RWTH BUSINESS SCHOOL

Mathematics & Statistics
M.Sc. Data Analytics and Decision Science
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AACSB ACCREDITED

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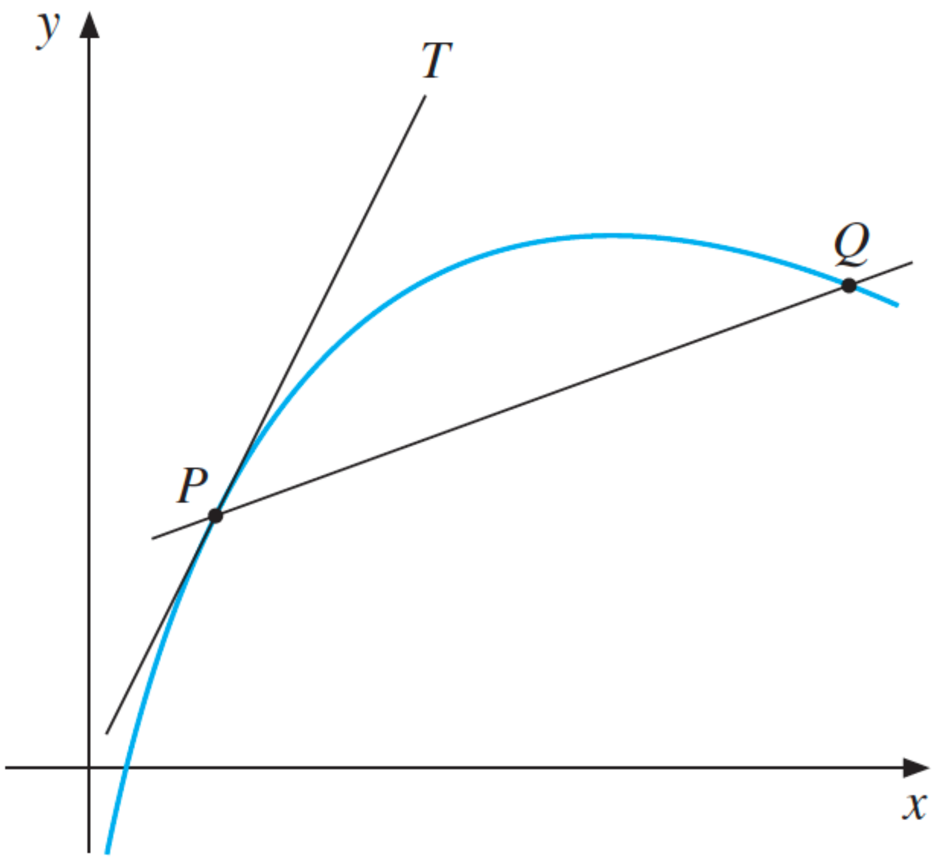
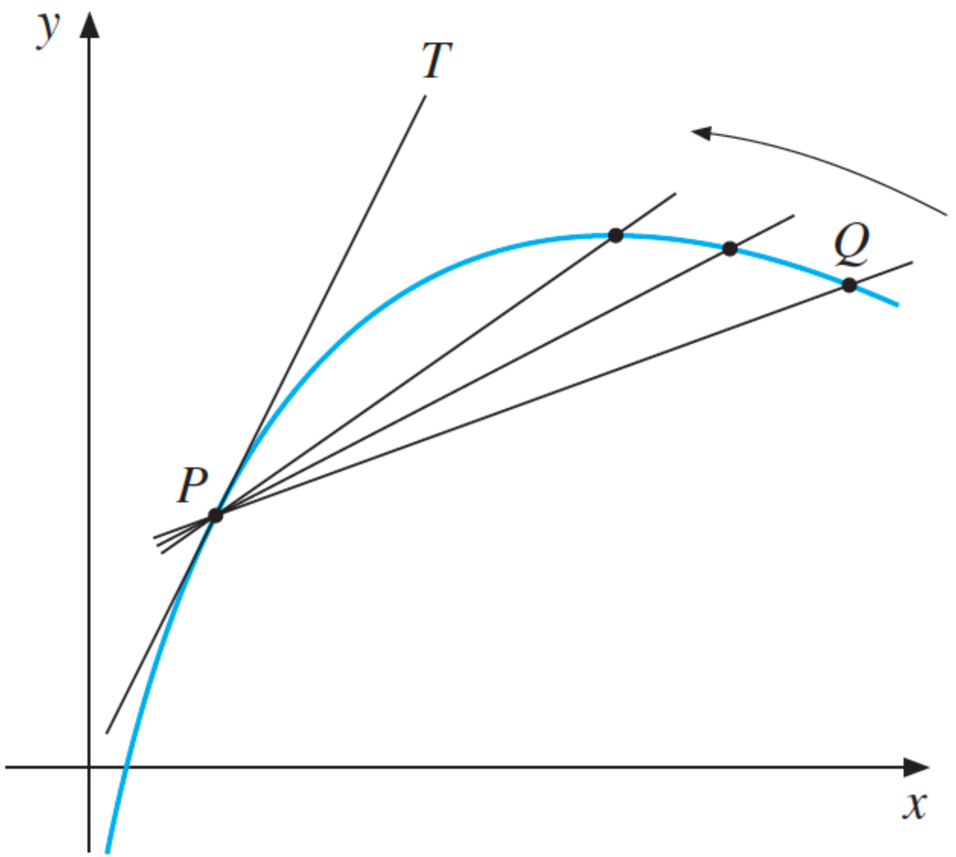
Outline

- Tangents and Derivatives
- Differentiation

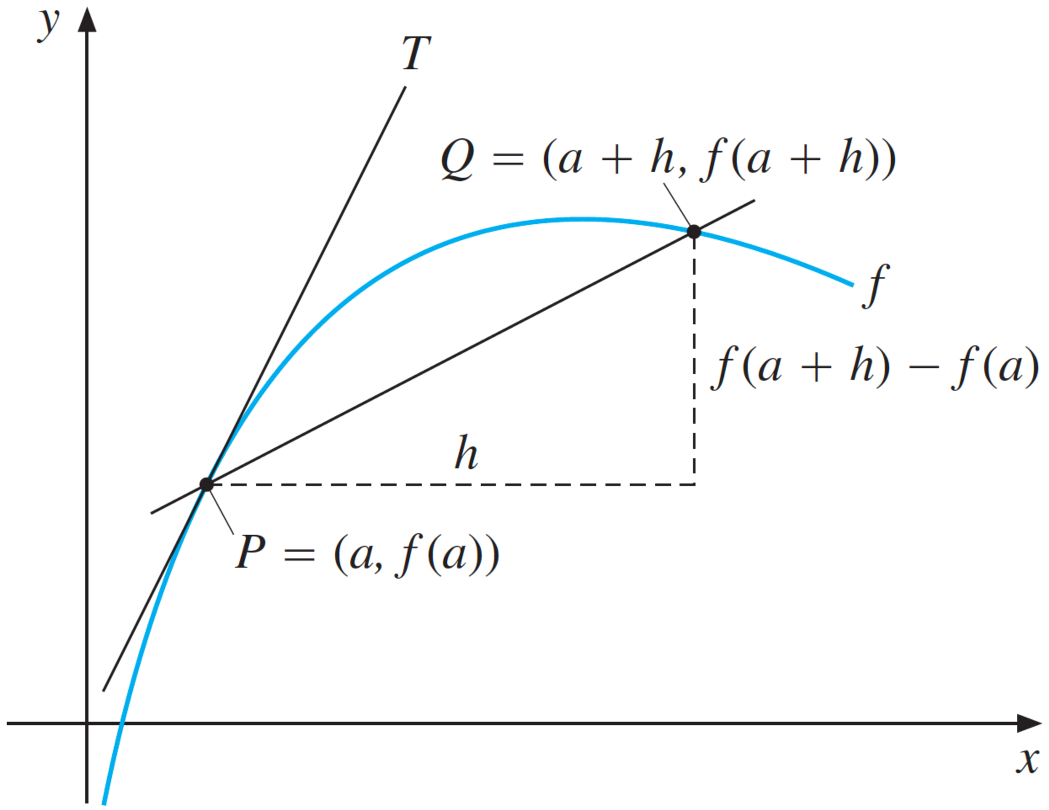
Tangents and Derivatives





Tangents and Derivatives

Suppose that the x-coordinate of Q is $a + h$, where h is a small number $\neq 0$. Then the x-coordinate of Q is not a (because $Q \neq P$), but a number close to a. Because Q lies on the graph of f, the y-coordinate of Q is equal to $f(a+h)$. Hence, the point Q has coordinates $\big(a+h, f(a+h)\big)$. The slope m_{PQ} of the secant PQ is therefore



$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

Tangents and Derivatives

This fraction is often called a **Newton quotient** of f. Note that when $h = 0$, the fraction becomes 0/0 and so is undefined. But choosing $h = 0$ corresponds to letting $Q = P$. When Q moves toward P along the graph of f, the x-coordinate of Q, which is $a + h$, must tend to a, and so h tends to 0. Simultaneously, the secant PQ tends to the tangent to the graph at P. This suggests that we ought to *define* the slope of the tangent at P as the number that m_{PQ} approaches as h tends to 0. In the previous section we called this number $f'(a)$. So we propose the following definition of $f'(a)$:

$$f'(a) = \begin{Bmatrix} \text{the limit as } h \\ \text{tends to 0 of} \end{Bmatrix} \frac{f(a+h) - f(a)}{h}$$

It is common to use the abbreviated notation $\lim_{h\to 0}$, or $\lim\limits_{h\to 0}$, for "the limit as h tends to zero" of an expression involving h. We therefore have the following definition:

The derivative of the function f at point a, denoted by $f'(a)$, is given by the formula

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Recipe for Computing f'(a)

(A) Add h to a and compute $f(a+h)$.

(B) Compute the corresponding change in the function value: $f(a+h) - f(a)$.

(C) For $h \neq 0$, form the Newton quotient $\dfrac{f(a+h) - f(a)}{h}$.

(D) Simplify the fraction in step (C) as much as possible. Wherever possible, cancel h from the numerator and denominator.

(E) Then $f'(a)$ is the limit of $\dfrac{f(a+h) - f(a)}{h}$ as h tends to 0.

Problem Set 1

Compute $f'(a)$ when $f(x) = x^3$.

Problem Set 1 - Solution

Compute $f'(a)$ when $f(x) = x^3$.

We follow the recipe in (3).

(A) $f(a+h) = (a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$

(B) $f(a+h) - f(a) = (a^3 + 3a^2h + 3ah^2 + h^3) - a^3 = 3a^2h + 3ah^2 + h^3$

(C)–(D) $\dfrac{f(a+h) - f(a)}{h} = \dfrac{3a^2h + 3ah^2 + h^3}{h} = 3a^2 + 3ah + h^2$

(E) As h tends to 0, so $3ah + h^2$ also tends to 0; hence, the entire expression $3a^2 + 3ah + h^2$ tends to $3a^2$. Therefore, $f'(a) = 3a^2$.

Tangents and Derivatives

If we use y to denote the typical value of the function $y = f(x)$, we often denote the derivative simply by y'. We can then write $y = x^3 \Longrightarrow y' = 3x^2$.

Several other forms of notation for the derivative are often used in mathematics and its applications. One of them, originally due to Leibniz, is called the **differential notation**. If $y = f(x)$, then in place of $f'(x)$, we write

$$\frac{dy}{dx} = dy/dx \qquad \text{or} \qquad \frac{df(x)}{dx} = df(x)/dx \qquad \text{or} \qquad \frac{d}{dx} f(x)$$

For instance, if $y = x^2$, then

$$\frac{dy}{dx} = 2x \qquad \text{or} \qquad \frac{d}{dx}(x^2) = 2x$$

Rules of Differentiation

$$y = A + f(x) \implies y' = f'(x) \qquad \text{(Additive constants disappear)}$$

$$y = Af(x) \implies y' = Af'(x) \qquad \text{(Multiplicative constants are preserved)}$$

$$f(x) = x^a \implies f'(x) = ax^{a-1} \qquad (a \text{ is an arbitrary constant})$$

If both f and g are differentiable at x, then the sum $f + g$ and the difference $f - g$ are both differentiable at x, and

$$F(x) = f(x) \pm g(x) \implies F'(x) = f'(x) \pm g'(x)$$

Rules of Differentiation

If both f and g are differentiable at the point x, then so is $F = f \cdot g$, and

$$F(x) = f(x) \cdot g(x) \Longrightarrow F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

If f and g are differentiable at x and $g(x) \neq 0$, then $F = f/g$ is differentiable at x, and

$$F(x) = \frac{f(x)}{g(x)} \Longrightarrow F'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$$

Differentiation – Problem Set 2

Differentiate the functions defined by the various formulas.

(a) $\dfrac{1}{x^6}$ (b) $x^{-1}(x^2+1)\sqrt{x}$ (c) $\dfrac{1}{\sqrt{x^3}}$ (d) $\dfrac{x+1}{x-1}$

(e) $\dfrac{x+1}{x^5}$ (f) $\dfrac{3x-5}{2x+8}$ (g) $3x^{-11}$ (h) $\dfrac{3x-1}{x^2+x+1}$

Differentiation – Problem Set 2 (Solution)

Differentiate the functions defined by the various formulas.

(a) $\dfrac{1}{x^6}$ (b) $x^{-1}(x^2+1)\sqrt{x}$ (c) $\dfrac{1}{\sqrt{x^3}}$ (d) $\dfrac{x+1}{x-1}$

(e) $\dfrac{x+1}{x^5}$ (f) $\dfrac{3x-5}{2x+8}$ (g) $3x^{-11}$ (h) $\dfrac{3x-1}{x^2+x+1}$

(a) $-6x^{-7}$ (b) $\frac{3}{2}x^{1/2}-\frac{1}{2}x^{-3/2}$ (c) $-(3/2)x^{-5/2}$

(d) $-2/(x-1)^2$ (e) $-4x^{-5}-5x^{-6}$ (f) $34/(2x+8)^2$

(g) $-33x^{-12}$ (h) $(-3x^2+2x+4)/(x^2+x+1)^2$

Differentiation – Chain Rule & Generalized Power Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$ **(Chain Rule)**

$$y = u^a \Longrightarrow y' = au^{a-1}u' \qquad (u = g(x))$$ **(Generalized Power Rule)**

Differentiation – Problem Set 2 (Solution)

Differentiate the functions

(a) $y = (x^3 + x^2)^{50}$ (b) $y = \left(\dfrac{x-1}{x+3}\right)^{1/3}$

(a) $y = (x^3 + x^2)^{50} = u^{50}$ where $u = x^3 + x^2$, so $u' = 3x^2 + 2x$. Then (2) gives

$$y' = 50u^{50-1} \cdot u' = 50(x^3 + x^2)^{49}(3x^2 + 2x)$$

(b) Again we use (2): $y = \left(\dfrac{x-1}{x+3}\right)^{1/3} = u^{1/3}$ where $u = \dfrac{x-1}{x+3}$. The quotient rule gives

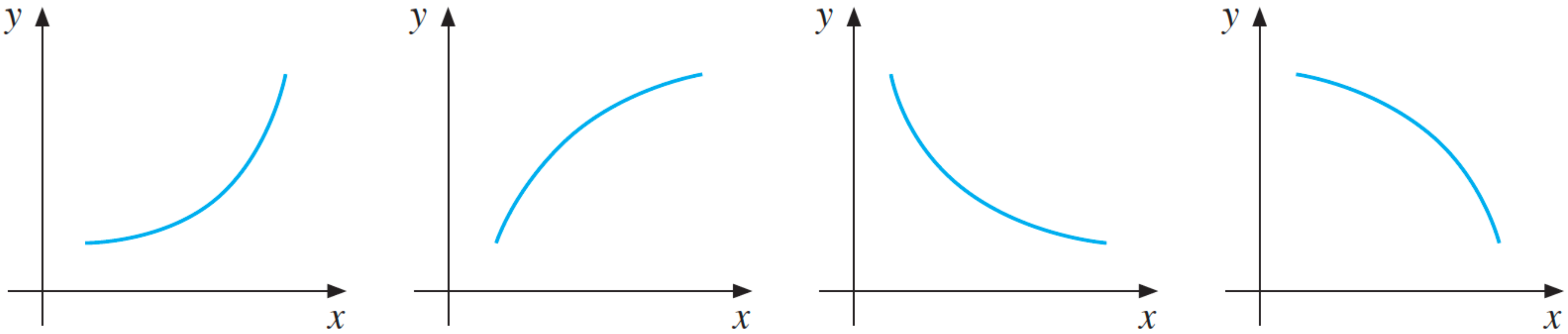
$$u' = \frac{1 \cdot (x+3) - (x-1) \cdot 1}{(x+3)^2} = \frac{4}{(x+3)^2}$$

and hence

$$y' = \frac{1}{3}u^{(1/3)-1} \cdot u' = \frac{1}{3}\left(\frac{x-1}{x+3}\right)^{-2/3} \cdot \frac{4}{(x+3)^2}$$

Differentiation – Concavity and Convexity (2nd Order Derivative)

$$f''(x) = \frac{d^2 f(x)}{dx^2} = d^2 f(x)/dx^2 \qquad \text{or} \qquad y'' = \frac{d^2 y}{dx^2} = d^2 y/dx^2$$



Increasing, convex

Increasing, concave

Decreasing, convex

Decreasing, concave

Differentiation – Problem Set 3

Check the convexity/concavity of the following functions:

(a) $f(x) = x^2 - 2x + 2$ and (b) $f(x) = ax^2 + bx + c$

Differentiation – Problem Set 3 (Solution)

Check the convexity/concavity of the following functions:

(a) $f(x) = x^2 - 2x + 2$ and (b) $f(x) = ax^2 + bx + c$

(a) Here $f'(x) = 2x - 2$ so $f''(x) = 2$. Because $f''(x) > 0$ for all x, f is convex.

(b) Here $f'(x) = 2ax + b$, so $f''(x) = 2a$. If $a = 0$, then f is linear. In this case, the function f meets both the definitions in (3), so it is both concave and convex. If $a > 0$, then $f''(x) > 0$, so f is convex. If $a < 0$, then $f''(x) < 0$, so f is concave. The last two cases are illustrated by the graphs in Fig. 4.6.1.

Differentiation – Exponential and Logarithmic Differentiation

The natural exponential function

$$f(x) = \exp(x) = e^x \qquad (e = 2.71828\ldots)$$

is differentiable, strictly increasing and convex. In fact,

$$f(x) = e^x \implies f'(x) = f(x) = e^x$$

The following properties hold for all exponents s and t:

$$\text{(a)}\ \ e^s e^t = e^{s+t} \qquad \text{(b)}\ \ e^s/e^t = e^{s-t} \qquad \text{(c)}\ \ (e^s)^t = e^{st}$$

Moreover,

$$e^x \to 0 \quad \text{as} \quad x \to -\infty\,, \qquad e^x \to \infty \quad \text{as} \quad x \to \infty$$

Differentiation – Problem Set 4

Find the first and second derivatives of (a) $y = x^3 + e^x$ (b) $y = x^5 e^x$ (c) $y = e^x / x$

Differentiation – Problem Set 4 (Solution)

Find the first and second derivatives of (a) $y = x^3 + e^x$ (b) $y = x^5 e^x$ (c) $y = e^x/x$

(a) We find that $y' = 3x^2 + e^x$ and $y'' = 6x + e^x$.

(b) We have to use the product rule: $y' = 5x^4 e^x + x^5 e^x = x^4 e^x (5 + x)$. To compute y'', we differentiate $y' = 5x^4 e^x + x^5 e^x$ once more to obtain

$$y'' = 20x^3 e^x + 5x^4 e^x + 5x^4 e^x + x^5 e^x = x^3 e^x (x^2 + 10x + 20)$$

(c) The quotient rule yields

$$y = \frac{e^x}{x} \implies y' = \frac{e^x x - e^x \cdot 1}{x^2} = \frac{e^x (x-1)}{x^2}$$

Differentiating $y' = \dfrac{e^x x - e^x}{x^2}$ once more w.r.t. x gives

$$y'' = \frac{(e^x x + e^x - e^x) x^2 - (e^x x - e^x) 2x}{(x^2)^2} = \frac{e^x (x^2 - 2x + 2)}{x^3}$$

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Differentiation – Exponential and Logarithmic Differentiation

$$g(x) = \ln x \implies g'(x) = \frac{1}{x}$$

$$y = a^x \Rightarrow y' = a^x \ln a$$

Differentiation – Problem Set 5

Compute y' and y'' when: (a) $y = x^3 + \ln x$ (b) $y = x^2 \ln x$ (c) $y = \ln x / x$.

Differentiation – Problem Set 5 (Solution)

Compute y' and y'' when: (a) $y = x^3 + \ln x$ (b) $y = x^2 \ln x$ (c) $y = \ln x / x$.

(a) We find easily that $y' = 3x^2 + 1/x$. Furthermore, $y'' = 6x - 1/x^2$.

(b) The product rule gives

$$y' = 2x \ln x + x^2(1/x) = 2x \ln x + x$$

Differentiating the last expression w.r.t. x gives $y'' = 2 \ln x + 2x(1/x) + 1 = 2 \ln x + 3$.

(c) Here we use the quotient rule:

$$y' = \frac{(1/x)x - (\ln x) \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

Differentiating again yields

$$y'' = \frac{-(1/x)x^2 - (1 - \ln x)2x}{(x^2)^2} = \frac{2 \ln x - 3}{x^3}$$

Differentiation – Problem Set 6

Find the derivative of:

(a) $\ln(\ln x)$ (b) $\ln\sqrt{1-x^2}$ (c) $e^x \ln x$ (d) $e^{x^3}\ln x^2$

(e) $\ln(e^x+1)$ (f) $\ln(x^2+3x-1)$ (g) $2(e^x-1)^{-1}$ (h) e^{2x^2-x}

Differentiation – Problem Set 6 (Solution)

Find the derivative of:

(a) $\ln(\ln x)$ (b) $\ln\sqrt{1-x^2}$ (c) $e^x \ln x$ (d) $e^{x^3}\ln x^2$

(e) $\ln(e^x+1)$ (f) $\ln(x^2+3x-1)$ (g) $2(e^x-1)^{-1}$ (h) e^{2x^2-x}

(a) $1/(x\ln x)$ (b) $-x/(1-x^2)$ (c) $e^x(\ln x + 1/x)$

(d) $e^{x^3}\left(3x^2\ln x^2 + 2/x\right)$ (e) $e^x/(e^x+1)$

(f) $(2x+3)/(x^2+3x-1)$ (g) $-2e^x(e^x-1)^{-2}$ (h $(4x-1)e^{2x^2-x}$

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