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integral

AQA AS Mathematics: Differentiation

Section 3: Extending the rule

Notes and Examples

These notes contain the following subsections:

Differentiating functions involving negative and fractional powers

You already know that the derivative, or gradient of x^n, where n is a positive integer, is given by nx^{n-1}.

In fact this formula for the derivative of x^n is true not only when n is a positive integer, but for all real values of n, including negative numbers and fractions.

Example 1

Differentiate the following functions.

(a) $y = \dfrac{1}{x}$

(b) $y = x^2\sqrt{x}$

(c) $y = \dfrac{1}{\sqrt{x}}$

Solution

(a) $y = \dfrac{1}{x} = x^{-1}$

First write the expression as a power of x

$\dfrac{dy}{dx} = -1x^{-2}$

Subtracting 1 from -1 gives -2

$= -\dfrac{1}{x^2}$

(b) $y = x^2\sqrt{x} = x^2x^{\frac{1}{2}} = x^{\frac{5}{2}}$

First write the expression as a power of x

$$\frac{dy}{dx} = \tfrac{5}{2}x^{\frac{3}{2}}$$

Subtracting 1 from $\frac{5}{2}$ gives $\frac{3}{2}$

(c) $y = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

First write the expression as a power of x

$$\frac{dy}{dx} = -\tfrac{1}{2}x^{-\frac{3}{2}}$$

Subtracting 1 from $-\frac{1}{2}$ gives $-\frac{3}{2}$

You can extend this idea to allow you to differentiate all functions of the form kx^n, where k is a constant, and sums and differences of such functions.

- The derivative of kx^n is knx^{n-1}, where k is a constant and n is any real number
- The derivative of sum (or difference) of two or more such functions is the sum (or difference) of the derivatives of the functions.

Example 2

Differentiate the following functions

(a) $y = (3 - 2x - x^2)\sqrt{x}$

(b) $y = \dfrac{3x - x^2}{x^5}$

Solution

(a) $y = (3 - 2x - x^2)\sqrt{x}$

$$= 3\sqrt{x} - 2x\sqrt{x} - x^2\sqrt{x}$$

$$= 3x^{\frac{1}{2}} - 2x^{\frac{3}{2}} - x^{\frac{5}{2}}$$

$$\frac{dy}{dx} = 3 \times \tfrac{1}{2}x^{-\frac{1}{2}} - 2 \times \tfrac{3}{2}x^{\frac{1}{2}} - \tfrac{5}{2}x^{\frac{3}{2}}$$

$$= \tfrac{3}{2}x^{-\frac{1}{2}} - 3x^{\frac{1}{2}} - \tfrac{5}{2}x^{\frac{3}{2}}$$

(b) $y = \dfrac{3x - x^2}{x^5}$

$= \dfrac{3}{x^4} - \dfrac{1}{x^3}$

$= 3x^{-4} - x^{-3}$

$\dfrac{dy}{dx} = 3 \times -4x^{-5} - (-3x^{-4})$

$= -12x^{-5} + 3x^{-4}$

Applications of differentiation

Now that you can differentiate a wider range of functions, you can also make use of various applications of differentiation in many more contexts. You already know how to use differentiation to find gradients of curves, find the equations of tangents and normals to curves and find maximum and minimum points on curves. The following examples cover these applications.

Example 3

For the graph $y = x - \sqrt{x}$

(a) find the gradient at the point (4, 2)
(b) find the equation of the tangent at this point
(c) find the equation of the normal at this point.

Solution

(a) $y = x - \sqrt{x} = x - x^{\frac{1}{2}}$

$\dfrac{dy}{dx} = 1 - \tfrac{1}{2}x^{-\frac{1}{2}} = 1 - \dfrac{1}{2\sqrt{x}}$

When $x = 4$, $\dfrac{dy}{dx} = 1 - \dfrac{1}{2\sqrt{4}} = 1 - \tfrac{1}{4} = \tfrac{3}{4}$

The gradient at (4, 2) is $\tfrac{3}{4}$.

(b) Gradient of tangent at (4, 2) = $\tfrac{3}{4}$

Using the equation of a line $y - y_1 = m(x - x_1)$ with $m = \frac{3}{4}$ and $(x_1, y_1) = (4, 2)$

Equation of tangent at (4, 2) is $y - 2 = \frac{3}{4}(x - 4)$

$$y = \tfrac{3}{4}x - 3 + 2$$

$$y = \tfrac{3}{4}x - 1$$

(c)

Remember that when two lines with gradients m_1 and m_2 are perpendicular, $m_1 m_2 = -1$

Gradient of normal at (4, 2) = $-\frac{4}{3}$

Using the equation of a line $y - y_1 = m(x - x_1)$ with $m = -\frac{4}{3}$ and $(x_1, y_1) = (4, 2)$

Equation of normal at (4, 2) is $y - 2 = -\frac{4}{3}(x - 4)$

$$y = -\tfrac{4}{3}x + \tfrac{16}{3} + 2$$

$$y = -\tfrac{4}{3}x + \tfrac{22}{3}$$

Example 4

Find the stationary points of the graph $y = \dfrac{x^2 - 3}{x^3}$ and determine their nature.

Solution

$$\begin{aligned} y &= \frac{x^2 - 3}{x^3} \\ &= \frac{1}{x} - \frac{3}{x^3} \\ &= x^{-1} - 3x^{-3} \end{aligned}$$

At stationary points, the gradient is zero. Differentiate the function and find the values of x for which $\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = -x^{-2} + 9x^{-4} = -\frac{1}{x^2} + \frac{9}{x^4}$$

At stationary points, $\dfrac{dy}{dx} = 0$

$$-\frac{1}{x^2}+\frac{9}{x^4}=0 \Rightarrow -x^2+9=0$$
$$\Rightarrow x^2=9$$
$$\Rightarrow x=\pm 3$$

Substitute the values of x into the original equation to find the y-coordinates of the stationary points.

When $x=3$, $y=\frac{9-3}{27}=\frac{6}{27}=\frac{2}{9}$

When $x=-3$, $y=\frac{9-3}{-27}=-\frac{6}{27}=-\frac{2}{9}$

The stationary points are $\left(3,\frac{2}{9}\right)$ and $\left(-3,-\frac{2}{9}\right)$.

Look at the gradient either side of each stationary point to determine the nature

When $x=2$, gradient is positive and when $x=4$, gradient is negative

so $\left(3,\frac{2}{9}\right)$ is a local maximum

When $x=-4$, gradient is negative and when $x=-2$, gradient is positive

so $\left(-3,-\frac{2}{9}\right)$ is a local minimum.