Electromagnetism and Relativity

Introduction: A Necessary Union

Historically, Maxwell's equations, describing electricity and magnetism, were discovered before Einstein's theory of special relativity. It was initially thought that light waves needed a medium, the "aether," to travel through, and that Maxwell's equations only held true in the rest frame of this aether.

However, we now understand the aether concept is unnecessary. Instead, Maxwell's equations are inherently consistent with special relativity and hold true in all inertial reference frames. In fact, studying Maxwell's equations was key to developing the Lorentz transformations, which form the foundation of special relativity. Viewing electromagnetism through the lens of relativity reveals its deeper structure and elegance, simplifying the equations and showing how electric and magnetic fields are intertwined.

A Brief Review of Special Relativity

Special relativity is built on two postulates:

1. The laws of physics are the same in all inertial reference frames.

2. The speed of light in a vacuum, *c*, is constant for all observers, regardless of their motion or the motion of the light source.

This leads to transformations between spacetime coordinates (*ct*, *x*, *y*, *z*) of an observer in frame S and (*ct'*, *x'*, *y'*, *z'*) of an observer in frame S' moving at velocity *v* relative to S (e.g., a boost in the x-direction):

$$x' = \gamma(x - (v/c)ct)$$

$$ct' = \gamma(ct - (v/c)x)$$

where γ is the Lorentz factor:

$$\gamma = 1 / \sqrt{(1 - v^2/c^2)}$$

(Note: Vectors below will be represented in bold: **x)**

4-Vectors: It's useful to group spacetime coordinates into 4-vectors: X^μ = (*ct*, *x*, *y*, *z*) = (*ct*, **x**). The index μ runs from 0 to 3. Lorentz transformations (rotations and boosts) are linear maps acting on these 4-vectors, represented by matrices $\Lambda^\mu{}_\nu$. These transformations preserve the Minkowski metric, $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$.

The invariant interval, the "distance" squared between spacetime points, is constant for all observers:

$$X \cdot X = X^\mu \eta_{\mu\nu} X^\nu = c^2t^2 - x^2 - y^2 - z^2 = c^2t^2 - |\mathbf{x}|^2$$

Proper Time, 4-Velocity, 4-Momentum: The proper time, τ, is the time experienced by a moving particle itself. It's used to define relativistic kinematic quantities in a frame-independent way.

The 4-velocity is $U^\mu = dX^\mu/d\tau$, which has components $U^\mu = \gamma$(*c*, **u**), where **u** is the 3-velocity.

The 4-momentum is $P^\mu = mU^\mu$, where *m* is the rest mass. Its components are P^μ = (*E/c*, **p**), where $E = \gamma mc^2$ is the total energy and **p** = γm**u** is the relativistic 3-momentum.

Indices Up and Down: We distinguish between vectors with upper indices (like X^μ) and covectors with lower indices, $X_\mu = \eta_{\mu\nu} X^\nu$. For X^μ, this means X_μ = (*ct*, -**x**). Inner products are formed by summing over one upper and one lower index: $X^\mu X_\mu$.

4-Derivative: The relativistic generalization of the gradient is the 4-derivative:

$$\partial_\mu = \partial/\partial X^\mu = \left((1/c)\partial/\partial t, \nabla \right)$$

Tensors: Objects with multiple upper and lower indices that transform predictably under Lorentz transformations are called tensors. Physical laws written using tensors with correctly contracted indices are automatically covariant (look the same in all inertial frames).

Conserved Currents and the 4-Current Vector

The charge density ρ and current density **J** combine into a 4-vector called the 4-current:

$J^\mu = (\rho c,$ **J**$)$

This transformation property explains relativistic effects like Lorentz contraction influencing observed charge density ($\rho' = \gamma\rho_0$) and how moving charges constitute a current (**J**' = $-\gamma\rho_0$**v**).

The continuity equation, expressing local charge conservation ($\partial\rho/\partial t + \nabla \cdot$ **J** $= 0$), takes a simple, manifestly covariant form:

$\partial_\mu J^\mu = 0$

Magnetism as a Relativistic Effect: Consider a neutral wire with positive charges moving right (+v) and negative charges moving left (-v). A test charge moving parallel to the wire with velocity **u** experiences no electric field in the wire's rest frame. However, if we boost to the test charge's rest frame, the velocities of positive and negative charges transform differently according to the relativistic velocity addition formula. This, combined with Lorentz contraction affecting the charge densities differently for the positive and negative charges, results in the wire appearing to have a net charge density in the test particle's frame. This net charge creates an electric field that exerts a force on the test particle. Transforming this force back to the original (wire's rest) frame reveals a force that exactly matches the magnetic Lorentz force we'd calculate using the magnetic field generated by the current. This demonstrates that magnetism can be understood as an electric force viewed from a different inertial frame, arising from relativistic length contraction.

Gauge Potentials and the Electromagnetic Tensor

To handle time-varying fields, the static potentials ($\mathbf{E} = -\nabla\Phi$, $\mathbf{B} = \nabla\times\mathbf{A}$) are generalized:

$$\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

These relations automatically satisfy the two source-free Maxwell equations ($\nabla\cdot\mathbf{B}=0$ and $\nabla\times\mathbf{E} + \partial\mathbf{B}/\partial t = 0$). These potentials are not unique; they can be changed by a gauge transformation:

$$\Phi' = \Phi - \partial\xi/\partial t$$

$$\mathbf{A}' = \mathbf{A} + \nabla\xi$$

where $\xi(x,t)$ is any scalar function. This gauge freedom can be expressed concisely using the 4-potential $A^\mu = (\Phi/c, \mathbf{A})$:

$$A'_\mu = A_\mu - \partial_\mu \xi$$

The Electromagnetic Tensor ($F_{\mu\nu}$): We can construct a gauge-invariant object from the 4-potential and the 4-derivative, called the electromagnetic tensor (or field strength tensor):

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This is an antisymmetric tensor ($F_{\mu\nu} = -F_{\nu\mu}$). Its components contain the electric and magnetic fields:

$F_{\mu\nu} =$

$$[\ 0 \quad E_x/c \quad E_y/c \quad E_z/c\ ]$$

$$[-E_x/c \quad 0 \quad -B_z \quad B_y\ ]$$

$$[-E_y/c \quad B_z \quad 0 \quad -B_x]$$

$$[-E_z/c \quad -B_y \quad B_x \quad 0]$$

Since $F_{\mu\nu}$ is a tensor, it transforms under Lorentz transformations (Λ) as:

$$F'_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu F_{\rho\sigma}$$

This transformation law explicitly mixes electric and magnetic fields. For a boost with velocity **v** = (v, 0, 0):

$$E'_x = E_x$$

$$E'_y = \gamma(E_y - vB_z)$$

$$E'_z = \gamma(E_z + vB_y)$$

$$B'_x = B_x$$

$$B'_y = \gamma(B_y + (v/c^2)E_z)$$

$$B'_z = \gamma(B_z - (v/c^2)E_y)$$

This confirms that what one observer sees as a magnetic field, another moving observer might see partially as an electric field, and vice versa. Examples like boosting a line charge or a point charge explicitly show how Gauss's Law transforms into Ampere's Law components and how the field lines of a moving charge get "squeezed".

Lorentz Invariants: Combinations of **E** and **B** that all observers agree on can be formed from $F_{\mu\nu}$:

1. $(1/2) F_{\mu\nu}F^{\mu\nu} = |\mathbf{B}|^2 - |\mathbf{E}|^2/c^2$

2. $(1/4) F_{\mu\nu} \tilde{F}^{\mu\nu} = (1/c)\, \mathbf{E} \cdot \mathbf{B}$, where $\tilde{F}$ is the dual tensor (obtained using the Levi-Civita symbol, effectively swapping **E**/c and -**B**).

Maxwell's Equations in Covariant Form

The four Maxwell equations can be written elegantly as two tensor equations:

1. $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$

2. $\partial_\mu \tilde{F}^{\mu\nu} = 0$ (or the Bianchi identity: $\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0$)

These equations are *covariant*, meaning they maintain the same form in all inertial frames, even though the components (**E**, **B**, ρ, **J**) transform. The first equation contains Gauss's law (for $\nu=0$) and the Ampere-Maxwell law (for $\nu=1,2,3$). The second equation contains the magnetic Gauss's law and Faraday's law.

Current conservation ($\partial_\mu J^\mu = 0$) follows automatically from the first equation because $F^{\mu\nu}$ is antisymmetric. The second equation is automatically satisfied if $F_{\mu\nu}$ is derived from a 4-potential A_μ. Therefore, all of Maxwell's dynamics can be captured by:

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \mu_0 J^\nu$$

The Lorentz Force Law in Covariant Form

The force on a charge *q* moving in electromagnetic fields is also written covariantly using the 4-momentum P^μ, proper time τ, 4-velocity U_ν, and the electromagnetic tensor $F^{\mu\nu}$:

$$dP^\mu / d\tau = q F^{\mu\nu} U_\nu$$

Unpacking this equation:

- The spatial components ($\mu=1,2,3$) give the relativistic version of the familiar Lorentz force law: $d\mathbf{p}/dt = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$, where **p** is the relativistic momentum $\mathbf{p} = \gamma m \mathbf{u}$.

- The time component ($\mu=0$) describes the rate of change of energy: $dE/dt = q\mathbf{E} \cdot \mathbf{u}$, which is the work done by the electric field.

Energy and Momentum of Electromagnetic Fields

Electromagnetic fields carry energy and momentum.

- **Energy Density:** $\mathscr{E} = (\varepsilon_0/2)|\mathbf{E}|^2 + (1/2\mu_0)|\mathbf{B}|^2$
- **Poynting Vector (Energy Flux):** $\mathbf{S} = (1/\mu_0)\, \mathbf{E} \times \mathbf{B}$
- **Momentum Density:** $\mathbf{P} = \varepsilon_0\, \mathbf{E} \times \mathbf{B} = \mathbf{S}/c^2$

These quantities obey local conservation laws. Poynting's theorem is the energy conservation equation:

$\partial\mathscr{E}/\partial t + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}$ (work done on charges)

There's a similar conservation law for momentum involving the **Maxwell Stress Tensor**, σ_{ij}:

$\partial P_i/\partial t + \partial_j \sigma_{ij} = -(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})_i$ (force density on charges)

where $\sigma_{ij} = \varepsilon_0(E_i E_j - (1/2)\delta_{ij}|\mathbf{E}|^2) + (1/\mu_0)(B_i B_j - (1/2)\delta_{ij}|\mathbf{B}|^2)$

The Energy-Momentum Tensor ($T^{\mu\nu}$): Energy density, momentum density, energy flux, and stress are components of a single, symmetric, rank-2 tensor called the energy-momentum tensor:

$$T^{\mu\nu} = (1/\mu_0) [F^{\mu\rho} F^\nu{}_\rho - (1/4)\eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}]$$

Its components are arranged roughly as:

$T^{\mu\nu}$ corresponds to a matrix containing:

(Energy Density, Energy Flux / c)

(Momentum Density * c, Stress Tensor)

It is symmetric ($T^{\mu\nu} = T^{\nu\mu}$), reflecting the relationship $\mathbf{P} = \mathbf{S}/c^2$. Its conservation law, in the presence of sources, combines energy and momentum conservation:

$$\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho$$

This tensor plays a crucial role in physics, particularly in General Relativity where it acts as the source of spacetime curvature. For electromagnetism (and light), this tensor is also traceless ($T^\mu{}_\mu = \eta_{\mu\nu} T^{\mu\nu} = 0$), which implies for a photon gas that the pressure $P = \mathcal{E} / 3$.