Vectors and matrices

January 25, 2017

Outline

Definition

- Think of matrices as as a class
- Data: 2d table of entries of a type that can be added and multiplied
- Methods: + and *, transpose, ...

Examples

Entries can be

- .Numbers$\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and$\mathbb{C}$
- Functions, e.g. polynomials
- Logical expressions
- Matrices

- $A[i][j]$ is the entry in row i and column j
- We use A_{ij} in maths.

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Examples

- Simple weighted networks = matrices (train network, internet ...)
  - $A[i][j]$ is the weight on the arrow from j to i.
- Binary relations
- Matrices = operations on vectors (=data).
- Linear systems of equations
- 2d data of numbers (e.g. pixels on the screen)

Linear system of equations

- A linear system of equations $A \cdot x = b$ can be thought of as a matrix
  - 0, 1 or solution with parameters.
- All algorithms for solving linear system of equations over the reals have issues with rounding.
- Gaussian elimination can be used to solve the system
  - Row operations preserve solution
  - Numerically unstable
- Many other algorithms

Operations I

- Addition: $A + B$
  - A and B has the same size
  - Addition is elementwise
- Multiplication: $A \cdot B$ orAB
  - num. of columns in A = num. of rows in B
- Scalar multiplication: $c \cdot A$ or cA.
  - Scalar multiplication is elementwise

Operations II

- Identity matrix denoted by I or I_n.
  - $I[i][i] = 1, I[i][j] = 0$ for $i \neq j$.
- A and B are square matrices
- If $AB = I$: B is the inverse of A, written as A^{-1}.
  - Fact: A is also the inverse of B.
- Inverses may not exist.
  - Calculated by solving a set of equations.
  - $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

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- A invertible for system $A \cdot x = b$.
  - Solution is $A^{-1} \cdot b$.
  - Invert A once, can solve $A \cdot x = b$ for many b.
- If no solution: least squares approximations
  - We will look at this more carefully later in the course.

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Operations III

- Vectors are matrices.
  - Row vectors
  - Column vectors
- $A \cdot v$ and $w \cdot A$ are vectors (if defined)
  - So matrices transform vectors
  - E.g. Rotation, Scaling, Projection,....

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Angles

- One time around the circle: $0^\circ..360^\circ$degrees (artificial)
- One time around the circle: $0..2\pi$ radians (natural)
- $rad = \frac{2\pi \cdot deg}{360} = \frac{\pi \cdot deg}{180}$, $deg = \frac{180 \cdot rad}{\pi}$
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Definitions

- We will use sine $sin(x)$, cosine $cos(x)$ and sometimes tangent $tan(x) = \frac{sin(x)}{tan(x)}$.
- Can be defined using right angled triangles.
- Can be defined using unit circles.
  - $(cos(x), sin(x))$ is the point on the unit circle when angle is x.
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Inverse function

- $sin^{-1}(x)$ or $arcsin(x)$. asin(x), acos(x), atan(x) in C
- $y = sin(x)$ has infinite number of solutions.
  - Gives one solution usually from half-plane $-\frac{\pi}{2} \dots \frac{\pi}{2}$
  - $atan2(x, y)$ is useful

Things we should know

- A vector can be described by coordinates
- A vector can be described by length and direction (angle in 2d)

- Scalar (dot) product
- Vector (cross) product
- Length of vector

- Length: $|v| = \sqrt{x^2 + y^2}$
  - Follows from Pythagoras' theorem
  - Generalises to higher dimensions

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Scalar product

- $v \cdot w = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = x_1 y_1 + x_2 y_2.$
  - Scalar product takes two vectors as input and produces a number
  - Generalises to higher dimensions
  - Really the matrix product $v^T \cdot w$ for column vectors.
- or, $cos(\theta) = \frac{v \cdot w}{|v||w|}$
  - $\theta$is the angle between the vectors
- v and w are orthogonal if $v \cdot w = 0$
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- $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 \cdot z_2 - y_2 \cdot z_1 \\ -(x_1 \cdot z_2 - x_2 \cdot z_1) \\ x_1 \cdot y_2 - x_2 \cdot y_2 \end{pmatrix}$
- Or.$|v \times w| = |v| \cdot |w| \cdot sin\theta$ and $v \times w$ is (right handed) orthogonal to v and w.
  - θ is angle between v and w.
  - $|v \times w|$ is the area of paralellogram spanned by v and w.
  - v and w are paralell if $v \times w = 0$
- Note: Two vectors as input and a vector as output.
- Generalisation to higher dimension is non-trivial
- 2d embedded in 3d: $\begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_1 \cdot y_2 - x_2 \cdot y_2 \end{pmatrix}$
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