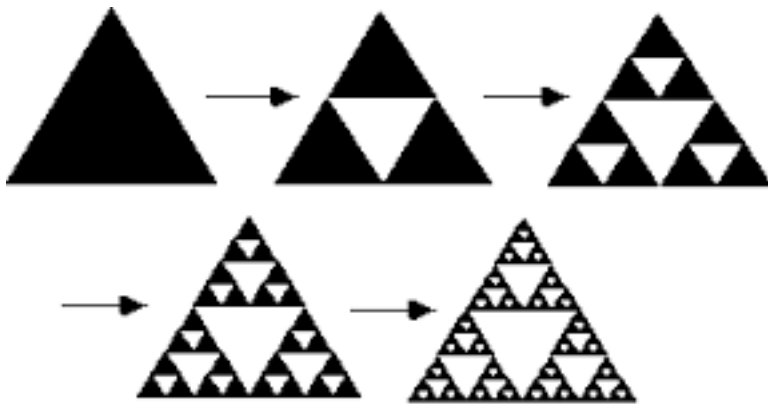




Nature has always been the source for man's motivation—it has always inspired scientific thinking. You must have come across stunning patterns such as the ones given above. Initially, these appear like highly complex shapes – but when you look closer, you might notice that they both follow a relatively simple pattern: all the individual units resemble

closely to their parent shape, albeit a bit tinier. The same pattern is repeated over and over again, at smaller scales. Shapes having this property of **self-similarity** are called **fractals**. While many fractals are *self-similar*, a better definition is that **fractals** are shapes which have a **non-integer dimension**. Find it strange? Not your fault; this is one of the weirdest properties of fractals.

Consider a straight line segment. It obviously has a **dimension 1**. When scaling it by a factor of 2, its length increases by a factor of $2^1=2$. Trivial, right? Next consider a square (**dimension 2**). When scaling it by a factor of 2, its area increases by a factor of $2^2=4$. A cube has **dimension 3**. When scaling it by a factor of 2, its volume increases by a factor of $2^3=8$. Now let's look at a fractal object, the Sierpinski triangle, named after the Polish mathematician Waław Sierpiński. It can be created by starting with one large, equilateral triangle, and then repeatedly cutting smaller triangles out of its center.



The Sierpinski Triangle

If we scale it by a factor of 2, you can see that its “area” increases by a factor of **3**. If d is the dimension then following the pattern above, we note that $2^d = 3$ which implies $d = \log_2 3 \approx 1.585$. What! Yes, a fractal does have a fractional dimension. In fact, the name fractal comes from fractional dimension. In hindsight this makes perfect sense as the Sierpinski triangle isn't a true 2D object (otherwise area would have scaled by a factor of 4 upon doubling the side length) and also not a 1D object—so something between 1 and 2 matches our intuition. You will find something similar for Menger sponge and Koch snowflake (the left pic in the second row).

The Mandelbrot set is also a fractal and has been an intriguing research topic in contemporary mathematics. It is generated by recursion, a technique where you start with a specific number, and then you apply the same recursive formula, again and again, to get the next number in the sequence. The sequence we are interested in is $x_n = x_{n-1}^2 + c$,

where c is a complex number and the sequence itself is in the complex plane. As an example let's list the sequence when the first term $x_0 = 0$ and $c = i$:

$$x_0 = 0$$

$$x_1 = i$$

$$x_2 = -1 + i$$

$$x_3 = -i$$

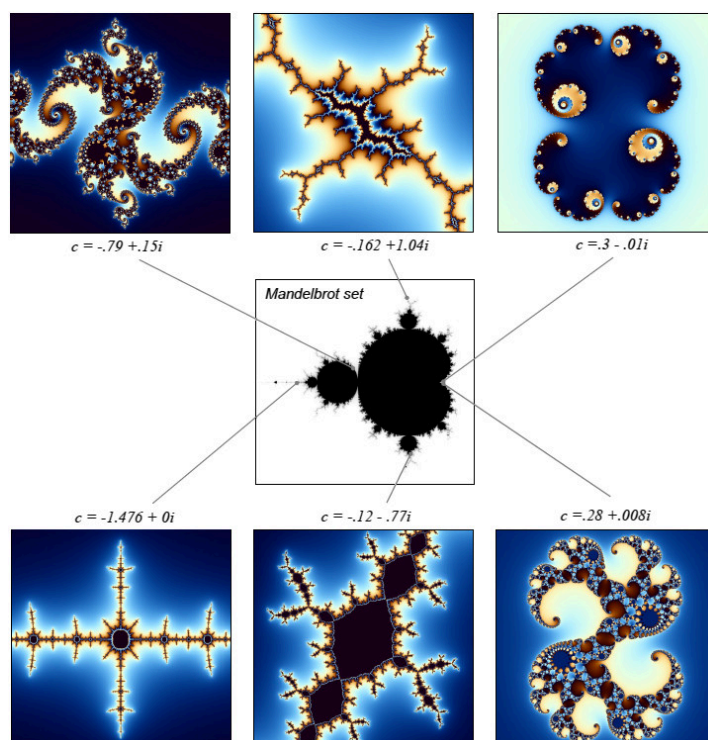
$$x_4 = -1 + i$$

$$x_5 = -i$$

\vdots

Formally: The Mandelbrot set **M** consists of all of those (complex) **c**-values for which the corresponding orbit of **0** ($x_0 = 0$) under $x^2 + c$ does not escape to infinity. One can easily check that $c = 0, i, -1, -1.3, -1.38$, and $-0.7i$ all lie in the Mandelbrot set, whereas $c = 1$ and $c = 2i$ do not. You must be wondering at this point how did we exactly generate this amazing image. To be honest, I cheated a bit: rather than checking whether the sequence diverged to infinity, I just checked whether the modulus of successive sequence elements crossed an arbitrarily chosen value (2, in this case). The colour coding (available in Python Pyplot using `.imshow` method) was based on the number of iterations it took for the sequence to diverge while the colour black represented the Mandelbrot set (region in the complex plane which provided c value which led to convergence of the chosen sequence).

Interestingly enough, if c is held constant and the initial value of **z** (i.e. x_0) is varied instead, one obtains the corresponding Julia set for the point c , like this:



As you move the value of c around the Mandelbrot set, you might notice a curious property:

- All sequences within the main body (the cardioid region) of the Mandelbrot set converge to a single point.
- The sequences within the large bulb at the top and bottom reach an orbit consisting of 3 points.
- Sequences in the smaller bulbs (at roughly $\pm 45^\circ$) have orbits of length 5.

Every bulb has a differently-sized orbit, with smaller bulbs having more and more points in their orbits. The size of these orbits are closely related to the **Logistic Map**, an important concept in Chaos theory.

What I have tried to present here is just a glimpse into the world of Complex Analysis, one of the numerous branches of mathematics. You are encouraged to refer to more resources for a more serious study. The list below is just a comprehensive one and is by no means complete. In case I have missed listing or have incorrectly cited a source, I would like to apologise and assure that it wasn't intentional.

References and Sources:

- <https://mathigon.org/course/fractals/mandelbrot>
- <https://math.bu.edu/DYSYS/FACGEOM/FACGEOM.html>
- <https://www2.math.upenn.edu/~rimmer/math170/notes/unit9fractalchaos.pdf>
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