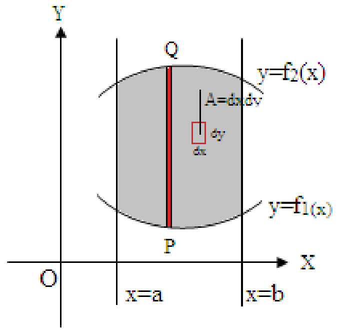
Applications of Double Integrals

In this section, we will study how to find out area and mass of lamina using double integrals.

Area in Cartesian coordinates:

R be the region bounded by the curves $y = f_1(x)$, $y = f_2(x)$ and the lines $x = a$ and $x = b$. The area of region R is given by

$$A = \iint\limits_R \mathrm{d}x\,\mathrm{d}y$$



Procedure to find area: To find area bounded by curves $y = f_1(x)$, $y = f_2(x)$ and the lines $x = a$ and $x = b$ we follow the steps given below.

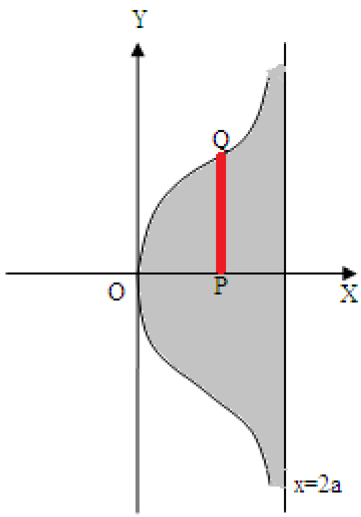
step-a) Using given limits sketch the region of integration for area

step-b) Take integrating strip either parallel to x-axis or parallel to y-axis.

step-c) Find the integration limits.

step-d) Using the formula $A = \iint\limits_R \mathrm{d}x\,\mathrm{d}y$, find area of bounded region.

Example 1. Find the area bounded by the curve $y^2(2a - x) = x^3$ and its asymptote.
Solution: The region bounded by the curve $y^2(2a - x) = x^3$ and its asymptote is shown (shaded region) in the following figure.



Here, the curve is symmetric about x-axis. Therefore, consider a strip PQ parallel to y-axis as shown in above figure. The point P lies on x-axis i.e. $y = 0$ and Q lies on $y^2(2a - x) = x^3$ i.e. $y = \sqrt{\dfrac{x^3}{2a - x}}$.

Therefore, y varies from 0 to $\sqrt{\dfrac{x^3}{2a-x}}$ and x varies from 0 to $2a$. Therefore, required area is given by

$$A = 2\int_0^{2a}\int_0^{\sqrt{x^3/2a-x}} \mathrm{d}y\mathrm{d}x = 2\int_0^{2a}\sqrt{\frac{x^3}{2a-x}}\mathrm{d}x$$

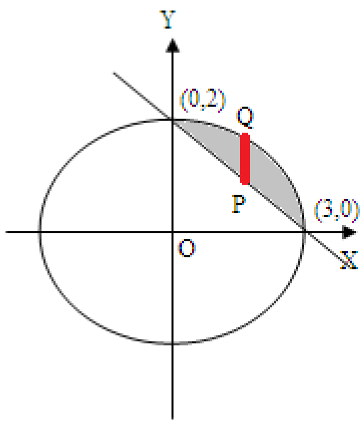
Put $x = 2at \Rightarrow\ \mathrm{d}x = 2a\mathrm{d}t$. When $x = 0$, we get $t = 0$ and for $x = 2a$ we get $t = 1$. Therefore,

$$\begin{aligned}
A &= 2\int_0^1 \sqrt{\frac{(2at)^3}{2a-2at}}2a\mathrm{d}t = 2\times 2a\times 2a\int_0^1 \frac{t^{3/2}}{(1-t)^{1/2}}\mathrm{d}t\\
&= 8a^2\int_0^1 t^{3/2}(1-t)^{-1/2}\mathrm{d}t = 8a^2\beta\left(\frac{3}{2}+1,-\frac{1}{2}+1\right)\\
&= 8a^2\beta\left(\frac{5}{2},\frac{1}{2}\right)\\
&= 8a^2\frac{\Gamma(5/2)\ \Gamma(1/2)}{\Gamma(3)} = 8a^2\frac{3/2\ 1/2\ \pi}{2}\\
&= 3a^2\pi \text{ sq.units}
\end{aligned}$$

Example 2. Find the smaller of the area bounded by the ellipse $4x^2+9y^2 = 36$ and the line $2x+3y = 6$.
Solution: First we shall find the point of intersection of ellipse $4x^2+9y^2 = 36$ and the line $2x+3y = 6$. Putting $3y = 6 - 2x$ in $4x^2 + 9y^2 = 36$, we get

$$\begin{aligned}
4x^2 + (6-2x)^2 = 36 &\Rightarrow 4x^2 + 36 - 24x + 4x^2 = 36\\
&\Rightarrow 8x^2 - 24x = 0\\
&\Rightarrow x = 0 \text{ or } x = 3
\end{aligned}$$

When $x = 0$, we get $y = 2$ and when $x = 3$, we get $y = 0$. Therefore, the ellipse $4x^2 + 9y^2 = 36$ and the line $2x + 3y = 6$ intersects at $(3, 0)$ and $(0, 2)$.
The smaller region bounded by the ellipse $4x^2 + 9y^2 = 36$ and the line $2x + 3y = 6$ is shown (shaded region) in the following figure.



To find the area of shaded region, consider a strip parallel to y-axis as shown in the above figure. The point P lies on a line $2x + 3y = 6$ i.e. $y = \dfrac{6-2x}{3}$ and the point Q lies on $4x^2 + 9y^2 = 36$ i.e.

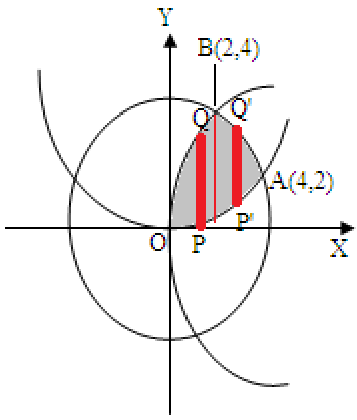
$y = \dfrac{2}{3}\sqrt{9-x^2}$. Therefore, y varies from $\dfrac{6-2x}{3}$ to $\dfrac{2}{3}\sqrt{9-x^2}$ and x varies from 0 to 3. Therefore,

$$
\begin{aligned}
A = \iint_R \mathrm{d}x\,\mathrm{d}y &= \int_0^3 \int_{\frac{6-2x}{3}}^{\frac{2}{3}\sqrt{9-x^2}} \mathrm{d}x\,\mathrm{d}y = \int_0^3 \left[\frac{2}{3}\sqrt{9-x^2} - \frac{6-2x}{3}\right] \mathrm{d}y \\
&= \frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 - \frac{1}{3}\left[6x - x^2\right]_0^3 \\
&= \frac{2}{3}\left[0 + \frac{9}{2}\frac{\pi}{2}\right] - \frac{1}{3}[18-9] \\
&= \frac{3\pi}{2} - 3 \\
&= \frac{3}{2}[\pi - 2] \text{ sq.units}
\end{aligned}
$$

Example 3. Find the area of curvilinear triangle lying in the first quadrant with one vertex at origin and bounded by the curves $y^2 = 8x$, $x^2 = 8y$ and $x^2 + y^2 = 20$.

Solution: First we shall find point of intersections. Solving $x^2 + y^2 = 20$ and $y^2 = 8x$, we get $x^2 + 8x - 20 = 0$. This gives $x = 2$ and $x = -10$. Here, we neglect $x = -10$ because we have to find area in first quadrant. For $x = 2$, we get $y = 4$. Thus $x^2 + y^2 = 20$ and $y^2 = 8x$ intersects at $(2, 4)$. Similarly, $x^2 + y^2 = 20$ and $x^2 = 8y$ intersects in first quadrant at $(4, 2)$.

Now consider the curvilinear triangle lying in the first quadrant with one vertex at origin and bounded by the curves $y^2 = 8x$, $x^2 = 8y$ and $x^2 + y^2 = 20$ as shown in following figure.

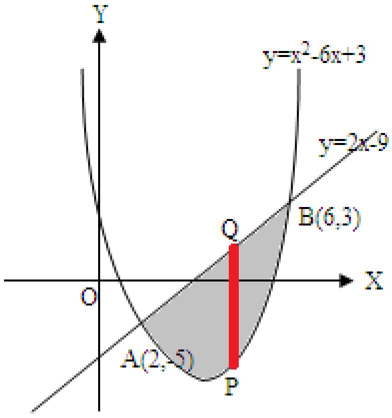


Consider the strips PQ and $P'Q'$ parallel to y-axis as shown in above figure. For the strip PQ, P lies on $x^2 = 8y$ i.e. $y = \dfrac{x^2}{8}$ and Q lies on $y^2 = 8x$ i.e. $y = \sqrt{8x}$. Therefore, y varies from $\dfrac{x^2}{8}$ to $\sqrt{8x}$ and x varies from 0 to 2. Now, fro strip $P'Q'$, y varies from $\dfrac{x^2}{8}$ to $\sqrt{20-x^2}$ and x varies from 2 to 4. Therefore, required area is given by

$$
\begin{aligned}
A &= \int_0^2 \int_{x^2/8}^{\sqrt{8x}} \mathrm{d}y\mathrm{d}x + \int_2^4 \int_{x^2/8}^{\sqrt{20-x^2}} \mathrm{d}y\mathrm{d}x \\
&= \int_0^2 \left[\sqrt{8x} - \frac{x^2}{8}\right] \mathrm{d}x + \int_2^4 \left[\sqrt{20-x^2} - \frac{x^2}{8}\right] \mathrm{d}x
\end{aligned}
$$

$$= \left[\frac{2\sqrt{8}}{3}x^{3/2} - \frac{x^3}{24}\right]_0^2 + \left[\frac{x}{2}\sqrt{20-x^2} + \frac{20}{2}\sin^{-1}\left(\frac{x}{\sqrt{20}}\right)\right]_2^4$$
$$= \left[\frac{16}{3} - \frac{1}{3}\right] + \left[\frac{4}{2}2 + 10\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - 4 - 10\sin^{-1}\left(\frac{1}{5}\right)\right]$$
$$= 5 + 10\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - 10\sin^{-1}\left(\frac{1}{5}\right)$$

Example 4. Find the area between the parabola $y = x^2 - 6x + 3$ and the line $y = 2x - 9$.
Solution: Here, $y = x^2 - 6x + 3$ i.e. $(x-3)^2 = y + 6$ is the parabola with vertex at $(3, -6)$ and it is symmetric about the line $x = 3$. Solving $y = x^2 - 6x + 3$ and $y = 2x - 9$, we get $2x - 9 = x^2 - 6x + 3$. Solving this, we get $x = 2$ and $x = 6$. For $x = 2$, we get $y = -5$ and for $x = 6$ we get $y = 3$. Therefore the parabola $y = x^2 - 6x + 3$ and the line $y = 2x - 9$ intersects at $(2, -5)$ and $(6, 3)$. The area between the parabola $y = x^2 - 6x + 3$ and the line $y = 2x - 9$ is shown in the following figure.



Now, consider an integrating strip parallel to y-axis as shown in above figure. The point P lies on $y = x^2 - 6x + 3$ and Q lies on $y = 2x - 9$. Therefore y varies from $x^2 - 6x + 3$ to $y = 2x - 9$ and x varies from 2 to 6. Therefore,

$$\text{Area} = \int_2^6 \int_{x^2-6x+3}^{2x-9} \mathrm{d}y\mathrm{d}x = \int_2^6 [2x - 9 - x^2 + 6x - 3]\mathrm{d}x = \int_2^6 [8x - x^2 - 12]\mathrm{d}x$$
$$= \left[4x^2 - \frac{x^3}{3} - 12x\right]_2^6 = \left(144 - \frac{216}{3} - 72\right) - \left(16 - \frac{8}{3} - 24\right)$$
$$= 8 + \frac{8}{3}$$
$$= \frac{32}{3}$$