Homework # 03

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(**Total**: **34 points**)

Submission Deadline: 28^{th} February (Thursday) 16:00 PM
Homeworks are accepted as .pdf files upload to Canvas

Question 01:

(**1 point**) Prove that if functions $f(.), g(.)$ are convex, then $h(x) = \max\{f(x), g(x)\}$ is convex as well.

Question 02:

(**1 point**) Derive gradient and Hessian matrix (both in vector form) for the quadratic form $f(x) = (Ax, x)$. Matrix $A \in \mathbf{R}^{n\times n}$ may be non-symmetric.

Question 03:

(**2 point**) Prove composition rule $f(x) = h(g(x))$ is convex if $\overline{h}$ is convex non-increasing, and g is concave" (here $h : \mathbb{R} \to \mathbb{R}$, while $g : \mathbb{R}^n \to \mathbb{R}$).

Question 04:

(**3 point**) Prove that all sub-level sets $Q(c) = \{x : f(x) \leq c\}$ of a strongly convex function $f(.)$ are bounded. (Hint: assume that there exists unbounded $Q(c)$ for some c, and show contradiction.)

Question 05:

(**1 point**) Assume a convex function $f : \mathbb{R}^n \to \mathbb{R}$ has non-empty epigraph. A support hyperplane to the epigraph set has equation $(c, x) + b = 0$. What are the dimensions of c and b?

Question 06:

(**2 point**) Find conjugate function $f^*(y)$ for $f(x) = (Qx, x)$ and $Q \succ 0$.

Question: 07

(**1 point**) Solve one-dimensional problem (approximation by constant)

$$\min_{z\in\mathbb{R}} \frac{1}{m}\sum_{i}^{m}(z - x_i)^2,$$

Question: 08

(**3 point**) Best approximating line (coefficients $a, b \in \mathbb{R}$). For points on plane $(x_i, y_i) \in \mathbb{R}^2, i = 1, 2, ...m$ there is a best matching line $ax + b$, which minimizes mean of residuals:

$$\min_{a,b} \frac{1}{m} \sum_{i}^{m} (ax_i + b - y_i)^2,$$

Solve this problem explicitly with respect to a, b. (Hint: it is an unconstrained optimization problem.)

Question: 09

(**3+3 point**) Check that BGFS update formulae for a)B_{k+1} and b) H_{k+1} satisfy quasi-Newton conditions ($H_{k+1}d = s$ and $d = B_{k+1}s$)

Question: 10

(**3+3 points**) Consider the Rosenbrock function,

$$f(x, y) = 100(y - x^2)^2 + (1 - x)^2$$

a) Implement Newton method. Is the Newton method is well-defined for this problem.

b) Implement the BFGS quasi-Newton method with the line search algorithm with the Wolfe conditions. Check the condition $y_k^\top s_k > 0$ at each iteration.

Question: 11

(**4+4 points**)

$$\min_{x} \; c^\top x - \sum_{i=1}^{n+1} \log(b_i - (a^i)^\top x)$$

Consider the above problem with $x \in \mathbb{R}^n$, where

a) $n = 2$;

b) $n = 10$.

Solve the problem using Newton method and BFGS quasi-Newton method.