# RWTH BUSINESS SCHOOL

Mathematics & Statistics
M.Sc. Data Analytics and Decision Science
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AACSB ACCREDITED

BUSINESS SCHOOL | RWTH AACHEN UNIVERSITY

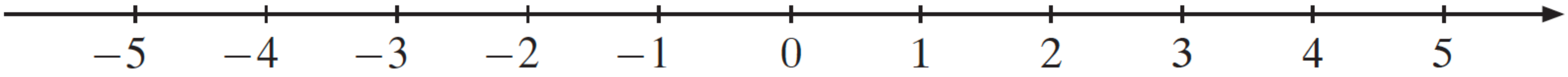
# Outline

- Real Numbers
- Integer Powers
- Intervals
- Summation Notation
- Set Theory

# The Real Numbers

- **Natural numbers**: 1, 2, 3, 4, 5, …’
  - Are also called *positive integers*
  - 1, 3, 5, 7, … are called *odd*
  - 2, 4, 6, 8, … are called *even*
- **Integers**: 0, +-1, +-2, +-3, +-4, +-5, …’
  - Consist of 0 and *positive and negative integers*
  - Can be represented on a *number line*

**Number line:**



# The Real Numbers

- **Rational numbers**: e.g., *3/5*
  - Can be written in the form *a/b,* where *a* and *b* are integers
  - An integer *n* is also a rational number, because *n=n/1*
  - They can be expressed by a point on the number line
  - Other examples of rational numbers are:

$$\frac{1}{2}, \quad \frac{11}{70}, \quad \frac{125}{7}, \quad -\frac{10}{11}, \quad 0 = \frac{0}{1}, \quad -19, \quad -1.26 = -\frac{126}{100}$$

- **Irrational numbers**:
- Cannot be written in the form *a/b,* where *a* and *b* are integers
- Examples include:

$$\sqrt{2}, -\sqrt{5}, \pi, 2^{\sqrt{2}}, \text{ and } 0.12112111211112\ldots.$$

## The Real Numbers: Problem Set 1

Which of the following statements are true?

(a) 1984 is a natural number.

(b) $-5$ is to the right of $-3$ on the number line.

(c) $-13$ is a natural number.

(d) There is no natural number that is not rational.

(e) 3.1415 is not rational.

(f) The sum of two irrational numbers is irrational.

(g) $-3/4$ is rational.

(h) All rational numbers are real.

# The Real Numbers: Problem Set 1 (Solution)

Which of the following statements are true?

(a) 1984 is a natural number. (b) $-5$ is to the right of $-3$ on the number line. (c) $-13$ is a natural number. (d) There is no natural number that is not rational. (e) 3.1415 is not rational. (f) The sum of two irrational numbers is irrational. (g) $-3/4$ is rational. (h) All rational numbers are real.

(a) True.

(b) False. $-5$ is smaller than $-3$, so on the number line it is to the left of $-3$.

(c) False. $-13$ is an integer, but not a natural number.

(d) True. Every natural number is rational. For example $5 = 5/1$.

(e) False, since $3.1415 = 31415/10000$, the quotient of two integers. (Note that 3.1415 is only an approximation to the irrational number pi).

(f) False. Counterexample: $\sqrt{2} + (-\sqrt{2}) = 0$.

(g) True.

(h) True.

# Integer Powers

If $a$ is any number and $n$ is any natural number, then $a^n$ is defined by

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$$

The expression $a^n$ is called the $n$th power of $a$ ; here $a$ is the base , and $n$ is the exponent .

We have, for example, $a^n = a \cdot a$, $x^4 = x \cdot x \cdot x \cdot x$ , and

$$\left(\frac{p}{q}\right)^5 = \frac{p}{q} \cdot \frac{p}{q} \cdot \frac{p}{q} \cdot \frac{p}{q} \cdot \frac{p}{q}$$

where $a = p/q$ , and $n = 5$.

# Integer Powers

Properties of powers:

$$a^{-n} = \frac{1}{a^n}$$

$$(a^r)^s = a^{rs}$$

$$a^r \cdot a^s = a^{r+s}$$

$$(a+b)^r \neq a^r + b^r$$

## Integer Powers: Problem Set 2

Simplify: (a) $x^p x^{2p}$ (b) $t^s \div t^{s-1}$ (c) $a^2 b^3 a^{-1} b^5$ (d) $\dfrac{t^p t^{q-1}}{t^r t^{s-1}}$

## Integer Powers: Problem Set 2 (Solution)

Simplify: (a) $x^p x^{2p}$ (b) $t^s \div t^{s-1}$ (c) $a^2 b^3 a^{-1} b^5$ (d) $\dfrac{t^p t^{q-1}}{t^r t^{s-1}}$

(a) $x^p x^{2p} = x^{p+2p} = x^{3p}$

(b) $t^s \div t^{s-1} = t^{s-(s-1)} = t^{s-s+1} = t^1 = t$

(c) $a^2 b^3 a^{-1} b^5 = a^2 a^{-1} b^3 b^5 = a^{2-1} b^{3+5} = a^1 b^8 = ab^8$

(d) $\dfrac{t^p \cdot t^{q-1}}{t^r \cdot t^{s-1}} = \dfrac{t^{p+q-1}}{t^{r+s-1}} = t^{p+q-1-(r+s-1)} = t^{p+q-1-r-s+1} = t^{p+q-r-s}$

## Integer Powers: Problem Set 3

If $x^{-2}y^3 = 5$, compute $x^2y^{-3} + 2x^{-10}y^{15}$

# Integer Powers: Problem Set 3 (Solution)

If $x^{-2}y^3 = 5$, compute $x^2y^{-3} + 2x^{-10}y^{15}$

$$x^2y^{-3} + 2x^{-10}y^{15} = (x^{-2}y^3)^{-1} + 2(x^{-2}y^3)^5 = 5^{-1} + 2 \cdot 5^5 = 6250.2$$

# Integer Powers: Application – Compound Interest

A quantity *K* which increases by *p%* per year will have increased after *t* years to

$$K\left(1+\frac{p}{100}\right)^t$$

Here, $1+\frac{p}{100}$ is called the *growth factor* for a growth of *p%.*

*Example:* A new car has been bought for \$15,000 and is assumed to decrease in value (depreciate) by 15% per year over a six-year period. What is its value after 6 years?

After six years we realize that its value must be $15{,}000(0.85)^6 \approx 5{,}657$

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# Integer Powers: Problem Set 4

(a) $12,000 deposit in an account earns 4% interest per year. What is the amount after 15 years?

(b) If the interest rate is 6% each year, how much money should you have deposited in a bank 5 years ago to have $50,000 today?

(c) A quantity increases by 25% each year for 3 years. How much is the combined percentage growth p over the three year period?

## Integer Powers: Problem Set 4 (Solution)

(a) \$12,000 deposit in an account earns 4% interest per year. What is the amount after 15 years?

(b) If the interest rate is 6% each year, how much money should you have deposited in a bank 5 years ago to have \$50,000 today?

(c) A quantity increases by 25% each year for 3 years. How much is the combined percentage growth p over the three year period?

(a) $12\,000 \cdot (1.04)^{15} \approx 21611.32$

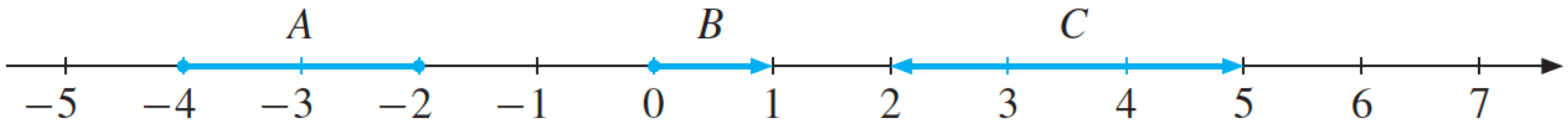
(b) $50\,000 \cdot (1.06)^{-5} \approx 37362.91$

(c) $p \approx 95.3\%$, since $(1.25)^3 = 1.9531$.

# Intervals

Let $a$ and $b$ be any two numbers on the real line. Then we call the set of all numbers that lie between $a$ and $b$ an interval . In many situations, it is important to distinguish between the intervals that include their endpoints and the intervals that do not. When $a < b$ , there are four different intervals that all have $a$ and $b$ as endpoints:

| | | |
|---|---|---|
| $(a, b)$ | The open interval from $a$ to $b$. | $a < x < b$ |
| $[a, b]$ | The closed interval from $a$ to $b$. | $a \leq x \leq b$ |
| $(a, b]$ | A half-open interval from $a$ to $b$. | $a < x \leq b$ |
| $[a, b)$ | A half-open interval from $a$ to $b$. | $a \leq x < b$ |




$A = [-4, -2]$, $B = [0, 1)$, and $C = (2, 5)$

# Summation Notation

Suppose $p$ and $q$ are integers with $q \geq p$. Then:

$$\sum_{i=p}^{q} a_i = a_p + a_{p+1} + \cdots + a_q$$

Example: Compute $\displaystyle\sum_{j=0}^{2} \frac{(-1)^j}{(j+1)(j+3)}$.

$$\sum_{j=0}^{2} \frac{(-1)^j}{(j+1)(j+3)} = \frac{1}{1 \cdot 3} + \frac{-1}{2 \cdot 4} + \frac{1}{3 \cdot 5} = \frac{40 - 15 + 8}{120} = \frac{33}{120} = \frac{11}{40}$$

# Summation Notation: Problem Set 5

Decide which of the following equalities are generally valid.

(a) $\sum_{k=1}^{n} ck^2 = c\sum_{k=1}^{n} k^2$

(b) $\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2$

(c) $\sum_{j=1}^{n} b_j + \sum_{j=n+1}^{N} b_j = \sum_{j=1}^{N} b_j$

(d) $\sum_{k=3}^{7} 5^{k-2} = \sum_{k=0}^{4} 5^{k+1}$

(e) $\sum_{i=0}^{n-1} a_{i,j}^2 = \sum_{k=1}^{n} a_{k-1,j}^2$

(f) $\sum_{k=1}^{n} \frac{a_k}{k} = \frac{1}{k}\sum_{k=1}^{n} a_k$

# Summation Notation: Problem Set 5

Decide which of the following equalities are generally valid.

(a) $\sum_{k=1}^{n} ck^2 = c\sum_{k=1}^{n} k^2$ (b) $\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2$ (c) $\sum_{j=1}^{n} b_j + \sum_{j=n+1}^{N} b_j = \sum_{j=1}^{N} b_j$

(d) $\sum_{k=3}^{7} 5^{k-2} = \sum_{k=0}^{4} 5^{k+1}$ (e) $\sum_{i=0}^{n-1} a_{i,j}^2 = \sum_{k=1}^{n} a_{k-1,j}^2$ (f) $\sum_{k=1}^{n} \frac{a_k}{k} = \frac{1}{k}\sum_{k=1}^{n} a_k$

(a), (c), (d), and (e) are always true; (b) and (f) are generally not true.

## Summation Operations

The following properties of the sigma notation are helpful when manipulating sums:

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \qquad \textbf{(additivity property)}$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i \qquad \textbf{(homogeneity property)}$$

Example combining both properties:

$$\sum_{i=1}^{n}(a_i + b_i - 2c_i + d) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i - 2\sum_{i=1}^{n} c_i + nd$$

## Summation Operations: Application – Mean and Variance

The **arithmetic mean** (or **mean**) $\mu_x$ of $T$ numbers $x_1, x_2, \ldots, x_T$ is their average, defined as the sum of all the numbers divided by $T$, the number of terms. That is, $\mu_x = \frac{1}{T}\sum_{i=1}^{T} x_i$.

Prove that $\sum_{i=1}^{T}(x_i - \mu_x) = 0$ and $\sum_{i=1}^{T}(x_i - \mu_x)^2 = \sum_{i=1}^{T} x_i^2 - T\mu_x^2$.

## Summation Operations: Application – Mean and Variance

Prove that $\sum_{i=1}^{T}(x_i - \mu_x) = 0$ and $\sum_{i=1}^{T}(x_i - \mu_x)^2 = \sum_{i=1}^{T} x_i^2 - T\mu_x^2$.

Solution: The difference $x_i - \mu_x$ is the deviation between $x_i$ and the mean. We prove first that the sum of these deviations is 0, using the foregoing definition of $\mu_x$:

$$\sum_{i=1}^{T}(x_i - \mu_x) = \sum_{i=1}^{T} x_i - \sum_{i=1}^{T} \mu_x = \sum_{i=1}^{T} x_i - T\mu_x = T\mu_x - T\mu_x = 0$$

## Summation Operations: Application – Mean and Variance

Prove that $\sum_{i=1}^{T}(x_i - \mu_x) = 0$ and $\sum_{i=1}^{T}(x_i - \mu_x)^2 = \sum_{i=1}^{T} x_i^2 - T\mu_x^2$.

Furthermore, the sum of the squares of the deviations is

$$\sum_{i=1}^{T}(x_i - \mu_x)^2 = \sum_{i=1}^{T}(x_i^2 - 2\mu_x x_i + \mu_x^2) = \sum_{i=1}^{T} x_i^2 - 2\mu_x \sum_{i=1}^{T} x_i + \sum_{i=1}^{T} \mu_x^2$$

$$= \sum_{i=1}^{T} x_i^2 - 2\mu_x T\mu_x + T\mu_x^2 = \sum_{i=1}^{T} x_i^2 - T\mu_x^2$$

Dividing by $T$, we see that the mean square deviation or **variance** $(1/T)\sum_{i=1}^{T}(x_i - \mu_x)^2$ must equal the mean square, $(1/T)\sum_{i=1}^{T} x_i^2$, minus the square of the mean, $\mu_x^2$.

# Summation Operations: Double-Sums

$$\text{Compute} \quad \sum_{i=1}^{3}\sum_{j=1}^{4}(i + 2j).$$

$$\sum_{i=1}^{3}\sum_{j=1}^{4}(i + 2j) = \sum_{i=1}^{3}\left[(i+2) + (i+4) + (i+6) + (i+8)\right]$$

$$= \sum_{i=1}^{3}(4i + 20) = 24 + 28 + 32 = 84$$

You should check that the result is the same by summing over $i$ first instead.

## Summation Operations: Problem Set 6

Consider a group of individuals each having a certain number of units of $m$ different goods. Let $a_{ij}$ denote the number of units of good $i$ owned by person $j$ $(i = 1, \ldots, m;\ j = 1, \ldots, n)$. Explain in words the meaning of the following sums:

(a) $\sum_{j=1}^{n} a_{ij}$ (b) $\sum_{i=1}^{m} a_{ij}$ (c) $\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}$

# Summation Operations: Problem Set 6 (Solution)

Consider a group of individuals each having a certain number of units of $m$ different goods. Let $a_{ij}$ denote the number of units of good $i$ owned by person $j$ ($i = 1, \dots, m$; $j = 1, \dots, n$). Explain in words the meaning of the following sums:

(a) $\sum_{j=1}^{n} a_{ij}$ (b) $\sum_{i=1}^{m} a_{ij}$ (c) $\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}$

(a) The total number of units of good *i*. (b) The total number of units of all goods owned by person *j*.

(c) The total number of units of goods owned by the group as a whole.

# Set Theory

Basic notation: $S = \{typical\ member : defining\ properties\}$

Set membership: $x \in S$ indicates that $x$ is an element of $S$

Let A and B be any two sets. Then A is a subset of B if it is true that every member of A is also a member of B . Then we write A ⊆ B . In particular, A ⊆ A . From the definitions we see that A = B if and only if A ⊆ B and B ⊆ A .

## Set Theory

| Notation | Name | The set consists of: |
|---|---|---|
| $A \cup B$ | $A$ **union** $B$ | The elements that belong to at least one of the sets $A$ and $B$ |
| $A \cap B$ | $A$ **intersection** $B$ | The elements that belong to both $A$ and $B$ |
| $A \setminus B$ | $A$ **minus** $B$ | The elements that belong to $A$, but not to $B$ |

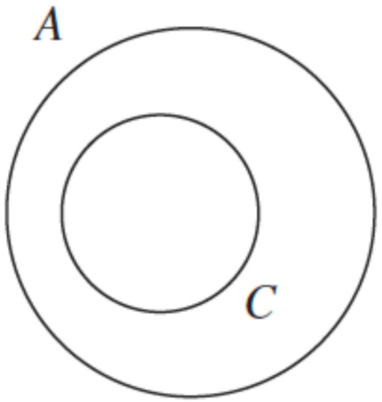
Thus,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$
$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

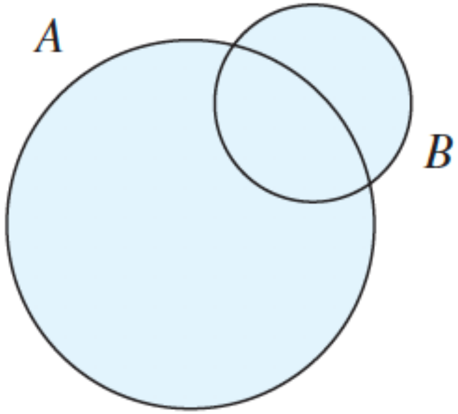
Example: Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 6\}$. Find $A \cup B$, $A \cap B$, $A \setminus B$, and $B \setminus A$.

$$A \cup B = \{1, 2, 3, 4, 5, 6\},\ A \cap B = \{3\},\ A \setminus B = \{1, 2, 4, 5\},\ B \setminus A = \{6\}.$$

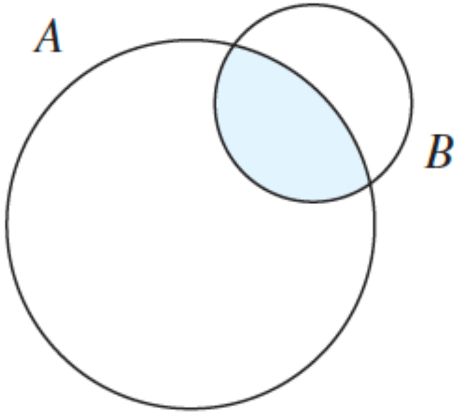
# Set Theory: Venn Diagrams



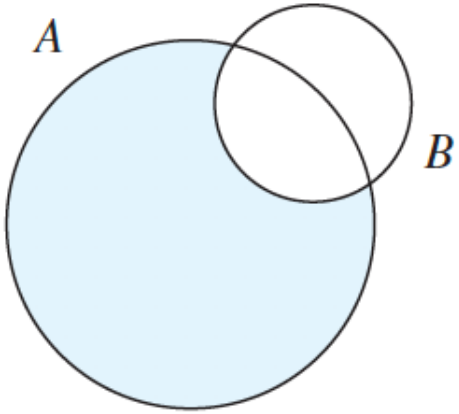

$C \subseteq A$




$A \cup B$




$A \cap B$



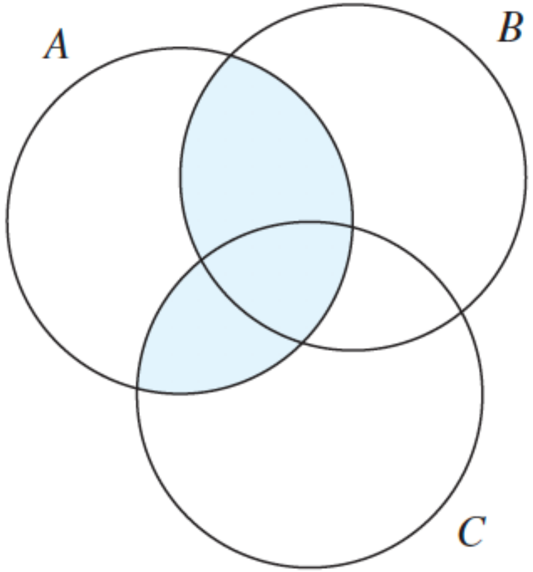

$A \setminus B$

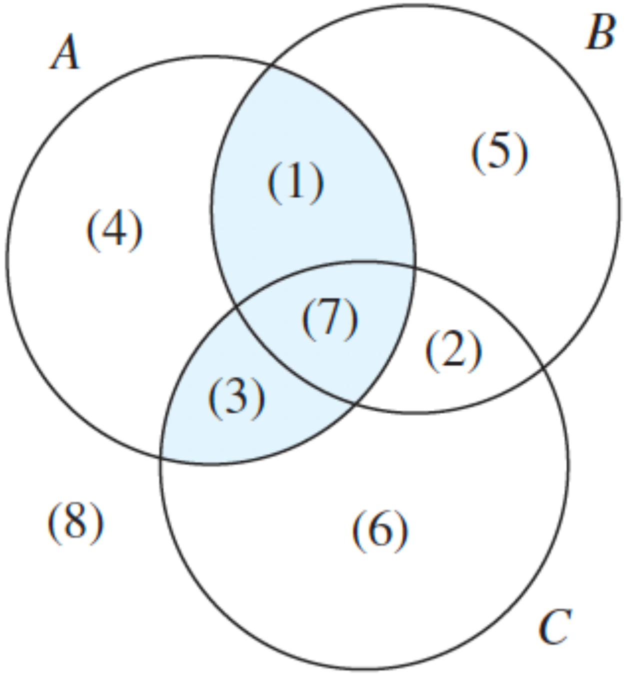
# Set Theory: Venn Diagrams

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(1) $(A \cap B) \setminus C$ (2) $(B \cap C) \setminus A$ (3) $(C \cap A) \setminus B$ (4) $A \setminus (B \cup C)$

(5) $B \setminus (C \cup A)$ (6) $C \setminus (A \cup B)$ (7) $A \cap B \cap C$ (8) $\complement(A \cup B \cup C)$



# Set Theory: Problem Set 7

Let $A = \{2, 3, 4\}$, $B = \{2, 5, 6\}$, $C = \{5, 6, 2\}$, and $D = \{6\}$.

(a) Determine which of the following statements are true: $4 \in C$; $5 \in C$; $A \subseteq B$; $D \subseteq C$; $B = C$; and $A = B$.

(b) Find $A \cap B$; $A \cup B$; $A \setminus B$; $B \setminus A$; $(A \cup B) \setminus (A \cap B)$; $A \cup B \cup C \cup D$; $A \cap B \cap C$; and $A \cap B \cap C \cap D$.

# Set Theory: Problem Set 7 - Solution

Let $A = \{2, 3, 4\}$, $B = \{2, 5, 6\}$, $C = \{5, 6, 2\}$, and $D = \{6\}$.

(a) Determine which of the following statements are true: $4 \in C$; $5 \in C$; $A \subseteq B$; $D \subseteq C$; $B = C$; and $A = B$.

(b) Find $A \cap B$; $A \cup B$; $A \setminus B$; $B \setminus A$; $(A \cup B) \setminus (A \cap B)$; $A \cup B \cup C \cup D$; $A \cap B \cap C$; and $A \cap B \cap C \cap D$.

(a) $5 \in C$, $D \subseteq C$, and $B = C$ are true. The three others are false. (b) $A \cap B = \{2\}$, $A \cup B = \{2, 3, 4, 5, 6\}$, $A \setminus B = \{3, 4\}$, $B \setminus A = \{5, 6\}$, $(A \cup B) \setminus (A \cap B) = \{3, 4, 5, 6\}$, $A \cup B \cup C \cup D = \{2, 3, 4, 5, 6\}$, $A \cap B \cap C = \{2\}$, and $A \cap B \cap C \cap D = \emptyset$.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