# 1 Probabilities are sensitive to the form of the question that was used to generate the answer

## 1.1

Possible outcomes for $X$, the genders of the two children, are

$$X \in \{(B, B), (B, G), (G, B), \cancel{(G, G)}\}$$

We have eliminated the possiblity of two girls. There are two remaining outcomes that include at least one girl, so the probability is $\frac{2}{3}$.

## 1.2

$X \in \{B, G\}$, so the probability is simply $\frac{1}{2}$.

# 2 Legal reasoning

## 2.1

Let $G$ = defendant is guilty

Let $B$ = defendant's blood type matches blood type at the scene

The probability that the defendant is guilty given that the defendant's blood type was found at the scene is then

$$P(G|B) = \frac{P(G \cap B)}{P(B)}$$

The prosecutor is confusing $P(B)$ for $P(G|B)$. The prosecutor is failing to account for $P(G)$, the prior probability that the defendant committed the crime, which is extremely low.

## 2.2

The defense attorney is ignoring all of the other evidence against the defendant. Not all of the 8000 people with the matching blood type are equally likely to have committed the crime.

# 3 Variance of a sum

Prove: $var[X + Y] = var[X] + var[Y] + 2cov[X, Y]$

$$var[X] = E[X^2] - \mu^2 \tag{1}$$
$$cov[X, Y] = E[XY] - E[X]E[Y] \tag{2}$$

$$
\begin{aligned}
var[X+Y] &= E[(X+Y)(X+Y)] - E^2[X+Y] \\
&= E[X^2 + 2XY + Y^2] - E^2[X+Y] \\
&= E[X^2 + 2XY + Y^2] - (\mu_x + \mu_y)^2 \\
&= E[X^2 + 2XY + Y^2] - (\mu_x^2 + 2\mu_x\mu_y + \mu_y^2) \\
&= E[X^2 + 2XY + Y^2] - \mu_x^2 - 2\mu_x\mu_y - \mu_y^2 \\
&= E[X^2] - \mu_x^2 + E[Y^2] - \mu_y^2 + 2(E[XY] - \mu_x\mu_y) \qquad \text{(By linearity of expectations)} \\
&= var[X] + var[Y] + 2cov[X,Y] \\
&= var[X+Y] \quad \square
\end{aligned}
$$

# 4 Bayes rule for medical diagnosis

Let D = You have the disease

Let T = You test positive

$$
P(D) = \frac{1}{10000} \qquad P(\bar{D}) = \frac{9999}{10000}
$$

$$
P(T|D) = \frac{99}{100} \qquad P(T|\bar{D}) = \frac{1}{100}
$$

$$
\begin{aligned}
P(D|T) &= \frac{P(D \cap T)}{P(T)} = \frac{P(T \cap D)}{P(T)} \\
P(D|T) &= \frac{P(T|D)P(D)}{P(T)} \\
P(D|T) &= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\bar{D})P(\bar{D})} \\
P(D|T) &= \frac{(.99)(.0001)}{(.99)(.0001) + (.01)(.9999)} \\
P(D|T) &= .98\%
\end{aligned}
$$

# 5 The Monty Hall problem

Let $A$ = the prize is behind door 1

Let $B$ = the prize is behind door 2

Let $C$ = the prize is behind door 3

Let $c$ = the host opens door 3

$$
\begin{aligned}
P(A) = P(B) = P(C) &= \frac{1}{3} \\
P(c|A) &= \frac{1}{2} \\
P(c|B) &= 1 \\
P(c|C) &= 0
\end{aligned}
$$

$$
\begin{aligned}
P(B|c) &= \frac{P(c|B)P(B)}{P(c)} \\
&= \frac{P(c|B)P(B)}{P(c|A)P(A) + P(c|B)P(B) + P(c|C)p(C)} \\
&= \frac{\frac{1}{3}}{\frac{1}{6} + \frac{2}{6} + 0} \\
&= \frac{2}{3}
\end{aligned}
$$

There is a $\frac{2}{3}$ chance that the prize is behind door 2, so the contestant should definitely switch.

# 6 Conditional independence

## 6.1

$$
\begin{aligned}
P(H = k|e_1, e_2) &= \frac{P(H = k \cap e_1, e_2)}{P(e_1, e_2)} \\
&= \frac{P(e_1 \cap e_2|H = k)P(H = k)}{P(e_1, e_2)} \qquad \text{(ii.)}
\end{aligned}
$$

(ii.) is sufficient.

## 6.2

$$E_1 \perp\!\!\!\perp E_2|H \implies P(e_1, e_2|H) = P(e_1|H) \cdot P(e_2|H)$$

$$
\begin{aligned}
P(H=k|e_1,e_2) &= \frac{P(e_1 \cap e_2|H=k)P(H=k)}{P(e_1,e_2)} \\
&= \frac{P(e_1|H=k)P(e_2|H=k)P(H=k)}{P(e_1,e_2)} && \text{(i.)} \\
&= \frac{P(e_1|H=k)P(e_2|H=k)P(H=k)}{\sum\limits_{j=1}^{K} P(e_1,e_2|H=j)P(H=j)} \\
&= \frac{P(e_1|H=k)P(e_2|H=k)P(H=k)}{\sum\limits_{j=1}^{K} P(e_1|H=j)P(e_2|H=j)P(H=j)} && \text{(iii.)}
\end{aligned}
$$

(i.), (ii.), and (iii.) are each sufficient.

# 7 Pairwise independence does not imply mutual independence

Example: Let X and Y represent two coin tosses, with value 0 or 1 for heads or tails, respectively. Let Z = 1 if two coin tosses result in exactly one heads.

$$
(X,Y,Z) = \left\{ \begin{matrix} (0, & 0, & 0) \\ (0, & 1, & 1) \\ (1, & 0, & 1) \\ (1, & 1, & 0) \end{matrix} \right\}
$$

$$
P(X=1) = P(Y=1) = P(Z=1) = \frac{1}{2}
$$
$$
P(X=1|Y=1) = P(X=1|Z=1) = P(Y=1|Z=1) = \frac{1}{2}
$$
$$
P(X=1,Y=1,Z=1) \stackrel{?}{=} P(X=1)P(Y=1)P(Z=1)
$$
$$
0 \neq \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}
$$

The state of any two variables $\in (X,Y,Z)$ determines the third. This means that $X$, $Y$, and $Z$ are pairwise independent, but **not** mutually independent.

# 8 Conditional independence iff joint factorizes

By definition, $X \perp\!\!\!\perp Y|Z$ iff

$$
p(x,y|z) = p(x|z)p(y|z) \tag{1}
$$

We wish to prove an alternate definition - $X \perp\!\!\!\perp Y|Z$ iff there exist functions $g$ and $h$ such that

$$p(x,y|z) = g(x,z)h(y,z) \tag{2}$$

for all $x, y$ and $z$ such that $p(z) > 0$.

To prove this we will show that given equation **??**, equation **??** follows, and $X \perp\!\!\!\perp Y|Z$.

We begin by integrating out $x$.

$$\begin{aligned}
\int_x p(x,y|z)dx &= \int_x g(x,z)h(y,z)dx \\
p(y|z) &= h(y,z)g(z) \\
p(y|z)\frac{1}{g(z)} &= h(y,z)
\end{aligned}$$

By a symmetric argument, we have $p(x|z)\frac{1}{h(z)} = g(x,z)$. Plugging into our original equation, we have

$$\begin{aligned}
p(x,y|z) &= p(y|z)p(x|z)\frac{1}{g(z)}\frac{1}{h(z)} \\
\int_x \int_y p(x,y|z)dxdy &= \int_x \int_y p(y|z)p(x|z)\frac{1}{g(z)}\frac{1}{h(z)}dx\,dy \\
1 &= \frac{1}{g(z)}\frac{1}{h(z)} \\
p(x,y|z) &= p(y|z)p(x|z) \cdot 1 \quad \square
\end{aligned}$$

# 9 Conditional independence

## 9.1

$$\overbrace{(X \perp\!\!\!\perp W|Z,Y)}^{A} \wedge \overbrace{(X \perp\!\!\!\perp Y|Z)}^{B} \overset{?}{\Longrightarrow} (X \perp\!\!\!\perp Y,W|Z)$$

By decomposition of $(X \perp\!\!\!\perp Y,W|Z)$, we have

$$X \perp\!\!\!\perp Y|Z \tag{1}$$

$$X \perp\!\!\!\perp W|Z \tag{2}$$

(**??**) is trivally true by (B). Now we must examine (**??**).

$$\overbrace{(X \perp\!\!\!\perp W|Z,Y)}^{A} \wedge \overbrace{(X \perp\!\!\!\perp Y|Z)}^{B} \overset{?}{\Longrightarrow} (X \perp\!\!\!\perp W|Z)$$