Cartesian Form

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Rectification means the process of finding length of curve whose eq^n is given between two given points.

1) Length of arc of curve given in Cartesian Form

If curve is $y = f(x)$

Then length of curve

$$S = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

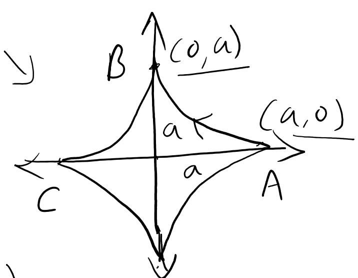
If curve is $x = g(y)$

Then length of curve

$$S = \int_{y_1}^{y_2} \left(\sqrt{1+\left(\frac{dx}{dy}\right)^2}\right) dy$$

Problems:

1) Find total length of curve $x^{2/3} + y^{2/3} = a^{2/3}$ — (1)



Sol^n The curve is asteroid with distance a, as shown in figure.

Let S be the length of arc AB from $A(a,0)$ to $B(0,a)$

diff (1) w.r. to x

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\cancel{2/3}\, x^{-1/3}}{\cancel{2/3}\, y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

By the formula,

$$S = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_a^0 \sqrt{1+\frac{y^{2/3}}{x^{2/3}}}\, dx = \int_a^0 \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}}\, dx$$

$$S = \int_a^0 \sqrt{\frac{a^{2/3}}{x^{2/3}}}\, dx = \int_a^0 a^{1/3} x^{-1/3}\, dx = a^{1/3}\left[\frac{x^{2/3}}{2/3}\right]_a^0$$

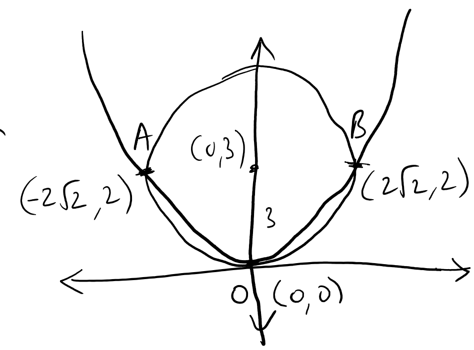
$$S = -\frac{3}{2} a^{1/3} a^{2/3} = -\frac{3}{2}a \text{ units} \quad \boxed{S = \frac{3}{2}a}$$ length is always pos^{ve}

Total length of given curve $= 4S = 4\left(\frac{3}{2}a\right) = 6a$

2) Find the length of parabola $x^2 = 4y$ which lies inside circle $x^2+y^2 = 6y$

Solⁿ: consider eqⁿ of circle, $x^2+y^2-6y+9 = 9$

$\boxed{x^2+(y-3)^2 = 3^2}$ Hence center is $(0,3)$ & radius is 3



Circle is touching x axis at origin

Parabola $x^2 = 4y$ is symmetric about y axis upward &

To find Point of intersection put $x^2 = 4y$

$4y + y^2 = 6y \quad \therefore \quad y^2 - 2y = 0, \; y(y-2) = 0$

$\therefore \; y = 0$ & $y = 2$

when, $y = 0$, $x^2 = 4(0) = 0 \; \therefore \; x = 0$

when, $y = 2$, $x^2 = 4(2) = 8$, $x = \pm 2\sqrt{2}$

Since $y = \frac{x^2}{4} \quad \therefore \quad \frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$

Then length of curve is given by,

$$S = \int_{x_1}^{x_2} \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}\, dx$$

$$S = 2\int_0^{2\sqrt{2}} \sqrt{\left(1+\frac{x^2}{4}\right)}\, dx$$

$$= \int_0^{2\sqrt{2}} \sqrt{4+x^2}\, dx$$

$$S = \left[\frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\log\left(x+\sqrt{x^2+4}\right)\right]_0^{2\sqrt{2}}$$

$$= \frac{2\sqrt{2}}{\cancel{2}}\sqrt{12} + 2\log(2\sqrt{2}+\sqrt{12}) - 2\log 2$$

$$S = 2\left[\sqrt{6} + \log\left(2(\sqrt{2}+\sqrt{3}\right) - \log 2\right]$$

$$= 2\left[\sqrt{6} + \cancel{\log 2} + \log(\sqrt{2}+\sqrt{3}) - \cancel{\log 2}\right]$$

$$S = 2\left[\sqrt{6} + \log(\sqrt{2}+\sqrt{3})\right]$$

Q3] Find the length of loop of curve $3ay^2 = x(x-a)^2$, $a > 0$

Since even powers of y are in eqⁿ $\therefore$

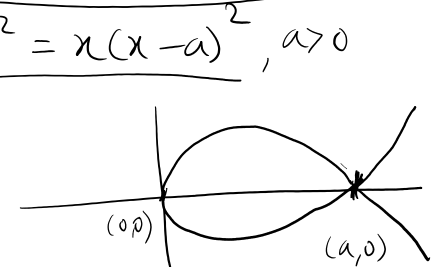
$\therefore$ Curve is symmetric about x axis,

check for negative x, Then $3ay^2$ is negative

$\Rightarrow y^2$ is negative, y is imaginary

if $x > a$ curve is present on right of $x = a$

point of intersection are $(0,0)$ & $(a,0)$



Since it is symmetric

$$S = 2\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$\therefore$ the given function

$\therefore$ length of Given Curve is

poi...

diff the given function

$$3ay^2 = x(x-a)^2$$

$$6ay\frac{dy}{dx} = (x-a)^2 + x\,2(x-a)$$

$$\frac{dy}{dx} = \frac{(x-a)[3x-a]}{6ay}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{(x-a)^2(3x-a)^2}{36a^2y^2}$$

$$= \frac{\cancel{(x-a)^2}(3x-a)^2}{12\,\cancel{36}\,a^{\cancel{2}}}\left[\frac{\cancel{3}a}{x\cancel{(x-a)^2}}\right]$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(3x-a)^2}{12ax} \qquad 9x^2-6ax+a^2$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(3x-a)^2}{12ax} = \frac{12ax+(3x-a)^2}{12ax}$$

$$1+\left(\frac{dy}{dx}\right)^2 = \frac{(3x+a)^2}{12ax}$$

$\therefore$ length of Given Curve is

$$S = 2\int_0^a \sqrt{\frac{(3x+a)^2}{12ax}}\,dx$$

$$= \cancel{2}\int_0^a \frac{3x+a}{\cancel{2}\sqrt{3}\sqrt{a}\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{3a}}\int_0^a\left(3\sqrt{x}+\frac{a}{\sqrt{x}}\right)dx$$

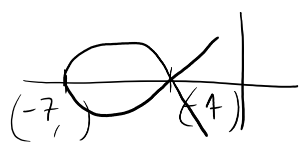
$$= \frac{1}{\sqrt{3a}}\left[\cancel{3}\frac{x^{3/2}}{\cancel{3}/2} + a\frac{x^{1/2}}{1/2}\right]_0^a$$

$$= \frac{1}{\sqrt{3a}}\left[2x^{3/2}+2ax^{1/2}\right]_0^a$$

$$S = \frac{1}{\sqrt{3a}}\left[4a^{3/2}+0\right] = \frac{4}{\sqrt{3}}a$$

$$\boxed{S = \frac{4}{\sqrt{3}}a \text{ units}}$$

HW: 3) Find total length of loop of curve $9y^2 = (x+7)(x+4)^2$



4) Find the length of parabola $y^2 = 4ax$ cut off by the line $3y = 8x$

Sol^n: To find points of intersection of line

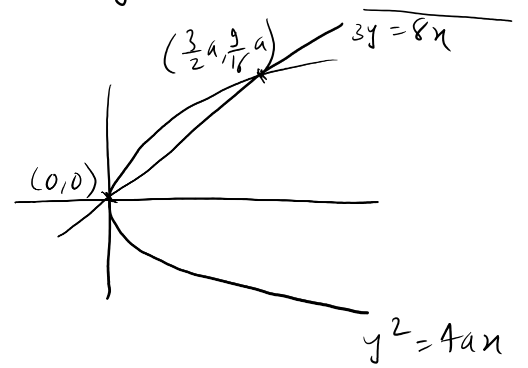
$3y = 8x$ [line passing through origin]

& parabola $y^2 = 4ax$

Put $4x = \frac{3y}{2}$, $y^2 = (4x)a = \left(\frac{3y}{2}\right)a$

$\therefore 2y^2 - 3ay = 0 \Rightarrow y(2y-3a) = 0$

$\therefore y = 0$ & $y = \frac{3}{2}a$



... $y = 0$, $x = 0$ ... $\frac{3}{8}\left(\frac{3}{2}a\right) = \frac{9}{16}a$

$\therefore S = \frac{1}{\ldots}\int \frac{y}{\ldots}\sqrt{y^2+4a^2} + \frac{4a^2}{2a}\ldots$

$\therefore\ y = 0 \quad \& \quad y = \frac{3}{2}a$

when $y = 0$, $x = 0$

when $y = \frac{3}{2}a$, $x = \frac{3}{8}y = \frac{3}{8}\left(\frac{3}{2}a\right) = \frac{9}{16}a$

Consider $x = \frac{y^2}{4a}$ $\therefore$ $\frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$

$\therefore$ length of curve is given by

$$S = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_0^{3a/2} \sqrt{1+\left(\frac{y}{2a}\right)^2}\, dy$$

$$= \int_0^{3a/2} \frac{1}{2a} \sqrt{4a^2+y^2}\, dy$$

$$\therefore S = \frac{1}{2a}\left[\frac{y}{2}\sqrt{y^2+4a^2} + \frac{4a^2}{2} \log\left(y+\sqrt{y^2+4a^2}\right)\right]_0^{3a/2}$$

$$S = \frac{1}{2a}\left[\frac{3a}{4}\sqrt{\frac{9a^2}{4}+4a^2} + \frac{4a^2}{2}\log\left(\frac{3a}{2}+\sqrt{\frac{9a^2}{4}+4a^2}\right) - 0 - 2a^2\log(2a)\right]$$

$$S = \frac{1}{2a}\left[\frac{3a}{4}\left[\frac{5a}{2}\right] + 2a^2\log\left(\frac{3a}{2}+\frac{5a}{2}\right) - 2a^2\log(2a)\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + 2a^2\log 4a - 2a^2\log 2a\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + 2a^2\log 2\right]$$

$$\boxed{S = a\left[\frac{15}{16} + \log 2\right]}$$