Decoherence Prelims 2021 - Subjective

Team Decoherence

June 2021

Instructions

1. The duration of this round is 24 hours. Submission deadline is 27th June 2021, 9 AM. Any submission after that **will not be considered**. Only one submission per team is allowed.

2. We highly encourage you to start the process of submitting your answers in google form at least 30 min before to avoid network related issues. Late submissions, in any case, will not be accepted. If you face persisting network issues during submission, inform us immediately by calling or texting to one of us using the contact details provided to you via email earlier.

3. This paper contains **8 questions**, including two data analysis type questions (as already mentioned previously).

4. Mention question number and title (which is given at the beginning of each question) clearly. Answer all sub parts of a question together and begin each question in **new page**. Clearly mention subpart number.

5. Submit only typed or partially typed answers by compiling all your answers in a **single** pdf or word file in **order**.

6. Each question carries 25 marks. There is no negative marking.

7. If you've used any code or software to plot, **attach that file separately** along with the pdf or word file that you will submit in the google form (For eg: .sciprj file for scidavis; .py/.ipynb for python etc). We should be able to see both the graph and the data you used.

8. Show your steps clearly. Do not skip steps to receive full credit. Partial credit may be awarded for an incomplete solution or progress towards a solution.

9. **In case of any clarification required kindly mail us.** We will try our best to clarify. Even after that, **if you feel you have any comments regarding the question (eg incompleteness, incorrectness etc) you can mention that in your answer clearly** with all the reasons why you think so. If we feel that you claim is correct, scoring will be done accordingly. But the first priority should be getting it clarified from us.

10. You are always free to do beyond what is asked to do in the question (though it will not be considered for evaluation). You are also free to add any comments in your answer regarding any question.

11. The exam is open book. You may consult any **non living** sources such as books, internet etc but you **must** cite the sources if you use them. In case you fail to do so, you will be penalised on grounds of plagiarism and it may even lead to disqualification of the team.

12. **If there are any corrections to any question, those will be informed to you over mail.**

Best of Luck!

1 The motion of a raindrop

The dynamics of a raindrop depends upon complex properties of the atmosphere. Here we discover various properties of gases in our atmosphere. We will restrict our attention to the troposphere only.

A raindrop of radius r is falling through the atmosphere. It has various forces, like buoyant and damping forces, acting on it. In the early stages of formation, it also experiences a force because it absorbs water vapours from the atmosphere. Assume that this is an isothermal process. The atmospheric pressure at the Earth's surface is $P_\circ$, the average molar mass of air is M and the radius of the Earth is R_e.

a. What is the atmospheric pressure at height h above the ground? Consider the acceleration due to gravity g to be constant all over. (1.5 marks)

b. What will be the atmospheric pressure if g is not constant?Acceleration due to gravity is g_0 at surface of the earth. (2.5 marks)

But obviously, the atmosphere is not isothermal. For small heights up to troposphere, where rain originates, the temperature T varies linearly as $T = T_\circ - \lambda h$, where $T_\circ$ is the temperature of the Earth's surface and λ is a constant. Let T_s be the surface temperature of the Sun, R_s be its radius and R_{se} be the distance between the centres of the Sun and the Earth. Consider the Sun and the Earth to be ideal black bodies in equilibrium.

c. Assume that the Earth gets heat only from the Sun's radiation and the atmosphere neither absorbs nor conducts it when it comes from the Sun. The earth emits a fraction of the energy acquired in form of radiation which does not contribute to the heating of atmosphere(assume this fraction to be η). Also assume that the troposphere gets only the energy radiated by the Earth's surface and conducts it to the upper layers of the atmosphere. No convection happens anywhere in the atmosphere and atmosphere does not radiate heat. Find the thermal conductivity of the troposphere as a function of height h. (3 marks)

d. Does the mathematical function reflect the general intuition about the physical world? (1 mark)

e. Considering this new law, find the atmospheric pressure and air density as functions of height h for small heights up to troposphere. Assume g is constant and make relevant approximations. (3 marks)

   By what percentage the pressure would fall when go to the top of mount Everest? Assume: temperature falls 6.4 K per each km when going upward. Atmosphere is 78 % nitrogen, 21 % oxygen and 1 % argon. g is assumed to be 9.81 m/s^2. Height of Mount Everest is 8850 m. universal gas constant value is 8.31 SI unit. Give answer in 3 figures after decimal. (3 marks)

In the beginning of the formation of the raindrop, it absorbs water droplets suspended in the air. Say water has density $\rho_\circ$ and moisture suspended in air has uniform density ρ_e independent of height. Say a water droplet starts to fall with initial radius r. Neglect buoyancy as $\rho_\circ >> \rho_e$.

f. Find the acceleration of the droplet in terms of g assuming that g is constant. (4 marks)

As the drop has grows in size (radius r), this effect gets negligible. Think of the water drop as a rigid ball. Suppose air has N (assumed to be constant) particles per unit volume, each of mass m. Consider all the collisions to be elastic.

g. How is the damping force related to v? Is it a familiar result? (4 marks)

h. You might have got an equation of the form $F \propto v^n$. What is the proportionality constant as a function of height, considering linear decrease of temperature model explained before? (Your answer should NOT contain N, m and consider g to be uniform) (3 marks)

2 A Mysterious Diary

It was night-time and I was roaming through Gulmohar marg. I got a call from Vinay who was my junior in IISc.

"Are you busy? Can you please come here in room 102, I found something important"!!

I reached there in 5 minutes, and he was sitting with a diary, and I remembered that was Ram's room and he is the batchmate of Vinay.

"This is his diary, and we will surely get some clues what had happened at that day."

We closed the door and started reading. "Seems like he is a crazy person, why his diary is with full of equations!!". We need to know what had happened on July 22, 2021. On that day Ram wrote like this in his diary.

July 22, 2021

Thursday

"He was very upset today and shared his concerns about grades in some courses. Seemed like he was totally confused with concepts of Physics course.

He asked, "does **E** (electric field) in equation connecting atomic polarisability and dipole moment is same as the **E** in equation for Polarisation of linear dielectrics (which shows **P** is directly proportional to **E**)? **(1)**" I derived some relation connecting both E's in terms of number of atoms per unit volume (say M) and some other parameters by assuming each atom is sphere of radius R **(2)**.

He was in a happy mood after my explanation, I just gave him some funny questions, "How much force acting on positively charged atom which is balanced by electrostatic field? **(3)**" He said he will send me his thoughts on this question tomorrow. So, I showed him that force on a neutral polarisable atom which is well balanced in electrostatic field is proportional to gradient of E^2 **(4)** and as an interesting fact also proved E^2 cannot have a local maxima in a charge free region **(5)**.

Finally, he asked some clarification regrading mismatches with theoretical equation and experimental result when he tried to calculate susceptibility of water at different temperatures. He got almost similar values at higher temperature while large deviation in lower temperature. I asked which formula he used for calculation, and he said,

$$\chi_e = \frac{MP^2}{3\epsilon_0 kT}$$

where k= Boltzman constant, M= number of atoms per unit volume, P= permanent molecular dipole moment, T= temperature

I wondered from where he got this cute looking formula which makes this trouble. He said some friends send me this and do not know from where this comes. I was in a deep thought about how to derive such result from scratch. "

"What do you think ??" Vinay asked me. "He gave enough clue for what had happened on that day, look I can assure you this incident is not happened on that day rather he is saying something through these problems!! Even order of problems is well planned. It is not arbitrary."

a. Ram did a lot of work through his diary. Those are marked as 1,2,3,4,5. Complete those with proper explanation. Write your assumptions clearly. You can use any fundamental results from electromagnetism. (1- Two marks 2-Two marks 3-One mark 4-Three marks 5- Five marks)

b. It is interesting to think how that formula mentioned in the diary come from, try to attempt to derive that from scratch.
(Use the fact that for a material in equilibrium at absolute temperature T, the probability of a given molecule having energy u is proportional to Boltzmann factor $\exp(-u/kT)$
where u is the energy of a dipole in an external field E, k= Boltzmann constant, T= absolute temperature. Show the relation between polarisation of a molecule having M molecules per unit volume and

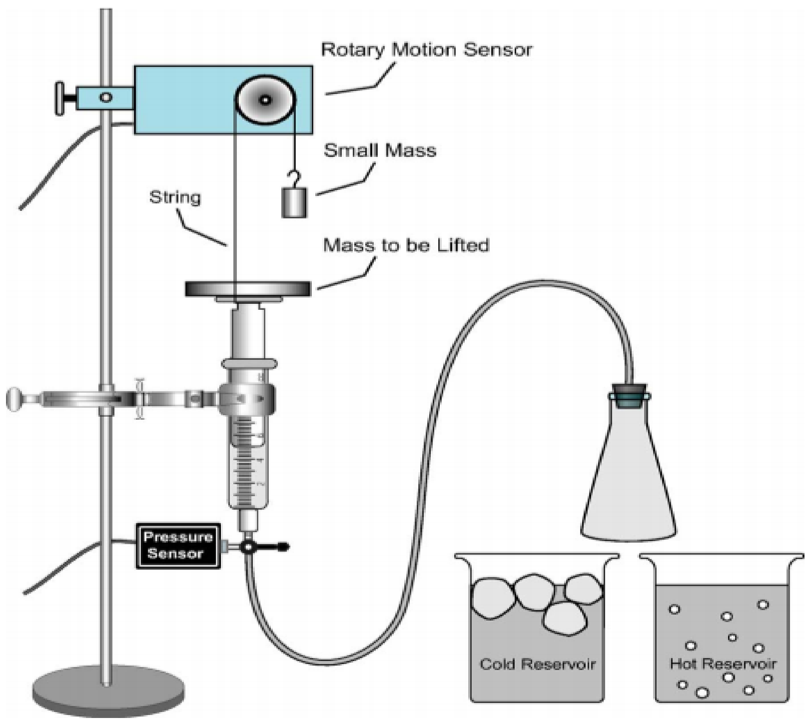
later arrive at above mentioned formula by suitable approximation (assume $kT >> pE$)) (8 marks)

c. Why there is a mismatch in the results from this formula at lower temperature while very less deviation at higher temperature? (4 marks)

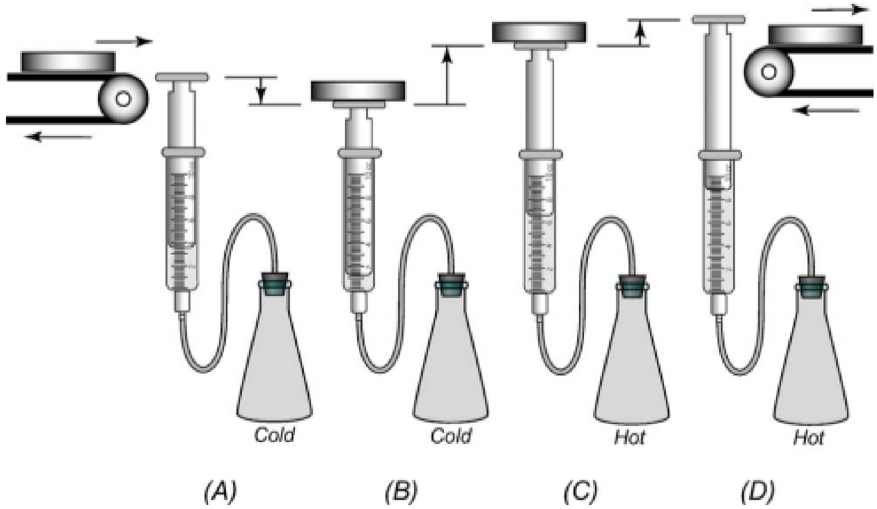
3 The Incredible Mass-Lifting Heat Engine

A typical nineteenth century heat engine was a complicated device full of chambers, pistons, levers and gears. With the advancement in the experimental techniques, we can design a simple but elegant engine whose function is to lift masses from one level to another.

The Mass-Lifting heat engine consists of a glass syringe connected to a flask that is transferred from cold water to hot water and back again. The basic mass-lifter engine cycle is shown in the figure. The flask begins in cold reservoir at 0°C with no mass on the piston. Then a mass of 10kg is quasi - statically placed on top of the syringe piston. The flask is then transferred to hot water chamber at 100°C where the expanding air causes the piston (with mass) to rise. When the piston stops rising, the mass is slowly removed. Finally, the flask is returned to the cold water and the piston descends to its original height, thus completing the cycle.



An illustration of the mass lifting heat engine is shown. A string attached to the piston is placed over a rotary motion sensor to measure volume changes of the system. The small mass is used to keep the string taut.



The basic cycle of the mass lifting heat engine is shown. The flask begins in cold water (A). A mass is then transferred to the piston (B). The flask is transferred to the hot reservoir and the piston lifts the mass (C). The mass is taken off the piston (D). The flask is then transferred back to the cold reservoir (A), and the cycle is repeated.

a. Assuming the piston to be massless, calculate the net work done in one complete cycle of the engine and hence calculate its thermodynamic efficiency.
   Given : The piston is initially at an height of 10 cm from the base of the syringe and has a surface area of 10^{-2} m^2.
   The air pressure outside the engine is 1 atm
   Note : Air can be treated as an ideal diatomic gas with Cv = (5/2)R and Cp = (7/2)R (5 marks)

b. The incredible mass - lifting heat engine is actually not so simple though understanding the stages of the engine cycle is reasonably straight forward. Practically, air does leak out after every cycle of the engine around the piston, especially when larger masses are added to the platform. This means that the number of moles of air decrease over time. (This can be observed by noting that in the transition from (D) to (A), the piston can actually end up in a lower position than it had at the beginning of the previous cycle.)

   Assuming the number of moles decrease by 10% after each complete cycle, calculate the maximum thermodynamic work that can be extracted from the engine. (10 marks)

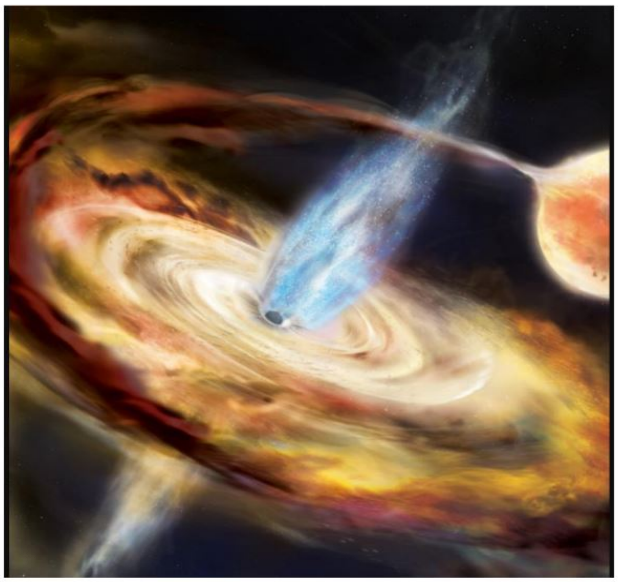
   (Hint : The maximum work done by the engine is the sum of work done in every complete cycle of the engine until the piston reaches the base of the syringe i.e. all the air inside the syringe has been leaked)

c. Using the results from part-(1) and part-(2), find the maximum height to which the mass of 10Kg can be lifted if every cycle of the engine is used to lift the same mass. Also calculate the total heat supplied to the engine during the entire process. (10 marks)

4 Black hole jets

Date: 2nd June 2365. Spacecraft Voyager III has travelled to a nearby black hole with mass $\approx 8M_{sun}$ at very high speeds. Upon arrival, it decelerated to almost zero velocity when it was at a distance 153.6 AU from a black hole. Hereon, it plans to get into a smaller orbit around the system to learn more about it.

The black hole is accreting matter from a star in close orbit around it. As the accretion occurs, the black hole shoots off neutral jets of nuclei electrons and positrons in opposite directions perpendicular to the plane of accretion, at roughly equal relativistic velocities. The jets are not uniform, they are shot with some irregularities, and tracking the irregularities gives us directly the speed of the jets. After Voyager III had decelerated, it pointed its camera at the black hole. It saw something of this sort.



Credit: Aurore Simonnet / NASA Goddard Spaceflight centre

The two jets (which we shall now on call J1 and J2) showed certain disturbances in them. These disturbances we tracked in the camera.

Camera specifications:

- *Aperture lens diameter = 0.1 m*
- *Aperture lens focal length = 0.1 m*
- *Pixel size = 10 microns*
- *CCD: 5000 pixel X 5000 pixel*

The pixels are numbered as follows:

(-2,2)	(-1,2)	(0,2)	(1,2)	(2,2)
(-2,1)	(-1,1)	(0,1)	(1,1)	(2,1)
(-2,0)	(-1,0)	(0,0)	(1,0)	(2,0)
(-2,-1)	(-1,-1)	(0,-1)	(1,-1)	(2,-1)
(-2,-2)	(-1,-2)	(0,-2)	(1,-2)	(2,-2)

And so on to cover all pixels. The camera is pointed such that the black hole centre always occupies position (0,0).

Four disturbances in the jets are tracked with time, taking images at successive times and noting down the pixel number of the observed disturbance. (D1, D2, D3, D4).

Voyager III	**D1**		**D2**		**D3**		**D4**	
Time (mins)	Pixel		Pixel		Pixel		Pixel	
0	16	24	68	102	-174	-264	-324	-486
10	70	107	123	185	-194	-294	-344	-516
20	124	190	177	267	-213	-323	-364	-546
30	179	273	233	350	-233	-353	-384	-576
40	233	356	288	433	-253	-384	-403	-604
50	287	439	342	515	-272	-413	-423	-634
60	342	522	397	598	-292	-443	-443	-664
70	396	605	452	681	-312	-473	-463	-694
80	451	689	507	763	-331	-502	-483	-724
90	505	772	562	846	-351	-532	-503	-754
100	559	855	617	929	-371	-563	-522	-783

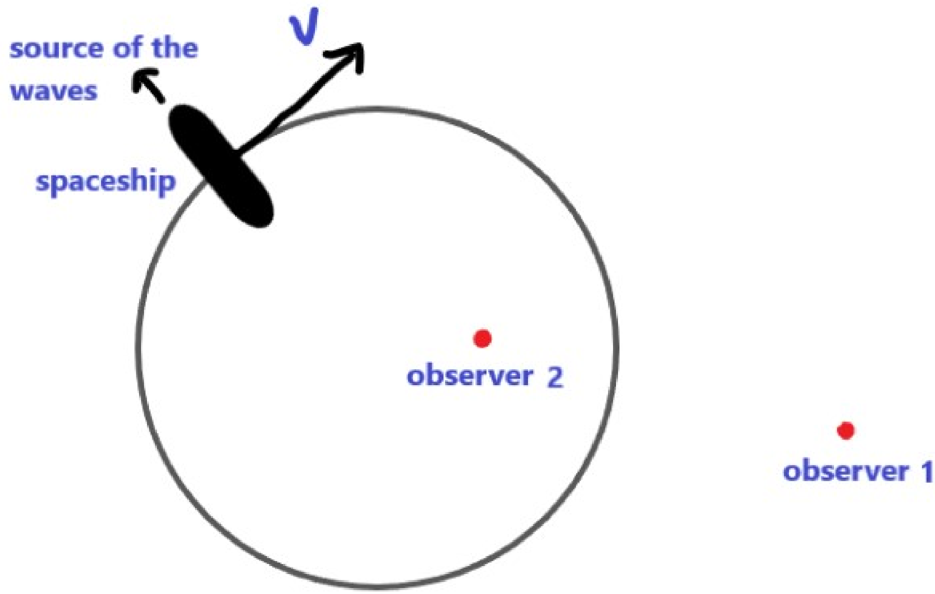
a. Justify the use of 10μm pixels in the CCD detector and find the resolving power of the camera. Can the event horizon of the black hole be resolved at this point? (4 marks)

b. Find the field of view of the camera. (3 marks)

c. Plot the given data of disturbances on a graph sheet, and find the directions of the jets along the plane perpendicular to the line of sight. (10 marks)

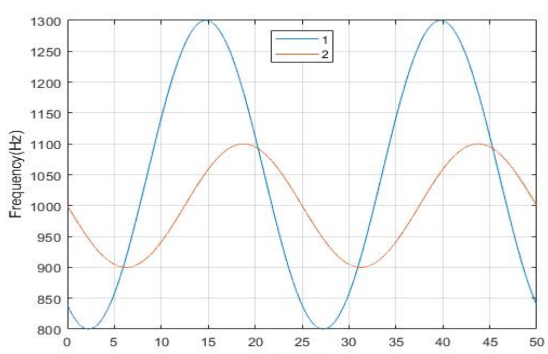
d. Analyse the data (carefully) and figure out the velocity at which the jet is shot out. (8 marks)
(Don't forget to include the motion along line of sight).

5 Space Ship at risk



A long rod-shaped spaceship is moving around an anonymous planet on a circle of a certain radius and also rotating around an axis passing through its midpoint. Suddenly emergency alarm rings! And Tim and Lee, the two astronauts, who needed tech support, had to connect with a troubleshooting section in the ISRO office. However, the rooms for seeking help from ISRO were *at the two different spaceship ends.* These rooms are containing all arrangements of a self-sufficient little spaceship. When needed, they can safely land on a terrestrial body. Now the main problem was that the rounding speed suddenly started increasing instead of being constant. They assumed with correct calculations what can be fatal results (including being lost in infinite space). So, as soon as possible, they went to the two end rooms to connect with Earth. But alas! Boooom! They found themselves detached from the main spaceship with their respective little ships. The excess rotation speed with respect axis made the rear wall of the ends broken, and they fell into space and **comes to rest at some point**.

The spaceship has a wave generator that is made for detecting floating terrestrial bodies in space. Assume the speed of the wave used in free space is 330 m/s, but unlike sound, it can move in free space.



1. This is the curve for frequency got by the two observers. *Assume clock of the two detectors are synchronized.* Determine which observer gets which curve. (1 mark)

2. What is the time period of rotation of space ship? (1 mark)
3. Obtain frequency of source wave. (3 marks)
4. Calculate intuitive distance of outer and inner observer from the calculation done. (3 marks)
5. What happened to these distances if the curve of received frequencies were not purely sin curve but slightly inclined towards right, i.e. rising arm is longer and lowering arm is shorter? (2 marks)

   Now consider a slightly changed case, where the used signal is replaced by an electromagnetic wave and the changed velocity of rounding space ship is n c where c is speed of light in vacuum.
6. Considering relativistic Doppler shift, what would be the maximum and minimum frequencies observed in reference frame of outer receiver? (2 marks)

   Let Lee is the inside observer. He is in a small part of the space-ship that was detached with him. It has all necessary arrangements to land safely on a terrestrial body. Now assume Lee's little ship will not be still after detaching but start rounding around a smaller radius n-fold smaller than previous one. Assume that this shifting to n-fold smaller radius orbit goes on until it reaches the land of the new anonymous planet *of mass M.*
7. Determine the converging sequence showing the radius of all steps afterwards. (5 marks)

   Now fortunately a near-by space-station got the emergency signal of Lee and Tim. They sent a space ship to take them safely.
8. What will be the velocity of the space-ship so that when it passes parallel to the rounding wrecked ship, it receives same frequency as f_0? (3 marks)
9. The recovery ship docks with Tim's ship outside. At that moment what are minimum and maximum distances of Lee's ship from their docked ship? Assume Lee's ship is surrounding at 3rd smaller orbit then and Tim's ship is at a finite distance D from center of the planet. (3 marks)
10. If $D >> r_k$, how much time will pass in Tim's clock in Lee's reference frame when Lee's clock passes 10 minutes? Express in terms of k and v. (2 marks)

Let's hope for the best that the recovery ship will safely land on the planet and pick up Lee!!

6 Doraemon: Nobita's Physics Day

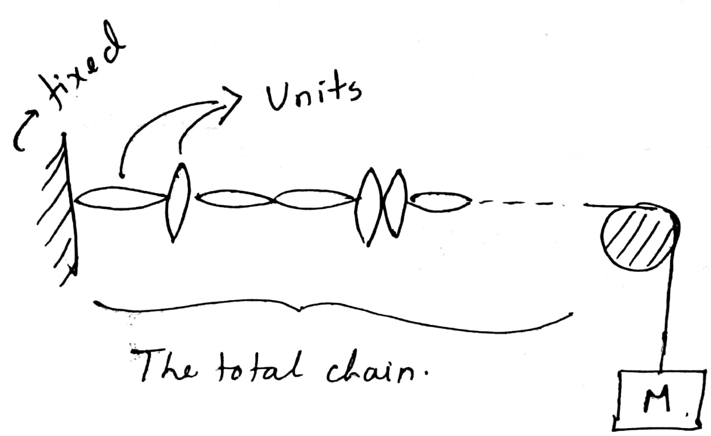
After regular scolding by Mom, Dad, Teacher and Doraemon, Nobita started studying and he improved a lot. He passed his middle school and joined high school and by then he has elevated himself from a boy getting 0 in tests to a high school student taking interest in advanced topics. His subject of interest became physics.

One day in school, teacher was teaching probabilities. He said that the sum of probabilities of all the events in a probability space is always 1 i.e. sum of probabilities of all possible events of concern is 1. He added to raise the curiosity of Nobita that probabilities play a great role in statistical physics where probability of a system attaining energy E is proportional to $e^{-\frac{E}{kT}}$ where T is the temperature at which the system exists and k is the Boltzmann constant. Nobita became interested and starts to work out some problems on his own:

Question 1 (1.5 marks)
Show that, if the constant of proportionality is $\frac{1}{Z}$, $Z = \sum_{n=1}^{\infty} e^{-\beta E_n}$ ($\beta = \frac{1}{kT}$) where n represents any configuration of the system and E_n represents corresponding energy. In the same context, suppose that the possible values of energy depends on a parameter l of the system. In such a case, in discrete energy limit, Z is given by $\sum_l N(l)e^{-\beta E(l)}$ where $N(l)$ represents number of system configuration with parameter $l = l$ and in continuous energy limit, $Z = \int_{l_1}^{l_2} g(l)e^{-\beta E(l)}\, dl$ where l_1 and l_2 are lower and upper possible values of l and $g(l)$ is the degeneracy defined as number of configurations possible per unit value of l.

Nobita shows his calculations to the teacher, who became very impressed. Happily Nobita went home, when he saw Doraemon doing something strange. Doraemon was pulling out his Dora Cake from where he hid it using a gadget, as he suspects some mice to be there. He is using a long arm, which gradually was becoming shorter and pulling out the cake. Nobita asks Doraemon how the gadget works. Doraemon says that it requires basic Statistical Physics to understand. Nobita recalled what he learnt/did in class and tries to understand it.

Question 2
The gadget is in a form of a chain, be made of a series of N units of length a each. The units are linked to one another in such a way that each can assume only two configurations: vertical (contributing nothing to total length of the chain) and horizontal (contributing a length a to the chain). Suppose one end of the chain is fixed and from other end a mass is hanged (as shown in the figure.)



Without the mass attached, the units have no preference between horizontal and vertical configurations.
Part A (3 marks): Calculate the Z for this system.
Part B (2 marks): Calculate the average length of the chain at temperature T.
Part C (1 marks): If the temperature of the chain is changed, how will the length change ? Mention why the two extremes are physically justified.
Part D (1.5 marks): What is the requirement on temperature T and F such that Hooke's law applies ? When temperature is raised, do we need more or less force to stretch the chain ?

Excited, Nobita borrows the gadget from Doraemon (as he always does) and went to Sizuka's house. But he became annoyed when he say Dekisugi already there. He had a weird box and was explaining something to Sizuka as he was holding it in hand. There was a small pinhole in the box and when he opened the pinhole for a moment, green light was seen to be coming out. The box was such that, one face of it was of form of a piston and initially it was held in its position. To teach them a lesson, Nobita uses Doraemon's gadget to pull the piston. And as he did that Dekisugi felt strange. He again opened the pinhole and he was surprised to see that the colour of light changed! Surprised, the three now sits down to analyze:

Question 3 (2.5 marks)
The box was contained photons initially at temperature T. The fundamental equation connecting the entropy, energy and volume of the photon gas is given by:

$$S = CE^{\frac{3}{4}}V^{\frac{1}{4}}$$

where C is a constant. Now suppose, due to pulling of the piston by Nobita, the volume increased by a factor α. If initially the wavelength of light having the maximum intensity is λ, calculate the new wavelength after what Nobita did. Assume that momentarily opening pinhole did not disturb equilibrium and the box is insulated.

This phenomenon raised the curiosity of Nobita many folds. He now wonders what the proportionality constant in the probability may mean. He had learn from his teacher that entropy of a system being in energy E depends on the total number of states of the system having the same energy (microstates). This is denoted by Ω and the relation is

$$S = kln\Omega$$

where S is the entropy. Ω is dependent on the energy of the system. Also, when two systems are in contact, the total number of micro states is multiplicative. With this knowledge, help Nobita understand the significance of Z

Questions 4
Suppose there is a reservoir with temperature T and a small subsystem in contact. The total energy of the reservoir and the subsystem is E_T and it is a constant. Suppose that the subsystem is in energy E_i.
Part A (2 marks): Express the probability of the subsystem being at energy E_i in terms of $\Omega_r es$ (number of microstates of reservoir), E_i and E_T and Ω_T (total number of microstates).
Part B (3 marks): Using the relation between Ω and S and using the approximation that the energies of the subsystem is much lower than energy of the reservoir, prove that the relation between Z and F is $F = -klnZ$ where F is the Helhmoltz Free Energy of the subsystem. Note: F is defined in terms of average energy.

Wondering over this beautiful result, Nobita reaches the field where he saw Gian and Sunio doing something. They had two magnets each and they were holding them apart. The magnets, due to their mutual forces were jiggling. Nobita readily thought that he can use his understanding to study the interaction energy of the magnets.

Question 5
The magnets can be though of as two point dipoles with dipole moment towards unit direction $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$.

When the two magnets are separated by a distance d the interaction energy is given by:

$$E(\overrightarrow{p_1}, \overrightarrow{p_2}, \overrightarrow{d}) \propto \frac{1}{d^3}(\overrightarrow{p_1} \cdot \overrightarrow{p_2} - \frac{3}{d^2}(\overrightarrow{p_1} \cdot \overrightarrow{d})(\overrightarrow{p_2} \cdot \overrightarrow{d}))$$

Part A (5.5 marks): Judiciously choosing the coordinate system and in the very high temperature limit, determine in the first non-zero approximation the distance dependence of the average interaction energy and thus the force.
Part B (3 marks): Argue whether the average interaction energy will have (in general) any odd order terms in d. This is a non-mathematical question. Next, verify your claim with mathematical working.

Gian and Sunio was staring at Nobita when he was explaining all these to them. Gian became angry on how Nobita can know so many things. He starts chasing Nobita. Nobita came back home, crying.

7 Quantum Demon pushes the wall: Fast or Slow?

Consider a standard quantum mechanics problem of a particle of mass m confined in a infinite square well of length a.

1. What is the nth normalised energy eigen function Ψ_n and energy eigen value E_n of the system? **(1 mark)**

Now, a quantum demon pushes one of the walls of the well. Let $w(t)$ be the separation between the walls at any time t such that $w(0) = a$. Hamiltonian of the system is:

$$H = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}; \qquad 0 \leq x \leq w(t)$$

This can be solved analytically without using any perturbation methods for a special case: $\frac{dw}{dt} =$ constant. For $w(t) = a + vt$ where v is a constant(sign of v depends on the direction in which demon pushes the wall), a complete set of solutions is given by $\{\Phi_n(x,t)\}_{n=1}^{\infty}$ where

$$\Phi_n(x,t) = u_n(x,t)\ \exp\left(\frac{-i}{\hbar}\int_0^t E(t')dt'\right)\ \exp\left(\frac{imvx^2}{2\hbar w(t)}\right)$$

with $u_n(x,t), E(t)$ being the instantaneous normalised energy eigenfuntion and energy eigen value respectively.

2. Write $u_n(x,t), E(t)$ and evaluate $\Phi_n(x,t)$. Explicitly show that $\Phi_n(x,t)$ satisfies time dependent schrodinger wave equation and boundary conditions. **(8 marks)**

Any general solution can be expanded in this basis as:

$$\Psi(x,t) = \sum_n c_n \Phi_n(x,t)$$

Now suppose that the particle starts out with a wave function $\Psi(x,0)$

3. Show that the expansion co-efficient can be written in the form

$$c_n = \sqrt{\frac{2}{a}}\int_0^a \exp\left(\frac{-imvx^2}{2\hbar a}\right)\sin\left(\frac{n\pi x}{a}\right)\Psi(x,0)\ dx$$

(3 marks)

However, the above integral cannot be evaluated in terms of elementary functions if the initial states are energy eigen functions.

So far the treatment has been exact and we have explored dynamics of the problem. We now consider two special situations for this problem - demon pushes the wall very fast, demon pushes the wall very slowly.

Suppose the demon pushes the wall such that it expands from a to $2a$ in time T_{ext}. Now, define another time scale T_{int} as the period of the time-dependent exponential factor in the ground state wave function of *usual* particle in a box.

4. Write T_{ext}, T_{int} in terms of the parameters defined in the problem. **(2 marks)**

Let us analyse the situation when the demon pushes very slowly i.e., in the limit $T_{ext} >> T_{int}$. Assume that the particle initially is in ground state.

5. **Demon pushes the wall very slowly** **(6 marks)**

   i. Using the expression obtained in (3.), obtain the wave function $\Psi(x,t)$ in the limit $T_{ext} >> T_{int}$.

ii. Is the above wave function consistent with the adiabatic theorem? Explain. Comment on the phase factor in wave function obtained above.

iii. When the demon pushes the wall such that it expands from a to $2a$, what is the change in energy of the particle? Compare it with the classical result by calculating from the mean force exerted on a wall by the bouncing wall.
(This problem can be thought classically as a ball of mass m bouncing elastically between two infinite plane walls separated by distance $w(t)$)

Let us now analyse the situation when the demon pushes the wall very fast that the distance between the walls doubled almost instantaneously. Now, energy of the system is measured. As above, assume that the particle initially is in ground state.

6. **Demon pushes the wall very fast** **(5 marks)**

   i. Compute the expectation value of energy.

   ii. Argue classically that there is no change in particle's energy

   iii. Compute the probability that the system is in nth eigen state of the new Hamiltonian.

8 The Diffracting Electrons

Note: In this question, you have to do error analysis. Clearly mention all the assumptions, error calculations etc. Graphs should be plotted with error bars. The data along with errors must be submitted in some form (so that we can clearly see). Henceforth, explicit mention about doing error analysis is not done. But this note stands valid throughout. Answers without proper error analysis will not receive full credit.

It is well known that particles also have wave nature as proposed by De-Broglie. One of the greatest evidence of this strange hypothesis came from the observation of electron diffraction from a crystal. A diffraction pattern gets visible following the Bragg's law. It is very easy to see from Bragg's Law that when the crystal plane is aligned at θ to the incident beam such that

$$2dsin\theta = n\lambda$$

the beam gets reflected by the crystal in a fashion such that we see constructive interference at an angle 2θ from the beam axis. We get circular fringes when electron beam is incident on a crystal.

Question 1 **(2.5 marks)**
Why do we use crystals for diffraction experiments with electrons ? Also, explain why do we get circular fringe pattern when performin electron diffraction with crystals.

The fringes obtained are not very sharp and are considerably broadened. The broadening angle are defined as $\beta = 2\theta$. The broadening is due to the following two causes:

Size Broadening: This is the broadening caused by the finite (non-infnite) size of the crystals involved in the diffraction. This broadening is characterised by the Scherrer Equation and is given by

$$\beta_{Scherrer} = \frac{K\lambda}{Dcos\theta}$$

where K is the Scherrer constant and D is the mean size of ordered crystalline domains.

Strain Broadening: This is the broadening caused by an inhomogeneous strain present in the specimen. This broadening can be approximately given by:

$$\beta_{strain} = k\epsilon \tan\theta$$

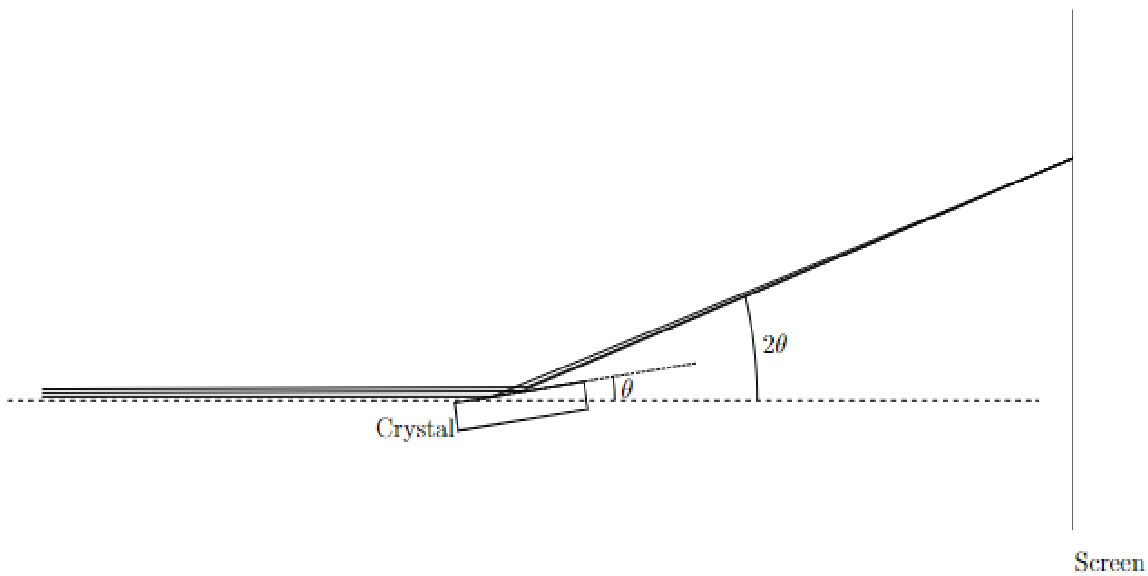
k is a constant dependent on the nature of the inhomogeneous strain ϵ is the mean of the inhomogeneous strain.

The net broadening β is the sum of the above two effects. **This is the Williamson-Hall method of estimating total broadening.**

Now, let's try to heuristically derive both the Scherrer equation and the strain broadening.

Question 2 and Question 3 (The next two questions) are bonus and will be used for tie breaking (if needed). Otherwise, scores for the questions will not be considered.

Question 2: Scherrer equation
Consider a small crystal made of one one kind of diffracting plane with a distance between planes 'a'. Because the crystal is small, we have to consider that it is made up of a finite number of planes, say N.



i. For a parallel beam of intensity I_λ and wavelength λ falling on this crystal, find the intensity on the screen as a function of the angle to the screen 2θ. Represent it as $I_\lambda(x)$ where $x = 2a\sin\theta$.

ii. For a given constructive interference at $x_\circ$, find Δx such that $I_\lambda(x_\circ + \Delta x) = I_\lambda(x_\circ)/2$ for a large enough N and assuming that $\Delta x << x_\circ$. Using this, you will get an equation that cannot be solved analytically, but has a numerical solution. Call the numerical solution η.

iii. Now, substitute $D = Na$ and $x = 2a\sin\theta$. Also, making the assumption that Δx is small, write the broadening
$$\beta_{\text{scherrer}} = 2\Delta x \frac{d\theta}{dx}$$

iv. Replace the constants in the obtained broadening by the Scherrer constant K so that other geometries be generalised to.

Question 3: Strain Broadening
As one might see from Fig.8, a constant strain on a crystal does not lead to broadening, but leads to a constant change in the angle where interference occurs. In a polycrystalline substance, there are multiple small crystals with different sizes, shapes, and hence inhomogeneous strains.

i. Defining the strain $\epsilon = \Delta a/a$, for a monocrystal, find the change in the angle for constructive interference($\Delta\theta$) for a small homogeneous strain.

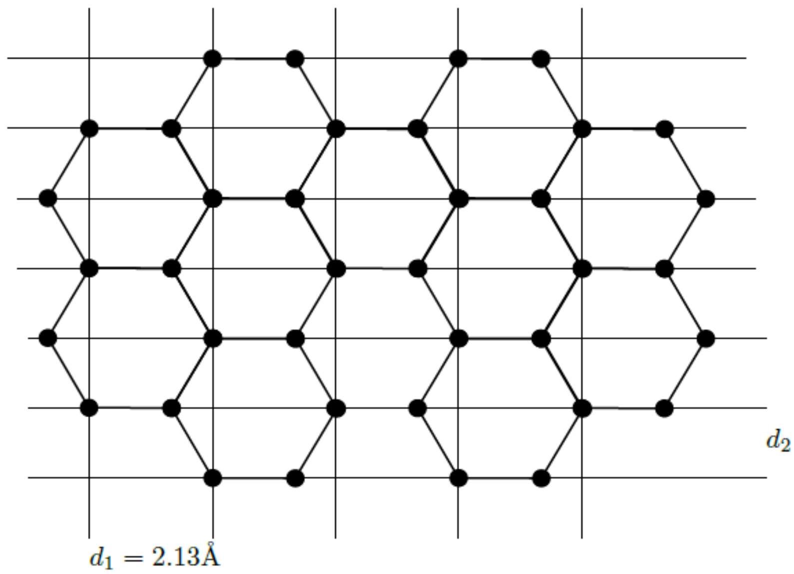
ii. Now, to take into consideration the effect of different strains in a polycrystal, assume that the mean strain in the polycrystal is 0(which would also imply that $\langle\Delta(2\theta)\rangle = 0$), but the strain has a distribution with FWHM, ϵ. Find the broadening $\beta_{\text{strain}} = \sigma_{\Delta(2\theta)}$.

iii. Now to consider the effect of kinds of strains, replace the constants in the equation by k. You finally get the broadening due to inhomogeneous strain in the polycrystal.

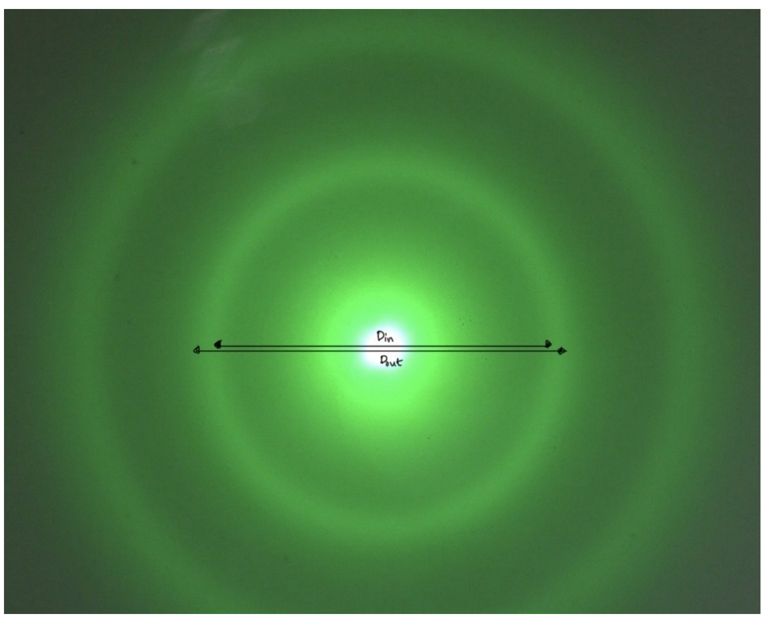
Now, when an electron is accelerated through a voltage, the electron gets a characteristic de-Broglie wavelength. The calculation can be done both in non-relativistic and relativistic way. Let's see how much difference does it make.

Question 4: (1.5 marks)
For a particle with charge q and mass m accelerated through a potential V, find the de Brogile wavelength, $\lambda(V)$ with and without relativistic corrections. What is the percentage correction at $V = 10kV$ for electrons?

Now we are ready to work with some data. A very thin slice of graphite was used for the diffraction experiment. The following diagram shows the various planes of the graphite.



Two diffraction fringes are observed on the screen corresponding to d_1 and d_2.



The fringes are broadened due to the previously mentioned reasons. The outer diameter and inner diameter of the fringes are recorded. Also recorded is the voltage by which the electron was accelerated for the measurement. The data is given here (Least count of voltage readings: 0.1kV, distance between graphite and screen was measured with a normal laboratory scale):

Distance between Graphite foil and screen, L=13.5cm

Least count of Vernier Calliper=0.01mm

V(kV)	$D_{1_{in}}$ (mm)	$D_{1_{out}}$ (mm)	$D_{2_{in}}$ (mm)	$D_{2_{out}}$ (mm)
2	33.14	37.32	59.78	65.26
2.5	29.36	34.11	52.93	57.94
3	26.04	30.88	48.44	52.26
3.5	24.96	29.02	44.75	48.65
4	23.40	26.58	41.21	44.47
4.5	22.34	25.42	38.61	42.34
5	20.59	23.13	37.00	40.93

With the following data, answer the following questions:

Question 5: Analysis of data

a. Tabulate θ and β as functions of the accelerating voltage (3 marks).

b. Theoretically obtain the wavelength of electrons impinged on the sample and compare it with the wavelength obtained by using θ and d_1. Compare the values for both relativistic and non-relativistic assumptions (3 marks).

c. Using the theoretically estimated relativistic values of λ and the value of θ, plot an appropriate graph to find the value of d_2. Compare the obtained value with the theoretical expectation (4 marks).

d. Plotting (an) appropriate graph/s in only measured values (you are not allowed to use derived values like θ or λ), find the value of the Planck's constant. Compare it with the literature value. Make the functions as simple as possible. Clearly mention the approximations. (3.5 marks)

e. Plotting $\beta cot\theta$ vs $\lambda cosec\theta$ for different accelerating voltages, verify the Williamson-Hall method for the diffraction from both d_1 and d_2. (4.5 marks)

f. Comment on why we won't be able to see diffraction pattern for inter planes of graphite here. (3 marks)