

# Gaussian Gamble: Problem Set 1

August 21, 2020

Maximum possible score : 25

The deadline to submit solutions is 11:59 PM, 28<sup>th</sup> August 2020.

**Partial solutions will receive partial marking, so mention any result you may have obtained, including any conjectures that you make.**

- Q1. Find all ordered pairs of integers  $(x, y)$  such that the sum of their cubes is equal to the square of their sum, i.e. (4)

$$x^3 + y^3 = (x + y)^2$$

- Q2. Consider 2 circles  $C_1$  and  $C_2$  tangent to each other and tangent to a line  $L$ ,  $C_1$  being to the left of  $C_2$ .

Now, construct a sequence of circles  $\{C_n\}$ ,  $n \geq 2$  with the following property :  $C_n$  is tangent to the circles  $C_{n-1}$  and  $C_{n-2}$ , and the line  $L$  for all  $n \geq 2$ .

Label the point of tangency of the circles  $C_1, C_2, \dots, C_n, \dots$  with the line  $L$  as  $P_1, P_2, \dots, P_n, \dots$

- (a) Prove that the sequence of points  $\{P_n\}$  approaches some limiting point,  $P_\infty$ . (2)

- (b) If  $P_\infty$  divides the line segment  $\overline{P_1 P_2}$  in the ratio 1 : 1, find the ratio of the radii of the initial circles  $C_1$  and  $C_2$ . (4)

- Q3. (a) Give an example of a 3-D object having 6 vertices such that there are only two possible values for the distance between any two vertices. (2)

- (b) Show that we can find a set  $S$  of 2016 distinct points in  $\mathbb{R}^{63}$  and two positive real numbers  $x$  and  $y$  such that the distance between any two distinct points in  $S$  is either  $x$  or  $y$ . (4)

- (c) Generalize the argument in b) to come up with a similar statement for  $\mathbb{R}^n$  for any arbitrary positive integer  $n$ . (1)

- Q4. A group of  $n$  students in a classroom are playing a game of ‘Society’. (8)  
Each student has some friends (possibly none), and  
friendship is mutual. Every student begins with an integral amount of  
dollars (possibly negative).  
A move consists of some student giving \$1 to each of their friends. We  
say that the game is *fair* if it is possible to transform the original dis-  
tribution of money into any other arbitrary one with the same amount  
of total money using some finite sequence of moves.  
Given that the game is fair, find the number of friendships among the  
students.