RWTH BUSINESS SCHOOL

Mathematics & Statistics
M.Sc. Data Analytics and Decision Science
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Outline

- Single variable optimization
- Integration
- Indefinite Integrals
- Definite Integrals
- Integration by Parts

Single Variable Optimization – Example

Suppose $Y(N)$ bushels of wheat are harvested per acre of land when N pounds of fertilizer per acre are used. If P is the dollar price per bushel of wheat and q is the dollar price per pound of fertilizer, then profits in dollars per acre are

$$\pi(N) = PY(N) - qN, \qquad N \geq 0$$

Suppose there exists N^* such that $\pi'(N) \geq 0$ for $N \leq N^*$, whereas $\pi'(N) \leq 0$ for $N \geq N^*$. Then N^* maximizes profits, and $\pi'(N^*) = 0$. That is, $PY'(N^*) - q = 0$, so

$$PY'(N^*) = q$$

Single Variable Optimization – Example

Let us give an economic interpretation of this condition. Suppose N^* units of fertilizer are used and we contemplate increasing N^* by one unit. What do we gain? If N^* increases by one unit, then $Y(N^*+1)-Y(N^*)$ more bushels are produced. Now $Y(N^*+1)-Y(N^*) \approx Y'(N^*)$. For each of these bushels, we get P dollars, so

$$\text{by increasing } N^* \text{ by one unit, we gain} \approx PY'(N^*) \text{ dollars}$$

On the other hand,

$$\text{by increasing } N^* \text{ by one unit, we lose } q \text{ dollars}$$

because this is the cost of one unit of fertilizer.

Single Variable Optimization – Problem Set 1

(a) In an (unrealistic) example $Y(N) = \sqrt{N}$, $P = 10$, and $q = 0.5$. Find the amount of fertilizer which maximizes profits in this case.

(b) An agricultural study in Iowa estimated the yield function $Y(N)$ for the year 1952 as

$$Y(N) = -13.62 + 0.984N - 0.05N^{1.5}$$

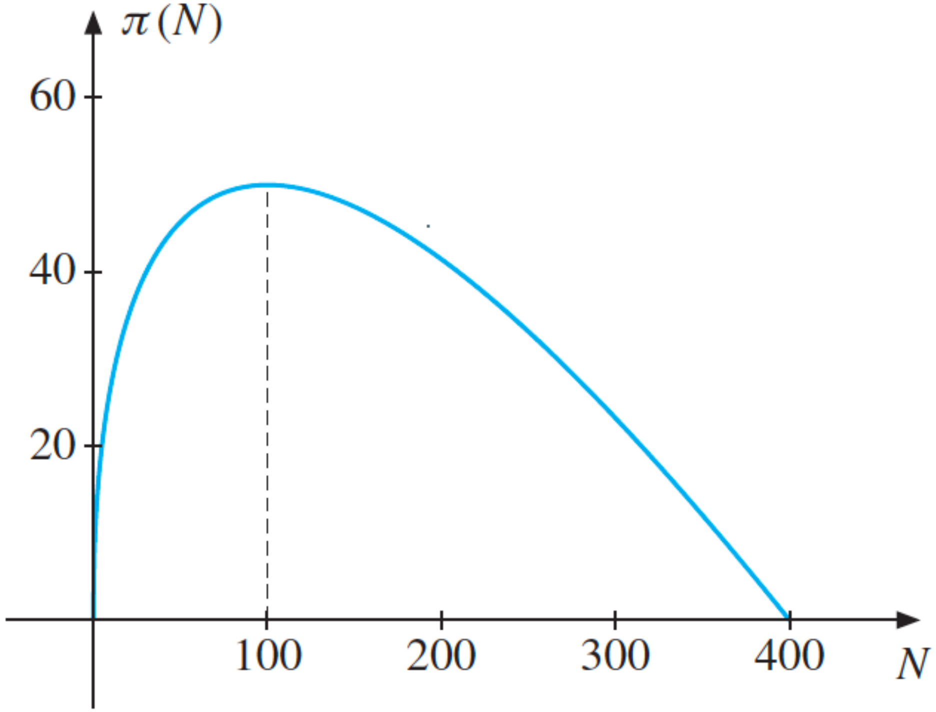
If the price of wheat is \$1.40 per bushel and the price of fertilizer is \$0.18 per pound, find the amount of fertilizer that maximizes profits.

Single Variable Optimization – Problem Set 1 (Solution)

(a) The profit function is

$$\pi(N) = PY(N) - qN = 10N^{1/2} - 0.5N, \qquad N \geq 0$$

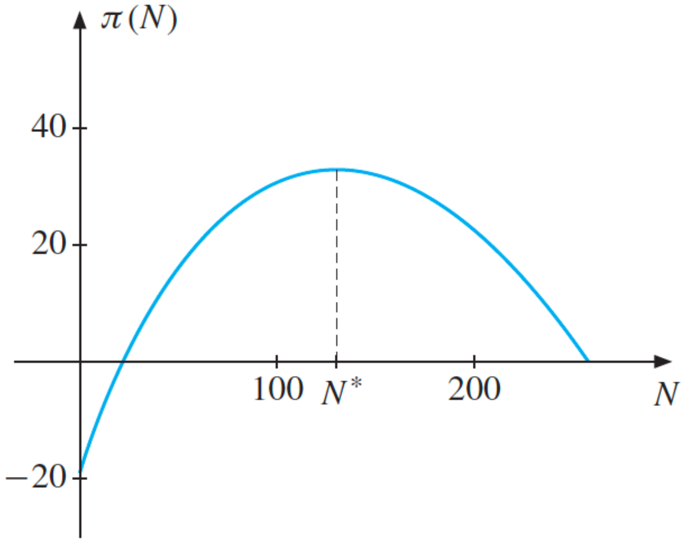
Then $\pi'(N) = 10(1/2)N^{-1/2} - 0.5 = 5N^{-1/2} - 0.5$. We see that $\pi'(N^*) = 0$ when $(N^*)^{-1/2} = 0.1$, hence $N^* = 100$. Moreover, it follows that $\pi'(N) \geq 0$ when $N \leq 100$ and $\pi'(N) \leq 0$ when $N \geq 100$. We conclude that $N^* = 100$ maximizes profits.



Single Variable Optimization – Problem Set 1 (Solution)

(b) In this case

$$\pi(N) = 1.4(-13.62 + 0.984N - 0.05N^{1.5}) - 0.18N$$
$$= -19.068 + 1.1976N - 0.07N^{1.5}$$



so that

$$\pi'(N) = 1.1976 - 0.07 \cdot 1.5N^{0.5} = 1.1976 - 0.105\sqrt{N}$$

Hence $\pi'(N^*) = 0$ when $0.105\sqrt{N^*} = 1.1976$. This implies that

$$\sqrt{N^*} = 1.1976/0.105 \approx 11.4 \quad \text{or} \quad N^* \approx (11.4)^2 \approx 130$$

By studying the expression for $\pi'(N)$, we see that $\pi'(N)$ is positive to the left of N^* and negative to the right of N^*. Hence, $N^* \approx 130$ maximizes profits.

Indefinite Integrals

Suppose we do not know the function F, but we have been told that its derivative is equal to x^2, so that $F'(x) = x^2$. What is F? Since the derivative of x^3 is $3x^2$, we see that $\frac{1}{3}x^3$ has x^2 as its derivative. But so does $\frac{1}{3}x^3 + C$ where C is an arbitrary constant, since additive constants disappear with differentiation.

In fact, let $G(x)$ denote an arbitrary function having x^2 as its derivative. Then the derivative of $G(x) - \frac{1}{3}x^3$ is equal to 0 for all x. But a function that has derivative equal to 0 for all x must be constant.

This shows that

$$F'(x) = x^2 \iff F(x) = \tfrac{1}{3}x^3 + C$$

with C as an arbitrary constant.

Problem Set 2

Assume that the marginal cost function of a firm is

$$C'(x) = 2x^2 + 2x + 5$$

and that the fixed costs are 100. Find the cost function $C(x)$.

Problem Set 2 - Solution

Assume that the marginal cost function of a firm is

$$C'(x) = 2x^2 + 2x + 5$$

and that the fixed costs are 100. Find the cost function $C(x)$.

Solution: Considering separately each of the three terms in the expression for $C'(x)$, we realize that the cost function must have the form $C(x) = \frac{2}{3}x^3 + x^2 + 5x + c$, because if we differentiate this function we obtain precisely $2x^2 + 2x + 5$. But the fixed costs are 100, which means that $C(0) = 100$. Inserting $x = 0$ into the proposed formula for $C(x)$ yields $c = 100$. Hence, the required cost function must be

$$C(x) = \frac{2}{3}x^3 + x^2 + 5x + 100$$

Indefinite Integrals

Suppose $f(x)$ and $F(x)$ are two functions of x having the property that $f(x) = F'(x)$ for all x in some interval I. We pass from F to f by taking the derivative, so the reverse process of passing from f to F could appropriately be called taking the **antiderivative**. But following usual mathematical practice, we call F an **indefinite integral** of f over the interval I, and denote it by $\int f(x)\,dx$. Two functions having the same derivative throughout an interval must differ by a constant, so:

$$\int f(x)\,dx = F(x) + C \quad \text{when} \quad F'(x) = f(x) \quad (C \text{ is an arbitrary constant})$$

Indefinite Integrals

The symbol $\int$ is the **integral sign**, and the function $f(x)$ appearing in (1) is the **integrand**. Then we write dx to indicate that x is the **variable of integration**. Finally, C is a **constant of integration**. We read it this way: The indefinite integral of $f(x)$ w.r.t. x is $F(x)$ plus a constant. We call it an *indefinite* integral because $F(x) + C$ is not to be regarded as one definite function, but as a whole class of functions, all having the same derivative f.

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

i.e., that the derivative of an indefinite integral equals the integrand.

Some Important Integrals

There are some important integration formulas which follow immediately from the corresponding rules for differentiation.

Let a be a fixed number $\neq -1$. Because the derivative of $x^{a+1}/(a+1)$ is x^a, one has

$$\int x^a \, dx = \frac{1}{a+1} x^{a+1} + C \qquad (a \neq -1)$$

This very important result states that the indefinite integral of any power of x (except x^{-1}) is obtained by increasing the exponent of x by 1, then dividing by the new exponent, and finally adding a constant of integration. Here are three prominent examples.

Some Important Integrals - Examples

$$\text{(a)} \int x\,dx = \int x^1\,dx = \frac{1}{1+1}x^{1+1} + C = \frac{1}{2}x^2 + C$$

$$\text{(b)} \int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{1}{-3+1}x^{-3+1} + C = -\frac{1}{2x^2} + C$$

$$\text{(c)} \int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + C = \frac{2}{3}x^{3/2} + C$$

Some Important Integrals

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

$$\int e^{ax}\, dx = \frac{1}{a}e^{ax} + C \qquad (a \neq 0)$$

$$\int a^x\, dx = \frac{1}{\ln a}a^x + C \qquad (a > 0 \text{ and } a \neq 1)$$

Some General Rules

$$\int af(x)\,dx = a\int f(x)\,dx \qquad (a \text{ is a constant})$$

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int [a_1 f_1(x)+\cdots+a_n f_n(x)]\,dx = a_1\int f_1(x)\,dx + \cdots + a_n\int f_n(x)\,dx$$

Problem Set 3

Evaluate: (a) $\int \frac{B}{r^{2.5}}\,dr$ (b) $\int (a + bq + cq^2)\,dq$ (c) $\int (1+t)^5\,dt$

Problem Set 3 - Solution

Evaluate: (a) $\int \frac{B}{r^{2.5}}\,dr$ (b) $\int (a + bq + cq^2)\,dq$ (c) $\int (1+t)^5\,dt$

$$\int \frac{B}{r^{2.5}}\,dr = B\int r^{-2.5}\,dr = B\frac{1}{-2.5+1}r^{-2.5+1} + C = -\frac{B}{1.5r^{1.5}} + C$$

$$\int (a + bq + cq^2)\,dq = aq + \tfrac{1}{2}bq^2 + \tfrac{1}{3}cq^3 + C$$

$$\int (1+t)^5\,dt = \tfrac{1}{6}(1+t)^6 + C$$

Definite Integrals

$$\int_a^b f(x)\,dx = \Big|_a^b F(x) = F(b) - F(a)$$

where F is any indefinite integral of f over an interval containing both a and b.

Definite Integrals Problem Set 4

Evaluate (a) $\int_{2}^{5} e^{2x}\,dx$ (b) $\int_{-2}^{2} (x - x^3 - x^5)\,dx$

Definite Integrals Problem Set 4 (Solution)

Evaluate (a) $\int_2^5 e^{2x}\,dx$ (b) $\int_{-2}^{2} (x - x^3 - x^5)\,dx$

(a) Because $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$, $\int_2^5 e^{2x}\,dx = \Big|_2^5 \frac{1}{2}e^{2x} = \frac{1}{2}e^{10} - \frac{1}{2}e^4 = \frac{1}{2}e^4(e^6 - 1)$.

(b) $\int_{-2}^{2} (x - x^3 - x^5)\,dx = \Big|_{-2}^{2} (\frac{1}{2}x^2 - \frac{1}{4}x^4 - \frac{1}{6}x^6) = (2 - 4 - \frac{64}{6}) - (2 - 4 - \frac{64}{6}) = 0.$

Properties of Definite Integrals

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx \qquad (\alpha \text{ an arbitrary number})$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Problem Set 5

Evaluate the following integrals

(a) $\displaystyle\int_0^5 (x + x^2)\,dx$ (b) $\displaystyle\int_{-2}^2 (e^x - e^{-x})\,dx$ (c) $\displaystyle\int_2^{10} \frac{dx}{x-1}$ (d) $\displaystyle\int_0^1 2xe^{x^2}\,dx$

(e) $\displaystyle\int_{-4}^4 (x-1)^3\,dx$ (f) $\displaystyle\int_1^2 (x^5 + x^{-5})\,dx$ (g) $\displaystyle\int_0^4 \tfrac{1}{2}\sqrt{x}\,dx$ (h) $\displaystyle\int_1^2 \frac{1+x^3}{x^2}\,dx$

Problem Set 5 (Solution)

Evaluate the following integrals

(a) $\displaystyle\int_0^5 (x+x^2)\,dx$ (b) $\displaystyle\int_{-2}^2 (e^x-e^{-x})\,dx$ (c) $\displaystyle\int_2^{10} \frac{dx}{x-1}$ (d) $\displaystyle\int_0^1 2xe^{x^2}\,dx$

(e) $\displaystyle\int_{-4}^4 (x-1)^3\,dx$ (f) $\displaystyle\int_1^2 (x^5+x^{-5})\,dx$ (g) $\displaystyle\int_0^4 \tfrac{1}{2}\sqrt{x}\,dx$ (h) $\displaystyle\int_1^2 \frac{1+x^3}{x^2}\,dx$

(a) $\displaystyle\Big|_0^5 (\frac{1}{2}x^2+\frac{1}{3}x^3) = 325/6$ (b) 0 (c) $\ln 9$ (d) $e-1$ (e) -136

(f) $687/64$ (g) $\displaystyle\int_0^4 \frac{1}{2}x^{1/2}\,dx = \Big|_0^4 \frac{1}{2}\cdot\frac{2}{3}x^{3/2} = \frac{8}{3}$

(h) $\displaystyle\int_1^2 \frac{1+x^3}{x^2}\,dx = \int_1^2 \left(\frac{1}{x^2}+x\right)dx = \Big|_1^2 \left(-\frac{1}{x}+\frac{1}{2}x^2\right) = 2$

Integration by Parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Integration by Parts – Problem Set 6

Evaluate the following: (a) $I = \int \frac{1}{x} \ln x \, dx$ (b) $J = \int x^3 e^{2x} \, dx$.

Integration by Parts – Problem Set 6 (Solution)

Evaluate the following: (a) $I = \int \frac{1}{x} \ln x \, dx$ (b) $J = \int x^3 e^{2x} \, dx$.

(a) Choosing $f(x) = 1/x$ and $g'(x) = \ln x$ leads nowhere. Choosing $f(x) = \ln x$ and $g'(x) = 1/x$ works better:

$$I = \int \frac{1}{x} \ln x \; dx = \int \underset{f(x)}{\underset{\downarrow}{\ln x}} \; \underset{g'(x)}{\underset{\downarrow}{\frac{1}{x}}} \; dx = \underset{f(x)}{\underset{\downarrow}{\ln x}} \, \underset{g(x)}{\underset{\downarrow}{\ln x}} - \int \underset{f'(x)}{\underset{\downarrow}{\frac{1}{x}}} \; \underset{g(x)}{\underset{\downarrow}{\ln x}} \; dx$$

In this case, the last integral is exactly the one we started with, namely I. So it must be true that $I = (\ln x)^2 - I + C_1$ for some constant C_1. Solving for I yields $I = \frac{1}{2}(\ln x)^2 + \frac{1}{2}C_1$. Putting $C = \frac{1}{2}C_1$, we conclude that

$$\int \frac{1}{x} \ln x \, dx = \tfrac{1}{2}(\ln x)^2 + C$$

Integration by Parts – Problem Set 6 (Solution)

(b) We begin by arguing rather loosely as follows. Differentiation makes x^3 simpler by reducing the power in the derivative $3x^2$ from 3 to 2. On the other hand, e^{2x} becomes about equally simple whether we differentiate or integrate it. Therefore, we choose $f(x) = x^3$ and $g'(x) = e^{2x}$, so that integration by parts tells us to differentiate f and integrate g'. This yields $f'(x) = 3x^2$ and we can choose $g(x) = \frac{1}{2}e^{2x}$. Therefore,

$$J = \int x^3 e^{2x}\, dx = x^3\left(\tfrac{1}{2}e^{2x}\right) - \int (3x^2)\left(\tfrac{1}{2}e^{2x}\right) dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{2}\int x^2 e^{2x}\, dx \qquad \text{(i)}$$

The last integral *is* somewhat simpler than the one we started with because the power of x has been reduced. Integrating by parts once more yields

Integration by Parts – Problem Set 6 (Solution)

$$J = \int x^3 e^{2x}\,dx = x^3(\tfrac{1}{2}e^{2x}) - \int (3x^2)(\tfrac{1}{2}e^{2x})\,dx = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{2}\int x^2e^{2x}\,dx \qquad \text{(i)}$$

The last integral *is* somewhat simpler than the one we started with because the power of x has been reduced. Integrating by parts once more yields

$$\int x^2e^{2x}\,dx = x^2(\tfrac{1}{2}e^{2x}) - \int (2x)(\tfrac{1}{2}e^{2x})\,dx = \tfrac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx \qquad \text{(ii)}$$

Using integration by parts a third and final time gives

$$\int xe^{2x}\,dx = x(\tfrac{1}{2}e^{2x}) - \int \tfrac{1}{2}e^{2x}\,dx = \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C \qquad \text{(iii)}$$

Successively inserting the results of (iii) and (ii) into (i) yields (with $3C/2 = c$):

$$J = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{4}xe^{2x} - \tfrac{3}{8}e^{2x} + c$$

It is a good idea to double-check your work by verifying that $dJ/dx = x^3e^{2x}$.

Thank you and see you next time !

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