INTERPOLATION

1. Parametric equation

1.1. Coordinate systems. In the Cartesian coordinate system, a vector P in the plane is represented by two numbers a and b,

$$P = \begin{pmatrix} a \\ b \end{pmatrix}$$

The vector is obtained by moving from the origin a distance a in the positive x direction and then a distance b in the positive y direction. In other words,

$$P = a \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In the polar coordinate system, a vector is represented by its distance r from the origin and the angle θ it makes with the positive x-axis. We write

$$P = r\angle\theta \text{ or } P = r \cdot e^{i\theta}$$

for the corresponding vector. Note here that r is a non-negative number and that

$$r\angle\theta = r\angle(\theta \pm 2 \cdot \pi).$$

We can convert between these coordinate systems as follows:

1. Given a vector in Cartesian coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$, then

$$r = \sqrt{x^2 + y^2}.$$

Some care is needed when computing the angle, for instance we can do

$$\theta = arccos(\frac{y}{r})$$

when $x \geq 0$ and

$$\theta = \pi - arccos(\frac{y}{r})$$

when $x < 0$. In C++ we have the option of using

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angle = atan2(x,y)
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without worrying about which quadrant the angle is in.

2. Given a vector in polar coordinates, then

$$x = r \cdot cos(\theta) \text{ and } y = r \cdot sin(\theta).$$

In 3-dimensions: common examples of coordinate systems are Cartesian, spherical and cylindrical coordinates.

1.2. **Some examples.** A line in cartesian coordinates can be described with one linear equation

$$a \cdot x + b \cdot y = c,$$

for example

1. $x = 4$ - a vertical line
2. $y = 0$ - the y-axis
3. $x - y = 0$

A circle has the equation

$$(x - a)^2 + (y - a)^2 = r^2$$

where r is the radius and $\begin{pmatrix} a \\ b \end{pmatrix}$ is the center of the circle.

In polar coordinates, the equation $\theta = 0$ is the positive x axis and $\theta = \frac{\pi}{2}$ is the positive y-axis. The equation $r = 10$ is the circle with center in the origin and radius 10.

1.3. **Parametrisation of curves.** Curves can be described by a function

$$P(t) : \mathbb{R} \to \mathbb{R}^2.$$

In practise, the domain of $P(t)$ is usually an interval, e.g. $[0, 1]$. The curve C is the collection of all points in the image of $P(t)$,

$$C = \{P(t) | t \in \mathbb{R}\}.$$

Sometimes $P(t)$ is thought of as the position of a point at a given value t. For example, $P(t)$ could describe the motion of an object depending on time. Note that

$$\begin{pmatrix} t^3 \\ t^3 \end{pmatrix} \text{ and } \begin{pmatrix} t \\ t \end{pmatrix}$$

describes the same curve, but the motion is different.

In cartesian coordinates we would have

$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

and in polar coordinates

$$P(t) = r(t)\angle\theta(t)$$

.

For example a line is given by

$$P(t) = \begin{pmatrix} a + x \cdot t \\ b + y \cdot t \end{pmatrix}$$

where $\begin{pmatrix} a \\ b \end{pmatrix}$ is a point of the line and $\begin{pmatrix} x \\ y \end{pmatrix}$ is the direction of the line.

A circle is given by

$$P(t) = \begin{pmatrix} r \cdot cost + a \\ r \cdot sint + b \end{pmatrix}$$

where r is the radius and $\begin{pmatrix} a \\ b \end{pmatrix}$ is the center of the circle.

Curves with symmetry around the origin are often easier in polar coordinates. For instance a circle is given by

$$\theta = t, r = a$$

where a is the radius of the curve. A spiral could be

$$\theta = t, r = 0.5t$$

where $t \geq 0$.

2. Interpolation

We wil now discuss how to make a smooth changes between points or vectors. This is called interpolation.

3. LERP - Linear interpolation

Given two points P_0 and P_1 in space. The linear interpolation (LERP) between these two points is

$$B_1(P_0, P_1, t) = (1 - t) \cdot P_0 + t \cdot P_1$$

At $t = 0$ this expression evaluates to P_0 and at $t = 1$ it evaluates to P_1. At intermediate values for t we are somewhere between these two points. We can think of B_1 as moving on a straight line from P_0 to P_1 in one unit of time (for instance one second). We refer to P_0 and P_1 as control points of the motion.

4. Quadratic and Cubic Bezier Curves

The quadratic Bezier curve on three points is given by

$$B_2(P_0, P_1, P_2, t) = (1-t)B_1(P_0, P_1, t) + tB_1(P_1, P_2, t)$$

In other words, the quadratic Bezier curve is a smooth change between two linear interpolations. At times close to 0, we are moving like the interpolation $B_1(P_0, P_1, t)$, and then for values of t close to 1 we are moving like the interpolation $B_1(P_1, P_2, t)$. If we expand the formula we have

$$B_2(P_0, P_1, P_2, t) = (1-t)^2 P_0 + 2(t-1)tP_1 + t^2 P_2$$

The expression is quadratic in t, and so we call this the quadratic Bezier curve.

A cubic Bezier curve is given by

$$B_3(P_0, P_1, P_2, P_3, t) = (1-t)B_2(P_0, P_1, P_2, t) + tB_2(P_1, P_2, P_3, t)$$

or

$$B_3(P_0, P_1, P_2, P_3, t) = (1-t)^3 P_0 + 3(1-t)^2 tP_1 + 3(1-t)t^2 P_2 + t^3 P_3$$

The recursive definition shows that a cubic Bezier curve is a smoothing between two quadratic Bezier curves.

5. SLERP - Rotational interpolation

Let u and v be two vectors. We want to interpolate between the vectors along an arc. In particular, if the lengths of u and v are equal, the interpolation should be a long a circular arc with each intermediate vector of the same length as u and v.

First find the angle between the two vectors

$$\Omega = acos(\frac{u \cdot v}{|u| \cdot |v|}).$$

The spherical linear interpolation (SLERP) is

$$P(t) = \frac{sin(\Omega(1-t))}{sin(\Omega)} \cdot u + \frac{sin(\Omega t)}{sin(\Omega)} \cdot v.$$

We need $0 < \Omega < \pi$ for the formula to be valid.

Alternatively, we can use polar coordinates. Let r and θ be the polar coordinates of u and s and ϕ be those of v. With linear interpolation we have

$$R(t) = (1-t) \cdot r + t \cdot s \text{ and } \Theta(t) = (1-t) \cdot \theta + t \cdot \phi$$

Converting to Cartesian cordinates gives us

$$P(t) = R(t)\angle\Theta(t) = \begin{pmatrix} ((1-t)\cdot r + t\cdot s)cos((1-t)\cdot\theta + t\cdot\phi) \\ ((1-t)\cdot r + t\cdot s)sin((1-t)\cdot\theta + t\cdot\phi) \end{pmatrix}$$

Unlike SLERP, this interpolation will not always choose the shortest arc between the two vectors.

6. Splines

Beyond the scope of this course.

Exercises.

Exercise. 1. *Convert* $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *and* $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ *to polar coordinates*

Exercise. 2. *Convert* $10\angle\frac{\pi}{3}$ *to cartesian coordinates.*

Exercise. 3. *Convert* $r = 4, \theta = 2t$ *cartesian coordinates. What is the curve when* $t = 0...2\pi$*? Describe the motion.*

Exercise. 4. *Write a program which illustrates the use of interpolation to move an object from one point to another on the screen. Try*

a) *Linear interpolation.*
b) *Quadratic interpolation.*
c) *Cubic interpolation.*
d) *Spherical interpolation in the plane (slerp).*

Exercise. 5. *Write a program which draws the curve created by the superformula.*