Gaussian Gamble: Problem Set 1

August 21, 2020

Maximum possible score : 25

The deadline to submit solutions is 11:59 PM, 28^{th} August 2020.
Partial solutions will receive partial marking, so mention any result you may have obtained, including any conjectures that you make.

Q1. Find all ordered pairs of integers (x, y) such that the sum of their cubes is equal to the square of their sum,i.e (4)

$$x^3 + y^3 = (x + y)^2$$

Q2. Consider 2 circles C_1 and C_2 tangent to each other and tangent to a line L , C_1 being to the left of C_2.
Now, construct a sequence of circles $\{C_n\}$, $n \geq 3$ with the following property :
C_n is tangent to the circles C_{n-1} and C_{n-2}, and the line L for all $n \geq 3$. It is also inside the region enclosed by C_{n-1}, C_{n-2} and the line L.
Label the point of tangency of the circles $C_1, C_2, ...C_n, ...$ with the line L as $P_1, P_2, ..P_n, ...$

(a) Prove that the sequence of points $\{P_n\}$ approaches some limiting point, P_∞. (2)

(b) If P_∞ divides the line segment $\overline{P_1P_2}$ in the ratio 1 : 1, find the ratio of the radii of the initial circles C_1 and C_2. (4)

Q3. (a) Give an example of a 3-D object having 6 vertices such that there are only two possible values for the distance between any two vertices. (2)

(b) Show that we can find a set S of 2016 distinct points in $\mathbb{R}^{63}$ and two positive real numbers x and y such that the distance between any two distinct points in S is either x or y. (4)

(c) Generalize the argument in b) to come up with a similar statement for $\mathbb{R}^n$ for any arbitrary positive integer n. (1)

Q4. A group of n students in a classroom are playing a game of 'Society'. (8)
Each student has some friends (possibly none), and
friendship is mutual. Every student begins with an integral amount of dollars (possibly negative).
A move consists of some student giving \$1 to each of their friends.We say that the game is *fair* if it is possible to transform the original distribution of money into any other arbitrary one with the same amount of total money using some finite sequence of moves.

Given that the game is fair, find the number of friendships among the students.