Matrix IV

REA1121

Mathematics for programming

Outline

- Roadmap
- Solving linear equations
- Exercises

ROADMAP

Roadmap

Vectors

- Linear combination $c\boldsymbol{v} + d\boldsymbol{w}$
- Dot product $\boldsymbol{v} \cdot \boldsymbol{w}$
- Length
- Unit vector
- Angle

Vector space

- Subspace
- Linear independence
- Basis
- Orthonormal basis
- Dimension

SOLVING LINEAR EQUATIONS

Linear equation

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Linear equation

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{cases}
$$

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ \vdots \\ b_n
\end{bmatrix}
$$

Linear equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\boldsymbol{x} = \boldsymbol{b}$$

Linear equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\boldsymbol{x} = ?$$

Linear equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\boldsymbol{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Linear equation

$$A\boldsymbol{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

- $A\boldsymbol{x}$ is a linear combination of the columns of A

Column space

DEFINITION The ***column space*** consists of *all linear combinations of the columns*. The combinations are all possible vectors $A\boldsymbol{x}$. They fill the column space $\boldsymbol{C}(A)$.

Column space

- $\boldsymbol{C}(A)$ contains not just the columns of A, but all their combinations $A\boldsymbol{x}$.
- To solve $A\boldsymbol{x} = \boldsymbol{b}$ is to express $\boldsymbol{b}$ as a linear combination of the columns.
- The coefficients in that combination gives us a solution $\boldsymbol{x}$ to the system $A\boldsymbol{x} = \boldsymbol{b}$.
- **The system $Ax = \boldsymbol{b}$ is solvable *if and only if b is in the column space of* A.**

Column space

- $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

- The column space of A is a subspace of ?

Column space

- $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

- The column space of A is a subspace of $\mathbf{R}^m$.

Example 1

$$Ax = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$$A\boldsymbol{x} = \boldsymbol{b}$$

Example 1

- The column space $\boldsymbol{C}(A)$ is ….
- The system $A\boldsymbol{x} = \boldsymbol{b}$ is solvable when ….
- Then $\boldsymbol{b}$ is ….

Example 1

- The column space $\boldsymbol{C}(A)$ is a plane containing the two columns.
- The system $A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is on that plane.
- Then $\boldsymbol{b}$ is a linear combination of the columns.

Example 1

- Most $\boldsymbol{b}$ in $\mathbf{R}^3$ is not in the column space
- For most $\boldsymbol{b}$, there is NO solution to the 3 equations with 2 unknowns.
- Is $\boldsymbol{b} = [0 \quad 0 \quad 0]^T$ in the column space?
- Is there a solution to $A\boldsymbol{x} = \mathbf{0}$?

Example 1

- Most $\boldsymbol{b}$ in $\mathbf{R}^3$ is not in the column space
- For most $\boldsymbol{b}$, there is NO solution to the 3 equations with 2 unknowns.
- $\boldsymbol{b} = [0 \quad 0 \quad 0]^T$ in the column space, as column space is a subspace, in this case, a plane passing through the origin.
- There is certainly a solution to $A\boldsymbol{x} = \mathbf{0}$, which, always available, is $\boldsymbol{x} = \mathbf{0}$.

Nullspace

$$A\boldsymbol{x} = \boldsymbol{0}$$

- When A is invertible, the only solution is $\boldsymbol{x} = \boldsymbol{0}$.
- When A is not invertible, there are non-zero solutions to $A\boldsymbol{x} = \boldsymbol{0}$. Each solution $\boldsymbol{x}$ belongs to the *nullspace* of A.

Nullspace

- The nullspace of A consists of all solutions to $A\boldsymbol{x} = \mathbf{0}$.
- These vectors $\boldsymbol{x}$ are in $\mathbf{R}^n$. (Distinct from column space that is a subspace of $\mathbf{R}^m$.)
- The nullspace containing all solutions to $A\boldsymbol{x} = \mathbf{0}$ is denoted by $\boldsymbol{N}(A)$.

Nullspace

- Nullspace is a subspace.
  - Suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ are in the nullspace
    - $A\boldsymbol{x} = \mathbf{0}$ and $A\boldsymbol{y} = \mathbf{0}$
  - $\boldsymbol{x} + \boldsymbol{y}$ is in the subspace
    - $A(\boldsymbol{x} + \boldsymbol{y}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$
  - $c\boldsymbol{x}$ is in the subspace
    - $A(c\boldsymbol{x}) = c\mathbf{0} = \mathbf{0}$

Example 2

Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Example 2

Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

- Apply elimination to $A\boldsymbol{x} = \mathbf{0}$
- There is only one equation.
- The line corresponding to the first equation is the nullspace, which contains all solutions.

Example 2

Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

- Choose one point on the line ("special solution"), e.g., $\begin{bmatrix} -2 & 1 \end{bmatrix}^T$.
- The nullspace $N(A)$ contains all multiples of $s = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
- The nullspace consists of all combinations of the special solutions.

Example 3

$x + 2y + 3z = 0$ comes from the 1×3 matrix $A = [1 \quad 2 \quad 3]$.

This equation $A\boldsymbol{x} = \mathbf{0}$ produces a plane through the origin $[0 \quad 0 \quad 0]^T$.

The plane is a subspace of $\mathbf{R}^3$. *It is the nullspace of A.*

The solutions to $x + 2y + 3z = 6$ also form a plane, but not a subspace.

Example 3

$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ has the special solutions $s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

The vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ lie on the plane $x + 2y + 3z = 0$, which is the nullspace of $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. All vectors on the plane are combinations of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.

Example 3

- The first column of $A = [1 \quad 2 \quad 3]$ contains the *pivot*, so the first component of $\boldsymbol{x}$ is *not free*.
- The last two components of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are *free* and correspond to columns without pivots.
- We choose them specially to be 1 and 0. Then the first components are determined by the equation $A\boldsymbol{x} = \boldsymbol{0}$.
- The special choice (1 or 0) is only for the free variables.

Example 4

Describe the nullspaces of these three matrices A, B, C

- $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$
- $B = \begin{bmatrix} A \\ 2A \end{bmatrix}$
- $C = \begin{bmatrix} A & 2A \end{bmatrix}$

Example 4

- $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$
- The equation $A\boldsymbol{x} = \mathbf{0}$ has only the zero solution $\boldsymbol{x} = \mathbf{0}$. The nullspace $N(A)$ is $\mathbf{Z}$, which contains only the single point $\boldsymbol{x} = \mathbf{0}$ in $\mathbf{R}^2$. This comes from elimination.
- A is invertible. There are no special solutions. All columns of A have pivots.

Example 4

- $B = \begin{bmatrix} A \\ 2A \end{bmatrix}$
- The rectangular matrix B has the same nullspace of $\mathbf{Z}$ as $N(A)$.
- The first two equations in $B\boldsymbol{x} = \mathbf{0}$ again require $\boldsymbol{x} = \mathbf{0}$. The last two equations would also force $\boldsymbol{x} = \mathbf{0}$.
- When we add extra equations, the nullspace certainly cannot become larger.

Example 4

- $C = [A \quad 2A]$
- C has extra columns instead of extra rows. The solution vector $\boldsymbol{x}$ has four components.
- Elimination produces pivots in the first two columns of C, but the last two columns are "free“ as they don't have pivots.

Example 4

$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$ becomes, after elimination,

$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$, an ***Echelon matrix***.

pivot columns　　**free** columns

Example 4

- For the free variables x_3 and x_4, we make special choices of 1s and 0s. First $x_3 = 1, x_4 = 0$ and then $x_3 = 0, x_4 = 1$.
- The pivot variables x_1 and x_2 are determined by $U\boldsymbol{x} = \mathbf{0}$
- Then we get two special solutions, $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, in the nullspace of C (which is also the nullspace of U).

Example 4

- The nullspace is formed by all combinations of the special solutions, $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.

$$\boldsymbol{s}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Nullspace

- For many matrices A, the only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$.
- Their nullspaces $N(A) = Z$, that contains only the zero vector.
- The only combination of the columns that produces $\boldsymbol{b} = \mathbf{0}$ is the "zero combination".
- It means the columns of A are independent.
- All columns of A have pivots and no columns are free.

Solving $A\boldsymbol{x} = \mathbf{0}$

- Forward elimination takes A to a triangular U.
- Back substitution in $U\boldsymbol{x} = \mathbf{0}$ produces $\boldsymbol{x}$.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

Solving $A\boldsymbol{x} = \mathbf{0}$

To find all solutions to $U\boldsymbol{x} = \mathbf{0}$, a good method is to separate the ***pivot variables*** from the ***free variables***.

pivots

$$\begin{bmatrix} \mathbf{1} & 1 & 2 & 3 \\ 0 & 0 & \mathbf{4} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

pivot variables

free variables

pivot columns

free columns

Solving $A\boldsymbol{x} = \mathbf{0}$

- The upper triangular U can be further simplified into the ***reduced row echelon matrix*** R:
  - Produce zeros above the pivots, by eliminating upward
  - Produce ones in the pivots, by dividing the whole row by its pivot.
- The reduced row echelon matrix R has zeros above the pivots as well as below.
- If A is invertible, its reduced row echelon form is the identity matrix $R = I$.

Solving $A\boldsymbol{x} = \boldsymbol{0}$

$$U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the pivot columns contain I

Solving $A\boldsymbol{x} = \boldsymbol{0}$

- The free variables x_2 and x_4 can be given any values. Then back substitution finds the pivot variables x_1 and x_3.
- The simplest choices for the free variables are ones and zeros.
- These choices give the special solutions.

Solving $A\boldsymbol{x} = \mathbf{0}$

- The free variables x_2 and x_4 can be given any values. Then back substitution finds the pivot variables x_1 and x_3.
- The simplest choices for the free variables are ones and zeros.
- These choices give the special solutions.

Rank

- The numbers m and n give the size of a matrix, but not necessarily the true size of a linear system.
  - If there are two identical rows in A, the second one disappears in elimination.
  - if row 3 is a combination of rows I and 2, then row 3 will become all zeros
  - We don't want to count rows of zeros
- The true size of A is given by its rank

Rank

DEFINITION The ***rank*** of A is the number of pivots. This number is r.

- The matrices A and U and R have r independent rows (the pivot rows). They also have r independent columns (the pivot columns).
- The rank r is the dimension of the column space. It is also the dimension of the row space.

Example 5

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} \text{->} U = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Two pivots, namely the rank of A (and U) is 2.

Rank

- The four possibilities of linear equations depend on the rank r
  - $r = m = n$, A is square and invertible, $A\boldsymbol{x} = \boldsymbol{b}$ has 1 solution
  - $r = m < n$, A is short and wide, $A\boldsymbol{x} = \boldsymbol{b}$ has ∞ solutions
  - $r = n < m$, A is tall and thin, $A\boldsymbol{x} = \boldsymbol{b}$ has 0 or 1 solution
  - $r < m, r < n$, A is not full rank, $A\boldsymbol{x} = \boldsymbol{b}$ has 0 or ∞ solutions

EXERCISES

Problem 1

Describe the column spaces for

- $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$
- $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

Problem 2

- Find the special solutions and describe the *complete solution* to $A\boldsymbol{x} = \mathbf{0}$ for
$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} A_2 & A_2 \end{bmatrix}$$
- Which are the pivot columns?
- Which are the free variables?
- Which is R in each case?

Problem 3

- Find the reduced echelon form of A. What is the rank? What is the special solution to $A\boldsymbol{x} = \mathbf{0}$?

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Solution 1

Describe the column space for

- $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- The column space of I is the whole space $\mathbf{R}^2$.
- Every vector is a combination of the columns of I. $\boldsymbol{C}(I)$ is $\mathbf{R}^2$.

Solution 1

Describe the column space for

- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$
- The column space of A is only a line.
- The column space contains all vectors $[c \quad 2c]^T$ along that line.
- The equation $A\boldsymbol{x} = \boldsymbol{b}$ is only solvable when $\boldsymbol{b}$ is on the line.

Solution 1

Describe the column space for

- $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$
- The column space $\boldsymbol{C}(B)$ is all of $\mathbf{R}^2$. Every $\boldsymbol{b}$ is attainable.
- B has the same column space as I or any $\boldsymbol{b}$ is allowed.
- $\boldsymbol{x}$ has extra components and there are more solutions, more combinations, that give $\boldsymbol{b}$.

Solution 2

- $A_1\boldsymbol{x} = \mathbf{0}$ has four special solutions. They are the columns $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4$ of the 4×4 identity matrix. The nullspace is all of $\mathbf{R}^4$. The complete solution to $A_1\boldsymbol{x} = \mathbf{0}$ is any $\boldsymbol{x} = c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2 + c_3\boldsymbol{s}_3 + c_4\boldsymbol{s}_4$ in $\mathbf{R}^4$.
- There are no pivot columns.
- All variables are free.
- The reduced R is the same zero matrix as A_1.

Solution 2

- $A_2\boldsymbol{x} = \boldsymbol{0}$ has only one special solution $s = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. The multiples $\boldsymbol{x} = c\boldsymbol{s}$ give the complete solution.
- The first column of A_2 is its pivot column.
- x_2 is the free variable.
- The row reduced matrix $R_2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

Solution 2

- There are three special solutions to $A_3\boldsymbol{x} = \boldsymbol{0}$.

$$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, s_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, s_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{x} = c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2 + c_3\boldsymbol{s}_3$$

- The first column of A_3 is its pivot column.
- All the variables x_2, x_3, x_4 are free.
- The row reduced matrix $R_3 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Solution 3

- Add row 1 to row 2.
- Then add row 2 to row 3.
- Then add row 3 to row 4.

$$U = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution 3

- Add row 3 to row 2.
- Then add row 2 to row 1.

$$R = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution 3

$$R = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The rank is $r = 3$.
- There is one free variable x_4.
- The special solution is $s = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$