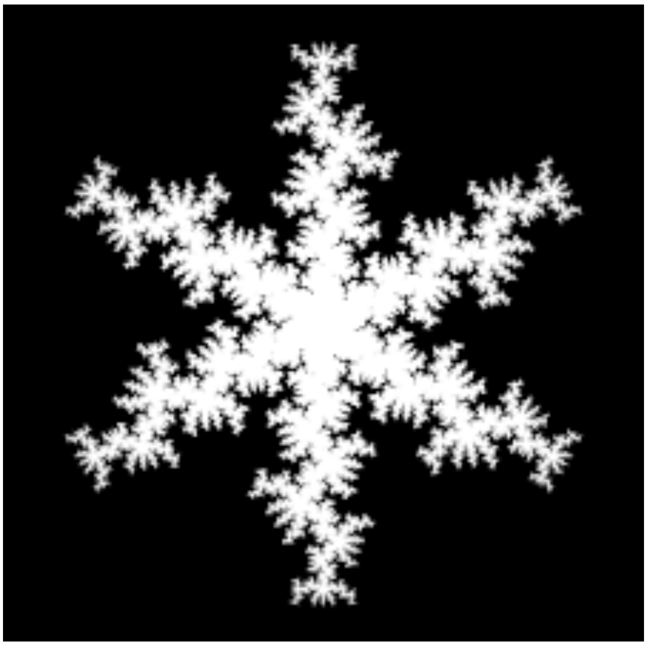
Nature has always been the source for man’s motivation—it has always inspired scientific thinking. You must have come across stunning patterns such as the ones given below.





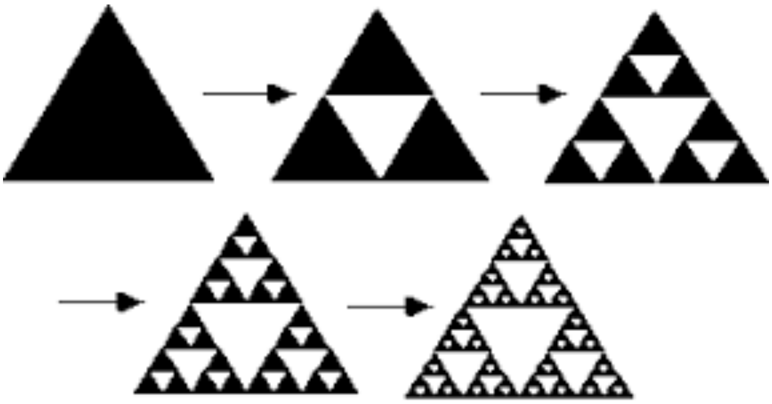
Initially, these appear like highly complex shapes – but when you look closer, you might notice that they both follow a relatively simple pattern: all the individual units resemble closely to their parent shape, albeit a bit tinier. The same pattern is repeated over and over again, at smaller scales. Shapes having this property of **self-similarity** are called **fractals**. While many fractals are *self-similar*, a better definition is that **fractals** are shapes which have a **non-integer dimension**. Find it strange? Not your fault; this is one of the weirdest properties of fractals.

To explain the concept of fractal dimension, it is necessary to understand what we mean by dimension in the first place. Obviously, a line has dimension 1, a plane dimension 2, and a cube dimension 3. But why is this?

People often say that a line has dimension 1 because there is only 1 way to move on a line. Similarly, the plane has dimension 2 because there are 2 directions in which to move. Of course, there really are 2 directions in a line -- backward and forward -- and infinitely many in the plane. What they really are trying to say is there are 2 linearly independent directions in the plane. Of course, they are right. But the notion of linear independence is quite sophisticated and difficult to articulate. An alternative way to specify the dimension of a self-similar object is: The dimension is simply the exponent of the scaling factor (N) so that the scaled figure may be broken into $N^{\text{dimension}}$ number of self similar pieces (each when scaled by the scaling factor, yields the scaled figure).

$$dimension = \frac{log\ Number\ of\ self\ similar\ pieces}{log\ N} = log_N Number\ of\ self\ similar\ pieces$$

Consider a straight line segment. It obviously has a dimension 1. When scaling it by a factor of 2, its length increases by a factor of 2^1=2. Trivial, right? Next consider a square (dimension 2). When scaling it by a factor of 2, its area increases by a factor of 2^2= 4. A cube has dimension 3. When scaling it by a factor of 2, its volume increases by a factor of 2^3= 8. Now let’s look at a fractal object, the Sierpinski triangle, named after the Polish mathematician Wacław Sierpiński. It can be created by starting with one large, equilateral triangle, and then repeatedly cutting smaller triangles out of its center.



The Sierpinski Triangle

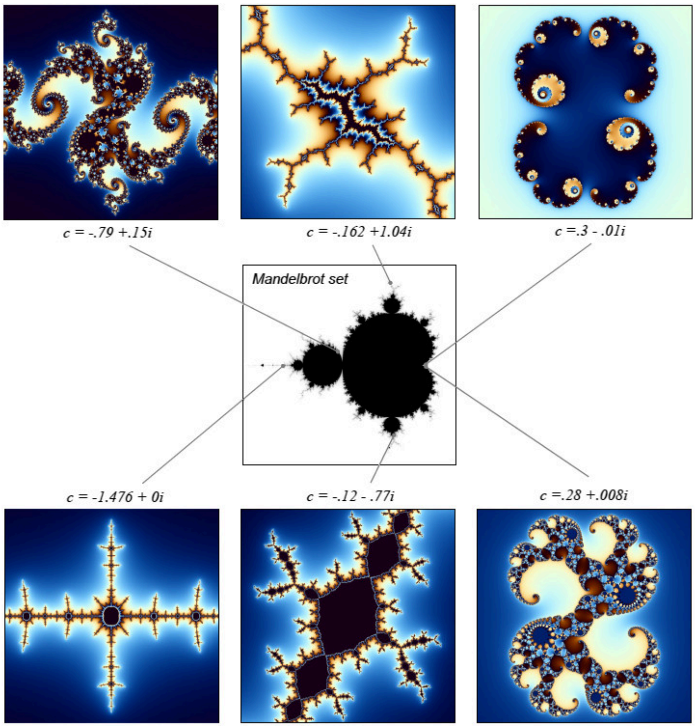
If we scale it by a factor of 2, you can see that it's "area" increases by a factor of 3. If *d* is the dimension then following the pattern above, we note that $2^d = 3$ which implies *d* $= log_2 3 \approx 1.585$. What! Yes, a fractal does have a fractional dimension. In fact, the name fractal comes from fractional dimension. In hindsight this makes perfect sense as the Sierpinski triangle isn't a true 2D object (otherwise area would have scaled by a factor of 4 upon doubling the side length) and also not a 1D object—so something between 1 and 2 matches our intuition. You will find something similar for Menger sponge and Koch snowflake (the left pic in the second row).

The Mandelbrot set is also a fractal and has been a intriguing research topic in contemporary mathematics. It is generated by recursion, a technique where you start with a specific number, and then you apply the same recursive formula, again and again, to get the next number in the sequence. The sequence we are interested in is $x_n = x_{n-1}^2 + c$, where *c* is a complex number and the sequence itself is in the complex plane. As an example let's list the sequence when the first term $x_0 = 0$ and $c = i$:

$$
\begin{aligned}
&x_0 = 0\\
&x_1 = i\\
&x_2 = -1 + i\\
&x_3 = -i\\
&x_4 = -1 + i\\
&x_5 = -i\\
&\vdots
\end{aligned}
$$

Formally: The Mandelbrot set **M** consists of all of those (complex) ***c***-values for which the corresponding orbit of **0** ($x_0 = 0$) under $x^2 + c$ does not escape to infinity. One can easily check that $c = 0,\ i,\ -1,\ -1.3,\ -1.38,$ and $-0.7i$ all lie in the Mandelbrot set, whereas $c = 1$ and $c = 2i$ do not. You must be wondering at this point how did we exactly generate this amazing image. To be honest, I cheated a bit: rather than checking whether the sequence diverged to infinity, I just checked whether the modulus of successive sequence elements crossed an arbitrarily chosen value (2, in this case). The colour coding (available in Python Pyplot using .imshow method) was based on the number of iterations it took for the sequence to diverge while the colour black represented the Mandelbrot set (region in the complex plane which provided *c* value which led to convergence of the chosen sequence).

Interestingly enough, if *c* is held constant and the initial value of **z** (i.e. x_0) is varied instead, one obtains the corresponding Julia set for the point *c*, like this:



As you move the value of *c* around the Mandelbrot set, you might notice a curious property:

- All sequences within the main body (the cardioid region) of the Mandelbrot set converge to a single point.
- The sequences within the large bulb at the top and bottom reach an orbit consisting of 3 points.
- Sequences in the smaller bulbs (at roughly $\pm 45^{\circ}$) have orbits of length 5.

Every bulb has a differently-sized orbit, with smaller bulbs having more and more points in their orbits. The size of these orbits are closely related to the **Logistic Map**, an important concept in Chaos theory. There is also a wonderful relation between the number of spokes radiating out from a bulb and the dynamics of $x^2 + c$ for *c* inside the primary bulb. It is known that if *c* lies in the interior of such a decoration, then the orbit of 0 is attracted to a cycle of a given period **n**. The number **n** is the same for any *c* inside this main decoration. Try zooming in and out of the Mandelbrot set simulation, keeping an eye on the recurring patterns, and look out for other interesting relations.

What I have tried to present here is just a glimpse into the world of Complex Analysis, one of the numerous branches of mathematics. You are encouraged to refer to more resources for a more serious study. The list below is just a comprehensive one and is by no means complete. In case I have missed listing or have incorrectly cited a source, I would like to apologise and assure that it wasn't intentional.

References and Sources:

- https://mathigon.org/course/fractals/mandelbrot
- https://math.bu.edu/DYSYS/FRACGEOM/FRACGEOM.html
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- https://u.math.biu.ac.il/~megereli/final_topology.pdf (Advanced Reading)
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