Decoherence subjective prelims solutions

Team Decoherence

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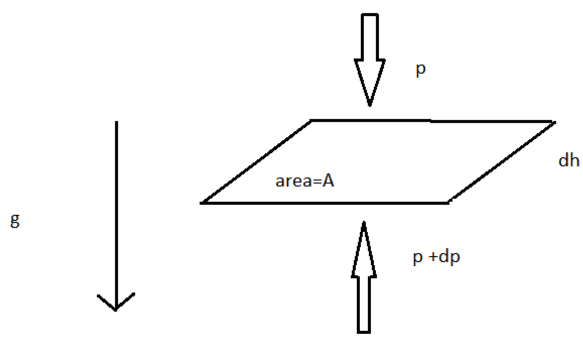
Note: These solutions are brief and some calculations are skipped which are expected to be worked out by the reader. However during grading, full credit is given only if all the details are worked out as instructed in the question paper.

1 The motion of a raindrop

1.1 Part a

We draw a small cuboid region of atmosphere. We assume that pressure will decrease as go upward by intuition . So, the pressures as as follows across the volume of area A and infinitesimal height dh is : p and p+dp as shown in the figure.



Let the density of the medium at that height is ρ which by the ideal gas law is $\rho = \frac{PM}{RT}$ Thus we know that the weight of the gas must be balanced by the extra upwards pressure. Thus we must have, for isothermal atmosphere:

$$dpA = Agdh\rho$$

thus,

$$\frac{dp}{p} = \frac{Mg}{RT}(-dh)$$

Integrating and putting boundary conditions we get that:

$$P(h) = P(0)e^{\frac{-Mg}{RT}h}$$

1.2 Part b

We have the same differential equation, just that now g is represented by $g = \frac{g_0 {R_e}^2}{(R_e+h)^2}$.

Thus we have :

$$\frac{dp}{p} = \frac{-Mg_0 {R_e}^2}{RT(R_e+h)^2}dh$$

clearly integrating this and putting boundary conditions we get:

$$P(h) = P(0)e^{\frac{-MgR_e h}{RT(R_e+h)}}$$

1.3 Part c

So, as the temperature is given by T(h)=T(0)- λh. Thus we have $\frac{dT}{dh} = -\lambda$. Now by Stefan's law the total heat radiated by the sun per second is $4\sigma\pi R_s^2 T_s^4$. The total heat coming to earth is

$$H_T = \frac{\pi R_e^2}{4\pi R_{se}^2} \times 4\sigma\pi R_s^2 T_s^4(1-\eta)$$

AS this amount of heat also must be radiated out of the earth through conduction in atmosphere must be the same with H_T. Thus, clearly :

$$\kappa(4\pi(R_e+h)^2)(\frac{dT}{dh}) = H_T$$

Thus, we have :

$$\kappa = \frac{H_T}{4\pi(R_e+h)^2\lambda}$$

Finally,

$$\kappa = \frac{\sigma R_e^2 R_s^2 T_s^4(1-\eta)}{4\lambda R_{se}^2(R_e+h)^2}$$

1.4 Part d

It does represent the real world as we can see the conductivity decreasing with increasing height. Conductivity depends upon the heat carrier molecules of air thus we may have a correlation between the density and conductivity. Clearly, the density decreases when height increases, naively we can expect the conductivity to also go down, as indicated by the equation. Although strictly speaking, the exact dependence may not be valid accurately.

1.5 Part e

WE again have :

$$\frac{dp}{p} = \frac{-Mg}{RT}dh$$

put $T = T(0) - \lambda h$ Integrating we have : (P(0) to p and 0 to h):

$$p = p_0(\frac{T_0 - \lambda h}{T_0})^{\frac{Mg}{\lambda R}}$$

now we have $\rho = \frac{PM}{RT}$ putting the expression for T, we get :

$$\rho = \frac{P_0 M}{R(T_0 - \lambda h)}(\frac{T_0 - \lambda h}{T_0})^{\frac{Mg}{\lambda R}}$$

1.5.1 2nd subpart of e

The average molecular weight of air can be found out by averaging : 29.969512 g/mol. linear constant is -6.4K/km. temperature at base is earth's average temperature 289.15K.

Thus we have : $\frac{Mg}{\lambda R}$= 5.3435, decrement in T is 6.4 x 8.85 = 56.64 K. Thus pressure decrease we can get by simply plugging in the values are:

$$\frac{P}{P_0} = 0.31194$$

Thus, decrease is 68.806 % (range : 67-69 will be considered as a team can take different convenient base temperatures.)

1.6 part f

Let us assume the radius is r and the mass is m . The height traversed is h and the the velocity is v.

In dt time the mass collected from the atmosphere is $\pi r^2 v\rho_e dt$ which is equal to differential increase in mass :

$$dm = 4\pi r^2\rho_0 dr = \pi r^2\rho_e dh$$

From this, integrating we have :

$$h = \frac{4\rho_0}{\rho_e}r \tag{1}$$

Now we have: F= ma + v (dm/dt) Thus indeed becomes :

$$Mgdt = \pi r^2 v\rho_e dh + \frac{4}{3}\pi r^3\rho_0 dv$$

this simplified to :

$$\frac{4}{3}\rho_0 gr = v\rho_e dh + \frac{4}{3}\rho_0 dv$$

Now we put (1) to get :

$$\frac{gh}{3} = v^2 + \frac{ah}{3}$$

fetching till this step would earn full marks as acceleration constant not mentioned.)

Taking acceleration constant gives us : $\boxed{\text{a=g/7}}$

1.7 part g

Let us assume that the ball is moving in the x direction. The force has to be found out by integrating over rings with make angle θ at the centre of the sphere.

From some geometric manipulation we can easily visualise that for such a ring the number of particles hitting the surface in dt time is = (surface area x velocity x particles per unit volume) = $(2\pi r^2 sin\theta cos\theta d\theta)$ x v x N = $2\pi vNr^2 sin\theta cos\theta d\theta = \frac{d_{Particle}}{dt}$

Also ,it is clear that momentum change of each particle is $\Delta p = (1 + cos2\theta)mv$where m is the mass of each particle. Thus to find the total force we must integrate the quantity $\Delta p \times \frac{d_{particle}}{dt} \times dt$ from $\theta = 0$ to $(\pi/2)radian$. Not π because particles can't hit from backside according to this model. Doing this we get ,

$$F = -\pi mv^2 Nr^2$$

This is clearly very familiar v^2 law that is commonly used in damping.

1.8 Part h

The damping constant is $-\pi Nmr^2$ We know that, $N \times m = \rho$ Thus the proportionality constant becomes: $-\rho r^2$ Thus we have the proportionality constant in terms of previous parameters as:

$$b = -\frac{P_0 Mr^2}{R(T_0 - \lambda h)}(\frac{T_0 - \lambda h}{T_0})^{\frac{Mg}{\lambda R}}$$

2 A Mysterious Diary

2.1 Part 1

1. Here, we have two simple equations

$$p = \alpha E \text{ (p=dipole moment)} \tag{2}$$
$$P = \epsilon_0 \chi E \tag{3}$$

These two electric fields are not equivalent. Field in equation 2 is the total macroscopic field in the medium. Where field in equation 1 is the field due to everything except the field due to atom under consideration.

2.

$$P = pM$$

P=Polarization
M=Number of atoms per unit volume.
p=dipole moment per atom.

As mentioned in the question, assume each atom has volume $\frac{4}{3}\pi R^3$
where R is the atomic radius,

$$M = \frac{1}{\frac{4}{3}\pi R^3}$$

Macroscopic field $E = E_{self} + E_{else}$

$$p = \alpha E \implies P = M\alpha E$$

Now, $E_{self} = -\frac{p}{4\pi\epsilon_0 R^3}$ Hence,$E = -\frac{\alpha E_{\text{else}}}{4\pi\epsilon_0 R^3} + E_{\text{else}}$
So,

$$E = \left(1 - \frac{M\alpha}{3\epsilon_0}\right) E_{else}$$

3. Earnshaw's theorem

4.

$$F = (p.\nabla)E$$

then $F = \alpha(E.\nabla)E$
We can use the product rule, $\nabla(E.E) = 2E \times (\nabla \times E) + 2(E.\nabla)E$
but $\nabla \times E = 0$

$$\therefore (E.\nabla)E = \frac{1}{2}\nabla E^2$$

$$F = \frac{1}{2}\alpha\nabla(E^2)$$

5. Suppose E^2 has a local maximum at point P, Then there will be a sphere of radius R about P such that

$$E^2(P') < E^2(P) \implies |E(P')| < |E(P)|$$

for all points on the surface
But if there is no charge inside the sphere average field over the spherical surface is equal to the value at the center.

$$= \frac{1}{4\pi R^2} \int \vec{E}.\vec{da}$$

Now we can write $|\vec{E}(P)| = \frac{1}{4\pi R^2}\left|\int \vec{E}.\vec{da}\right| \leq \frac{1}{4\pi R^2}\int |\vec{E}|.\vec{da} \implies |\vec{E}(P)| \leq \frac{1}{4\pi R^2}\int |\vec{E}|.\vec{da}$
Here we used the fact that absolute value of an integral is less than or equal to integral of absolute value of function.
We can say $|\vec{E}(P)|$ must be surely larger than the largest value of $|\vec{E}(P')|$, but maximum value of $\vec{E}(P')$ will be surely larger than average value of $|\vec{E}(P')|$ on the sphere.
then $|\vec{E}(P')|$ should be larger than average since $|\vec{E}(P)| > |\vec{E}(P')|$
but this will lead to contradiction. Hence no maxima.

2.2 Part 2

First step is the derivation of Langevin Equation from the assumptions given:
The energy of a dipole in an external field E is

$$u = -p.E = -pE\cos\theta$$

if we orient the z axis along E. Statistical mechanics says that for a material in equilibrium at absolute temperature T , the probability of a given molecule having energy u is proportional to the Boltzmann factor $e^{\frac{-u}{kT}}$
The average energy of the dipoles is therefore

$$\langle u\rangle = \frac{\int ue^{\frac{-u}{kT}}\,d\Omega}{\int e^{\frac{-u}{kT}}\,d\Omega}$$

where $d\Omega = \sin\theta\,d\theta\,d\phi$ also $\theta : 0 \to \pi$ $\phi : 0 \to 2\pi$
Doing the integral with ϕ and changing the remaining integration variable from θ to u (du = pE $sin\,\theta\,d\theta$), we get

$$\langle u\rangle = (kT - pE)\coth\frac{pE}{kT}$$

$P = M\langle p\rangle$; $p = \langle p\cos\theta\rangle\hat{E} = \langle p.E\rangle\left(\frac{\hat{E}}{E}\right) = -\langle u\rangle\left(\frac{\hat{E}}{E}\right)$ $P = Mp\frac{-\langle u\rangle}{pE} = Mp\left[\coth\frac{pE}{kT} - \frac{kT}{pE}\right]$
hence we reached at Langevin Formula (M=molecules per unit volume)
Now, take $y = \frac{P}{Mp}$ and $x = \frac{pE}{kT}$
Equation become

$$y = \coth x - \frac{1}{x}$$

as $x \to 0$ $y = \frac{x}{3} - \frac{x^3}{45}\cdots$ then for small x, $y \approx \frac{x}{3} \implies \frac{P}{Mp} \approx \frac{pE}{3kT} \implies P \approx \frac{Mp^2E}{3kT} = \epsilon_0\chi_e E$

$$\therefore \chi_e = \frac{Mp^2}{3\epsilon_0 kT} \tag{4}$$

2.3 Part 3

This is due to the fact that for large fields or low temperature, virtually all the molecules are lined up and the material is non-linear.
But assuming material is linear and using equation 3, we are getting off values at low temperature. Here we are neglecting the distinction between E and E_{else}
The agreement is better in low density gases for which the difference between E and E_{else} is negligible. Hence we get better result at higher temperature.

3 The Incredible Mass-Lifting Heat Engine

a. **Processes:**
A $\rightarrow$ B - Isothermal Compression
B $\rightarrow$ C - Isobaric Expansion
C $\rightarrow$ D - Isothermal Expansion
D $\rightarrow$ A - Isobaric Compression

PV diagram:

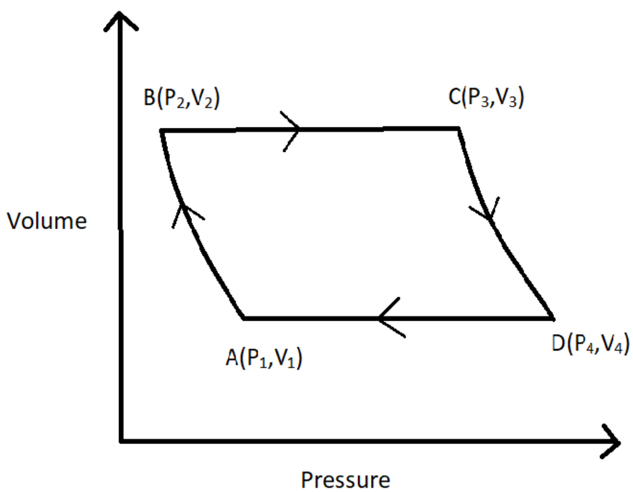


Figure 1: PV diagram

Determining corner points of PV diagram:

Since the piston is massless,

$P_1 = P_{atm} = 1atm$

$V_1 =$ Area of piston $\times$ Initial length

$P_2 = P_1 + \frac{mg}{A} = 1.1atm$

Now, $P_1V_1 = P_2V_2$

$V_2 = \frac{P_1V_1}{P_2} = \frac{1atm\times 1L}{1.1atm} = 0.91L$

$P_3 = P_2 = 1.1atm$

$V_3 = \frac{V_2T_3}{T_2} = \frac{0.91L\times 373K}{273K} = 1.24L$

$P_4 = P_1 = 1atm$

$V_4 = \frac{P_3V_3}{P_4} = \frac{1.1atm\times 1.24L}{1atm} = 1.364L$

Calculation of total work done:

$W_{A\rightarrow B} = nRT\log_e(\frac{V_2}{V_1}) = P_1V_1\log_e(\frac{V_2}{V_1}) = \log_e(0.91)\ Latm = -0.094\ Latm$

$W_{B\rightarrow C} = P_2(V_3 - V_2) = 1.1\ atm(1.24\ L - 0.91\ L) = 0.363\ Latm$

$W_{C\rightarrow D} = nRT\log_e(\frac{V_4}{V_3}) = P_3V_3\log_e(\frac{V_4}{V_3}) = 0.13\ Latm$

$W_{D\to A} = P_4(V_1 - V_4) = 1\ atm(0.364\ L) = -0.364\ Latm$

Total work done $= (-0.094 + 0.363 + 0.13 - 0.364)\ Latm \approx 3.55\ J$

Calculation of total heat absorbed:

$Q_{B\to C} = nC_p\Delta T = \frac{P_2V_2}{RT}\frac{7R}{2}\Delta T \approx 1.28\ Latm$

$Q_{C\to D} = W_{C\to D} = 0.13\ Latm$

Total heat absorbed = (1.28 + 0.13) Latm = 1.41 Latm = 142.87 J

Therefore, Thermodynamic efficiency $= \frac{3.55}{142.87} \times 100 = 2.5\%$

b. Consider an arbitrary cycle starting with n moles of air

$W_{total} = nRT_1\log_e(\frac{V_2}{V_1}) + P_2(V_3 - V_2) + nRT_3\log_e(\frac{V_4}{V_3}) + P_4(V_1 - V_4)$

$W_{total} = nRT_1\log_e(\frac{P_1}{P_2}) + P_2(\frac{V_1T_3}{T_2} - V_2) + nRT_3\log_e(\frac{P_3}{P_4}) + P_4(V_1 - \frac{P_3V_3}{P_4})$

Since $P_1 = P_4, P_2 = P_3$, after simplification, we get

$W_{total} = nR\log_e(\frac{P_1}{P_2})(T_1 - T_3)$

Since, $P_2 = P_1 + \frac{mg}{A}, T_1 = 273K, T_3 = 373K, P_1 = 1atm$ for every cycle, we have:

$W_{total} = 9.53nR$

Maximum work that can be extracted from the engine if 10% of moles leak after each cycle $= \sum_{i=1}^{\infty} W_i$

We get a geometric progression:

$W_{max} = 9.53nR(1 + \frac{9}{10} + \frac{9^2}{10^2} + ...) = 95.3nR$

But, $n =$ initial number of moles $= \frac{P_1V_1}{RT_1}$

Therefore, $W_{max} = 9.53\frac{1L\ 1atm}{273K} = 0.35Latm \approx 35.4J$

c. The height to which the mass is raised in the first cycle $= \frac{V_3 - V_1}{A} = 0.024$ m

Total work done in the first cycle = 3.55 J

Useful mechanical work done in the first cycle = mgh = 2.4 J

Mechanical efficiency of the engine $= \frac{2.4}{3.55} \times 100 = 67.6\%$

Maximum useful mechanical work done = Mechanical efficiency $\times$ $\mathrm{W_{max}}$ = 0.676 $\times$ 35.4 J = 23.93 J

Maximum height achieved by the mass $= \frac{\mathrm{W}}{\mathrm{mg}} = \frac{23.93}{10\times10} \approx 24$ cm

Total heat absorbed by the engine $= \frac{\mathrm{W_{max}}}{\text{Thermodynamic efficiency}} = \frac{35.4}{0.025}$ J = 1416 J

4 Black hole jets

a.
Maximum possible resolution possible with given aperture is given by:

$$\Delta\theta = \frac{1.22\lambda}{aperture} \approx \frac{1.22 \times 500nm}{0.1m} \approx 6 \times 10^{-6}rad \sim 10^{-5}rad$$

Now, angle recorded by a pixel is given by:

$$\delta\theta = \frac{Pixel\ size}{focal\ length} = \frac{10\mu m}{0.1m} = 10^{-4}rad$$

If the pixel size were made smaller significantly, we would run into the diffraction limit. If the pixel size were significantly larger, the resolution would be compromised for no reason.
The black hole with mass $\sim 8M_{sun}$ will have radius

$$r = \frac{2G(8M_{sun})}{c^2} = 24km$$

Angular size of the black hole is thus,

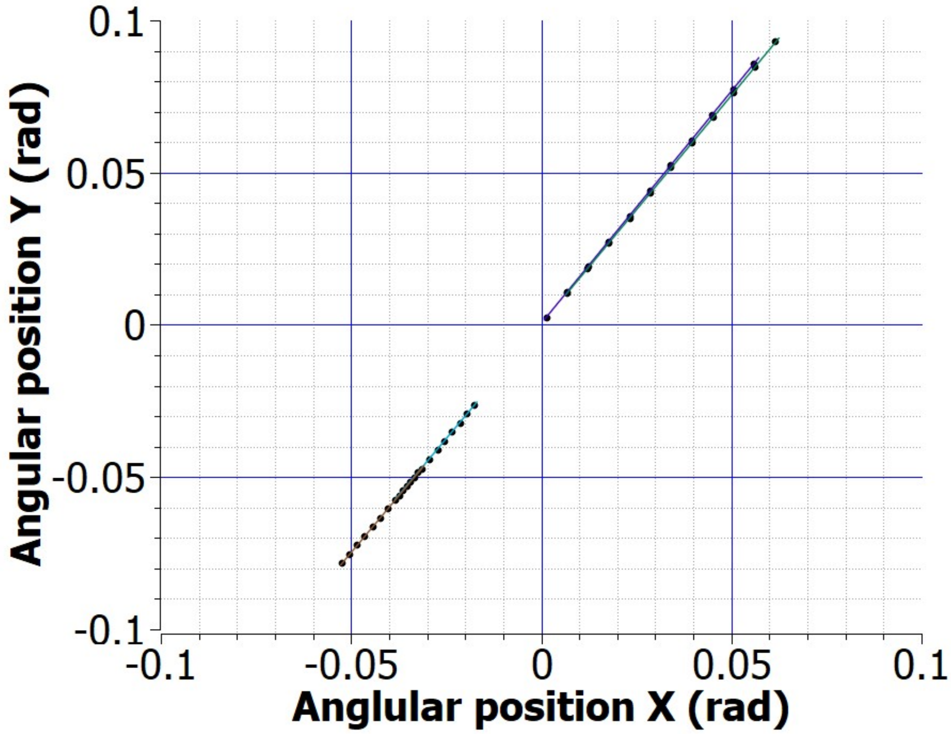
$$\frac{24km}{153.6AU} = 10^{-9}rad << 10^{-4}rad$$

The black hole can't be resolved by the camera.

b.
Each pixel holds a $10^{-4}rad$ angle. Since we have 5000 pixels,the FOV is given by:

$$FOV = 5000 \times 10^{-4} = 0.5rad$$

c.
Since each pixel holds $10^{-4}rad$ of angle, we plot the angular position of the disturbances with respect to the black hole on a graph sheet.



We see that the directions of the jet (perpendicular to the line of sight) are roughly

$$0.555\hat{\imath} + 0.835\hat{\jmath} \quad and \quad -0.555\hat{\imath} - 0.835\hat{\jmath}$$

d.
The angular speed of the two jets are found from the data to be:
$1.654 \times 10^{-5} rad/s$ and $0.596 \times 10^{-5} rad/s$
Now, if the jets have velocity v and are oriented at an angle θ to the line of sight, we have the angular velocity seen of the jets to be (note: Reason this yourself. Note: Light reaching us from the jets has a velocity 'c'. So, the light emitted a time t from a jet directed toward or away from us reaches us with a time gap of less than or more than t.

$$\frac{v}{d} \frac{\sin\theta}{1 - (v/c)\cos\theta}$$

and

$$\frac{v}{d} \frac{\sin\theta}{1 - (v/c)\cos\theta}$$

Equating these with the observed angular velocities, we see:

$$\frac{\beta \sin\theta}{1 - \beta\cos\theta} = 1.268$$

$$\frac{\beta \sin\theta}{1 - \beta\cos\theta} = 0.457$$

where v/c is denoted by β.
Solving these gives:

$$\beta = 0.82 \quad \theta = 55^\circ$$

Thus, the jets have velocity $2.46 \times 10^8 m/s$ and are directed 55° to the line of sight.

5 Space Ship at risk

1. Observer 1 $\rightarrow$ Curve 1; Observer 2 $\rightarrow$ Curve 2
2. Time period= distance between two peaks = 25s
3. If freq of source is f_0 then if velocity of used signal in free space is v_0 then $f_{out}^{max/min} = \frac{f_0}{1 \mp \frac{\omega R}{v_0}}$. Also, $f_{in}^{max/min} = \frac{f_0}{1 \mp \frac{\omega D}{v_0}}$. If $v = \omega R$ then $f_0 = 991$ Hz and v=78.6 m/s. As $R\frac{2\pi}{T} = 78.6$ hence R=313 m.
4. $\omega = \frac{\pi}{12.5}$. Time difference between maximum and minimum frequency is 12.5 s. Hence angular difference between them is π radian.Hence outer object must be at infinity or the distance is much much greater than the radius of circle.Also inner observer distance from center is 126 m.
5. The distance of outer observer would have been finite.
6. $f_0(1+n)$ is maximum; $f_0(1-n)$ is minimum.
7. Next radii are $\frac{C}{n^i(1+n)}$ where $C = \frac{2GM}{v^2}$ where v is the initial velocity and i is odd numbers.
8. Point of null shift: same velocity of wrecked ship i.e. v as observed from a far place; let's say Earth.
9. D+0.001*C; D-0.001*C
10. $10\sqrt{1 - \frac{v^2 n^{4k-2}}{c^2}}$

6 Doraemon: Nobita's Physics Day

Question 1
First expression comes directly from sum of all the probabilities is 1. Second expression is a corollary of the first expression. For the integral, use the limit of sum as integral.

Question 2:
Part A: The units each have two possible configurations: horizontal and vertical. There is only one energy consideration - the hanging mass. The chains have themselves no preferred orientation. Let us consider that when all the chains are horizontal, the energy is 0. When n units are vertical, the total energy is $mgan$ (as the mass acquires potential energy). Number of such configurations are $\binom{N}{n}$. Now referring to the second expression from question 1 we get:

$$Z = \sum_n \binom{N}{n} e^{-mgna/kT} = (1 + e^{-mga/kT})^N$$

Part B: The probability that the chain has length na is $p_n = \frac{\binom{N}{n} e^{-mgna}}{Z}$. The average length is given by: $\sum_n (na)p_n$. The result is: $L_{avg} = \frac{Na}{1+e^{-mgNa/kT}}$

Part C: It is evident from the relation obtained in previous part that as temperature increases, the length of the chain decreases. When temperature is 0, the length is Na. This is because the system acquires the state with minimum energy. When the temperature is infinity, the system is in the state of highest disorder. This leads to the expectation that on average, N/2 units will be horizontal and rest will be vertical, thus a length of Na/2. This is confirmed from the relation obtained.

Part D: (The quantity F was meant to be force = mg. But F is a standard symbol for Helmholtz free energy. Due to lack of clarity, who ever attempted this sub part was awarded the marks.)

Question 3:
As the box is insulated, the process is adiabatic thus isentropic. From the standard relation $\frac{\partial S}{\partial E} = \frac{1}{T}$. Computing this partial derivative and rewriting the equation in terms of V and T we get:

$$S \propto T^3 V$$

where. From here when S is constant and V is changed to $(1+\alpha)V$, the new temperature is $\frac{T}{(1+\alpha)^{\frac{1}{3}}}$. Now from Wein's law, the final wavelength is found to be $\lambda_f = \lambda(1+\alpha)^{\frac{1}{3}}$.

Question 4:
Part A: Suppose that the subsystem has energy E_j. As subsystem and reservoir forms a closed system, principle of equal a priori probability holds. Thus, probability is: $\frac{\Omega_{res}(E_T - E_j)}{\Omega_T(E_T)}$

Part B: Using relation between Ω and S we can write the probability as: $\frac{exp(S_{res}(E_T - E_j)/k)}{exp(S_T(E_T)/k)}$. Now, we see that the entropy is additive. Thus, $S_T(E_T) = S(U) + S_{res}(E_T - U)$ where U is the average energy of the subsystem, S is entropy of the subsystem and $S_{res}(E_T - E_j) = S_{res}(E_T - U) + (U - E_j)/T$ where we have used that $U - E_j$ is very small $\frac{\partial S}{\partial E} = \frac{1}{T}$. Substituting these we get the probability as: $e^{\frac{U - TS(U)}{kT}} e^{-\frac{E_j}{kT}}$. From this and comparing with $\frac{1}{Z} e^{-\frac{E_j}{kT}}$, we get the desired result.

Question 5:
Part A: The coordinate system to be used is polar coordinate. Place the two dipoles along z axis. The partition function will be:

$$Z \sim \int\int\int\int e^{-\frac{\beta}{a^3}(sin\theta_1 sin\theta_2 cos(\phi_1 - \phi_2) - 2cos\theta_1 cos\theta_2)} sin\theta_1 sin\theta_2 \, d\theta_1 d\theta_2 d\phi_1 d\phi_2$$

Now, on high temperature limit, the exponential can be expanded and the integral can be computed by neglecting higher terms. The first non-zero term is $\frac{1}{a^6}$ dependence, thus so is the energy. The force thus has $\frac{1}{a^7}$ dependence.

Part B: The average energy is a scalar value and is independent of any choice of coordinate system chosen. Here, the change from d to -d can be thought of as flipping the coordinate system for one dipole where p goes to $-p$ (this negative can be absorbed in distance). Now, as the the final result should be independent, the odd powers of the expansion will all vanish. To be precise, any odd powers of the energy term in the expansion in the exponential will go to zero on integration and as the energy term has odd power in d, only even powers in d will remain.

Mathematically, looking at the odd powers of the expansion of the exponential, it looks like

$$(sin\theta_1 sin\theta_2 cos(\phi_1 - \phi_2) - 2cos\theta_1 cos\theta_2)^n sin\theta_1 sin\theta_2 \sim (sin\theta_1 sin\theta_2 cos(\phi_1 - \phi_2))^p (cos\theta_1 cos\theta_2)^q sin\theta_1 sin\theta_2$$

where $p+q = n$. Now when n is odd, either p is odd or q is odd. When p is odd, $\int_0^{2\pi} \int_0^{2\pi} cos^p(\phi_1 - \phi_2)\, d\phi_1 d\phi_2 = 0$ and when q is odd, then $\int_0^{\pi} sin^p\theta_1 cos^q\theta_1\, d\theta_1 = 0$. Thus, no odd terms remains.

7 Quantum Demon pushes the wall: Fast or Slow?

1. n^{th} normalised energy eigen function is

$$\Psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) & 0 \le x \le a \\ 0 & otherwise \end{cases}$$

   and energy eigen value is

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

2. Instantaneous normalised energy eigen function:

$$u_n(x,t) = \sqrt{\frac{2}{w(t)}}\sin\left(\frac{n\pi x}{w(t)}\right)$$

   Instantaneous energy eigen value:

$$E_n(t) = \frac{n^2\pi^2\hbar^2}{2mw(t)^2}$$

$$\begin{aligned}\int_0^t E(t')dt' &= \int_0^t \frac{n^2\pi^2\hbar^2}{2m(a+vt')^2}dt' \\ &= \frac{n^2\pi^2\hbar^2}{2ma^2}\frac{at}{a+vt} \\ &= E_n^i\frac{at}{w(t)}\end{aligned}$$

   where E_n^i is the nth energy eigen value of initial particle in a box
   Substituting these in the given expression of $\Phi_n(x,t)$, we get:

$$\Phi_n(x,t) = \sqrt{\frac{2}{w(t)}}\sin\left(\frac{n\pi x}{w(t)}\right)\exp\left(\frac{i(mvx^2 - 2E_n^i at)}{2\hbar w(t)}\right)$$

   Boundary conditions:

   Coordinates of wall are $x = 0, x = w(t)$. For $x = 0, \sin\left(\frac{n\pi x}{w(t)}\right) = 0$, for $x = w(t), \sin\left(\frac{n\pi x}{w(t)}\right) = \sin(n\pi) = 0$. Therefore, $\Phi_n(0,t) = 0, \Phi_n(w(t),t) = 0$. Clearly, boundary conditions are satisfied.

   TDSE:

   Let $\phi(x,t) = \frac{(mvx^2 - 2E_n^i at)}{2\hbar w(t)}$ then $\Phi_n(x,t) = \sqrt{\frac{2}{w(t)}}\sin\left(\frac{n\pi x}{w(t)}\right)\exp(i\phi)$

   TDSE reads:

$$H\Phi_n = i\hbar\frac{\partial\Phi_n}{\partial t}$$

$$\text{RHS} = i\hbar\frac{\partial\Phi_n}{\partial t} = -i\hbar\left[\frac{v}{2w} + \frac{n\pi xv}{w^2}\cot\frac{n\pi}{w}x + i\frac{E_n^i a}{\hbar w} + i\frac{v}{w}\phi\right]\Phi_n$$

$$\text{LHS} = H\Phi_n = -\frac{\hbar^2}{2m}\frac{\partial^2\Phi_n}{\partial x^2} = -\frac{\hbar^2}{2m}\left\{-\left(\frac{n\pi}{w}\right)^2\csc^2\frac{n\pi}{w}x + \frac{imv}{\hbar w} + \left[\frac{n\pi}{w}\cot\frac{n\pi}{w}x + i\frac{mvx}{\hbar w}\right]^2\right\}$$

   It is easy to check that the trigonometric terms and the other coefficients in both LHS and RHS match. Hence, $\Phi_n(x,t)$ satisfies Schrodinger wave equation.

3. Consider the expression,

$$\Psi(x,t) = \sum_n c_n\Phi_n(x,t)$$

But at t = 0,

$$\Psi(x,0) = \sum_n c_n \Phi_n(x,0)$$
$$= \sqrt{\frac{2}{a}} \sum_n c_n \sin\left(\frac{n\pi x}{a}\right) \exp\left(\frac{imvx^2}{2\hbar a}\right)$$

Multiply by $\Phi^*_{n'}(x,0)$ on both sides of the above equation and integrate from $x = 0$ to a to get:

$$\int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n'\pi x}{a}\right) \exp\left(\frac{-imvx^2}{2\hbar a}\right) \Psi(x,0)\ dx = \sum_n c_n \int_0^a (\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right))(\sqrt{\frac{2}{a}} \sin\left(\frac{n'\pi x}{a}\right))\ dx$$
$$= \sum_n c_n \delta_{nn'}$$
$$= c_{n'}$$

We used orthonormality of usual particle in a box energy eigen functions above i.e. $\int_0^a (\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right))(\sqrt{\frac{2}{a}} \sin\left(\frac{n'\pi x}{a}\right))dx = \delta_{nn'}$

Therefore, (replacing n' by n since its a dummy variable)

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \exp\left(\frac{-imvx^2}{2\hbar a}\right) \sin\left(\frac{n\pi x}{a}\right) \Psi(x,0)\ dx$$

Define $z = \frac{\pi x}{a}$, a dimensionless constant $\alpha = \frac{mva}{2\hbar\pi^2}$ and rewriting c_n in terms of these,

$$c_n = \frac{\sqrt{2a}}{\pi} \int_0^\pi \sin(nz)\Psi(\frac{za}{\pi},0) \exp\left(-i\alpha z^2\right)\ dz$$

4. By definition of T_{ext},

$$vT_{ext} = 2a - a$$
$$\implies T_{ext} = \frac{a}{v}$$

Time-dependent exponential factor in the ground state wave function of *usual* particle in a box is $\exp(-iE_1 t/\hbar)$ with $E_1 = \frac{\hbar^2\pi^2}{2ma^2}$

Therefore, time period $= 2\pi\hbar/E_1$

Hence, by definition,

$$T_{int} = \frac{4ma^2}{\hbar\pi}$$

5. i. Consider the limit,

$$T_{ext} >> T_{int}$$
$$\frac{a}{v} >> \frac{4ma^2}{\hbar\pi}$$
$$1 >> \frac{4mva}{\hbar\pi} = 8\pi\alpha$$
$$\implies 1 >> \alpha$$

It is given that $\Psi(x,0) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$ Using the expression in (3.), we get

$$c_n = \frac{2}{\pi} \int_0^\pi \sin(nz)\sin(z) \exp\left(-i\alpha z^2\right)\ dz$$

In the limit, $\alpha << 1$, $\exp\bigl(-i\alpha z^2\bigr) \approx 1$. Therefore,

$$c_n = \frac{2}{\pi}\int_0^{\pi} \sin(nz)\sin(z)\ dz = \delta_{n1}$$

Now,

$$\begin{aligned}
\Psi(x,t) &= \sum_n c_n \Phi_n(x,t) \\
&= \sum_n \delta_{n1}\Phi_n(x,t) \\
&= \Phi_1(x,t) \\
&= \sqrt{\frac{2}{w(t)}}\sin\left(\frac{\pi x}{w(t)}\right)\exp\left(\frac{i(mvx^2 - 2E_1^i at)}{2\hbar w(t)}\right)
\end{aligned}$$

Consider, $\frac{mvx^2}{2\hbar w}$ is atmost $\frac{mva^2}{2\hbar a} = \frac{mva}{2\hbar} = \alpha\pi^2 << 1$. Therefore,

$$\Psi(x,t) \approx \sqrt{\frac{2}{w(t)}}\sin\left(\frac{\pi x}{w(t)}\right)\exp\left(\frac{-iE_1^i at}{\hbar w(t)}\right)$$

ii. Initially the particle is in its ground state. In the adiabatic limit, the wave function remains in its instantaneous energy eigen state(ground state) which is evident from first two terms and the phase factor is accumulation over the time as follows:

$$\begin{aligned}
E_1(t) &= \frac{\pi^2\hbar^2}{2mw(t)^2} \\
\int_0^t E_1(t')dt' &= \int_0^t \frac{\pi^2\hbar^2}{2m(a+vt')^2}dt' \\
&= \frac{\pi^2\hbar^2}{2ma^2}\frac{at}{a+vt} \\
&= E_1^i\frac{at}{w(t)}
\end{aligned}$$

$$\Psi(x,t) \approx \sqrt{\frac{2}{w(t)}}\sin\left(\frac{\pi x}{w(t)}\right)\exp(i\phi)$$

The phase factor ϕ is therefore:

$$\phi = \frac{-1}{\hbar}\int_0^t E(t')\ dt'$$

Which is accumulation of the phase from t=0 to t because energy is time dependent. Therefore, it is clear that the wave function is consistent with the adiabatic theorem.

iii. Change in energy of the particle quantum mechanically is

$$\Delta E = \frac{\hbar^2\pi^2}{2ma^2} - \frac{\hbar^2\pi^2}{2m(2a)^2} = \frac{3}{4}\frac{\hbar^2\pi^2}{2ma^2} = \frac{3}{4}E_i$$

Classically, calculating the change in energy of the particle as follows. In a time period of $2s/v$, particle undergoes a momentum change of $2mv$ where s is the instantaneous separation between the walls. Hence, average force acting on the particle is $\frac{2mv}{(2s/v)} = mv^2/s$ and the force is retarding in nature. Writing newton's second law of motion for the particle,

$$F = m\frac{dv}{dt}$$

$$F = -\frac{mv^2}{s}$$

$$\text{But, } m\frac{dv}{dt} = mv\frac{dv}{ds}$$

$$\implies vds + sdv = 0$$

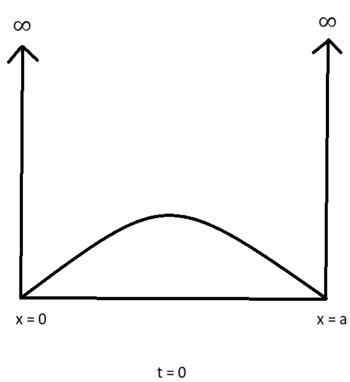
$$\implies v(s) = \frac{v_0 a}{s}$$

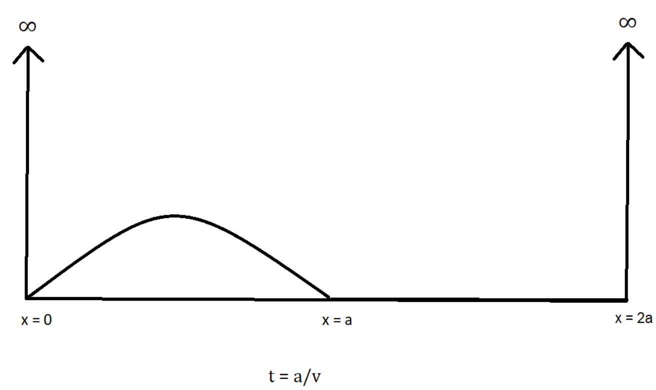
At $s = 2a$, speed of the particle is half of its initial speed. Hence, change in kinetic energy is $\frac{3}{4}E_i$ which agrees with the quantum mechanical result.

6. i. Expectation value of energy $= \langle\Psi| H |\Psi\rangle = \int \Psi^* H \Psi dx$

$H = -\hbar^2 \frac{d^2}{dx^2}$

Hamiltonian at t=0 is the above expression between x=0 and x=a and at t=a/v, Hamiltonian is the same expression between x=0 and x=2a. However, the wave function remains same at t=0 an t=a/v between x=0 and x=a and it is zero between x=a and x=2a at t=a/v. Hence, the expectation value of energy remains same and is equal to the ground state energy.





ii. Since the wall moves fast enough without allowing any collision between the wall and the particle, energy of the particle remains same.

iii.

$$\Psi_f(x) = \sum_n c_n \phi_n(x)$$

where $\{\phi_n(x)\}_{n=1}^{\infty}$ are the normalised eigenfunctions of the new hamiltonian. Since we know that they are orthonormal,

$$c_n = \langle\phi_n|\Psi_f\rangle$$

$$\Psi_f(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) & 0 \le x \le a \\ 0 & otherwise \end{cases}$$

$$\phi_n(x) = \begin{cases} \sqrt{\frac{1}{a}} \sin\left(\frac{n\pi x}{2a}\right) & 0 \le x \le 2a \\ 0 & otherwise \end{cases}$$

Therefore,

$$\begin{aligned} c_n &= \int_0^a \sqrt{\frac{1}{a}} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{n\pi x}{2a}\right) \; dx \\ &= \frac{\sqrt{2}}{\pi} \int_0^{\pi} \sin\left(\frac{nt}{2}\right) \sin(t) \; dt \end{aligned}$$

Evaluating the above integral, we get:

$$c_n = \begin{cases} \frac{1}{\sqrt{2}} & n = 2 \\ \frac{4\sqrt{2}}{\pi} \frac{\sin(n\pi/2)}{n^2-4} & otherwise \end{cases}$$

$$P_n = |c_n|^2 = \begin{cases} \frac{1}{2} & n = 2 \\ \frac{32}{\pi^2} \frac{1}{(n^2-4)^2} & n = \text{odd number} \\ 0 & n = \text{even number except 2} \end{cases}$$

8 The Diffracting Electrons

Question 1
The wavelength of the electrons are comparable to the crystal spacing.

The fringes are circular due to different orinetation of the crystal.

Question 4
Use relation $p = \frac{h}{\lambda}$. For relativistic particle, use $E = \sqrt(p^2c^2 + m^2c^4) = qV$. The values are to be inserted for the numerical to get the answer.

Question 5
Note: A model solution is given. The numbers may vary based on the errors considered etc. Shorter but correct answers were also considered and awarded score.

Analysis

8.1 a.

8.1.1 The steps

1. The angle was obtained from the relation $\phi_{in} = \tan\left(\frac{D_{i,in}/2}{L}\right)$ and $\phi_{out} = \tan\left(\frac{D_{i,out}/2}{L}\right)$ where $l = 13.5cm$ is the length of the screen from the graphite crystal and i can be 1 or 2.
2. Now this angle is from the direction of the main light beam. Thus $\phi = 2\theta$.
3. The ϕ_{in} and ϕ_{out} corresponds to the inner and outer diameter of the received diffracted ray. The spread is due to the dispersion caused by several reasons like crystal finiteness, impurities etc. For θ, the average of the two are taken. Thus, $\theta = \frac{\phi_{in}+\phi_{out}}{4}$.
4. The angles are averaged and not the average diameter was taken as the spread is angular. The peak in the whole signal will lie approximately in the middle of the angular spread and not the diameter spread on the screen.
5. The value of beta is given by $\delta 2\theta$. Thus, $\beta = \frac{\phi_{out}-\phi_{in}}{2}$.

8.1.2 Error analysis

1. $\phi = \tan\left(\frac{D/2}{L}\right)$. Let $D/2 = R$ Now, the error in D has been taken as 0.01mm. Thus, the error in R is 0.005mm. The error in the L is taken as 0.1cm = 1mm. Therefore, $\delta\phi = \frac{1}{1+(\frac{R}{L})^2}(\frac{\delta R}{L} + \frac{R\delta L}{L^2})$. This has been used for both ϕ_{in} and ϕ_{out}.
2. From above we get, $\delta\theta = \frac{\delta\phi_{in}+\phi_{out}}{4}$.
3. Similarly, $\delta\beta = \frac{\delta\phi_{in}+\phi_{out}}{2}$.
4. This has been followed for both 1 and 2 (i.e. d_1 and d_2 data).

8.1.3 The tables

V(kV)	δV (kV)	θ (radians)	$\delta\theta$ (radians)	β (radians)	$\delta\beta$ (radians)
2	0.1	0.0649	0.0005	0.0076	0.0010
2.5	0.1	0.0585	0.0004	0.0087	0.0009
3	0.1	0.0525	0.0004	0.0089	0.0008
3.5	0.1	0.0498	0.0004	0.0074	0.0008
4	0.1	0.0461	0.0004	0.0058	0.0007
4.5	0.1	0.0441	0.0003	0.0057	0.0007
5	0.1	0.0404	0.0003	0.0047	0.0006

Table 1: Table for d_1

V(kV)	δV (kV)	θ (radians)	$\delta\theta$ (radians)	β (radians)	$\delta\beta$ (radians)
2	0.1	0.1138	0.0008	0.0096	0.0017
2.5	0.1	0.1012	0.0007	0.0089	0.0015
3	0.1	0.0922	0.0007	0.0068	0.0014
3.5	0.1	0.0856	0.0006	0.0070	0.0013
4	0.1	0.0787	0.0006	0.0059	0.0012
4.5	0.1	0.0744	0.0006	0.0068	0.0011
5	0.1	0.0717	0.0005	0.0071	0.0011

Table 2: Table for d_2

8.2 b.

8.2.1 Method

1. To calculate the wavelength theoretically by considering and not considering relativistic effect is done.
2. Given d_1, and the values of θ obtained, the values for wavelengths can be calculated as $\lambda = 2d\sin\theta$. Here $n = 1$ as the primary peak is taken.

8.2.2 Error analysis

1. The value of d_1 is taken to be a constant as given in the manual.
2. From the relation, $\lambda = 2d\sin\theta$, we get $\delta\lambda = 2d\sin\theta\delta\theta$. The values of $\delta\theta$ has already been found and tabulated earlier.
3. In the theoretical calculation, the error in the voltage measurement has been taken as 0.1kV. Others are taken to be constants.
4. For the non relativistic case, $\frac{\delta\lambda}{\lambda} = \frac{\delta V}{2V}$.
5. For the relativistic case, upon differentiation and division, we get: $\frac{\delta\lambda}{\lambda} = \frac{\delta V}{V}\frac{qV+c^2m}{qV+2c^2m}$.

8.2.3 Table containing the experimental and theoretical wavelength values

All wavelengths are in $\mathring{A}$.

λ(exp)	$\delta\lambda$(exp)	λ(th,non relativistic)	$\delta\lambda$(th,non relativistic)	λ(th,relativistic)	$\delta\lambda$(th,relativistic)
0.276	0.002	0.275	0.007	0.274	0.007
0.249	0.002	0.246	0.005	0.245	0.005
0.224	0.002	0.224	0.004	0.224	0.004
0.212	0.002	0.208	0.003	0.207	0.003
0.197	0.002	0.194	0.002	0.194	0.002
0.188	0.001	0.183	0.002	0.183	0.002
0.172	0.001	0.174	0.002	0.173	0.002

8.2.4 Comments

We see that the values dont actually match, the relativistic value being further away. Though they have the same orders of magnetide and in a sense they match very well till around the first significant figure, from the subsequent figures they tend to differ. The experimental value is always larger than the theoretical value. The reason for this error may be the following:

1. The experimental errors including error in data collection or the instrumental errors.
2. The voltage was lower than the displayed voltage and thus the electrons were accelerated by some lower voltage, gaining less momentum and thus increasing the wavelength.
3. There are energy losses from friction due to air or when colliding with the graphite layers. These drag take away some of the energy, reduce the momentum of the particle and thus increases its wavelength from the expected theoretical value.

8.3 c.

8.3.1 Method and Error anslysis

1. The graph plotted is between the wavelength (using the theoretical relativistic value as instructed) and the sin of the angle. The wavelength λ is in the y axis and the $\sin\theta$ in the x axis.
2. The slope is $2d_2$. The value of d_2 is thus half the slope.
3. The error of θ has already been listed. The error for $\sin\theta$ is determined by using relation: $\delta(\sin\theta) = \delta\theta\sin\theta$. Error calculation for relativistic wavelength have already been discussed.
4. The error in the d_2 is half of error in slope of the straight line fit.

8.3.2 Graph and the fit

The wavelengths are plotted in angstorms.

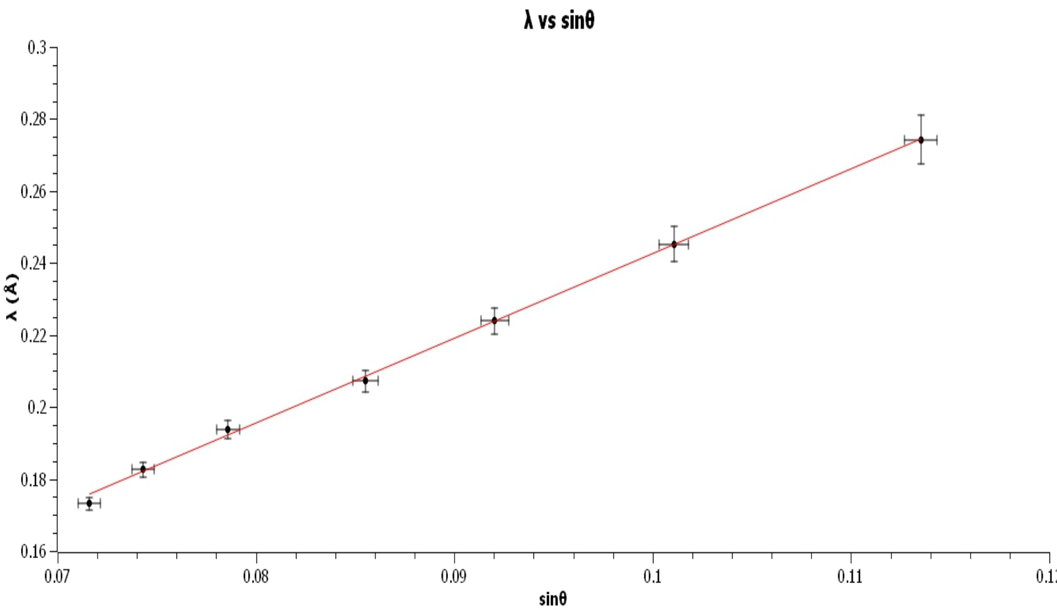


Figure 2: Wavelength vs $\sin\theta$

$a * x + b$
$a = 2.35529124733156 \pm 0.0767110046594202$
$b = 0.00699168024244256 \pm 0.00762437795633649$
$\chi^2 = 0.521779938835952$
$R^2 = 0.999446811466867$

8.3.3 Determination of d_2

As we see from the graph, the intercept is very small. This is actually expected as the fit ideally should be of type y=mx. The straight line fit verifies the Bragg's relation for the electrons.
The slope is: 2.36 ± 0.08
Thus, $d_2 = 1.18 \pm 0.04$ angstorms.

The expected (literature) value is around 1.23 angstorm. The obtained value is quite close. The error is: $\frac{1.23-1.18}{1.23} \times 100 = 4.07\%$ which is well within the experimental error limits.

8.4 d.

8.4.1 Method

Here an approximation method have been used. As the θ values are coming small, the following approximation have been done: $\sin\theta = \tan\theta = \theta$. Now, from Bragg's law: $2d\sin\theta = \lambda$, where $\theta = 0.5 \times \tan^{-}1\frac{D}{2L}$. Now, from the previous approximation: $\theta = \frac{D}{2L}$. Now we have seen that the correction for the relativistic speed is very little. Thus, for ease of plotting and fitting, we here use the non-relativistic wavelength of the electron. Therefore, the final relation is:

$$D = \frac{2Lh}{d\sqrt{2mqV}} \tag{5}$$

Thus, the slope of D vs $\frac{1}{\sqrt{V}}$ can give the value of data. This has been done with d_1.

8.4.2 Graph and fit

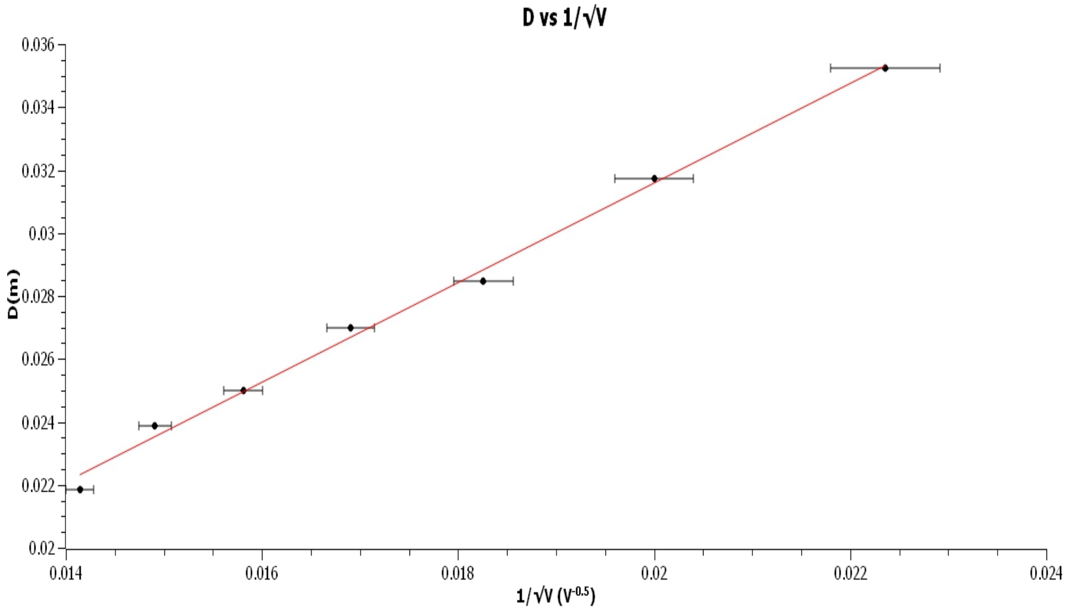


Figure 3: D vs $\frac{1}{\sqrt{V}}$

$a * x + b$
$a = 1.58295886714368 \pm 0.00139138869154946$
$b = -8.29140838924684e - 05 \pm 0.00139138869154946$
$\chi^2 = 5,922.96378423223$
$R^2 = 0.995444732254607$

8.4.3 Value of h

We fine the intercept is really small. This says the approximation is well valid and also that the expected $y = mx$ fit is what is true. Still the error in the intercept is large. We see that the slope is: 1.583 ± 0.001. The slope in turn is: $\frac{Lh}{\sqrt{2mq}}$. Substituting the other values we get:
$h = (6.7 \pm 0.4) \times 10^{-34}$
The literature value is: 6.626×10^{-34}
We see that there is quite close to the literature value.

8.5 e.

The Williamson hall method gives the following relation:

$$\beta = \frac{K\lambda}{D\cos\theta} + k\epsilon\tan\theta \tag{6}$$

Now, multiplying oth sides by $\cot\theta$ we get:

$$\beta cot\theta = \frac{K\lambda cosec\theta}{D} + k\epsilon \tag{7}$$

Now, we find that $\frac{\lambda}{cosec\theta}$ is a constant (as from Bragg's law, $\lambda cosec\theta = 2d$ where d is the crystal plane spacing). Thus finally we get that both $\lambda cosec\theta$ and $\beta cot\theta$ should be same for all θ given a d. Thus, the plot between them should theoretically be a point.

8.5.1 Data and Graph

$\beta cosec\theta$	$\delta\beta cot\theta$	$\lambda cosec\theta$	$\delta\lambda cosec\theta$
1.80	0.26	4.23	0.06
2.53	0.30	4.20	0.06
3.21	0.34	4.27	0.07
3.00	0.36	4.16	0.07
2.74	0.38	4.20	0.07
2.91	0.40	4.14	0.07
2.86	0.43	4.29	0.07

Table 3: d_1 data

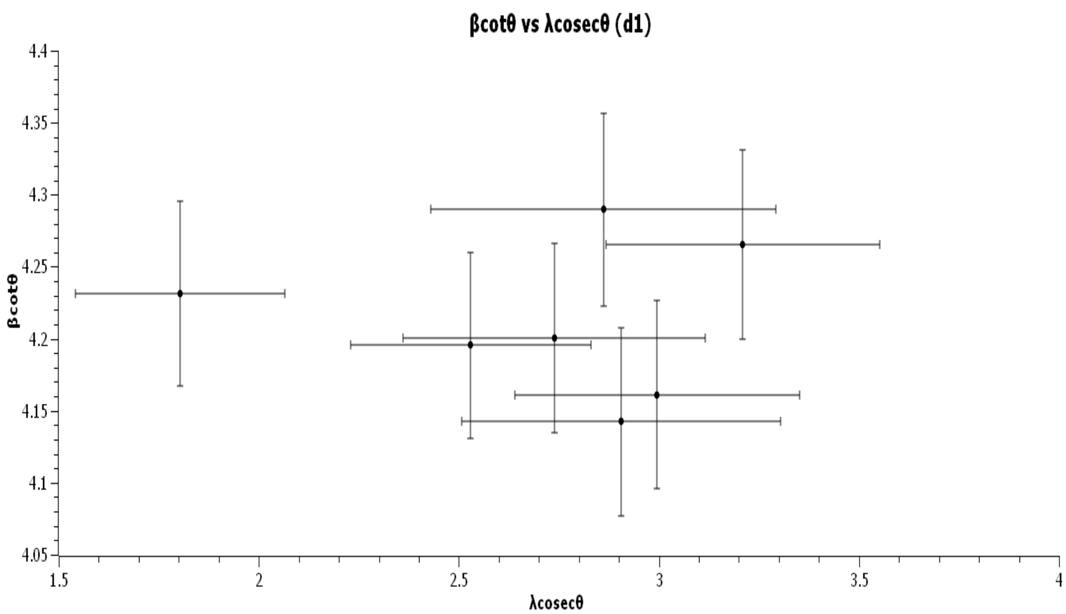


Figure 4: $\beta cot\theta$ vs $\lambda cosec\theta$ for d_1

$\beta cot\theta$	$\delta\beta cot\theta$	$\lambda cosec\theta$	$\delta\lambda cosec\theta$
0.084	0.015	2.42	0.08
0.088	0.015	2.43	0.07
0.074	0.015	2.43	0.06
0.082	0.016	2.42	0.05
0.075	0.016	2.47	0.05
0.091	0.016	2.46	0.05
0.099	0.016	2.42	0.04

Table 4: d_2 data

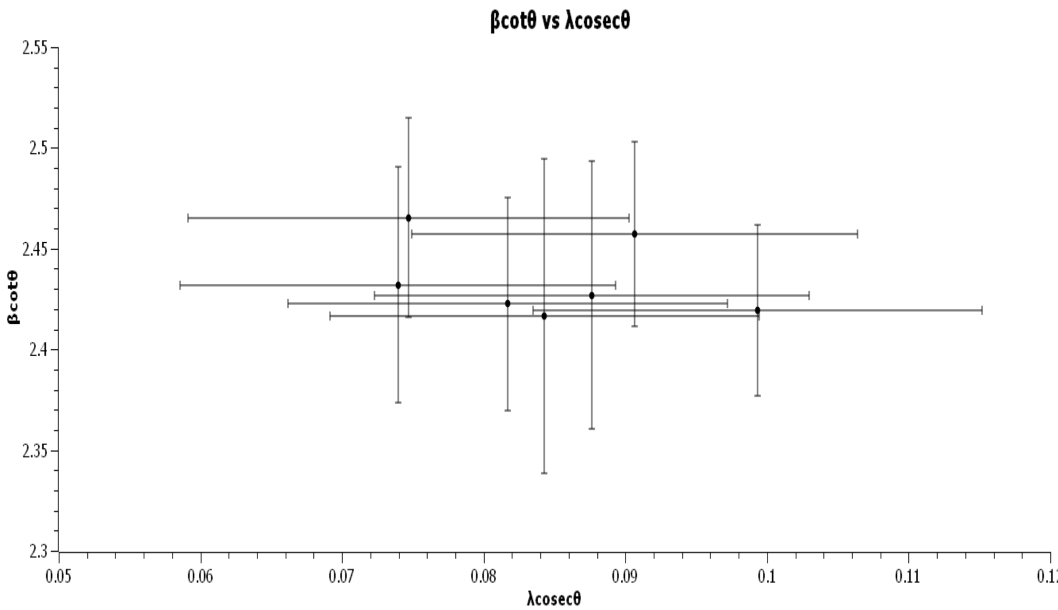


Figure 5: $\beta cot\theta$ vs $\lambda cosec\theta$ for d_2

8.5.2 Comments

We see that in both the plots, the data are very clustered in a small region. Thus, the Williamson Hall method is verified. Also, the error bars have some common intersection too. This shows that the data is well clustered and qualifies to be called a point data. The deviation from the perfect point may be because of following reasons:

1. The relativistic wavelength value used is not the actual one as we saw that there is deviation from the experimentally determined value.
2. The values are not perfect in itself (instrument error, precision error etc).

8.6 f.

The sample is quite thin and the number of interplanes are small in number. This leads to too much broadening of the peaks and thus we cannot see them.