RWTH BUSINESS SCHOOL

Mathematics & Statistics
M.Sc. Data Analytics and Decision Science
Prof. Dr. Thomas S. Lontzek

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Outline

- Functions of one variable
- Quadratic functions
- Polynomials
- Power functions
- Exponentials
- Logarithmic functions
- Properties of functions

Functions of One Variable

A (real-valued) **function** of a real variable x with **domain** D is a rule that assigns a unique real number to each real number x in D. As x varies over the whole domain, the set of all possible resulting values $f(x)$ is called the **range** of f.

If f is a function, we sometimes let y denote the value of f at x, so

$$y = f(x)$$

Then we call x the **independent variable**, or the **argument** of f, whereas y is called the **dependent variable**, because the value y (in general) depends on the value of x. The domain of the function f is then the set of all possible values of the independent variable, whereas the range is the set of corresponding values of the dependent variable.

Functions of One Variable - Example

The total dollar cost of producing x units of a product is given by

$$C(x) = 100x\sqrt{x} + 500$$

for each nonnegative integer x. Find the cost of producing 16 units. Suppose the firm produces a units; find the *increase* in the cost from producing one additional unit.

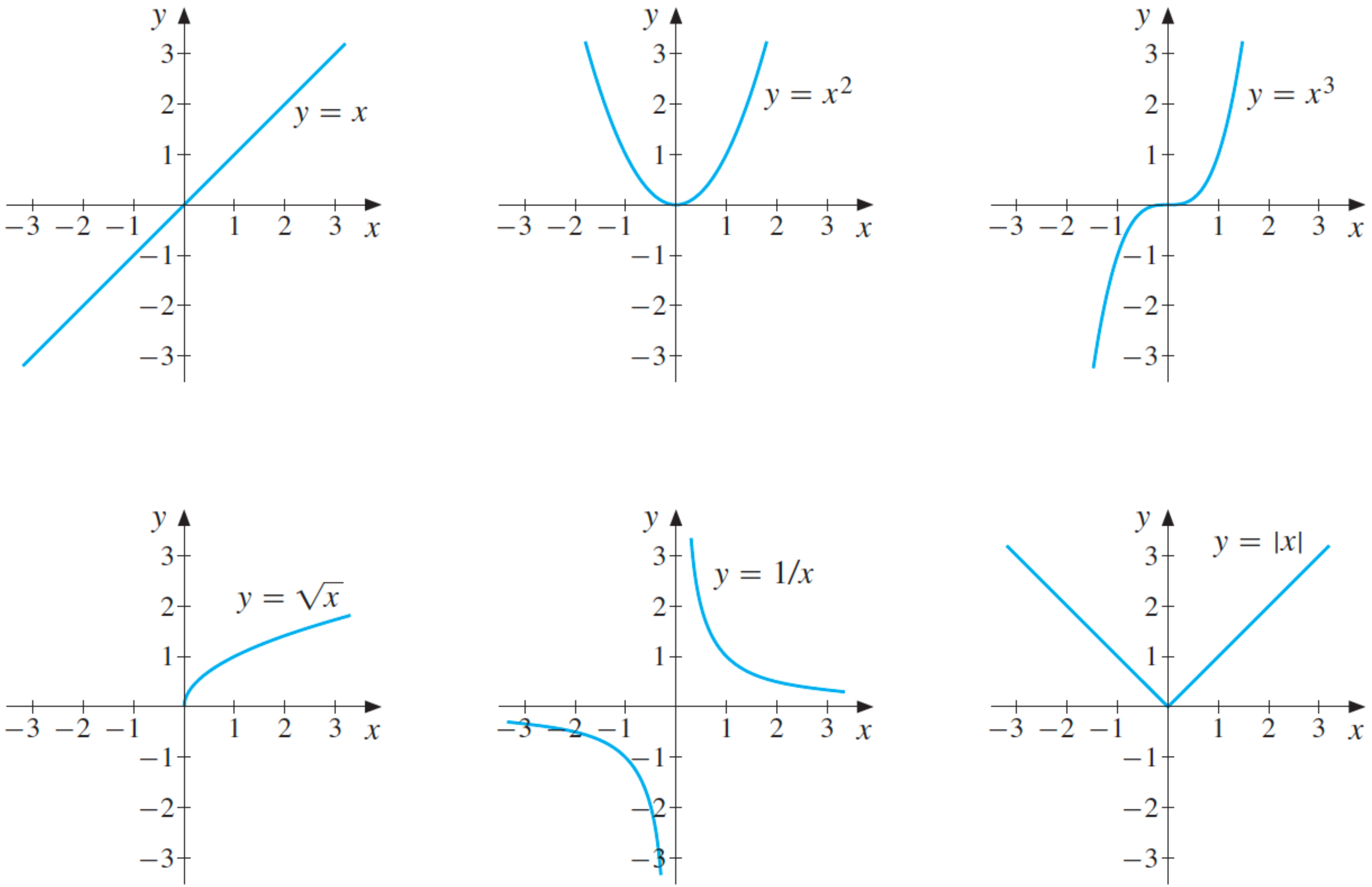
Solution: The cost of producing 16 units is found by substituting 16 for x in the formula for $C(x)$:

$$C(16) = 100 \cdot 16\sqrt{16} + 500 = 100 \cdot 16 \cdot 4 + 500 = 6900$$

The cost of producing a units is $C(a) = 100a\sqrt{a} + 500$, and the cost of producing $a + 1$ units is $C(a + 1)$. Thus the increase in cost is

$$\begin{aligned} C(a+1) - C(a) &= 100(a+1)\sqrt{a+1} + 500 - 100a\sqrt{a} - 500 \\ &= 100\left[(a+1)\sqrt{a+1} - a\sqrt{a}\right] \end{aligned}$$

Graphs of Functions



Linear Functions

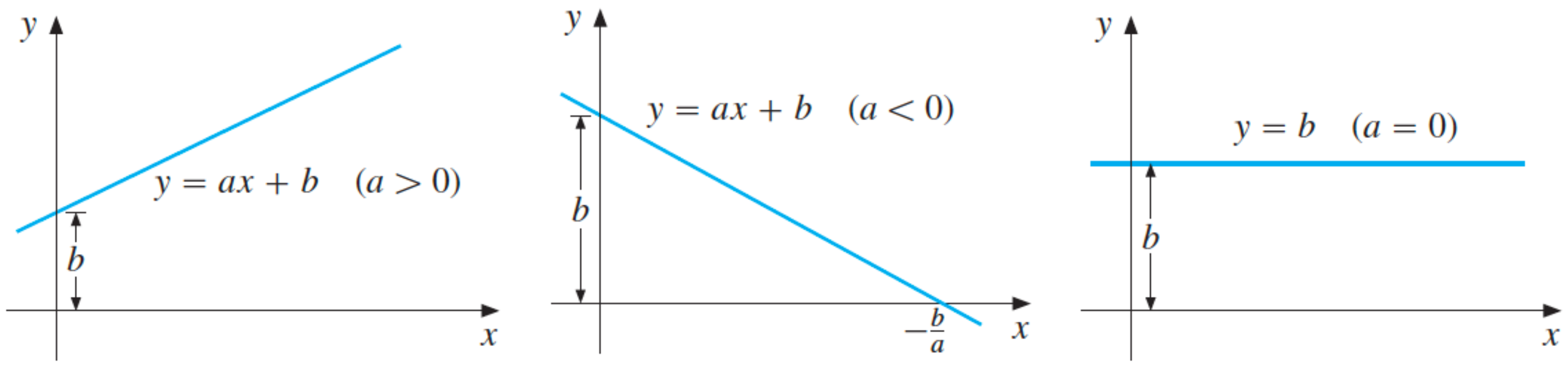
$$y = ax + b \qquad (a \text{ and } b \text{ are constants})$$

The graph of the equation is a straight line. If we let f denote the function that assigns y to x, then $f(x) = ax + b$, and f is called a **linear** function.

Take an arbitrary value of x. Then

$$f(x+1) - f(x) = a(x+1) + b - ax - b = a$$

This shows that a measures the change in the value of the function when x increases by 1 unit. For this reason, the number a is the **slope** of the line (or the function).



Graphical Solution to Linear Equations – Problem Set 1

Solve each of the following three pairs of equations graphically:

(a) $$\begin{aligned} x + y &= 5 \\ x - y &= -1 \end{aligned}$$

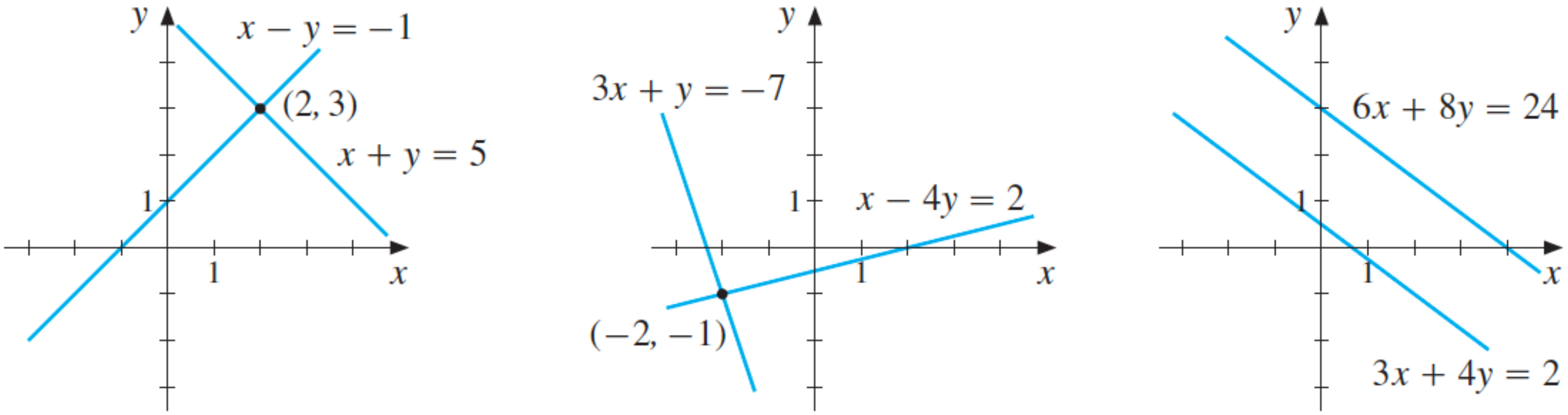
(b) $$\begin{aligned} 3x + y &= -7 \\ x - 4y &= 2 \end{aligned}$$

(c) $$\begin{aligned} 3x + 4y &= 2 \\ 6x + 8y &= 24 \end{aligned}$$

Graphical Solution to Linear Equations – Problem Set 1 (Solution)

Solve each of the following three pairs of equations graphically:

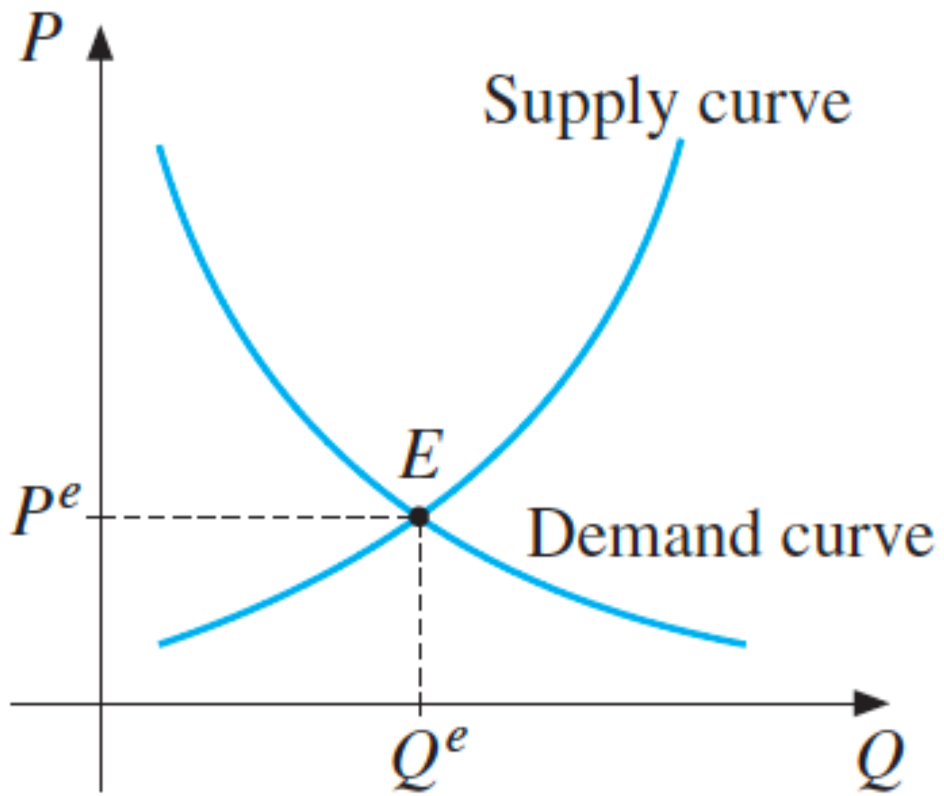
(a) $\begin{aligned} x + y &= 5 \\ x - y &= -1 \end{aligned}$ (b) $\begin{aligned} 3x + y &= -7 \\ x - 4y &= 2 \end{aligned}$ (c) $\begin{aligned} 3x + 4y &= 2 \\ 6x + 8y &= 24 \end{aligned}$



Functions of One Variable : Application – Supply and Demand in Economics

Over a fixed period of time such as a week, the quantity of a specific good that consumers demand (that is, are willing to buy) will depend on the price of that good. Usually, as the price increases the demand will decrease. Also, the number of units that the producers are willing to supply to the market during a certain period depends on the price they are able to obtain. Usually, the supply will increase as the price increases. Typical shapes of demand and supply curves are given in the plot.

Here, *P* denotes the price, *Q*, the quantity and *E*, the resulting equilibrium.

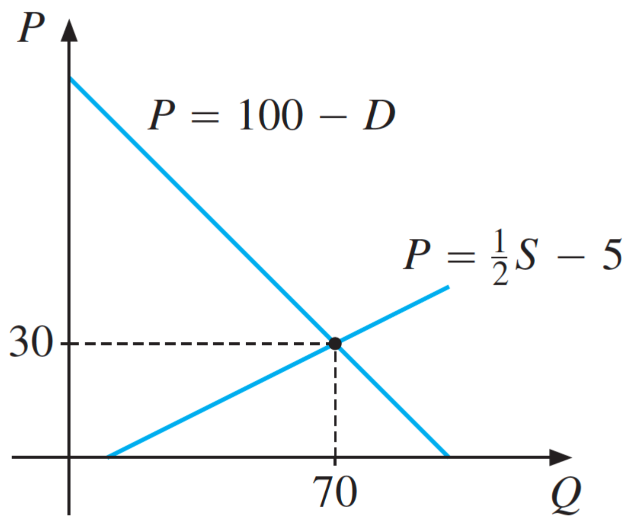


Functions of One Variable : Application – Supply and Demand in Economics

As a very simple example, consider the following linear demand and supply functions:

$$\text{(i)}\quad D = 100 - P \qquad\qquad \text{(ii)}\quad S = 10 + 2P$$

or in inverse form, $P = 100 - D$ and $P = \frac{1}{2}S - 5$, as in Fig. 2. The quantity demanded D equals the quantity supplied S provided $100 - P = 10 + 2P$, that is, $3P = 90$. So the equilibrium price is $P^e = 30$, with equilibrium quantity $Q^e = 70$.



Functions of One Variable: Problem Set 2

Find the equilibrium price for each of the two linear models of supply and demand:

(a) $D = 75 - 3P, \quad S = 20 + 2P$

(b) $D = 100 - 0.5P, \quad S = 10 + 0.5P$

Functions of One Variable: Problem Set 2 - Solution

Find the equilibrium price for each of the two linear models of supply and demand:

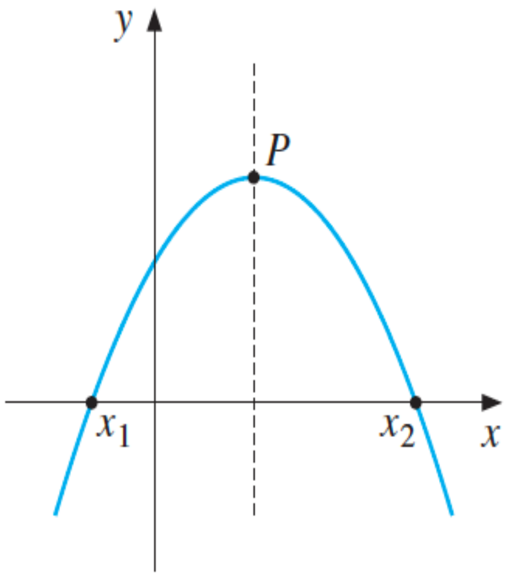
(a) $D = 75 - 3P, \quad S = 20 + 2P$ (b) $D = 100 - 0.5P, \quad S = 10 + 0.5P$

(a) $75 - 3P^e = 20 + 2P^e$, and hence $P^e = 11$. (b) $P^e = 90$

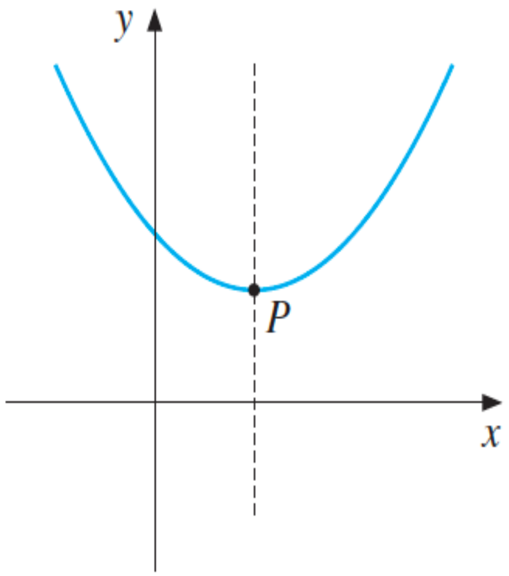
Quadratic Functions

$$f(x) = ax^2 + bx + c \qquad (a, b, \text{ and } c \text{ are constants, } a \neq 0)$$

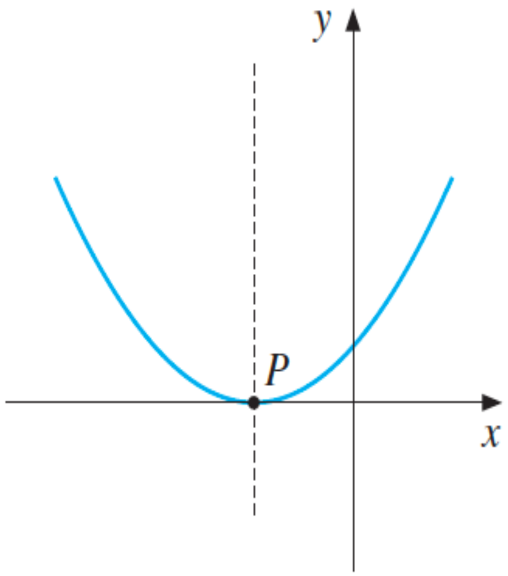
(If $a = 0$, the function is linear; hence, the restriction $a \neq 0$.) In general, the graph of $f(x) = ax^2 + bx + c$ is called a **parabola**. The shape of this parabola roughly resembles $\cap$ when $a < 0$ and $\cup$ when $a > 0$.



(a) $a < 0,\ b^2 > 4ac$



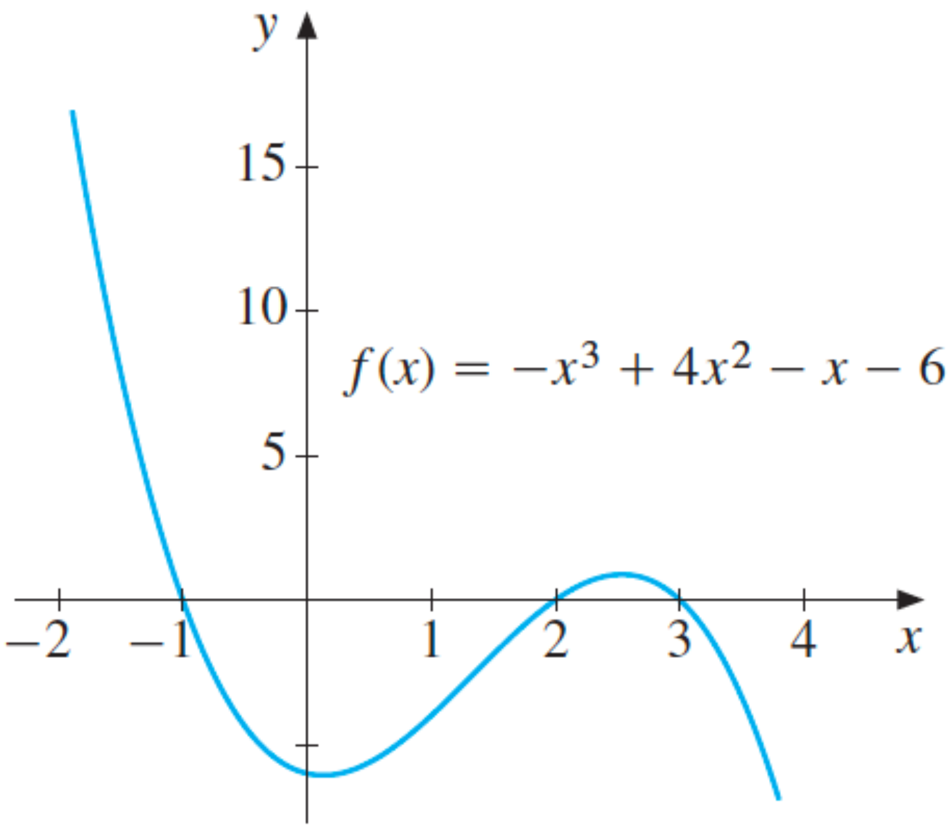
(b) $a > 0,\ b^2 < 4ac$



(c) $a > 0,\ b^2 = 4ac$

Polynomials - Cubic

$$f(x) = ax^3 + bx^2 + cx + d \qquad (a, b, c, \text{ and } d \text{ are constants; } a \neq 0)$$



General Polynomials

Linear, quadratic, and cubic functions are all examples of **polynomials**. The function P defined for all x by

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad (a\text{'s are constants; } a_n \neq 0)$$

is called the **general polynomial of degree *n*** with **coefficients** $a_n, a_{n-1}, \ldots, a_0$. When $n = 4$, we obtain $P(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$, which is the general quartic function, or polynomial of degree 4. Neither $5 + \frac{1}{x^2}$ nor $\frac{1}{x^3 - x + 2}$ are polynomials, however.

Power Functions

Consider the general **power function** f defined by the formula

$$f(x) = Ax^r \qquad (x > 0,\ r \text{ and } A \text{ are any constants})$$

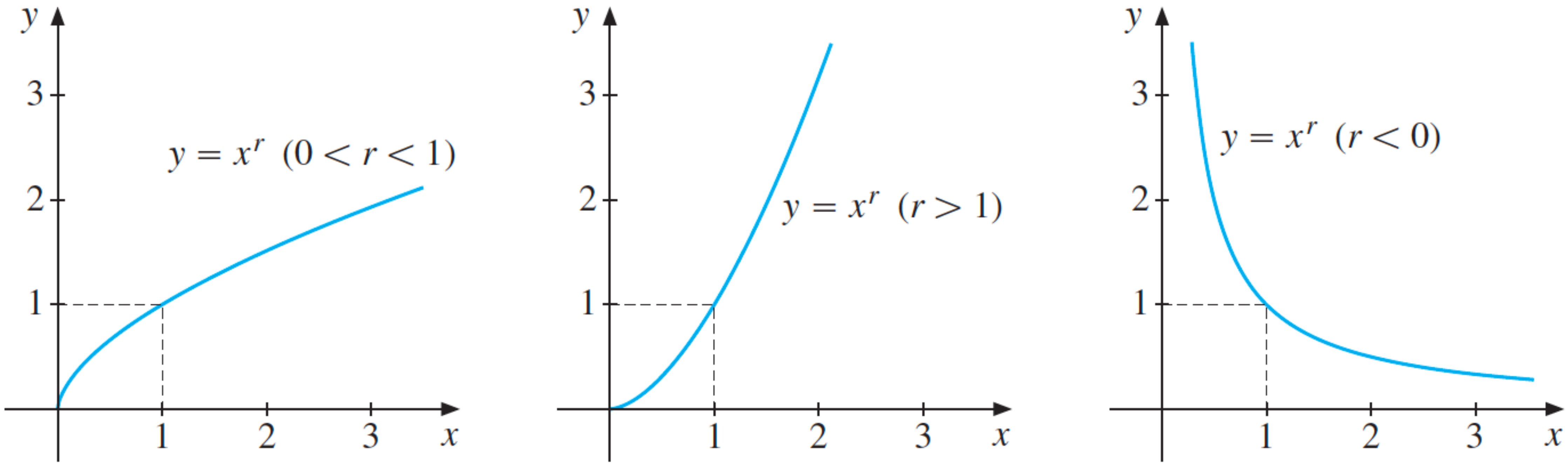
Here are three examples of why powers with rational exponents are needed:

1. The formula $S \approx 4.84V^{2/3}$ gives the approximate surface area of a ball as a function of its volume.

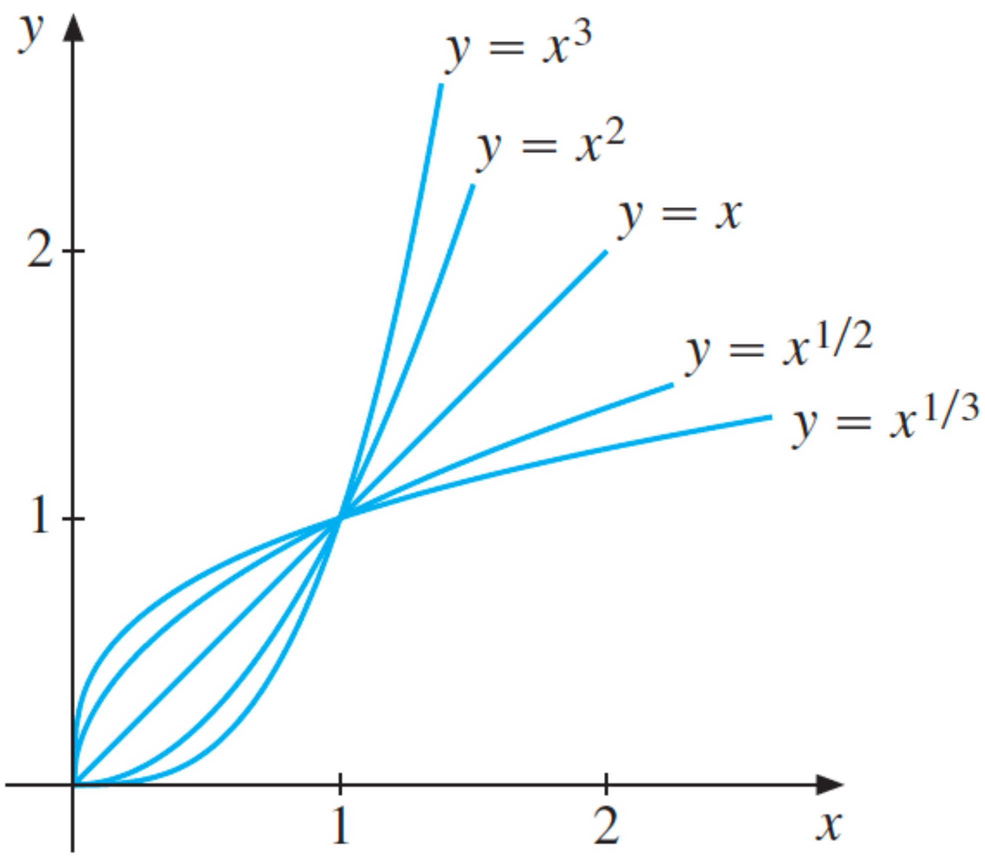
2. The flow of blood (in litres per second) through the heart of an individual is approximately proportional to $x^{0.7}$, where x is the body weight.

3. The formula $2.262K^{0.203}L^{0.762}(1.02)^t$ appears in a study of the growth of U.S. national product and shows how powers with fractional exponents can arise in economics. (Here Y is the net national product, K is capital stock, L is labour, and t is time.)

Power Functions - Graphs



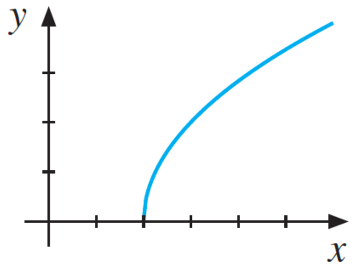
Power Functions - Graphs



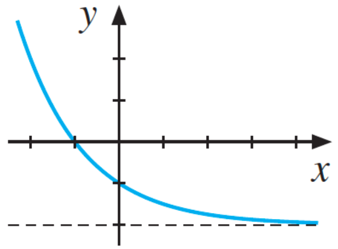
Power Functions – Problem Set 3

Match each of the graphs A–F with one of the functions (a)–(f) in the following table. (In (f) try to find a suitable function which has the remaining graph.)

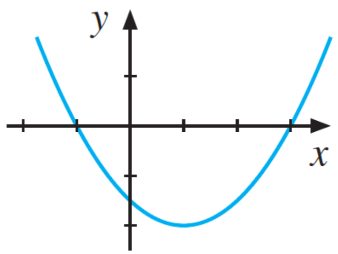
(a) $y = \frac{1}{2}x^2 - x - \frac{3}{2}$ has graph ___	(b) $y = 2\sqrt{2-x}$ has graph ___
(c) $y = -\frac{1}{2}x^2 + x + \frac{3}{2}$ has graph ___	(d) $y = \left(\frac{1}{2}\right)^x - 2$ has graph ___
(e) $y = 2\sqrt{x-2}$ has graph ___	(f) $y =$ has graph ___



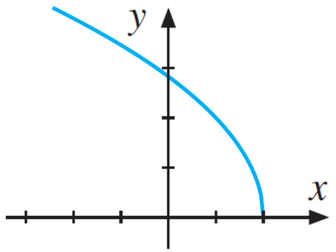
A



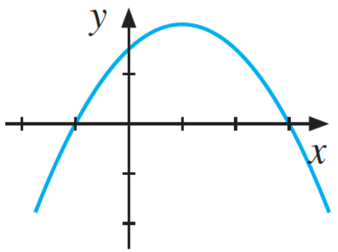
B



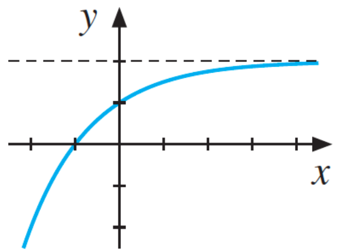
C



D



E



F

Power Functions – Problem Set 3 (Solution)

Match each of the graphs A–F with one of the functions (a)–(f) in the following table. (In (f) try to find a suitable function which has the remaining graph.)

(a) $y = \frac{1}{2}x^2 - x - \frac{3}{2}$ has graph C

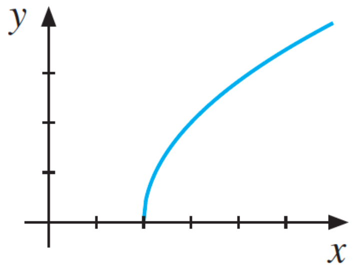
(b) $y = 2\sqrt{2 - x}$ has graph D

(c) $y = -\frac{1}{2}x^2 + x + \frac{3}{2}$ has graph E

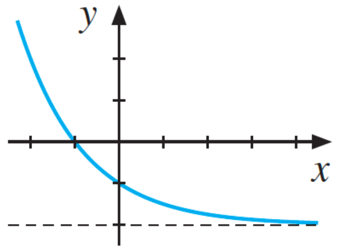
(d) $y = \left(\frac{1}{2}\right)^x - 2$ has graph B

(e) $y = 2\sqrt{x - 2}$ has graph A

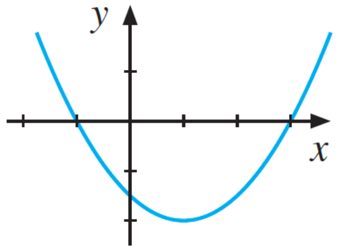
(f) $y = 2 - (1/2)^x$ has graph F



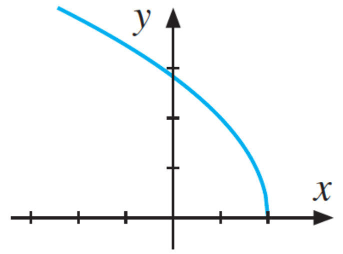
A



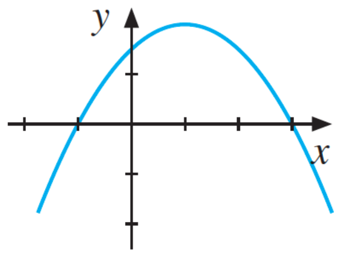
B



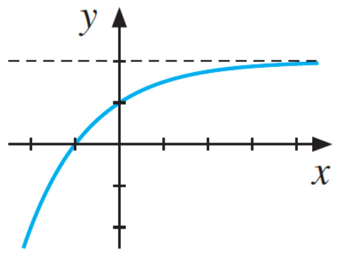
C



D



E



F

The General Exponential Function

The **general exponential function** with base $a > 0$ is

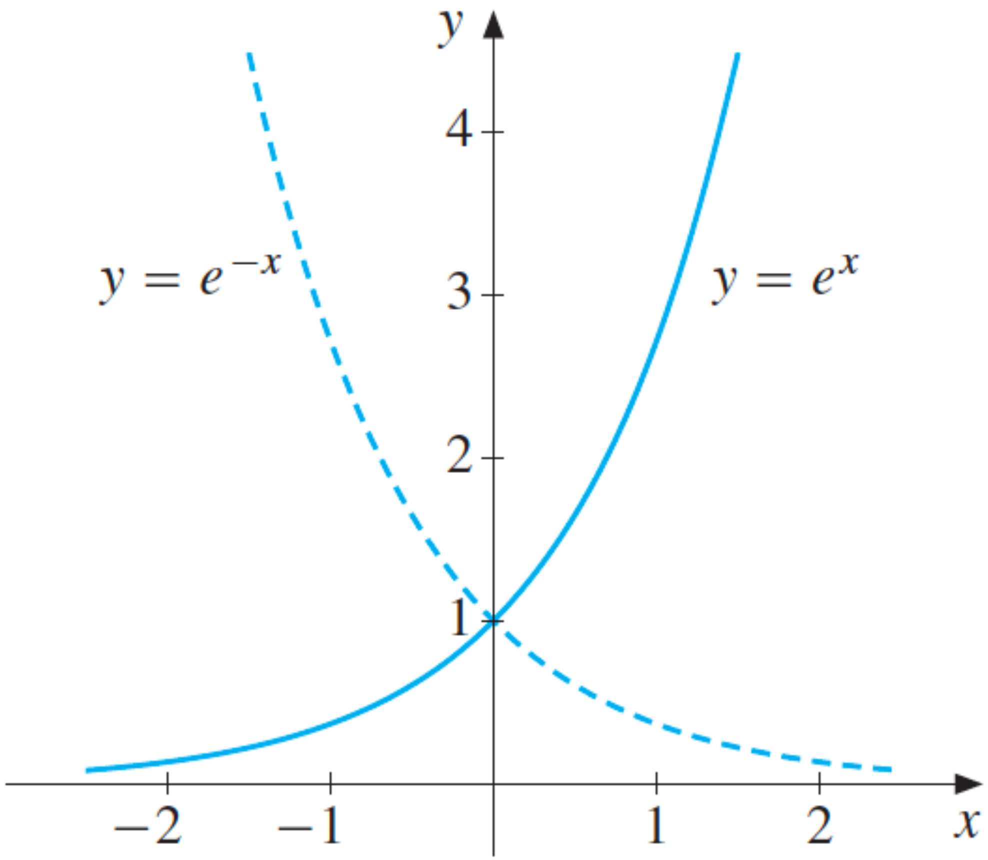
$$f(x) = Aa^x$$

where a is the factor by which $f(x)$ changes when x increases by 1.

If $a = 1 + p/100$, where $p > 0$ and $A > 0$, then $f(x)$ will increase by $p\%$ for each unit increase in x.

If $a = 1 - p/100$, where $0 < p < 100$ and $A > 0$, then $f(x)$ will decrease by $p\%$ for each unit increase in x.

The Natural Exponential Function



$e = 2.718281828459045\ldots$

Logarithmic Functions

We begin with equations in which the base of the exponential is e, which was, as you recall, the irrational number 2.718 . . . . Here are some examples:

$$\text{(i)} \quad e^x = 4 \qquad \text{(ii)} \quad 5e^{-3x} = 16 \qquad \text{(iii)} \quad A\alpha e^{-\alpha x} = k$$

In all these equations, the unknown x occurs as an exponent. We therefore introduce the following useful definition. If $e^u = a$, we call u the **natural logarithm** of a, and we write $u = \ln a$. Hence, we have the following definition of the symbol $\ln a$:

$$e^{\ln a} = a \qquad (a \text{ is any positive number})$$

Thus, $\ln a$ is the power of e you need to get a. In particular, if $e^x = 4$, then x must be $\ln 4$.

Logarithmic Functions: Rules

(a) $\ln(xy) = \ln x + \ln y$ $\qquad$ (x and y positive)

(The logarithm of a *product* is the *sum* of the logarithms of the factors.)

(b) $\ln \dfrac{x}{y} = \ln x - \ln y$ $\qquad$ (x and y positive)

(The logarithm of a *quotient* is the *difference* between the logarithms of its numerator and denominator.)

(c) $\ln x^p = p \ln x$ $\qquad$ (x positive)

(The logarithm of a *power* is the exponent multiplied by the logarithm of the base.)

(d) $\ln 1 = 0, \quad \ln e = 1, \quad x = e^{\ln x},\ (x > 0)$ and $\ln e^x = x$

Logarithmic Functions: Problem Set 4

Solve the following equations for x:

(a) $5e^{-3x} = 16$ (b) $A\alpha e^{-\alpha x} = k$ (c) $(1.08)^x = 10$ (d) $e^x + 4e^{-x} = 4$

Logarithmic Functions: Problem Set 4 - Solution

Solve the following equations for x:

(a) $5e^{-3x} = 16$ (b) $A\alpha e^{-\alpha x} = k$ (c) $(1.08)^x = 10$ (d) $e^x + 4e^{-x} = 4$

(a) Take ln of each side of the equation to obtain $\ln(5e^{-3x}) = \ln 16$. The product rule gives $\ln(5e^{-3x}) = \ln 5 + \ln e^{-3x}$. Here $\ln e^{-3x} = -3x$, by rule (d). Hence, $\ln 5 - 3x = \ln 16$, which gives

$$x = \frac{1}{3}(\ln 5 - \ln 16) = \frac{1}{3}\ln\frac{5}{16}$$

(b) We argue as in (a) and obtain $\ln(A\alpha e^{-\alpha x}) = \ln k$, or $\ln(A\alpha) + \ln e^{-\alpha x} = \ln k$, so $\ln(A\alpha) - \alpha x = \ln k$. Finally, therefore,

$$x = \frac{1}{\alpha}[\ln(A\alpha) - \ln k] = \frac{1}{\alpha}\ln\frac{A\alpha}{k}$$

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Logarithmic Functions: Problem Set 4 - Solution

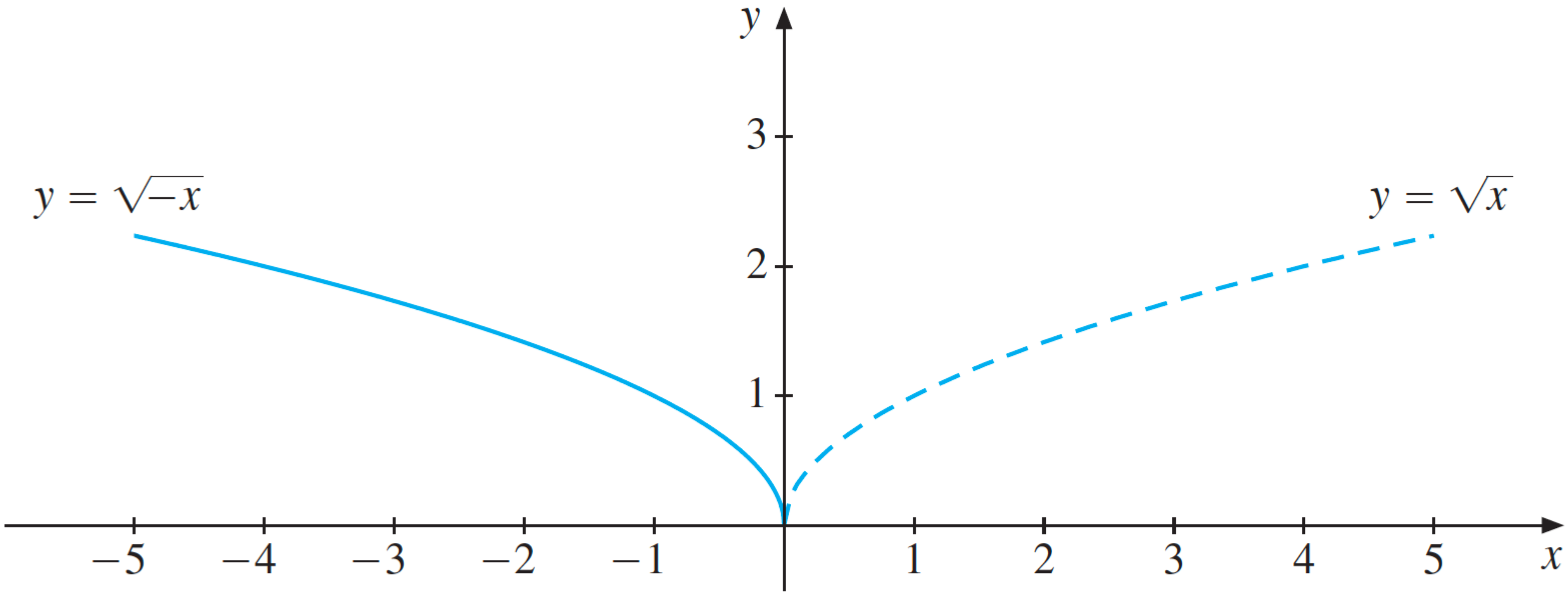
Solve the following equations for x:

(a) $5e^{-3x} = 16$ (b) $A\alpha e^{-\alpha x} = k$ (c) $(1.08)^x = 10$ (d) $e^x + 4e^{-x} = 4$

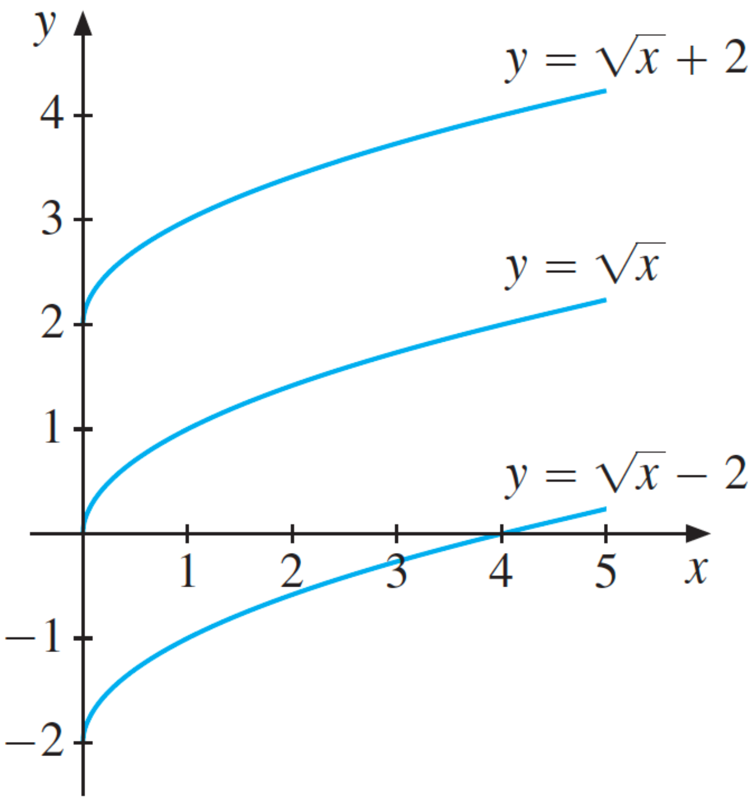
(c) Again we take the ln of each side of the equation and obtain $x \ln 1.08 = \ln 10$. So the solution is $x = \ln 10 / \ln 1.08$, which is ≈ 29.9. Thus, it takes just short of 30 years for \$1 to increase to \$10 when the interest rate is 8%.

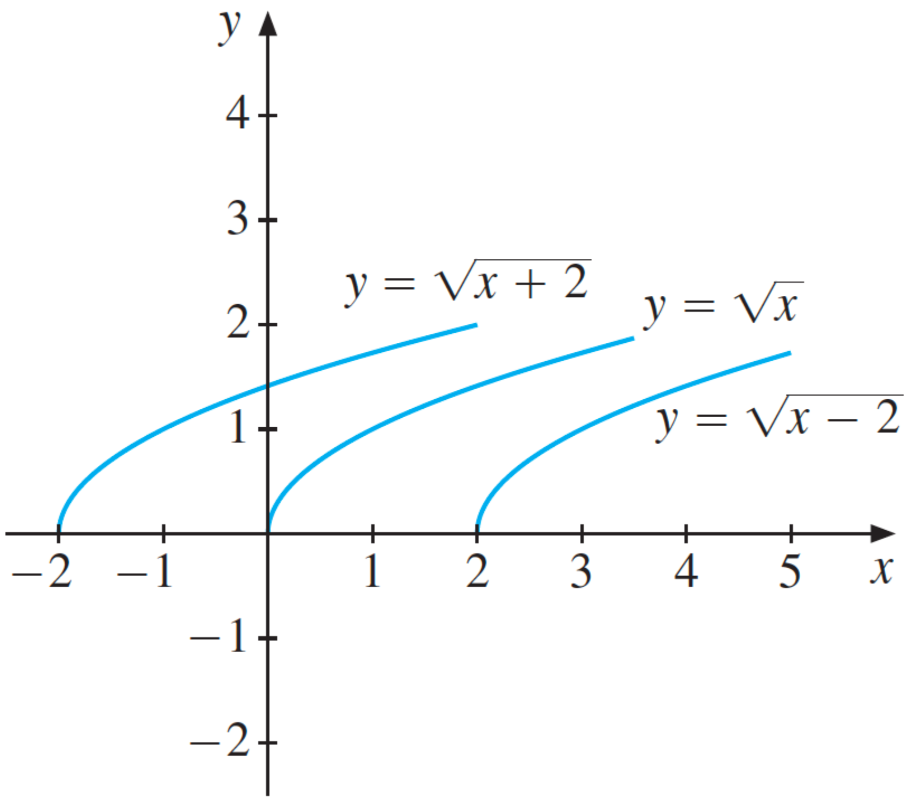
(d) It is very tempting to begin with $\ln(e^x + 4e^{-x}) = \ln 4$, but this leads nowhere, because $\ln(e^x + 4e^{-x})$ cannot be further evaluated. Instead, we argue like this: Putting $u = e^x$ gives $e^{-x} = 1/e^x = 1/u$, so the equation is $u + 4/u = 4$, or $u^2 + 4 = 4u$. Solving this quadratic equation for u yields $u = 2$ as the only solution. Hence, $e^x = 2$, and so $x = \ln 2$.

Properties of Functions

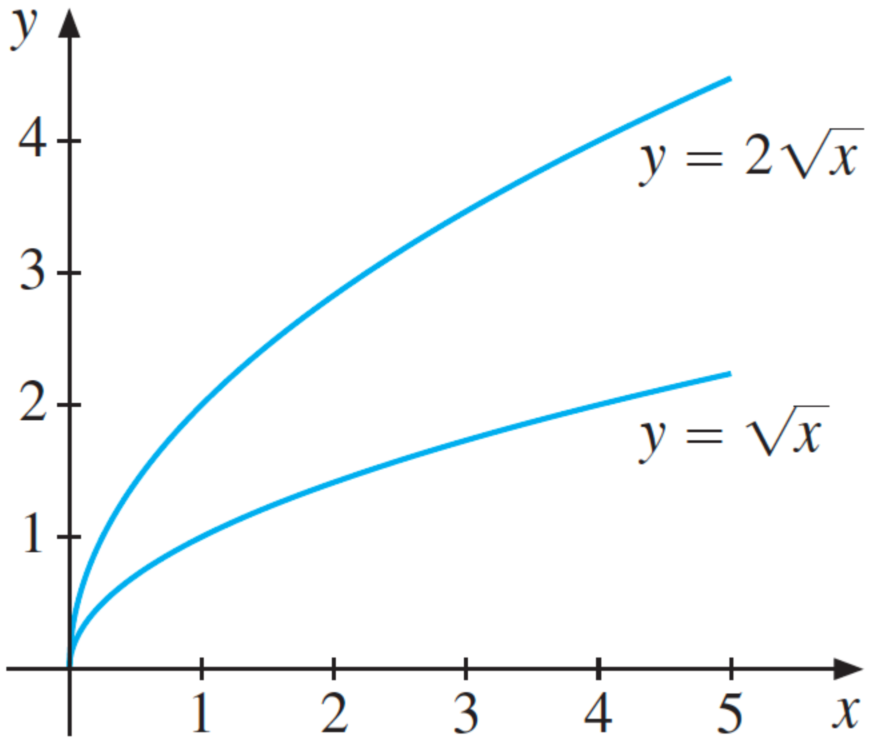


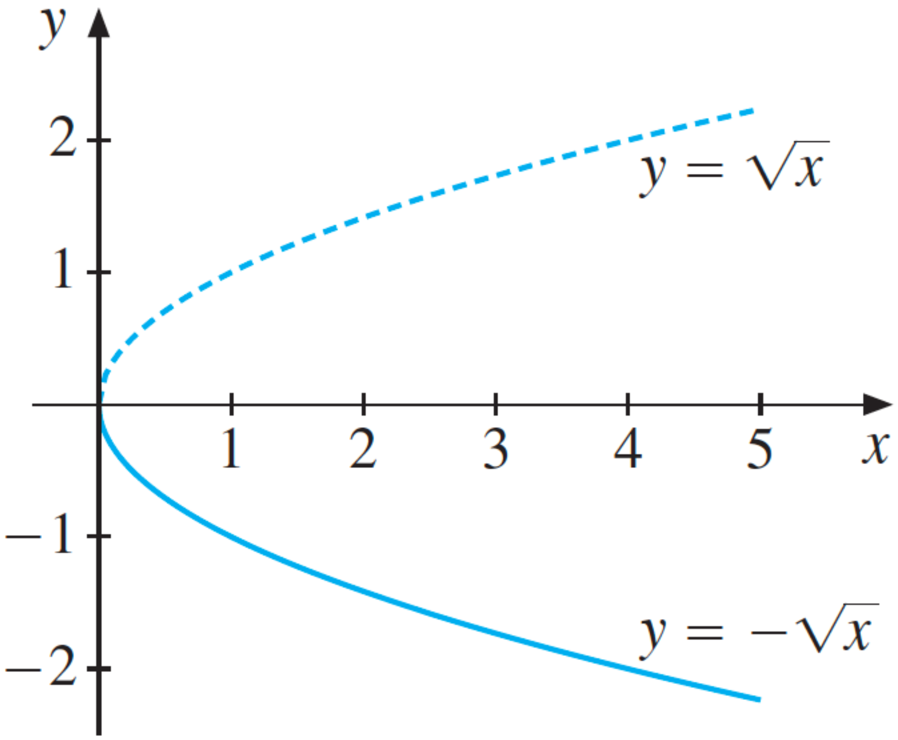
Properties of Functions



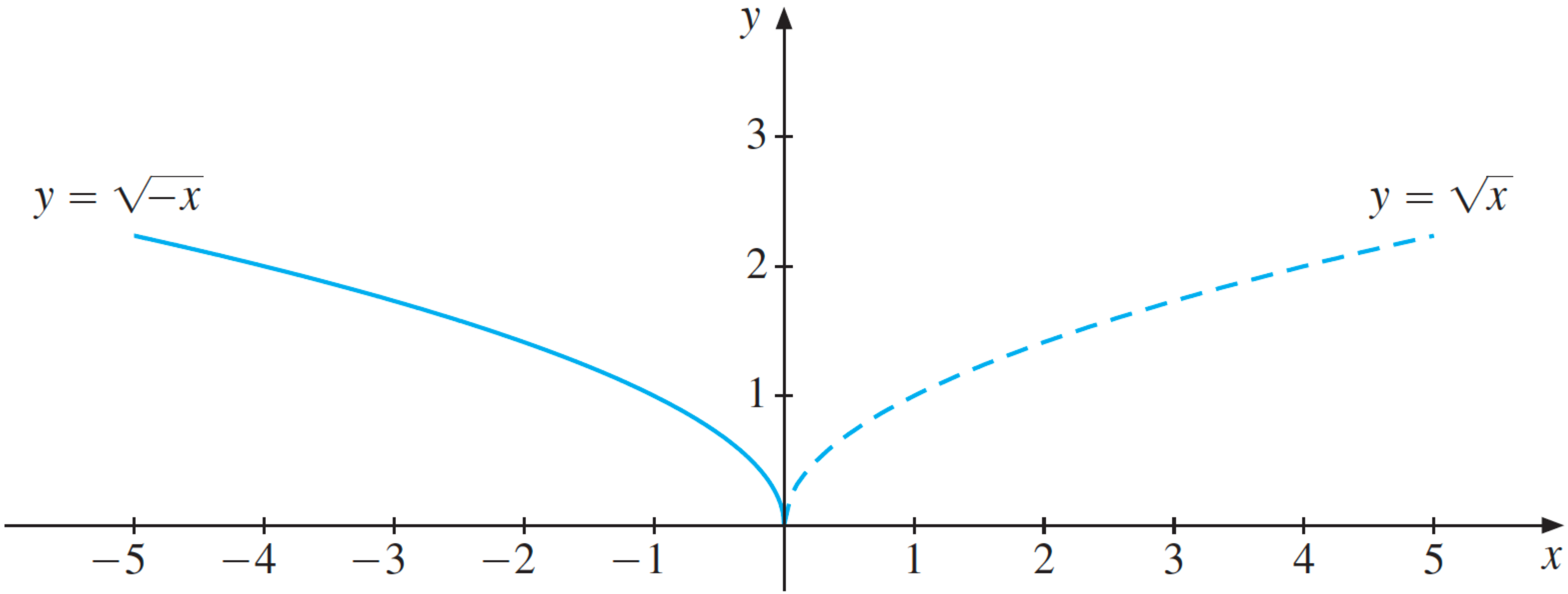


Properties of Functions





Properties of Functions



Properties of Functions

General rules for shifting the graph of $y = f(x)$:

(i) If $y = f(x)$ is replaced by $y = f(x) + c$, the graph is moved upwards by c units if $c > 0$ (downwards if $c < 0$).

(ii) If $y = f(x)$ is replaced by $y = f(x + c)$, the graph is moved c units to the left if $c > 0$ (to the right if $c < 0$).

(iii) If $y = f(x)$ is replaced by $y = cf(x)$, the graph is stretched vertically if $c > 0$ (stretched vertically and reflected about the x-axis if $c < 0$).

(iv) If $y = f(x)$ is replaced by $y = f(-x)$, the graph is reflected about the y-axis.

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