Màster universitari en Estadística i Investigació Operativa

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH — UNIVERSITAT DE BARCELONA

Lifetime Data Analysis, Course 2020/21

Exercises Topics 1 and 2

Delivery of exercises: October 19, 2021.

Exercise 1 (2 points)

The time (in days) until the first failure of a vending machine, T, is supposed to follow a Weibull distribution with parameters $k = 2$ y $\rho = 0.03$.

(a) Which are the probabilities that the machine works well during 40 and 60 days, respectively?

(b) Which is the 80% quantile of T? Interpret its value.

The 3-parameter Weibull distribution can be used to model survival times, if there is a minimum survival time (G). Its survival function has the following expression:

$$S(t) = \begin{cases} 1 & \text{if } t < G, \\ \exp\big(-\rho^k (t-G)^k\big) & \text{if } t \geq G. \end{cases}$$

(c) Give the expressions of the risk and density functions of this 3-parameter Weibull distribution.

(d) Given $k = 2, \rho = 0.03$, and assuming a minimum time without any failure of $G = 15$ (days), find the mean and median survival times.

Exercise 2 (2 points)

The hazard function of a piecewise exponential distribution is given as follows:

$$\lambda(t) = \lambda_j, \quad \forall t \in [\pi_{j-1}, \pi_j),$$

where $0 < \pi_1 < \pi_2 < \cdots < \pi_k < \pi_{k+1} = \infty$ and $j \in \{1, \ldots, k+1\}$.

(a) What are the expressions of the density and survival functions at a time $t \in [\pi_{j-1}, \pi_j)$?

(b) Given $k = 2$, draw the hazard, density, and survival functions for $\lambda_1 = 0.15$, $\lambda_2 = 0.1$, $\lambda_3 = 0.2$, $\pi_1 = 7$, and $\pi_2 = 13$.

Exercise 3 (1.5 points)

In an animal experiment on the development of breast cancer, mice are injected a carcinogenic substance at the age of seven weeks. Six weeks after the cancer induction, the mice are examined for the first time and from then, weekly examinations are carried out during a period of 14 weeks in order to study the possible cancer development. The objective of the study consists of estimating the distribution function of the time (T) from cancer induction until the development of a palpable breast cancer in mice.

In each of the following situations, identify the type of censoring and indicate the time interval that contains T taking into account that T is measured in days. Which is the contribution to the likelihood function in each case?

(a) The first mouse is detected a palpable cancer at the first examination.

(b) In the case of the second mouse, a cancer is detected in the 13th week. One week before, the cancer was not palpable yet.

(c) One mouse dies without any cancer symptoms before it is examined for the first time. It is unknown which day this mouse died.
(d) Another mouse dies, also without any cancer symptoms the 37th day after the cancer induction.
(e) The last mouse survives the study without any signs of breast cancer.

Exercise 4 (2 points)
Let T and C be, respectively, the survival time of interest and the censoring time, which follow exponential distributions with parameters ρ (T) λ (C), respectively. We assume that T and C are independent and define the random variables Y and δ

$$Y = \min(T, C) \quad \text{and} \quad \delta = \begin{cases} 1 & \text{if } T \leq C, \\ 0 & \text{if } T > C. \end{cases}$$

(a) Find the value of $P(\delta = 1)$ and provide an interpretation of the result.
(b) Which is the distribution of Y?

Exercise 5 (2.5 points)
Table 1 contains the times from bone marrow transplant until relapse and death, respectively, in ten patients with leukemia. By the end of the study, six patients were alive, four of whom had not suffered any relapse neither.

Table 1: Times (in months) until relapse and death in patients with a bone marrow transplant.

	Time until			Time until	
Patient	**Relapse**	**Death**	**Patient**	**Relapse**	**Death**
1	5	11	6	17	28+
2	8	12	7	16+	16+
3	12	15	8	17+	17+
4	24	33+[a]	9	19+	19+
5	32	45	10	30+	30+

[a] '+' indicates right censoring

We assume that the time until relapse follows an exponential distribution with parameter λ, $T_1 \sim \mathcal{E}(\lambda)$, and that the time until death follows a Weibull distribution with parameters ρ and k, $T_2 \sim \mathcal{W}(\rho, k)$.
(a) Which is the value of the maximum likelihood estimator of λ? Give an interpretation of that value.
(b) Find the estimated mean residual lifetime of T_1 after 10 months.
(c) Give the expression of the likelihood function of the parameters ρ and k.
(d) Use the function `fitdistcens` of the `fitdistrplus` package[1] to estimate the parameters of the Weibull distribution.

Note: The shape and scale parameters of the R functions `dweibull`, `qweibull`, or `pweibull` are related as follows to the parameters ρ and k: `shape` $= k$ and `shape` $= 1/\rho$.

[1] M.L. Delignette-Muller & C. Dutang (2015). fitdistrplus: An R Package for Fitting Distributions. *Journal of Statistical Software*, 64(4), 1–34. URL: `https://www.jstatsoft.org/article/view/v064i04`.