Developed by MEI Mathematics Education Innovation

integral

AQA AS Mathematics: Integration

Section 2: Finding the area under a curve

Notes and Examples

These notes contain the following subsections:

Definite integration

When you learned about differentiation, you saw that the value of a derivative at a particular point gives the rate of change (or gradient) of a function at that point.

Working out the value of an integral at a particular point doesn't have any meaning, because of the arbitrary constant. However, you can work out the value of an integral between two points by subtracting the value at the first point from the value at the second point, which means that the arbitrary constant cancels out.

This is called a **definite integral**. The values of x at the two points are called the **limits** of the definite integral.

The definite integral from a to b of a function $\mathrm{f}(x)$, which is written as $\int_a^b \mathrm{f}(x)\mathrm{d}x$, is found as follows:

- Integrate $\mathrm{f}(x)$ – suppose we call the integral $\mathrm{g}(x)$
- Write the integral in square brackets, with the limits on the right hand side: $[\mathrm{g}(x)]_a^b$
- Work out the value of $\mathrm{g}(x)$ with $x = a$ and $x = b$, and subtract:

$$[\mathrm{g}(x)]_a^b = \mathrm{g}(b) - \mathrm{g}(a).$$

Example 1

Evaluate $\int_1^2 (2x - x^2)\mathrm{d}x$.

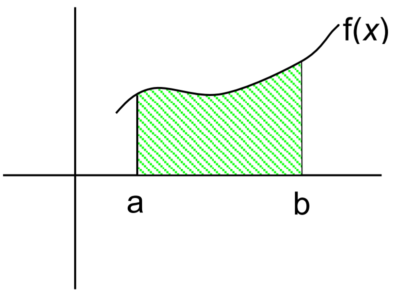
Solution

$$\int_1^2 \left(2x - x^2\right) \mathrm{d}x = \left[x^2 - \tfrac{1}{3}x^3\right]_1^2$$
$$= \left(2^2 - \tfrac{1}{3}\times 2^3\right) - \left(1^2 - \tfrac{1}{3}\times 1^3\right)$$
$$= 4 - \tfrac{8}{3} - 1 + \tfrac{1}{3}$$
$$= \tfrac{2}{3}$$

Be careful with signs!

The definite integral as an area

The definite integral $\int_a^b \mathrm{f}(x)\mathrm{d}x$ calculates the area between the curve $y = \mathrm{f}(x)$ and the x-axis.



Example 2

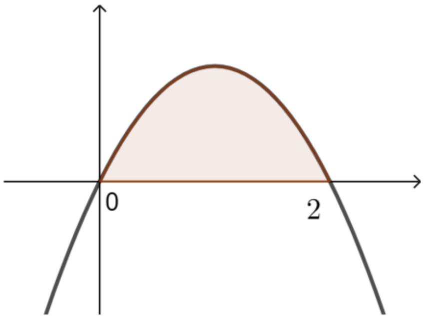
Find the area enclosed by the curve $y = 2x - x^2$ and the x-axis.

Solution

You need to know where the curve crosses the x-axis, so that you know what the limits of integration are.

$$2x - x^2 = 0$$
$$x(2 - x) = 0$$
$$x = 0 \text{ or } x = 2$$

It is helpful to sketch the curve.



Area $= \int_0^2 (2x - x^2)\,dx$

$= \left[x^2 - \tfrac{1}{3}x^3\right]_0^2$

$= 2^2 - \tfrac{8}{3} - 0$

$= \tfrac{4}{3}$

Area of a region below the *x*-axis

In Example 2, the curve was above the x-axis, so that the value of y is positive, and the definite integral works out to be positive. However, if the curve is below the x-axis, so that y is negative, the integral works out to be negative. This is shown in Example 3.

Example 3

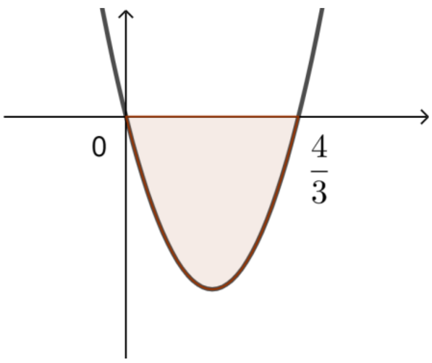
Find the area enclosed by the curve $y = 3x^2 - 4x$ and the x-axis.

Solution

The curve crosses the x-axis when $3x^2 - 4x = 0$

$\Rightarrow x(3x-4) = 0$

$\Rightarrow x = 0 \text{ or } x = \tfrac{4}{3}$



$$\text{Area} = \int_0^{\frac{4}{3}} (3x^2 - 4x)\,\mathrm{d}x = \left[x^3 - 2x^2\right]_0^{\frac{4}{3}}$$
$$= \left(\left(\tfrac{4}{3}\right)^3 - 2\left(\tfrac{4}{3}\right)^2\right) - \left(0^3 - 2\times 0^2\right)$$
$$= -\tfrac{32}{27}$$

> This integral works out to be negative because the curve is below the x-axis.
> Sketching the curve is important because this means you expect the integral to be negative.

The area is $\frac{32}{27}$ square units.

Notice that you should give your final answer as positive, since an area cannot be negative. However, remember that this only applies to definite integrals which are being used to find an area – if you are just asked to work out the value of a definite integral, then the answer may be positive or negative.

Area of a region partly below and partly above the *x*-axis

Sometimes you may need to find the area of a region which is partly above and partly below the x-axis. It is particularly important to sketch a graph in these situations. This is illustrated in the next example.

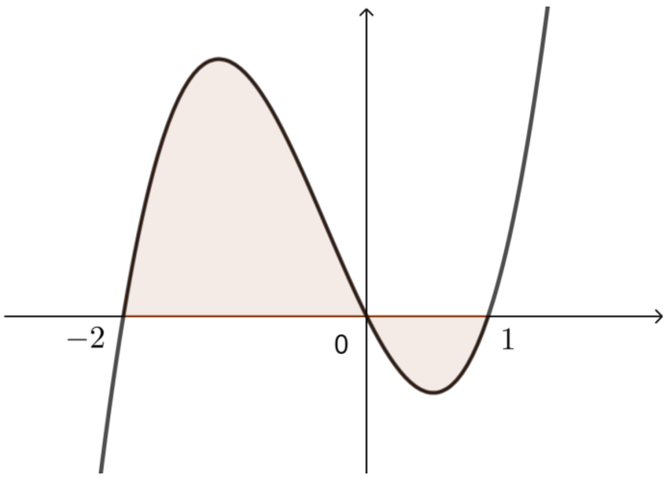
Example 4

(a) Find the total area enclosed by the curve $y = x(x-1)(x+2)$ and the x-axis.

(b) Find the value of $\int_{-2}^{1} x(x-1)(x+2)\,\mathrm{d}x$

Solution

(a) $y = x(x-1)(x+2) = x^3 + x^2 - 2x$



Part of the region is above the x-axis, and part of it is below. You need to calculate these separately.

Area between $x=-2$ and $x=0$ is given by $\int_{-2}^{0}(x^3+x^2-2x)\,\mathrm{d}x$

$$=\left[\tfrac{1}{4}x^4+\tfrac{1}{3}x^3-x^2\right]_{-2}^{0}$$
$$=0-\left(\tfrac{(-2)^4}{4}+\tfrac{(-2)^3}{3}-(-2)^2\right)$$
$$=-4+\tfrac{8}{3}+4$$
$$=\tfrac{8}{3}$$

Area between $x=0$ and $x=1$ is given by $\int_{0}^{1}(x^3+x^2-2x)\,\mathrm{d}x$

$$=\left[\tfrac{1}{4}x^4+\tfrac{1}{3}x^3-x^2\right]_{0}^{1}$$
$$=\left(\tfrac{1^4}{4}+\tfrac{1^3}{3}-(1)^2\right)-0$$
$$=\tfrac{1}{4}+\tfrac{1}{3}-1$$
$$=-\tfrac{5}{12}$$

This is negative, as expected from the graph.

Total area $=\frac{8}{3}+\frac{5}{12}=\frac{37}{12}$

(b) $\int_{-2}^{1}x(x-1)(x+2)\,\mathrm{d}x=\int_{-2}^{1}(x^3+x^2-2x)\,\mathrm{d}x$

$$=\left[\tfrac{1}{4}x^4+\tfrac{1}{3}x^3-x^2\right]_{-2}^{1}$$
$$=\left(\tfrac{1^4}{4}+\tfrac{1^3}{3}-(1)^2\right)-\left(\tfrac{(-2)^4}{4}+\tfrac{(-2)^3}{3}-(-2)^2\right)$$
$$=-\tfrac{5}{12}+\tfrac{8}{3}$$
$$=\tfrac{9}{4}$$

Notice in the example above that when the definite integral is worked out, you are adding a negative value and a positive value. So the result is the difference between the two areas, and not the total area.

So if you are just calculating a definite integral, you can do it in one calculation, but if you are finding the area of a region partly above and partly below the x-axis, you must split it up and find the areas separately.

Area between a line and a curve

Sometimes you may need to find the area between a line and a curve.

You can do this by subtracting the area under a curve from a rectangle, triangle or trapezium as appropriate.

This is shown in the example below.

Example 5

Find the area between the curve $y = x^2 - 2x + 2$ and

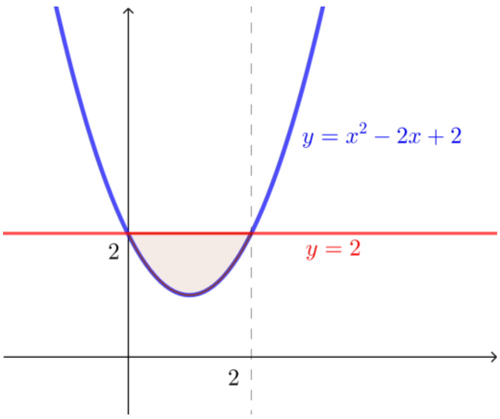
(a) the line $y = 2$
(b) the line $y = x + 2$

Solution

(a) The line and the curve intersect where $x^2 - 2x + 2 = 2$

$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x = 0 \text{ or } x = 2$$

So the intersection points are at (0, 2) and (2, 2).



The area of the rectangle bounded by the lines $y = 2$, $x = 2$ and the coordinate axes is $2 \times 2 = 4$.

The area under the curve is given by $\int_0^2 (x^2 - 2x + 2)\,\mathrm{d}x$

$$= \left[\tfrac{1}{3}x^2 - x^2 + 2x\right]_0^2$$

$$= \tfrac{8}{3} - 4 + 4 - 0$$

$$= \tfrac{8}{3}$$

So the area between the line and the curve is $4 - \frac{8}{3} = \frac{4}{3}$

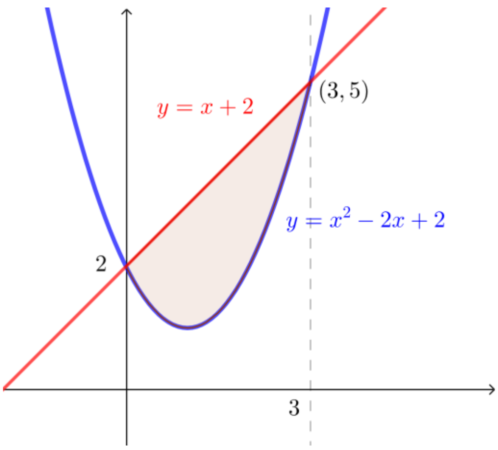
(b) The line and the curve intersect where $x^2 - 2x + 2 = x + 2$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$x = 0 \text{ or } x = 3$$

So the intersection points are (0, 2) and (3, 5).



The area of the trapezium bounded by the lines $y = x + 2$, $x = 3$ and the coordinate axes is $\frac{1}{2} \times 3(2+5) = 10.5$

The area under the curve is given by $\int_0^3 (x^2 - 2x + 2)\,\mathrm{d}x$

$$= \left[\tfrac{1}{3}x^3 - x^2 + 2x\right]_0^3$$

$$= 9 - 9 + 6 - 0$$

$$= 6$$

So the area between the line and the curve is $10.5 - 6 = 4.5$