

Year 12

## Physics Assignment 01

### Vectors in Physics

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This is the first Physics assignment from COMPOS Y12. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Very similar problems will be discussed on webinars every Tuesday at 6pm, so think of the questions you would like to ask. We hope that eventually you will be able to solve most of the problems. Good luck!

## 1 Introduction to vectors and basic operations with them

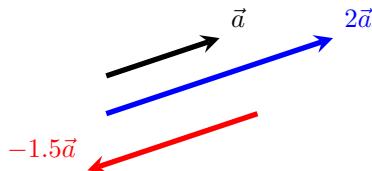
Please watch the Khan Academy [Vectors and scalars intro](#) and [Vector intro for linear algebra](#) videos.

There are two basic operations defined for vectors are multiplication by a number and addition. We review them separately below. When studying these operations, it is important to understand that a vector does not have a particular location in space. It is *displacement-invariant*: displacing a vector without changing its direction results in the same vector<sup>1</sup>.

### 1.1 Multiplying a vector by a scalar

Watch the video: [Multiplying a vector by a scalar](#)

When a vector  $\vec{a}$  is multiplied by a scalar  $k$ , the resulting vector  $k\vec{a}$  is *collinear* to  $\vec{a}$ . If  $k < 0$  the vector  $k\vec{a}$  is *antiparallel* to vector  $\vec{a}$ .



<sup>1</sup>In physics, a vector can sometimes be ascribed a specific location. For example, a force can be exerted at a particular point, and its location may matter for the resulting motion — a door is easier to open when pushing on the handle rather than near the hinge. However, when studying operations with vectors, we abstract from this feature and consider vectors displacement-invariant.

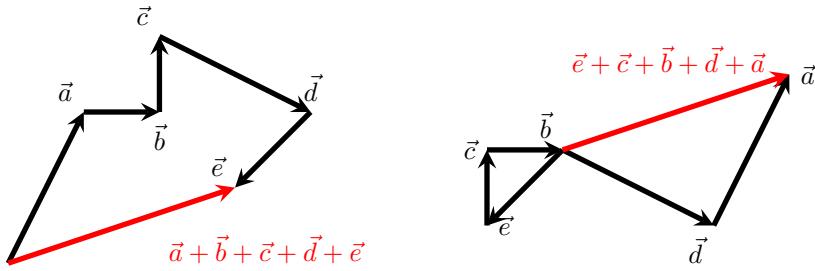
## 1.2 Adding vectors

Please watch the following Khan Academy videos: [Adding & Subtracting vectors](#), [Parallelogram rule](#) and [Subtracting vectors with parallelogram rule](#). Review an Isaac Physics example on [Describing and adding vectors](#).

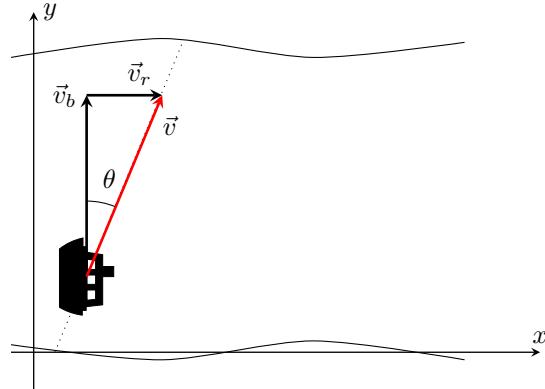
As we can see, there are two ways of adding vectors (both used equally frequently):



To add three or more vectors, we use the “tip-to-end” method (which, again, takes advantage of the displacement invariance property). Draw each vector from the tip of the previous. Connect the beginning of the first vector with the end of the last vector. This is the resultant vector (shown in red). Note: the order, in which you add vectors, does not matter, as is shown in the diagram below.



**Example 1.** A motorboat is crossing a river. The driver directs the boat at a right angle to the river bank. The speed of the boat in still water is  $v_b = 12$  mph, the speed of the river flow is  $v_r = 5$  mph. What is the speed and direction of the boat relative to the shore and, if the river is  $h = 0.5$  mile wide, how far downstream will the boat land relative to the starting point?



*Solution.* The boat's velocity is the vector sum of the velocity in still water and flow velocity,  $\vec{v} = \vec{v}_b + \vec{v}_r$ . Using the triangle rule for addition of vectors and noticing that  $\vec{v}_b \perp \vec{v}_r$ , we evaluate the magnitude according to the Pythagorean theorem:  $|\vec{v}| = \sqrt{v_b^2 + v_r^2} = \sqrt{12^2 + 5^2} = 13$  mph. The direction is found using  $\tan \theta = \frac{v_r}{v_b}$ ; so  $\theta = \arctan \frac{v_r}{v_b} \approx 22.6^\circ$ .

To find the time to cross the river, we can work in the reference frame of the water, in which there is no flow. We have  $t = \frac{h}{v_b} = \frac{1}{24}$  hours = 150 seconds.

Returning to the reference frame of the earth, we find that, during the time  $t$ , the water will travel  $v_r t = 0.208$  miles  $\approx 333$  meters.

Answer: 13 mph;  $22.6^\circ$  to the normal to the riverbank; 333 m downstream.

**Problem 1.** (1 marks) Find the magnitude and direction of the resultant of the following 4 forces:

- 13 N to the north,
- 3 N to the east,
- 1 N to the south,
- 12 N to the west.

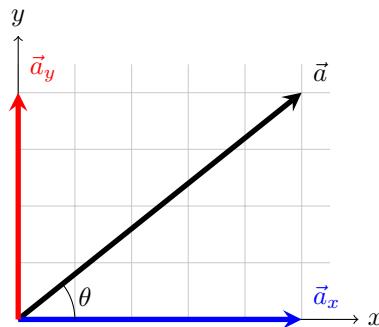
**Hint:** Recall that the order, in which you add vectors, does not matter.

## 2 Components of vectors

Please watch the Khan Academy videos: [Introduction to vector components](#), [Finding the components of a vector](#) [Unit vector notation](#) and [Direction of vectors from components](#). You can also study Isaac Physics' page on [Resolving vectors](#).

Let us sum up these materials. A vector in a plane can be represented as a sum of its projections onto the two coordinate axes:

$$\vec{a} = \vec{a}_x + \vec{a}_y.$$



Here,  $\vec{a}_x$  is the projection of vector  $\vec{a}$  onto the  $x$ -axis and  $\vec{a}_y$  its projection onto the  $y$ -axis. Further, each projection can be written as a product  $\vec{a}_x = a_x \vec{i}$  and  $\vec{a}_y = a_y \vec{j}$ , where  $\vec{i}$  and  $\vec{j}$  are the *coordinate unit vectors*<sup>2</sup> and the scalars  $a_x, a_y$  are the *coordinates* of the vector  $\vec{a}$ . Hence we can write  $\vec{a} = a_x \vec{i} + a_y \vec{j}$ . Alternatively, we can represent the vector as just a pair of numbers — a column  $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ .

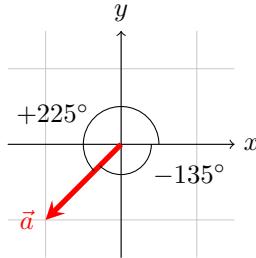
There is one-to-one correspondence between the representations of a vector in terms of its magnitude+direction and in terms of its coordinates. The coordinates are calculated from the magnitude and angle  $\theta$  of the vector with respect to the  $x$  axis according to

$$a_x = |\vec{a}| \cos \theta; \quad a_y = |\vec{a}| \sin \theta. \quad (1)$$

On the other hand, we can find the magnitude and direction of a vector if we know its coordinates:

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2}; \quad \tan \theta = \frac{a_y}{a_x}. \quad (2)$$

Importantly, the angle  $\theta$  is considered positive if measured from the  $x$ -axis in the *counterclockwise* direction. If it is measured in the clockwise direction, it must be taken with the negative sign. For example, in the diagram below, we can use either  $\theta = +225^\circ$  or  $\theta = -135^\circ$ .



Note that computing the angle using Eq. (2) is ambiguous: for example, in the case of the above diagram, we have  $a_x = a_y = -1$ , so  $\tan \theta = 1$ . Taking the inverse tangent gives us  $\theta = 45^\circ$ , whereas the correct answer is  $225^\circ$  (whose tangent is also 1). To deal with this ambiguity, you should always look at the signs of the two coordinates and figure in which quadrant the vector lies.

### Notation variation

There are different notations for vectors in different communities. For example, a vector  $\vec{a}$  can be written as  $\mathbf{a}$  or  $\underline{a}$ , whereas the coordinate unit vectors are sometimes written as  $\hat{i}, \hat{j}$  and  $\hat{k}$ . The magnitude (length)  $|\vec{a}|$  of a vector is sometimes written as  $\|\vec{a}\|$ . A row  $\vec{a} = (a_x, a_y)$  can be used instead of the column  $\begin{pmatrix} a_x \\ a_y \end{pmatrix}$ . As long as you are aware of these variations, you should not get confused.

For two vectors,  $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$  and  $\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$ , the coordinates of the sum are given by  $\vec{a} + \vec{b} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \end{pmatrix}$ . We can see this by writing the vectors in terms of the coordinate unit vectors:

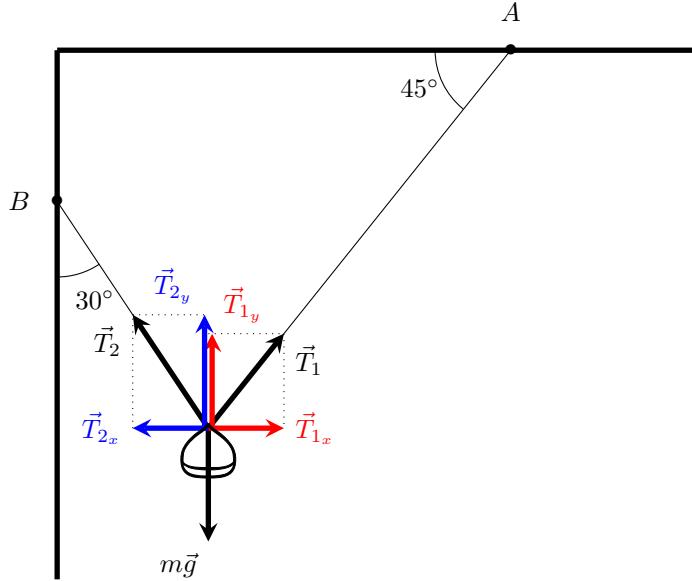
$$\vec{a} + \vec{b} = (a_x \vec{i} + a_y \vec{j}) + (b_x \vec{i} + b_y \vec{j}) = (a_x + b_x) \vec{i} + (a_y + b_y) \vec{j}.$$

Similarly, multiplying a vector by a scalar is equivalent to multiplying its both coordinates by that scalar:

$$k\vec{a} = k(a_x \vec{i} + a_y \vec{j}) = (ka_x) \vec{i} + (ka_y) \vec{j}.$$

<sup>2</sup>In the three-dimensional space, coordinate unit vector  $\vec{k}$  is additionally used for the  $z$  axis.

**Example 2.** A street lamp of mass  $m$  is suspended from two light strings at  $A$  and  $B$  as shown on the diagram. Find the ratio of the tensions of the two strings. Find the tension in each string in terms of  $m$  and  $g$ .



*Solution.* We start by resolving the vectors in the horizontal and vertical components:

$$T_{1x} = T_1 \cos 45^\circ, \quad T_{1y} = T_1 \sin 45^\circ;$$

$$T_{2x} = -T_2 \sin 30^\circ, \quad T_{2y} = T_2 \cos 30^\circ.$$

The lamp is at rest, so the forces must be balanced, hence  $T_{1x} + T_{2x} = 0$  and  $T_{1y} + T_{2y} = mg$ .

From the first of these equations, we find

$$T_1 \cos 45^\circ = T_2 \sin 30^\circ \Rightarrow \frac{\sqrt{2}}{2} T_1 = \frac{1}{2} T_2 \Rightarrow T_2 = \sqrt{2} T_1.$$

Substituting this into the second equation, we obtain

$$T_1 \sin 45^\circ + T_2 \cos 30^\circ = mg \Rightarrow \frac{\sqrt{2}}{2} T_1 + \frac{\sqrt{3}}{2} T_2 = mg \Rightarrow \frac{\sqrt{2}}{2} T_1 + \frac{\sqrt{3}}{2} \sqrt{2} T_1 = mg \Rightarrow (\sqrt{2} + \sqrt{6}) T_1 = 2mg.$$

Therefore

$$T_1 = \frac{\sqrt{2}mg}{1 + \sqrt{3}} \approx 0.52mg;$$

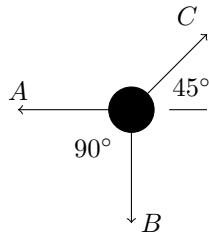
$$T_2 = \sqrt{2} T_1 = \frac{2mg}{1 + \sqrt{3}} \approx 0.73mg.$$

**Problem 2.** (3 marks) The magnetic field of the Earth can be represented by the magnetic induction vector  $\vec{B}$ . This vector can be split into a horizontal and a vertical components  $\vec{B}_H$  and  $\vec{B}_V$ ,  $\vec{B} = \vec{B}_H + \vec{B}_V$ . Near Edinburgh the vertical component is approximately 94% of the full magnetic induction:  $|\vec{B}_V| \approx 0.94 |\vec{B}|$ .

What percentage of the full induction is the horizontal component? What is the direction of  $\vec{B}$  relative to the horizontal (also known as the angle of magnetic inclination)?

**Problem 3.** (2 marks) Vector  $\vec{p}$  is of length 2 and directed at angle  $45^\circ$  clockwise from the  $y$  axis. Vector  $\vec{q}$  is of length 4 and directed at  $60^\circ$  anticlockwise from the  $y$  axis. Find the magnitude and direction of  $\vec{p} - \vec{q}$ . Give the answer to 3 significant figures (s.f.).

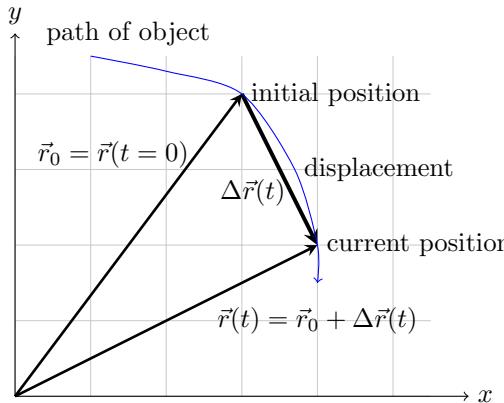
**Problem 4** (PAT 2015, 2 marks). Consider a mass and three strings, all lying on a horizontal table. The strings exert forces outwards on the mass as shown below. The mass does not move. What is the force on string  $C$  in terms of the force on string  $A$ ? What is the relationship between the force exerted by string  $A$  and the force exerted by string  $B$ ?



**Problem 5.** (3 marks)  $|22\vec{a} - \vec{b}| = |22\vec{b} - \vec{a}|$ . Show that  $|\vec{a}| = |\vec{b}|$ . Do not use the dot product.

### 3 Position vector $\vec{r}(t)$

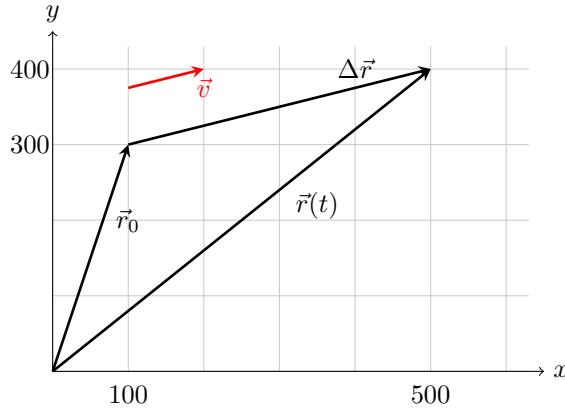
When describing the movement of a *material point* (a body of negligible size) we use the position vector (also known as the *radius-vector*), denoted as  $\vec{r}(t)$ . It is a vector drawn from the origin to the current position of the body. The position vector gives the position of an object (spatial coordinates) at any given moment of time  $t$ . The behaviour of this vector as a function of time is the *equation of motion* of the object. The change in the position vector  $\Delta\vec{r} = \vec{r} - \vec{r}_0$  from its initial position is known as the *displacement* ( $\vec{s} = \Delta\vec{r}$ ).



**Example 3.** An aeroplane at  $t = 0$  s is at a point with the position vector  $\vec{r}_0 = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$  m; the velocity of

the aeroplane is  $\vec{v} = \begin{pmatrix} 100 \\ 25 \end{pmatrix}$  m/s. Find the displacement vector  $\Delta\vec{r}$  and the position vector of the aeroplane when  $t = 4$  s. Write down the aeroplane's equation of motion.

*Solution.* We start by drawing a diagram. One square represents 100 m.



The displacement is equal to velocity multiplied by time  $\Delta\vec{r} = \vec{v} \times t = \begin{pmatrix} 100 \\ 25 \end{pmatrix}$  m/s  $\times$  4s =  $\begin{pmatrix} 400 \\ 100 \end{pmatrix}$  m.

We know that  $\Delta\vec{r} = \vec{r} - \vec{r}_0$ . Hence  $\vec{r} = \vec{r}_0 + \Delta\vec{r} = \vec{r}_0 + \vec{v}t$ . Substituting the given values, we calculate  $\vec{r}(t = 4s) = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$  m +  $\begin{pmatrix} 100 \\ 25 \end{pmatrix}$  m/s  $\times$  4s =  $\begin{pmatrix} 500 \\ 400 \end{pmatrix}$  m.

The equation of motion takes the form  $\vec{r}(t) = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$  m +  $\left[ \begin{pmatrix} 100 \\ 25 \end{pmatrix}$  m/s  $\right] t$ .

## 4 Scalar (dot) product

Of major importance for the application of vectors in physics is the notion of the scalar product. It has two equivalent definitions.

Definition 1. For two vectors with the known coordinates  $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$  and  $\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$ , the dot product is the number given by

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y. \quad (3)$$

Definition 2. For two vectors  $\vec{a}$  and  $\vec{b}$  with known magnitudes and directions, the dot product is the number given by

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta, \quad (4)$$

where  $\theta$  is the angle between vectors.

The equivalence of the two definitions is derived from the law of cosines, as shown in this [video by Virtually](#)

Passed.

**Example 4.** Vector  $\vec{a}$  has length 1 and is directed at  $30^\circ$  clockwise from the positive  $x$  axis. Vector  $\vec{b}$  has length 2 and is directed at  $30^\circ$  counterclockwise from the negative  $x$  axis. Find the coordinates of both vectors and their scalar product according to both definitions.

*Solution.* The coordinates of the vectors are

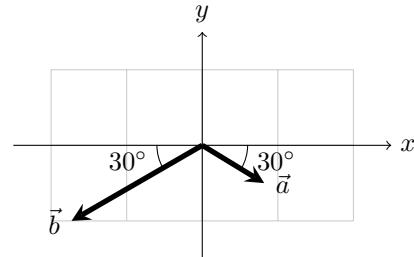
$$\vec{a} = 1 \times \begin{pmatrix} \cos 30^\circ \\ -\sin 30^\circ \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix} \text{ and } \vec{b} = 2 \times \begin{pmatrix} -\cos 30^\circ \\ -\sin 30^\circ \end{pmatrix} = \begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix}.$$

According to Definition 1, we find

$$\vec{a} \cdot \vec{b} = -\frac{\sqrt{3}}{2} \times \sqrt{3} + \frac{1}{2} \times 1 = -1.$$

To use Definition 2, notice that the angle between the vectors is  $120^\circ$ . Hence

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos 120^\circ = 1 \times 2 \times \left(-\frac{1}{2}\right) = -1.$$



As expected, the results are the same.

Before we proceed to solving further problems, let us mention a few important properties of the scalar product.

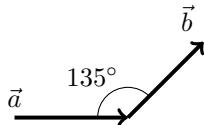
- For any vector  $\vec{a}$ ,  $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$ .
- For any three vectors,  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$  (this property is known as *distributivity*).
- Two nonzero vectors are orthogonal (perpendicular to each other) if and only if they have a zero scalar product.
- The  $x$ - and  $y$ -coordinates of a vector are obtained by multiplying that vector by the corresponding coordinate unit vector:  $a_x = \vec{a} \cdot \vec{i}$ ;  $a_y = \vec{a} \cdot \vec{j}$ .

Please convince yourself that the above statements are true as an independent exercise. You can, of course, use any one of the above definitions.

A few additional points:

- Scalar product is straightforwardly generalized to three-dimensional space. Definition 1 becomes  $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$ .

- The angle  $\theta$  in Eq. (4) is formed by the vectors when they are drawn from the same point. For example, in the diagram below, the angle between vectors  $\vec{a}$  and  $\vec{b}$  is  $45^\circ$  rather than  $135^\circ$ .



**Example 5.** What is the angle between the initial and final position vectors of the aeroplane in Example 3?

*Solution.* We apply Definition 2 of the scalar product to the initial and final position vectors. We have  $|\vec{r}_0| = \sqrt{100^2 + 300^2} = 317$  m and  $|\vec{r}(t)| = \sqrt{500^2 + 400^2} = 640$  m. The scalar product is then  $\vec{r}_0 \cdot \vec{r}(t) = 100 \times 500 + 300 \times 400 = 170000$  m<sup>2</sup>. Substituting all this into Eq. 4, we find  $\cos \theta = \frac{170000}{317 \times 640} = 0.84$ , so  $\theta = 33^\circ$ .

**Problem 6.** (3 marks) Vectors  $\vec{v} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$  and  $\vec{w} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$  represent the sides of a parallelogram. Find the angle between the diagonals of the parallelogram.

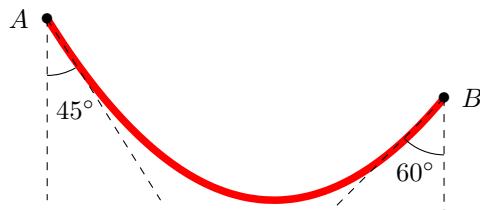
**Problem 7.** (3 marks) A motorboat is crossing a river. The motorboat's speed in still water is 24 mph. The speed of the river flow is 10 mph. At what angle to the riverbank should the driver direct the motorboat in order to cross the river at  $90^\circ$  to the riverbank? What is the speed of the boat relative to the shore?

**Problem 8.** (2 marks)  $|22\vec{a} - \vec{b}| = |22\vec{a} + \vec{b}|$ . Show that  $\vec{a} \perp \vec{b}$ .

**Problem 9** (PAT 2018, 3 marks). A pilot takes off from an airfield 5 km west of her house and flies in a direction  $60^\circ$  east of north. After 5 minutes she sees that direction to her house is now at angle of  $135^\circ$  to her course. How far away is she from her house? (Use  $\sqrt{2} \approx 1.4$ ).

**Problem 10.** (4 marks) Vector  $\vec{v} + 3\vec{w}$  is perpendicular to vector  $7\vec{v} - 5\vec{w}$ . Vector  $\vec{v} - 4\vec{w}$  is perpendicular to vector  $7\vec{v} - 2\vec{w}$ . Find the angle between vectors  $\vec{v}$  and  $\vec{w}$ .

**Problem 11.** (4 marks) A heavy rope of mass  $m$  is attached at A and B (see diagram). Find the tension of the rope  $T_A$  and  $T_B$  at A and B respectively. Give your answer in terms of  $m$  and gravitational field strength  $g$ .



**Problem 12.** (4 marks) In the setting of Problem 11, determine the ratio between the masses of the parts of the rope located to the left and to the right of its lowest point. The rope is uniform, i.e. its mass density per unit length is constant.

**Problem 13\*.** (6 marks) A motorboat is crossing a river. The motorboat's speed in still water is 6 mph. The speed of the river flow is 10 mph. At what angle to the riverbank should the driver direct the motorboat to minimize the drift of the boat due to the stream? What is the angle and speed of the boat relative to the shore? Do not use calculus.