RWTH BUSINESS SCHOOL

Mathematics & Statistics
M.Sc. Data Analytics and Decision Science
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AACSB ACCREDITED

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The Job Interview

You have one position to fill and you want to hire a new employee. There are n applicants are waiting in the lobby and you may interview them one-by-one, individually. Here are some assumptions and rules:

- After each interview, you must make a binary decision: *hire* or *pass*.
- If you select *hire*, the interview process immediately stops.
- If you *pass* on a candidate, you won't get a second chance with them.
- You have no idea of the distribution of talent in the pool, but you do know the total number of candidates. (Just assume an ordinal ranking from 1… n).

Your objective is to try to select the very best candidate from the applicant group.

1. What is the optimal strategy to maximize your chances of selecting the top candidate?
2. What is the probability of success with this strategy?
3. What happens to this strategy as n changes?

The Job Interview - Some Trivial strategies

Suppose you select totally at random. What is the chance of hiring the best applicant? $\boxed{1/n}$

Suppose you select the first candidate you see. What is the chance of hiring the best applicant? $\boxed{1/n}$

The Job Interview - Some Trivial strategies

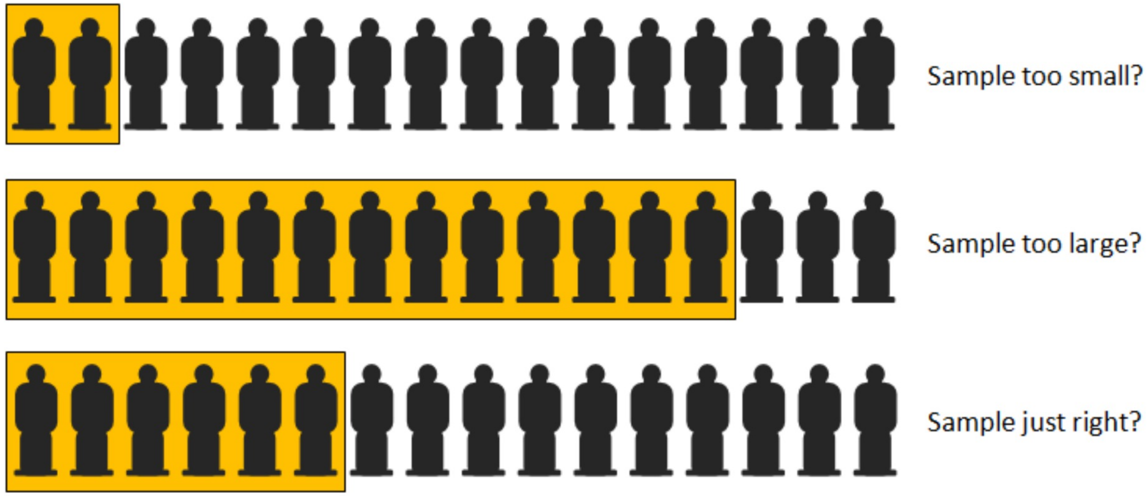
Suppose you select totally at random. What is the chance of hiring the best applicant?

1/n

Suppose you select the first candidate you see. What is the chance of hiring the best applicant?

1/n

It seems like taking a sample first and then hiring someone better than the sample is a good strategy. But what is the best sample size?



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The Job Interview - Some Non-Trivial strategies

If the sample size is too small, then the number of candidates not in the sample is very large

If the sample size is too large, it is quite probable that this sample will include the best candidate

The Job Interview - The Best Strategy (Trivial Cases)

For n =1:

- We will always end up with the best candidate

For n =2:

- The chances are 50% to pick the best candidate, regardless of the sample size (i.e., 1 or 2).

Solve for n = 3

The Job Interview - The Best Strategy With 3 Candidates

Sample size: 0

1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

Success: 0.334%

Sample size: 1

1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

Success: 0.5%

Sample size: 2

1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

Success: 0.334%

1 Candidate hired

n Sample candidate

n Best candidate chosen

The Job Interview - The Best Strategy With 4 Candidates

Sample size: 0

1	2	3	4
1	2	4	3
1	3	2	4
1	3	4	2
1	4	2	3
1	4	3	2
2	1	3	4
2	1	4	3
2	3	1	4
2	3	4	1
2	4	1	3
2	4	3	1
3	1	2	4
3	1	4	2
3	2	1	4
3	2	4	1
3	4	1	2
3	4	2	1
4	1	2	3
4	1	3	2
4	2	1	3
4	2	3	1
4	3	1	2
4	3	2	1

Success: 0.25%

Sample size: 1

1	2	3	4
1	2	4	3
1	3	2	4
1	3	4	2
1	4	2	3
1	4	3	2
2	1	3	4
2	1	4	3
2	3	1	4
2	3	4	1
2	4	1	3
2	4	3	1
3	1	2	4
3	1	4	2
3	2	1	4
3	2	4	1
3	4	1	2
3	4	2	1
4	1	2	3
4	1	3	2
4	2	1	3
4	2	3	1
4	3	1	2
4	3	2	1

Success: 0.46%

Sample size: 2

1	2	3	4
1	2	4	3
1	3	2	4
1	3	4	2
1	4	2	3
1	4	3	2
2	1	3	4
2	1	4	3
2	3	1	4
2	3	4	1
2	4	1	3
2	4	3	1
3	1	2	4
3	1	4	2
3	2	1	4
3	2	4	1
3	4	1	2
3	4	2	1
4	1	2	3
4	1	3	2
4	2	1	3
4	2	3	1
4	3	1	2
4	3	2	1

Success: 0.42%

Sample size: 3

1	2	3	4
1	2	4	3
1	3	2	4
1	3	4	2
1	4	2	3
1	4	3	2
2	1	3	4
2	1	4	3
2	3	1	4
2	3	4	1
2	4	1	3
2	4	3	1
3	1	2	4
3	1	4	2
3	2	1	4
3	2	4	1
3	4	1	2
3	4	2	1
4	1	2	3
4	1	3	2
4	2	1	3
4	2	3	1
4	3	1	2
4	3	2	1

Success: 0.25%

The Job Interview - The Best Strategy With n Candidates

Candidates	Sample	Candidates /Sample	Success
3	1	0.33	0.50
4	2	0.25	0.49
5	2	0.40	0.44
6	2	0.44	0.43
7	2	0.28	0.41
8	3	0.38	0.41
8	3	0.33	0.41
10	3	0.3	0.40
11	4	0.36	0.40
12	4	0.33	0.40
13	5	0.38	0.39
14	5	0.36	0.39
15	5	0.33	0.39
16	6	0.38	0.39
17	6	0.35	0.39
18	6	0.33	0.39
19	7	0.37	0.39
20	7	0.35	0.38

For n = 100:

- Candidates/Sample = 0.3679
- Success = 0.3679

0.3679 = 1 / e

Sample Exercises:

1. Explain intuitively the difference between “classical probability”, “empirical probability” and “subjective probability”
2. What is the concept behind Benford’s Law and why is it useful?
3. Explain intuitively why the Schlitz brewing company dared to conduct a live beer tasting with testers who originally favored the competitor’s beer.
4. The following table represents the distribution of women and men among the programs EMBA, MME, MBA.

	EMBA	MME	MBA	Total
Women	0.1	0.2		
Men				0.6
Total	0.3		0.2	

a) Fill in the intersection and marginal probabilities in the empty cells.
b) What is the probability that a man is enrolled in the EMBA Program?
c) What is the probability that an EMBA student is a woman?

In Class Exercise: Return on an Investment Fund

Consider the following data on an investment fund:

Return Scenarios	P(R=r)	F(R)
-8.0	0.100	0.1000
-5.0	0.100	0.2000
-2.0	0.200	0.4000
5.0	**0.300**	0.7000
16.0	0.200	0.9000
20.0	0.100	1.0000

E.g., the probability of a 16% return is 20%. The probability of a -8% return Is 10%.

a) Compute the Expected return on investing in this fund.

b) Compute the variance and standard deviation of the return.

In Class Exercise: Return on an Investment Fund

a) Compute the Expected return on investing in this fund.

b) Compute the variance and standard deviation of the return.

Return Scenarios	P(R=r)	$F(R)$	$r \times P(R = r)$
-8.0	0.100	0.1000	-0.8
-5.0	0.100	0.2000	-0.5
-2.0	0.200	0.4000	-0.4
5.0	**0.300**	0.7000	1.5
16.0	0.200	0.9000	3.2
20.0	0.100	1.0000	2
			5
			E(R)

In Class Exercise: Return on an Investment Fund

a) Compute the Expected return on investing in this fund.

b) Compute the variance and standard deviation of the return.

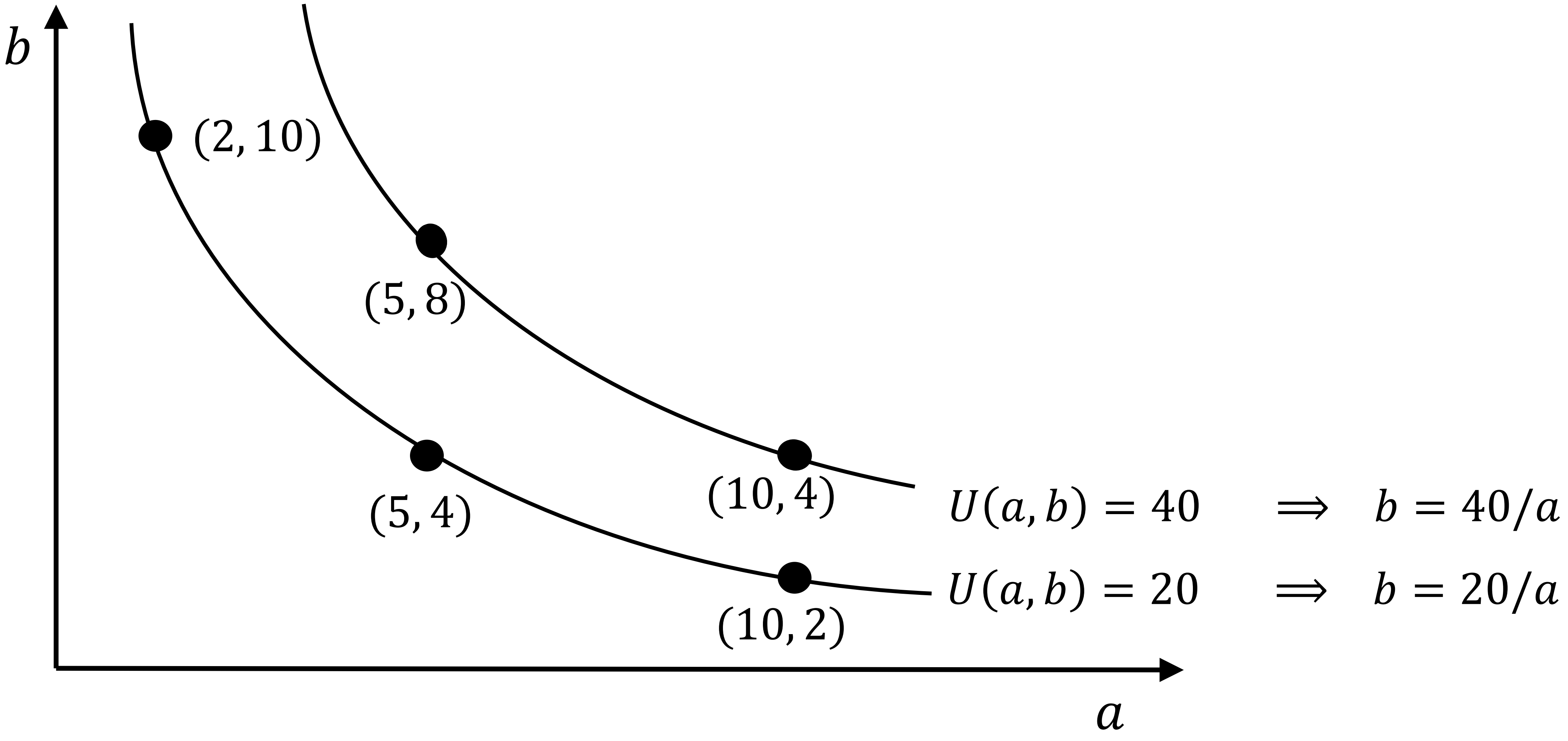
Return Scenarios	**P(R=r)**	$F(R)$	$r \times P(R=r)$	$(r - E(R))^2 \times P(R=r)$
-8.0	0.100	0.1000	-0.8	16.9
-5.0	0.100	0.2000	-0.5	10
-2.0	0.200	0.4000	-0.4	9.8
5.0	**0.300**	0.7000	1.5	0
16.0	0.200	0.9000	3.2	24.2
20.0	0.100	1.0000	2	22.5
			5	83.4000
			E(R)	**Var(R)**
				9.132360045
				SD(R)

Decisions under Risk: The Utility Function (2 Goods)

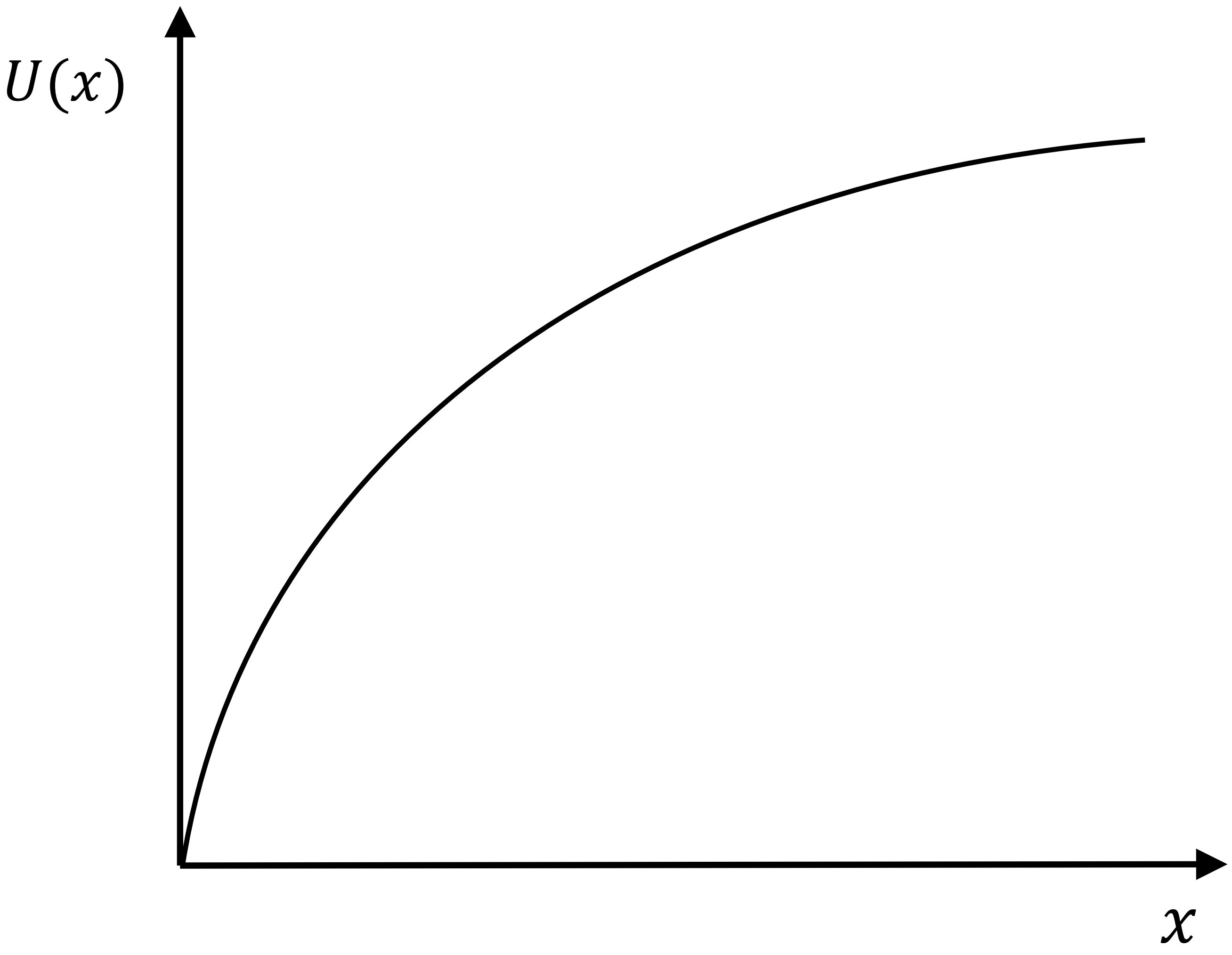
Utility Function:

- Assigns a level of utility to each consumption bundle.
- Example: $U(a,b) = a \cdot b$
- With $U(a,b)$ the consumption bundle $(2,10)$ would be equally preferred as bundle $(4,5)$.
- The consumer is said to be *indifferent* between those bundles and e.g., $(10,2)$ and $(20,1)$.
- Since all bundles yield $U(a,b) = 20$, those bundles can be represented by an iso-utility curve, which in economics is also called *indifference curve.*

Specifying Preferences and Objectives: Indifference curves



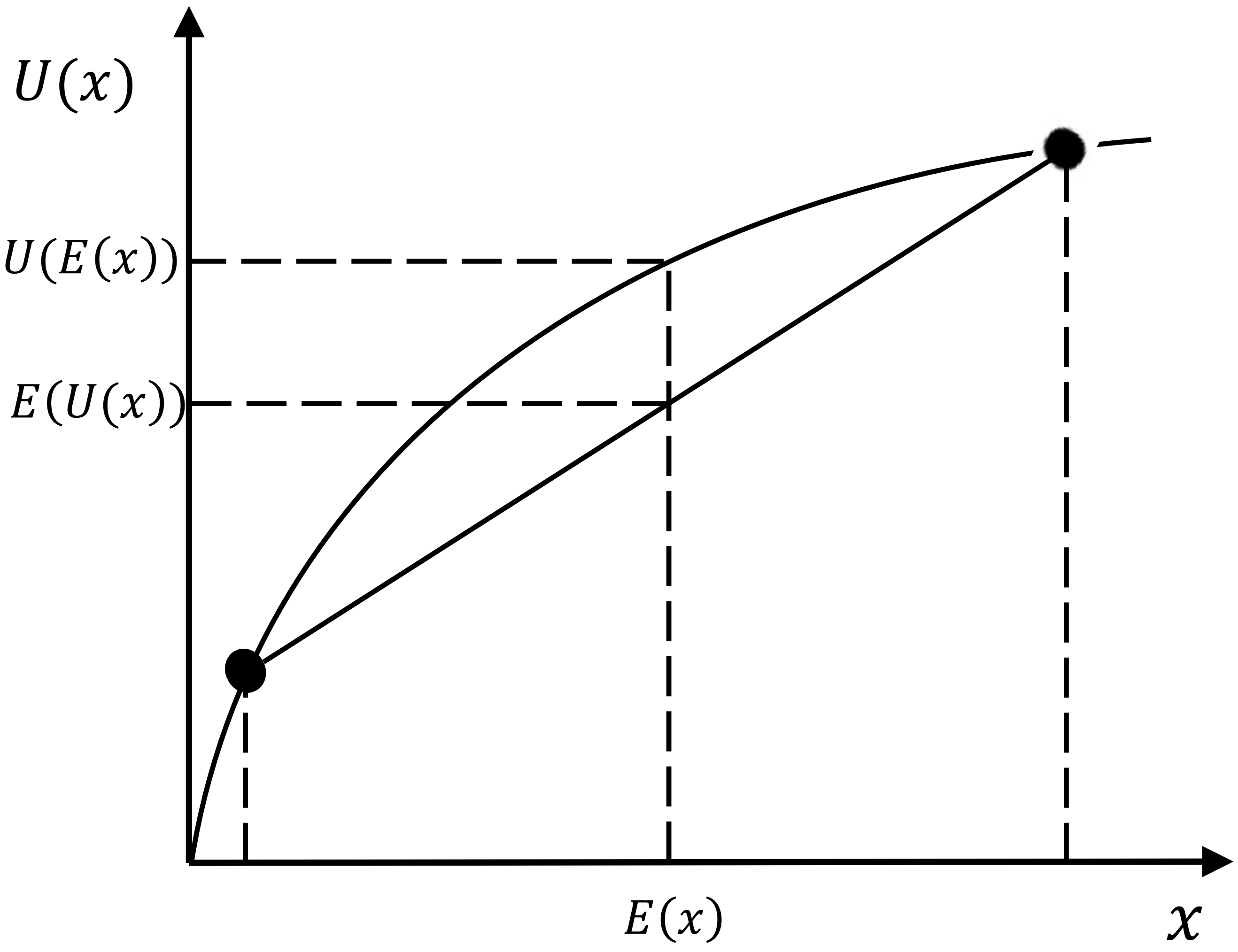
Specifying Preferences and Objectives: Utility Function of Income



Some general properties of a standard utility function on are:

- $U(0) = 0$
- $U_x > 0$, the marginal utility is positive
- $U_x(0) = \infty$
- $U_{xx} < 0$, the marginal utility is diminishing

Specifying Preferences and Objectives: Risk Aversion



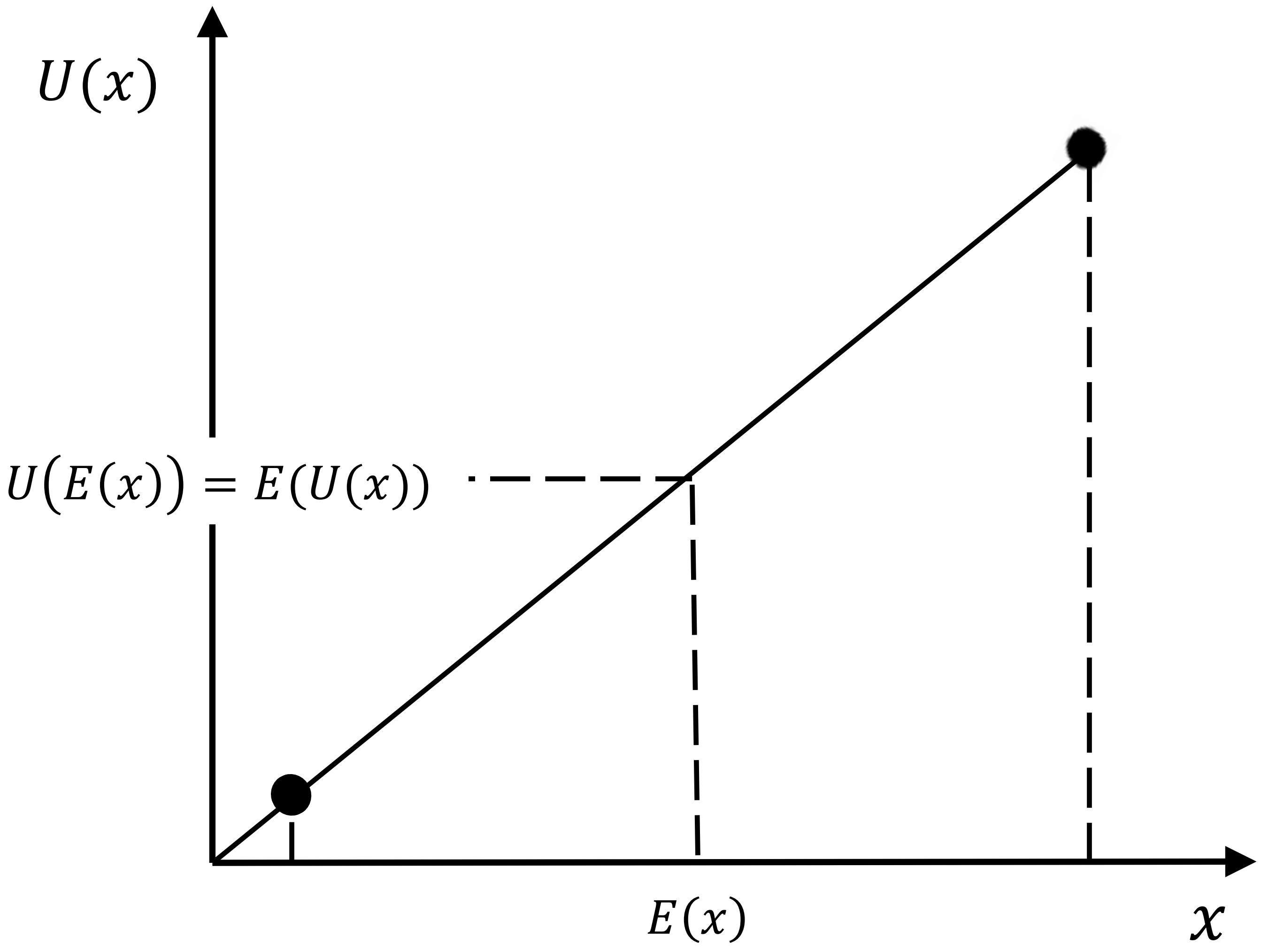
Suppose you face two payoffs with the same expected value:

- Payoff 1: $x = 1$ or $x = 11$, each with 50%
- Payoff 2: $x = 6$ with 100%

The individual is said to be *risk averse* if she/he prefers the less risky payoff 2

- Given the expected value $E(x)$, the utility of expected value is $U(E(x))$ while the expected utility is $E(U(x))$
- It holds that: $U\big(E(x)\big) > E(U(x))$
- Example: $U(x) = \ln(x)$, $U(x) = \sqrt{x}$

Specifying Preferences and Objectives: Risk Neutrality & Utility of Income



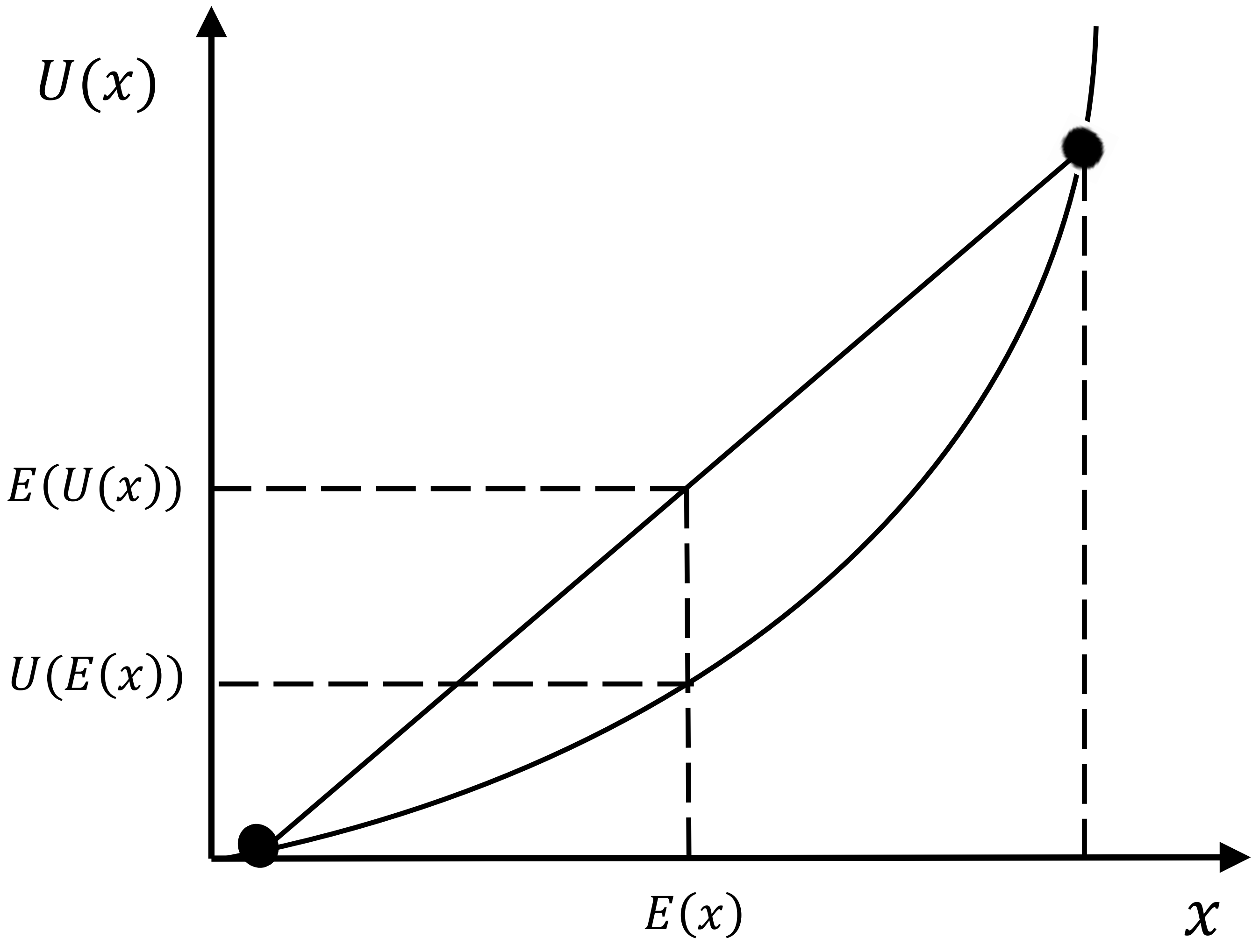
Suppose you face two payoffs with the same expected value:

- Payoff 1: $x = 1$ or $x = 11$, each with 50%
- Payoff 2: $x = 6$ with 100%

The individual is said to be *risk neutral* if she/he is indifferent between these Payoffs

- Given the expected value $E(x)$, the utility of expected value is $U(E(x))$ while the expected utility is $E(U(x))$
- It holds that: $U\big(E(x)\big) = E(U(x))$
- Example: $U(x) = \mathrm{x}$

Specifying Preferences and Objectives: Risk Aversion & Utility of Income



Suppose you face two payoffs with the same expected value:

- Payoff 1: $x = 1$ or $x = 11$, each with 50%
- Payoff 2: $x = 6$ with 100%

The individual is said to be *risk seeking* if she/he prefers the risky payoff 1

- Given the expected value $E(x)$, the utility of expected value is $U(E(x))$ while the expected utility is $E(U(x))$
- It holds that: $U\big(E(x)\big) < E(U(x))$
- Example: $U(x) = e^x$, $U(x) = x^2$

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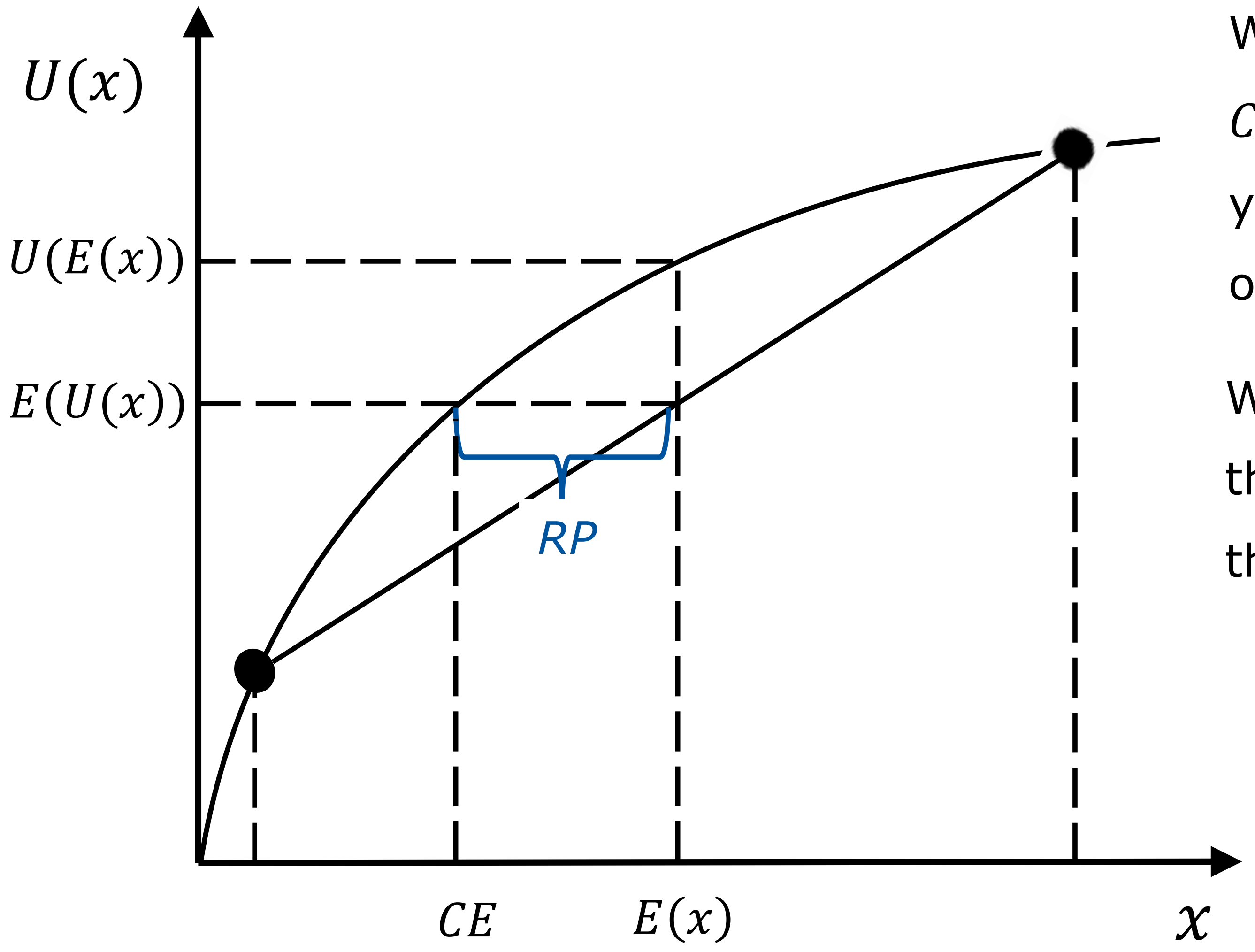
Specifying Preferences and Objectives: Certainty Equivalent & Risk Premium



With risk aversion, the *certainty equivalent* CE is the payoff which which (with certainty) yields the same utility as the expected utility of the gamble.

With risk aversion, the *risk premium (RP)* is the maximum willingness to pay to eliminate the risk.

$$CE + \mathrm{RP} = \mathrm{E(x)}$$

$$U(CE) = E(U(x))$$

Specifying Preferences and Objectives — The Parking Ticket

You are on you way downtown and need to park your car for the day. If you do not buy a parking ticket, there is a 50% chance of being fined with €16. Your disposable budget for the day is €25. Your utility function is $U(x) = \sqrt{x}$, where x denotes your disposable income. What is the maximum amount you are willing to pay for the parking ticket?

Specifying Preferences and Objectives : The Parking Ticket

You are on you way downtown and need to park your car for the day. If you do not buy a parking ticket, there is a 50% chance of being fined with €16. Your disposable budget for the day is €25. Your utility function is $U(x) = \sqrt{x}$, where x denotes your disposable income. What is the maximum amount you are willing to pay for the parking ticket?

If you buy the ticket, your utility for the day will be $\mathrm{U}(25 - t) = \sqrt{25 - t}$

If you don't buy the ticket, your expected utility for the day will be $\underbrace{\frac{1}{2}\mathrm{U}(25 - 16) + \frac{1}{2}U(25) = 4}_{E(U(x))}$

$$U(CE) = E\big(U(x)\big) \implies \sqrt{25 - t} = 4 \implies \mathrm{t} = 9$$

$$E(x) = \frac{1}{2}(25 - 16) + \frac{1}{2}(25) = 17$$

Managing Constraints and Tradeoffs

Your utility function is given by $U(x) = \sqrt{x}$. Suppose you have the option to choose from the following 3 jobs. Compute the expected utility and the expected Income for each job.

Income	Probability: 0.5	Probability: 0.5
Job 1	144	49
Job 2	121	64
Job 3	100	81

Your utility function is given by $U(x) = \sqrt{x}$. Suppose you have the option to choose from the following 3 jobs. Compute the expected utility and the expected Income for each job.

Income	Probability: 0.5	Probability: 0.5
Job 1	144	49
Job 2	121	64
Job 3	100	81

Job 1: $E(U(x)) = 0.5(\sqrt{144} + \sqrt{49}) = 9.5$

$E(x) = 0.5\,(144 + 49) = 96.5$

Job 2: $E(U(x)) = 0.5(\sqrt{121} + \sqrt{64}) = 9.5$

$E(x) = 0.5\,(121 + 64) = 92.5$

Job 3: $E(U(x)) = 0.5(\sqrt{100} + \sqrt{81}) = 9.5$

$E(x) = 0.5\,(100 + 81) = 90.5$

Compute the standard deviation for each job offer

Income	Probability: 0.5	Probability: 0.5	$E(U(x))$	$E(x)$
Job 1	144	49	9.5	96.5
Job 2	121	64	9.5	92.5
Job 3	100	81	9.5	90.5

Compute the standard deviation for each job offer

Income	Probability: 0.5	Probability: 0.5	$E(U(x))$	$E(x)$
Job 1	144	49	9.5	96.5
Job 2	121	64	9.5	92.5
Job 3	100	81	9.5	90.5

Job 1: $\sigma(x) = \sqrt{0.5(144 - 96.5)^2 + 0.5(49 - 96.5)^2} = 47.5$

Job 2: $\sigma(x) = \sqrt{0.5(121 - 92.5)^2 + (64 - 92.5)^2} = 28.5$

Job 1: $\sigma(x) = \sqrt{0.5(100 - 90.5)^2 + 0.5(81 - 90.5)^2} = 9.5$

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Compute the standard deviation for each job offer

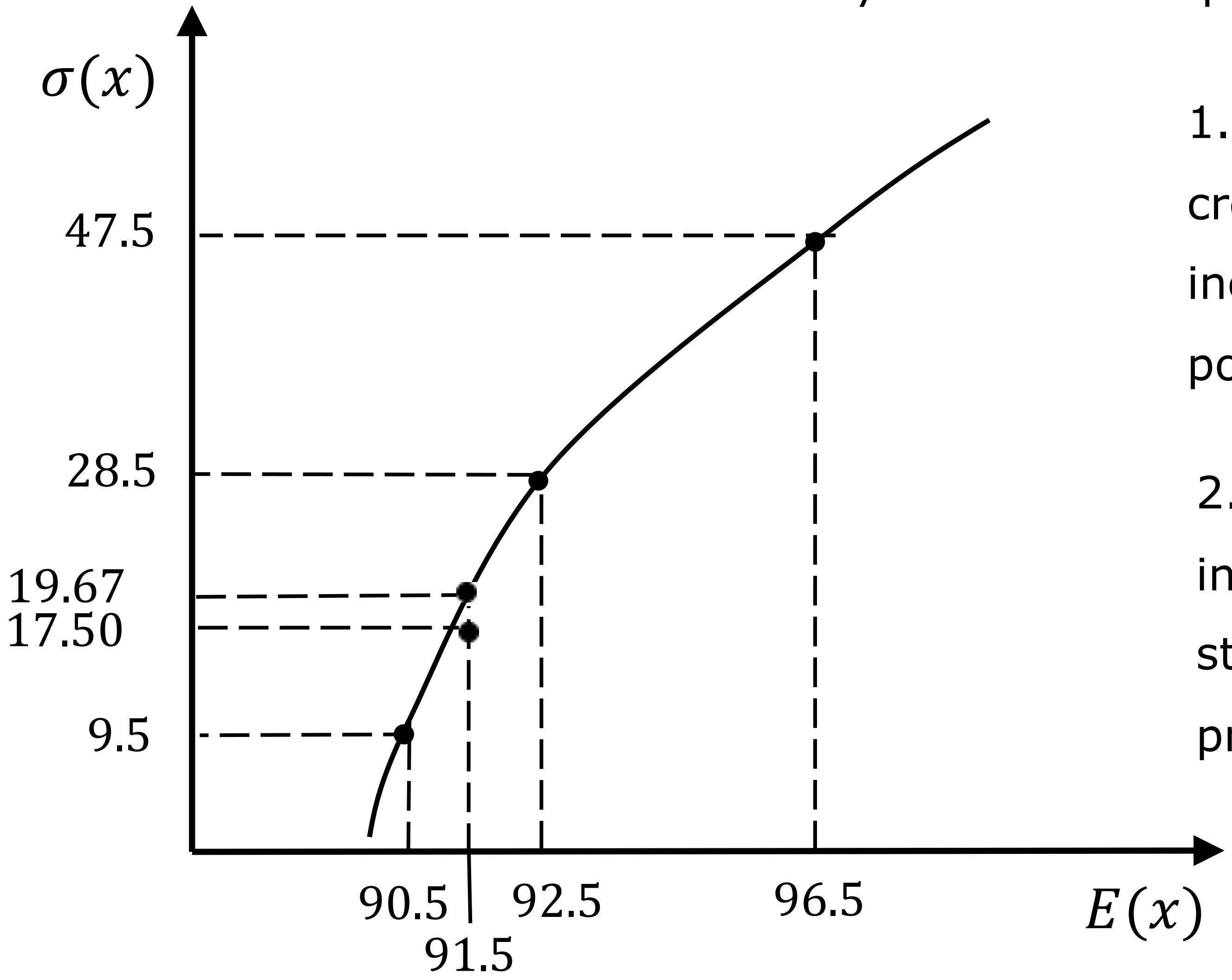
Income	Probability: 0.5	Probability: 0.5	$E(U(x))$	$E(x)$	$\sigma(x)$
Job 1	144	49	9.5	96.5	47.5
Job 2	121	64	9.5	92.5	28.5
Job 3	100	81	9.5	90.5	9.5

Suppose you are offered job 4 with an expected income of 91.5 and a standard deviation of 17.5.

Should you rather accept job 4?

Managing Constraints and Tradeoffs: Expected Income & Variation

Suppose you are offered job 4 with an expected income of 91.5 and a standard deviation of 17.5. Should you rather accept job 4?



1. We can use the information on Jobs 1-3 to create an indifference curve of expected income and standard deviation bundles. All points along this cure yield the same utility.

2. Since Job 4 is situated below the indifference curve of expected income and standard deviation bundles, it is strictly preferred to Jobs 1-3.

Managing Constraints and Tradeoffs: Portfolio Diversification

The following table summarizes possible returns on a fixed investment budget into each stock, respectively. There are two possible return scenarios, given economic conditions, each with a 50% chance.

Returns	Economy 1	Economy 2
Stock 1	4	3
Stock 2	3	4

The expected return and standard deviation of investing only in one of the stocks are given by

$$E(S_1) = 3.5 \qquad E(S_2) = 3.5$$

$$\sigma(S_1) = 0.5 \qquad \sigma(S_2) = 0.5$$

Analyze if a diversification strategy of investing half of the research budget into each stock would a good alternative

Managing Constraints and Tradeoffs: Portfolio Diversification

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$$\sigma(S_1) = 0.5 \qquad \sigma(S_2) = 0.5$$

Analyze if a diversification strategy of investing half of the research budget into each stock would a good alternative

$$E(D) = 0.5(0.5(4+3) + 0.5(4+3)) = 3.5$$

$$\sigma(D) = \sqrt{0.5(3.5-3.5)^2 + 0.5(3.5-3.5)^2} = 0$$

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Introduction: Two Systems of Thinking — Count the Numbers

Which number has been tossed?

Introduction: Two Systems of Thinking — Count the Numbers

Which number has been tossed?

Introduction: Two Systems of Thinking

Kahnemann (2011) elaborates on the idea that human thinking is channeled via two very distinct systems: System 1 and System 2

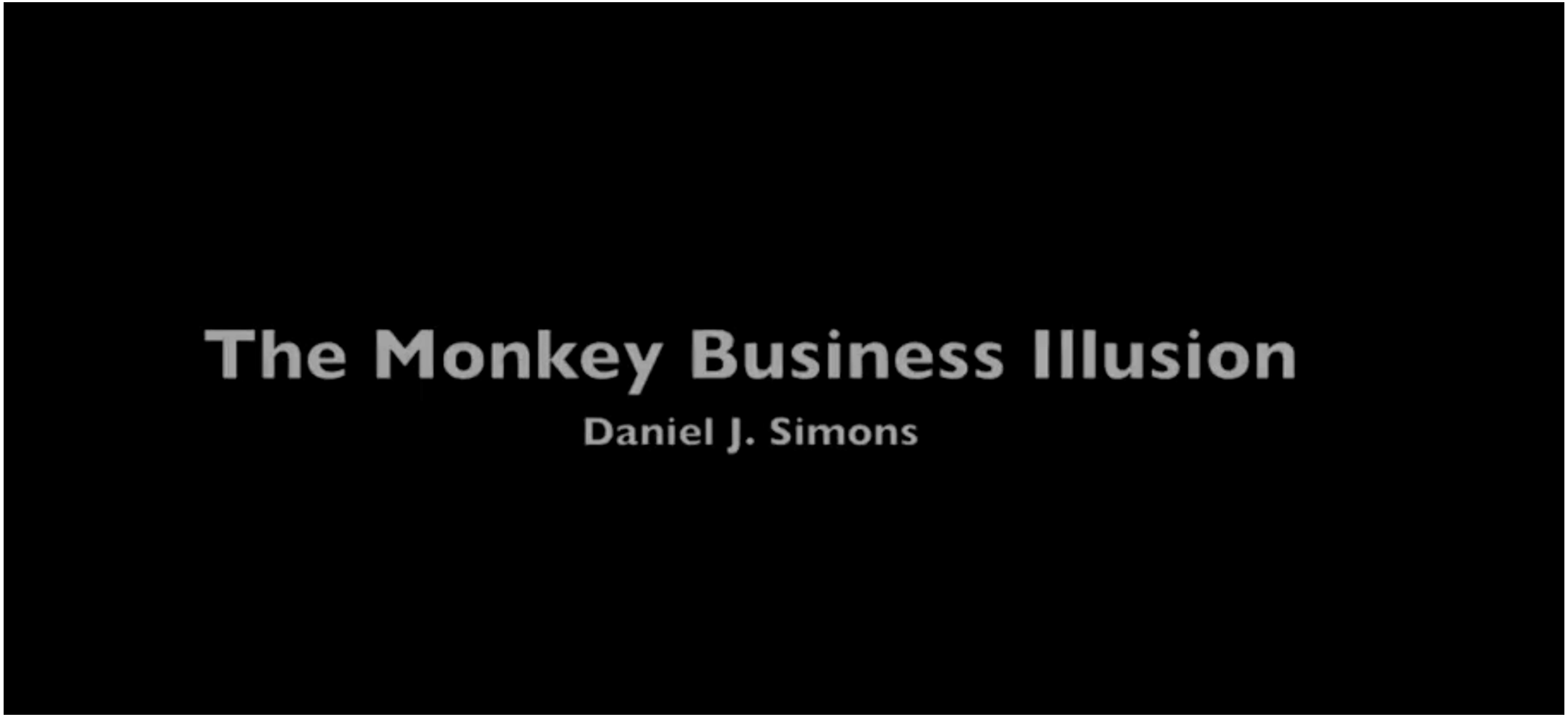
System 1:

- Fast
- Intuitive
- Automatic
- Emotional

System 2:

- Slow
- rational
- Controlled
- Logical

Introduction: Two Systems of Thinking — Video



Introduction: Two Systems of Thinking

About half of the people watching the monkey illusion video do not see the gorilla.

- System 2 has a limited ability to process information. Everything else is blurred or ignored.
- System 2 also gets tired pretty fast.
- System 1 is constantly monitoring everything and tries to suggest solutions without much effort.
- System 2 is called for help if System 1 notices something unusual.

In decision-making tasks we need to decide what to focus on.

Introduction: Bounded Rationality

Simon (1957) argues that decision makers cannot actually make optimal decision

People still try to decide rationally. However, there might be several reasons preventing them from making optimal decisions:

Limited resources

- Time: People lack the time necessary to acquire full information, to exactly specify their goals and to study all possible alternatives
- Cost: Some information or access to optimal strategic plans must be purchased

Limited abilities

- intellect: people might lack the intellectual ability to process information
- Subjective bias: people make errors in accessing their situation and the consequences of their decision.

Managing Hidden Information: Moral Hazard – Example: Principal & Agent

Principal

- Hires the agent
- Cannot completely observe the effort of the agent, car, life
- Wants the agent to perform the best work possible with the highest effort

Agent

- Is aware of his/her level of skills
- Needs proper incentives to perform the best work possible with the highest effort

Managing Hidden Information: Moral Hazard – Example: Principal & Agent

Suppose you (principal) are the owner of an old-fashioned travel agency. You employ one worker (agent) who runs the agency. If the worker's effort is high, the daily profit is €1.000 with a 75% probability and €400 with a 25% probability. If the worker's effort level is low, the profit probabilities are reversed.

The worker has a preference for low effort (basically shirking the whole day) and perceives the costs of high effort to be €100. Unfortunately, you cannot observe your agent's effort. Currently, you are paying a daily wage of €150. What is your expected profit?

Managing Hidden Information: Moral Hazard – Example: Principal & Agent

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The worker has a preference for low effort (basically shirking the whole day) and perceives the costs of high effort to be €100. Unfortunately, you cannot observe your agent's effort. Currently, you are paying a daily wage of €150. What is your expected profit?

Given the fixed wage, the worker has no incentives for high effort as the worker's profit is €150 with low effort and €50 with high effort. Thus your expected profit will be:

0.25(€1.000) + 0.75(€400) - €150 = €400

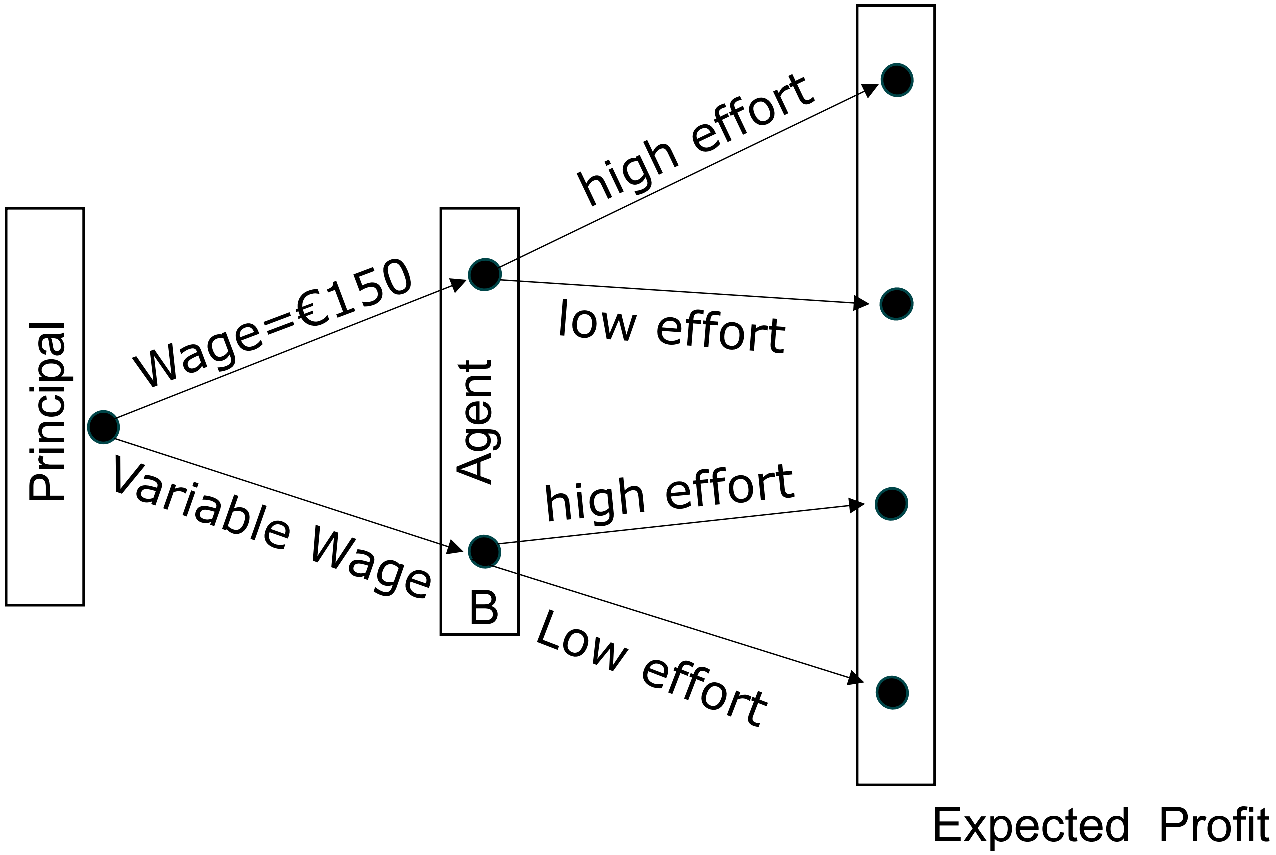
Managing Hidden Information: Moral Hazard – Example: Principal & Agent

How could you incentivize the worker to have a high effort?

Managing Hidden Information: Moral Hazard – Example: Principal & Agent

How could you incentivize the worker to have a high effort?

Suppose you offer a variable wage: €200 if profit is high and €0 if profit is low.



Managing Hidden Information: Moral Hazard – Example: Principal & Agent

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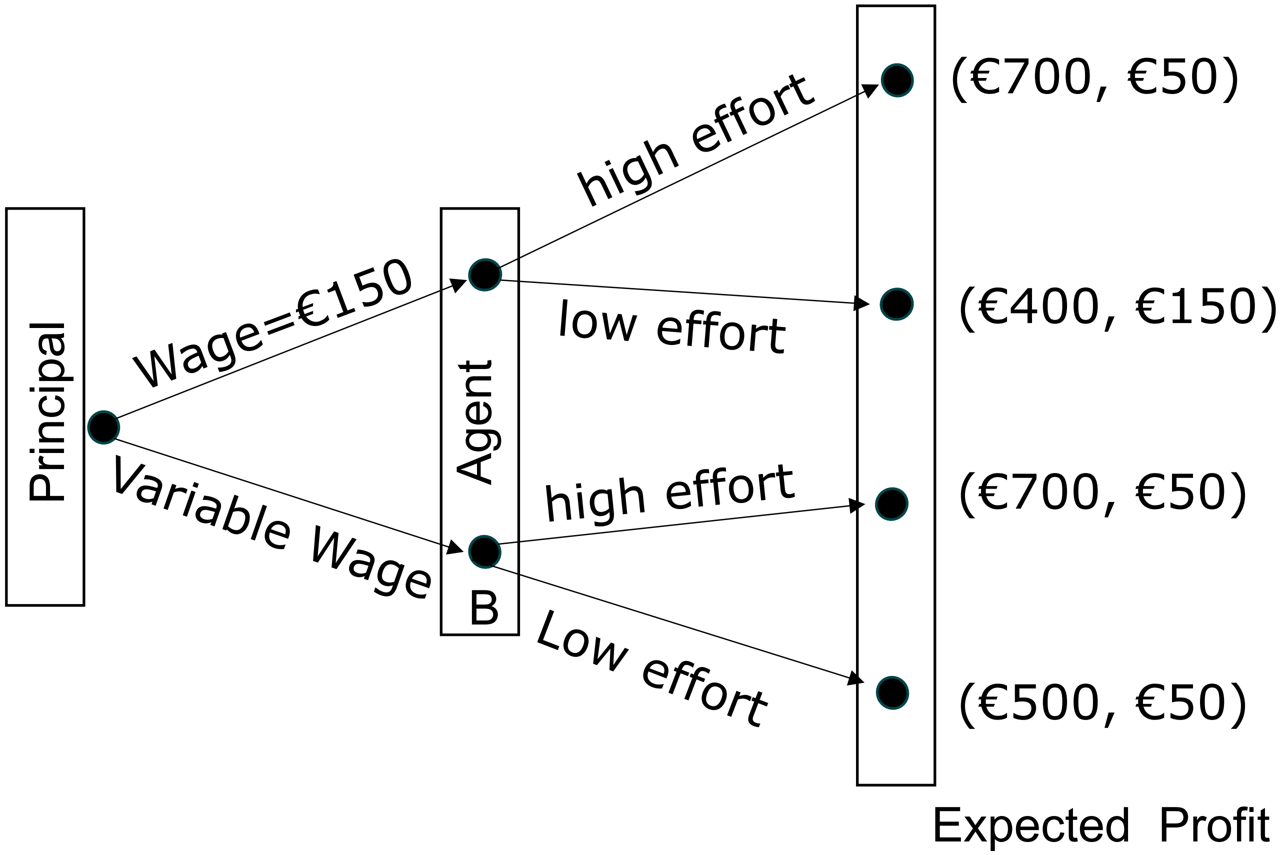


.75(1.000) + .25(400) – 150, 150-100

.25(1.000) + .75(400) – 150, 150

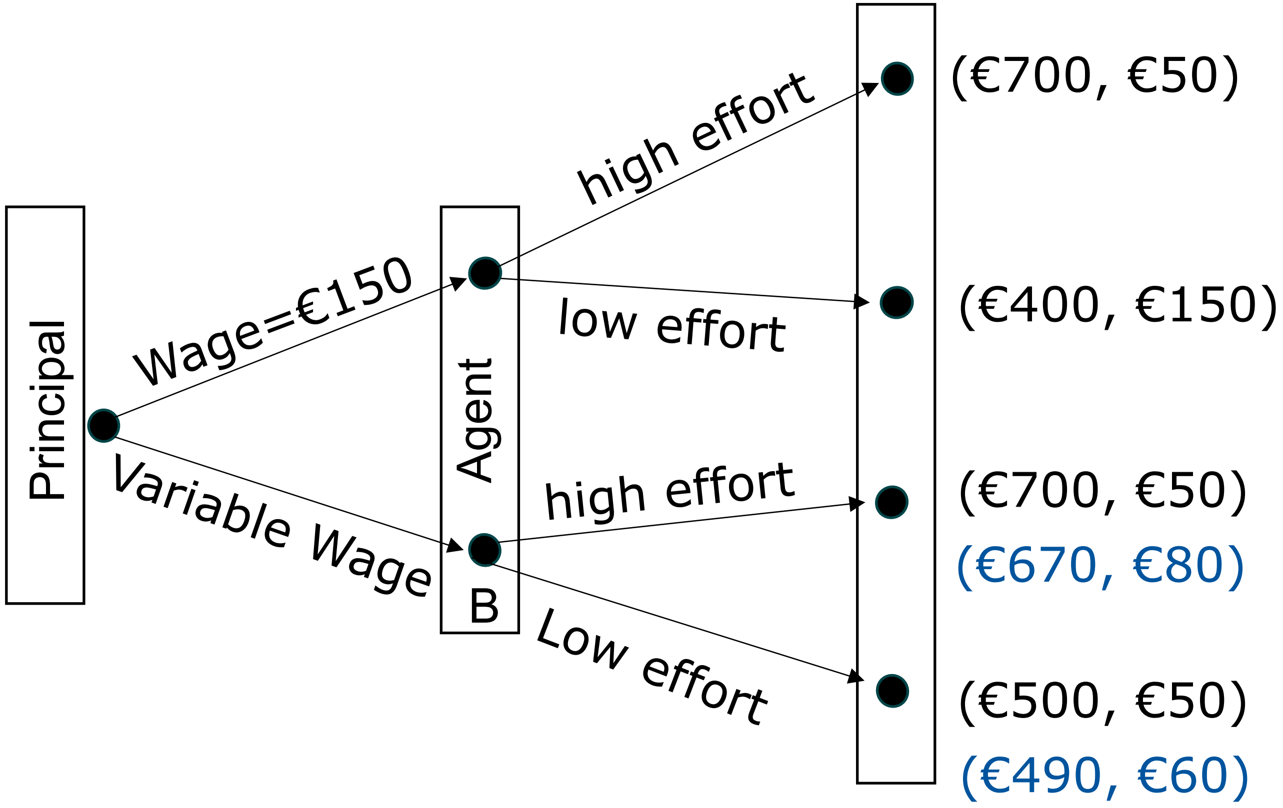
.75(100-200)+.25(400), .75(200)+.25(0) -100

.25(1000-200)+.75(400), .25(200)+.75(0)

Managing Hidden Information: Moral Hazard – Example: Principal & Agent

How could you incentivize the worker to have a high effort?

Suppose you offer a variable wage: €200 if profit is high and €0 if profit is low.



.75(1.000) + .25(400) – 150, 150-100

.25(1.000) + .75(400) – 150, 150

.75(100-200)+.25(400), .75(200)+.25(0) -100

.75(100-240)+.25(400), .75(240)+.25(0) -100

.25(1000-200)+.75(400), .25(200)+.75(0)

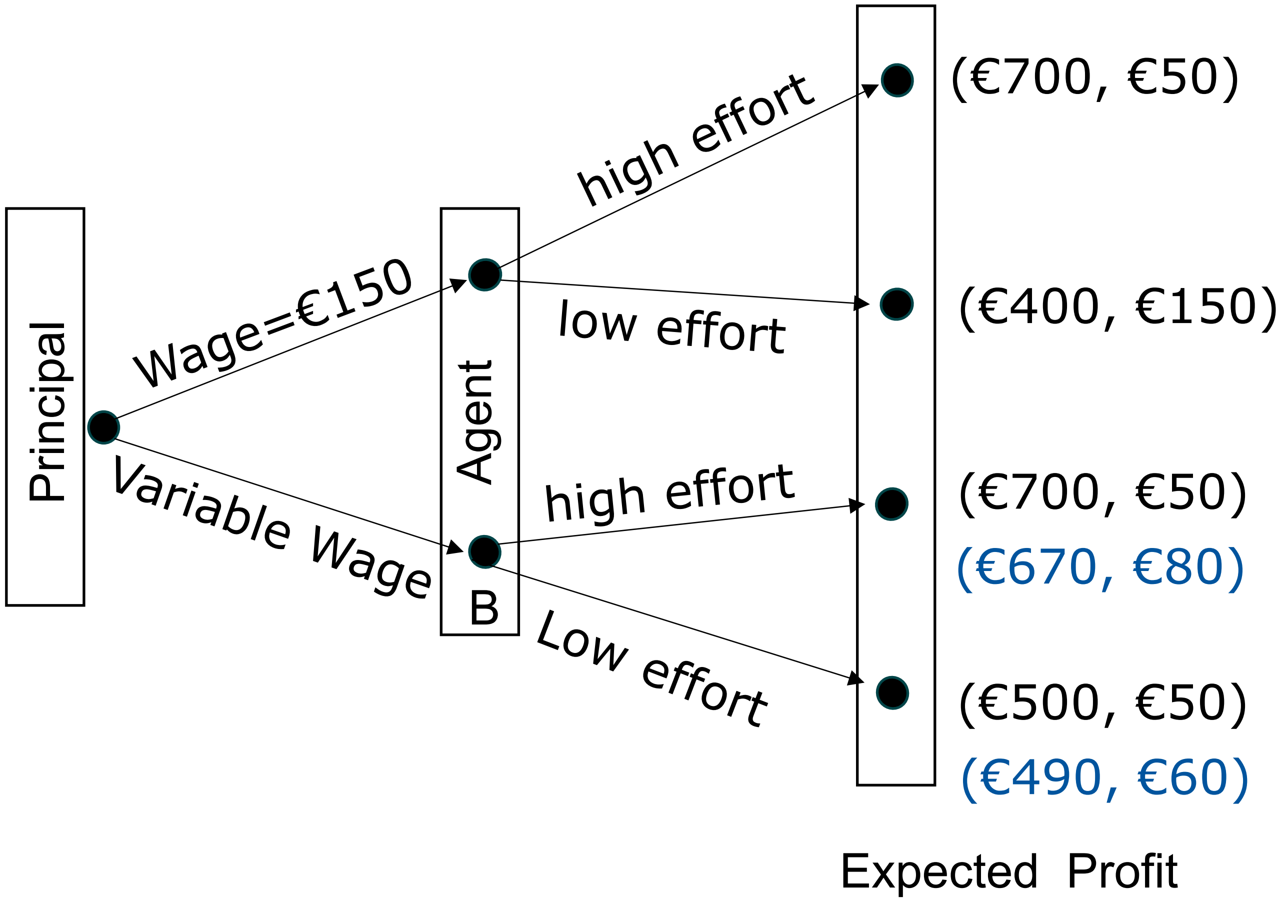
.25(1000-240)+.75(400), .25(240)+.75(0)

Expected Profit

Managing Hidden Information: Moral Hazard – Example: Principal & Agent

How could you incentivize the worker to have a high effort?

Suppose you offer a variable wage: €200 if profit is high and €0 if profit is low.



Pay more than €200 to ensure the effort is high. E.g., 240€

800 (0.75)+100, 0.75(200)+0.25(0)-100

760 (0.75)+100, 0.75(240)+0.25(0)-100

800(0.25) +300, 0.25(200)+0.75(0)

760(0.25) +300, 0.25(240)+0.75(0)

Managing Hidden Information: Asymmetric Information — Signaling

Signaling the level of education

- High-skilled workers typically have higher life-time earnings thank low-skilled workers.
- If not observable, high-skilled workers have an incentive to signal their skills to employers
- One way doing so, is by obtaining a university degree.
- This is much more costly for the low-skilled
- The university serves as a screening authority – a third party that verifies the skill level

Managing Hidden Information: Asymmetric Information — Signaling

Other scenarios of signaling

- Warranty
- Engagement
- Wearing expensive clothes and consuming expensive products
- Special food labels

Managing Constraints and Tradeoffs: Optimization – The Cake Eating Problem

The scenario: you are given a cake, which will stay fresh for 3 days in the fridge

The objective: you want to maximize your utility (joy) from eating that cake over those 3 days

The assumptions:

- The size of the cake is $\bar{X}$
- Define consumption (of the cake) in period t by C_t.
- Denote by $U(C)$ the utility of consuming C units of the cake.
- Assume that marginal utility is positive, but decreasing, i.e.: $U_C > 0, U_{CC} < 0$.
- Denote by β your time preference rate, i.e.: the present-value factor of next period's utility.

Managing Constraints and Tradeoffs: Optimization – The Cake Eating Problem

We can solve the 3-period cake eating problem as a constrained optimization problem

$$\max_{C_1,C_2,C_3} U(C_1) + \beta U(C_2) + \beta^2 U(C_3) \qquad \text{subject to: } C_1 + C_2 + C_3 = \bar{X}$$

Without discounting utility (e.g., $\beta = 1$), the solution is trivial:

$$C_1 = C_2 = C_3 = \frac{\bar{X}}{3}$$

We simply divide the cake in 3 pieces, one for each period

With the utility discount rate β (e.g., $\beta = 0.95$), the solution is:

$$C_1 = \frac{\bar{X}}{1+\beta+\beta^2} \qquad C_2 = \frac{\beta\bar{X}}{1+\beta+\beta^2} \qquad C_3 = \frac{\beta^2\bar{X}}{1+\beta+\beta^2}$$

Managing Constraints and Tradeoffs: Optimization – The Cake Eating Problem

```
###############################################################################
### Cake eating  ####
################################################################################

param T:=3 ;                                 # number of time periods
param beta:=1;                                       # discount factor
param X1 = 9;                                        # cake size

var X{t in 1..T}>=0;                                 #size of cake at time t
let X[1]:=X1
var C{t in 1..T}>=0  ;                               # cons. of cake at time t
var U{t in 1..T} = log(C[t]);                # Utility function

maximize obj:sum {t in 1..T} beta^(t-1)*U[t];        # objective function

subject to Xnext{t in 1..T-1}: X[t+1] <= X[t]-C[t];  # state transition rule
subject to cakesize{t in 1..T}: C[t]<=X[t];          # constraint on control
subject to Xini: X[1] = X1;                          # initial condition on state

solve;

option display_round 6, display_width 400;           # output options
display U, C, X;
```

https://neos-server.org/neos/

:	U	C	X
1	1.254389	3.505697	10.000000
2	1.203096	3.330412	6.494303
3	1.151803	3.163891	3.163891

Recursive Problem Solving : Dynamic Programming — Introduction

Richard Bellman

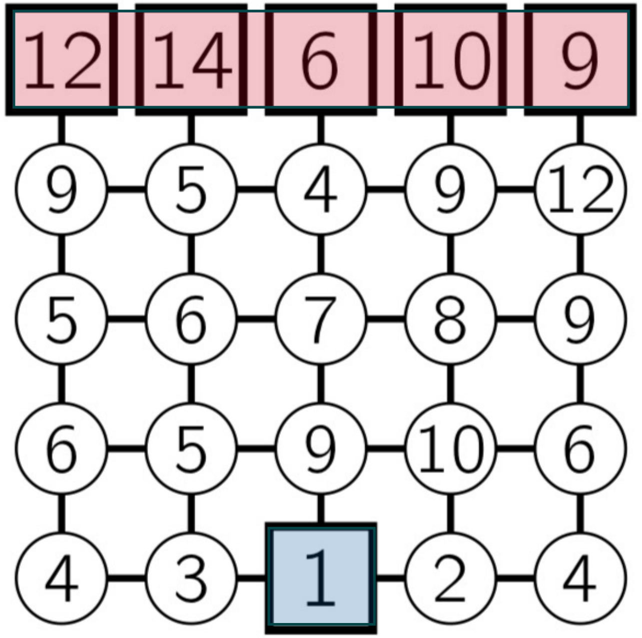
Bellman's Principle of Optimality (Bellman, 1957):

"An optimal policy has the property that, whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

Recursive Problem Solving: Introduction

Objective: Find a way from the blue field (1) to any of the red fields (12, 14, 6, 10, 9).

Constraint: you may only visit fields with a higher number than your current field



Recursive Problem Solving : Dynamic Programming — The Cake Eating Problem

We can also solve the 3-period cake eating problem with dynamic programming:

According to Bellman's principle of Optimality a dynamic program can be expressed by the Bellman Equation. Its solution is the value function.

$$V_t(X_t) = \max_{c_t} U(C_t) + \beta V_{t+1}(X_{t+1})$$

subject to:

$$X_{t+1} = X_t - C_t$$

$$X_1 = \bar{X}$$

$$V_T(X_T) = U(X_T) \qquad \text{for } t = 1 \dots T$$

$V_t(X_t)$: Value function

$V_T = U(X_T)$: Terminal value function

X_{t+1}: Transition rule for the state

The Bellman Equation approach allow us to split a problem involving multiple periods into multiple two-period problems.

Recursive Problem Solving : The Cake Eating Problem—Period 3 Solution

We can start solving the Cake Eating Problem recursively—beginning in the last period, period 3.

To simplify the analysis, let us assume that the utility function is given by $\mathrm{U}(C) = \ln(C)$.

Period 3 solution:

- The optimal solution is trivial: consume what is left of the cake in the last period
- The terminal value function $V_T(X_T) = U(X_T)$ will therefore be $V_3(X_3) = U(X_3) = U(C_3) = \ln(C_3)$

Interpretation:

The Terminal Value Function $V_3\ (X_3) = \ln(C_3)$ implies that it is optimal to consume what is left of the cake in the last decision period. Note, that this "optimal program" is independent of how much has been consumed so far, e.g., C_1 and C_2.

Recursive Problem Solving : The Cake Eating Problem—Period 2

- Recall the general Bellman Equation: $V_t(X_t) = \max_{c_t} U(C_t) + \beta V_{t+1}(X_{t+1})$
- For period 2, this implies: $V_2(X_2) = \max_{c_2} U(C_2) + \beta V_3(X_3)$
- We can use the transition rule: $X_3 = X_2 - C_2$, the utility function $U(C_t) = \ln(C_t)$ and the solution to period 3, $V_3\ (X_3) = \ln(C_3)$ to restate the Bellman Equation for period 2 as:

$$V_2(X_2) = \max_{c_2} \underbrace{ln(C_2)}_{\text{If } C_2 \text{ is chosen optimally…}} + \overbrace{\beta \underbrace{\ln(X_2 - C_2)}_{V_3}}^{V_3 \text{ in present value terms}}$$

…then $V_2(X_2)$ satisfies the Bellman Equation and can be called a Value Function

Recursive Problem Solving : The Cake Eating Problem—Period 2 Solution

$$V_2(X_2) = \max_{c_2} ln(C_2) + \beta \ln(X_2 - C_2)$$

The first order condtion w.r.t C_2 is:

$$\frac{1}{C_2} + \frac{-\beta}{X_2 - C_2} = 0$$

$$\Rightarrow \qquad X_2 - C_2 = \beta C_2$$

$$\Rightarrow \qquad C_2 = \frac{X_2}{1+\beta}$$

This is the optimal program in period 2.

Using the optimal program, we can write the Bellman Equation for period 2 as:

$$V_2(X_2) = ln\left(\frac{X_2}{1+\beta}\right) + \beta ln\left(\frac{\beta X_2}{1+\beta}\right)$$

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Recursive Problem Solving : The Cake Eating Problem—Period 2 Interpretation

$$V_2(X_2) = \underbrace{ln\left(\frac{X_2}{1+\beta}\right)}_{\text{Utility value of today's optimal decision}} + \overbrace{\beta ln\left(\frac{\beta X_2}{1+\beta}\right)}^{\text{Present Value of utility from all future decisions}}$$

Interpretation:

- Note that the Value Function depends only on the state variable X_2. All past do not matter for the optimal program in period 2.
- Note also, that the problem in period 2 involves only two periods (i.e., 2 and 3).

Recursive Problem Solving : The Cake Eating Problem—Period 1

- Recall the general Bellman Equation: $V_t(X_t) = \max_{c_t} U(C_t) + \beta V_{t+1}(X_{t+1})$
- For period 1, this implies: $V_1(X_1) = \max_{c_1} U(C_1) + \beta V_2(X_2)$
- With the solution to period 2, $V_2(X_2) = ln\left(\frac{X_2}{1+\beta}\right) + \beta ln\left(\frac{\beta X_2}{1+\beta}\right)$ and transition rule $X_2 = X_1 - C_1$ we can write: $V_2(X_2) = ln\left(\frac{X_1-C_1}{1+\beta}\right) + \beta ln\left(\frac{\beta(X_1-C_1)}{1+\beta}\right)$ and obtain the Bellman Equation for period 1:

$$V_1(X_1) = \underbrace{\max_{c_1} ln(C_1)}_{\text{If } C_1 \text{ is chosen optimally...}} + \overbrace{\beta\left(\underbrace{ln\left(\frac{X_1 - C_1}{1+\beta}\right) + \beta ln\left(\frac{\beta(X_1 - C_1)}{1+\beta}\right)}_{V_2}\right)}^{V_2 \text{ in present value terms}}$$

...then $V_1(X_1)$ satisfies the Bellmann Equation and can be called a Value Function

Recursive Problem Solving : The Cake Eating Problem—Period 1 Solution

$$V_1(X_1) = \max_{c_1} ln(C_1) + \beta\left(ln\left(\frac{X_1 - C_1}{1+\beta}\right) + \beta ln\left(\frac{\beta(X_1 - C_1)}{1+\beta}\right)\right)$$

The first order condtion $\mathrm{w.r.t}$ C_1 is:

$$\frac{1}{C_1} = \frac{\beta(1+\beta)}{X_1 - C_1}$$

$$C_1 = \frac{X_1 - C_1}{\beta + \beta^2}$$

$$C_1 = \frac{X_1}{1+\beta+\beta^2}$$

This is the optimal program in period 1.

Recursive Problem Solving : The Cake Eating Problem—Period 1 Solution

Using the optimal program, $C_1 = \dfrac{X_1}{1+\beta+\beta^2}$, the fact that $X_1 = \bar{X}$, and Bellman equation

$$V_1(X_1) = \max_{c_1} ln(C_1) + \beta\left(ln\left(\frac{X_1 - C_1}{1+\beta}\right) + \beta ln\left(\frac{\beta(X_1 - C_1)}{1+\beta}\right)\right)$$

we can write the Value Function for period 1 as:

Present Value of utility from all future decisions

Depends only on parameters

$$V_1(\bar{X}) = \underbrace{\ln\left(\frac{\bar{X}}{1+\beta+\beta^2}\right)}_{\text{Utility value of today's optimal decision}} + \overbrace{\beta\left(ln\left(\frac{\bar{X}-\left(\frac{\bar{X}}{1+\beta+\beta^2}\right)}{1+\beta}\right) + \beta ln\left(\frac{\beta\left(\bar{X}-\left(\frac{\bar{X}}{1+\beta+\beta^2}\right)\right)}{1+\beta}\right)\right)}^{\text{Present Value of utility from all future decisions}}$$

Utility value of today's optimal decision

Recursive Problem Solving : The Cake Eating Problem—Checking the Solution

We know the intuitive and trivial solution of consuming 1/3 of the cake each period. By assuming $\beta = 1$, we can see that our solution is correct.

$$V_1(\bar{X}) = \ln\left(\frac{\bar{X}}{3}\right) + \left(ln\left(\frac{\bar{X}-\left(\frac{\bar{X}}{3}\right)}{2}\right) + ln\left(\frac{\left(\bar{X}-\left(\frac{\bar{X}}{3}\right)\right)}{2}\right)\right) = \ln\left(\frac{\bar{X}}{3}\right) + \ln\left(\frac{\bar{X}}{3}\right) + \ln\left(\frac{\bar{X}}{3}\right) = 3\ln\left(\frac{\bar{X}}{3}\right)$$

Recursive Problem Solving : The Cake Eating Problem—Stochastic Extension

- Same example as before, but now $T = 2$
- Also, the size of the cake in period 2 is uncertain.
- With probability p the cake size is reduced by γ
- With probability 1-p the cake size is increased by γ.
- For simplicity. we assume that γ is small relative to the cake size and that the consumption-smoothing motive ensures that the next-period cake size is positive.

The general stochastic formulation of the Bellmann Equation is

$$V_t(X_t) = \max_{c_t} U(C_t) + \beta[E\{V_{t+1}(X_{t+1})\}]$$

where $E\{\cdot\}$ is an expectations operator.

Recursive Problem Solving : The Cake Eating Problem—Stochastic Extension

$$V_t(X_t) = \max_{c_t} U(C_t) + \beta[E\{V_{t+1}(X_{t+1})\}]$$

In our example, the (stochastic) terminal value function can be written as

$$E\{V_2(X_2)\} = p \cdot ln(X_2^L) + (1-p)ln(X_2^H)$$

with:

$$X_2^L = X_1 - C_1 - \gamma$$

$$X_2^H = X_1 - C_1 + \gamma$$

Recursive Problem Solving : The Cake Eating Problem—Stochastic Extension

$$V_t(X_t) = \max_{c_t} U(C_t) + \beta[E\{V_{t+1}(X_{t+1})\}]$$

Given the optimal solution to consume all of the cake in the last period, we obtain the following optimal program:

$$C_2 = \begin{cases} X_2^H & if\ state: H \\ X_2^L & if\ state: L \end{cases}$$

In our example, the (stochastic) terminal value function can be written as

$$E\{V_2(X_2)\} = p \cdot ln(X_2^L) + (1-p)ln(X_2^H)$$

with:

$X_2^L = X_1 - C_1 - \gamma$

$X_2^H = X_1 - C_1 + \gamma$

Recursive Problem Solving : The Cake Eating Problem—Stochastic Extension

The Bellman Equation in period 1 is given by

$$V_1(X_1) = \max_{c_1} U(C_1) + \beta[p \cdot ln(X_2^L) + (1-p)ln(X_2^H)]$$

Using the low of motion for the uncertain state X,

$$X_2^L = X_1 - C_1 - \gamma$$

$$X_2^H = X_1 - C_1 + \gamma$$

We can formulate the Bellman Equation as:

$$V_1(X_1) = \max_{c_1} U(C_1) + \beta[p \cdot ln(X_1 - C_1 - \gamma\ ) + (1-p) \cdot ln(X_1 - C_1 + \gamma\ )]$$

Recursive Problem Solving : The Cake Eating Problem—Stochastic Solution

$$V_1(X_1) = \max_{C_1} U(C_1) + \beta[p \cdot ln(X_1 - C_1 - \gamma) + (1-p) \cdot ln(X_1 - C_1 + \gamma)]$$

The first order condtion w.r.t C_1 is:

$$\frac{1}{C_1} = \frac{\beta p}{X_1 - C_1 - \gamma} + \frac{\beta(1-p)}{X_1 - C_1 + \gamma}$$

With: p=0.5, beta=1 $C_1 = \frac{3}{4} - \sqrt{\frac{X_1^2}{2} + \gamma^2}$

With: p=0.5, beta=1, X1=10, gamma=1 $C_1 = 4.90192$

With: p=0.9, beta=1, X1=10, gamma=1 $C_1 = 4.5711$

How does risk affect the optimal decision?

C_1	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$
$p = 0.25$	5.16	5.11	4.83
$p = 0.50$	4.90	4.63	4.22
$p = 0.75$	4.69	4.28	3.81

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