Lorenz Attractor

You might have heard of the term ‘butterfly effect’ in pop culture, which is usually described as ‘A butterfly flapping it’s wings in one part of the world can be the cause of a thunderstorm in some other part.’ Dynamical systems, where even a small change in initial conditions can lead to drastically different evolutions are called chaotic. The Lorenz system of equations is one of the widely known examples of a system that demonstrates chaos. Edward Lorenz came up with this system of non-linear differential equations as a simplified manner to describe atmospheric convection. It soon found many applications in biology, chemistry, meteorology, and electronics, to say the least. This should not be surprising, as real life is almost always unpredictable and chaotic…

You may have dealt with many linear systems such as oscillators of different forms and wave mechanics. Unlike those, this is a non-linear system. [A non-linear differential equation is a differential equation that is not a linear equation in the unknown function and its derivatives.]
I should mention that although discovered in the context of meteorological phenomena, Lorenz systems have inspired a lot of mathematical research in ‘Dynamical Systems’ and ‘Chaos Theory’.

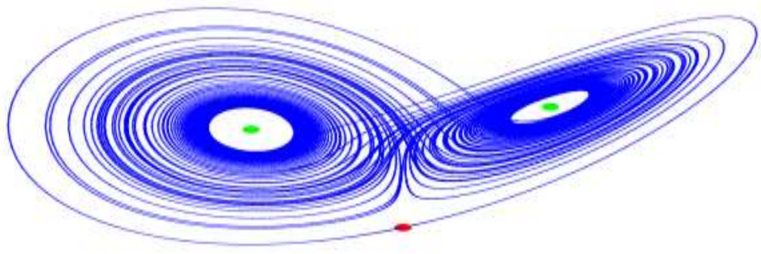
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x),$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y,$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z.$$

We will soon see why. But first, let’s go over the actual system of equations.

It is important to note that here x, y and z are not necessarily our usual spatial coordinates! For eg. In the context of 2-dimensional (one horizontal and one vertical direction) convection, x is proportional to the convection rate and y and z are proportional to temperature gradients in the horizontal and vertical directions respectively.
Abstract spaces made up of such general coordinates are called ‘phase spaces’, to differentiate them from our usual physical space. We can study the trajectories traced by our system in the phase space just like we study the motion of balls, pendulums etc in physical space.
You will soon observe as you try the simulation for yourself that, the Lorenz system is indeed chaotic. We seem to get random and irregular patterns from a well described deterministic system of these nonlinear differential equations. In the field of Chaos Theory, Butterfly effect is the term used to describe the drastic difference in the later state of a system for a slight deviation in its initial state. An attractor in a dynamical system is the set of states to which a system tends to evolve for a large set of initial states. We have talked about how even the slightest change in initial states can lead to great difference in later states in such systems. Now let’s look at the Lorenz attractor which appears contrary to that behaviour.



Ponder about the following points while experimenting with the simulation-

1)The system is deterministic! - If you know the phase space coordinates at a particular instant of time, you know how fast these coordinates are changing and thus their values at the next instant of time.

1) Why is the attractor symmetric about the z axis?
   Answer-
   Replace x with -x and y with -y in the Lorenz equations. Do you recover the original equations?

2) There are three equilibrium points- The system doesn't change at all if it starts at one of these points. Find these points. Can you find the coordinates of the green points in the above picture?
   (Hint- Set all three derivatives in the Lorenz equations to 0 (that is, set the RHS of the three equations equal to 0). Find all possible (x,y,z) from the resulting equations.)

3) Keeping the other parameters fixed, decrease Rho until it becomes 1. You will observe that the green points move closer and closer to the origin as Rho is decreased – If Rho becomes less than 1, the green points become imaginary! (All these properties are easy to see if you correctly calculated the coordinates in 2).)
   This means that the origin is the only equilibrium point if Rho is less than 1- in fact all trajectories ultimately (that is, as t-> infinity) converge to the origin in this case, thus no chaos! (Recall the definition of chaos from the first paragraph.).
   So whether the system is chaotic or not depends on the values of Sigma, Rho and Beta-
   Our atmosphere has the correct values to make it chaotic! Lorenz used the values 10, 28 and 8/3 (Sigma, Rho and Beta respectively) for his analysis, and in this case we do see chaos.

4) Observer the trajectories carefully in this case- Almost all of them eventually settle on a fixed set. What's the shape of this set?
   This is where the term 'butterfly effect' comes from!

This set has many amazing properties- having a fractional 'dimension' (just like the Sierpinski Triangle) not being the most surprising one! This might make you wonder- How can something be so chaotic yet orderly at the same time?!

5) As a final exercise, focus on any one general trajectory within the attractor. Let n_1 be the number of times it loops around the first green point before beginning to loop around the second, then let n_2 be the number of times it loops around the second green point before moving to loop back around the first, then let n_3 be the number of times it loops around the first green point before moving back to the second and so on…
   Write down the sequence n_1, n_2, n_3, n_4,…… Can you spot any pattern? Write down more terms if necessary. Do you see something now?
   If your answer is no, then it's not your fault- This is an example of a 'Random sequence' (Yes, there are ways to formalise the notion of a 'Random Sequence' but trying to do so will get us too deep into Theoretical Math and Computer Science!). This means that this sequence has no structure whatsoever, in pretty much the literal sense of the word! (Entropy, anyone?!).
6) The chaotic nature of the Lorenz system is the reason why weather forecasting is hard, and we get progressively worse at predicting it the further in time we need to predict it.

This is a brief look into the fields of chaos theory and dynamical systems.
You can also have a look at similar physical systems such as the double pendulum. I hope that by now I have convinced you that the Lorenz attractor is something which deserves your attention and is worth studying much more about! Be sure to check out the following amazing resources-

http://legacy-www.math.harvard.edu/archive/118r_spring_05/handouts/lorentz.pdf

https://www2.physics.ox.ac.uk/sites/default/files/profiles/read/lect6-43147.pdf

https://en.m.wikipedia.org/wiki/Lorenz_system

https://ocw.mit.edu/resources/res-18-009-learn-differential-equations-up-close-with-gilbert-strang-and-cleve-moler-fall-2015/solving-odes-in-matlab/lorenz-attractor-and-chaos/

https://books.google.co.in/books/about/Nonlinear_Dynamics_and_Chaos.html?id=1kpnDwAAQBAJ&printsec=frontcover&source=kp_read_button&hl=en&newbks=1&newbks_redir=1&redir_esc=y

http://www.csun.edu/climate/math483/lorenzmodel.pdf

https://youtu.be/aAJkLh76QnM

https://youtube.com/playlist?list=PLbN57C5Zdl6j_qJA-pARJnKsmROzPnO9V

https://www.maa.org/sites/default/files/pdf/upload_library/22/Ford/Volchan46-63.pdf