AQA AS Mathematics: Differentiation

Section 1: Introduction to differentiation

Notes and Examples

These notes contain the following subsections:

What is differentiation?

In this section, you will be studying the relationship between the position of a point on a curve and the gradient of the curve.

Straight lines are, by definition, lines of constant gradient. Curves, on the other hand, have varying gradient – the gradient depends on whereabouts you are on the curve. Differentiation is the process of finding the gradient at any point on a curve from the equation of the curve.

Differentiation, together with its reverse process, called integration, form the branch of mathematics called calculus. The discovery of calculus (Made in the 17th century by Isaac Newton in England and, independently, by Gottfried von Leibnitz in Germany) was one of the most significant advances in the history of mathematics and science, and was crucial to unlocking the mathematical basis of our planetary system.

Differentiation is the process of finding the gradient function, or derivative, or derived function. Given an equation for y in terms of x, the gradient function or derivative is written $\frac{\mathrm{d}y}{\mathrm{d}x}$, and gives the gradient of the curve in terms of x.

Investigating gradients

You can investigate how the gradients of chords approach the gradient of a tangent using graphing software, or the Explore resource **The gradient of a curve** available on Integral. You can then go on to investigate the pattern in the value of the gradient at different points on a curve.

Rules for finding derivatives

If y is a polynomial function (made up of powers of x), you may have been able to spot patterns in the gradient function by investigating tangents.

This leads to the following rules will enable you to find the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$:

The derivative of x^n is nx^{n-1},

The derivative of kx^n is knx^{n-1},

The derivative of a constant is zero.

The derivative of a sum (or difference) is the sum (or difference) of the derivatives.

You will learn how these results are formally proved in a later section.

Example 1

Differentiate $y = 2x^3 - 5x^2 + 4$

Solution

The derivative of $2x^3$ is $2 \times 3x^2 = 6x^2$

The derivative of $5x^2$ is $5 \times 2x = 10x$

The derivative of 4 is 0.

So $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 10x$.

The next example involves an expression which is the product of two functions. You cannot differentiate this by differentiating each function separately and then multiplying the results, i.e. the derivative of a product of two functions is not the product of the derivatives! So with examples involving brackets, you will need to multiply out the brackets first. (There is a rule for differentiating the product of two functions, but you do not need to know this yet.)

Example 2

(a) Find the derivative of $y = (x-2)(x^2+1)$.
(b) Hence find the gradient of the curve at the point (2, 0).
(c) Find the coordinates of the points where the gradient is zero.

Solution

(a) $y=(x-2)(x^2+1)=x^3-2x^2+x-2$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2-4x+1$$

(b) Substituting $x=2$ into the gradient function,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3\times2^2-4\times2+1=5$$

so the gradient of the curve at (2, 0) is 5.

(c) The gradient of the curve is zero when $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$$\Rightarrow 3x^2-4x+1=0$$
$$\Rightarrow (3x-1)(x-1)=0$$
$$\Rightarrow x=\tfrac{1}{3} \text{ or } x=1$$

Now calculate the y coordinates for these values of x.

When $x=\frac{1}{3}$, $y=\left(\frac{1}{3}-2\right)\left(\frac{1}{9}+1\right)=-\frac{5}{3}\times\frac{10}{9}=-\frac{50}{27}$

When $x=1$, $y=(1-2)(1^2+1)=-2$.

So the points on the curve with gradient zero are (1, –2) and $\left(\frac{1}{3},-\frac{50}{27}\right)$

The points where the gradient is zero are called the turning points or stationary points of the curve. You will look at such points in more detail in Section 2.

The next example involves the quotient of two functions (i.e. one function divided by another). As with products, the derivative of a quotient is not the quotient of the derivatives. You need to divide the fraction first.

Example 3

Differentiate $\dfrac{3x^2-4x^4}{2x}$.

Solution

$$y=\frac{3x^2-4x^4}{2x}=\frac{3x^2}{2x}-\frac{4x^4}{2x}$$
$$=\tfrac{3}{2}x-2x^3$$

$$\frac{dy}{dx} = \tfrac{3}{2} - 6x^2$$

Rates of change

In the work you have done so far, you have looked at the derivative of a function as representing the gradient of the graph of that function. However, more generally, the derivative $\frac{dy}{dx}$ represents the rate of change of y with respect to x. The variables need not be y and x – they can be any letter, representing any quantity.

Finding a rate of change is no different from finding the gradient of a graph – in both cases you differentiate. However, you must make sure that if letters other than y and x are being used, you use the same letters.

e.g. if you are given that $s = t^2 + 3t - 1$, and you are asked to find the rate of change of s with respect to t, you are finding $\frac{ds}{dt}$, not $\frac{dy}{dx}$.

One important application of rates of change is in the motion of a particle. If you study Mechanics you will learn more about this.

Finding tangents and normals to curves

The gradient of a tangent to a curve at a particular point is the same as the gradient of the curve at that point. So to find the equation of a tangent to a curve, you first need to find the gradient m of the curve via differentiation. You can then substitute m and the coordinates (x_1, y_1) of the point on the curve into the formula:

$$y - y_1 = m(x - x_1)$$

Example 4

Find the equation of the tangent to the curve $y = 2x^3 - 3x$ at the point with x-coordinate 1.

Solution

To find the gradient of the tangent, first differentiate the equation of the curve and substitute the x-coordinate

$$y = 2x^3 - 3x \Rightarrow \frac{dy}{dx} = 6x^2 - 3$$

At the point with x-coordinate 1

$$\frac{dy}{dx} = (6 \times 1^2) - 3 = 3$$

The gradient of the tangent is therefore 3.

Find the y-coordinate of the point where $x = 1$ by substituting into the equation of the curve.

$y = 2x^3 - 3x$

When $x = 1$, $y = (2 \times 1^3) - (3 \times 1) = 2 - 3 = -1$

Use the formula for the equation of a line with $m = 3$, $x_1 = 1$ and $y_1 = -1$

$$y - y_1 = m(x - x_1)$$
$$y - (-1) = 3(x - 1)$$
$$y + 1 = 3x - 3$$
$$y = 3x - 4$$

This is the required equation of the tangent.

The normal to a curve is the line perpendicular to the tangent. Remember that the gradient of a line perpendicular to a line with gradient m is m', where

$$m' = -\frac{1}{m}.$$

Example 5

Show that the normal to the curve $y = 2x^2 - 3x$ at the point (1, –1) passes through the origin.

Solution

First, find the gradient of the tangent by differentiating y.

$$y = 2x^2 - 3x \Rightarrow \frac{dy}{dx} = 4x - 3$$

At the point with x–coordinate 1, $\frac{dy}{dx} = (4 \times 1) - 3 = 1$

Gradient of the tangent = 1, so gradient of the normal = $-\frac{1}{1}=-1$

Use the formula for the equation of a line with $m=-1$, $x_1=1$ and $y_1=-1$

$$y-y_1=m(x-x_1)$$
$$y-(-1)=-1(x-1)$$
$$y+1=-x+1$$
$$y=-x$$

This line passes through the origin.

Example 6

A curve has equation $y=x^3-x^2+x+2$.

(a) Find the x-coordinates of the points on the curve with gradient 6.
(b) Find the x-coordinates of the points on the curve for which the gradient of the normal is $-\frac{1}{2}$.

Solution

$$y=x^3-x^2+x+2 \Rightarrow \frac{dy}{dx}=3x^2-2x+1$$

(a) When gradient = 6

$$3x^2-2x+1=6$$
$$\Rightarrow 3x^2-2x-5=0$$
$$\Rightarrow (3x-5)(x+1)=0$$
$$\Rightarrow x=\tfrac{5}{3} \text{ or } x=-1$$

(b) Gradient of normal = $-\frac{1}{2}$ $\Rightarrow$ gradient of curve = 2

When gradient = 2

$$3x^2-2x+1=2$$
$$\Rightarrow 3x^2-2x-1=0$$
$$\Rightarrow (3x+1)(x-1)=0$$
$$\Rightarrow x=-\tfrac{1}{3} \text{ or } x=1$$