

Gaussian Gamble: Problem Set 2

September 4, 2020

Maximum possible score : 25

The deadline to submit solutions is 11:59 PM, 11th September 2020.

Partial solutions will receive partial marking, so mention any result you may have obtained, including any conjectures that you make.

- Q1. (a) Define a sequence x_n as $x_0 = 0$ and $x_n = \sqrt{2 + x_{n-1}}$. Define another sequence $\{v_i\}$ as $v_i = \frac{x_i}{2}$. Prove that the following limit exists and find its value. (1)

$$\lim_{k \rightarrow +\infty} \prod_{n=1}^k v_n$$

- (b) Using part (a) or otherwise find the value of (3)

$$\frac{\sqrt{2}}{1 + \sqrt{2}} \cdot \frac{\sqrt{2 + \sqrt{2}}}{1 + \sqrt{\sqrt{2}}} \cdot \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{1 + \sqrt{\sqrt{\sqrt{2}}}} \dots$$

- Q2. A simple graph is an unweighted, undirected graph containing no self loops or multiple edges among its vertices. How many connected simple graphs exist such that

- (a) The sum of the degrees of the vertices is equal to the product of the degrees of the vertices. (2)

- (b) The sum of the squares of the degrees of the vertices is equal to the product of the degrees of the vertices. (4)

- Q3. Let A and B be two $n \times n$ matrices such that every entry is real. If $C = AB - BA$ is an invertible matrix and $A^2 + B^2 = \sqrt{3}C$, show that n is not a prime. (7)

Q4. The community of ‘Mathomaniacs’ is spread over a large area of land, which is in the shape of a convex polygon having n sides. To oversee the land, a lighthouse is present at every vertex. For better administration, their leader decides to break up the territory into smaller, mutually disjoint regions. A region must be in the shape of a convex polygon with its vertices at the lighthouses.

Their leader forms two teams A and B in order to enumerate the various possibilities for division. Team A is asked to present various ways in which the territory can be broken up into an odd number of regions, and B is asked to do the same for an even number of regions.

(a) Is it possible for any positive integer n that the two teams come up with an equal number of ways to split the territory? (2)

(b) Let the number of ways team A comes up with be S_A , with S_B being defined similarly. Find $S_A - S_B$ in terms of n . (6)

PLEASE NOTE: Submit your solutions in a file named as per the format “Team name_pset2.pdf”. For example, if your team name is 123, your file should be named “123_pset 2.pdf”.