2.2 PROBABILITY I

1. Probability

Let S be the **sample space**, that is, the set of all possible outcomes of an experiment, and E is the **event**, the set of outcomes of a certain type. We assume that all outcomes in S has the same probability. The probability that E happens is

$$P(E) = \frac{|E|}{|S|}$$

where $|E|$ and $|S|$ denotes the number of elements in E and S, respectively.

We will look at some methods for computing probabilities:

1. List all possibilities.
2. Use enumerative combinatorics (counting). This was Module 6 in REA1101.
3. Use networks and matrices (Markov chains).
4. Run simulations.

1.1. Examples of lists.

We calculate the probability of getting two heads with two throws with a coin. The sample space is

$$S = \{HH, HT, TH, TT\}$$

and the event is

$$E = \{HH\}$$

The probability is therefore

$$P(E) = \frac{|E|}{|S|} = \frac{1}{4} = 25\%$$

We calculate the probability of getting at least two heads with three throws with a coin. The sample space is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

and the event is

$$E = \{HHH, HHT, HTH, THH\}$$

which gives the probability

$$P(E) = \frac{4}{8} = 50\%$$

We calculate the probability of getting at least one six thowing two dice. We view the sample space as a 6 by 6 matrix. Outcomes with at least one 6 are marked with a 1.

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

The sample space has 36 entries, 11 of which are in the event. The probability of at least one 6 is therefore

$$\frac{11}{36} \approx 31\%$$

Note that sometimes it is easier to calculate the probability of not an event happening. The relevant rule is

$$P(\overline{A}) = 1 - P(A).$$

1.2. **Independence.** Two events A and B are called **independent** if

$$P(A \cap B) = P(A) \cdot P(B).$$

Example 1.

1. *The event that the first roll of a die is* 6 *is independent with getting* 3 *on the second roll.*
2. *The event that the first card you draw from a deck of cards is diamond, is not independent with the event of drawing a spade on the second card.*

Independence simplifies many calculations in probability theory. Similar to the product rule for enumeration.

1.3. **Dice.** We discuss some examples with $n = 5$ fair dice, each having $m = 6$ sides. The sample space consist of

$$|S| = m^n = 6^5$$

elements. Using this sample space implies that we are calculating with ordered selection allowing repeats.

For example, what is the probability of roll 5 dice with exactly 1 dice showing 1. We first choose which dice which should show 1. There are 5 possibilities. The other dice could show any number except 1, altogether 5^4 possibilities. This gives us probability

$$P = \frac{5 \cdot 5^4}{6^5} \approx 40\%$$

What is the probability of rolling 5 dice with exactly 3 dice showing 1?
We first choose which dice which should show 1. There are $C(5, 3)$ possibilities. The other dice could have number except 1 giving us 5^2 possibilities. The probability is

$$P = \frac{C(5, 3) \cdot 5^2}{6^5}.$$

What is the probability of rolling 5 dice with at at least three dice showing 3.
We can divide up according to the number of 3.

With 3 dice showing 3: $C(5, 3) \cdot 5^2$
With 4 dice showing 3: $C(5, 4) \cdot 5^1$
With 5 dice showing 3: 1

This gives us the proability

$$P = \frac{C(5, 3) \cdot 5^2 + C(5, 4) \cdot 5 + 1}{6^5} \approx 3.6\%$$

All of these examples can be generalised to any m and n.

Exercises:

Exercise. 1. *You are throwing two dice. Use lists to calculate probabilities of*

a) *throwing two fives.*
b) *throwing a six and a five.*
c) *throwing the sum is* 7
d) *throwing the two dice show different numbers.*
e) *throwing the sum is even.*

Exercise. 2. *You are throwing four dice. Calculate probabilities of*

a) *throwing four sixes.*
b) *throwing three sixes.*
c) *throwing at least two sixes.*
d) *throwing the four dice have different numbers.*
e) *throwing the sum is 6.*

Exercise. 3. *You are throwing five dice. Calculate probabilities of*

a) *throwing an odd number of sixes.*
b) *throwing two five, two six and a four.*
c) *throwing sum is less than* 28*,*
d) *throwing yahtzee.*
e) *throwing a two pairs.*
f) *throwing a pair.*

Solutions/answers:

Exercise. 1.

a) $1/36$.

b) $1/18$.

c) $1/6$.

d) $5/6$.

e) $1/2$.

Exercise. 2.

a) $1/6^4$

b) $\frac{C(4,3)\cdot 5}{6^4}$

c) $\frac{C(4,2)\cdot 5^2 + C(4,3)\cdot 5 + C(4,4)}{6^4}$

d) $\frac{6\cdot 5\cdot 4\cdot 3}{6^4}$

e) Possible sums are: $1+1+1+3$, $1+1+2+2$. We need to calculate separately and use the Rule of Sum.

Adding up we get $\frac{C(4,1)+C(4,2)}{6^4}$.

Exercise. 3.

a) We would throw 1,3 or 5 sixes. We calculate individually and add using the Rule of Sum.

In total we get $\frac{C(5,1)\cdot 5^4+C(5,3)\cdot 5^2+C(5,5)}{6^5}$.

b) $\frac{C(5,2)\cdot C(3,2)\cdot C(1,1)}{6^5}$

c) Here it is easyer to first calculate sum greater than or equal to 28. There are 4 cases.

Sum 30: $6+6+6+6+6$

Sum 29: $5+6+6+6+6$

Sum 28: $5+5+6+6+6$ or $4+6+6+6+6$

which gives $\frac{1+C(5,1)+C(5,2)+C(5,1)}{6^5}$ and the answer is $1-\frac{1+C(5,1)+C(5,2)+C(5,1)}{6^5}$.

d) $6/6^5 = 1/6^4$.

e) There are $\frac{6\cdot 5}{2}$ different two pair combinations. Any given pair can be obtained in $C(5,2)\cdot C(3,2)\cdot 4$ different ways. So the answer is $\frac{6\cdot 5\cdot C(5,2)\cdot C(3,2)\cdot 4}{2\cdot 6^5}$.

f) $\frac{6\cdot C(5,2)\cdot 5\cdot 4\cdot 3}{6^5}$