4.3 MARKOV CHAINS I

1. Markov Chains

We are now going to use networks and probabilities to study probabilities. In this context, vertices are called **states**, and weights on the arrows encode probabilities of moving between states.

For example, consider the following network (which is called the Drunkards Walk)

$$\begin{tikzcd}
1 \arrow[loop left, "1"] & 2 \arrow[l, "0.5"'] \arrow[r, bend right, "0.5"'] & 3 \arrow[l, bend right, "0.5"'] \arrow[r, "0.5"] & 4 \arrow[loop right, "1"]
\end{tikzcd}$$

Such a network is called a **Markov Chain** if:

1. Each weight is a probability, i.e. in the interval $[0, 1]$
2. The sum of values leaving a state is equal to 1.

Note that there are no restrictions on the values going into a node. The nodes in a Markov Chain are usually called **states**, and the value on an arrow indicates the probability of jumping between states. A state is called **absorbing** if it has a loop with probability 1, otherwise it is called **transient**. If there is a path in the chain from any node to an absorbing state, then the chain is an **absorbing Markov Chain**. If there is a path from any node to any other node, the chain is called **ergodic Markov Chain**.

Any Markov Chain gives rise to a matrix where the entry in row j and column i is the probability of going from state i to state j. Conversely, any matrix where each entry has values in $[0, 1]$ and the column sum to 1 is the matrix of a Markov Chain. Such a matrix is called a **stochastic matrix**.

For the chain above we have the matrix

$$\begin{pmatrix} 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 \end{pmatrix}$$

where columns and rows are ordered according to the state labels in the picture.

Let M be a stochastic matrix. The entry $M[i][j]$ then encodes the probability of jumping from state j to state i in 1 step. Furthermore, the entry $M^t[i][j]$ encodes the probability of jumping from state j to state i in t steps.

1.1. **Absorbing Markov Chains.** For now we will only discuss absorbing Markov Chains. That is, chains where there is a path from any vertex to an absorbing vertex. For these chains it is convenient to number the transient states before the absorbing states. So for the Drunkards walk we may choose the numbering,

$$\begin{tikzcd} 3 \arrow[loop left, "1"] & 1 \arrow[l, "0.5"'] \arrow[r, bend right, "0.5"'] & 2 \arrow[l, bend right, "0.5"'] \arrow[r, "0.5"] & 4 \arrow[loop right, "1"] \end{tikzcd}$$

which gives us the matrix

$$P = \begin{pmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 1 & 0 \\ 0 & 0.5 & 0 & 1 \end{pmatrix}.$$

We view this matrix as consisting of four matrix blocks

$$P = \begin{pmatrix} Q & 0 \\ R & I \end{pmatrix},$$

where in this case

$$Q = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}, R = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that, for an arbitrary chain, Q is always a square matrix, I is an identity matrix, and that R need not be square.

Using this blocking we can compute

$$P^n = \begin{pmatrix} Q^n & 0 \\ R(I + Q + \cdots + Q^{n-1}) & I \end{pmatrix}$$

which in the limit $n \to \infty$ is equal to

$$lim_{n\to\infty} P^n = \begin{pmatrix} 0 & 0 \\ RN & I \end{pmatrix}$$

where $N = I + Q + Q^2 + \cdots = (I - Q)^{-1}$ is called the **fundamental matrix** of the absorbing chain.

From this set of data we can for instance compute

1. $N[i][j]$ which is the expected number of times we vil visit i before absorbsion, when we start at j.
2. $RN[i][j]$ which is the probability that we will end up in state i at absorbsion, when we start at j.

3. $cN[j]$, where $c = (1, 1, \cdots, 1)$, which is the expected number of steps before absorbsion, when we start at j.

I refer to links on Fronter for computation of other quantities.

Exercises:

Exercise. 1. *You are repeatedly throwing two dice with the aim of getting two sixes. A dice with six are left on the table and are not thrown again.*

a) *Draw the Markov Chain (with probabilities) and write down the corresponding matrix.*
b) *Identify absorbing and transient states. Is this an absorbing chain?*
c) *What is the probability that you get two sixes in two throws?*
d) *What is the expected (average) number of throws you need until you get three sixes?*

Exercise. 2. a) *Write down the matrix of the following Markov chain*

$$\begin{tikzcd} 1 \arrow[loop left, "1"] & 2 \arrow[l, "0.3"] \arrow[r, bend right, "0.7"'] & 3 \arrow[l, bend right, "0.1"'] \arrow[r, "0.9"] & 4 \arrow[loop right, "1"] \end{tikzcd}$$

What are the transient and absorbing states in this Markov chain? Is this an absorbing Markov chain? Explain!

b) *Using the Markov chain in a) and assuming the starting state is state 2, calculate the expected number of moves needed before absorbsion.*

Answers/Solutions:

Exercise 1.

a)

I use state 1: no six, state 2: 1 six, state 3: 2 six, with matrix (I skip the picture here)

$$\begin{pmatrix} \frac{25}{36} & 0 & 0 \\ \frac{10}{36} & \frac{5}{6} & 0 \\ \frac{1}{36} & \frac{1}{6} & 1 \end{pmatrix}$$

b)

State 1 and 2 are transient and state 3 is absorbing. Both state 1 and 2 has an arrow to state 3, so the chain is absorbing.

c)

I square the matrix and read off $\frac{121}{6^4}$ from the lower left corner.

d)

$$\begin{pmatrix} 1 & 1 \end{pmatrix} N = \begin{pmatrix} 1 & 1 \end{pmatrix} (I - Q)^{-1} = \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{0.0508} \begin{pmatrix} 0.1667 & 0 \\ 0.278 & 0.305 \end{pmatrix}$$

The expected number is approximately 8.73.

Exercise 2.

a)

The matrix (vertex ordering 2,3,1,4) is

$$\begin{pmatrix} 0 & 0.1 & 0 & 0 \\ 0.7 & 0 & 0 & 0 \\ 0.3 & 0 & 1 & 0 \\ 0 & 0.9 & 0 & 1 \end{pmatrix}$$

Transient states are 2 and 3. Absorbing states are 1 and 4. There is a path from any state to an absorbing state, so the chain is absorbing.

b)

$$\begin{pmatrix} 1 & 1 \end{pmatrix} N = \begin{pmatrix} 1 & 1 \end{pmatrix} (I - Q)^{-1} = \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{0.93} \begin{pmatrix} 1 & 0.1 \\ 0.7 & 1 \end{pmatrix}$$

The expected number is $\frac{1}{0.93} \cdot 1.7 = 1.828$.