Homework 5

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Problem 1: Signal reconstruction with l_1-penalization (4 points).

1. Propose three test signals $x_i, i = 1, \dots, 1000$: e.g., piece-wise linear continuous, piece-wise linear discontinuous, quadratic spline.

2. Corrupt your signal adding Gaussian and/or random bounded noise

$$y = x + \xi.$$

3. For each of three test signals run a recovery procedure with penalties $\|D_1\hat{x}\|_2, \|D_1\hat{x}_1\|_1, \|D_2\hat{x}\|_1$.

4. Explore dependency on the regularizer weight γ. Plot penalty (e.g. $\|D_1\hat{x}\|_2$) vs $\|y-x\|_2$ for different γ, you'll get the curve implicitly parameters by γ. Which γ looks like the optimal choice for each case?

$$D_1 = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 \end{pmatrix}.$$

Problem 2: SDP and Schur complement (1 points)

For symmetric matrix

$$X = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}$$

the following holds:

$$X \succ 0 \Leftrightarrow A \succ 0, C - B^\top A^{-1} B \succ 0;$$

$$X \succ 0 \Leftrightarrow C \succ 0, A - BC^{-1}B^\top \succ 0;$$

The expressions $C - B^\top A^{-1} B$ and $A - BC^{-1}B^\top$ are called Schur complement.

Use this structure to describe the fact:

$$a \in \mathcal{E} = \{x : (x-c)^\top P^{-1}(x-c) \leq 1\}$$

as a Linear Matrix Inequality (LMI) with respect to a.

Problem 3: Goemans-Willamson relaxation (4 points).

Create or take data for MAX CUT problem, $n \sim 20$. Construct a random Laplacian matrix L with number of nonzero elements being over 80%.

$$\max x^T L x$$
$$x_i^2 - 1 = 0, \quad i = 1, \ldots, n$$

Demonstrate four figures in one plot: x-axis – sample number; y-axis – objective value.

1. Naive randomization. Choose randomly $N = 1000$ random cuts x and plot function value for them.
2. Solution of SDP relaxation. Plot as horizontal line — the same value for all $N = 1000$ samples.
3. Goemans-Williamson approximation. Depict the objective function values for $\tilde{x}$ obtained by $N = 1000$ different random ξ.
4. Mathematical expectation (mean value) $\mathbf{E}_\xi(\tilde{x}^T L \tilde{x})$. Plot as horizontal line. Plot as horizontal line — the same value for all $N = 1000$ samples.

Problem 4: Linear Programming Example (3 points).

A cargo plane has three compartments for storing cargo: front, centre and rear. These compartments have the following limits on both weight and space:

Compartment	Weight capacity (tonnes)	Space capacity (cubic metres)
Front	10	6800
Centre	16	8700
Rear	8	5300

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the plane.

The following four cargoes are available for shipment on the next flight:

Cargo	Weight (tonnes)	Volume (cubic metres/tonne)	Profit ($/tonne)
C1	18	480	310
C2	15	650	380
C3	23	580	350
C4	12	390	285

Any proportion of these cargoes can be accepted. The objective is to determine how much (if any) of each cargo C1, C2, C3 and C4 should be accepted and how to distribute each among the compartments so that the total profit for the flight is maximized.

1. Formulate the above problem as a linear program.
2. What assumptions you made in formulating this problem as a linear program?
3. Solve the problem (any appropriate package is ok, e.g. `linprog` in MatLab Optimization toolbox, `scipy.optimize.linprog`; see also `CVX_byNikita.pdf`)
4. Sensitivity to the constraint values. What if the available cargoes weight increases/decreases by 10%? Which cargo's weight influence more to the total profit? Explain dependence between dual variables and changes in total profit.