Questions for Deco

September 9, 2020

Consider a uniform sphere of mass M_s that is constant with time. Imagine that we have an observer sitting at the edge of this sphere, and is glued to the surface of the sphere, so that *any motion of the observer corresponds to a change in the radius of the sphere itself,* described by a function $R_s(t)$ Suppose at some time t_0 the radius of the sphere is R_0. Let the function $R_s(t)$ be such that it obeys

$$R_s(t) = a(t)R_0$$

(a) Assume that gravity is the only force at play here. Then show that

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho(t) + \frac{\kappa c^2}{R_0^2 a^2}$$

where $\dot{a} \equiv \frac{da}{dt}$, $\rho(t)$ is the mass density of the sphere and κ is some arbitrary dimensionless constant, as required. What is the physical interpretation of κ (that is how does κ come into this problem)?

(b) Suppose we set $\kappa = 0$. Then show, by dimensional arguments, that the equation can be generalised to

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3c^2}\varepsilon(t)$$

where $\varepsilon(t)$ is the energy density of the body, while c is the speed of light in vacuum.
If you're wondering why we used c and not some arbitrary velocity, please hang on for a moment.

(c) Consider the sphere to be made out of a fluid, undergoing *adiabatic* evolution. Beginning from the first law of thermodynamics, show that the energy density $\varepsilon(t)$ satisfies the following relation :

$$a(t)\dot{\varepsilon} + 3\dot{a}(\varepsilon + p) = 0$$

Suppose the fluid is such that the pressure p is related to the energy density ε as $P = \omega\varepsilon$, which is typically what we see for very dilute fluids. Then show that the following is satisfied :

$$\varepsilon(a) = \varepsilon_0 a^{-3(1+\omega)}$$

From this, write down the exact functional form of both a(t) and $\varepsilon(t)$ in terms of ω and other constants. **State the necessary assumptions for the value of ω.**

(d) (i) For a moment, assume that the particles of the fluid making up the sphere are extremely non-relativistic. Then show that $\omega \approx 0$, where ω is defined as above. Hence state the form of a(t).
(ii) What happens when $\omega = -1$? Write down the form of a(t).
(iii) On the same graph paper, plot the approximate graphs of a(t) as in (i) and (ii), clearly comparing the behaviour at $t \to 0$, t_0 and as $t \to \infty$ (use appropriate scaling).

Now it is finally time to reward you after all this gymnastics. The above exercise was a very very crude way to look at the evolution of our universe, the large scale structure of which is governed by the energy density of all its different components, and the universal constants G and c. The function a(t) is called the *scale factor* and describes how the universe expands (or contracts) with time. Different components have different values for ω - for example, as you saw, nonrelativistic matter has $\omega = 0$. The one having the value of ω as given in (d)-(ii), is called the *dark energy*, which is said to be responsible for the accelerated expansion of our universe (You may have already got a sense of this if you have solved subpart (d)(ii)).

(e) However, you might not be completely convinced, so as a final exercise, show that the instantaneous velocity of the sphere's radius $v(t_*)$ is related to the radius $R_s(t_*)$ by

$$v(t_*) = H_0 R_s(t)$$

where $H_0 \equiv \frac{\dot{a}}{a}|_{t=t_*}$ is a constant. This is the famous Hubble's constant, and the relationship above is analogous to the famous Hubble's Law, that relates the velocity of galaxies to their distance from us.

Distribution of marks : [4+1+6+3+1 = 15 points]