Matrix II

REA1121

Mathematics for programming

Outline

- Introduction
- Vector space
  - Linear combination
  - Vector space and subspace
  - Linear independence
- Exercises

INTRODUCTION

Introduction

- What can matrices represent?
- What can matrix computation be used for?

Vector & matrix

vector

$$\begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ -2 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$$

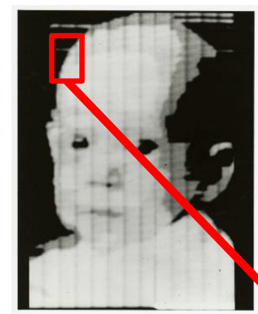
What can a matrix represent?

$$
\begin{array}{cc}
 & n \text{ columns} \\
m \text{ rows} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}
\end{array}
$$

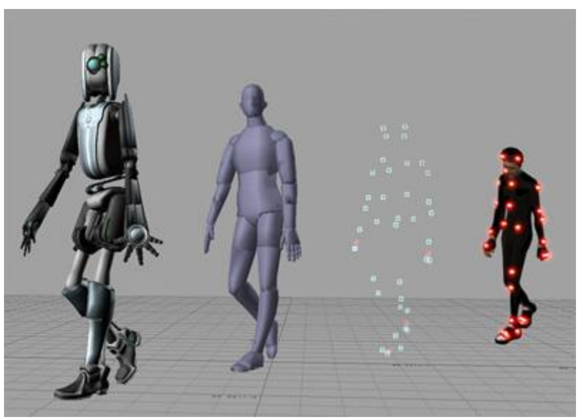
$m \times n$ matrix

Representation of images

- Raster images
  - Image data
  - Non-image data



72	73	89	92	97	102	107	117	130
72	74	100	105	111	117	123	134	148
68	76	99	106	115	122	130	143	160
56	72	90	97	108	116	125	139	157
44	65	88	96	106	114	122	136	154
37	54	83	102	112	116	125	136	148
36	57	77	95	107	115	129	142	154
34	55	74	93	106	117	133	147	158
33	51	78	99	115	124	137	148	158
38	57	83	107	124	132	140	148	157
48	72	88	111	127	134	142	149	157

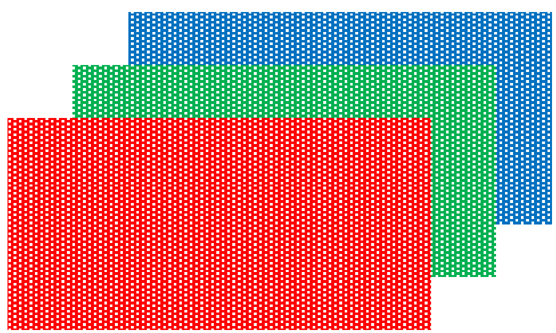


https://en.wikipedia.org/wiki/Digital_image#/media/File:NBSFirstScanImage.jpg
https://en.wikipedia.org/wiki/Computer-generated_imagery#/media/File:Activemarker2.PNG

Representation of images

- Image processing
- Original Image I: $m \times n \times 3$ matrix

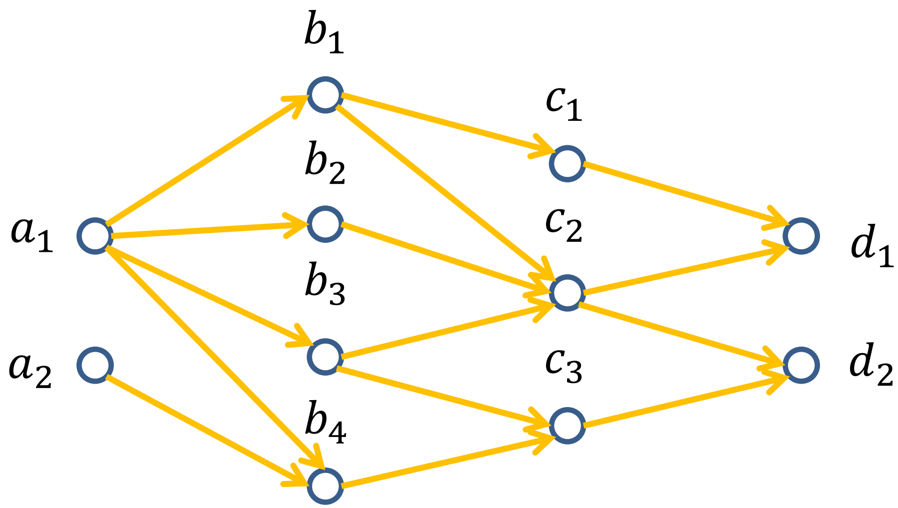
$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & r_{22} & r_{23} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & r_{m3} & \cdots & r_{mn} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & g_{13} & \cdots & g_{1n} \\ g_{21} & g_{22} & g_{23} & \cdots & g_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{m1} & g_{m2} & g_{m3} & \cdots & g_{mn} \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_{11} & \boldsymbol{b}_{12} & \boldsymbol{b}_{13} & \cdots & \boldsymbol{b}_{1n} \\ \boldsymbol{b}_{21} & \boldsymbol{b}_{22} & \boldsymbol{b}_{23} & \cdots & \boldsymbol{b}_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{b}_{m1} & \boldsymbol{b}_{m2} & \boldsymbol{b}_{m3} & \cdots & \boldsymbol{b}_{mn} \end{bmatrix}$$



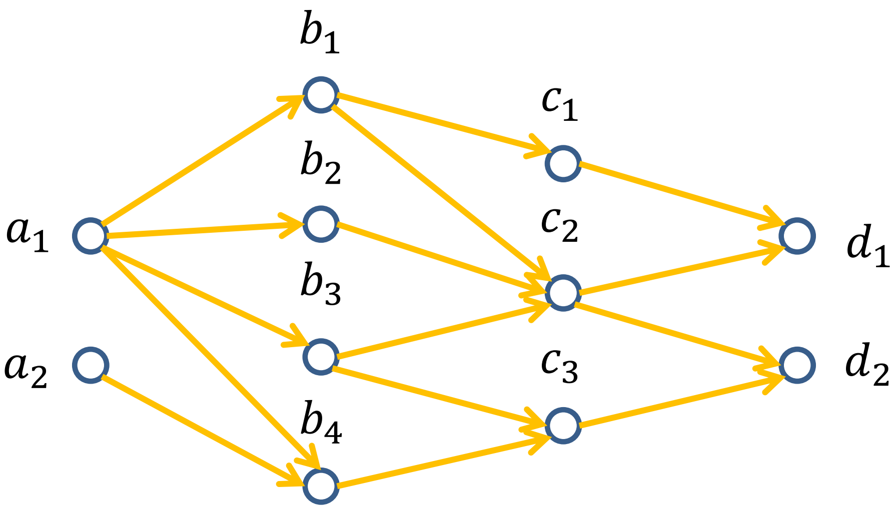
Representation of images

- Photometric processing
  - $I \times 0.5$
- Geometric processing
  - $Transpose\ I(:,:,1), I(:,:,2), I(:,:,3)$
- Colour processing
  - Reshape I into a $(m \times n) \times 3$ matrix
  - Multiply the resulting matrix with a 3×3 matrix
  - Reshape the resulting matrix into a $m \times n \times 3$ matrix

Representation of graph



Representation of graph

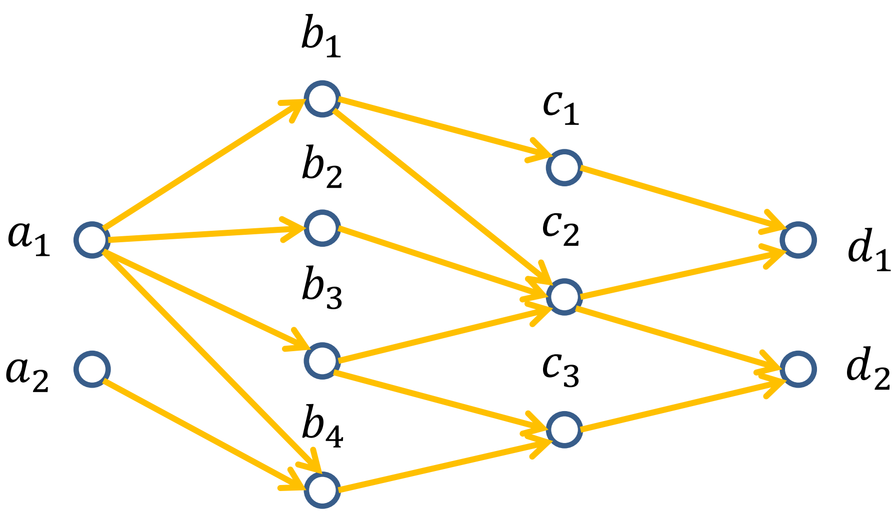


	b_1	b_2	b_3	b_4
a_1				
a_2				

	c_1	c_2	c_3
b_1			
b_2			
b_3			
b_4			

	d_1	d_2
c_1		
c_2		
c_3		

Representation of graph

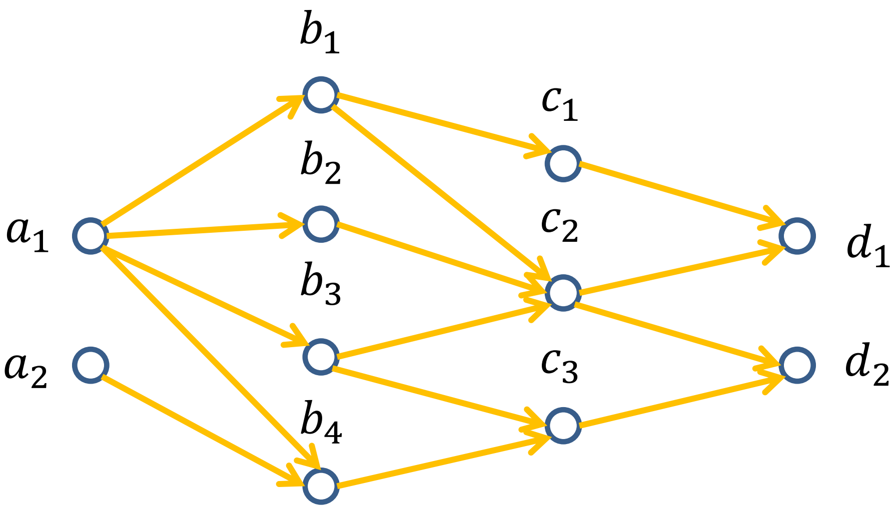


	b_1	b_2	b_3	b_4
a_1	1	1	1	1
a_2	0	0	0	1

	c_1	c_2	c_3
b_1	1	1	0
b_2	0	1	0
b_3	0	1	1
b_4	0	0	1

	d_1	d_2
c_1	1	0
c_2	1	1
c_3	0	1

Representation of graphs

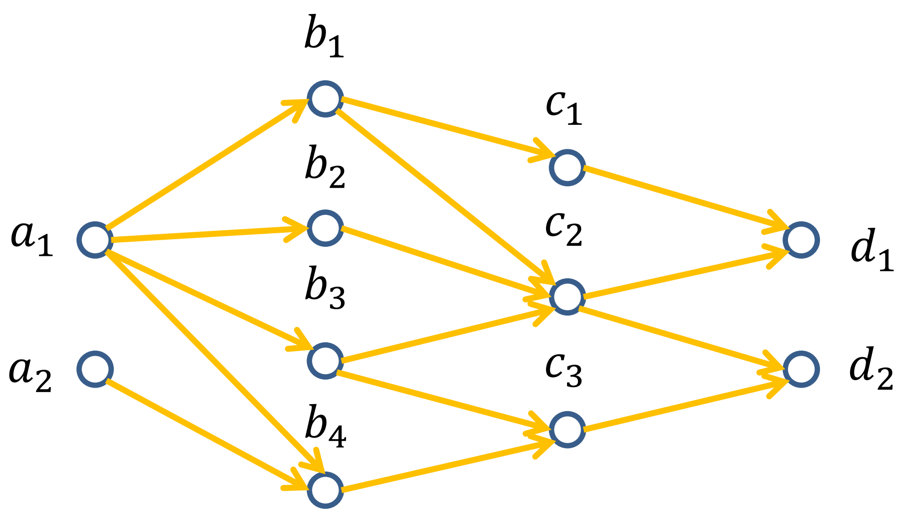


$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Representation of graphs



$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

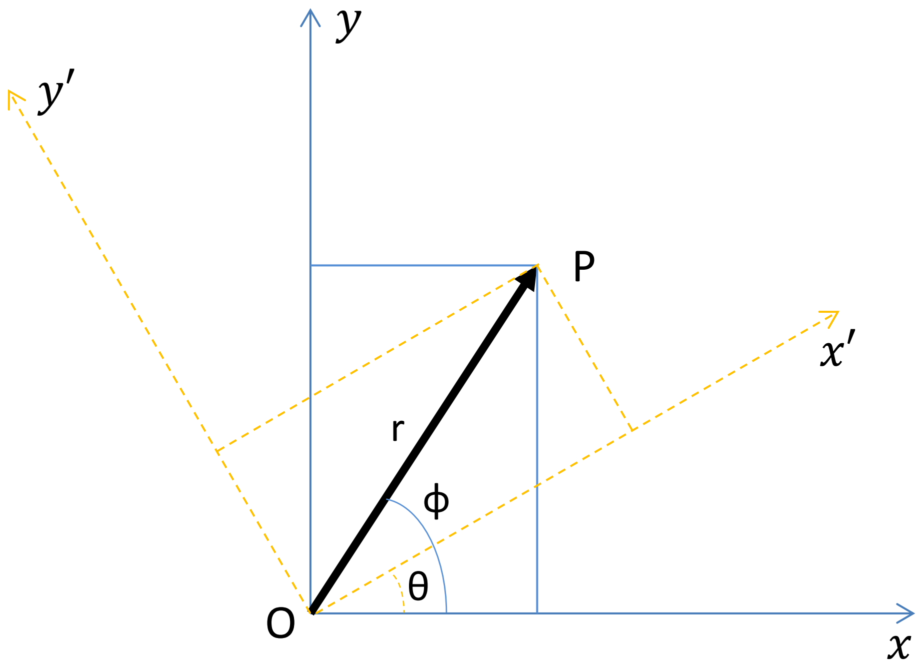
$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

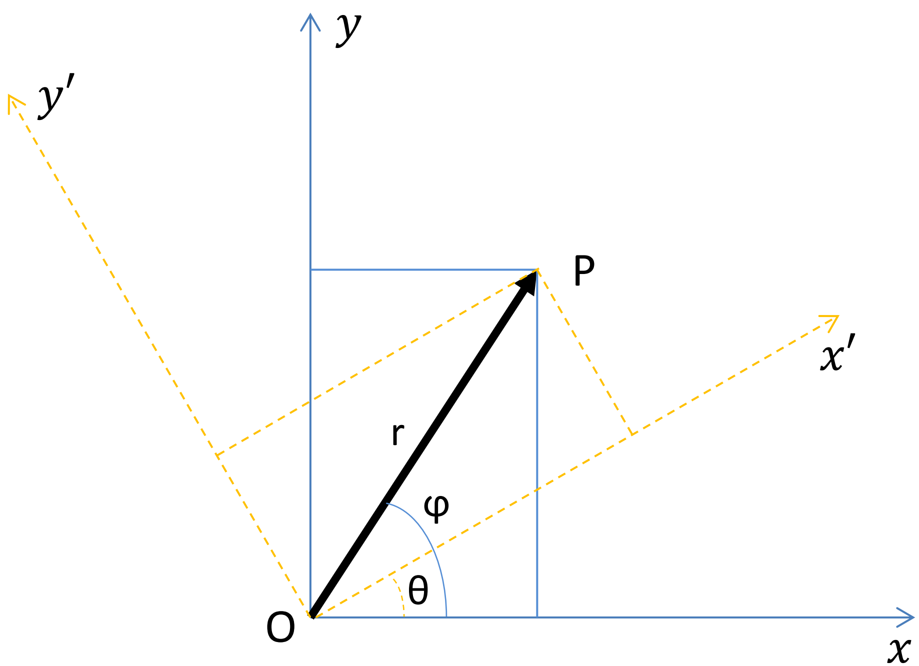
What do you get from PQ and PQR?

Representation of transform

The relationship between the coordinates of P
in the Oxy system
and
the Ox’y’ system



Representation of transform



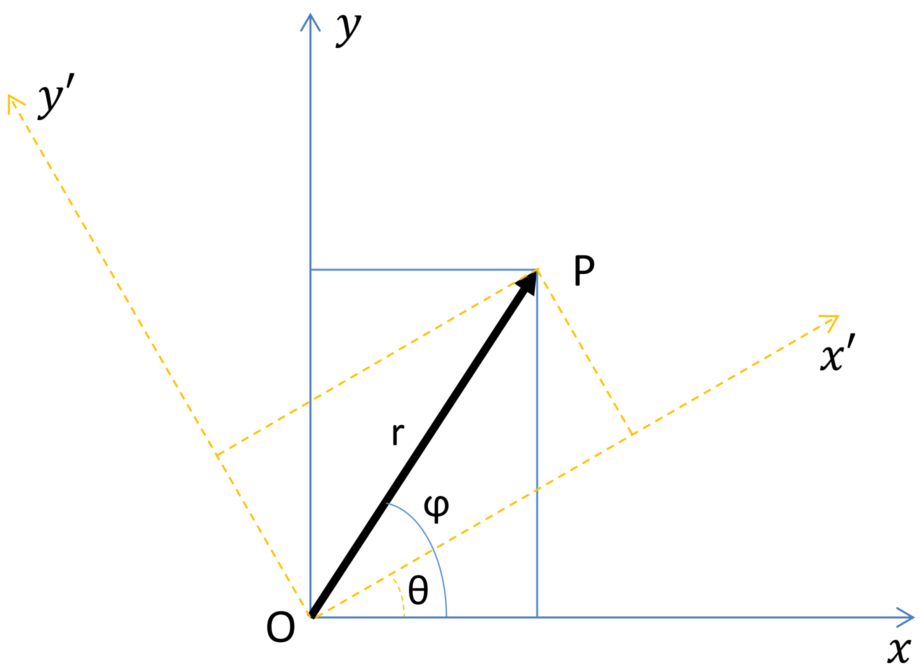
In the Oxy system, the coordinates of P: (x, y)

$$x = r\cos\varphi\,, \qquad y = r\sin\varphi$$

In the Ox'y' system, the coordinates of P: (x', y')

$$x' = r\cos(\varphi - \theta)\,, \qquad y' = r\sin(\varphi - \theta)$$

Representation of transform



In the Oxy system, the coordinates of P: (x, y)

$$x = r\cos\varphi\,, \qquad y = r\sin\varphi$$

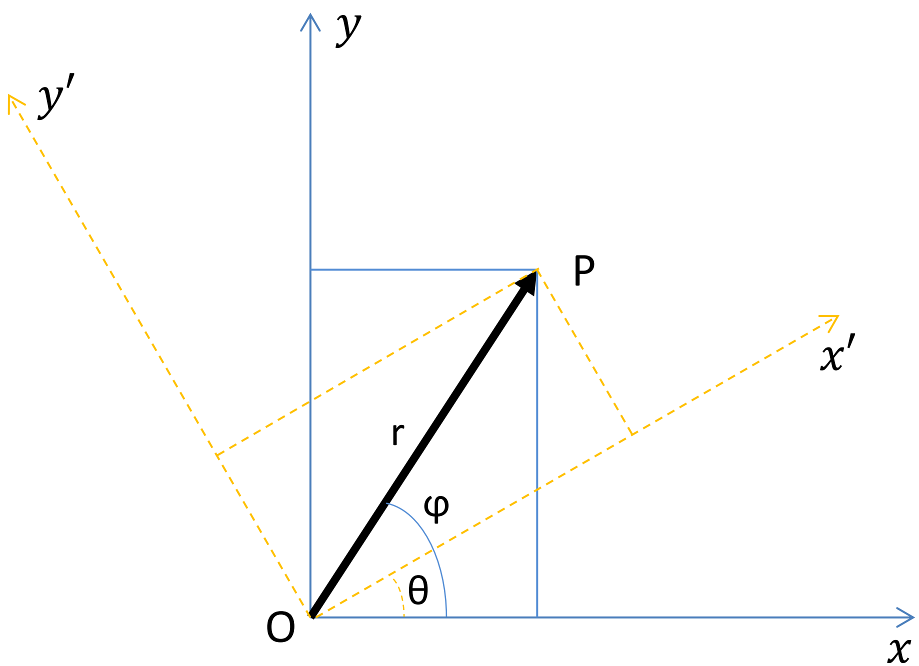
In the Ox'y' system, the coordinates of P: (x', y')

$$x' = r\cos(\varphi-\theta)\,, \qquad y' = r\sin(\varphi-\theta)$$

$$\begin{aligned} x' &= r\cos\varphi\cos\theta + r\sin\varphi\sin\theta \\ &= x\cos\theta + y\sin\theta \end{aligned}$$

$$\begin{aligned} y' &= r\sin\varphi\cos\theta - r\cos\varphi\sin\theta \\ &= y\cos\theta - x\sin\theta \end{aligned}$$

Representation of transform



$$x' = r\cos\varphi\cos\theta + r\sin\varphi\sin\theta$$
$$= x\cos\theta + y\sin\theta$$

$$y' = r\sin\varphi\cos\theta - r\cos\varphi\sin\theta$$
$$= y\cos\theta - x\sin\theta$$

$$B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},\quad X = \begin{bmatrix} x \\ y \end{bmatrix},\quad X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$X' = BX,\quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Representation of transform

- For a commercial flight, a matrix may transform the number of passengers to the weight of baggage and passengers

$$\begin{array}{r c}
 & \begin{array}{ccc} \text{Checked baggage} & \text{Cabin baggage} & \text{Passenger body} \end{array} \\
\begin{array}{r} \text{First class} \\ \text{Business} \\ \text{Economy} \end{array} & \begin{bmatrix} 96 & 15 & 90 \\ 46 & 12 & 90 \\ 23 & 8 & 90 \end{bmatrix}
\end{array}$$

Introduction

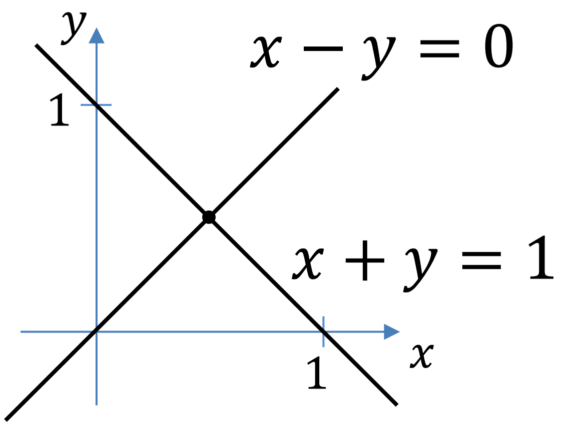
- What can matrices represent?
- What can matrix computation be used for?

Linear equations

- In a linear system, we know the system and the output, how do we get the input?
- $\begin{cases} x + y = 1 \\ x - y = 0 \end{cases}$
- $A\boldsymbol{x} = b$
- $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Linear equations

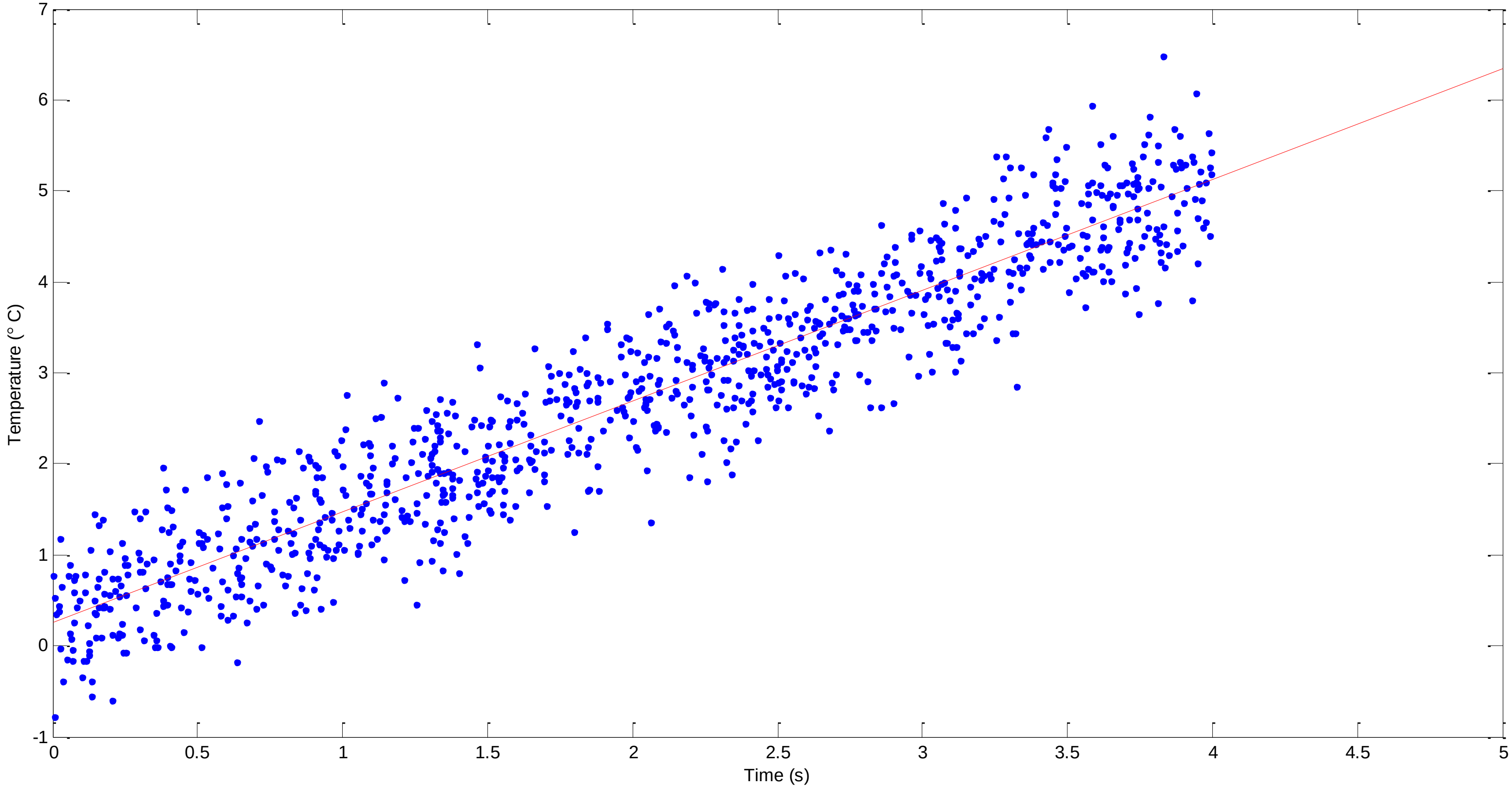
- In a linear system, we know the system and the output, how do we get the input?



Linear regression

- In a physical experiment, you observed two variables
  - time
  - temperature
- You did plenty of measurements
- How do you find the relationship between the two variables?

Linear regression

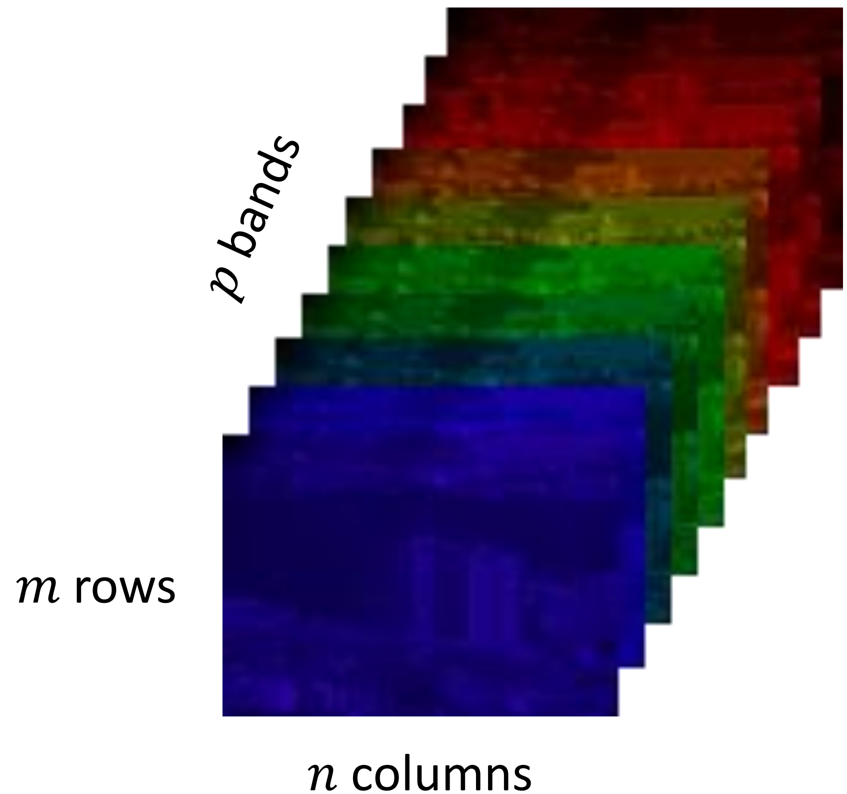


Linear regression

- Least squares
- Essentially a type of curve fitting

Data compression

- A spectral image
- Multi-band
- Visible band
  - 380 nm - 780 nm
- Divide the spectrum by 5 nm intervals
  - 81 bands
- Reduce size of data



Data compression

- A: $m \times n$ matrix
- Singular value decomposition (SVD)

$$A = USV'$$

- U: $m \times m$ matrix
- S: $m \times n$ matrix
- V: $n \times n$ matrix

Data compression

- A: $m \times n$ matrix
- Singular value decomposition (SVD)

$$A = USV'$$

- U: $m \times m$ matrix reduces to U1: $m \times 1$ vector
- S: $m \times n$ matrix reduces to S1: scalar
- V: $n \times n$ matrix reduces to V1: $n \times 1$ vector
- $U_1 S_1 V_1{}'$ is satisfactorily close to A.

VECTOR SPACE

Linear combination

- In linear algebra, there are two basic vector operations
  - Addition: $\boldsymbol{v} + \boldsymbol{w}$
  - Scalar multiplication: $c\boldsymbol{v}, d\boldsymbol{w}$

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\boldsymbol{v} + \boldsymbol{w} = ? \qquad \boldsymbol{v} - \boldsymbol{w} = ?$$

$$2\boldsymbol{v} = ? \qquad -\boldsymbol{w} = ?$$

Linear combination

- Combining addition with scalar multiplication gives rise to the **linear combination**

DEFINITION *$c\boldsymbol{v} + d\boldsymbol{w}$ is a linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$.*

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$c\boldsymbol{v} + d\boldsymbol{w} = ?$$

Linear combination

- Four special linear combinations
  - sum: $1\boldsymbol{v} + 1\boldsymbol{w}$
  - difference: $1\boldsymbol{v} - 1\boldsymbol{w}$
  - zero: $0\boldsymbol{v} - 0\boldsymbol{w}$
  - scalar multiple: $c\boldsymbol{v} + 0\boldsymbol{w}$

Linear combination

- Vector representation

Linear combination

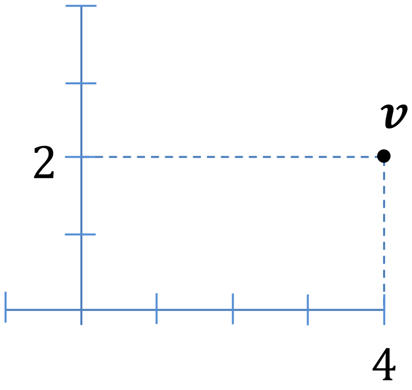
- Vector representation
  - numbers

$$v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Linear combination

- Vector representation
  - numbers
  - point

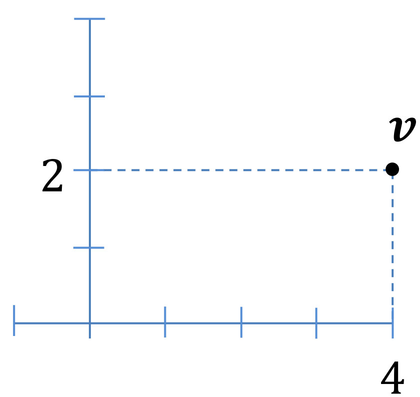
$$v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

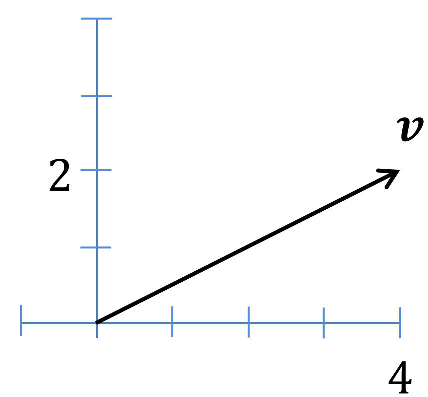


Linear combination

- Vector representation
  - numbers
  - point
  - arrow

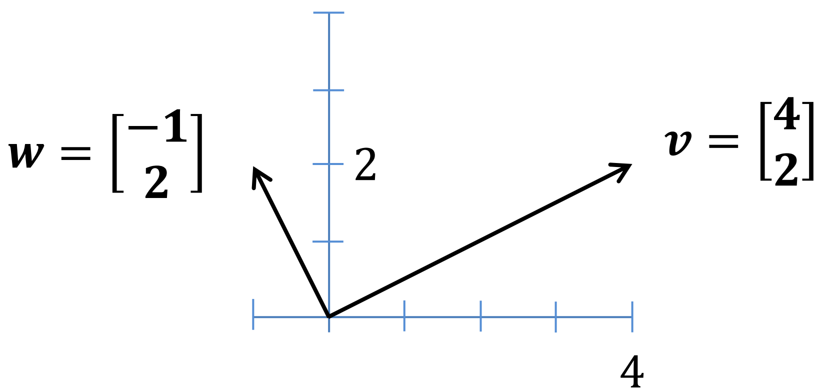
$$v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$





Linear combination

- Visualise **linear combination** with arrows



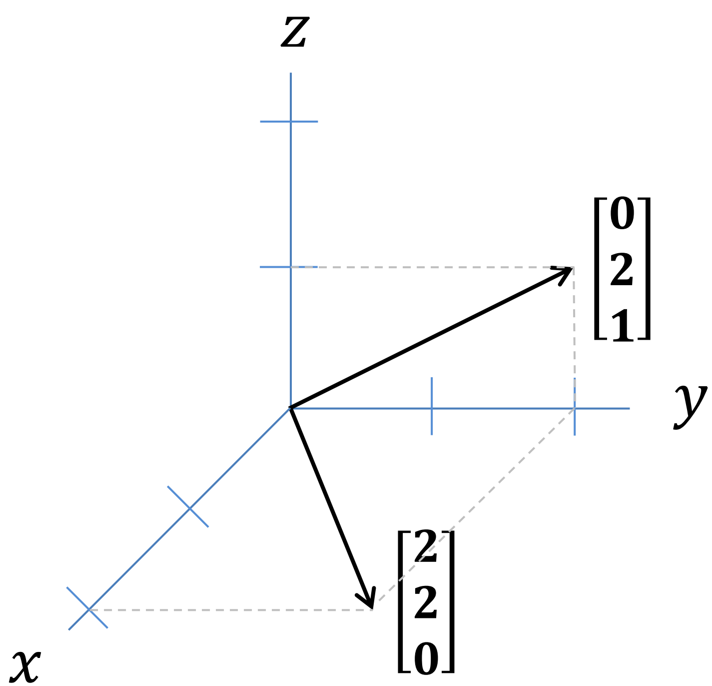
$\boldsymbol{v} + \boldsymbol{w}$

$\boldsymbol{v} - \boldsymbol{w}$

$\boldsymbol{v} - \mathbf{2}\boldsymbol{w}$

Linear combination

- Vectors in three dimensions



Linear combination

- Linear combinations in three dimensions

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

$$\boldsymbol{u} + 4\boldsymbol{v} - 2\boldsymbol{w} = ?$$

Linear combination

- Suppose the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in three-dimensional space
- What is the picture of all combinations
  - $c\boldsymbol{u}$
  - $c\boldsymbol{u} + d\boldsymbol{v}$
  - $c\boldsymbol{u} + d\boldsymbol{v} + ew$

Example 1

- The linear combinations of

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ fill a plane.}$$

- Describe that plane. Find a vector that is not a combination of v and w.

Example 2

- For $\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, describe all points $c\boldsymbol{v}$ with
  - whole numbers (integers) c
  - nonnegative $c > 0$
- Then add all vectors $d\boldsymbol{w}$ and describe all $c\boldsymbol{v} + d\boldsymbol{w}$.

Example 3

- Find two equations for the unknowns c and d so that the linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ equals the vector $\boldsymbol{b}$:

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Vector space

DEFINITION *The space* R^n *consists of all column vectors* $\boldsymbol{v}$ *with* n *components*.

- The components of $\boldsymbol{v}$ are real numbers
- A real vector space is a set of "vectors“ together with rules for vector addition and for multiplication by real numbers.
- The addition and the multiplication must produce vectors that are in the space.

Vector space

- 8 conditions are required of every vector space
  - $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$
  - $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$
  - There is a unique "zero vector" such that $\boldsymbol{x} + 0 = \boldsymbol{x}$ for all $\boldsymbol{x}$
  - For each x there is a unique vector $-\boldsymbol{x}$ such that $\boldsymbol{x} + (-\boldsymbol{x}) = 0$
  - 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$
  - $(c_1 c_2)\boldsymbol{x} = c_1(c_2\boldsymbol{x})$
  - $c(\boldsymbol{x} + \boldsymbol{y}) = c\boldsymbol{x} + c\boldsymbol{y}$
  - $(c_1 + c_2)\boldsymbol{x} = c_1\boldsymbol{x} + c_2\boldsymbol{x}$

Vector space

- One-dimensional space R^1
- Two-dimensional space R^2
- Three-dimensional space R^3
- Four-dimensional space R^4
- Five-dimensional space R^5

$$[1] \qquad \begin{bmatrix} 4 \\ \pi \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Vector space

- Any vector spaces other than R^n?

Vector space

- Vector spaces other than R^n
  - The vector space of all real 2 by 2 matrices.
  - The vector space of all real functions $f(x)$.
  - The vector space that consists only of a zero vector.

Vector space

DEFINITION A ***subspace*** of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements: *If* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *are vectors in the subspace and* c *is any scalar, then*

i. $\boldsymbol{v}+\boldsymbol{w}$ is in the subspace

ii. $c\boldsymbol{v}$ is in the subspace

Vector space

- Possible subspaces in R^3

Vector space

- Possible subspaces in R^3
  - Any line through the origin
  - Any plane through the origin
  - The whole space
  - The zero vector

Example 4

- Keep only the vectors (x, y) whose components are positive or zero (this is a quarter-plane).
- It this quarter-plane a subspace?

Example 4

- Keep only the vectors (x, y) whose components are positive or zero (this is a quarter-plane).
- It this quarter-plane a subspace?
- Include also the vectors whose components are both negative. Now we have two quarter-planes.
- Do these two quarter-planes form a subspace?

Linear independence

DEFINITION The sequence of vectors $v_1, \cdots, v_n$ is ***linearly independent*** if the only combination that gives the zero vector is $0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \cdots + 0\boldsymbol{v}_n$.

- Linear independence

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n = 0$$

only happens when all x's are zero.

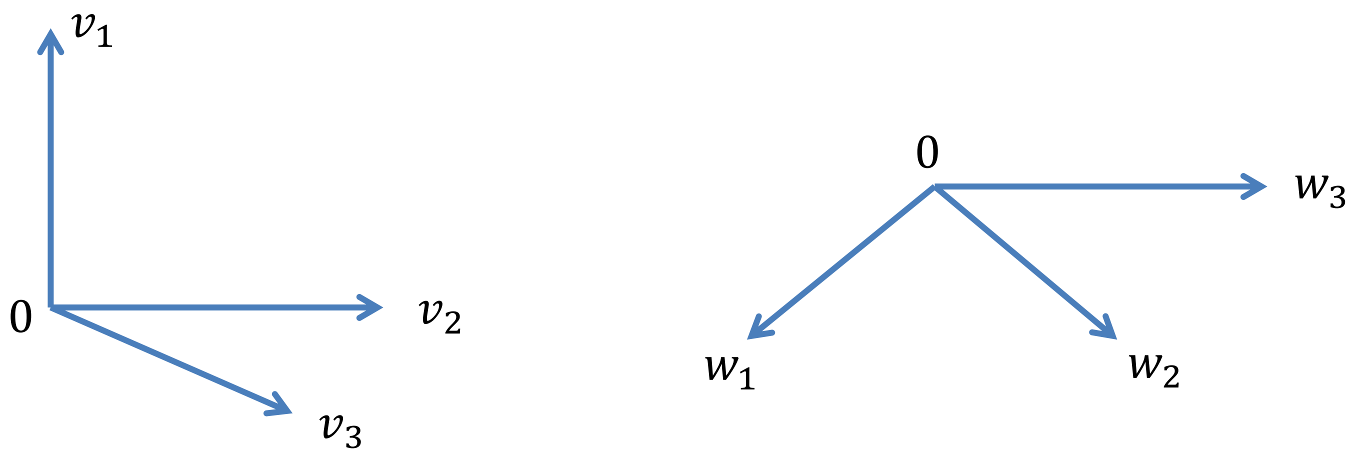
- If a combination gives 0, when the x's are not all zero, the vectors are dependent.

Linear independence

- Are the following vectors dependent or independent?
  - $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$
  - $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0.00001\end{bmatrix}$
  - $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\-1\end{bmatrix}$
  - $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\0\end{bmatrix}$
  - Any three vectors v_1, v_2, v_3 in R^2

Linear independence

- Linear independence in R^3



Linear independence

- Linear independence in R^3

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, w_3 = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}$$

Linear independence

- Linear independence in R^3

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, a_3 = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Linear independence

- **DEFINITION** The columns of A are *linearly independent* when the only solution to $A\boldsymbol{x} = 0$ is $x = 0$. No other combination $A\boldsymbol{x}$ of the columns gives the zero vector.
- A systematic way of solving linear equations, such as $A\boldsymbol{x} = 0$, is elimination.

Example 5

- Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

EXERCISE

Problem 1

- Find two different combinations of the three vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ that produce $\boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Will there always be two different combinations of any three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the plane that produce $\boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

Problem 2

- Suppose $[x_1 \quad x_2]^T + [y_1 \quad y_2]^T$ is defined to be $[x_1 + y_2 \quad x_2 + y_1]^T$. With the usual multiplication $c\boldsymbol{x} = [cx_1 \quad cx_2]^T$, which of the eight conditions (see Slide 45) are not satisfied?
- Suppose the multiplication $c\boldsymbol{x}$ is defined to produce $[c\boldsymbol{x_1} \quad 0]^T$ instead of $[cx_1 \quad cx_2]^T$. With the usual addition in $\mathbf{R}^2$, are the eight conditions satisfied?

Problem 3

- Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

a) The plane of vectors $[b_1 \quad b_2 \quad b_3]^T$ with $b_1 = b_2$.
b) The plane of vectors with $b_1 = 1$.
c) The vectors with $b_1 b_2 b_3 = 0$.
d) All linear combinations of $\boldsymbol{v} = [1 \quad 4 \quad 0]^T$ and $\boldsymbol{w} = [2 \quad 2 \quad 2]^T$.
e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.
f) All vectors with $b_1 \leq b_2 \leq b_3$.

Problem 4

- Find the largest possible number of independent vectors among

$$
\begin{aligned}
\boldsymbol{v}_1 &= [1 \quad -1 \quad 0 \quad 0]^T \\
\boldsymbol{v}_2 &= [1 \quad 0 \quad -1 \quad 0]^T \\
\boldsymbol{v}_3 &= [1 \quad 0 \quad 0 \quad -1]^T \\
\boldsymbol{v}_4 &= [0 \quad 1 \quad -1 \quad 0]^T \\
\boldsymbol{v}_5 &= [0 \quad 1 \quad 0 \quad -1]^T \\
\boldsymbol{v}_6 &= [0 \quad 0 \quad 1 \quad -1]^T
\end{aligned}
$$

Solution 1

- Two combinations out of infinitely many that produce $\boldsymbol{b} = [0\ 1]^T$ are $-2\boldsymbol{u} + \boldsymbol{v}$ and $\frac{1}{2}\boldsymbol{w} - \frac{1}{2}\boldsymbol{u}$.
- No, three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the *x*-*y* plane could fail to produce $\boldsymbol{b}$ if all three lie on a line that does not contain $\boldsymbol{b}$. Yes, if one combination produces $\boldsymbol{b}$ then two (and infinitely many) combinations will produce $\boldsymbol{b}$. This is true even if $\boldsymbol{u} = 0$; the combinations can have different $c\boldsymbol{u}$.

Solution 2

- The following conditions are not satisfied:
  - $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$
  - $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$
  - $(c_1 + c_2)\boldsymbol{x} = c_1\boldsymbol{x} + c_2\boldsymbol{x}$
- When $c[\boldsymbol{x_1} \quad \boldsymbol{x_2}]^T = [c\boldsymbol{x_1} \quad 0]^T$, the only broken rule is "1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$". Rules (1)-(4) for addition $\boldsymbol{x} + \boldsymbol{y}$ still hold since addition is not changed.

Solution 3

- The only subspaces are

  a) The plane of vectors $[b_1 \quad b_2 \quad b_3]^T$ with $b_1 = b_2$.

  d) All linear combinations of $\boldsymbol{v} = [1 \quad 4 \quad 0]^T$ and $\boldsymbol{w} = [2 \quad 2 \quad 2]^T$.

  e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

Solution 4

- $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent (the -1's are in different positions). All six vectors are on the *hyperplane* $[1 \quad 1 \quad 1 \quad 1]\boldsymbol{v} = 0$,so no four of these six vectors can be independent.