbf{Ax} = \mathbf{b}$$

Matrices – Systems of Equation

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

can be written as $\mathbf{A}\mathbf{x} = \mathbf{b}$

Matrices – Multiplication

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \qquad \text{(associative law)}$$

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC} \qquad \text{(left distributive law)}$$

$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC} \qquad \text{(right distributive law)}$$

$$(\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B}) = \alpha(\mathbf{AB})$$

$$\mathbf{A}^n = \mathbf{AA}\cdots\mathbf{A} \qquad (\mathbf{A} \text{ is repeated } n \text{ times})$$

Matrices – Identity Matrix

The **identity matrix** of order n, denoted by $\mathbf{I}_n$ (or often just by $\mathbf{I}$), is the $n \times n$ matrix having ones along the main diagonal and zeros elsewhere:

$$\mathbf{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}_{n \times n} \qquad \textbf{(identity matrix)}$$

If $\mathbf{A}$ is any $m \times n$ matrix, it is easy to verify that $\mathbf{A}\mathbf{I}_n = \mathbf{A}$. Likewise, if $\mathbf{B}$ is any $n \times m$ matrix, then $\mathbf{I}_n\mathbf{B} = \mathbf{B}$. In particular,

$$\mathbf{A}\mathbf{I}_n = \mathbf{I}_n\mathbf{A} = \mathbf{A} \qquad (\text{for every } n \times n \text{ matrix } \mathbf{A})$$

Matrices - Transpose

Consider any $m \times n$ matrix $\mathbf{A}$. The **transpose** of $\mathbf{A}$, denoted by $\mathbf{A}'$, is defined as the $n \times m$ matrix whose first column is the first row of $\mathbf{A}$, whose second column is the second row of $\mathbf{A}$, and so on. Thus,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \Longrightarrow \mathbf{A}' = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

So we can write $\mathbf{A}' = (a'_{ij})$, where $a'_{ij} = a_{ji}$. The subscripts i and j have to be interchanged because the jth row of $\mathbf{A}$ becomes the jth column of $\mathbf{A}'$, whereas the ith column of $\mathbf{A}$ becomes the ith row of $\mathbf{A}'$.

Matrices - Transpose

$$(\mathbf{A}')' = \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$$

$$(\alpha \mathbf{A})' = \alpha \mathbf{A}'$$

$$(\mathbf{A}\mathbf{B})' = \mathbf{B}'\mathbf{A}'$$

Matrices – Problem Set 5

Let $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 2 & 3 \\ 5 & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \end{pmatrix}$. Find $\mathbf{A}'$ and $\mathbf{B}'$.

Matrices – Problem Set 5 (Solution)

Let $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 2 & 3 \\ 5 & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \end{pmatrix}$. Find $\mathbf{A}'$ and $\mathbf{B}'$.

Solution: $\mathbf{A}' = \begin{pmatrix} -1 & 2 & 5 \\ 0 & 3 & -1 \end{pmatrix}$, $\qquad \mathbf{B}' = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \\ 4 & 1 \end{pmatrix}$.

Gaussian Elimination - Example

Find all possible solutions of the system

$$\begin{aligned} 2x_2 - x_3 &= -7 \\ x_1 + x_2 + 3x_3 &= 2 \\ -3x_1 + 2x_2 + 2x_3 &= -10 \end{aligned} \qquad \text{(i)}$$

Solution: The idea will be to eliminate one unknown x_1 from both the second and third equations, and then to eliminate x_2 from the third equation, which remains with only the unknown x_3. We begin, however, by interchanging the first two equations, which certainly will not alter the set of solutions. We obtain

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 2x_2 - x_3 &= -7 \\ -3x_1 + 2x_2 + 2x_3 &= -10 \end{aligned} \qquad \text{(ii)}$$

Gaussian Elimination - Example

This has removed x_1 from the second equation. The next step is to use the first equation in (ii) to eliminate x_1 from the third equation. This is done by adding three times the first equation to the last equation. (The same result is obtained if we solve the first equation for x_1 to obtain $x_1 = -x_2 - 3x_3 + 2$, and then substitute this into the last equation.) This gives

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 2x_2 - x_3 &= -7 \\ 5x_2 + 11x_3 &= -4 \end{aligned} \tag{iii}$$

Having eliminated x_1, the next step in the systematic procedure is to multiply the second equation in (iii) by 1/2, so that the coefficient of x_2 becomes 1. Thus,

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ x_2 - \tfrac{1}{2}x_3 &= -\tfrac{7}{2} \\ 5x_2 + 11x_3 &= -4 \end{aligned} \tag{iv}$$

Gaussian Elimination - Example

Next, eliminate x_2 from the last equation by multiplying the second equation by -5 and adding the result to the last equation. This gives:

$$\begin{aligned} x_1 + x_2 + \quad 3x_2 &= \quad 2 \\ x_2 - \quad \tfrac{1}{2}x_3 &= -\tfrac{7}{2} \\ \tfrac{27}{2}x_3 &= \quad \tfrac{27}{2} \end{aligned} \qquad \text{(v)}$$

Finally, multiply the last equation by $\frac{2}{27}$ to obtain $x_3 = 1$. Now the other two unknowns can easily be found by "back substitution": Inserting $x_3 = 1$ into the second equation in (v) gives $x_2 = -3$, and the first equation in (v) subsequently yields $x_1 = 2$. Therefore the only solution of the given system is $(x_1, x_2, x_3) = (2, -3, 1)$.

Gaussian Elimination – Problem Set 6

Find all possible solutions of the following system of equations:

$$
\begin{aligned}
x_1 + 3x_2 - x_3 &= 4 \\
2x_1 + x_2 + x_3 &= 7 \\
2x_1 - 4x_2 + 4x_3 &= 6 \\
3x_1 + 4x_2 &= 11
\end{aligned}
$$

Gaussian Elimination – Problem Set 6 (Solution)

Solution: We begin with three operations to remove x_1 from equations 2, 3, and 4:

$$
\begin{array}{rcrcrcrcccc}
x_1 & + & 3x_2 & - & x_3 & = & 4 & & -2 & -2 & -3 \\
2x_1 & + & x_2 & + & x_3 & = & 7 & \leftarrow & & & \\
2x_1 & - & 4x_2 & + & 4x_3 & = & 6 & \leftarrow & & & \\
3x_1 & + & 4x_2 & & & = & 11 & \leftarrow & & &
\end{array}
$$

The result is

$$
\begin{array}{rcrcrcrc}
x_1 & + & 3x_2 & - & x_3 & = & 4 & \\
 & & -5x_2 & + & 3x_3 & = & -1 & \quad -\frac{1}{5} \\
 & & -10x_2 & + & 6x_3 & = & -2 & \\
 & & -5x_2 & + & 3x_3 & = & -1 &
\end{array}
$$

Gaussian Elimination – Problem Set 6 (Solution)

where we have also indicated the next operation of multiplying row 2 by $-\frac{1}{5}$. Further operations on the result lead to

$$
\begin{array}{rcrcc}
x_1 + 3x_2 - x_3 &=& 4 & & \\
x_2 - \frac{3}{5}x_3 &=& \frac{1}{5} & 10 & 5 \\
-10x_2 + 6x_3 &=& -2 & \leftarrow & \\
-5x_2 + 3x_3 &=& -1 & \leftarrow &
\end{array}
\qquad \text{then} \qquad
\begin{array}{rcrc}
x_1 + 3x_2 - x_3 &=& 4 & \leftarrow \\
x_2 - \frac{3}{5}x_3 &=& \frac{1}{5} & -3 \\
0 &=& 0 & \\
0 &=& 0 &
\end{array}
$$

We have now constructed the staircase. The last two equations are superfluous, and we continue by creating zeros above the leading entry x_2:

$$
\begin{array}{rcl}
x_1 \qquad + \frac{4}{5}x_3 &=& \frac{17}{5} \\
x_2 - \frac{3}{5}x_3 &=& \frac{1}{5}
\end{array}
\qquad \text{or} \qquad
\begin{array}{rcl}
x_1 &=& -\frac{4}{5}x_3 + \frac{17}{5} \\
x_2 &=& \frac{3}{5}x_3 + \frac{1}{5}
\end{array}
\qquad\qquad (*)
$$

Gaussian Elimination – Problem Set 6 (Solution)

$$\begin{aligned} x_1 \qquad + \tfrac{4}{5}x_3 &= \tfrac{17}{5} \\ x_2 - \tfrac{3}{5}x_3 &= \tfrac{1}{5} \end{aligned} \quad \text{or} \quad \begin{aligned} x_1 &= -\tfrac{4}{5}x_3 + \tfrac{17}{5} \\ x_2 &= \quad \tfrac{3}{5}x_3 + \tfrac{1}{5} \end{aligned} \qquad (*)$$

Clearly, x_3 can be chosen freely, after which x_1 and x_2 are uniquely determined by $(*)$. Putting $x_3 = t$, we can represent the solution set as:

$$(x_1, x_2, x_3) = \left(-\tfrac{4}{5}t + \tfrac{17}{5},\ \tfrac{3}{5}t + \tfrac{1}{5},\ t\right) \qquad (t \text{ is any real number})$$

We say that the solution set of the system has *one degree of freedom*, since one of the variables can be freely chosen. If this variable is given a fixed value, then the other two variables are uniquely determined.

Vectors

Since a vector is just a special types of matrix, the algebraic operations introduced for matrices are equally valid for vectors. So:

(A) Two n-vectors $\mathbf{a}$ and $\mathbf{b}$ are **equal** if and only if all their corresponding components are equal; we then write $\mathbf{a} = \mathbf{b}$.

(B) If $\mathbf{a}$ and $\mathbf{b}$ are two n-vectors, their **sum**, denoted by $\mathbf{a} + \mathbf{b}$, is the n-vector obtained by adding each component of $\mathbf{a}$ to the corresponding component of $\mathbf{b}$.[2]

(C) If $\mathbf{a}$ is an n-vector and t is a real number, we define $t\mathbf{a}$ as the n-vector whose components are t times the corresponding components in $\mathbf{a}$.

(D) The **difference** between two n-vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}$.

Vectors

If $\mathbf{a}$ and $\mathbf{b}$ are two n-vectors and t and s are real numbers, the n-vector $t\mathbf{a} + s\mathbf{b}$ is said to be a **linear combination** of $\mathbf{a}$ and $\mathbf{b}$. In symbols, using column vectors,

$$t\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + s\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} ta_1 + sb_1 \\ ta_2 + sb_2 \\ \vdots \\ ta_n + sb_n \end{pmatrix}$$

Vectors: Inner Product

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are n-vectors and α is a scalar, then

(a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(b) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(c) $(\alpha \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\alpha \mathbf{b}) = \alpha (\mathbf{a} \cdot \mathbf{b})$

(d) $\mathbf{a} \cdot \mathbf{a} > 0 \iff \mathbf{a} \neq \mathbf{0}$

If $\mathbf{a} = (1, -2, 3)$ and $\mathbf{b} = (-3, 2, 5)$, compute $\mathbf{a} \cdot \mathbf{b}$.

Solution: We get $\mathbf{a} \cdot \mathbf{b} = 1 \cdot (-3) + (-2) \cdot 2 + 3 \cdot 5 = 8$.

Vectors: Problem Set 7

(a) Let $\mathbf{a} = (1, 2, 1)$, $\mathbf{b} = (-3, 0, -2)$. Find numbers x_1 and x_2 such that $x_1\mathbf{a} + x_2\mathbf{b} = (5, 4, 4)$.

(b) Prove that there are no real numbers x_1 and x_2 satisfying $x_1\mathbf{a} + x_2\mathbf{b} = (-3, 6, 1)$.

Vectors: Problem Set 7 - Solution

(a) Let $\mathbf{a} = (1, 2, 1)$, $\mathbf{b} = (-3, 0, -2)$. Find numbers x_1 and x_2 such that $x_1\mathbf{a}+x_2\mathbf{b} = (5, 4, 4)$.

(b) Prove that there are no real numbers x_1 and x_2 satisfying $x_1\mathbf{a} + x_2\mathbf{b} = (-3, 6, 1)$.

(a) $x_1(1, 2, 1) + x_2(-3, 0, -2) = (x_1 - 3x_2, 2x_1, x_1 - 2x_2) = (5, 4, 4)$

when $x_1 = 2$ and $x_2 = -1$.

(b) x_1 and x_2 would have to satisfy $x_1(1, 2, 1) + x_2(-3, 0, -2) = (-3, 6, 1)$.

Then $x_1 - 3x_2 = -3$, $2x_1 = 6$, and $x_1 - 2x_2 = 1$.

The first two equations yield $x_1 = 3$ and $x_2 = 2$; then the last equation is not satisfied.

Determinants

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{cases} a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \\ \qquad - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{cases}$$

$$|\mathbf{A}| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinants: Problem Set 7

Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + ab + ac + bc.$

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