Matrix VI

REA1121

Mathematics for programming

Outline

- Roadmap
- Eigenvalues and eigenvectors
- Diagonalization of matrices
- Exercises

ROADMAP

Roadmap

- Vectors
  - Linear combination $c\boldsymbol{v} + d\boldsymbol{w}$
  - Dot product $\boldsymbol{v} \cdot \boldsymbol{w}$
  - Length
  - Unit vector
  - Angle
- Vector space
  - Subspace
  - Linear independence
  - Basis
  - Orthonormal basis
  - Dimension
  - Column space
  - Nullspace
  - Rank
- Orthogonality
  - Projection
  - Least-squares approximation
  - Pseudoinverse

EIGENVALUES AND EIGENVECTORS

Example 1

Suppose $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$, then $A^{100} = ?$

Example 1

Suppose $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$, then $A^{100} =?$

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0.65 & 0.525 \\ 0.35 & 0.475 \end{bmatrix}$$

$$\vdots$$

$$A^{100} \approx \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

Example 1

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

There exist two vectors $\boldsymbol{x}_1, \boldsymbol{x}_2$ and two scalers λ_1, λ_2 such that $A\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$ and $A\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2$.

$$\lambda_1 = 1, \boldsymbol{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$\lambda_1 = \frac{1}{2}, \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Example 1

$$\begin{bmatrix}0.8\\0.2\end{bmatrix} = \boldsymbol{x}_1 + (0.2)\boldsymbol{x}_2 = \begin{bmatrix}0.6\\0.4\end{bmatrix} + \begin{bmatrix}0.2\\-0.2\end{bmatrix}$$

$$A\begin{bmatrix}0.8\\0.2\end{bmatrix} = A\boldsymbol{x}_1 + A(0.2)\boldsymbol{x}_2 = \boldsymbol{x}_1 + \frac{1}{2}(0.2)\boldsymbol{x}_2$$

$$= \begin{bmatrix}0.6\\0.4\end{bmatrix} + \begin{bmatrix}0.1\\-0.1\end{bmatrix} = \begin{bmatrix}0.7\\0.3\end{bmatrix}$$

$$A^{99}\begin{bmatrix}0.8\\0.2\end{bmatrix} = \boldsymbol{x}_1 + \left(\frac{1}{2}\right)^{99}(0.2)\boldsymbol{x}_2 = \begin{bmatrix}0.6\\0.4\end{bmatrix} + \begin{bmatrix}small\\vector\end{bmatrix}$$

Example 1

$$\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \boldsymbol{x}_1 - (0.3)\boldsymbol{x}_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} - \begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix}$$

$$A\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = A\boldsymbol{x}_1 - A(0.3)\boldsymbol{x}_2 = \boldsymbol{x}_1 - \frac{1}{2}(0.3)\boldsymbol{x}_2$$

$$= \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} - \begin{bmatrix} 0.15 \\ -0.15 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix}$$

$$A^{99}\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \boldsymbol{x}_1 + \left(\frac{1}{2}\right)^{99}(0.3)\boldsymbol{x}_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} small \\ vector \end{bmatrix}$$

Eigenvalue and eigenvector

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

- The number λ is an ***eigenvalue*** of A.
- The vector $\boldsymbol{x}$ is an ***eigenvector*** of A.
- $\boldsymbol{x}$ is in the same direction as $A\boldsymbol{x}$.
- An eigenvalue of 0 means the eigenvector $\boldsymbol{x}$ is in the nullspace.
- If A is the identity matrix, every vector has $A\boldsymbol{x} = \boldsymbol{x}$. All vectors are eigenvectors of I. All eigenvalues $\lambda = 1$.

Example 2

$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 0$.

- Suppose $\boldsymbol{x}$ is an eigenvector of P, then we have $P\boldsymbol{x} = \lambda\boldsymbol{x}$.
- Substitute into the linear equation, we may get two eigenvectors

$$\boldsymbol{x}_1 = [1,1]^T, \boldsymbol{x}_2 = [1,-1]^T$$

Example 2

$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 0$.

- P is singular (not invertible), so $\lambda = 0$ is an eigenvalue.
- P is symmetric, so its eigenvectors are perpendicular.

$$\boldsymbol{x}_1 = [1,1]^T, \boldsymbol{x}_2 = [1,-1]^T$$

Example 3

$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = -1$.

- Suppose $\boldsymbol{x}$ is an eigenvector of R, then we have $R\boldsymbol{x} = \lambda\boldsymbol{x}$.
- Substitute into the linear equation, we may get two eigenvectors

$$\boldsymbol{x}_1 = [1,1]^T, \boldsymbol{x}_2 = [1,-1]^T$$

Example 3

$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = -1$.

- It can be found that $R = 2P - I$ (P in Example 2) while P and R share the same eigenvectors.
- As $P\boldsymbol{x} = \lambda\boldsymbol{x}$, $2P\boldsymbol{x} = 2\lambda\boldsymbol{x}$, subtract $I\boldsymbol{x} = \boldsymbol{x},$ we then have $(2P - I)\boldsymbol{x} = (2\lambda - 1)\boldsymbol{x}$
- Thus when a matrix is shifted by I, each λ is shifted by 1. No change in eigenvectors.

Example 4

Try to find eigenvalues and eigenvectors of Q

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Suppose $\boldsymbol{x}$ is an eigenvector of Q, then we have $Q\boldsymbol{x} = \lambda\boldsymbol{x}, (Q - \lambda I)\boldsymbol{x} = 0.$
- To find the eigenvalues, we let: $\det(Q - \lambda I) = \det\left(\begin{bmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{bmatrix}\right) = \lambda^2 + 1 = 0, \lambda^2 = -1.$
- Thus Q has no real eigenvalues, λ is imaginary.

Example 4

Try to find eigenvalues and eigenvectors of Q

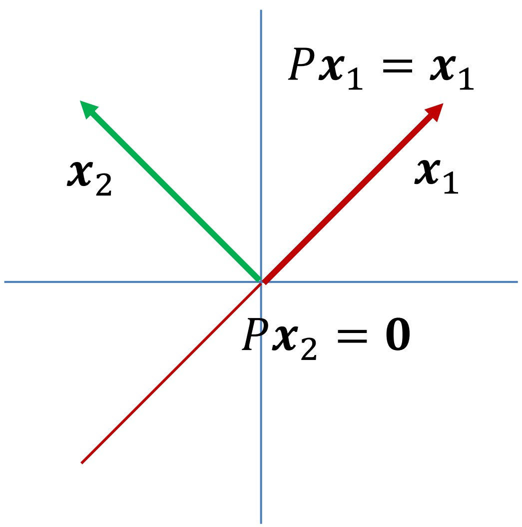
$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- $\lambda_1 = i, \lambda_2 = -i$
- Substitute λ into the equation $(Q - \lambda I)\boldsymbol{x} = \boldsymbol{0}$, we will find two eigenvectors

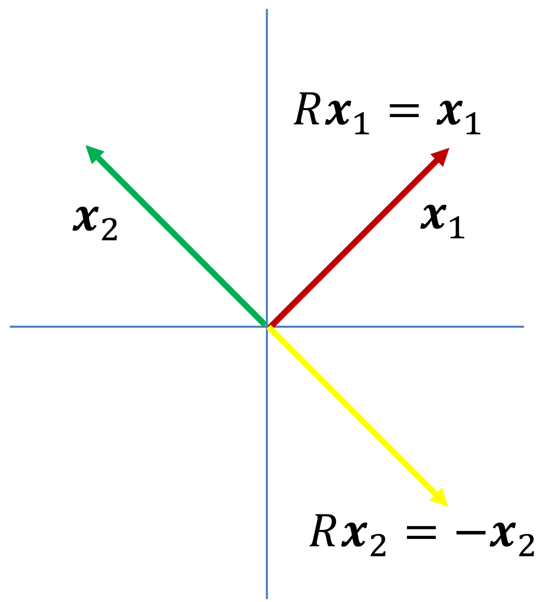
$$\boldsymbol{x}_1 = [i, 1]^T, \boldsymbol{x}_2 = [1, i]^T$$

Property of Matrix and eigenvectors

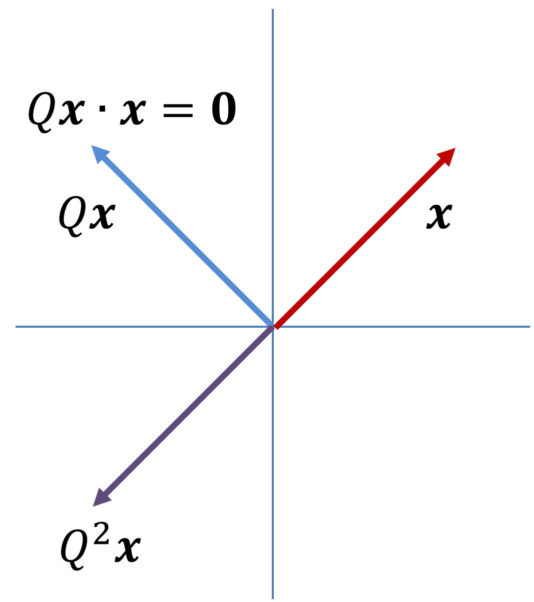
- Special properties of a matrix lead to special eigenvalues and eigenvectors.



P: projection matrix
Projects to a line



R: reflection matrix
Reflects about a line



Q: rotation matrix
Rotates by 90 degrees

Computation of eigenvalue

- For small (e.g., 2-by-2) matrices, it is convenient to make use of determinants to compute eigenvalues.

1. Compute the determinant of $A - \lambda I$.
2. Find the roots of this polynomial by solving $\det(A - \lambda I) = 0$.
3. For each eigenvalue λ, solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ to find an eigenvector $\boldsymbol{x}$.

DIAGONALIZATION OF MATRICES

Diagonalization

Diagonalization Suppose the n by n matrix A has n linearly independent eigenvectors $x_1, \cdots, x_n$. Put them into the columns of an eigenvector matrix S. Then $S^{-1}AS$ is the eigenvalue matrix Λ:

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Diagonalization

- $AS = S\Lambda$
- $S^{-1}AS = \Lambda$
- $A = S\Lambda S^{-1}$
- The matrix S has an inverse, because its columns (the eigenvectors of A) were assumed to be linearly independent.
- Without n independent eigenvectors, we can't diagonalize.

Example 5

$A = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and eigenvalues $\lambda = 1$ and $\lambda = 6$.

$$\underset{S^{-1}}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}} \underset{S}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} = \underset{\Lambda}{\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}}$$

Power of A: $A^k = S\Lambda^k S^{-1}$

$$\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}^k = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6^k - 1 \\ 0 & 6^k \end{bmatrix}$$

Symmetric matrix

(**Spectral Theorem**) Every symmetric matrix has the factorization $A = Q\Lambda Q^T$ with real eigenvalues in Λ and orthonormal eigenvectors in $S = Q$:

Symmetric diagonalization

$A = Q\Lambda Q^{-1} = Q\Lambda Q^T$ with $Q^{-1} = Q^T$

Example 6

Find the eigenvalues and eigenvectors of

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

- The determinant of $A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix}$ is $\lambda^2 - 5\lambda$.
- The eigenvalues are 0 and 5, and $0 + 5 = 1 + 4$ (sum of the diagonal components, or the trace of A).
- Two eigenvectors are $\begin{bmatrix} 2 & -1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 2 \end{bmatrix}^T$, which are orthogonal (but not yet orthonormal) as A is symmetric.

Example 6

- $Q = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$
- $Q^{-1}AQ = Q^T AQ = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \Lambda$

Orthogonal eigenvectors Eigenvectors of a real symmetric matrix (when they correspond to different λ's) are always perpendicular.

Similar matrix

- When S is the eigenvector matrix, the diagonal matrix $S^{-1}AS = \Lambda$ is the eigenvalue matrix.
- Diagonalization is not possible for every A, as some matrices have too few eigenvectors.

Similar matrix

DEFINITION Let M be any invertible matrix. Then $B = M^{-1}AM$ is similar to A.

- If B is similar to A, then A is similar to B.
- If A is diagonalizable, $M = S$.

Example 7

The projection matrix $A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ is similar to $\Lambda = S^{-1}AS = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Now choose $M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$. The similar matrix $M^{-1}AM = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Also choose $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The similar matrix $M^{-1}AM = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$.

Example 7

- These matrices all have the same eigenvalues 1 and 0.
- All 2×2 matrices with those eigenvalues 1 and 0 are similar to each other.
- The eigenvectors change with M, but the eigenvalues do not change.

EXERCISES

Problem 1

Computer the eigenvalues and eigenvectors of A and A^{-1}. Check the trace.

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

Problem 2

a) Factor these two matrices into $A = S\Lambda S^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

b) If $A = S\Lambda S^{-1}$ then

$$A^3 = (\quad)(\quad)(\quad)$$
$$A^{-1} = (\quad)(\quad)(\quad)$$

Problem 3

Find an orthogonal matrix Q that diagonalizes the symmetric matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

Problem 4

Which of the six matrices are similar?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Solution 1

A has $\lambda_1 = 2$ and $\lambda_2 = -1$, with $\boldsymbol{x}_1 = [1 \quad 1]^T$ and $\boldsymbol{x}_2 = [2 \quad -1]^T$. The sum of eigenvalues $\lambda_1 + \lambda_2$ is equal to the trace of A.

A^{-1} has its eigenvalues as the reciprocal of the eigenvalues of A. $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -1$, with the same eigenvectors. Again the sum of eigenvalues $\lambda_1 + \lambda_2$ is equal to the trace of A.

Solution 2

$a)\ A = S\Lambda S^{-1}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

b) If $A = S\Lambda S^{-1}$ then

$$A^3 = S\Lambda^3 S^{-1}$$

$$A^{-1} = S\Lambda^{-1} S^{-1}$$

Solution 3

The orthogonal matrix Q that diagonalizes the symmetric matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \text{ is } Q = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

Solution 4

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are similar. They all have eigenvalues 1 and 0.