1 MLE for the Bernoulli/ binomial model

$$X_i \sim Ber(\theta) \tag{1}$$
$$p(D|\theta) = \theta^{N_1}(1-\theta)^{N_0} \tag{2}$$

$$\begin{aligned}
\ln\big(p(D|\theta)\big) &= \ln\left(\theta^{N_1}(1-\theta)^{N_0}\right) \\
&= \ln\left(\theta^{N-1}\right) + \ln(1-\theta)^{N_0} \\
&= N_1 \ln\theta + N_0 \ln(1-\theta) \\
\tfrac{\mathrm{d}}{\mathrm{d}\theta} \ln p(D|\theta) &= \frac{N_1}{\theta} - \frac{N_0}{1-\theta}
\end{aligned}$$

The log-likelihood will take a maximum when the derivative equals 0.

$$\begin{aligned}
0 &= \frac{N_1}{\theta} - \frac{N-N_1}{1-\theta} \\
0 &= N_1(1-\theta) - \theta(N-N_1) \\
0 &= N_1 - \theta N_1 - \theta N + \theta N_1 \\
0 &= N_1 - \theta(N_1 + N - N_1) \\
0 &= N_1 - \theta N \\
\hat{\theta} &= \frac{N_1}{N}
\end{aligned}$$

2 Marginal likelihood for Beta-Bernoulli model

$$p(X_{1:N}) = p(x_1)p(x_2|x_1)p(x_3|x_{1:2})...p(x_N|x_{N-1}) \tag{3}$$

$$p(X=k|D_{1:N}) = \frac{N_k + \alpha_k}{\sum_i N_i + \alpha_i} \tag{4}$$

$$(\alpha-1)! = \Gamma(\alpha) \tag{5}$$

Given $D = H,T,T,H,H \triangleq 1,0,0,1,1$

$$
\begin{aligned}
p(X=1|\alpha) &= \frac{\alpha_1}{\alpha} \\
p(X=0|\alpha, D_1) &= \frac{\alpha_0}{\alpha+1} \\
p(X=0|\alpha, D_{1:2}) &= \frac{\alpha_0+1}{\alpha+2} \\
p(X=1|\alpha, D_{1:3}) &= \frac{\alpha_0+1}{\alpha+3} \\
p(X=1|\alpha, D_{1:4}) &= \frac{\alpha_0+2}{\alpha+4}
\end{aligned}
$$

$$
\begin{aligned}
p(D) &= p(D_{1:5}) \\
&= p(D_1) \cdot p(D_2|D_1) \cdot p(D_3|D_{1:2}) \cdot p(D_4|D_{1:3}) \cdot p(D_5|D_{1:4}) && \text{by (3)} \\
&= \frac{\alpha_1}{\alpha} \cdot \frac{\alpha_0}{\alpha+1} \cdot \frac{\alpha_0+1}{\alpha+2} \cdot \frac{\alpha_1+1}{\alpha+3} \cdot \frac{\alpha_1+2}{\alpha+4} \qquad \text{by (4)} \\
&= \frac{\left[\alpha_1(\alpha_1+1)(\alpha_1+2)\right]\left[\alpha_0(\alpha_0+1)\right]}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} \\
&= \frac{\left[(\alpha_1)...(\alpha_1+N_1-1)\right]\left[(\alpha_0)...(\alpha_0+N_0-1)\right]}{(\alpha)...(\alpha+N-1)} \\
&= \frac{(\alpha_1+N_1-1)!}{(\alpha_1-1)!} \cdot \frac{(\alpha_0+N_0-1)!}{(\alpha_0-1)!} \cdot \frac{(\alpha-1)!}{(\alpha+N-1)!} \\
&= \frac{\Gamma(\alpha_1+N_1)}{\Gamma(\alpha_1)} \cdot \frac{\Gamma(\alpha_0+N_0)}{\Gamma(\alpha_0)} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+N)} \\
&= \frac{\Gamma(\alpha_1+N_1)\,\Gamma(\alpha_0+N_0)}{\Gamma(\alpha_1)\Gamma(\alpha_0)} \frac{\Gamma(\alpha_0+\alpha_1)}{\Gamma(\alpha_0+\alpha_1+N)}
\end{aligned}
$$

3 Posterior predictive for a Beta-Binomial model

$$
\begin{aligned}
p(x|n, D) &= Bb(x|\alpha_0', \alpha_1', n) \\
&= \frac{B(x+\alpha_1', n-x+\alpha_0')}{B(\alpha_1', \alpha_0')}\binom{n}{x}
\end{aligned}
$$

Given $n = 1$

$$
\begin{aligned}
Bb(1|\alpha_0, \alpha_1, 1) &= \frac{B(1+\alpha_1, \alpha_0)}{B(\alpha_1, \alpha_0)} \binom{1}{1} \\
&= \frac{\Gamma(1+\alpha_1)\Gamma(\alpha_0)}{\Gamma(\alpha_0+\alpha_1+1)} \cdot \frac{\Gamma(\alpha_0+\alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)} \\
&= \frac{\alpha_1\Gamma(\alpha_1)\Gamma(\alpha_0)}{(\alpha_0+\alpha_1)\Gamma(\alpha_0+\alpha_1)} \cdot \frac{\Gamma(\alpha_0+\alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)} \\
&= \frac{\alpha_1}{\alpha_0+\alpha_1} \\
&= \frac{\alpha_1}{\alpha}
\end{aligned}
$$

4 Beta updating from censored likelihood

Let n represent the number of coin tosses. Let X represent the number of heads. Given $n = 5$ and $X < 3$, we need to compute the posterior $p(\theta|X < 3)$ under a $B(1, 1)$ prior up to normalization constants.

$$
\begin{aligned}
P(\theta) &= \frac{P(\theta)P(D|\theta)}{P(D)} \\
&= \frac{P(\theta) \cdot P(X < 3|\theta)}{P(X < 3)} \\
P(\theta) &\propto P(\theta) \cdot P(X < 3) \\
&\propto B(1, 1) \cdot \sum_{k=0}^{2} P(k|\theta, 5) \\
&\propto \sum_{k=0}^{2} \binom{5}{k} \theta^k (1-\theta)^{5-k}
\end{aligned}
$$

5 Uninformative prior for log-odds ratio

Let $\phi = log\frac{\theta}{1-\theta}$. $p(\phi) = 1$ is equivalent to $p(\phi) = k$, where k is a constant and $0 < k < 1$.

$$
\begin{aligned}
\int_\phi p(\phi) d\phi = \int_\phi k d\phi = 1 \\
d\phi = \frac{d\phi}{d\theta} d\theta \\
\frac{d\phi}{d\theta} = \frac{d}{d\theta} \left(ln \frac{\theta}{1-\theta} \right) \\
= \frac{d}{d\theta} \left(\ln\theta - \ln(1-\theta) \right) \\
= \frac{1}{\theta} - \frac{1}{1-\theta} \cdot -1 \\
= \frac{1}{\theta} + \frac{1}{1-\theta} \\
\int_\phi k d\phi = \int_\theta k(\frac{1}{\theta} + \frac{1}{1-\theta}) d\theta \\
1 = k \int_\theta \theta^{-1} (1-\theta)^{-1} d\theta
\end{aligned}
$$

We recognize the final integral as the normalization constant for a $Beta(\theta|0,0)$ distribution.

6 MLE for the Poisson distribution

$$
P(X = k|\lambda) = e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{for } k \in \{0, 1, 2...\}
$$

$$
\begin{aligned}
p(x|\lambda) &= e^{-\lambda} \cdot \frac{\lambda^{x_1}}{x_1!} \cdot e^{-\lambda} \cdot \frac{\lambda^{x_2}}{x_2!} \cdot ... \\
&= \prod_{i=1}^{N} e^{-lambda} \cdot \frac{lambda^{x_i}}{x_i!}
\end{aligned}
$$

Now we take the log-likelihood and find its maximum.

$$
\begin{aligned}
\ell(\lambda) &= \ln p(x|lambda) \\
&= \ln\left(\prod_{i=1}^{N} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right) \\
&= \ln\left(\frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \cdot \frac{e^{-\lambda}\lambda^{x_2}}{x_2!}\right) ... \\
&= \sum_{i=1}^{N} \ln\left(\frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right) \\
&= \sum_{i=1}^{N} \ln(e^{-\lambda}\lambda^{x_i}) - \ln(x_i!) \\
&= \sum_{i=1}^{N} \ln(e^{-\lambda}) + \ln(\lambda^{x_i}) - \ln(x_i!) \\
&= \sum_{i=1}^{N} -\lambda + x_i \ln\lambda - \ln(x_i!) \\
&= -N\lambda + (\ln\lambda)\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} \ln(x_i!)
\end{aligned}
$$

$$
\begin{aligned}
\ell'(\lambda) &= \frac{1}{\lambda}\sum_{i=1}^{N} x_i - N = 0 \\
\frac{\sum_{i=1}^{N} x_i}{\lambda} &= N \\
\lambda_{MLE} &= \frac{\sum_{i=1}^{N} x_i}{N}
\end{aligned}
$$

7 Bayesian analysis of the Poisson distribution

7.1 a.

$$\begin{aligned}
p(\lambda|D) &= \frac{p(D|\lambda)p(\lambda)}{p(D)} \\
&\propto p(D|\lambda)p(\lambda) \\
&= \frac{e^{-N\lambda} \cdot \lambda^{\sum_{i=1}^{N} x_i}}{\prod_{i=1}^{N} x_i!} \cdot \frac{\lambda^{a-1}e^{-\lambda b}}{k} \\
&\propto e^{-N\lambda-\lambda b} \cdot \lambda^{a-1+\sum_{i=1}^{N} x_i} \\
&= e^{-\lambda(N+b)} \cdot \lambda^{\left[a+\sum_{i=1}^{N} x_i\right]-1} \\
&= Ga(\lambda|a+\sum_{i=1}^{N} x_i, N+b)
\end{aligned}$$

7.2 b.

$$\begin{aligned}
\frac{a+\sum_{i=1}^{N} x_i}{N+b} asa \to 0, b \to 0 \\
= \frac{\sum_{i=1}^{N} x_i}{N} \\
= \lambda_{MLE}
\end{aligned}$$

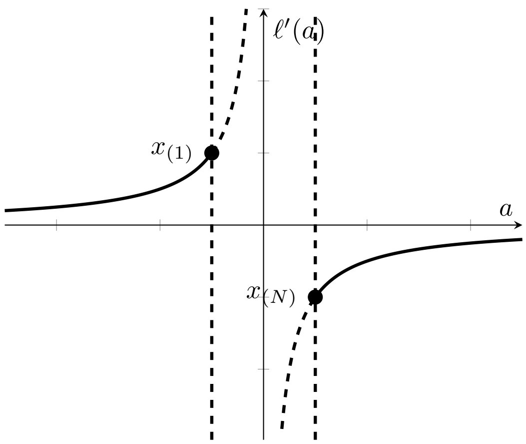
8 MLE for the uniform distribution

8.1 a.

$$\begin{aligned}
p(D|a) &= \prod_{i=1}^{N} p(x) \\
&= \left(\frac{1}{2a}\right)^N I(x_1, x_2, ..., x_N \in [-a, a]) \\
&= \begin{cases} \left(\frac{1}{2a}\right)^N & \text{if } -a \leq x_i \leq a, \forall i \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Now we take the derivative of the log-likelihood.

$$
\begin{aligned}
\ell(a) &= \ln\left(\frac{1}{2a}\right)^N \\
&= \ln 1 - \ln(2a)^N \\
&= -N\ln(2a) \\
\ell'(a) &= -N\frac{1}{2a}\cdot 2 \\
&= -\frac{N}{a}
\end{aligned}
$$



For $a < 0$, the likelihood is increasing, so it will be maximized where $a = x_{(1)}$, assuming that $|x_{(1)}| \geq |x_{(N)}|$. Similarly, for $a > 0$, the likelihood is decreasing, so it is maximized where $a = x_{(n)}$, assuming that $|x_{(N)}| \geq |x_{(1)}|$. This function is only defined when $\max_i |x_i| \leq a$. This means that ℓ is maximized at $a = \max(|x_{(1)}|, |x_{(n)}|)$.

8.2 b.

$p(x_{n+1}) = \frac{1}{b-a} = \frac{1}{2a}$

8.3 c.

Our approach is not Bayesian, so we will assign zero probability to $x_{n+1} > a$ and $x_{n+1} < -a$. A better solution would be to derive $\hat{a}_{MAP}$ and give a plug-in approximation.

9 Bayesian analysis of the uniform distribution

We must derive the posterior, $p(D|\theta)$ given the following:

$$p(D,\theta) = \frac{Kb^K}{\theta^{N+K+1}} \mathbb{I}(\theta \geq \max(D,b)) \tag{6}$$

Let $m = \max(D)$.

$$\begin{aligned} p(D) &= \int_m^\infty \frac{Kb^K}{\theta^{N+K+1}}\, d\theta \\ &= \begin{cases} \frac{K}{(N+K)b^N} & \text{if } m \leq b \\ \frac{Kb^K}{(N+K)m^{N+K}} & \text{if } m > b \end{cases} \end{aligned} \tag{7}$$

$$\begin{aligned} p(\theta|D) &= \frac{p(\theta,D)}{p(D)} \\ &= \begin{cases} \frac{Kb^K}{\theta^{N+K+1}} \cdot \frac{(N+K)b^N}{K} & \text{if } m \leq b \leq \theta \\ \frac{Kb^K}{\theta^{N+K+1}} \cdot \frac{(N+K)m^{N+K}}{Kb^K} & \text{if } b < m \leq \theta \end{cases} \\ &= \begin{cases} (N+K) \cdot b^{N+K} \cdot \theta^{-(N+K+1)} & \text{if } m \leq b \leq \theta \\ (N+K) \cdot m^{N+K} \cdot \theta^{-(N+K+1)} & \text{if } b < m \leq \theta \end{cases} \\ &\propto \begin{cases} \text{Pareto}(\theta|N+K,b) & \text{if } m \leq b \leq \theta \\ \text{Pareto}(\theta|N+K,m) & \text{if } b < m \leq \theta \end{cases} \\ &= \text{Pareto}(\theta|N+K,\max(m,b)) \end{aligned}$$

10 Taxicab (tramcar) problem

$$\text{Pareto}(\theta|N+K,\max(m,b)) \tag{8}$$

10.1 a.

Given a non-informative prior, $\text{Pareto}(\theta|0,0)$:

$$\begin{aligned} p(\theta|D) &= \text{Pareto}(\theta|1+0,\max(100,0)) \\ &= \text{Pareto}(\theta|1,100) \end{aligned}$$

10.2 b.

$E[\theta|D] = \frac{km}{k-1}$ and $k = 1$, so the posterior mean is not defined.
The mode is $\max(D) = 100$.

$$
\begin{aligned}
\int_{100}^{x} km^k \theta^{-(k+1)}\, d\theta &= \frac{1}{2} \\
100 \int_{100}^{x} \theta^{-2}\, d\theta &= \frac{1}{2} \\
\left[-\frac{1}{\theta}\right]_{100}^{x} &= \frac{1}{200} \\
-\frac{1}{x} + \frac{1}{100} &= \frac{1}{200} \\
x &= 200
\end{aligned}
$$

The median is 200.

10.3 c.

$$
\begin{aligned}
p(D'|D,\alpha) &= \int_\theta p(D'|\theta) p(\theta|D,\alpha)\, d\theta) \\
&= \int_\theta \frac{1}{\theta} \mathbb{I}(x \le \theta) \cdot N \cdot m^N \cdot \theta^{-(N+1)}\, d\theta \\
&= N m^N \int_\theta \theta^{-(N+2)}\, d\theta \\
&= N m^N \left[-\frac{1}{N-1} \theta^{-N-1}\right]_{\max(x,m)}^{\infty} \\
&= N m^N \left[0 - \left(-\frac{1}{N-1} \max(x,m)^{-N-1}\right)\right] \\
&= \frac{N m^N}{(N+1)\max(x,m)^{N+1}} \\
&= \begin{cases} \frac{N}{m(N+1)} & \text{if } x < m \\ \frac{N m^N}{x^{N+1}(N+1)} & \text{if } x \ge m \end{cases}
\end{aligned}
$$

10.4 d.

$$
\begin{aligned}
p(x = 100|D,\alpha) &= \frac{1 \cdot 100^1}{(1+1)100^{1+1}} = \frac{1}{200} \\
p(x = 50|D,\alpha) &= \frac{1}{(1+1)100} = \frac{1}{200} \\
p(x = 150|D,\alpha) &= \frac{1 \cdot 100^1}{(1+1)150^{1+1}} = \frac{1}{450}
\end{aligned}
$$

10.5 e.

To improve the accuracy we could use a more informative prior. For example, we could estimate the number of cabs based on the city's population. Accuracy will also improve with more observations.

11 Bayesian analysis of the exponential distribution

11.1 a.

$$\begin{aligned} p(x|\theta) &= \theta e^{-\theta x} \quad \text{for } x \geq 0, \theta \geq 0 \\ p(D|\theta) &= \prod_{i=1}^{N} p(x_i|\theta) = \prod_{i=1}^{N} \theta e^{-\theta x_i} \end{aligned}$$

$$\begin{aligned} \ln(p(D|\theta)) &= \sum_{i=1}^{N} \ln \theta e^{-\theta x_i} \\ &= \sum_{i=1}^{N} \ln \theta + \ln e^{-\theta x_i} \\ &= \sum_{i=1}^{N} \ln \theta + -\theta x_i \\ \frac{d}{d\theta}(\sum_{i=1}^{N} \ln \theta + -\theta x_i) &= \sum_{i=1}^{N} \frac{1}{\theta} - x_i \\ 0 &= \frac{N}{\theta} - \sum_{i=1}^{N} x_i \\ \hat{\theta} &= \frac{N}{\sum_{i=1}^{N} x_i} \\ \hat{\theta} &= \frac{1}{\frac{1}{N}\sum_{i=1}^{N} x_i} \end{aligned}$$

11.2 b.

$$\begin{aligned}\hat{\theta} &= \frac{1}{\frac{1}{N}\sum_{i=1}^{N} x_i} \\ &= \frac{1}{\frac{1}{3}(5+6+4)} \\ &= \frac{1}{5}\end{aligned}$$

11.3 c.

$$\begin{aligned}p(\theta) &= \mathrm{Expon}(\theta|\lambda) \\ &= \lambda e^{-\lambda\theta} \\ &= \frac{\lambda^1}{\Gamma(1)}\theta^{1-1}e^{-\lambda\theta} \\ &= \mathrm{Ga}(\theta|1,\lambda)\end{aligned}$$

The mean of the Gamma distribution is $\frac{1}{\lambda}$, so $\hat{\lambda} = 3$

11.4 d.

$$p(\theta|D,\hat{\lambda}) \propto p(D|\theta)p(\theta|\hat{\lambda})$$

$$\begin{aligned}p(D|\theta) &= \prod_{x=1}^{N} \theta e^{-\theta x_i} \\ &= \theta e^{-\theta x_1} \cdot \theta e^{-\theta x_2} \cdot \theta e^{-\theta x_3} \ldots \\ &= \theta^N e^{(-\theta x_1 - \theta x_2 - \theta x_3 \ldots)} \\ &= \theta^N e^{-\theta \sum x_i}\end{aligned}$$

$$p(\theta|\hat{\lambda}) = \hat{\lambda}e^{-\theta\hat{\lambda}}$$

$$\begin{aligned}p(\theta|D,\hat{\lambda}) &\propto \theta^N e^{-\theta\sum x_i}\hat{\lambda}e^{-\theta\hat{\lambda}} \\ &\propto \theta^N e^{-\theta(\hat{\lambda}+\sum x_i)} \\ &= \mathrm{Ga}(\theta|N+1, \hat{\lambda} + \sum_{i=1}^{N} x_i)\end{aligned}$$

11.5 e.

Yes, the prior is equivalent to a Gamma distribution and the posterior is also a Gamma distribution.

11.6 f.

The mean of a $\text{Ga}(\theta|a,b)$ distribution is $\frac{a}{b}$. The mean of $\text{Ga}(\theta|N+1, \hat{\lambda}+\sum_{i=1}^{N} x_i)$ $= \frac{N+1}{\hat{\lambda}+\sum_{i=1}^{N} x_i}$

11.7 g.

The posterior accounts for the exponential prior. The prior accounts for expert knowledge, thus it is more reasonable.

12 MAP estimate for the Bernoulli with non-conjugate priors

12.1 a.

We're looking for the MAP estimate for θ, defined as $\hat{\theta}_{MAP} = \underset{\theta}{\text{argmax}} p(\theta|D)$

We can calculate the posterior up to normalization constants given the number of occurrences of heads and tails and the piecewise function that defines the prior.

$$\begin{aligned}
p(\theta|D) &\propto p(D|\theta) \cdot p(\theta) \\
&= \theta^{N_1}(1-\theta)^{N_0} \cdot \begin{cases} .5 & \text{if } \theta = .4 \text{ or } .5 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} \theta^{N_1}(1-\theta)^{N-N_1} & \text{if } \theta \in \{.4, .5\} \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} .4^{N_1}(.6)^{N-N_1} & \text{if } \theta = .4 \\ .5^{N} & \text{if } \theta = .5 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

The MAP estimate is the value of θ that maximizes the equation above.

$$\hat{\theta}_{MAP} = \begin{cases} .4 & \text{if } (.4)^{N_1}(.6)^{N-N_1} \geq .5^{N} \\ .5 & \text{else} \end{cases}$$

12.2 b.

If N is small, $\theta = .4$ will lead to a better estimate since the prior is close to the true value. As N grows, the data will overwhelm the prior.

13 Posterior predictive distribution for a batch of data with the Dirichlet-multinomial model

$$
\begin{aligned}
p(D'|D,\alpha) &= \int_\theta p(D'|\theta)\cdot p(\theta|D)d\theta \\
&= \int_\theta \mathrm{Mu}(N_{new_1}, N_{new_2}, ...|\theta)\cdot \mathrm{Dir}(\theta|N_{old_1}+\alpha_1, N_{old_2}+\alpha_2...) \\
&= \frac{1}{\mathrm{B}(\alpha+N_{old})}\binom{N_{new}!}{N_{new_1}!...N_{new_k}!}\int_\theta \prod^k \theta_k^{N_{new_k}}\cdot\prod^k \theta_k^{\alpha_k+N_{old_k}-1}\,d\theta \\
&= \frac{1}{\mathrm{B}(\alpha+N_{old})}\binom{N_{new}!}{N_{new_1}!...N_{new_k}!}\int_\theta \underbrace{\prod^k \theta_k^{\alpha_k+N_{new_k}+N_{old_k}-1}\,d\theta}_{\text{normalization constant for } \mathrm{Dir}(\vec{\alpha}+\vec{N_{new}}+\vec{N_{old}})} \\
&= \binom{N_{new}!}{N_{new_1}!...N_{new_k}!}\frac{B(\alpha+N_{old}+N_{new})}{B(\alpha+N_{old})} \\
&= \frac{N_0'!}{\prod N_k'!}\cdot\frac{\prod\Gamma(\alpha_k+N_k+N_k')}{\Gamma(\alpha_0+N_0+N_0')}\cdot\frac{\Gamma(\alpha_0+N_0)}{\prod\Gamma(\alpha_k+N_k)} \quad \text{where } \alpha_0=\sum_{i=1}^k\alpha_i, N_0=\sum_{i=1}^k N_i, \text{ and } N_0'=\sum_{i=1}^k N \\
&= \frac{N_0'!}{\prod N_k'!}\cdot\frac{\Gamma(\alpha_0+N_0)}{\Gamma(\alpha_0+N_0+N_0')}\cdot\frac{\prod\Gamma(\alpha_k+N_k+N_k')}{\prod\Gamma(\alpha_k+N_k)}
\end{aligned}
$$

Notice that this looks like the formula we derived in exercise 2.

14 Posterior predictive for Dirichlet-multinomial

14.1 a.

$$
\begin{aligned}
p(X=j|D) &= E[\theta_j|D] \qquad \text{Equation (3.51) in the textbook} \\
&= \frac{\alpha_j+N_j}{\alpha_0+N} \\
&= \frac{10+260}{(10\cdot 27)+2000} \\
&= .119
\end{aligned}
$$

14.2 b.

We use the equation from 13.

$$p(D'|D,\alpha) = \frac{N_0'!}{\prod N_k'!} \cdot \frac{\Gamma(\alpha_0 + N_0)}{\Gamma(\alpha_0 + N_0 + N_0')} \cdot \frac{\prod \Gamma(\alpha_k + N_k + N_k')}{\prod \Gamma(\alpha_k + N_k)}$$

For a single trial, $N_0' = N_j' = 1$, so the first term is equal to 1. Now we examine the last term.

$$\begin{aligned}
p(x=j|D,\alpha) &= \frac{\Gamma(\alpha_0 + N_0)}{\Gamma(\alpha_0 + N_0 + N_0')} \cdot \frac{\prod_{k\neq j} \Gamma(\alpha_k + N_k + N_k')}{\prod_{k\neq j} \Gamma(\alpha_k + N_k)} \cdot \frac{\Gamma(\alpha_j + N_j + N_j')}{\Gamma(\alpha_j + N_j)} \\
&= \frac{\Gamma(\alpha_0 + N_0)}{\Gamma(\alpha_0 + N_0 + N_0')} \cdot \prod_{k\neq j} \frac{\Gamma(\alpha_k + N_k + 0)}{\Gamma(\alpha_k + N_k)} \cdot \frac{\Gamma(\alpha_j + N_j + N_j')}{\Gamma(\alpha_j + N_j)} \\
&= \frac{\Gamma(\alpha_0 + N_0)}{\Gamma(\alpha_0 + N_0 + N_0')} \cdot 1 \cdot \frac{\Gamma(\alpha_j + N_j + N_j')}{\Gamma(\alpha_j + N_j)} \\
&= \frac{\Gamma(\alpha_0 + N_0)}{\Gamma(\alpha_0 + N_0 + 1)} \cdot \frac{\Gamma(\alpha_j + N_j + 1)}{\Gamma(\alpha_j + N_j)} \\
&= \frac{\Gamma(\alpha_0 + N_0)}{(\alpha_0 + N_0)\Gamma(\alpha_0 + N_0)} \cdot \frac{(\alpha_j + N_j)\Gamma(\alpha_j + N_j)}{\Gamma(\alpha_j + N_j)} \\
&= \frac{\alpha_j + N_j}{\alpha_0 + N_0}
\end{aligned}$$

For independent samples

$$\begin{aligned}
p(x_{2001} = a, x_{2002} = p|D,\alpha) &= p(x_{2001} = a|D,\alpha) \cdot p(x_{2002} = p|D',\alpha) \\
&= \frac{10 + 100}{270 + 2000} \cdot \frac{10 + 87}{270 + 2001} \\
&= .0021
\end{aligned}$$