Homework # 04

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(**Total**: **36 points**)

Submission Deadline: 7^{th} March (Saturday) 16:00 PM
Homeworks are accepted as .pdf files upload to Canvas,
no online links are allowed for submission!

Question: 1

(**4 points**) Solve the following "projection on a Euclidean ball" problem via dual function.The $c, z \in \mathbb{R}^n$ and $r > 0$ are parameters.

$$\min_{x \in \mathbb{R}^n : ||x-c|| \leq r} ||z - x||_2,$$

(Hint: make the target function differentiable and rewrite constraints in more convenient form.)

Question: 2

(**4 points**) Find projection of a point to hyperplane $Q = \{(a, x) = b\}$, $a \neq 0$ via solving dual problem, with projection defined as a closest point on the set

$$\boldsymbol{Proj}_Q(z) = \min_{x \in Q} ||x - z||_2$$

i.e. solve

$$\min_{(a,x)=b} ||x - z||_2$$

Question: 3

(**4 points**) Write down dual function and its domain for the constrained optimization problem ($f : \mathbb{R}^n \to \mathbb{R}, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$)

$$\min_{Ax=b} f(x)$$

via conjugate function $f^*(y) = \sup_{x \in \mathbb{R}^n} ((x, y) - f(x))$

Question: 4

(**4+4 points**) a) Derive (Lagrange) dual function $d(\cdot)$, and write down dual problem for the following primal problem:

$$\min_{(Ax,x) \leq 1} (c, x),$$

where $x, c \in \mathbb{R}^n, c \neq 0, A \in \mathbb{S}_n, A \succ 0$.
b) solve the dual problem and find solution to the primal one.

Question 5

(**6 points**) Consider the optimization problem

$$\begin{array}{rl} \min & x^2 + 1 \\ \text{subject to} & (x-2)(x-4) \leq 0 \end{array}$$

with variable $x \in \mathbb{R}$.

1. Analysis of primal problem: Give the feasible set, the optimal solution x^* and the optimal value p^*
2. Lagrangian and dual function: Plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \geq \inf_x L(x, \lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.
3. State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal solution λ^*? and the dual optimal value d^*. Does strong duality hold?

Question 6

(**10 points**) Consider the general quadratic programming problem with linear constraints

$$\begin{array}{rl} \underset{x}{min} & \frac{1}{2} x^\top P x + (x, q) \\ \text{subject to} & Ax \leq b \end{array}$$

We will assume that P is symmetric positive definite (and so has full rank).

1. Find the dual.
2. Show how the primal optimal solution x^* can be computed from the dual optimal solution λ^*. (You should be able to convince yourself that both programs have unique solutions.)
3. Describe an implementation of ADMM to solve the dual problem (in a non-distributed manner, to start). Why is ADMM easier on the dual than the primal?
4. Describe how we can write the dual as

   $$\underset{\alpha,\beta}{min}\ g(\alpha) + h(\beta)$$

   where $g(.)$ is separable (a sum of individual components of α) and $h(.)$ is an indicator of a convex set which we can easily project onto. Write down explicitly how this projection operator works.
5. Using your results from part (4), describe how we can use distributed ADMM to solve the dual program.