4 Change The Order Of Integral

In evaluating double integrals, if the limits of both the variables (i.e. inner and outer integral limits) are constants then the order of integration is immaterial provided the limits of integration are changed accordingly. But if the limits of integration are variable, a change in the order of integration requires a change in the limit of integration. Some integrals are easily evaluated by changing the order of integration. Following is the procedure to change the order of integration.

Procedure To Change The Order Of Integration

Step-1: Sketch the region of integration using equation of curve(s) and line(s). Mark the region by cross lines over which we have to integrate.

Step-2: Obtain the original integrating strip (rectangle) which is depends on order of integral.

a) If order is first w.r.t. y and then w.r.t. x then integrating strip is parallel to y-axis

b) If order is first w.r.t. x and then w.r.t. y then integrating strip is parallel to x-axis

Step-3: To changing order of integral, change the integrating strip i.e. if originally strip is parallel to y-axis then take new strip parallel to x-axis and vice-versa

Step-4: Write new limits for integrating strip.

Type-I: Change the order of integral

Example 1. Change the order of integral of $\int_0^4 \int_x^{16/x} \phi(x, y)\, \mathrm{d}y\mathrm{d}x$
Solution: Consider,

$$I = \int_0^4 \int_x^{16/x} \phi(x, y)\, \mathrm{d}y\mathrm{d}x \tag{1}$$

From the limits of integration, it is clear that we have integrate w.r.t. y first and then w.r.t.x. Therefore, originally the integrating strip is parallel to y-axis.
Now, we plot the curves $y = x$, $y = 16/x$ i.e. $xy = 16$, the boundary $x = 0$ and $x = 4$ (if required) and consider the region of integration (shown shaded) as shown in following fig-a.

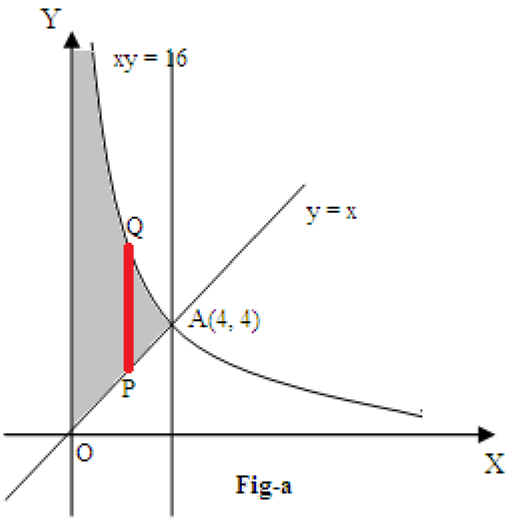


Fig-a

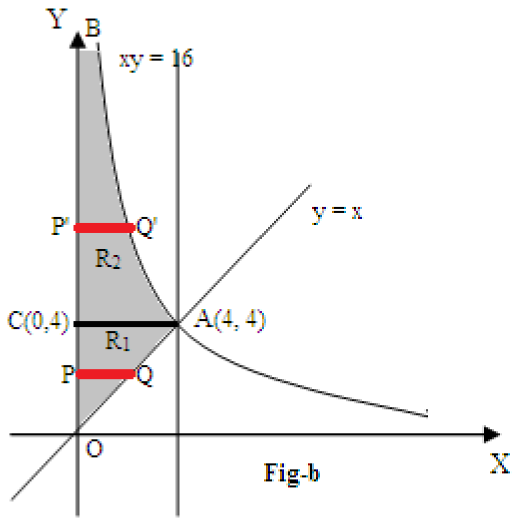


Fig-b

Solving $xy = 16$ and $y = x$, we get $y^2 = 16$. This implies $y = 4$. Therefore $x = 4$. Thus, $xy = 16$ and $y = x$ intersect at $A(4, 4)$.
Now, to change the order of integration, take an integrating strip parallel to x-axis. The whole region is divided into two regions R_1 and R_2. Separation line CA as shown in Fig-b.
For region R_1 : In the region R_1, the strip is parallel to x-axis (see Fig.-b). The point P lies on y-axis i.e. $x = 0$ and the point Q lies on the line $y = x$ i.e. $x = y$. Therefore x varies from 0 to y. Also, the region R_1 is bounded between $y = 0$ and $y = 4$. Therefore y varies from 0 to 4. Thus, in the region R_1, we get

$$I = \int_0^4 \mathrm{d}y \int_0^y \phi(x, y)\, \mathrm{d}x$$

For region R_2 : In the region R_2, the strip is parallel to x-axis (see Fig.-b). The point P' lies on y-axis i.e. $x = 0$ and the point Q lies on the line $xy = 16$ i.e. $x = \dfrac{16}{y}$. Therefore x varies from 0 to $16/y$.
Also, the region R_2 is unbounded between $y = 4$ and $y = \infty$. Therefore y varies from 0 to ∞. Thus, in the region R_2, we get

$$I = \int_0^\infty \mathrm{d}y \int_0^{16/y} \phi(x, y)\, \mathrm{d}x$$

Therefore, by (1), we get

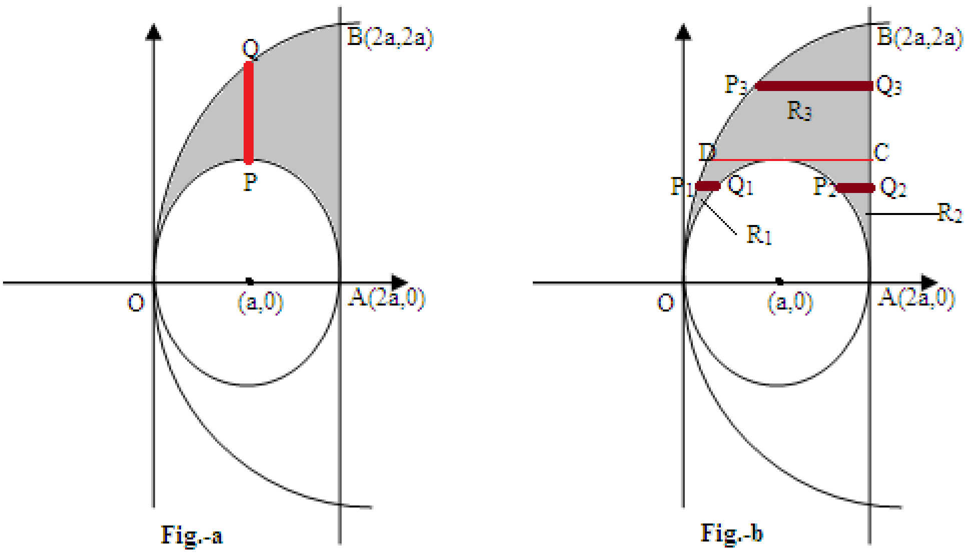
$$I = \int_0^4 \mathrm{d}y \int_0^y \phi(x, y)\, \mathrm{d}x + \int_0^\infty \mathrm{d}y \int_0^{16/y} \phi(x, y)\, \mathrm{d}x$$

Example 2. Change the order of integration of $\displaystyle\int_0^{2a} \int_{\sqrt{2ax - x^2}}^{\sqrt{2ax}} \phi(x, y)\, \mathrm{d}x\mathrm{d}y$

Solution: Consider

$$I = \int_0^{2a} \int_{\sqrt{2ax - x^2}}^{\sqrt{2ax}} \phi(x, y)\, \mathrm{d}x\mathrm{d}y \tag{1}$$

From the limits of integration, it is clear that we have integrate w.r.t. y first and then w.r.t.x. Therefore, originally the integrating strip is parallel to y-axis.
Now, we plot the curves $y = \sqrt{2ax - x^2}$ i.e. $x^2 + y^2 - 2ax = 0$ (circle with radius a and centre at $(a, 0)$), $y = \sqrt{2ax}$ i.e. $y^2 = 2ax$, the boundary $x = 0$, $x = 2a$ and consider the region of integration (shown shaded) as shown in following fig-a.



Solving $x^2+y^2-2ax=0$ and $y^2=2ax$, we get $x=0$ and $y=0$. Thus parabola and circle intersects at $(0,0)$ only. Therefore parabola is outside the circle. Again solving $x=2a$ and $y^2 2ax$, we get $y=\pm 2a$. Thus a line $x=2a$ and parabola $y^2=2ax$ intersects at $(2a,2a)$.

To change the order of integration we divide the region of integration into three regions R_1, R_2 and R_3 and then consider a integrating strip parallel to x-axis in each region.

For region R_1**:** In the region R_1, the strip is parallel to x-axis (see Fig.-b). The point P_1 lies on $y^2=2ax$ i.e. $x=y^2/2a$ and the point Q_1 lies on the circle $x^2+y^2-2ax=0$. Therefore, $x^2-2ax+a^2=a^2-y^2$ this gives $(x-a)^2=a^2-y^2$. Thus, $x=a-\sqrt{a^2-y^2}$ ($\because$ Point Q_1 lies on LHS of symmetric line $x=a$ and point P_2 lies on RHS of symmetric line $x=a$). Therefore x varies from $y^2/2a$ to $a-\sqrt{a^2-y^2}$.

Also, the region R_1 is bounded between $y=0$ and $y=a$ i.e. line CD. Therefore y varies from 0 to a. Thus, in the region R_1, we get

$$I_1=\int_0^a \mathrm{d}y\int_{y^2/2a}^{a-\sqrt{a^2-y^2}}\phi(x,y)\mathrm{d}x$$

For region R_2**:** In the region R_2, the strip is parallel to x-axis (see Fig.-b). The point P_2 lies on RHS symmetric line $x=a$. Therefore, $x-a=\sqrt{a^2-y^2}$ i.e $x=a+\sqrt{a^2-y^2}$ and Q_2 lies on $x=2a$. Therefore x varies from $x=a+\sqrt{a^2-y^2}$ to $2a$.

Also, the region R_2 is bounded between $y=0$ and $y=a$ i.e. line CD. Therefore y varies from 0 to a. Thus, in the region R_2, we get

$$I_2=\int_0^a \mathrm{d}y\int_{a+\sqrt{a^2-y^2}}^{2a}\phi(x,y)\mathrm{d}x$$

For region R_3**:** In the region R_3, the strip is parallel to x-axis (see Fig.-b). The point P_3 lies $y^2=2ax$ i.e. $x=y^2/2a$ and the point Q_3 lies on $x=2a$. Therefore x varies from $x=y^2/2a$ to $2a$.

Also, the region R_3 is bounded between the line CD i.e $y=a$ and $y=2a$. Therefore y varies from a to $2a$. Thus, in the region R_3, we get

$$I_3 = \int_a^{2a} \mathrm{d}y \int_{y^2/2a}^{2a} \phi(x,y)\mathrm{d}x$$

Therefore, by (1), we get

$$I = \int_0^a \mathrm{d}y \int_{y^2/2a}^{a-\sqrt{a^2-y^2}} \phi(x,y)\mathrm{d}x + \int_0^a \mathrm{d}y \int_{a+\sqrt{a^2-y^2}}^{2a} \phi(x,y)\mathrm{d}x + \int_a^{2a} \mathrm{d}y \int_{y^2/2a}^{2a} \phi(x,y)\mathrm{d}x$$

Example 3. Change the order of the integral $\int_{-1}^{2}\int_{x^2+1}^{x+3} \phi(x,y)\ \mathrm{d}x\mathrm{d}y$

Solution: Consider

$$I = \int_{-1}^{2} \mathrm{d}x \int_{x^2+1}^{x+3} \phi(x,y)\ \mathrm{d}y \tag{1}$$

Solving $y = x^2+1$ and $y = x+3$, we get $x=-1$ and $x=2$. When $x=-1$, we get $y=2$ and when $x=2$ we get $y=5$. Thus $y=x^2+1$ and $y=x+3$ intersects at $(-1,2)$ and $(2,5)$.
From the limits of integration, it is clear that we have integrate w.r.t. y first and then w.r.t.x. Therefore, originally the integrating strip is parallel to y-axis.
Now, we plot the curves $y=x^2+1$ i.e. $x^2=y-1$ (parabola), $y=x+3$, the boundary $x=-1$, $x=2$ and consider the region of integration (shown shaded) as shown in following fig-a.

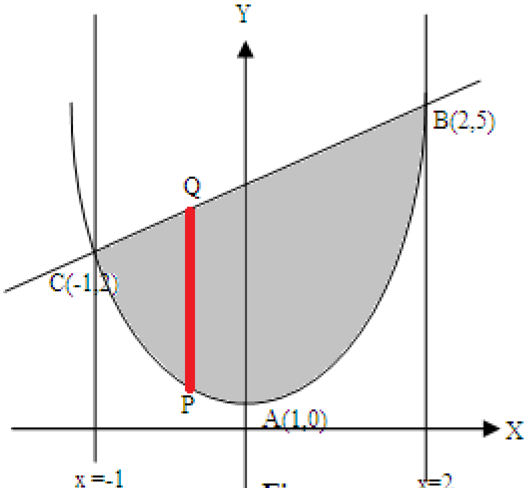


Fig-a

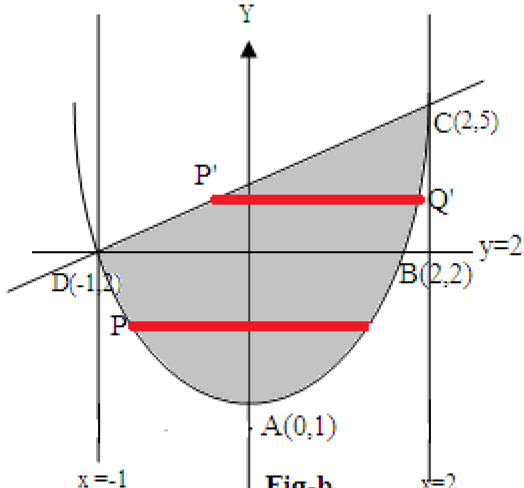


Fig-b

To change the order of integration we divide the region of integration into two regions R_1 and R_2 and then consider a integrating strip parallel to x-axis in each region as shown in Fig-b.
For region $ABDA$**:** In the region $ABDA$, The point P lies on $y=x^2+1$ i.e. $x=-\sqrt{y-1}$($\because P$ lies on LHS of symmetric line y-axis) and Q lies on $y=x^2+1$ i.e. $x=\sqrt{y-1}$ ($\because Q$ lies on RHS of symmetric line y-axis). Therefore x varies from $-\sqrt{y-1}$ to $\sqrt{y-1}$.
Also, the region $ABDA$ is bounded between $y=1$ and $y=2$. Therefore y varies from 1 to 2. Thus, in the region $ABDA$, we get

$$I = \int_1^2 \mathrm{d}y \int_{-\sqrt{y-1}}^{\sqrt{y-1}} \phi(x,y)\ \mathrm{d}x \tag{2}$$

For region $DBCD$**:** In the region $DBCD$, The point P' lies on $y = x + 3$ i.e. $x = y - 3$ and Q lies on $y = x^2 + 1$ i.e. $x = \sqrt{y-1}$ ($\because Q$ lies on RHS of symmetric line y-axis). Therefore x varies from $y - 3$ to $\sqrt{y-1}$.
Also, the region $DBCD$ is bounded between $y = 2$ and $y = 5$. Therefore y varies from 2 to 5. Thus, in the region $DBCD$, we get

$$I = \int_2^5 \mathrm{d}y \int_{y-3}^{\sqrt{y-1}} \phi(x, y)\, \mathrm{d}x \tag{3}$$

Therefore, by (2) and (3), we get

$$I = \int_1^2 \mathrm{d}y \int_{-\sqrt{y-1}}^{\sqrt{y-1}} \phi(x, y)\, \mathrm{d}x + \int_2^5 \mathrm{d}y \int_{y-3}^{\sqrt{y-1}} \phi(x, y)\, \mathrm{d}x$$

Example 4. Change the order of integration of $\int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} \phi(x, y)\, \mathrm{d}x\mathrm{d}y$
Solution: Consider,

$$I = \int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} \phi(x, y)\, \mathrm{d}x\mathrm{d}y = \int_0^a \mathrm{d}y \int_{\sqrt{a^2-y^2}}^{y+a} \phi(x, y)\, \mathrm{d}x \tag{1}$$

From the limits of integration, it is clear that we have integrate w.r.t. x first and then w.r.t.y. Therefore, originally the integrating strip is parallel to x-axis.
Now, we plot the curves $x = \sqrt{a^2 - y^2}$ i.e. $x^2 + y^2 = a^2$, $x = y + a$, $y = 0$, $y = a$ and consider the region of integration (shown shaded) as shown in following fig-a.

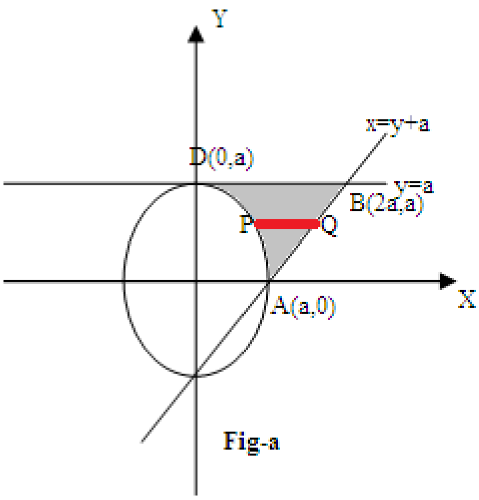


Fig-a

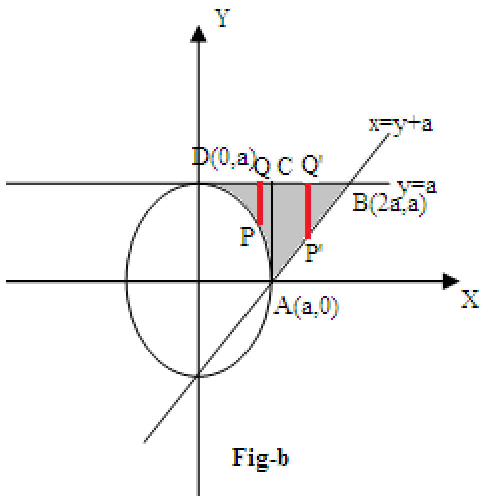


Fig-b

To change the order of integration we divide the region of integration into two regions $ACDA$ and $ABCA$ and then consider a integrating strip parallel to y-axis in each region as shown in Fig-b.
For region $ACDA$**:** In the region $ACDA$, The point P lies on $x^2 + y^2 = a^2$ i.e. $y = \sqrt{a^2 - x^2}$($\because P$ lies on LHS of symmetric line $x = 0$) and Q lies on $y = a$. Therefore y varies from $\sqrt{a^2 - x^2}$ to a.
Also, the region $ACDA$ is bounded between $x = 0$ and $x = a$. Therefore x varies from 0 to a. Thus, in the region $ACDA$, we get

$$I_1 = \int_0^a \mathrm{d}x \int_{\sqrt{a^2-x^2}}^{a} \phi(x, y)\, \mathrm{d}y \tag{2}$$

For region $ABCA$**:** In the region $ABCA$, The point P' lies on $y = x - a$ and Q lies on $y = a$. Therefore y varies from $x - a$ to a.
Also, the region $ABCA$ is bounded between $x = a$ and $x = 2a$. Therefore x varies from a to $2a$. Thus, in the region $ABCA$, we get

$$I_2 = \int_a^{2a} \mathrm{d}x \int_{x-a}^{a} \phi(x, y)\, \mathrm{d}y \tag{3}$$

Therefore, by (2) and (3),we get

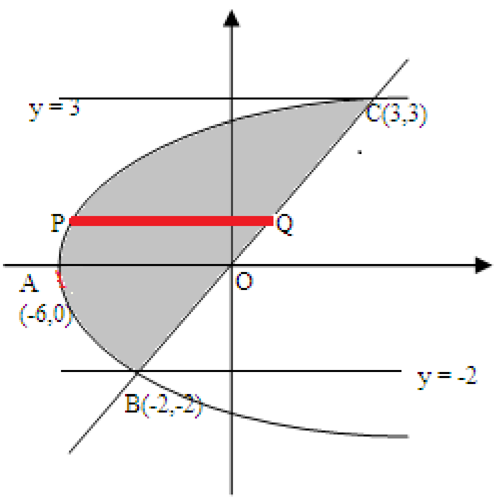
$$I = \int_0^a \mathrm{d}x \int_{\sqrt{a^2-x^2}}^{a} \phi(x, y)\, \mathrm{d}y + \int_a^{2a} \mathrm{d}x \int_{x-a}^{a} \phi(x, y)\, \mathrm{d}y$$

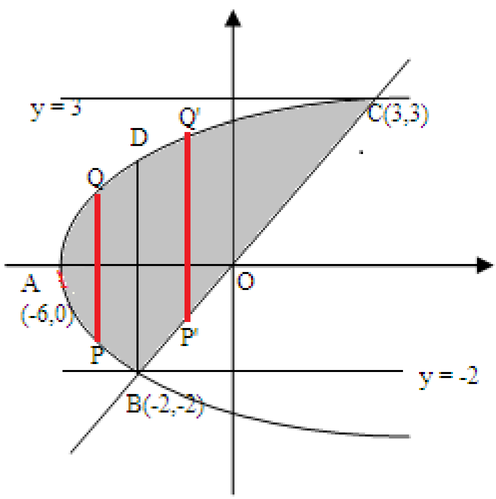
Example 5. Change the order of integration $\displaystyle\int_{-2}^{3} \int_{y^2-6}^{y} \phi(x, y)\mathrm{d}x\mathrm{d}y$

Solution: Consider,

$$I = \int_{-2}^{3} \int_{y^2-6}^{y} \phi(x, y)\mathrm{d}x\mathrm{d}y = \int_{-2}^{3} \mathrm{d}y \int_{y^2-6}^{y} \phi(x, y)\mathrm{d}x \tag{1}$$

Solving $x = y^2 - 6$ and $x = y$, we get $y = -2$ and $y = 3$. When $y = -2$,we get $x = -2$ and $y = 3$ gives $x = 3$. Thus $y^2 = x + 6$ and $x = y$ intersects at $(-2, -2)$ and $(3, 3)$.
From the limits of integration, it is clear that we have integrate w.r.t. x first and then w.r.t.y. Therefore, originally the integrating strip is parallel to x-axis.
Now, we plot the curves $x = y^2 - 6$ i.e. $y^2 = x + 6$, $x = y$, $y = -2$, $y = 3$ and consider the region of integration (shown shaded) as shown in following fig-a.





To change the order of integration we divide the region of integration into two regions $ABDA$ and $BCDB$ and then consider a integrating strip parallel to y-axis in each region as shown in Fig-b.
For region $ABDA$**:** In the region $ABDA$, The point P lies on parabola $y^2 = x + 6 \therefore y = \pm\sqrt{x + 6}$. But P lies below the symmetric line $y = 0$, therefore $y = -\sqrt{x + 6}$ and Q lies on $y = \sqrt{x + 6}$ ($\because Q$ lies above symmetric line). Therefore y varies from $-\sqrt{x + 6}$ to $\sqrt{x + 6}$.

Also, the region $ABDA$ is bounded between $x = -6$ and $x = -2$. Therefore x varies from -6 to -2. Thus, in the region $ABDA$, we get

$$I_1 = \int_{-6}^{-2} \mathrm{d}x \int_{-\sqrt{x+6}}^{\sqrt{x+6}} \phi(x,y)\ \mathrm{d}y \tag{2}$$

For region $BCDB$**:** In the region $BCDB$, The point P' lies on $y = x$ and Q' lies on $y = \sqrt{x+6}$ ($\because$ Q lies above symmetric line). Therefore y varies from x to $\sqrt{x+6}$.
Also, the region $BCDB$ is bounded between $x = -2$ and $x = 3$. Therefore x varies from -2 to 3. Thus, in the region $BCDB$, we get

$$I_2 = \int_{-2}^{3} \mathrm{d}x \int_{x}^{\sqrt{x+6}} \phi(x,y)\ \mathrm{d}y \tag{3}$$

Therefore, by (2) and (3), we get

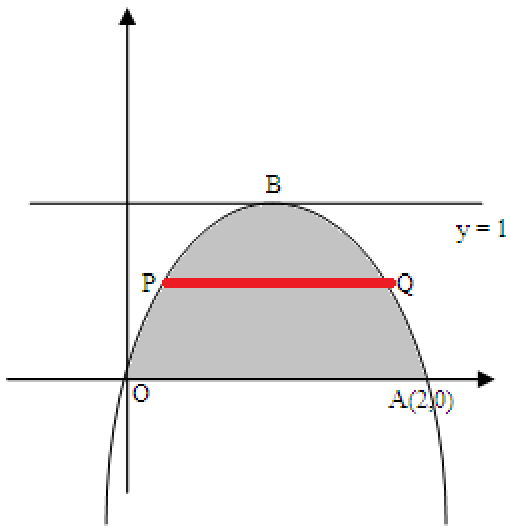
$$I = \int_{-6}^{-2} \mathrm{d}x \int_{-\sqrt{x+6}}^{\sqrt{x+6}} \phi(x,y)\ \mathrm{d}y + \int_{-2}^{3} \mathrm{d}x \int_{x}^{\sqrt{x+6}} \phi(x,y)\ \mathrm{d}y$$

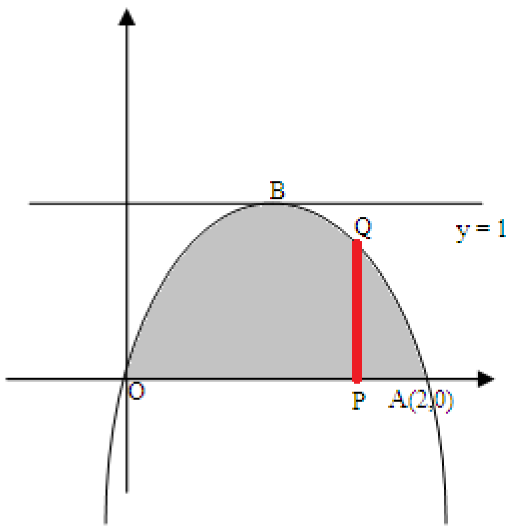
Example 6. Change the order of integration of $\displaystyle\int_0^1 \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} \phi(x,y)\ \mathrm{d}x\mathrm{d}y$

Solution: Consider

$$I = \int_0^1 \mathrm{d}y \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} \phi(x,y)\ \mathrm{d}x$$

Here x varies from $1 - \sqrt{1-y}$ i.e. $x - 1 = -\sqrt{1-y}$ to $1 + \sqrt{1-y}$ i.e. $x - 1 = \sqrt{1-y}$. Note that RHS of equality is $\pm$. Thus, one point lies on LHS of symmetric line $x = 1$ and other on RHS of line.
Now, sketch the curve $x - 1 = \pm\sqrt{1-y}$ i.e. $(x-1)^2 = -(y-1)$ (parabola with vertex at $(1,1)$ and symmetric about the line $x = 1$ along negative direction) and consider the region of integration (shown shaded) as shown in following fig-a.
From the limits of integration, it is clear that we have integrate w.r.t. x first and then w.r.t.y. Therefore, originally the integrating strip is parallel to x-axis.





To change the order of integration consider a integrating strip parallel to y-axis as shown in Fig-b. In the region $OABO$, The point P lies on $y = 0$ and Q lies on $(x-1)^2 = 1 - y$ i.e. $y = 2x - x^2$. Therefore y varies from 0 to $2x - x^2$.
Also, the region $OABO$ is bounded between $x = 0$ and $x = 2$. Therefore x varies from 0 to 2. Thus, in the region $OABO$, we get

$$I = \int_0^2 \mathrm{d}x \int_0^{2x-x^2} \phi(x,y)\,\mathrm{d}y$$

Example 7. Change the order of integration of $\displaystyle\int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} \phi(x,y)\,\mathrm{d}x\mathrm{d}y$

Solution: Consider,

$$I = \int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} \mathrm{d}x\mathrm{d}y = \int_0^a \mathrm{d}x \int_{\sqrt{a^2-x^2}}^{x+3a} \mathrm{d}y \qquad (1)$$

From the limits of integration, it is clear that we have integrate w.r.t. y first and then w.r.t.x. Therefore, originally the integrating strip is parallel to y-axis.
Now, we plot the curves $y = \sqrt{a^2 - x^2}$ i.e. $x^2 + y^2 = a^2$, $x = 0$, $x = a$ and consider the region of integration (shown shaded) as shown in following fig-a.

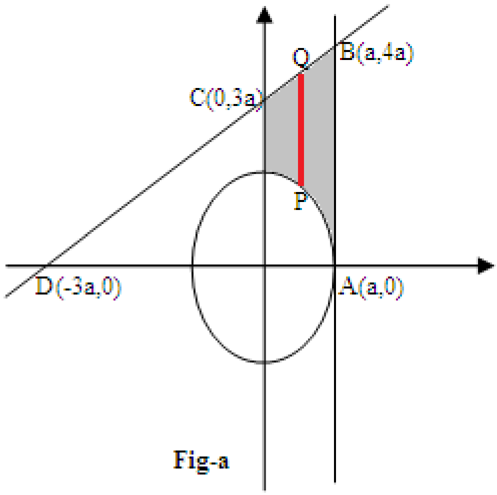


Fig-a

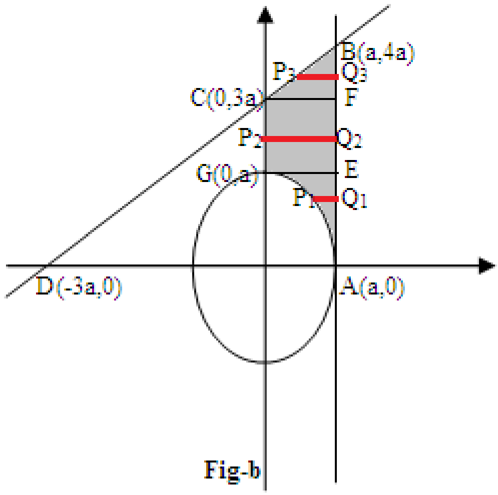


Fig-b

To change the order of integration we divide the region of integration into three regions $AEGA$, $EFCG$ and $FBCF$ and then consider a integrating strip parallel to x-axis in each region as shown in Fig-b.
For region $AEGA$**:** In the region $AEGA$, The point P_1 lies on $x^2 + y^2 = a^2$ i.e. $x = \sqrt{a^2 - y^2}$ ($\because P_1$ lies on RHS of symmetric line $x = 0$) and Q_1 lies on $x = a$. Therefore x varies from $\sqrt{a^2 - y^2}$ to a.
Also, the region $AEGA$ is bounded between $y = 0$ and $y = a$. Therefore y varies from 0 to a. Thus, in the region $AEGA$, we get

$$I_1 = \int_0^a \mathrm{d}y \int_{\sqrt{a^2-y^2}}^{a} \phi(x,y)\,\mathrm{d}x \qquad (2)$$

For region $EFCG$**:** In the region $EFCG$, The point P_2 lies on $x = 0$ and Q_2 lies on $x = a$. Therefore x varies from 0 to a.

Also, the region $EFCG$ is bounded between $y = a$ and $y = 3a$. Therefore y varies from a to $3a$. Thus, in the region $EFCG$, we get

$$I_2 = \int_a^{3a} \mathrm{d}y \int_0^a \phi(x,y)\,\mathrm{d}x \tag{3}$$

For region $FBCF$: In the region $FBCF$, The point P_3 lies on $y = x + 3a$ i.e $x = y - 3a$ and Q_3 lies on $x = a$. Therefore x varies from $y - 3a$ to a.
Also, the region $FBCF$ is bounded between $y = 3a$ and $y = 4a$. Therefore y varies from $3a$ to $4a$. Thus, in the region $FBCF$, we get

$$I_2 = \int_{3a}^{4a} \mathrm{d}y \int_{y-3a}^{a} \phi(x,y)\,\mathrm{d}x \tag{4}$$

Therefore by (2), (3) and (4), we get

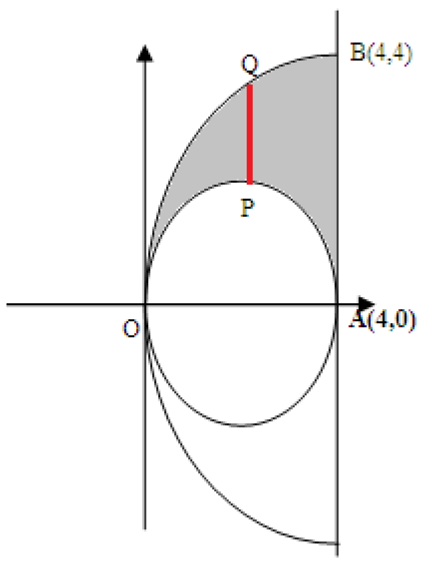
$$I = \int_0^a \mathrm{d}y \int_{\sqrt{a^2-y^2}}^{a} \phi(x,y)\,\mathrm{d}x + \int_a^{3a} \mathrm{d}y \int_0^a \phi(x,y)\,\mathrm{d}x + \int_{3a}^{4a} \mathrm{d}y \int_{y-3a}^{a} \phi(x,y)\,\mathrm{d}x$$

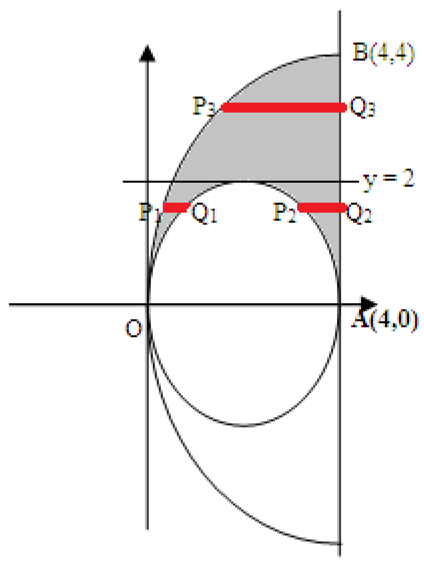
Example 8. Change the order of the integral of $\displaystyle\int_0^4 \int_{\sqrt{4x-x^2}}^{2\sqrt{x}} \phi(x,y)\,\mathrm{d}x\mathrm{d}y$

Solution: Consider,

$$I = \int_0^4 \int_{\sqrt{4x-x^2}}^{2\sqrt{x}} \phi(x,y)\,\mathrm{d}x\mathrm{d}y = \int_0^4 \mathrm{d}x \int_{\sqrt{4x-x^2}}^{2\sqrt{x}} \phi(x,y)\,\mathrm{d}y \tag{1}$$

From the limits of integration, it is clear that we have integrate w.r.t. y first and then w.r.t.x. Therefore, originally the integrating strip is parallel to y-axis.
Solving $y^2 = 4x$ and $x^2 + y^2 - 4x = 0$, we get $x = 0$. This implies $y = 0$. Thus circle and parabola intersects only at $(0,0)$. Therefore, parabola is outside the circle.
Now, we plot the curves $y = \sqrt{4x - x^2}$ i.e. $x^2 + y^2 - 4x = 0$ (circle: centre $(2,0)$ and radius 2), $y = 2\sqrt{x}$ i.e. $y^2 = 4x$, $x = 0$, $x = 4$ and consider the region of integration (shown shaded) as shown in following fig-a.





To change the order of integration we divide the region of integration into three regions as shown in Fig-b.

Remaining part is left to reader (see Example 2).

Example 9. Change the order of integration of $\int_0^1 \int_{2y}^{2(1+\sqrt{1-y})} \phi(x, y)\, dxdy$

Solution: Consider,

$$I = \int_0^1 \int_{2y}^{2(1+\sqrt{1-y})} \phi(x, y)\, dy\, dx = \int_0^1 dy \int_{2y}^{2(1+\sqrt{1-y})} \phi(x, y)\, dx$$

From the limits of integration, it is clear that we have integrate w.r.t. x first and then w.r.t.y. Therefore, originally the integrating strip is parallel to y-axis.

Now, we plot the curves $x = 2y$, $x = 2(1 + \sqrt{1-y})$ i.e. $(x-2)^2 = -4(y-1)$ (parabola with vertex at $(2, 1)$ and symmetric about $x = 2$ along negative direction), $y = 0$, $y = 1$ and consider the region of integration (shown shaded) as shown in following fig-a.

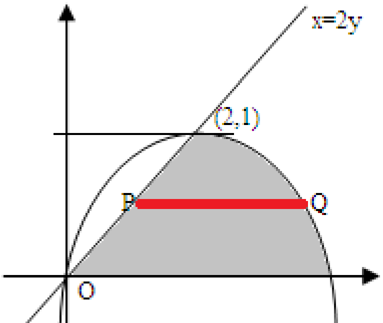


Fig-a

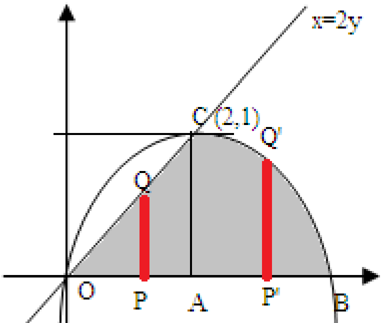


Fig-a

To change the order of integration we divide the region of integration into two regions $OACO$ and $ABCA$ and then consider a integrating strip parallel to y-axis in each region as shown in Fig-b.

For region $OACO$**:** In the region $OACO$, The point P lies on x-axis i.e. $y = 0$ and Q lies on $x = 2y$ i.e. $y = x/2$. Therefore y varies from 0 to $x/2$.

Also, the region $OACO$ is bounded between $x = 0$ and $x = 2$. Therefore x varies from 0 to 2. Thus, in the region $OACO$, we get

$$I_1 = \int_0^2 dx \int_0^{x/2} \phi(x, y)\, dy \tag{1}$$

For region $ABCA$**:** In the region $ABCA$, The point P' lies on x-axis i.e. $y = 0$ and Q' lies on $x = 2(1 + \sqrt{1-y})$ i.e. $y = \dfrac{4x - x^2}{4}$. Therefore y varies from 0 to $y = \dfrac{4x - x^2}{4}$.

Also, the region $ABCA$ is bounded between $x = 2$ and $x = 4$. Therefore x varies from 2 to 4. Thus, in the region $ABCA$, we get

$$I_2 = \int_2^4 dx \int_0^{\frac{4x-x^2}{4}} \phi(x, y)\, dy \tag{2}$$

By (1) and (2), we get

$$I = \int_0^2 dx \int_0^{x/2} \phi(x, y)\, dy + \int_2^4 dx \int_0^{\frac{4x-x^2}{4}} \phi(x, y)\, dy$$