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integral

AQA AS Mathematics: Integration

Section 3: Further integration

Notes and Examples

These notes contain the following subsections:

Integrating functions involving negative and fractional powers

In Section 1 you saw that the integral of x^n, where n is a positive integer, is given by

$$\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1} + c \qquad \text{where } c \text{ is an arbitrary constant}$$

In fact this formula is true not only when n is a positive integer, but for all real values of n, including negative numbers and fractions, except for $n = -1$.

The formula does not work for $n = -1$, since this would give a denominator of 0. There is a different way to integrate $\frac{1}{x}$, which is covered in later in A level Mathematics.

Example 1

Find the following indefinite integrals

(a) $\int \sqrt{x}\,\mathrm{d}x$

(b) $\int \frac{1}{x^3}\mathrm{d}x$

(c) $\int \left(\frac{2}{x^2} - \frac{3}{\sqrt{x}}\right)\mathrm{d}x$

Solution

(a) $$\int \sqrt{x}\,\mathrm{d}x = \int x^{\frac{1}{2}}\,\mathrm{d}x$$
$$= \tfrac{2}{3}x^{\frac{3}{2}} + c$$

$n = \frac{1}{2}$, so $n+1 = \frac{3}{2}$, so divide by $\frac{3}{2}$, i.e. multiply by $\frac{2}{3}$.

(b) $$\int \frac{1}{x^3}\,\mathrm{d}x = \int x^{-3}\,\mathrm{d}x$$
$$= -\tfrac{1}{2}x^{-2} + c$$

$n = -3$, so $n+1 = -2$, so divide by -2.

(c) $$\int \left(\frac{2}{x^2} - \frac{3}{\sqrt{x}}\right)\mathrm{d}x = \int \left(2x^{-2} - 3x^{-\frac{1}{2}}\right)\mathrm{d}x$$
$$= -2x^{-1} - 3 \times 2x^{\frac{1}{2}} + c$$
$$= -\frac{2}{x} - 6\sqrt{x} + c$$

For the first term, $n = -2$, so $n+1 = -1$, so divide by -1.
For the second term, $n = -\frac{1}{2}$, so $n+1 = \frac{1}{2}$, so divide by $\frac{1}{2}$, i.e. multiply by 2.

Example 2

Find the following definite integrals.

(a) $$\int_1^2 \left(\frac{4x-1}{x^4}\right)\mathrm{d}x$$

(b) $$\int_1^4 (3-x)\sqrt{x}\;\mathrm{d}x$$

Solution

(a) $$\int_1^2 \left(\frac{4x-1}{x^4}\right)\mathrm{d}x = \int_1^2 \left(4x^{-3} - x^{-4}\right)\mathrm{d}x$$
$$= \left[4 \times -\tfrac{1}{2}x^{-2} + \tfrac{1}{3}x^{-3}\right]_1^2$$
$$= \left[-\frac{2}{x^2} + \frac{1}{3x^3}\right]_1^2$$
$$= \left(-\tfrac{1}{2} + \tfrac{1}{24}\right) - \left(-2 + \tfrac{1}{3}\right)$$
$$= \tfrac{29}{24}$$

(b) $\int_1^4 (3-x)\sqrt{x}\ \mathrm{d}x = \int_1^4 \left(3x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)\mathrm{d}x$

$$= \left[3 \times \tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{2}{5}x^{\frac{5}{2}}\right]_1^4$$

$$= \left[2x\sqrt{x} - \tfrac{2}{5}x^2\sqrt{x}\right]_1^4$$

$$= \left(2\times4\times2 - \tfrac{2}{5}\times16\times2\right) - \left(2\times1\times1 - \tfrac{2}{5}\times1\times1\right)$$

$$= \tfrac{8}{5}$$

Applications of integration

Now that you can integrate a wider range of functions, you can also solve problems which involve integrating these functions, such as finding functions given their gradient function, and finding the area under a curve.

Example 3

The gradient function of a curve is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sqrt{x} - \frac{1}{\sqrt{x}}$$

and the curve passes through the point (4, 9).

Find the equation of the curve.

Solution

Integrate to find an expression for y in terms of x

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sqrt{x} - \frac{1}{\sqrt{x}} \Rightarrow y = \int\left(3\sqrt{x} - \frac{1}{\sqrt{x}}\right)\mathrm{d}x$$

$$= \int\left(3x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\mathrm{d}x$$

$$= 3\times\tfrac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$$

$$= 2x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$$

Substitute the coordinates of the given point to find the value of the constant c.

When $x=4, y=9$ $\Rightarrow 9=2\times 8-2\times 2+c$

$\Rightarrow c=9-16+4$

$\Rightarrow c=-3$

The equation of the curve is $y=2x^{\frac{3}{2}}-2x^{\frac{1}{2}}-3$

The next two examples are about finding the area under a curve.

Example 4

Find the area under the curve $y=1+\sqrt{x}$ between $x=0$ and $x=4$.

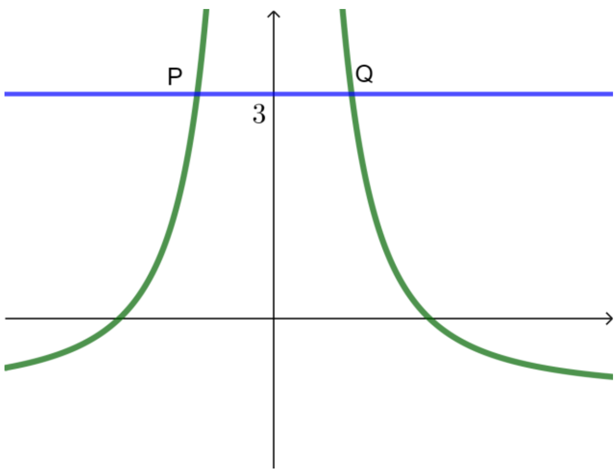
Solution

$y=1+\sqrt{x}$ is positive for all values of x

$$\begin{aligned}\text{Area under curve } &= \int_0^4 \left(1+\sqrt{x}\right)\,\mathrm{d}x \\ &= \int_0^4 \left(1+x^{\frac{1}{2}}\right)\,\mathrm{d}x \\ &= \left[x+\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^4 \\ &= \left(4+\tfrac{2}{3}\times 8\right)-0 \\ &= \tfrac{28}{3}\end{aligned}$$

Example 5

The diagram shows the curve $y=\dfrac{1}{x^2}-1$ and the line $y=3$.



(a) Find the coordinates of points P and Q.

(b) Find the area bounded by the curve, the line $y=3$ and the x-axis.

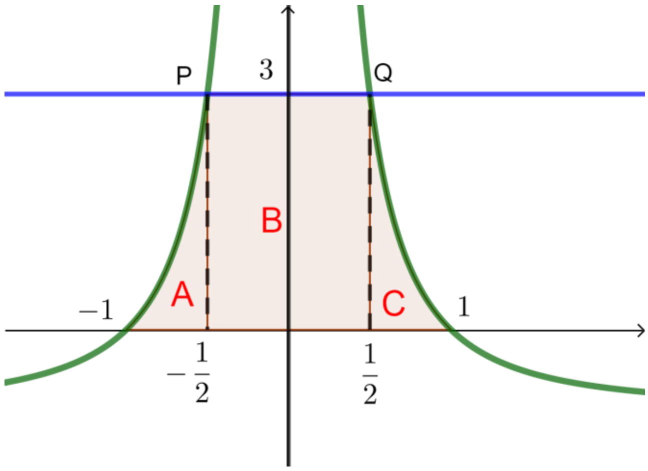
Solution

(a) At P and Q, $\frac{1}{x^2}-1=3 \Rightarrow \frac{1}{x^2}=4 \Rightarrow x^2=\frac{1}{4} \Rightarrow x=\pm\frac{1}{2}$.

The coordinates of P are $\left(-\frac{1}{2},3\right)$ and the coordinates of Q are $\left(\frac{1}{2},3\right)$.

(b)

The curve crosses the axes at $x=-1$ and $x=1$. You need to find the areas of A, B and C separately



Area C is given by $\int_{\frac{1}{2}}^{1}\left(\frac{1}{x^2}-1\right)\,\mathrm{d}x=\left[-\frac{1}{x}-x\right]_{\frac{1}{2}}^{1}$

$$=(-1-1)-\left(-2-\tfrac{1}{2}\right)$$
$$=\tfrac{1}{2}$$

By symmetry area A is also $\frac{1}{2}$.

Area B $=3\times1=3$

Total area = $\frac{1}{2}+\frac{1}{2}+3=4$.