Qubit...err...TLS

Team Decoherence

July 2021

1 Superconducting qubit

1.1 A primer on Circuit Lagrangians

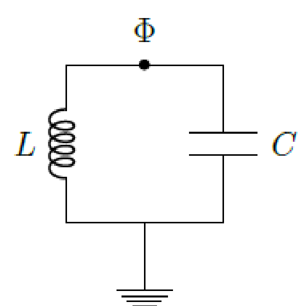
It is possible to extend Lagrangian method in classical mechanics to circuits. Just like in classical mechanics, we have a lagrangian for the system and an action that is extremised. However, we use node flux(defined below) as generalised coordinate. The recipe for constructing lagrangians and Hamiltonians is as follows:

i. Identify all the non-ground nodes in the circuit. If there is no node attached to ground, arbitrarily choose a node as ground.

ii. For each non-ground node, definte node flux as $\Phi_i = \int_{-\infty}^{t} V_i \, d\tau$

iii. For each capacitor, capacitor energy acts as kinetic energy term(T) and for each inductor, inductor energy acts as potential energy term(V). Lagrangian is then $L(\Phi, \dot{\Phi}, t) = T - V$

iv. Define charge, $Q_i = \frac{\partial L}{\partial \dot{\Phi}_i}$ (analogous to generalised momentum). Then the standard recipe for converting lagrangian to hamiltonian applies. $H(\Phi, Q, t)$



Questions:

a. Construct lagrangian and hamiltonian for standard LC circuit. **(2 marks)**

b. Verify that the equations governing the circuit obtained by Euler Lagrange equations on the above lagrangian are same as those obtained using Kirchoff's laws. **(2 marks)**

1.2 Artificial atom

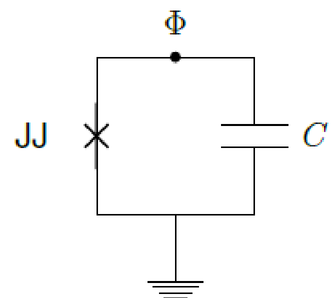
Consider an artificial atom made using Josephson Junction. A superconducting Josephson Junction(JJ) acts as a non-linear inductor. Due to *Josephson effect*, the tunneling current is given by:

$$I = I_0 \sin\left(\frac{2\pi\Phi}{\Phi_0}\right)$$

Where Φ is the node-flux difference across the Josephson junction and I_0, Φ_0 are constants.

a. Show that Josephson junction is a non-linear inductor i.e. inductance of Josephson Junction(L_J) depends on current flowing through it. **(2 marks)**

Josephson junction is usually shunted with a capacitor. This device – a capacitively shunted Josephson junction – is called a Transmon(see fig below)



b. Write hamiltonian for this circuit. **(4 marks)**

Since the charge carriers in a superconductor are "pairs of electrons"– termed Cooper-pairs, the charge on a superconducting capacitor comes in quanta of 2e (cooper pairs). Therefore we can write $\hat{Q} = 2e\hat{n}$ where n is the number of cooper pairs. Define two energy scales, $E_J = \frac{I_0\Phi_0}{2\pi}$ and $E_c = \frac{e^2}{2C}$ and $\hat{\theta} = \frac{2\pi\hat{\Phi}}{\Phi_0}$.

c. Rewrite above Hamiltonian in terms of given energy scales and operators $\hat{\theta}, \hat{n}$ **(2 marks)**

We now treat transmon as a weakly anharmonic oscillator i.e. in the limit $\Phi << \Phi_0$.

d. In the limit $\Phi << \Phi_0$, re write the hamiltonian upto fourth power of $\hat{\theta}$ **(2 marks)**

We now define annihilation and creator operators for the above hamiltonian as folows:

$$\hat{b} = \left(\frac{E_J}{32E_c}\right)^{1/4} \left\{\hat{\theta} + i\ \hat{n}\sqrt{\frac{8E_c}{E_J}}\right\}$$

$$\hat{b}^\dagger = \left(\frac{E_J}{32E_c}\right)^{1/4} \left\{\hat{\theta} - i\ \hat{n}\sqrt{\frac{8E_c}{E_J}}\right\}$$

Define $\hat{m} = \hat{b}^\dagger\hat{b}$ and $\hbar\omega_0 = \sqrt{8E_JE_c}$

e. Re-write the Hamiltonian in terms of above stated creation and annihilation operators. Find the energy eigen values of the hamiltonian i.e. $H|m\rangle = E_m|m\rangle$ in the limit $E_c << \hbar\omega_0$ upto first order perturbation. **(6 marks)**

Notice that the energy difference between the levels m and m+1 to observe that this difference depends on m just similar to a natural atom and isn't equal for all consecutive levels like in the case of harmonic oscillator

1.3 Artificial atom as a qubit

Now that we have an artificial atom, we make a two-level approximation. When the artificial atom is not driven, it is most likely in the ground state and when we drive the atom, likely transition is from ground state to first excited state. Therefore, we restrict ourselves to Ground state($|m = 0\rangle$, denote by $|0\rangle$) and Excited state($|m = 1\rangle$, denote by $|1\rangle$). Truncate the Hilbert space to this two level subspace. Define $E_1 - E_0 = \hbar\omega_0$

a. Write down the Hamiltonian of this Two Level System(TLS) in the basis $\{|0\rangle, |1\rangle\}$ **(2 marks)**

b. Re-write the above Hamiltonian in terms of Identity and Pauli-z operator **(2 marks)**

1.4 Qubit control

Now that we have realised qubit using an artificial atom in the two level approximation, we can control its state by interacting it with a harmonic drive field. Hamiltonian for this two-level system interacting with harmonic drive field is given by $H = H_0 + H_I$
where H_0 is the Hamiltonian of the two level system obtained in 1.3 b. and

$$H_I = \frac{\hbar\Omega}{2}\left(\hat{\sigma}_- \exp\{i\omega_d t + i\phi\} + \hat{\sigma}_+ \exp\{-i\omega_d t - i\phi\}\right)$$

Where $\hat{\sigma}$s are pauli matrices, $\hat{\sigma}_\pm = \frac{\hat{\sigma}_x \pm i\hat{\sigma}_y}{2}$, ϕ is the initial phase of the field, ω_d is the driving frequency.

a. Suppose that at any time t, general ket of the system can be written as $|\Psi_t\rangle = \alpha_t \exp\{\frac{-iE_0 t}{\hbar}\}|0\rangle + \beta_t \exp\{\frac{-iE_1 t}{\hbar}\}|1\rangle$. If the initial state of the system is given by $|\Psi_0\rangle = |0\rangle$, solve the schrodinger wave equation for the full hamiltonian and explicitly compute α_t, β_t **(12 marks)**

b. For the case when $\omega_d = \omega_0$ i.e. when the system is driven on resonance", write down the expressions for $|\alpha_t|^2$ and $|\beta_t|^2$ and roughly sketch them as functions of time **(4 marks)**

c. Write down the kets Ψ_t for $t = \frac{\pi}{2\Omega}, t = \frac{\pi}{\Omega}, t = \frac{3\pi}{2\Omega}, t = \frac{2\pi}{\Omega}$ when $\omega_d = \omega_0$ for two different values of initial phase of the drive $\phi = 0, \pi/2$ **(3 marks)**

d. Explain how a general qubit superposition can be prepared using this harmonic drive field. **(3 marks)**

2 Decoherence

2.1 A primer on Pure and mixed states

So far we have described Quantum mechanical systems with a state vector $|\Psi\rangle$. This description works only for pure states. If we have an ensemble of systems where some systems have their state as $|\Psi_1\rangle$, some systems have $|\Psi_2\rangle$ and so on. To describe such a mixed ensemble, we need to introduce density operator.

The density operator $\hat{\rho}$ is defined as

$$\hat{\rho} = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$$

where p_i is the probability to be in state $|\Psi_i\rangle$

For a pure state, density operator can be written as $\hat{\rho} = |\Psi\rangle\langle\Psi|$

Let us explore some mathematical properties of density operator.

a. Prove the following: (where Tr is trace operation)

   i. If $\hat{O}$ is an operator, $\langle\hat{O}\rangle = Tr(\hat{O}\hat{\rho}) = Tr(\hat{\rho}\hat{O})$ **(2 marks)**

   ii. For a pure state, $Tr(\hat{\rho}^2) = 1$ and for a mixed state, $Tr(\hat{\rho}^2) < 1$ **(2 marks)**

   iii. $\partial_t \hat{\rho}(t) = \frac{i}{\hbar}[\hat{\rho}, \hat{H}]$ **(1 mark)**

b. Suppose you've two ensembles A and B. A is a pure ensemble consisting of qubits in equal superpositions of $|0\rangle$ and $|1\rangle$. B is a mixed ensemble consisting of qubits in states $|0\rangle$ and $|1\rangle$ with equal probabilities. Explain how you can use a Hadamard gate to distinguish between the two ensembles. Given that you can measure the state of the qubits in the ensembles only in $|0\rangle$, $|1\rangle$ basis. **(4 marks)**

It is important to understand the concept of partial trace. Suppose $\rho_{\hat{A}B}$ is the joint density operator of systems A and B with basis $\{\chi_m\}$ and $\{\phi_n\}$ respectively, then we say that density matrix of A irrespective of the state of the system B is given by the partial trace of $\rho_{\hat{A}B}$ over system B which is defined as follows:

$$\hat{\rho_A} = Tr_B(\rho_{\hat{A}B}) = \sum_n \langle\phi_n|\rho_{\hat{A}B}|\phi_n\rangle$$

2.2 Decoherence

Let us begin by understanding the effect of Decoherence. Consider a system consisting of single qubit interacting with environment (very large system that, also called as thermal heat bath). For simplicity, assume that environment is also a qubit.
To begin with, the wave function of the system is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_{sys} + |1\rangle_{sys}\}$$

However, due to a weak interaction with environment gives rise to a joint wave function after some time t_0

$$|\Psi\rangle = \Big\{(1+\epsilon)\,|0\rangle_{sys} + (1-\epsilon)\,|1\rangle_{sys}\Big\} \otimes |0\rangle_{env} + \Big\{(1-\epsilon)\,|0\rangle_{sys} + (1+\epsilon)\,|1\rangle_{sys}\Big\} \otimes |1\rangle_{env}$$

a. Suppose an observer wants to measure the state of the system qubit after time t_0, what is the density operator of the system that he obtains for the system qubit? Is it a pure state or a mixed state? [Hint: Take partial trace over environment] **(5 marks)**

Because of this interaction of the qubit with environment howsoever weak it is, the qubit is no longer in the intial state it was prepared in. This uncontrolled change in the system state is what we call "Decoherence".