Introduction and standard Formulae

Monday, March 15, 2021 9:42 AM

$y = f(x)$, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, ..., $\frac{d^ny}{dx^n}$

y, y', y'', y''', $y^{(4)}$, ..., $y^{(n)}$

f, f', f'', , $f^{(4)}$, ..., $f^{(n)}$

y_1, y_2, y_3, , y_n

Use: $\log(3x+2)$

Derivative of n^{th} order for standard function

Exponential

1) $y = e^{mx}$, $y' = me^{mx}$

$y'' = m^2 e^{mx}$

$$\boxed{y^{(n)} = m^n e^{mx}}$$

2) $y = a^{mx}$, $y' = m \lg a \; a^{mx}$

$y'' = m^2 (\lg a)^2 a^{mx}$

$$\boxed{y^{(n)} = m^n (\lg a)^n a^{mx}}$$

Algebraic

3) If $y = x^m$ (m positive integer)

Example: x^5, $y' = 5x^4$, $y'' = 5.4x^3$

$y''' = 5.4.3\,x^2$ / $y^{(5)} = 5!$ / $y^{(6)} = 0$

m = 5
n = 3

$y''' = \frac{5.4.3\,(2.1)}{(2.1)} x^2 = \frac{5!}{2!} x^2 = \frac{5!}{(5-3)!} x^{(m-n)}$

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for $y = x^m \Rightarrow$

$$\boxed{\begin{array}{ll} y^{(n)} = \frac{m!}{(m-n)!} x^{(m-n)} & \text{if } n < m \\ = m! & n = m \\ = 0 & n > m \end{array}}$$

$$= m! \qquad n > m$$

$$= 0$$

4) (m positive power)

$y = (ax+b)^m$, $y' = m(ax+b)^{m-1}(a)$

$$\boxed{\begin{aligned} y^{(n)} &= \frac{m!}{(m-n)!} a^n (ax+b)^{m-n} && \text{if } n < m \\ &= m!\, a^n && \text{if } n = m \\ &= 0 && \text{if } n > m \end{aligned}}$$

5) (m negative power)

$$y = \frac{1}{(ax+b)^m} = \frac{(m+n-1)!}{(m-1)!} \frac{(-1)^n a^n}{(ax+b)^{(m+n)}}$$

$$y = (ax+b)^m \Rightarrow y^{(n)} = m(m-1)(m-2)\cdots(m-n+1)(ax+b)^{m-n}(a^n)$$

Replace $\underline{m}$ by $(-m)$

$$y = (ax+b)^{-m} \Rightarrow y^{(n)} = (-m)(-m-1)(-m-2)\cdots(-m-n+1)(ax+b)^{-m-n} a^n$$

$$= (-1)^n \underline{(m)(m+1)(m+2)\cdots(m+n-1)} \frac{a^n}{(ax+b)^{m+n}}$$

$$\boxed{y^{(n)} = (-1)^n \frac{(m+n-1)!}{(m-1)!} \frac{a^n}{(ax+b)^{m+n}}}$$

6) Corollaries:

$y = \dfrac{1}{x^m}$, $a = 1,\ b = 0$

$$\boxed{y^{(n)} = (-1)^n \frac{(m+n-1)!}{(m-1)!} \frac{1}{x^{m+n}}}$$

7) $y = \dfrac{1}{ax+b}$, $m = 1$

$$\boxed{y^{(n)} = \frac{n!\,(-1)^n a^n}{(ax+b)^{n+1}}}$$

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$$y^{(n)} = \frac{\quad}{(ax+b)^{n+1}}$$

8) Logarithm:

$y = \log(ax+b)$, $\quad y' = \left(\frac{1}{(ax+b)}\right)(a)$,

$$y'' = a\left[\frac{1}{(ax+b)}\right]'$$

$$\boxed{y^{(n)} = (a)\left[\frac{1}{(ax+b)}\right]^{(n-1)} = \frac{(n-1)!\,(-1)^{n-1}\,a^n}{(ax+b)^n}}$$

9) Trigonometric

$y = \sin(ax+b)$

Then $y' = a\cos(ax+b)$

$y' = a\sin\left(ax+b+\frac{\pi}{2}\right)$

$y'' = a^2\cos\left(ax+b+\frac{\pi}{2}\right)$

$= a^2\sin\left(\left(ax+b+\frac{\pi}{2}\right)+\frac{\pi}{2}\right)$

$y'' = a^2\sin\left((ax+b)+2\frac{\pi}{2}\right)$

$$\boxed{y^{(n)} = a^n\sin\left((ax+b)+n\frac{\pi}{2}\right)}$$

10) $y = \cos(ax+b)$

Then $\boxed{y^{(n)} = a^n\cos\left(ax+b+n\frac{\pi}{2}\right)}$

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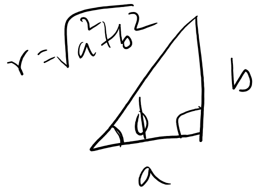
11) $y = e^{ax}\sin(bx+c)$

Then $y' = ae^{ax}\sin(bx+c) + e^{ax}\cos(bx+c)(b)$

$= e^{ax}\left[a\sin(bx+c) + b\cos(bx+c)\right]$

$\sqrt{a^2+b^2}$ $\quad = e^{ax}\sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin(bx+c) + \frac{b}{\sqrt{a^2+b^2}}\cos(bx+c)\right]$



$$= e^{ax}\sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin(bx+c)+\frac{b}{\sqrt{a^2+b^2}}\cos(bx+c)\right]$$

where $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi$, $r = \sqrt{a^2+b^2}$
$\phi = \tan^{-1}\left(\frac{b}{a}\right)$

$$y' = r\,e^{ax}\left[\sin(bx+c)\cos\phi + \cos(bx+c)\sin\phi\right]$$

$$y' = r\,e^{ax}\sin(\underline{(bx+c)+\phi})$$

$$y'' = \underline{r^2}\,e^{ax}\sin((bx+c)+\underline{2\phi})$$

$$\boxed{y^{(n)} = r^n e^{ax}\sin((bx+c)+n\phi), \quad r=\sqrt{a^2+b^2}, \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)}$$

12) $y = e^{ax}\cos(bx+c)$

$$\boxed{y^{(n)} = r^n e^{ax}\cos((bx+c)+n\phi)}, \quad r=?, \quad \phi=?$$

$\longrightarrow e^{(a\log k)x}$

13) $y = k^{ax}\sin(bx+c)$

$$\boxed{y^{(n)} = r^n k^{ax}\sin((bx+c)+n\phi), \quad r=\sqrt{(a\log k)^2+b^2}, \quad \phi = \tan^{-1}\left(\frac{b}{a\log k}\right)}$$

14) $y = k^{ax}\cos(bx+c)$

$y^{(n)} =$ H.W.

Algebraic Problems

Monday, March 15, 2021 11:51 AM

(Solved by partial fraction)

1) $y = \dfrac{x}{x^3 - 6x^2 + 11x - 6} = \dfrac{x}{(x-1)(x-2)(x-3)}$ (distinct roots)

$$y = \frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)} \quad \checkmark \quad (1)$$

$$\Rightarrow \frac{x}{(x-1)\cancel{(x-2)}(x-3)} = \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)\cancel{(x-2)}(x-3)}$$

comparing, $\checkmark$

$$\Rightarrow x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

put $x = 1$

$\rightarrow 1 = A(-1)(-2) + 0$

$\boxed{A = \frac{1}{2}}$

put $x = 2$ $\checkmark$

$\boxed{B = -2}$

put $x = 3$ $\checkmark$

$\boxed{C = \frac{3}{2}}$

put in (1)

$$y = \frac{1/2}{(x-1)} + \frac{-2}{(x-2)} + \frac{3/2}{(x-3)}$$

$$= \frac{1}{2}\left(\frac{1}{(x-1)}\right)^{(n)} - 2\left(\frac{1}{x-2}\right)^{(n)} + \frac{3}{2}\left(\frac{1}{x-3}\right)^{(n)}$$

$$\left[\text{Formula for } \left(\frac{1}{ax+b}\right)^{(n)} = \frac{n!\,(-1)^n a^n}{(ax+b)^{n+1}}\right]$$

$$= \frac{1}{2}\,\frac{(-1)^n n!}{(x-1)^{n+1}} - 2\,\frac{(-1)^n n!}{(x-2)^{n+1}} + \frac{3}{2}\,\frac{(-1)^n n!}{(x-3)^{n+1}}$$

$$\boxed{y^{(n)} = (-1)^n n!\left[\frac{1}{2(x-1)^{n+1}} - \frac{2}{(x-2)^{n+1}} + \frac{3/2}{(x-3)^{n+1}}\right]}$$

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2) $y = \dfrac{4x}{(x-1)^2(x+1)}$ (repeated roots)

$$y = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$\rightarrow \underline{Ax^2 + Cx^2}$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$(A+C) = 0$ $\checkmark$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)$$

put $x = 1$: $\boxed{B = 2}$ | put $x = -1$: $\boxed{C = -1}$ | $(A+C) = 0$, $\boxed{A = 1}$

$$y = \frac{1}{(x-1)} + \frac{2}{(x-1)^2} - \frac{1}{(x+1)}$$

$$\left[\left[\frac{1}{ax+b}\right]^{(n)} = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}, \quad \left[\frac{1}{(ax+b)^m}\right]^{(n)} = \frac{(m+n-1)!}{(m-1)!} \frac{(-1)^n a^n}{(ax+b)^{m+n}}\right]$$

$$y^{(n)} = \left[\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{2(n+1)!(-1)^n}{(x-1)^{n+2}} - \frac{(-1)^n n!}{(x+1)^{n+1}}\right]$$

$$= (-1)^n n! \left[\frac{1}{(x-1)^{n+1}} + \frac{2(n+1)}{(x-1)^{n+2}} - \frac{1}{(x+1)^{n+1}}\right]$$

3) $$y = \frac{x^2}{1-x^4} = \frac{x^2}{(1-x^2)(1+x^2)} = \frac{x^2}{(1-x)(1+x)(x+i)(x-i)}$$

$$y = \frac{A}{(1-x)} + \frac{B}{(1+x)} + \frac{C}{(x+i)} + \frac{D}{(x-i)}$$

$$y = \frac{1/4}{(1-x)} + \frac{1/4}{1+x} + \frac{1/4i}{x+i} + \frac{-1/4i}{x-i} \quad [\quad]$$

$$y^{(n)} = \frac{(-1)^n n!}{4}\left[\frac{(-1)^n}{(1-x)^{n+1}} + \frac{1}{(1+x)^{n+1}}\right] + \frac{(-1)^n n!}{4i}\left[\frac{1}{(x-i)^{n+1}} - \frac{1}{(x+i)^{n+1}}\right]$$

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4) $$y = \frac{x}{(x+1)^5} = \frac{x+1-1}{(x+1)^5} = \frac{1}{(x+1)^4} - \frac{1}{(x+1)^5}$$

H.W.

(If Deg N^r < Deg D^r) Apply Partial fraction

5) $$y = \left(\frac{x^3}{x^2-1}\right)$$

$$= Q + \frac{R}{Div} = x + \frac{x}{x^2-1}$$

$$x^2-1 \overline{)\, x^3} \quad \text{quotient } x$$
$$\underline{-x^3 \mp x}$$
$$x$$

$$y = x + \frac{x}{(x-1)(x+1)}$$

$$y = \underline{x} + \frac{x}{(x-1)(x+1)}$$

$$= \underline{x} + \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x+1}\right]$$

$$\frac{-x^3 \mp x}{\underline{\underline{x}}}$$

Apply nth order derivative. $\underline{(n > 1)}$

$$y^{(n)} = 0 + \frac{(-1)^n n!}{2}\left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right]$$

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r) If $y = x \log\left(\frac{x-1}{x+1}\right)$ Then Prove that

$$y^{(n)} = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$$

$\underline{\text{Sol}^n}$: $y = x\log(x-1) - x\log(x+1)$

$$y' = \frac{x}{x-1} + \underline{\log(x-1)} - \frac{x}{x+1} - \underline{\log(x+1)}$$

$$= \frac{x-1+1}{x-1} + \log(x-1) - \frac{x+1-1}{x+1} - \log(x+1)$$

$$= \cancel{1} + \frac{1}{x-1} + \log(x-1) - \cancel{1} + \frac{1}{x+1} - \log(x+1)$$

Apply $(n-1)^{th}$ order derivative.

$$\left[\quad \middle| \log(ax+b) = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n} \right]$$

$$y^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} + \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} + \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} - \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}}$$

$$= (-1)^{n-2}(n-2)!\left[\frac{-1(n-1)}{(x-1)^n} + \frac{(x-1)}{(x-1)^n} + \frac{-1(n-1)}{(x+1)^n} - \frac{x+1}{(x+1)^n}\right]$$

$$\boxed{y^{(n)} = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]}$$