Matrix III

REA1121

Mathematics for programming

Outline

- Review of Matrix II
- Dot product of vectors
- Basis for and dimension of vector spaces
- Exercises

REVIEW OF MATRIX II

Matrix II

- Linear combination
- Vector space and subspace
- Linear independence

Matrix II

- Linear combination
  - Combine scalar multiplication and addition of vectors
  - $c\boldsymbol{v} + d\boldsymbol{w}$ is an linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$, when c and d are any scalars.

$$
\begin{aligned}
c\boldsymbol{v} + d\boldsymbol{w} &= c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
&= [\boldsymbol{v} \quad \boldsymbol{w}]\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}
\end{aligned}
$$

Matrix II

- Vector space
  - A ***real vector space*** R^n, is a set of vectors together with rules for vector addition and for multiplication by real numbers.
  - The space R^n consists of all column vectors $\boldsymbol{v}$ with n components.
  - 8 conditions required of every vector space
- Subspace
  - A ***subspace*** of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements: If $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in the subspace and c is any scalar, then
    - $\boldsymbol{v}+\boldsymbol{w}$ is in the subspace
    - $c\boldsymbol{v}$ is in the subspace

Matrix II

- Linear independence
  - The sequence of vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ is ***linearly independent*** if the only combination that gives the zero vector is $0v_1 + 0v_2 + \cdots + 0v_n$.
  - $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n = 0$ only happens when all x's are zero.
  - If a combination gives 0, when the x's are not all zero, the vectors are dependent.
  - The columns of a matrix A are ***linearly independent*** when the only solution to $A\boldsymbol{x} = 0$ is $x = 0$. No other combination $A\boldsymbol{x}$ of the columns gives the zero vector.

DOT PRODUCT

Dot product

DEFINITION The ***dot product*** of $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is the number $\boldsymbol{v} \cdot \boldsymbol{w}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 = \boldsymbol{v}^T \boldsymbol{w}$$

- Dot product is also known as
  - scalar product
  - inner product
  - projection product

Example 1

The ***dot product*** of $\boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$:

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The ***dot product*** of $\boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0$$

$$\boldsymbol{w} \cdot \boldsymbol{v} = ?$$

Example 1

The ***dot product*** of $\boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0$$

$$\boldsymbol{w} \cdot \boldsymbol{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = -4 + 4 = 0$$

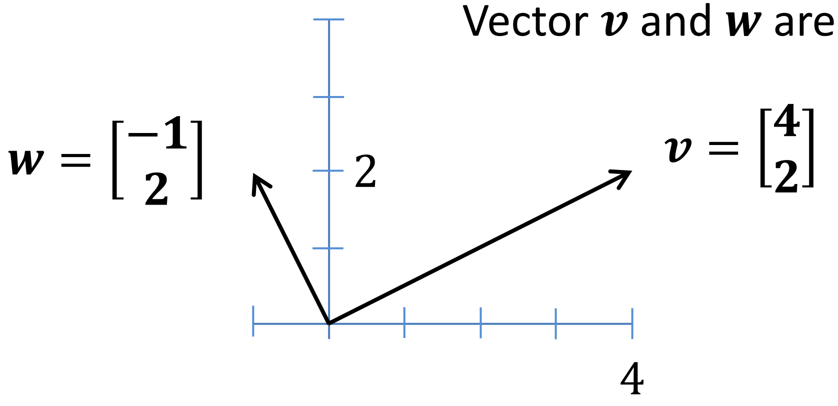
$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$$

Example 1

The ***dot product*** of $\boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0$$

Vector $\boldsymbol{v}$ and $\boldsymbol{w}$ are **orthogonal**



Example 2

- We have three types of goods to buy and sell.
- Their prices are (p_1, p_2, p_3) for each unit - the "price vector" p.
- The quantities we buy or sell are (q_1, q_2, q_3) -positive when we sell, negative when we buy – the “quantity vector” q.
- How much is the total income?

Example 2

- We have three types of goods to buy and sell.
- Their prices are (p_1, p_2, p_3) for each unit - the "price vector" $\boldsymbol{p}$.
- The quantities we buy or sell are (q_1, q_2, q_3) -positive when we sell, negative when we buy – the “quantity vector” $\boldsymbol{q}$.
- How much is the total income?

$$\boldsymbol{q} \cdot \boldsymbol{p} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = q_1 p_1 + q_2 p_2 + q_3 p_3$$

Dot product

$$\boldsymbol{v} \cdot \boldsymbol{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = v_1 w_1 + v_2 w_2$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{i=1}^{n} v_i w_i$$

Length of vectors

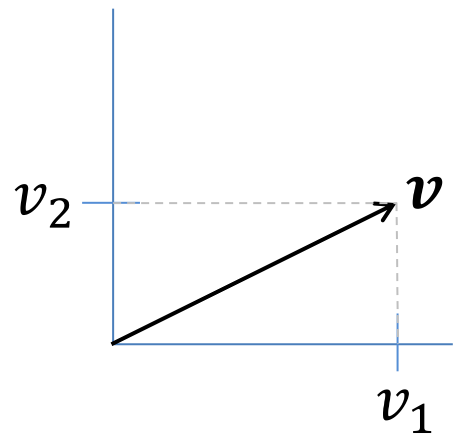
The dot product of a vector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ with itself

$$\boldsymbol{v} \cdot \boldsymbol{v} = v_1 v_1 + v_2 v_2 = v_1^2 + v_2^2$$

Length of vectors

The dot product of a vector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ with itself

$$\boldsymbol{v} \cdot \boldsymbol{v} = v_1 v_1 + v_2 v_2 = v_1^2 + v_2^2 = \left(\sqrt{v_1^2 + v_2^2}\right)^2$$

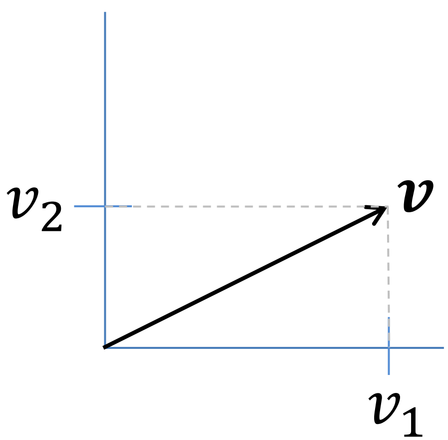


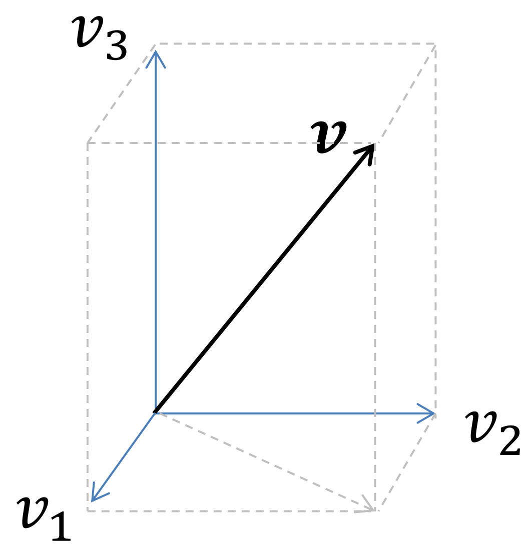
Length of vectors

DEFINITION The ***length*** $\|\boldsymbol{v}\|$ of a vector $\boldsymbol{v}$ is the square root of $\boldsymbol{v} \cdot \boldsymbol{v}$

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$$

Length of vectors





Length of vectors

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1 v_1 + v_2 v_2} = \sqrt{v_1^2 + v_2^2}$$

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1 v_1 + v_2 v_2 + v_3 v_3}$$

$$= \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1 v_1 + v_2 v_2 + \cdots + v_n v_n}$$

$$= \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

Example 3

Determine $\|\boldsymbol{v}\|$

1. $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
2. $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
3. $v = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$

Unit vector

DEFINITION A ***unit vector*** $\boldsymbol{u}$ is a vector whose length equals one.

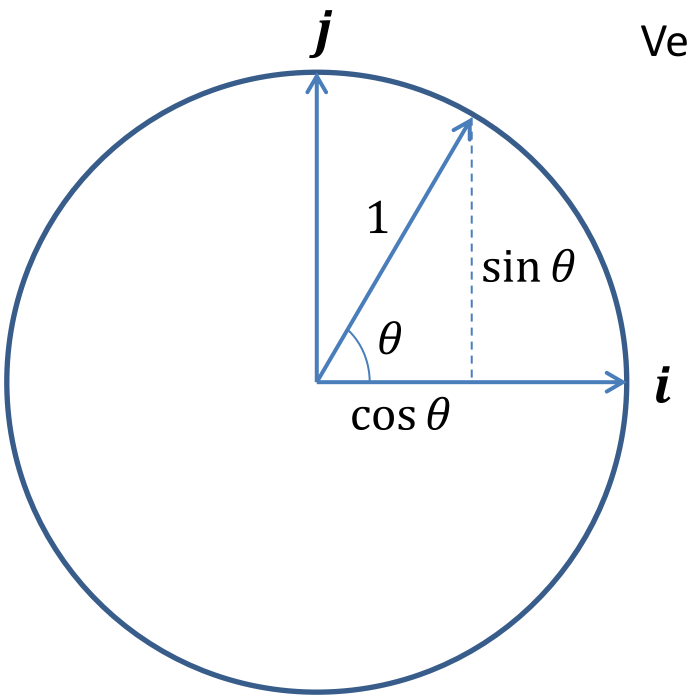
$$\boldsymbol{u} \cdot \boldsymbol{u} = 1$$

Unit vector

DEFINITION A ***unit vector*** $\boldsymbol{u}$ is a vector whose length equals one.

$$\boldsymbol{u} \cdot \boldsymbol{u} = 1$$

unit circle



Vector $\boldsymbol{i}$ and $\boldsymbol{j}$ are **orthonormal**

$$\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Unit vector

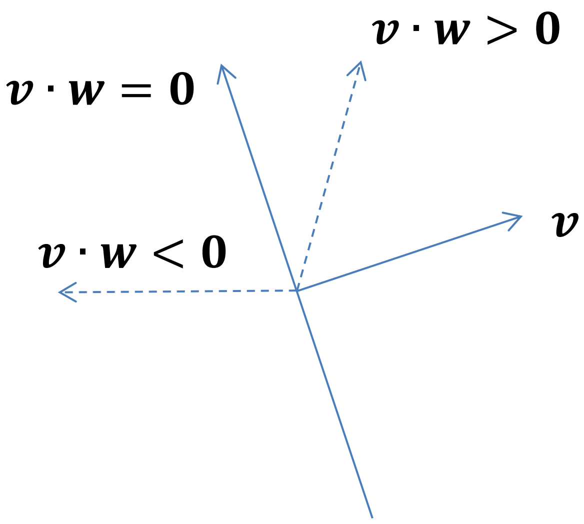
- $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ is a *unit vector* in the same direction as $\boldsymbol{v}$.

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \boldsymbol{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \boldsymbol{u} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

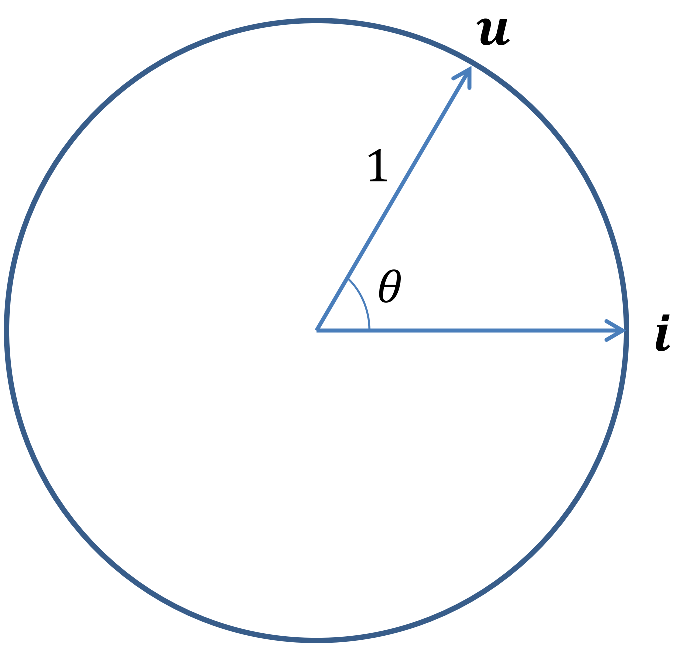
$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \boldsymbol{u} = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ 3/\sqrt{30} \\ 4/\sqrt{30} \end{bmatrix}$$

Angle between two vectors



Angle between two vectors

unit circle



$$\boldsymbol{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

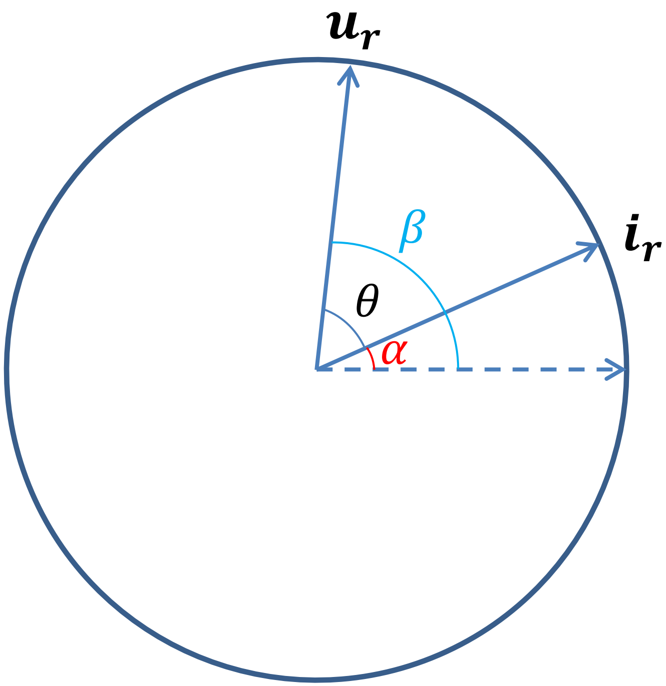
$$\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{u} \cdot \boldsymbol{i} = \cos\theta$$

Angle between two vectors

Rotate $\boldsymbol{u}, \boldsymbol{r}$ by α

unit circle



$$\beta = \alpha + \theta$$

$$\boldsymbol{u_r} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$$

$$\boldsymbol{i_r} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

$$\begin{aligned} &\boldsymbol{u_r} \cdot \boldsymbol{i_r} \\ &= \cos\beta\cos\alpha + \sin\beta\sin\alpha \\ &= \cos(\beta - \alpha) \\ &= \cos\theta \end{aligned}$$

Angle between two vectors

- For any two unit vectors $\boldsymbol{i}$ and $\boldsymbol{u}$,

$$\boldsymbol{u} \cdot \boldsymbol{i} = \cos\theta$$

θ is the angle between them

- For any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$,

$$\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \cos\theta$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta$$

Example 4

- Determine the angles between the pairs of vectors

$$\begin{bmatrix} \sqrt{3}/_2 \\ 1/_2 \end{bmatrix} \text{ and } \begin{bmatrix} -1/_2 \\ \sqrt{3}/_2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2}/_2 \\ 1/_2 \\ -1/_2 \end{bmatrix} \text{ and } \begin{bmatrix} -\sqrt{2}/_2 \\ 1/_2 \\ -1/_2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

BASIS FOR AND DIMENSION OF VECTOR SPACES

Span

DEFINITION A set of vectors ***spans*** a space if their linear combination fill the space.

Example 5

- $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span …?
- $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ span …?
- $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ span …?

Example 5

- $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span the full two-dimensional space $\mathbf{R}^2$.
- $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ also span the full space $\mathbf{R}^2$.
- $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ only span a line in $\mathbf{R}^2$.

Example 6

- In 3-dimensional space $\mathbf{R}^3$,
  - two non-zero vectors from the origin may span
    - a ___, if they are *linearly dependent*
    - a ___, if they are *linearly independent*
  - three non-zero vectors from the origin may span
    - a ___, if any of them is *linearly dependent* on any other one
    - a ___, if two of them are *linearly independent* and another one is *linearly dependent* on the two
    - a ___, if the set is *linearly independent*

Example 6

- In 3-dimensional space $\mathbf{R}^3$,
  - two non-zero vectors from the origin may span
    - a line, if they are *linearly dependent*
    - a plane, if they are *linearly independent*
  - three non-zero vectors from the origin may span
    - a line, if any of them is *linearly dependent* on any other one
    - a plane, if two of them are *linearly independent* and another one is *linearly dependent* on the two
    - a space, if the set is *linearly independent*

Basis for vector spaces

- Two vectors can’t span all of $\mathbf{R}^3$, even if they are independent.
- Four vectors can’t be independent, even if they span $\mathbf{R}^3$.
- We need enough (but not more) independent vectors to span the space.

Basis for vector spaces

DEFINITION A ***basis*** for a vector space is a sequence of vectors with two properties:

- The basis vectors are linearly independent,
- They span the space.

- Any vector $\boldsymbol{v}$ in the space is a combination of the basis vectors, because they span the space.
- The combination that produces $\boldsymbol{v}$ is unique, because the basis vectors are independent.

Example 7

- The columns of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ produce the "standard basis" for $\mathbf{R}^2$.
- The basis vectors $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are independent. They span $\mathbf{R}^2$.

Example 7

- The columns of the 3×3 identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are the “standard basis” $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.
- The columns of the $n \times n$ identity matrix $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ give the “standard basis” for $\mathbf{R}^n$.

Example 8

How many basis are there for any subspace of $\mathbf{R}^n$?

Example 8

How many bases are there for any subspace of $\mathbf{R}^n$?

- Suppose a matrix A consisting of n basis vectors
- The only solution to $A\boldsymbol{x} = 0$ is $x = A^{-1}0 = 0$
- A is necessarily invertible.
- The solution to $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{b}$ is any vector in the subspace, is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

The columns of *every invertible* n by n matrix give a basis for any subspace of $\mathbf{R}^n$.

Dimension of vector spaces

- As there are many choices for the basis vectors, does the *number* of basis vectors change?

Dimension of vector spaces

DEFINITION The ***dimension of a space*** is the number of vectors in every basis.

- All bases for a vector space contains the same number of vectors.
- The dimension of the space $\mathbf{R}^n$ is n.

Example 9

For vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$,

- What space $\mathbf{V}$ do they span?
- What is the dimension of $\mathbf{V}$?

Example 9

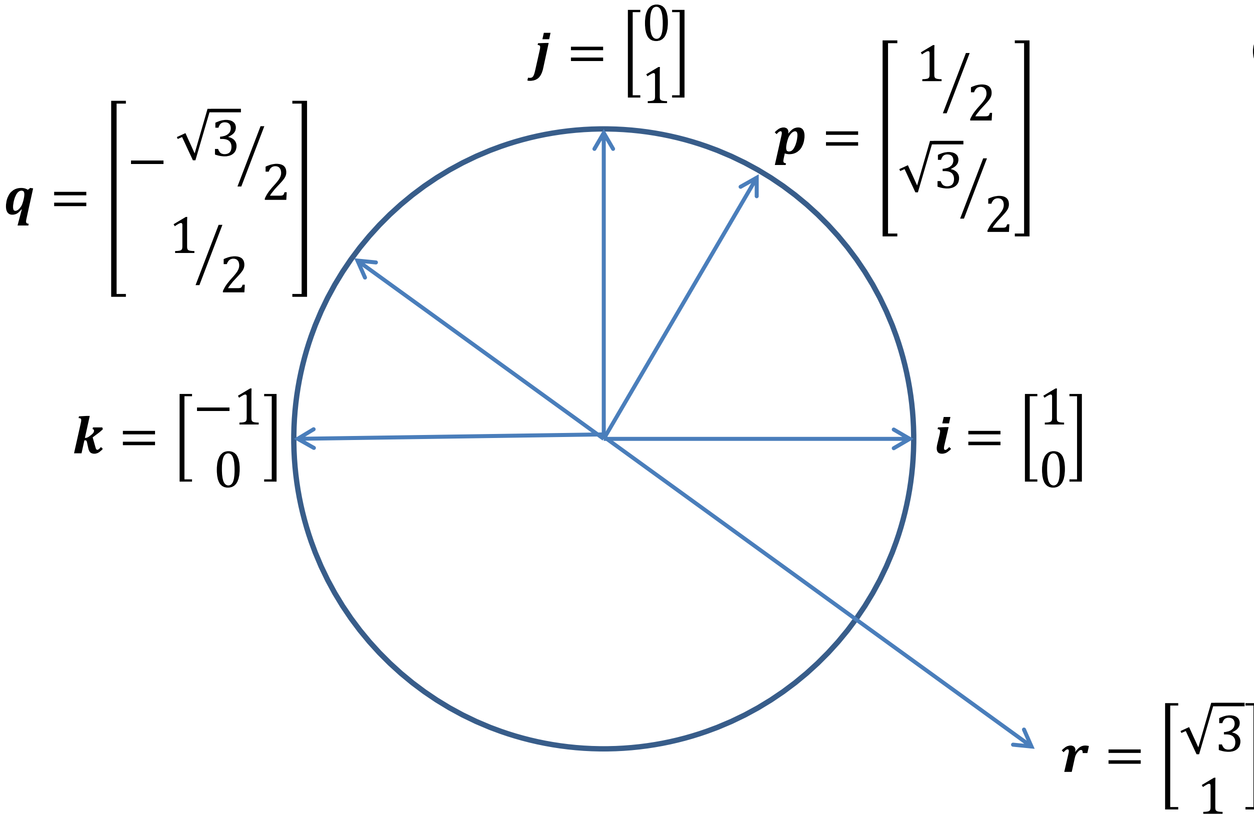
For vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$,

- What space $\mathbf{V}$ do they span?
  - The space $\mathbf{V}$ contains all vectors $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$, it is the xy plane in $\mathbf{R}^3$.
- What is the dimension of $\mathbf{V}$?
  - The dimension of $\mathbf{V}$ is 2, as the basis contains 2 vectors.

Orthonormal basis

- A set of vectors $v_1, \cdots, v_n$ that meet the following requirements:
  - Basis
    - The basis vectors are linearly independent,
    - They span the space.
  - Orthogonal to each other
    - $v_i \cdot v_j = 0, when\ i \neq j, 1 \leq i, j \leq n$
  - Unit vectors
    - $\|v_i\| = 1, 1 \leq i \leq n$

Example 10



unit circle

Orthonormal basis?

$\boldsymbol{i}, \boldsymbol{j}$

$\boldsymbol{p}, \boldsymbol{q}$

$\boldsymbol{i}, \boldsymbol{k}$

$\boldsymbol{p}, \boldsymbol{k}$

$\boldsymbol{p}, \boldsymbol{r}$

Example 10

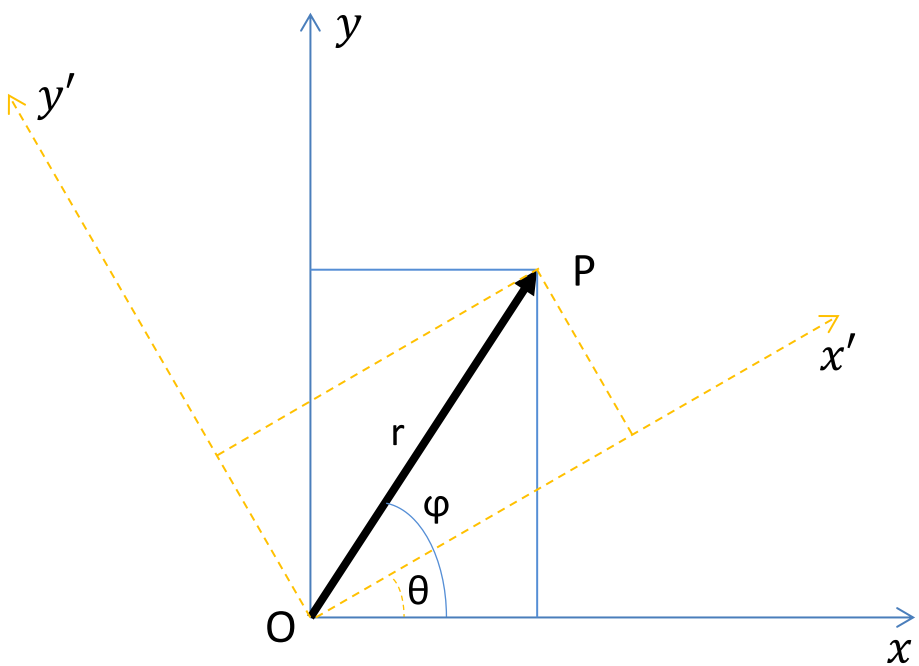
$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \begin{bmatrix}-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\end{bmatrix}$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Orthonormal basis

- Determining the coordinates of a vector with respect to a basis is *generally* NOT easy.
- With orthonormal basis, it is as easy as dot product of vectors
  - Projection of vectors

Example 11



In the Oxy system, the coordinates of P: (x, y)

$$x = r\cos\varphi\,, \qquad y = r\sin\varphi$$

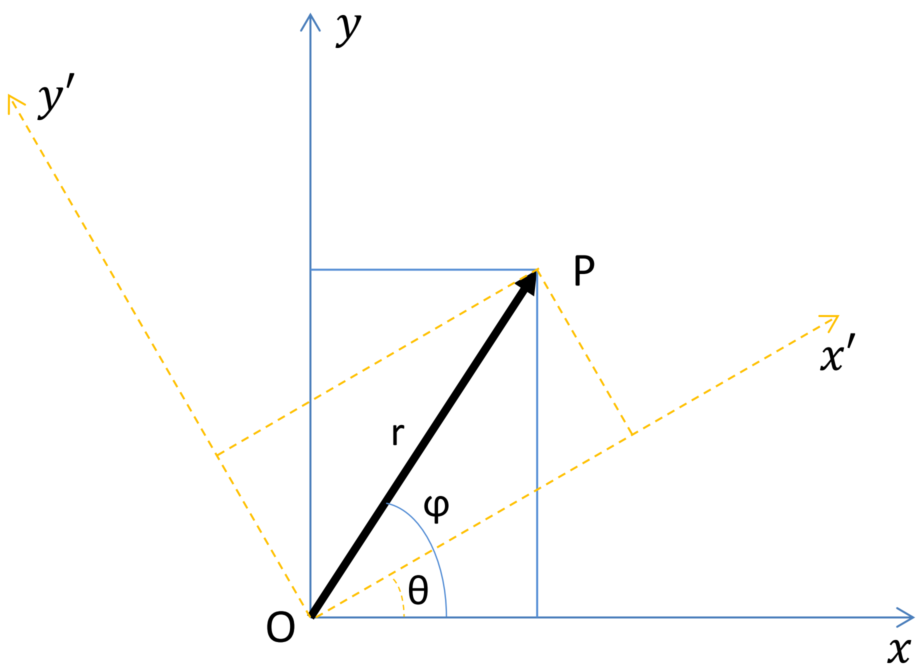
In the Ox'y' system, the coordinates of P: (x', y')

$$x' = r\cos(\varphi - \theta)\,, \qquad y' = r\sin(\varphi - \theta)$$

$$\begin{aligned} x' &= r\cos\varphi\cos\theta + r\sin\varphi\sin\theta \\ &= x\cos\theta + y\sin\theta \end{aligned}$$

$$\begin{aligned} y' &= r\sin\varphi\cos\theta - r\cos\varphi\sin\theta \\ &= y\cos\theta - x\sin\theta \end{aligned}$$

Example 11



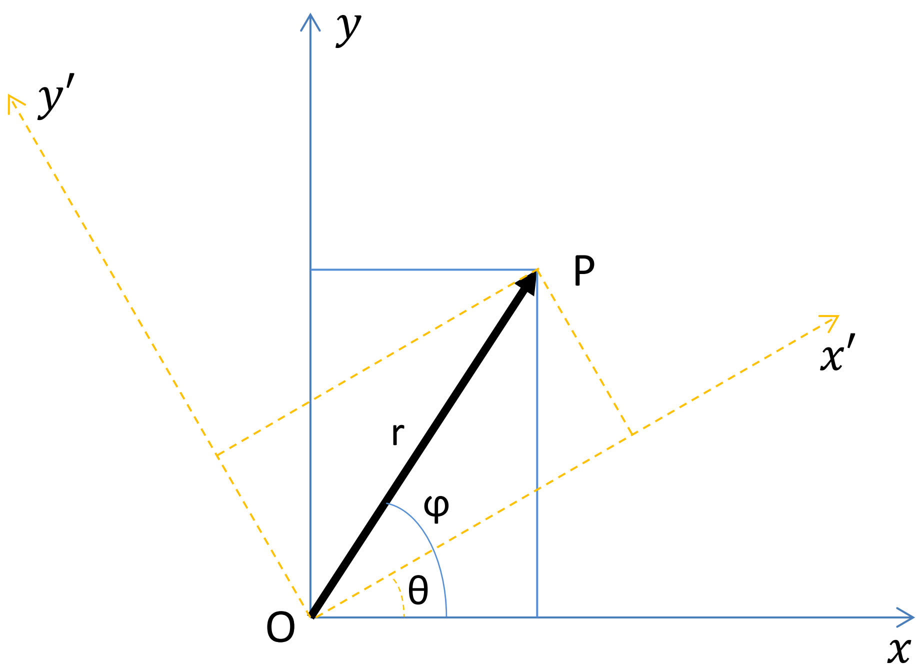
$$x' = r\cos\varphi\cos\theta + r\sin\varphi\sin\theta$$
$$= x\cos\theta + y\sin\theta$$

$$y' = r\sin\varphi\cos\theta - r\cos\varphi\sin\theta$$
$$= y\cos\theta - x\sin\theta$$

$$B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

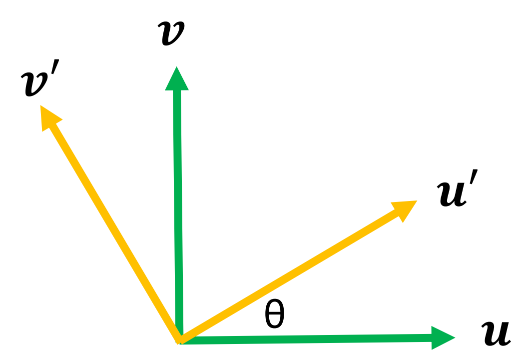
$$X' = BX, \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 11

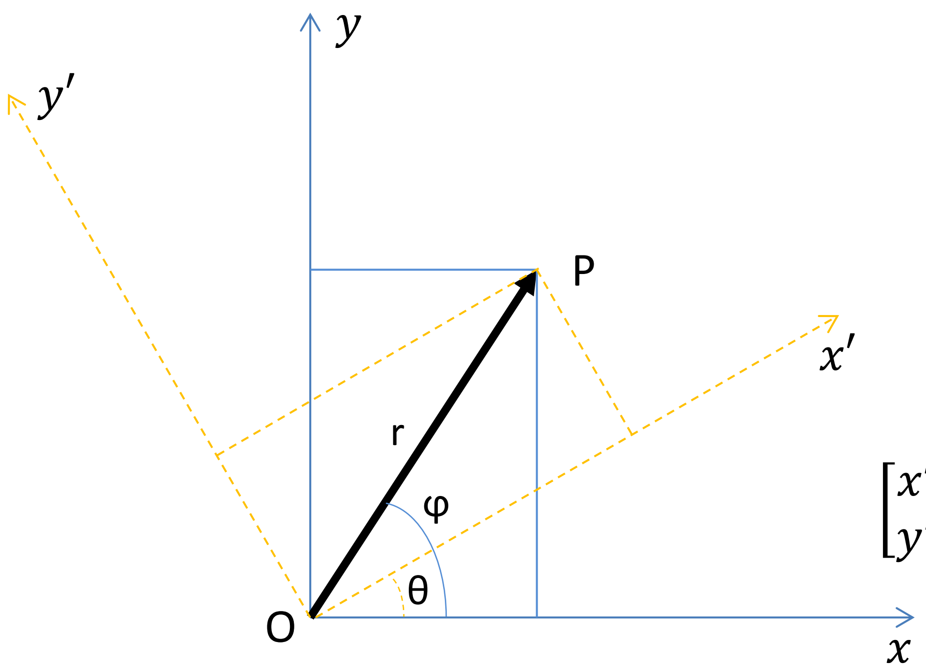


Coordinate system	Orthonormal basis
Oxy	$\boldsymbol{u}, \boldsymbol{v}$
Ox'y'	$\boldsymbol{u}', \boldsymbol{v}'$

$$\boldsymbol{u}' = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \boldsymbol{v}' = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



Example 11



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$$\boldsymbol{u}' = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \boldsymbol{v}' = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}'^T \cdot \boldsymbol{P} \\ \boldsymbol{u}'^T \cdot \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}'^T \\ \boldsymbol{v}'^T \end{bmatrix} \boldsymbol{P} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

EXERCISE

Problem 1

- Pick any numbers that add to $x + y + z = 0$. Find the angle between your vector $\boldsymbol{v} = [x\ y\ z]^T$ and the vector $\boldsymbol{w} = [z\ x\ y]^T$.
- Explain why $\frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|}$ is always $-\frac{1}{2}$.

Problem 2

Describe the subspace of $\mathbf{R}^3$ (is it a line or plane or $\mathbf{R}^3$?) spanned by

(a) the two vectors $[1 \quad 1 \quad -1]^T$ and $[-1 \quad -1 \quad 1]^T$

(b) the three vectors $[0 \quad 1 \quad 1]^T$ and $[1 \quad 1 \quad 0]^T$ and $[0 \quad 0 \quad 0]^T$

(c) all vectors in $\mathbf{R}^3$ with whole number components

(d) all vectors with positive components.

Problem 3

Find a basis for each of these subspaces of $\mathbf{R}^4$

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $\begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^T$ and $\begin{bmatrix}1 & 0 & 1 & 1\end{bmatrix}^T$.

Problem 4

- Find a basis for the space **S** of vectors $[a \quad b \quad c \quad d]^T$ with $a + c + d = 0$ and also for the space **T** with $a + b = 0$ and $c = 2d$.
- What is the dimension of the intersection $S \cap T$?

Problem 5

- The columns of matrix A form a set of vectors

$$A = \begin{bmatrix} \frac{2\sqrt{5}}{5} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{30}}{30} \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{6}}{3} & \frac{\sqrt{30}}{15} \\ 0 & \frac{\sqrt{6}}{6} & -\frac{\sqrt{30}}{6} \end{bmatrix}$$

a) Are they basis vectors?
b) Are they orthogonal to each other?
c) Do they form a orthonormal basis?
d) If c) is correct, try to find the coordinates of $\boldsymbol{p} = [1 \quad 1 \quad 1]^T$ with respect to the bases given in A.

Solution 1

- For a specific example, pick $\boldsymbol{v} = [1 \quad 2 \quad -3]^T$ and then $\boldsymbol{w} = [-3 \quad 1 \quad 2]^T$. In this example,
$$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|[\boldsymbol{w}]} = \frac{-7}{\sqrt{14}\sqrt{14}} = -\frac{1}{2}$$
$$\theta = 120°$$
- This always happens when $x + y + z = 0$:
$$\begin{aligned}\boldsymbol{v}\cdot\boldsymbol{w} &= xy + xz + yz \\ &= \frac{1}{2}\left((x+y+z)^2 - (x^2+y^2+z^2)\right) \\ &= 0 - \frac{1}{2}\|\boldsymbol{v}\|\|\boldsymbol{w}\|\end{aligned}$$

Solution 2

a) Line in $\mathbf{R}^3$
b) Plane in $\mathbf{R}^3$
c) All of $\mathbf{R}^3$
d) All of $\mathbf{R}^3$

Solution 3

- Note. The bases are not unique!

a) $[1 \quad 1 \quad 1 \quad 1]^T$ for the space of all constant vectors $[c \quad c \quad c \quad c]^T$

b) $[1 \quad -1 \quad 0 \quad 0]^T, [1 \quad 0 \quad -1 \quad 0]^T, [1 \quad 0 \quad 0 \quad -1]^T$ for the space of vectors with sum of components = 0

c) $[1 \quad -1 \quad -1 \quad 0]^T, [1 \quad -1 \quad 0 \quad -1]^T$ for the space perpendicular to $[1 \quad 1 \quad 0 \quad 0]^T$, $[1 \quad 0 \quad 1 \quad 1]^T$

Solution 4

- Bases for **S**:

$$[1 \quad 0 \quad -1 \quad 0]^T$$
$$[0 \quad 1 \quad 0 \quad 0]^T$$
$$[1 \quad 0 \quad 0 \quad -1]^T$$

- Bases for **T**:

$$[1 \quad -1 \quad 0 \quad 0]^T$$
$$[0 \quad 0 \quad 2 \quad 1]^T$$

- Solutions to the three equations simultaneously: $c[3 \quad -3 \quad -2 \quad -1]^T$, c is a number, so the dimension of $S \cap T$ is 1.

Solution 5

- Yes, the columns vectors of A form a orthonormal basis for $\mathbf{R}^3$.
- The coordinates with respect to the this basis:

$$A^T\boldsymbol{p} = \begin{bmatrix} \frac{3\sqrt{5}}{5} \\ \frac{\sqrt{6}}{3} \\ -\frac{2\sqrt{30}}{15} \end{bmatrix}$$