# My Favorite Plot

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## 1 Background Physics

As opposed to the galilean relativity that preceded it, special relativity requires the speed of light ($c$) to be constant in all inertial frames of reference. To impose this restriction, as well as the requirement that all laws of physics behave the same regardless of position and velocity, the transformations between inertial (non-accelerating) frames of reference are altered. These transformations are referred to as Lorentz transformations, which are described by the following equations [2]:

$$\begin{aligned} t' &= \gamma\left(t - u^2 x/c^2\right) \\ x' &= \gamma\left(x - ut\right) \end{aligned} \tag{1}$$

This transformation has an invariant between all inertial frames, the space-time interval. All observers in inertial frames of reference will measure the space-time interval between two events to be the same. The space-time interval between an event **and the origin** (the event at (0, 0)) is calculated as follows (w.r.t. the origin implies $x = \Delta x$, $t = \Delta t$):

$$\Delta s^2 = c^2 t^2 - x^2 = c^2 \tau^2 \tag{2}$$

Notice that the space-time interval is nothing more than a scaled proper time interval (the elapsed time measured by an inertial clock on the world-line between both events) for the case that the space-time interval is calculated for two causally related events [1] [1]. This is relevant to the plot generation, as two events with the same proper time interval (w.r.t. the origin) will have the same space-time interval (w.r.t. the origin). A closer look at the equation for the space-time interval reveals that it has the same form as a hyperbola for which the y-axis is the major axis, where $y = ct$, and $c^2 = \Delta s^2$. This implies that holding the space-time interval constant (w.r.t. the origin) produces a hyperbola.

$$y^2 - x^2 = c^2 \rightarrow c^2 t^2 - x^2 = \Delta s^2 \tag{3}$$

[1] Note that for cases where the space-time interval is negative and hence space-like, we deal with proper length and not proper time.

## 2 Plot Generation & Results

This plot (Figure 1) was generated using Python and the `matplotlib` library alongside `NumPy` and a small module I have written for automating the generation of space-time diagrams. This code (which should also be fully documented) is available on GitHub at: https://github.com/ArvinSKushwaha/my_favorite_plot.

Figure 1 displays a set of world-lines representing the path of particles traveling with 50 different velocities relative to the observer. The left-most path represents a particle traveling at $-0.9c$ and the right-most path represents a particle traveling at $0.9c$, which each intermediate particle having a measured velocity of $1.8c/49 \approx 0.0367c$ more than the particle to its left (on the diagram).

The 9 markers on each world-line represent the points at proper times of -4, -3, -2, -1, 0, 1, 2, 3, and 4 light-seconds. Because of the invariance of the space-time interval, every marker on the hyperbola will have the same space-time interval. As more and more markers are added, it becomes apparent that the set of all events with a constant space-time interval between the event and the origin forms a hyperbola.

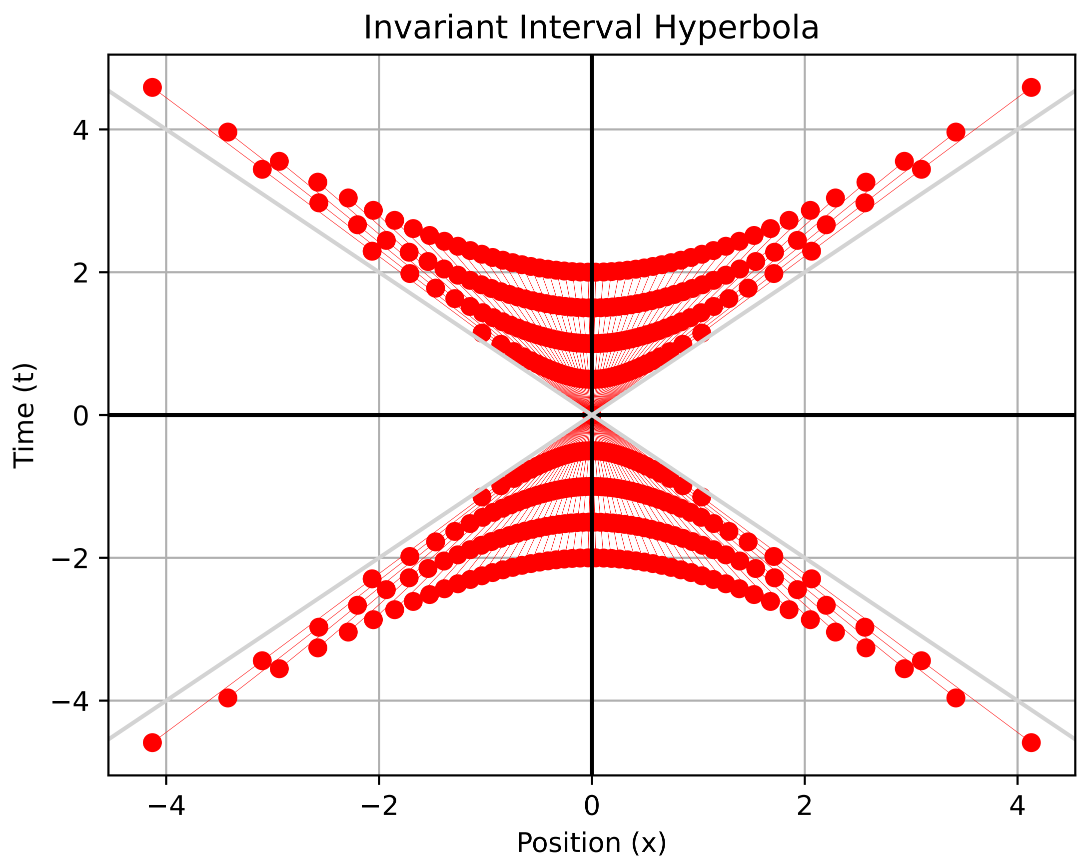


Figure 1: A plot of the world-lines of particles moving with different velocities relative to the observer frame, where every marker in the hyperbolae formed has the same proper time (and hence space-time interval relative to the origin).

# 3 Discussion/Conclusion

The limits for low speeds ($|u| << c$, $\gamma \approx 1$) for the Lorentz transformations are the Galilean transformations [1]:

$$\begin{aligned} t' &= t \\ x' &= x - ut \end{aligned} \tag{4}$$

As most of the dynamical systems we deal with on a day-to-day basis is non-relativistic, we have developed intuition for galilean relativity. On the other hand, special relativity has a reputation for being strange and counter-intuitive. Instead of the invariant time and length we see in galilean relativity, special relativity's space-time interval is not as easy to visualize. With the knowledge that every event along a world-line is transformed along the hyperbola corresponding to its space-time interval w.r.t. the origin, imagining the transformation of events between reference frames becomes more intuitive.

# References


[1] Peter Collier. *A most incomprehensible thing: Notes towards a very gentle introduction to the mathematics of Relativity*. Incomprehensible Books, 2017.

[2] Kenneth S. Krane. *Modern physics*. Wiley, 2020.