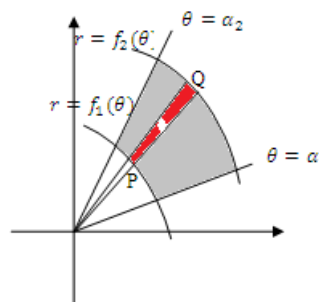


Area in Polar Coordinates

Let R be the region bounded by the curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$. The area of region R is given by

$$A = \iint_R r dr d\theta$$



Procedure to find area: To find area bounded by curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ we follow the steps given below.

step-a) Using given limits sketch the region of integration for area

step-b) Take integrating strip starting from origin.

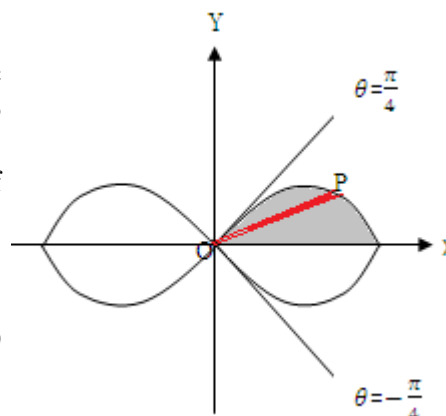
step-c) Find the integration limits for r and θ

step-d) Using the formula $A = \iint_R r dr d\theta = \int_{\alpha}^{\beta} \int_{f_1(\theta)}^{f_2(\theta)} r dr d\theta$, find area of bounded region.

Example 1. Find the area of one loop of lemniscate $r^2 = a^2 \cos 2\theta$.

Solution: The curve of $r^2 = a^2 \cos 2\theta$ is symmetric about initial line i.e. x -axis. For the area of one loop of the curve first we will obtain area of loop above the x -axis and then area of one loop is equal to double of area of loop above the x -axis i.e. $A = 2 \iint_R r dr d\theta$.

Now consider integrating strip in the region above the x -axis starting from origin as shown in above figure. At origin, we have $r = 0$ and P lies on $r^2 = a^2 \cos 2\theta$ i.e. $r = a\sqrt{\cos 2\theta}$.

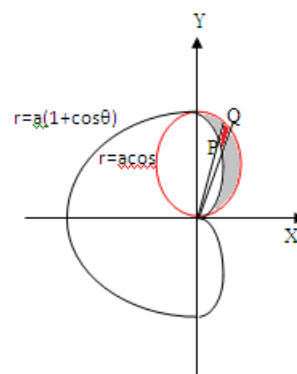


Therefore, r varies from 0 to $2\sqrt{\cos 2\theta}$ and θ varies from 0 to $\pi/4$. Therefore, the area of one loop is given by

$$\begin{aligned} A &= 2 \int_0^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r dr d\theta = 2 \int_0^{\pi/4} \left[\frac{r^2}{2} \right]_0^{a\sqrt{\cos 2\theta}} d\theta = \int_0^{\pi/4} a^2 \cos 2\theta d\theta = a^2 \int_0^{\pi/4} \cos 2\theta d\theta \\ &= a^2 \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/4} = \frac{a^2}{2} [1 - 0] = \frac{a^2}{2} \end{aligned}$$

Example 2. Find the area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$.

Solution: Given that $r = a \sin \theta$ is a circle. Using $y = r \sin \theta$, we get $r^2 = ay$ i.e. $x^2 + y^2 = ay$. One can write it as $x^2 + (y - a/2)^2 = a^2/4$. Thus $r = a \sin \theta$ is a circle with centre at $(0, a/2)$ and radius $a/2$. Consider the area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$ as shown in figure. Now, consider the integrating strip starting from origin as shown in figure. The point P lies on Cardioid $r = a(1 - \cos \theta)$ and Q lies on circle $r = a \sin \theta$. Therefore r varies from $a(1 - \cos \theta)$ to $a \sin \theta$ and θ varies from 0 to $\pi/2$



The area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$ is given by

$$\begin{aligned}
 A &= \int_0^{\pi/2} \int_{a(1-\cos \theta)}^{a \sin \theta} r dr d\theta = \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{a(1-\cos \theta)}^{a \sin \theta} d\theta = \frac{1}{2} \int_0^{\pi/2} [a^2 \sin^2 \theta - a^2(1 - \cos \theta)^2] d\theta \\
 &= \frac{a^2}{2} \int_0^{\pi/2} [\sin^2 \theta - 1 + 2 \cos \theta - \cos^2 \theta] d\theta = \frac{a^2}{2} \int_0^{\pi/2} [2 \cos \theta - 1 - \cos 2\theta] d\theta \\
 &= \frac{a^2}{2} \left[2 \sin \theta - \theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = \frac{a^2}{2} \left[2 - \frac{\pi}{2} - 0 \right] \\
 &= \frac{a^2}{4} [4 - \pi]
 \end{aligned}$$

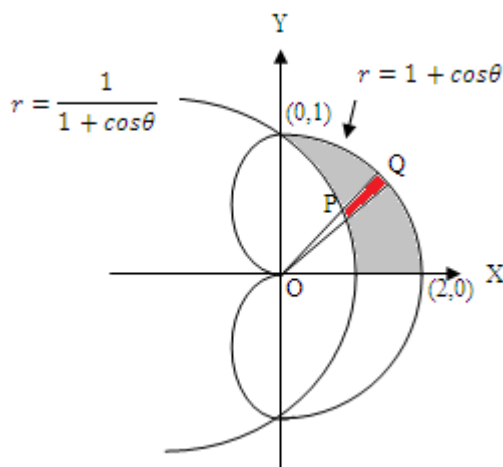
Example 3. Find the area inside the cardioid $r = 1 + \cos \theta$ and outside the curve $r = \frac{1}{1 + \cos \theta}$

Solution: First we shall convert the curve $r = \frac{1}{1 + \cos \theta}$ into Cartesian coordinates. We have $x = r \cos \theta$, this gives $\cos \theta = \frac{x}{r}$. Putting in above equation, we get

$$\begin{aligned}
 r &= \frac{1}{1 + x/r} \Rightarrow r = \frac{r}{r + x} \Rightarrow r = 1 - x \Rightarrow r^2 = (1 - x)^2 \\
 &\Rightarrow x^2 + y^2 = 1 - 2x + x^2 \\
 &\Rightarrow y^2 = -2 \left(x - \frac{1}{2} \right)
 \end{aligned}$$

Thus, $r = \frac{1}{1 + \cos \theta}$ represents a parabola with vertex at $(1/2, 0)$ and symmetric about x -axis along -ve direction.

Now, consider the area inside the cardioid $r = 1 + \cos \theta$ and outside the parabola $r = \frac{1}{1 + \cos \theta}$ as shown in the following figure.



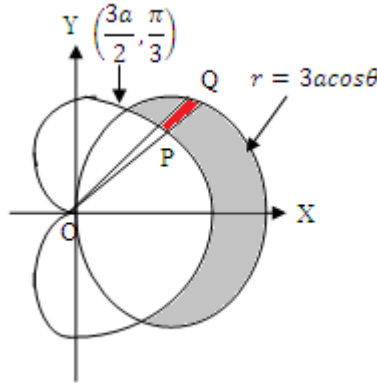
To find area of shaded region, consider a integrating strip starting from origin as shown in above figure. The point P lies on $r = \frac{1}{1 + \cos \theta}$ and Q lies on $r = 1 + \cos \theta$. Therefore r varies from $\frac{1}{1 + \cos \theta}$ to $1 + \cos \theta$ and θ varies from 0 to $\pi/2$. Since the area is symmetric about x -axis. Therefore required area is equal to twice the area of shaded region. Therefore,

$$\begin{aligned}
 A &= 2 \int_0^{\pi/2} \int_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} r dr d\theta = 2 \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} d\theta = \int_0^{\pi/2} \left[(1 + \cos \theta)^2 - \frac{1}{(1 + \cos \theta)^2} \right] d\theta \\
 &= \int_0^{\pi/2} \left[(1 + \cos^2 \theta + 2 \cos \theta) - \frac{1}{4} \sec^4 \left(\frac{\theta}{2} \right) \right] d\theta = \int_0^{\pi/2} \left[\left(1 + \frac{1 + \cos 2\theta}{2} + 2 \cos \theta \right) - \frac{1}{4} \sec^2 \left(\frac{\theta}{2} \right) \sec^2 \left(\frac{\theta}{2} \right) \right] d\theta \\
 &= \int_0^{\pi/2} \left\{ \frac{3}{2} + \frac{\cos 2\theta}{2} + 2 \cos \theta - \frac{1}{4} \sec^2 \left(\frac{\theta}{2} \right) \left[1 + \tan^2 \left(\frac{\theta}{2} \right) \right] \right\} d\theta \\
 &= \int_0^{\pi/2} \left\{ \frac{3}{2} + \frac{\cos 2\theta}{2} + 2 \cos \theta - \frac{1}{4} \left[\sec^2 \left(\frac{\theta}{2} \right) + \sec^2 \left(\frac{\theta}{2} \right) \tan^2 \left(\frac{\theta}{2} \right) \right] \right\} d\theta \\
 &= \left[\frac{3\theta}{2} + \frac{\sin 2\theta}{4} + 2 \sin \theta \right]_0^{\pi/2} - \frac{1}{4} \left[2 \tan \left(\frac{\theta}{2} \right) \right]_0^{\pi/2} - \frac{1}{4} \int_0^{\pi/2} \sec^2 \left(\frac{\theta}{2} \right) \tan^2 \left(\frac{\theta}{2} \right) d\theta \\
 &= \frac{3\pi}{4} + 2 - \frac{1}{2} - \frac{1}{4} \int_0^{\pi/2} \sec^2 \left(\frac{\theta}{2} \right) \tan^2 \left(\frac{\theta}{2} \right) d\theta \\
 &= \frac{3\pi}{4} + 2 - \frac{1}{2} - \frac{1}{4} \int_0^1 t^2 2 dt \quad (\text{put } \tan(\theta/2) = t) \\
 &= \frac{3\pi}{4} + 2 - \frac{1}{2} - \frac{1}{2} \left(\frac{1}{3} \right) = \frac{3\pi}{4} + \frac{4}{3}
 \end{aligned}$$

Example 4. Find the area which is inside the circle $r = 2a \cos \theta$ and outside the cardioid $r = a(1 + \cos \theta)$.

Solution: We have $x = r \cos \theta$. Then $r = 2a \cos \theta$ reduces to $r^2 = 2ax$ i.e. $x^2 + y^2 = 2ax$. One can rewrite as $\left(x - \frac{a}{2} \right)^2 + y^2 = \frac{a^2}{4}$. Thus $r = 2a \cos \theta$ is a circle with center at $\left(\frac{a}{2}, 0 \right)$ and radius

$\frac{3a}{2}$. Solving $r = 3a \cos \theta$ and $r = a(1 + \cos \theta)$ we get, $3a \cos \theta = a(1 + \cos \theta)$. This gives $2 \cos \theta = 1$. Thus, $\theta = \frac{\pi}{3}$. Therefore, the circle $r = 3a \cos \theta$ and the cardioid $r = a(1 + \cos \theta)$ intersects at $\left(\frac{3a}{2}, \frac{\pi}{3}\right)$. Consider the area inside the circle $r = 2a \cos \theta$ and outside the cardioid $r = a(1 + \cos \theta)$ as shown in the following figure.



Since both the curves are symmetric about x -axis, therefore we shall find area above the x -axis first and then the required area will be double of this. Now, consider integrating strip starting from origin as shown in above figure. The point P lies on cardioid and Q lies on the circle. Therefore r varies from $a(1 + \cos \theta)$ to $3a \cos \theta$ and θ varies from 0 to $\frac{\pi}{3}$. Therefore, the required area of shaded portion is

$$\begin{aligned}
 A &= 2 \int_0^{\pi/3} \int_{a(1+\cos \theta)}^{3a \cos \theta} r dr d\theta = 2 \int_0^{\pi/3} \left[\frac{r^2}{2} \right]_{a(1+\cos \theta)}^{3a \cos \theta} d\theta \\
 &= \int_0^{\pi/3} [9a^2 \cos^2 \theta - a^2(1 + \cos \theta)^2] d\theta = a^2 \int_0^{\pi/3} [9 \cos^2 \theta - 1 - 2 \cos \theta - \cos^2 \theta] d\theta \\
 &= a^2 \int_0^{\pi/3} [8 \cos^2 \theta - 1 - 2 \cos \theta] d\theta = a^2 \int_0^{\pi/3} [4 + 4 \cos 2\theta - 1 - 2 \cos \theta] d\theta \\
 &= a^2 \int_0^{\pi/3} [3 + 4 \cos 2\theta - 2 \cos \theta] d\theta = a^2 \left[3\theta + \frac{4 \sin 2\theta}{2} - 2 \sin \theta \right]_0^{\pi/3} \\
 &= a^2 \left[\frac{3\pi}{3} + 2 \sin \left(\frac{2\pi}{3} \right) - 2 \sin \left(\frac{\pi}{3} \right) - 0 \right] \\
 &= a^2 \pi
 \end{aligned}$$