Enumeration Prelims Solutions

July 9, 2021

1 Objective Solutions

1. B
2. C
3. B
4. B
5. B
6. 864
7. 2022
8. 1015
9. 55
10. 970399
11. 41
12. 46
13. 5185
14. 429
15. 24

2 Subjective Solutions

1. Number the houses from 1 to n clockwise. Let $f(n,k)$ denote the answer for n houses and k colours. We first observe that clearly $f(2,k) = k(k-1)$ for all k. We now induct on n, for fixed k and claim that the general answer is

$$f(n,k) = (k-1)^n + (k-1)(-1)^n$$

Consider this problem instead where houses $n, 1$ could have the same colour with everything else the same. Now we colour the houses step by step: house 1 first, then house 2, and so on. Clearly we have $k(k-1)^{n-1}$ ways of doing so. Now $f(n,k)$ of these ways would have houses $1, n$ of different colours. For the remaining cases, since houses $1, n$ have the same colour, we can treat them as a single house and hence this is, by definition, $f(n-1,k)$. Thus we have

$$\begin{aligned}
&f(n,k) = k(k-1)^{n-1} - f(n-1,k) \\
\implies &f(n,k) = k(k-1)^{n-1} - \left[(k-1)^{n-1} + (k-1)(-1)^{n-1}\right] \\
\implies &f(n,k) = (k-1)(k-1)^{n-1} - (k-1)(-1)^{n-1} = \boxed{(k-1)^n + (k-1)(-1)^n}
\end{aligned}$$

2. Let A, B, C, D be four points on a circle. The projections of A, B on CD are A', B'. The projections of C, D on AB are C', D'. We use directed angled modulo $180°$. Then clearly, $\measuredangle A'D'A = \measuredangle A'DA = \measuredangle CBA = \measuredangle CB'C'$. Therefore, $\measuredangle A'D'C' = \measuredangle A'B'C' \implies A', B', C', D'$ are concyclic.
   Alternatively, let $\sim$ denote the relation defined as follows: Lines $\ell_1 \sim \ell_2 \iff \ell_1, \ell_2$ are antiparallel in lines AB, CD. Observe that $A'D' \sim AD \sim BC \sim B'C' \implies A'D' \sim B'C' \implies A', B', C', D'$ are concyclic.

3. Without loss of generality consider all the a_i's to be distinct and b_i's to be non-zero. Assume to the contrary that there are only finitely many primes $p_1, p_2, ..., p_j$ dividing some element of B. We denote, $B_k \equiv \sum_{i=1}^{n} b_i a_i^k$. Take a prime $p \in \{p_1, p_2, ..., p_j\}$. It is easy to see that there exists $q_p \in \mathbb{N}$ such that, $Q_p = \sum_{i:p\nmid a_i} b_i a_i^q \neq 0$,for all $q \geq q_p$, since one of the terms grow much faster than the others. Consider, δ such that $p^\delta | Q_p$ and $p^{\delta+1} \nmid Q_p$. Observe that for, $k = \{q + p^\delta(p-1)t\}$,$t \in \mathbb{N}$ we have, $p^\delta | B_k$ and $p^{\delta+1} \nmid B_k$ since, $B_k = Q_p(\mathrm{mod} p^{\delta+1})$ by Euler's theorem and the fact that $p^{\delta+1} | a^{p^\delta(p-1)t}$ whenever, $p|a$ as $p^\delta(p-1)t \geq \delta + 1$. Now, fix $q \geq$max$\{q_{p_1}, q_{p_2}, ..., q_{p_j}\}$ and hence for $k = \{q + t \prod_{i=1}^{j} p_i^{\delta_i}(p_i - 1)\}$,$t \in \mathbb{N}$ we have, $p_i^{\delta_i} | B_k$ and $p_i^{\delta_i+1} \nmid B_k$ i.e, $\chi = \prod_{i=1}^{j} p_i^{\delta_i} | B_k$ and $p \nmid \frac{B_k}{\chi}$,for every $p \in \{p_1, p_2, ..., p_j\}$ which gives us a contradiction since $|B_k| > \chi$,for all sufficiently large k.

4. We would solve the problem in a general setting where each side has been divided internally in the ratio $1 : r$, iteratively. The polygon can be represented as a column matrix with complex entries where the vertices are taken in cyclic order.

$$P_0 = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

where z_i corresponds to the ith vertex.

The act of dividing each side internally is equivalent to a linear transformation to the polygon.

$$P_k = AP_{k-1} = \begin{pmatrix} \frac{1}{(r+1)} & \frac{r}{(r+1)} & 0 & \cdots & 0 \\ 0 & \frac{1}{(r+1)} & \frac{r}{(r+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{r}{(r+1)} & 0 & 0 & \cdots & \frac{1}{(r+1)} \end{pmatrix} P_{k-1}$$

Upon iteration,

$$P_k = A^k P_0 = \begin{pmatrix} \frac{1}{(r+1)} & \frac{r}{(r+1)} & 0 & \cdots & 0 \\ 0 & \frac{1}{(r+1)} & \frac{r}{(r+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{r}{(r+1)} & 0 & 0 & \cdots & \frac{1}{(r+1)} \end{pmatrix}^k \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

Now, since $|A - \lambda I_n| = \left(\frac{1}{(r+1)} - \lambda\right)^n - \left(-\frac{r}{(1+r)}\right)^n$, the eigenvalues of the transformation matrix A are $\lambda_j = \frac{1}{(r+1)} + \frac{r}{(r+1)}\omega^j$, $\omega = e^{\frac{2\pi i}{n}}$ and $j \in \{1, 2, 3, ...n\}$

Since all the eigenvalues are distinct, the eigenvectors form a basis. Consider, Γ_j be an eigenvector corresponding to the eigen value λ_j. It follows that

$$P_0 = \sum_1^n c_j \Gamma_j \implies P_k = A^k P_0 = \sum_1^n c_j \lambda_j^k \Gamma_j$$

Notice, $\lambda_n = 1$ and $|\lambda_j| < 1$, for $j \neq n$. Thus, all the coefficients of Γ_j $(j \neq n)$ vanish as $k \to \infty$, since $|\lambda| < 1 \implies \lim_{k\to\infty} \lambda^k = 0$, and hence, the polygon shrinks to $C_n \Gamma_n$.

We can see that

$$\Gamma_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to the eigenvalue 1 and it just corresponds to the point $z = 1$ in the complex plane. Thus, the iterations shrink the polygon to the point C_n.

Now, observe that under the linear transformation A the centroid of the polygon remains invariant. In other words, centroid of P_k = centroid of $P_{k-1} = \ldots =$ centroid of P_0. Hence, we must have

$$\lim_{k\to\infty} \text{centroid}(P_k) = \frac{C_n + C_n + \cdots + C_n}{n} = C_n = \frac{z_1 + z_2 + z_3 + \cdots + z_n}{n}$$

as required.

5.

(a) Alice only wins if $n = 2^k$. First, we show that this is sufficient. Suppose Alice adopts the strategy: Each round, she picks the positions with coins having "H" facing up. If at some point all coins have "T" facing up, Alice can simply write down all the positions and wins in the next round regardless of what Bob does.

Working modulo 2, i.e, in GF_2, Let

$$b(x) = b_0 + b_1 x + \ldots . b_{n-1} x^{n-1}$$

denote the state polynomial of the table, i.e, $b_i = 1$ if the coin at position i has "H" facing up, and 0 otherwise. If Bob cyclically shifts the coins by j positions, then the new state polynomial is given by

$$b(x) \to b(x)(1 + x^j)$$

After m rounds, the state polynomial is given by

$$b(x) \prod_{i=1}^m (1 + x^{j_i})$$

If $m \geq n(n-1) + 1$, by PHP there is an entry repeating at least n times in $\{j_1, j_2 \ldots j_m\}$

If $n = 2^k$, it is easy to see by induction that

$$(1 + x^j)^n = (1 + x^j)^{2^k} = 1 + x^{j \cdot 2^k} = 1 + 1 = 0$$

meaning that $b(x) = 0$ after m rounds, as desired.

(b) Now, we show that if $n \neq 2^k$, Bob can stall forever if the initial configuration is not all heads or all tails. Suppose for now that n is odd. Let $\mathbf{b}$ denote the state vector of the coins as discussed above. We claim that if

$$\mathbf{b} \notin \{0^n, 1^n\} - - - - - (1)$$

initially, then Bob can ensure that $\mathbf{b} \neq \{0^n, 1^n\}$ in all future rounds as well. Let $\mathbf{p}$ denote the state vector of positions chosen by Alice. If $\mathbf{p} \in \{0^n, 1^n\}$, then Bob doesn't rotate the table. Otherwise, there exists an i such that $b_i = 0$ and $b_{i+1} = 1$, and since n is odd there must be a j such that $p_j = p_{j+1}$. Now Bob can simply rotate the coins such that i is aligned with j and the new state $\mathbf{b}'$ satisfies $b'_i = 1, b'_{i+1} = 0$ if $p_j = p_{j+1} = 1$ and $b'_i = 0, b'_{i+1} = 1$ if $p_j = p_{j+1} = 0$. This implies (1) for $\mathbf{b}'$,

Now in general, we have $n = 2^k \cdot m$, where $m > 1$ is odd. Divide the set $S = \{0, 1, 2..., n-1\}$ into 2^k classes $S_0, S_1..., S_{2^k-1}$ as follows.

$$S_i = \{j \in S : j \pmod{2^k} = i\}$$

Let $\mathbf{b}^{(0)}, \mathbf{b}^{(1)}, ..., \mathbf{b}^{(2^k-1)}$ be the associated bit vectors. If (1) holds, then we must have

$$\mathbf{b}^{(i)} \notin \{0^m, 1^m\} - - - - (2)$$

for some i. Similarly, let $\mathbf{p}^{(0)}, \mathbf{p}^{(1)}, ..., \mathbf{p}^{(2^k-1)}$ denote the positions chosen by Alice. Now, Bob can simply restrict himself to multiples of 2^k for rotations, so that $\mathbf{b}^{(i)}$ is aligned with some rotation of $\mathbf{p}^{(i)}$, and apply the strategy for n odd for m instead to ensure that (2) always holds.

6. Notice that $a_0 \in \left(0, \frac{\pi}{2}\right)$ so, $sin(a_0) < a_0 = 1$ and $sin(a_i) < 1 \, \forall i \in \mathbb{N}$.

$$\therefore a_{i+1} = sin(a_i) < a_i \, \forall i \in \mathbb{N}$$

So, $\{a_n\}$ is a decreasing sequence and $a_i \in (0,1) \, \forall i \in \mathbb{N}$. $\therefore L = \lim_{n\to\infty}(a_n)$ exists. Clearly, $sin(L) = L$ and we have: $0 \leq L \leq 1$ ($\because a_i \in (0,1) \, \forall i \in \mathbb{N}$). So, $\lim_{n\to\infty}(a_n) = L = 0$.

Claim: $\lim_{n\to\infty}(\sqrt{n}a_n) = \sqrt{3}$. Let us compute $\lim_{n\to\infty}\left(\frac{1}{na_n^2}\right)$. Using Stolz-Cesàro Theorem on the sequences: $x_n = \frac{1}{a_n^2}$ and $y_n = n$ (notice that $\{y_n\}$ is strictly increasing and divergent). We have

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) &= \lim_{n\to\infty}\left(\frac{x_{n+1}-x_n}{y_{n+1}-y_n}\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}\right) \\
&= \lim_{n\to\infty}\left(\frac{a_n^2 - sin^2(a_n)}{a_n^2 sin^2(a_n)}\right) \\
&= \lim_{n\to\infty}\left(\frac{a_n^2 - \frac{1}{2}(1-cos(2a_n))}{\frac{1}{2}a_n^2(1-cos(2a_n))}\right) \\
&= \lim_{n\to\infty}\left(\frac{2a_n^2 - \left(\frac{(2a_n)^2}{2!} - \frac{(2a_n)^4}{4!} + ...\right)}{a_n^2\left(\frac{(2a_n)^2}{2!} - \frac{(2a_n)^4}{4!} + ...\right)}\right) \\
&= \lim_{a_n\to 0}\left(\frac{\frac{(2a_n)^4}{4!} - \frac{(2a_n)^6}{6!} + ...}{\frac{2^2a_n^4}{2!} - \frac{2^4a_n^6}{4!} + ...}\right) \qquad \left[\because \lim_{n\to\infty} a_n = 0\right] \\
\therefore \lim_{n\to\infty}\left(\frac{1}{na_n^2}\right) &= \frac{\frac{2^4}{4!}}{\frac{2^2}{2!}} = \frac{1}{3}
\end{aligned}$$

$\implies \lim_{n\to\infty}(\sqrt{n}a_n) = \sqrt{3}$. So, $\alpha = \frac{1}{2}$ and $\beta = \sqrt{3}$.

7. First choose integers a and b such that $2a + 6b = (2n+2)^n$ and $a + 2b = (2n)^n$, both n^{th} powers of integers. Solving this linear system yields $a = 3(2n)^n - (2n+2)^n = 2^n(3n^n - (n+1)^n)$ and $b = \frac{1}{2}((2n+2)^n - 2(2n)^n) = 2^{n-1}((n+1)^n - (2n)^n)$ Note that

$$2 < \left(1 + \frac{1}{n}\right)^n < 3$$

for $n > 1$ and thus it follows that a and b are positive integers. Express $2a$ and $2b$ as sums of distinct powers of 2 as follows:

$$2a = 2^{r_1} + 2^{r_2} + 2^{r_3} + ... + 2^{r_l} \qquad 1 \le r_1 < r_2 < ... < r_l;$$

$$2b = 2^{s_1} + 2^{s_2} + 2^{s_3} + ... + 2^{s_m} \qquad 1 \le s_1 < s_2 < ... < s_m;$$

Let $a_i = 2^{r_i}$ for $1 \le i \le l$ and $a_{l+j} = 3 * 2^{s_j}$ for $1 \le j \le m$. Let $k = l + m$ and consider $a_1, ...a_k$ which are clearly distinct. $\sum_{j=1}^{k} a_j = 2a + 6b = (2n+2)^n$. $\phi(2^r) = 2^{r-1}$ and $\phi(3 * 2^r) = 2^r$

$$\sum_{h=1}^{k} \phi(a_h) = \sum_{i=1}^{l} 2^{r_i - 1} + \sum_{j=1}^{m} 2^{s_i} = a + 2b = (2n)^n$$

8. Call the first type of partition a *balls in bin partition* and the second type of partition a *powers of 2 partition*. Let $\sum_{i=1}^{s} b_i$ be a *balls in bin partition* (b_i's are in increasing order). Map it to the *powers of 2 partition* $\sum_{k=0}^{s-1} 2^k a_k$ where $a_k = b_{s-k} - \sum_{j<s-k} b_j$. We can also reverse this map by simply mapping a *powers of 2 partition* $\sum_{k=0}^{s-1} 2^k a_k$ to a *balls in bin partition* $\sum_{i=1}^{s} b_i$ where $b_1 = a_{s-1}$ and $b_l = a_{s-l} + \sum_{m<l} b_m$ for $l > 1$. Thus the set of *powers of 2 partitions* and the set of *balls in bin partitions* are in bijection, hence the number of these partitions are equal.