AQA AS Mathematics: Differentiation

Section 2: Maximum and minimum points

Notes and Examples

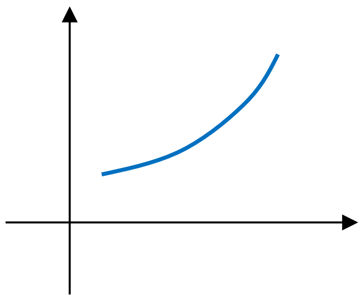
These notes contain the following subsections:

Increasing and decreasing functions
Turning points
Sketching the graph of a derivative

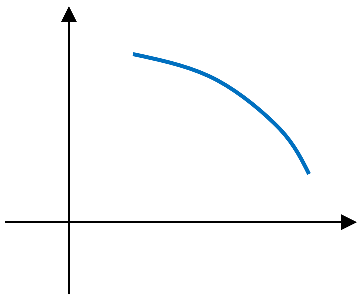
Increasing and decreasing functions

When the gradient $\frac{\mathrm{d}y}{\mathrm{d}x}$ of a graph is positive, the value of y is increasing.

Similarly, when the gradient is negative, the value of y is decreasing.



$\frac{\mathrm{d}y}{\mathrm{d}x} > 0$
Increasing function



$\frac{\mathrm{d}y}{\mathrm{d}x} < 0$
Decreasing function

Example 1

Find the range of values of x for which $y = x^3 - 3x^2 - 9x + 4$ is increasing.

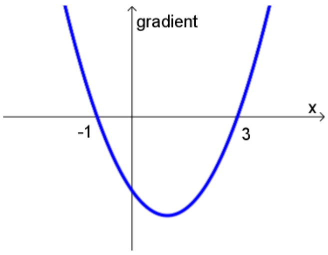
Solution

$y = x^3 - 3x^2 - 9x + 4$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x - 9$

The function is increasing if $\frac{dy}{dx} > 0$

$$\Rightarrow 3x^2 - 6x - 9 > 0$$
$$\Rightarrow x^2 - 2x - 3 > 0$$
$$\Rightarrow (x-3)(x+1) > 0$$
$$\Rightarrow x < -1 \text{ or } x > 3$$



So the function is increasing for $x < -1$ and for $x > 3$

Turning points

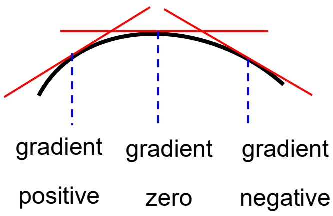
Points on a curve where the tangent is horizontal are called stationary points, or turning points.

At these points, the gradient of the curve is zero, so $\frac{dy}{dx} = 0$.

You will be looking at two types of stationary point:

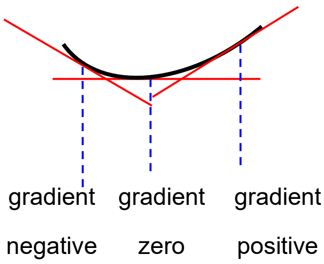
Local maximum

The gradient is positive to the left, zero at the point, and negative to the right



Local minimum

The gradient is negative to the left, zero at the point, and positive to the right.



To distinguish between these, you can test the value of the derivative either side of the stationary point, to see whether the gradient is positive or negative.

This is shown in the next example.

Example 2

Find the stationary points on the curve $y = x^3 - 3x^2 + 1$, investigate their nature, and sketch the curve.

Solution

Step 1: Differentiate the function.

$$y = x^3 - 3x^2 + 1$$
$$\frac{dy}{dx} = 3x^2 - 6x$$

Step 2: Solve $\frac{dy}{dx} = 0$

$$3x^2 - 6x = 0$$
$$3x(x-2) = 0$$
$$x = 0 \text{ or } x = 2$$

Step 3: Calculate the y–coordinates for these values of x (called the stationary values).

When $x = 0$, $y = 0^3 - 3\times 0^2 + 1 = 1$

When $x = 2$, $y = 2^3 - 3\times 2^2 + 1 = 8 - 12 + 1 = -3$

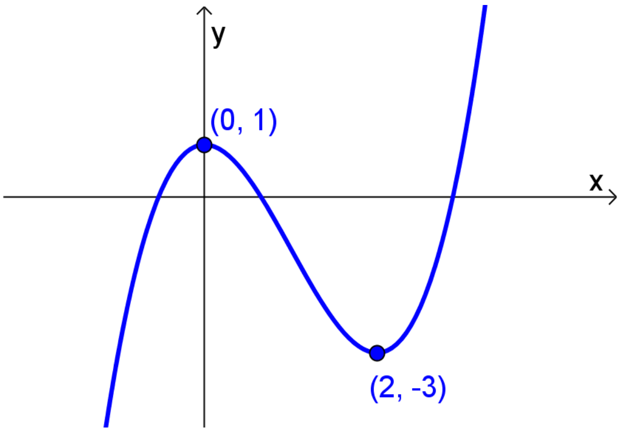
So the stationary points are (0, 1) and (2, –3).

Step 4: Use a table to investigate the sign of $\frac{dy}{dx}$ for values of x either side of the stationary values

x	-1	0	1	2	3
$\frac{dy}{dx}$	9 positive	0	-3 negative	0	9 positive

So (0, 1) is a local maximum and (2, -3) is a local minimum

Step 5: Sketch the curve.



Sketching the graph of a derivative

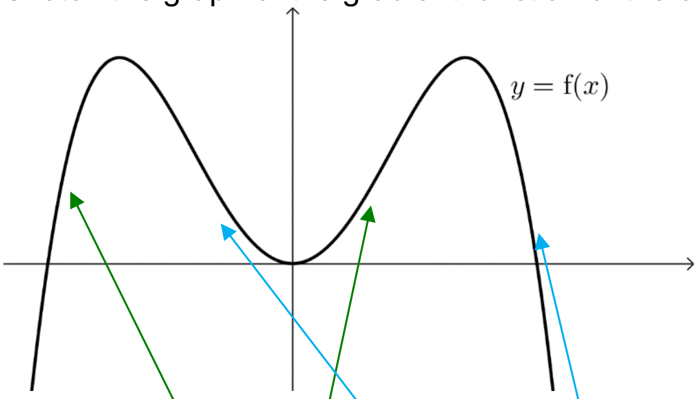
If you have the graph of a function $y = \mathrm{f}(x)$, you can sketch the graph of the corresponding gradient function, $y = \mathrm{f}'(x)$, by thinking about what is happening at different points on the graph.

- Where there is a turning point, the gradient of $y = \mathrm{f}(x)$ is zero so the gradient graph $y = \mathrm{f}'(x)$ crosses the x-axis
- Where the graph is increasing, the gradient of $y = \mathrm{f}(x)$ is positive so the gradient graph will be above the x-axis
- Where the graph is decreasing, the gradient of $y = \mathrm{f}(x)$ is negative so the gradient graph will be below the x-axis

The next example shows how this is done.

Example 3

Sketch the graph of the gradient function of the curve shown below.



Solution