DE MOIVRE'S THEOREM

Tuesday, October 12, 2021 1:00 PM

DE MOIVRE'S THEOREM:

Statement : For any rational number n the value or one of the values of $(\cos\theta + i\sin\theta)^n = \cos\, n\,\theta + i\,\sin n\,\theta$

$(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$

1. If $z = \cos\theta + i\sin\theta$ then

$\frac{1}{z} = z^{-1} = (\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

i.e. $\frac{1}{z} = \cos\theta - i\sin\theta$

2. $(\cos\theta - i\sin\theta)^n = \cos\, n\,\theta - i\,\sin n\,\theta$

For, $(\cos\theta - i\sin\theta)^n = \{\cos\,(-\theta) + i\,\sin\,(-\theta)\}^n$

$= \cos(-n\theta) + i\sin(-n\theta).$

$= \cos n\,\theta - i\sin\,n\,\theta$

Note : Note carefully that ,

(1) $(\sin\theta + i\cos\theta)^n \neq \sin n\,\theta + i\cos n\,\theta$

But $(\sin\theta + i\cos\theta)^n = [\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)]^n$

$= \cos\, n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$

(2) $(\cos\,\theta + i\sin\phi)^n \neq \cos\,n\,\theta + i\sin n$.

$(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$

SOME SOLVED EXAMPLES:

1. Simplify $\frac{(\cos 2\theta - i\sin 2\theta)^7(\cos 3\theta + i\sin 3\theta)^5}{(\cos 3\theta + i\sin 3\theta)^{12}(\cos 5\theta - i\sin 5\theta)^7}$

$(\cos 2\theta - i\sin 2\theta) = (\cos\theta + i\sin\theta)^{-2}$

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

$\cos 5\theta - i\sin 5\theta = (\cos\theta + i\sin\theta)^{-5}$

Given expression $= \dfrac{(\cos\theta + i\sin\theta)^{-14}(\cos\theta + i\sin\theta)^{15}}{(\cos\theta + i\sin\theta)^{36}(\cos\theta + i\sin\theta)^{-35}}$

$= \dfrac{\cos\theta + i\sin\theta}{\cos\theta + i\sin\theta} = 1.$

2.

Prove that $\frac{(1+i)^8\left(\sqrt{3}-i\right)^4}{(1-i)^4\left(\sqrt{3}+i\right)^8} = -\frac{1}{4}$

for $1+i$, $r = \sqrt{2}$ $\theta = \pi/4$ $\Rightarrow 1+i = \sqrt{2}e^{i\pi/4}$

$\sqrt{3}-i$, $r = 2$ $\theta = -\pi/6$ $\Rightarrow \sqrt{3}-i = 2e^{-i\pi/6}$

$1-i$, $r = \sqrt{2}$ $\theta = -\frac{\pi}{4}$ $\Rightarrow 1-i = \sqrt{2}e^{-i\pi/4}$

$\sqrt{3}+i$, $r = 2$ $\theta = \frac{\pi}{6}$ $\Rightarrow \sqrt{3}+i = 2e^{i\pi/6}$

$\sqrt{3}+i$, $r = 2$ $\theta = \frac{\pi}{6}$ $\Rightarrow \sqrt{3}+i = 2e^{i\pi/6}$

$$\therefore LHS = \frac{(\sqrt{2})^8 e^{i2\pi} \times (2)^4 e^{-i2\pi/3}}{(\sqrt{2})^4 e^{-i\pi} \times 2^8 e^{i4\pi/3}} = \frac{2^8}{2^{10}} e^{i\left(2\pi - \frac{2\pi}{3} + \pi - \frac{4\pi}{3}\right)}$$

$$= \frac{1}{4} e^{i\pi} = \frac{1}{4}(\cos\pi + i\sin\pi)$$

$$= \frac{-1}{4}$$

3.

Find the modulus and the principal value of the argument of $\frac{\left(1+i\sqrt{3}\right)^{16}}{\left(\sqrt{3}-i\right)^{17}}$

Soln:- $1+i\sqrt{3}$, $r = 2$, $\theta = \frac{\pi}{3}$ $\Rightarrow 1+i\sqrt{3} = 2e^{i\pi/3}$

$\sqrt{3}-i$, $r = 2$, $\theta = -\frac{\pi}{6}$ $\Rightarrow \sqrt{3}-i = 2e^{-i\pi/6}$

$$\frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}} = \frac{2^{16} e^{i16\pi/3}}{2^{17} e^{-i17\pi/6}} = \frac{1}{2} e^{i\left(\frac{16\pi}{3} + \frac{17\pi}{6}\right)}$$

$$= \frac{1}{2} e^{i\left(\frac{49\pi}{6}\right)} = \frac{1}{2}\left(\cos\frac{49\pi}{6} + i\sin\frac{49\pi}{6}\right)$$

$$= \frac{1}{2}\left[\cos\left(8\pi + \frac{\pi}{6}\right) + i\sin\left(8\pi + \frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

Hence the modulus is $\frac{1}{2}$ and principal value of argument is $\frac{\pi}{6}$.

4. Simplify $\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n$

we have $1 = \sin^2\alpha + \cos^2\alpha = (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha)$

$1 + \sin\alpha + i\cos\alpha = (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha) + (\sin\alpha + i\cos\alpha)$

$$1+\sin\alpha+i\cos\alpha = (\sin\alpha+i\cos\alpha)(\sin\alpha-i\cos\alpha)+(\sin\alpha+i\cos\alpha)$$

$$= (\sin\alpha+i\cos\alpha)(1+\sin\alpha-i\cos\alpha)$$

$$\Rightarrow \frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha} = \sin\alpha+i\cos\alpha$$

$$\Rightarrow \left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n = (\sin\alpha+i\cos\alpha)^n$$

$$= \left[\cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)\right]^n$$

$$= \cos n\left(\frac{\pi}{2}-\alpha\right)+i\sin n\left(\frac{\pi}{2}-\alpha\right)$$

5. If $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ *and* $\bar{z}$ is the conjugate of z prove that $(z)^{10} + (\bar{z})^{10} = 0$.

Soln :- $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$\therefore \bar{z} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$

$$(z)^{10}+(\bar{z})^{10} = \left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{10} + \left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)^{10}$$

$$= \cos\frac{10\pi}{4}+i\sin\frac{10\pi}{4}+\cos\frac{10\pi}{4}-i\sin\frac{10\pi}{4}$$

$$= 2\cos\frac{5\pi}{2}$$

$$= 2\times 0$$

$$= 0$$

Q.5 **(ii)** $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos(n\pi/3)$.

$$1+i\sqrt{3} = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$1-i\sqrt{3} = 2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$$

6. If α, β are the roots of the equation $x^2 - 2x + 2 = 0$, prove that $\alpha^n + \beta^n = 2.2^{n/2} \cos n\pi/4$, Hence, deduce that $\alpha^8 + \beta^8 = 32$

Soln :- α, β are roots of $x^2 - 2x + 2 = 0$

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

let $\alpha = 1+i$ and $\beta = 1-i$

$\therefore \alpha = 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\beta = 1-i = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

Now $\alpha^n + \beta^n = (\sqrt{2})^n\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n + (\sqrt{2})^n\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^n$

$= (\sqrt{2})^n\left\{\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right\}$

$= (\sqrt{2})^n\left\{2\cos\frac{n\pi}{4}\right\}$

$= 2 \cdot 2^{n/2} \cos n\pi/4$

$=$ RHS.

Now put $n = 8$,

$\alpha^8 + \beta^8 = 2 \cdot 2^{8/2} \cos\frac{8\pi}{4} = 2 \cdot 2^4 \cos 2\pi$

$= 2^5(1) = 32$

7. If α, β are the roots of the equation $x^2 - 2\sqrt{3}x + 4 = 0$, Prove that $\alpha^3 + \beta^3 = 0$ and $\alpha^3 - \beta^3 = 16\,i$ (HW)

8. If $a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$, $c = \cos 2\gamma + i\sin 2\gamma$, prove that $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2\cos(\alpha + \beta - \gamma)$

Soln :- $a = \cos 2\alpha + i\sin 2\alpha = e^{i2\alpha}$

$b = \cos 2\beta + i\sin 2\beta = e^{i2\beta}$

$$b = \cos 2\beta + i\sin 2\beta = e^{i2\beta}$$

$$c = \cos 2\gamma + i\sin 2\gamma = e^{i2\gamma}$$

Now $\frac{ab}{c} = \frac{e^{i2\alpha} \cdot e^{i2\beta}}{e^{i2\gamma}} = e^{i(2\alpha+2\beta-2\gamma)} = e^{i2(\alpha+\beta-\gamma)}$

$\therefore \frac{c}{ab} = e^{-i2(\alpha+\beta-\gamma)}$ (taking reciprocal)

$$LHS = \sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = \sqrt{e^{i2(\alpha+\beta-\gamma)}} + \sqrt{e^{-i2(\alpha+\beta-\gamma)}}$$

$$= e^{i(\alpha+\beta-\gamma)} + e^{-i(\alpha+\beta-\gamma)}$$

$$= \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)$$
$$+ \cos(\alpha+\beta-\gamma) - i\sin(\alpha+\beta-\gamma)$$

$$= 2\cos(\alpha+\beta-\gamma)$$
$$= RHS.$$

9. If $x - \frac{1}{x} = 2i\sin\theta, y - \frac{1}{y} = 2\,i\sin\Phi, z - \frac{1}{z} = 2i\sin\psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2\cos(\theta + \Phi + \psi)$ **(ii)** $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Soln :- Given $x - \frac{1}{x} = 2i\sin\theta$

$$x^2 - 1 = 2i\sin\theta\, x$$

$$x^2 - 2i\sin\theta x - 1 = 0$$

This is a quadratic eqn in x

$ax^2 + bx + c = 0$ $a = 1$, $b = -2i\sin\theta$
$c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2i\sin\theta \pm \sqrt{-4\sin^2\theta + 4}}{2}$$

$$= \frac{2i\sin\theta \pm 2\sqrt{1 - \sin^2\theta}}{2} = i\sin\theta \pm \cos\theta.$$

Let $x = \cos\theta + i\sin\theta$

Similarly $y = \cos\phi + i\sin\phi$ $z = \cos\psi + i\sin\psi$

Let $x = \cos\theta + i\sin\theta$

similarly $y = \cos\phi + i\sin\phi$, $z = \cos\psi + i\sin\psi$

$\therefore x = e^{i\theta}, \quad y = e^{i\phi}, \quad z = e^{i\psi}$

(i) $LHS = xyz + \frac{1}{xyz}$

$$= \left(e^{i\theta}\cdot e^{i\phi}\cdot e^{i\psi}\right) + \frac{1}{\left(e^{i\theta}\cdot e^{i\phi}\cdot e^{i\psi}\right)}$$

$$= e^{i(\theta+\phi+\psi)} + \frac{1}{e^{i(\theta+\phi+\psi)}}$$

$$= e^{i(\theta+\phi+\psi)} + e^{-i(\theta+\phi+\psi)}$$

$$= \cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi) + \cos(\theta+\phi+\psi) - i\sin(\theta+\phi+\psi)$$

$$= 2\cos(\theta+\phi+\psi)$$

(ii) $LHS = \frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = \frac{\sqrt[m]{e^{i\theta}}}{\sqrt[n]{e^{i\phi}}} + \frac{\sqrt[n]{e^{i\phi}}}{\sqrt[m]{e^{i\theta}}}$

$$= \frac{e^{i\theta/m}}{e^{i\phi/n}} + \frac{e^{i\phi/n}}{e^{i\theta/m}}$$

$$= e^{i\left(\frac{\theta}{m}-\frac{\phi}{n}\right)} \quad i\left(\phi \quad -\right)$$

$$= e^{i\left(\frac{\theta}{m}-\frac{\phi}{n}\right)} + e^{-i\left(\frac{\theta}{m}-\frac{\phi}{n}\right)}$$

$$= \cos\left(\frac{\theta}{m}-\frac{\phi}{n}\right) + i\sin\left(\frac{\theta}{m}-\frac{\phi}{n}\right) + \cos\left(\frac{\theta}{m}-\frac{\phi}{n}\right) - i\sin\left(\frac{\theta}{m}-\frac{\phi}{n}\right)$$

$$= 2\cos\left(\frac{\theta}{m}-\frac{\phi}{n}\right)$$

$= RHS.$

10. If $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, Prove that $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$.

Soln :- we consider

$$(\cos\alpha + 2\cos\beta + 3\cos\gamma) + i(\sin\alpha + 2\sin\beta + 3\sin\gamma) = 0$$

$$\therefore (\cos\alpha + i\sin\alpha) + 2(\cos\beta + i\sin\beta) + 3(\cos\gamma + i\sin\gamma) = 0$$

Let $x = \cos\alpha + i\sin\alpha$

$y = 2(\cos\beta + i\sin\beta)$ Note that $\boxed{x+y+z=0}$

$z = 3(\cos\gamma + i\sin\gamma)$

We have $(x+y+z)^3 = x^3+y^3+z^3 + 3(x+y+z)(xy+yz+zx) - 3xyz$

$$0 = x^3+y^3+z^3 + 0 - 3xyz$$

$$\Rightarrow x^3+y^3+z^3 = 3xyz$$

$$(\cos\alpha + i\sin\alpha)^3 + 2^3(\cos\beta + i\sin\beta)^3 + 3^3(\cos\gamma + i\sin\gamma)^3$$
$$= 3(\cos\alpha + i\sin\alpha)\cdot 2(\cos\beta + i\sin\beta)\cdot 3(\cos\gamma + i\sin\gamma)$$

$$\Rightarrow (\cos 3\alpha + i\sin 3\alpha) + 8(\cos 3\beta + i\sin 3\beta) + 27(\cos 3\gamma + i\sin 3\gamma)$$
$$= 18[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$$

$$\Rightarrow (\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma) + i(\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma)$$
$$= 18[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$$

comparing the imaginary parts on both sides

$$\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$$

11.

If $x_r = cos\frac{\pi}{3^r} + isin\frac{\pi}{3^r}$, prove that **(i)** $x_1x_2x_3 \ldots ad.inf. = i$ **(ii)** $x_0x_1x_2 \ldots ad.inf. = -i$

Soln :- $x_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$

$$x_0 = \cos\frac{\pi}{3^0} + i\sin\frac{\pi}{3^0} = \cos\pi + i\sin\pi = -1$$

$x_1 = \cos\frac{\pi}{} + i\sin\pi$

3^0 $\quad$ 3^0

$x_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$x_2 = \cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}$ and so on

(i) Now $x_1 x_2 x_3 \cdots\cdots$ ad.inf

$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\left(\cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3}\right)\cdots\cdots$

$= \cos\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \cdots\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \cdots\right)$

This is sum of an infinite G.P.

where $a = \frac{\pi}{3}$ & $r = \frac{1}{3}$ $\quad S_n = \frac{a}{1-r} = \frac{\pi/3}{1-1/3} = \frac{\pi}{2}$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$ = RHS

(ii) $x_0 x_1 x_2 \cdots\cdots$ ad inf $= x_0\,(x_1 x_2 x_3 \cdots$ ad inf$)$

$= (-1)(i)$

$= -i$ = RHS.

12. If $(\cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)\ldots\ldots[\cos(2n-1)\theta + i\sin(2n-1)\theta] = 1$ then show that the general value of θ is $\frac{2r\pi}{n^2}$

Soln:- LHS:- $(\cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)\cdots\cdots$

$[\cos(2n-1)\theta + i\sin(2n-1)\theta] = 1$

$\Rightarrow \cos[\theta + 3\theta + \cdots + (2n-1)\theta] + i\sin[\theta + 3\theta + \cdots + (2n-1)\theta] = 1$

$\Rightarrow \cos[1 + 3 + \cdots + (2n-1)]\theta + i\sin[1 + 3 + \cdots + (2n-1)]\theta = 1.$

$1 + 3 + \cdots + (2n-1)$ is an A.P. with first term 1, the common difference $= 2$, no of terms $= n$

The sum, $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[2 + (n-1)2] = n^2$

$\therefore \cos(n^2\theta) + i\sin(n^2\theta) = 1$

$= \cos 0 + i\sin 0$ (principal value)

$$\therefore \cos(n^2\theta) + i\sin(n^2\theta) = 1$$

$$= \cos 0 + i \sin 0 \quad \text{(principal value)}$$

$$= \cos(2r\pi) + i\sin(2r\pi)$$

$$\therefore n^2\theta = 2r\pi$$

$$\Rightarrow \theta = \frac{2r\pi}{n^2} \Rightarrow \text{general value of } \theta.$$

13. By using De Moivre's Theorem show that $\sin\alpha + \sin 2\alpha + \cdots + \sin 5\alpha = \frac{\sin 3\alpha \sin(5\alpha/2)}{\sin\alpha/2}$

Soln:- $$\frac{1-z^6}{1-z} = 1+z+z^2+z^3+z^4+z^5 \quad \text{—— (i)}$$

Let $z = \cos\alpha + i\sin\alpha$, $z^n = \cos n\alpha + i \sin n\alpha$

$$\therefore 1+z+z^2+z^3+z^4+z^5$$

$$= 1 + (\cos\alpha + i\sin\alpha) + (\cos 2\alpha + i\sin 2\alpha) + \cdots + (\cos 5\alpha + i \sin 5\alpha)$$

$$= (1+\cos\alpha + \cdots + \cos 5\alpha) + i(\sin\alpha + \cdots + \sin 5\alpha) \quad \text{—— (ii)}$$

$$\frac{1-z^6}{1-z} = \frac{1-(\cos 6\alpha + i\sin 6\alpha)}{1-(\cos\alpha + i\sin\alpha)} = \frac{(1-\cos 6\alpha) - i\sin 6\alpha}{(1-\cos\alpha) - i\sin\alpha}$$

$$= \frac{2\sin^2 3\alpha - 2i\sin 3\alpha\cos 3\alpha}{2\sin^2\alpha/2 - 2i\sin\alpha/2\cos\alpha/2}$$

$$= \frac{2\sin 3\alpha\,(\sin 3\alpha - i\cos 3\alpha)}{2\sin\alpha/2\,(\sin\alpha/2 - i\cos\alpha/2)}$$

$$= \frac{\sin 3\alpha}{\sin\alpha/2} \cdot \frac{[\sin 3\alpha - i\cos 3\alpha][\sin\alpha/2 + i\cos\alpha/2]}{[\sin\alpha/2 - i\cos\alpha/2][\sin\alpha/2 + i\cos\alpha/2]}$$

$$= \frac{\sin 3\alpha}{\sin\alpha/2} \cdot \frac{[\cos(\frac{\pi}{2} - 3\alpha) - i\sin(\frac{\pi}{2} - 3\alpha)][\cos(\frac{\pi}{2} - \frac{\alpha}{2}) + i\sin(\frac{\pi}{2} - \frac{\alpha}{2})]}{\sin^2\alpha/2 + \cos^2\alpha/2}$$

$$= \frac{\sin 3\alpha}{\sin \alpha/2}\left[\cos\left(-\frac{\pi}{2}+3\alpha\right)+i\sin\left(-\frac{\pi}{2}+3\alpha\right)\right]\left[\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right]$$

$$= \frac{\sin 3\alpha}{\sin \alpha/2}\left[\cos\left(-\frac{\pi}{2}+3\alpha+\frac{\pi}{2}-\frac{\alpha}{2}\right)+i\sin\left(-\frac{\pi}{2}+3\alpha+\frac{\pi}{2}-\frac{\alpha}{2}\right)\right]$$

$$= \frac{\sin 3\alpha}{\sin \alpha/2}\left(\cos\frac{5\alpha}{2}+i\sin\frac{5\alpha}{2}\right) \quad \text{--- (iii)}$$

comparing (i), (ii) and (iii)
we get the answer.

Applications of De-Moivre's Theorem

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ROOTS OF ALGEBRAIC EQUATIONS:

De Moivre's theorem can be used to find the roots of an algebraic equation.
General values of $\cos\theta = \cos(2k\pi + \theta)$ and $\sin\theta = \sin(2k\pi + \theta)$ where k is an integer.

To solve the equation of the type $z^n = \cos\theta + i\sin\theta$, we apply De Moivre's theorem

$$z = (\cos\theta + i\sin\theta)^{\frac{1}{n}} = \cos\frac{\theta}{n} + i\sin\frac{\theta}{n}$$

n^{th} deg $\rightarrow$ n roots.

z^{10}

This shows that $\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$ is one of the n roots of $z^n = \cos\theta + i\sin\theta$.

The other roots are obtain by expressing the number in the general form

$$z = \{\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)\}^{\frac{1}{n}} = \cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)$$

Taking k = 0, 1, 2,..............,(n – 1). We get n roots of the equation.

Note: (i) Complex roots always occur in conjugate pair if coefficients of different powers of x including constant terms in the equation are real.

(ii) Continued products mean products of all the roots of the equation.

SOME SOLVED EXAMPLES:

1. If ω is a cube root of unity, prove that $(1-\omega)^6 = -27$

Soln consider $z^3 = 1$

$\therefore z = \sqrt[3]{1} = (1)^{1/3}$

$z = (\cos 0 + i\sin 0)^{1/3}$

$= [\cos(2k\pi) + i\sin(2k\pi)]^{1/3}$

$z = \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)$

where $k = 0, 1, 2$

$1 = 1 + i0$

$r = \sqrt{x^2+y^2}$

$= \sqrt{1+0} = 1$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$= \tan^{-1}\left(\frac{0}{1}\right) = 0$

$\cos 0 + i\sin 0$

when $k = 0$, $z_0 = \cos(0) + i\sin(0) = 1$

$k = 1$, $z_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \omega$

$$k=2, \quad z_2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$$

$$= \left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]^2 = \omega^2$$

To prove $(1-\omega)^6 = -27$

$$1+\omega+\omega^2 = 1 + \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$$

$$= 1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$1+\omega+\omega^2 = 0$.

(sum of all the nth roots of unity is always 0)

$\therefore 1+\omega^2 = -\omega$

$$\text{Now } (1-\omega)^6 = \left[(1-\omega)^2\right]^3 = \left[1-2\omega+\omega^2\right]^3$$

$$= \left[-\omega-2\omega\right]^3 = (-3\omega)^3 = -27\omega^3$$

$$= -27(1)$$

$$= -27.$$

nth root. $n=3$.

2. Find all the values of $\sqrt[3]{(1+i)/\sqrt{2}} + \sqrt[3]{(1-i)/\sqrt{2}}$

Soln :- $$\sqrt[3]{\frac{1+i}{\sqrt{2}}} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{1/3} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/3}$$

$$= \left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi+\frac{\pi}{4}\right)\right]^{1/3}$$

$$= \left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]$$

$$= \left[\cos\left(\frac{8k+1}{4}\right)\pi + i\sin\left(\frac{8k+1}{4}\right)\pi\right]^{1/3}$$

$$= \cos\left(\frac{8k+1}{12}\right)\pi + i\sin\left(\frac{8k+1}{12}\right)\pi$$

where $k = 0, 1, 2$

Similarly $\left(\frac{1-i}{\sqrt{2}}\right)^{1/3} = \cos\left(\frac{8k+1}{12}\right)\pi - i\sin\left(\frac{8k+1}{12}\right)\pi$

$k = 0, 1, 2$

Now $\left(\frac{1+i}{\sqrt{2}}\right)^{1/3} + \left(\frac{1-i}{\sqrt{2}}\right)^{1/3} = 2\cos\left(\frac{8k+1}{12}\right)\pi$

$k = 0, 1, 2$

$\therefore$ The roots are

$2\cos\frac{\pi}{12}$, $2\cos\frac{9\pi}{12}$, $2\cos\frac{17\pi}{12}$

3. Find the cube roots of $(1 - cos\theta - i\ sin\theta)$.

Soln:- $(1 - \cos\theta - i\sin\theta)^{1/3}$

$$= \left[2\sin^2\left(\frac{\theta}{2}\right) - 2i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{1/3}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]^{1/3}$$

$$= \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]^{1/3}$$

$$= \left[2\sin\left(\frac{\theta}{2}\right)\right]\left[\cos\left(\frac{\theta}{2}-\frac{\pi}{2}\right)+i\sin\left(\frac{\theta}{2}-\frac{\pi}{2}\right)\right]$$

$$= \left[2\sin\left(\frac{\theta}{2}\right)\right]^{1/3}\left[\cos\left(\frac{\theta}{2}-\frac{\pi}{2}\right)+i\sin\left(\frac{\theta}{2}-\frac{\pi}{2}\right)\right]^{1/3}$$

$$= \left[2\sin\left(\frac{\theta}{2}\right)\right]^{1/3}\left[\cos\left(2k\pi+\frac{\theta}{2}-\frac{\pi}{2}\right)+i\sin\left(2k\pi+\frac{\theta}{2}-\frac{\pi}{2}\right)\right]^{1/3}$$

$$= \left[2\sin\frac{\theta}{2}\right]^{1/3}\left[\cos\left(\frac{(4k-1)\pi+\theta}{2}\right)+i\sin\left(\frac{(4k-1)\pi+\theta}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{1/3}\left[\cos\left(\frac{(4k-1)\pi+\theta}{6}\right)+i\sin\left(\frac{(4k-1)\pi+\theta}{6}\right)\right]$$

where $k = 0, 1, 2$

4. Find the continued product of all the value of $(-i)^{2/3}$

Soln :- $z = (-i)^{2/3} = \left[(-i)^2\right]^{1/3} = [-1]^{1/3}$

$\theta = \tan^{-1}\left(\frac{0}{-1}\right)$ $= (\cos\pi + i\sin\pi)^{1/3}$

$= \left[\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)\right]^{1/3}$

$$z = \cos\left(\frac{2k+1}{3}\right)\pi + i\sin\left(\frac{2k+1}{3}\right)\pi$$

$k = 0, 1, 2$

Now $z_0 = \cos\left(\frac{\pi}{3}\right) + i\sin\frac{\pi}{3}$

$z_1 = \cos\pi + i\sin\pi$

$$z_2 = \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)$$

Now the continued product

$$z_0 \cdot z_1 \cdot z_2 = \left[\cos\left(\frac{\pi}{3}\right) + i\sin\frac{\pi}{3}\right]\left[\cos\pi + i\sin\pi\right]\left[\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right]$$

$$= \cos\left(\frac{\pi}{3} + \pi + \frac{5\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \pi + \frac{5\pi}{3}\right)$$

$$= \cos(3\pi) + i\sin(3\pi) = -1$$

5. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is 1.(HW)

6. SOLVE: $x^7 + x^4 + x^3 + 1 = 0$

Soln :- no. of roots = deg of eq^n = 7.

$$x^7 + x^4 + x^3 + 1 = 0$$

$$x^4(x^3+1) + 1(x^3+1) = 0$$

$$(x^3+1)(x^4+1) = 0$$

$\therefore x^3 + 1 = 0$ and. $x^4 + 1 = 0$.

Now $x^3 + 1 = 0$

$$x^3 = -1$$

$$x = (-1)^{1/3} = [-1 + i0]^{1/3} = [\cos\pi + i\sin\pi]^{1/3}$$

$$= \left[\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)\right]^{1/3}$$

$$x = \cos\frac{(2k+1)\pi}{3} + i\sin\frac{(2k+1)\pi}{3}$$

where $k = 0, 1, 2$

$$\therefore x = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3},\ \cos\pi + i\sin\pi,\ \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

For $x^4 + 1 = 0 \Rightarrow x^4 = -1 = -1 + i0$

$$\therefore x = \left[-1 + i0\right]^{1/4} = \left[\cos\pi + i\sin\pi\right]^{1/4}$$

$$= \left[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\right]^{1/4}$$

$$= \cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}$$

where $k = 0, 1, 2, 3$

$$x = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4},\ \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4},\ \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$$

$$\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$$

7. SOLVE: $\underline{\underline{x}}^4 + x^3 + x^2 + x + 1 = 0$ $\rightarrow$

Multiply the given eqn by $x-1$, we get

$$(x-1)(x^4 + x^3 + x^2 + x + 1) = 0$$

$$\Rightarrow x^5 - 1 = 0.$$

$$\Rightarrow x^5 = 1 = \cos 0 + i\sin 0$$

$$x = (\cos 0 + i\sin 0)^{1/5}$$

$$= \left[\cos(2k\pi) + i\sin(2k\pi)\right]^{1/5}$$

$$= \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)$$

where $k = 0,1,2,3,4$

$k=0$, $x_0 = \cos 0 + i\sin 0 = \underline{1}$

$k=1$, $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$k=2$, $x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$k=3$, $x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$

$k=4$, $x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

x_0 is the extra root, introduced because of multiplication by $x-1$.

$\therefore$ x_1, x_2, x_3, x_4 are roots of given eqn.

Note:- $x^4 - x^3 + x^2 - x + 1 = 0$.

multiply by $\underline{x+1}$ } H.w.

$\Rightarrow$ $x^5 + 1 = 0 \Rightarrow x^5 = -1 \rightarrow$

8. SOLVE: $x^4 - x^2 + 1 = 0$

multiply by $\boxed{x^2+1}$ on both sides

$(x^2+1)(x^4 - x^2 + 1) = 0 \Rightarrow x^6 + 1 = 0$.

ex:- $x^4 + x^2 + 1 = 0$

multiply by $\underline{x^2 - 1}$.

$\therefore$ $x^6 = -1 = \cos\pi + i\sin\pi$

$$x = \left[\cos(2k\pi + \pi) + i\sin(2k\pi+\pi)\right]^{1/6}$$

$$= \cos\frac{(2k+1)\pi}{6} + i\sin\frac{(2k+1)\pi}{6}$$

$x^2 + 1 = 0$

$x^2 = -1$

$x = \pm i$

Putting $k = 0,1,2,3,4,5$ we get the six roots

Putting $k = 0, 1, 2, 3, 4, 5$ we get the six roots

$x_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$, $\boxed{x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i}$

$x_2 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$, $x_3 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

$x_4 = \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}$, $x_5 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$

$= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

$x_4 = -i$

It is clear that i and $-i$ are the roots of $x^2+1=0$ and the remaining roots x_0, x_2, x_3, x_5 are roots of $x^4 - x^2 + 1 = 0$.

9. Find the roots common to $x^4 + 1 = 0$ and $x^6 - i = 0$.

Soln:- $x^4 + 1 = 0 \Rightarrow x^4 = -1 = [\cos\pi + i\sin\pi]$

$x = [\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)]^{1/4}$

$= \left[\cos\left(\frac{2k+1}{4}\right)\pi + i\sin\left(\frac{2k+1}{4}\right)\pi\right]$

$k = 0, 1, 2, 3$

$x_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$, $x_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$x_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$, $x_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

$x^6 - i = 0 \Rightarrow x^6 = i = 0 + i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$\Rightarrow x^6 = \cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)$

$$x^6 = \cos\left(\frac{4k+1}{2}\right)\pi + i\sin\left(\frac{4k+1}{2}\right)\pi$$

$$\Rightarrow x = \cos\left(\frac{4k+1}{12}\right)\pi + i\sin\left(\frac{4k+1}{12}\right)\pi$$

$$k = 0, 1, 2, 3, 4, 5$$

$x_0 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$, $x_1 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$

$x_2 = \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}$, $x_3 = \cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}$

$x_4 = \cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}$, $x_5 = \cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12}$

$\rightarrow$ roots common are $\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$ & $\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$.

10. If $(1+x)^6 + x^6 = 0$ $show\ that\ x = -\frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2}\ where$ $\theta = (2k+1)\,\pi/6\,, k = 0,1,2,3,4,5.$

Soln :- $(1+x)^6 + x^6 = 0$

$$\Rightarrow \frac{(1+x)^6}{x^6} + 1 = 0 \quad \Rightarrow \left(\frac{1+x}{x}\right)^6 = -1$$

$$\therefore \frac{1+x}{x} = (-1)^{1/6} = (\cos\pi + i\sin\pi)^{1/6}$$

$$= \left[\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)\right]^{1/6}$$

$$= \cos\frac{(2k+1)\pi}{6} + i\sin\frac{(2k+1)\pi}{6} \quad \text{where } k = 0, 1, 2, \ldots 5$$

$$\text{Let } \frac{(2k+1)\pi}{6} = \theta$$

$$\therefore \frac{1+x}{x} = \cos\theta + i\sin\theta$$

$$\therefore \frac{1}{x}+1 = \cos\theta + i\sin\theta$$

$$\frac{1}{x} = (\cos\theta - 1) + i\sin\theta$$

$$x = \frac{1}{(\cos\theta-1)+i\sin\theta} \times \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)-i\sin\theta}$$

$$= \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)^2+\sin^2\theta} = \frac{(\cos\theta-1)-i\sin\theta}{2(1-\cos\theta)}$$

$$= \frac{(\cos\theta-1)}{2(1-\cos\theta)} - i\frac{\sin\theta}{2(1-\cos\theta)}$$

$$= -\frac{1}{2} - i\left[\frac{2\sin\theta/2\cos\theta/2}{2(2\sin^2\theta/2)}\right]$$

$$x = -\frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2} \quad \text{where } \theta = \frac{(2k+1)\pi}{6}$$

$$k = 0, 1, 2, 3, 4, 5.$$

11. If one root of $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$ is $1+i$, find all other roots.

Soln: The given eqn is $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$

since one of the root is $1+i$

$\therefore$ other root must be $1-i$ (since roots always occur in pairs)

$\therefore x = 1 \pm i$ are the two roots

$$x - 1 = \pm i \Rightarrow (x-1)^2 = -1 \Rightarrow x^2 - 2x + 1 = -1$$

$$\therefore x^2 - 2x + 2 = 0$$

Now we want to find other two remaining roots for that we divide

$x^4 - 6x^3 + 15x^2 - 18x + 10$ by $x^2 - 2x + 2$

we get $\leftrightarrow$ (actual polynomial division)

$$(x^4 - 6x^3 + 15x^2 - 18x + 10) = (x^2 - 2x + 2)(x^2 - 4x + 5)$$

The remaining two roots are roots of

$$x^2 - 4x + 5 = 0$$

$$\therefore x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = 2 \pm i$$

$\therefore$ The remaining roots are $1-i$, $2 \pm i$

12. If $\alpha, \alpha^2, \alpha^3, \alpha^4$, are the roots of $x^5 - 1 = 0$, find them & show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$.

Soln.

$$x^5 - 1 = 0 \Rightarrow x^5 = 1 = x = (1)^{1/5}$$

$$x = (\cos 0 + i \sin 0)^{1/5} = (\cos 2k\pi + i \sin 2k\pi)^{1/5}$$

$$x = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5} \quad \text{where } k = 0, 1, 2, 3, 4$$

$$x_0 = \cos 0 + i \sin 0 = 1$$

$$x_1 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} = \alpha$$

$$x_2 = \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5} = \left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\right)^2 = \alpha^2$$

$$x_3 = \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5} = \alpha^3$$

$$x_4 = \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5} = \alpha^4$$

$x^2 - 5x + 6 = 0$
roots are
2, 3

$x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \alpha^4$

2, 3

$(x-2)(x-3)$

$\therefore$ the roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$

$$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = \frac{x^5-1}{x-1}$$

$$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = x^4+x^3+x^2+x+1$$

put $x=1$

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 1+1+1+1+1 = 5$$

13. Solve the equation $z^4 = i(z-1)^4$ and show that the real part of all the roots is $1/2$.

Soln :- we have $z^4 = i(z-1)^4$

$$\left(\frac{z}{z-1}\right)^4 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$= \cos\left(2k\pi+\frac{\pi}{2}\right) + i\sin\left(2k\pi+\frac{\pi}{2}\right)$$

$$\left(\frac{z}{z-1}\right)^4 = \cos\left(\frac{4k+1}{2}\right)\pi + i\sin\left(\frac{4k+1}{2}\right)\pi$$

$$\frac{z}{z-1} = \cos\left(\frac{4k+1}{8}\right)\pi + i\sin\left(\frac{4k+1}{8}\right)\pi$$

where $k = 0, 1, 2, 3$ let $\left(\frac{4k+1}{8}\right)\pi = \theta$

$$\frac{z}{z-1} = \cos\theta + i\sin\theta$$

$\downarrow$ H.W

$$z = \frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2}$$

14. If ω is a 7th root of unity, prove that $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n} = 7$ if n is a multiple of 7 and is equal to zero otherwise.

Sol^n:- we have $x = (1)^{1/7} = (\cos 2k\pi + i\sin 2k\pi)^{1/7}$

$$= \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}$$

$k = 0, 1, 2, 3, 4, 5, 6$

we take $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$$\therefore \omega^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7 = \cos 2\pi + i\sin 2\pi = 1.$$

$$\omega^{7n} = 1^n = 1.$$

If n is not a multiple of 7 then $\omega^n \neq 1$.

Now $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n}$

$$= \frac{1-\omega^{7n}}{1-\omega^n}$$ sum of 7 terms of G.P. $\left(\frac{a-r^n}{a-r}\right)$ $a=1,\ r=\omega^n$

$$= \frac{1-(\omega^7)^n}{1-\omega^n} = \frac{1-1}{1-\omega^n} = 0$$ (n not a multiple of 7)

If n is a multiple of 7, say $n = 7k$.

$$S = 1 + \omega^{n} + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n}$$

$$= 1 + (\omega^7)^{k} + (\omega^7)^{2k} + (\omega^7)^{3k} + (\omega^7)^{4k} + (\omega^7)^{5k} + (\omega^7)^{6k}$$

$$= 1 + (1)^{k} + (1)^{2k} + (1)^{3k} + (1)^{4k} + (1)^{5k} + (1)^{6k}$$

$$= 1+1+1+1+1+1+1$$

$$= 7.$$

15. Prove that $\sqrt{1 + sec(\theta/2)} = (1 + e^{i\theta})^{-1/2} + (1 + e^{-i\theta})^{-1/2}$

Soln:- TPT $\sqrt{1 + \sec(\frac{\theta}{2})} = \frac{1}{\sqrt{1+e^{i\theta}}} + \frac{1}{\sqrt{1+e^{-i\theta}}}$

Squaring both sides,

TPT $1 + \sec\frac{\theta}{2} = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$

RHS:- $\frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$

$$= \frac{1}{1+e^{i\theta}} + \frac{e^{i\theta}}{1+e^{i\theta}} + \frac{2}{\sqrt{1+e^{-i\theta}+e^{i\theta}+1}}$$

$$= \frac{1+e^{i\theta}}{1+e^{i\theta}} + \frac{2}{\sqrt{2+e^{i\theta}+e^{-i\theta}}}$$

but $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$$RHS = 1 + \frac{2}{\sqrt{2+2\cos\theta}}$$

$$= 1 + \frac{2}{\sqrt{2(1+\cos\theta)}} = 1 + \frac{2}{\sqrt{2(2\cos^2\theta/2)}}$$

$$= 1 + \frac{2}{2\cos\theta/2}$$

$$= 1 + \sec\frac{\theta}{2}$$

$= LHS.$

HYPERBOLIC FUNCTIONS

Monday, October 25, 2021 2:30 PM

CIRCULAR FUNCTIONS:

From Euler's formula, we have $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

If $z = x + iy$ is complex number, then $\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$

These are called circular function of complex numbers.

HYPERBOLIC FUNCTIONS:

If x is real or complex, then sine hyperbolic of x is denoted by $sinh\ x$ and is given as, $\sinh x = \frac{e^x - e^{-x}}{2}$ and

Cosine hyperbolic of x is denoted by $cosh\ x$ and is given as, $\cosh x = \frac{e^x + e^{-x}}{2}$

From above expressions, other hyperbolic functions can also be obtained as

$\tan \mathrm{hx} = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \mathrm{cosechx} = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \qquad \mathrm{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$, and

$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

TABLE OF VALUES OF HYPERBOLIC FUNCTION:

From the definitions of sinhx, cos x, tanhx, we can obtain the following values of hyperbolic function.

x	$-\infty$	0	∞
$\sinh x$	$-\infty$	0	∞
$\cosh x$	∞	1	∞
$\tanh x$	-1	0	1

Note: since $\tanh(-\infty) = -1, \tanh(0) = 0, \tanh(\infty) = 1$

$\therefore |\tanh x| \le 1$

RELATION BETWEEN CIRCULAR AND HYPERBOLIC FUNCTIONS :

(i)	$\sin ix = i\sinh x$ & $\sinh x = -i\sin ix$	$\sinh ix = i\sin x$ & $\sin x = -i\sinh ix$
(ii)	$\cos ix = \cosh x$	$\cosh ix = \cos x$
(iii)	$\tan ix = i\tanh x$ & $\tanh x = -i\tan ix$	$\tanh ix = i\tan x$ & $\tan x = -i\tanh ix$

FORMULAE ON HYPERBOLIC FUNCTIONS :

	CIRCULAR FUNCTIONS	HYPERBOLIC FUNCTIONS
1	$\sin(-x) = -(\sin x)$	$\sinh(-x) = -\sinh x$,
2	$\cos(-x) = (cosx)$	$\cosh(-x) = \cosh x$
3	$e^{ix} = \cos x + i\sin x$	$e^x = \cosh x + \sinh x$
4	$e^{-ix} = \cos x - i\sin x$	$e^{-x} = \cosh x - \sinh x$
5	$sin^2x + cos^2x = 1$	$\cosh^2 x - \sinh^2 x = 1$
6	$1 + \tan^2 x = \sec^2 x$	$\mathrm{sech}^2 x + \tanh^2 x = 1$
7	$1 + \cot^2 x = \mathrm{cosec}^2 x$	$\coth^2 x - \mathrm{cosech}^2 x = 1$
8	$\sin 2x = 2\sin x\cos x$ $= \frac{2\tan x}{1+\tan^2 x}$	$\sinh 2x = 2\sinh x\cosh x$ $= \frac{2\tanh x}{1-\tanh^2 x}$
9	$\cos 2x = \cos^2 x - \sin^2 x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$

	$= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ $= \frac{1-\tan^2 x}{1+\tan^2 x}$	$= 2\cosh^2 x - 1$ $= 1 + 2\sinh^2 x$ $= \frac{1+\tanh^2 x}{1-\tanh^2 x}$
10	$\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$	$\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$
11	$\sin 3x = 3\sin x - 4\sin^3 x$	$\sinh 3x = 3\sinh x + 4\sinh^3 x$
12	$\cos 3x = 4\cos^3 x - 3\cos x$	$\cosh 3x = 4\cosh^3 x - 3\cosh x$
13	$\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$	$\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$
14	$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
15	$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
16	$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$	$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
17	$\cot(x \pm y) = \dfrac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$	$\coth(x \pm y) = \dfrac{-\coth x \coth y \mp 1}{\coth y \pm \coth x}$
18	$\sin x + \sin y = 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$	$\sinh x + \sinh y = 2\sinh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}$
19	$\sin x - \sin y = 2\cos\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$	$\sinh x - \sinh y = 2\cosh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$
20	$\cos x + \cos y$ $= 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$	$\cosh x + \cosh y = 2\cosh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}$
21	$\cos x - \cos y$ $= -2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$	$\cosh x - \cosh y = 2\sinh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$
22	$2\sin x \cos y = \sin(x+y) + \sin(x-y)$	$2\sinh x \cosh y = \sinh(x+y) + \sinh(x-y)$
23	$2\cos x \sin y = \sin(x+y) - \sin(x-y)$	$2\cosh x \sinh y = \sinh(x+y) - \sinh(x-y)$
24	$2\cos x \cos y = \cos(x+y) + \cos(x-y)$	$2\cosh x \cosh y = \cosh(x+y) + \cosh(x-y)$
25	$2\sin x \sin y = \cos(x-y) - \cos(x+y)$	$2\sinh x \sinh y = \cos h(x+y) - \cos h(x-y)$

PERIOD OF HYPERBOLIC FUNTIONS:

$\sinh(2\pi i + x) = \sinh(2\pi i)\cosh x + \cosh(2\pi i)\sinh x$

$= i\sin 2\pi \cosh x + \cos 2\pi \sinh x$

$= 0 + \sinh x \qquad = \sinh x$

Hence $sinh\ x$ is a periodic function of period 2πi

Similarly we can prove that $cosh\ x$ and $tanh\ x$ are periodic functions of period 2πi and πi .

DIFFERENTIATION AND INTRGRATION :

(i) If $y = \sinh x$,

$y = \frac{e^x - e^{-x}}{2}$

$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{\cancel{2}} = \cosh x$

If $y = \sinh x$, $\frac{dy}{dx} = \cosh x$

(ii) If $y = cosh\,x$,

$$y = \frac{e^x+e^{-x}}{2},$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{e^x-e^{-x}}{2} = \sinh x$$

If $y = cosh\,x, \frac{dy}{dx} = \sinh x$

(iii) If $y = \tanh x$,

$$y = \frac{\sinh x}{\cosh x}$$

$$\therefore \frac{dy}{dx} = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{cosh^2 x} = \frac{1}{cosh^2 x} = sech^2 x$$

If $y = \tanh x, \frac{dy}{dx} = sech^2 x$

Hence, we get the following three results

$\int \cosh x\,dx = \sinh x$, $\int \sinh x\,dx = \cosh x$, $\int sech^2 x dx = \tanh x$

SOME SOLVED EXAMPLES:

1. If $\tanh x = \frac{1}{2}$, *find* $\sinh 2x$ *and* $\cosh 2x$

$\tanh x = \frac{1}{2} \Rightarrow \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{2}$

$\Rightarrow \frac{e^{2x}-1}{e^{2x}+1} = \frac{1}{2}$ $\therefore 2e^{2x} - 2 = e^{2x}+1$

$\Rightarrow e^{2x} = 3$

$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2} = \frac{3 - \frac{1}{3}}{2} = \frac{8}{6} = \frac{4}{3}$

$\cosh 2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{3 + \frac{1}{3}}{2} = \frac{10}{6} = \frac{5}{3}$

2. Solve the equation $7cosh\,x + 8sinh\,x = 1$ for real values of x.

Sol:- $7\cosh x + 8\sinh x = 1$

$7\left(\frac{e^x + e^{-x}}{2}\right) + 8\left(\frac{e^x - e^{-x}}{2}\right) = 1$

$$15e^{x} - e^{-x} = 2$$

$$15e^{2x} - 2e^{x} - 1 = 0$$

quadratic in e^x

$$e^x = \frac{-(-2) \pm \sqrt{4 - 4(15)(-1)}}{2(15)}$$

$$e^x = \frac{1}{3} \text{ or } -\frac{1}{5}$$

$$\Rightarrow x = \log\left(\frac{1}{3}\right) \text{ or } x = \log\left(-\frac{1}{5}\right)$$

since x is real

$$x = \log\left(\frac{1}{3}\right) = -\log 3$$

3. If $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x$ then prove that $x = a\sqrt{1+b^2} + b\sqrt{1+a^2}$

Soln:- Let $\alpha = \sinh^{-1} a \Rightarrow \sinh\alpha = a$

$\beta = \sinh^{-1} b \Rightarrow \sinh\beta = b$

$y = \sinh^{-1} x \Rightarrow \sinh y = x$

$\therefore \quad y = \alpha + \beta$

$$\sinh(y) = \sinh(\alpha+\beta)$$

$$= \sinh\alpha \cosh\beta + \cosh\alpha \sinh\beta$$

$$x = a\cosh\beta + \cosh\alpha \cdot b$$

$\Big($ Now $\cosh^2 z - \sinh^2 z = 1$

$\cosh^2 z = 1 + \sinh^2 z \Big)$

$\cosh z = \sqrt{1+\sinh^2 z}$

$$\cosh z = \sqrt{1+\sinh^2 z}$$

$$\therefore x = a\sqrt{1+\sinh^2\beta} + \sqrt{1+\sinh^2\alpha}\cdot b$$

$$= a\sqrt{1+b^2} + \sqrt{1+a^2}\cdot b$$

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4. Prove that $16\, sinh^5 x = \sinh 5x - 5\sinh 3x + 10\sinh x$

$$LHS = 16\sinh^5 x = 16\left(\frac{e^x - e^{-x}}{2}\right)^5$$

$$\therefore LHS = \frac{16}{2^5}\Big[(e^x)^5 - 5(e^x)^4(e^{-x}) + 10(e^x)^3(e^{-x})^2 - 10(e^x)^2(e^{-x})^3 + 5(e^x)(e^{-x})^4 - (e^{-x})^5\Big]$$

$$\left(\text{using } (a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \cdots + {}^nC_{n-1} a b^{n-1} + {}^nC_n b^n\right)$$

$$\therefore LHS = \frac{1}{2}\left[e^{5x} - 5e^{3x} + 10e^{x} - 10e^{-x} + 5e^{-3x} - e^{-5x}\right]$$

$$= \frac{1}{2}\left[\left(e^{5x} - e^{-5x}\right) - 5\left(e^{3x} - e^{-3x}\right) + 10\left(e^{x} - e^{-x}\right)\right]$$

$$= \left[\left(\frac{e^{5x} - e^{-5x}}{2}\right) - 5\left(\frac{e^{3x} - e^{-3x}}{2}\right) + 10\left(\frac{e^{x} - e^{-x}}{2}\right)\right]$$

$$16\sinh^5 x = \sinh 5x - 5\sinh 3x + 10\sinh x$$

Hence proved

5. Prove that $16 cosh^5 x = \cosh 5x + 5\cosh 3x + 10\cosh x$ (HW)

6. Prove that $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-cosh^2 x}}} = cosh^2 x$

Soln:

$1-\frac{}{1-\cosh^2 x}$

Sol^n:- $LHS = \cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{1-\cosh^2 x}}}$

$(1-\cosh^2 x = -\sinh^2 x)$

$LHS = \cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{-\sinh^2 x}}} = \cfrac{1}{1-\cfrac{1}{1+\operatorname{cosech}^2 x}}$

$\left(1+\operatorname{cosech}^2 x = 1+\frac{1}{\sinh^2 x} = \frac{\sinh^2 x+1}{\sinh^2 x} = \coth^2 x\right)$

$LHS = \cfrac{1}{1-\cfrac{1}{\coth^2 x}} = \frac{1}{1-\tanh^2 x} = \frac{1}{\operatorname{sech}^2 x}$

$= \cosh^2 x = RHS.$

7. If $u = \log tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, Prove that

(i) $\cosh u = \sec\theta$ **(ii)** $\sinh u = \tan\theta$ **(iii)** $\tanh u = \sin\theta$ **(iv)** $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

Sol^n:- $u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$\therefore e^u = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}} = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$

$\therefore e^{-u} = \frac{1-\tan\frac{\theta}{2}}{}$

$$\therefore e^{-u} = \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

(i) $$\cosh u = \frac{e^{u}+e^{-u}}{2} = \frac{1}{2}\left[\frac{1+\tan\theta/2}{1-\tan\theta/2} + \frac{1-\tan\theta/2}{1+\tan\theta/2}\right]$$

$$= \frac{1}{2}\left[\frac{(1+\tan\theta/2)^2 + (1-\tan\theta/2)^2}{1-\tan^2\theta/2}\right]$$

$$= \frac{1}{2}\left[\frac{2(1+\tan^2\theta/2)}{1-\tan^2\theta/2}\right] = \frac{1+\tan^2\theta/2}{1-\tan^2\theta/2}$$

$$\therefore \cosh u = \frac{1}{\cos\theta} = \sec\theta.$$ $\left[\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$

(ii) $$\sinh u = \sqrt{\cosh^2 u - 1}$$ $(\cosh^2 u - \sinh^2 u = 1)$

$$= \sqrt{\sec^2\theta - 1}$$

$$= \sqrt{\tan^2\theta}$$

$$\therefore \sinh u = \tan\theta$$

(iii) $$\tanh u = \frac{\sinh u}{\cosh u} = \frac{\tan\theta}{\sec\theta} = \sin\theta$$

(iv) $$\tanh\frac{u}{2} = \frac{\sinh\frac{u}{2}}{\cosh\frac{u}{2}} = \frac{2\sinh\frac{u}{2}\cosh\frac{u}{2}}{2\cosh^2\frac{u}{2}}$$

(multi & div by $2\cosh\frac{u}{2}$)

$$LHS = \frac{\sinh u}{1+\cosh u} = \frac{\tan\theta}{1+\sec\theta} \quad \text{(using (i) \& (ii))}$$

$$= \frac{\sin\theta/\cos\theta}{1+\frac{1}{\cos\theta}} = \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{2\sin\theta/2\,\cos\theta/2}{2\cos^2\theta/2} = \frac{\sin\theta/2}{\cos\theta/2}$$

$$\therefore \tanh\left(\frac{u}{2}\right) = \tan\frac{\theta}{2}$$

8. If $\cosh x = \sec\theta$, Prove that

(i) $x = \log(\sec\theta + \tan\theta)$ **(ii)** $\theta = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$ **(iii)** $\tanh\frac{x}{2} = tan\frac{\theta}{2}$

Soln:- $\cosh x = \sec\theta$

$$\frac{e^x + e^{-x}}{2} = \sec\theta$$

$$e^x + e^{-x} = 2\sec\theta$$

$$e^{2x} + 1 = 2\sec\theta\, e^x$$

$$e^{2x} - (2\sec\theta)e^x + 1 = 0 \rightarrow y^2 - (2\sec\theta)y + 1 = 0$$

This is a quadratic in e^x

$$\therefore y = e^x = \frac{-(-2\sec\theta) \pm \sqrt{(-2\sec\theta)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2\sec\theta \pm 2\sqrt{\sec^2\theta - 1}}{2}$$

$$\frac{\quad\sqrt{\;\;\;}\quad}{2}$$

$$e^x = \sec\theta \pm \tan\theta$$

$$x = \log(\sec\theta \pm \tan\theta)$$

$$= \pm \log(\sec\theta + \tan\theta)$$

$$\left[\log(\sec\theta - \tan\theta) = -\log(\sec\theta + \tan\theta) \quad \text{H.W.}\right]$$

$$\therefore \boxed{x = \log(\sec\theta + \tan\theta)}$$

(ii) T.P.T. $\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$

Let $\tan^{-1}(e^{-x}) = \alpha$

$$\therefore e^{-x} = \tan\alpha \quad \Rightarrow e^{x} = \cot\alpha$$

by given data

$$\sec\theta = \cosh x = \frac{e^x + e^{-x}}{2} = \frac{\cot\alpha + \tan\alpha}{2}$$

$$= \frac{\frac{\cos\alpha}{\sin\alpha} + \frac{\sin\alpha}{\cos\alpha}}{2} = \frac{1}{2\sin\alpha\cos\alpha}$$

$$\therefore \sec\theta = \frac{1}{\sin 2\alpha}$$

$$\therefore \cos\theta = \sin 2\alpha = \cos\left(\frac{\pi}{2} - 2\alpha\right)$$

$$\Rightarrow \theta = \frac{\pi}{2} - 2\alpha = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$$

(iii) $\underline{\text{Pt}}$ $\tanh\frac{x}{2} = \tan\frac{\theta}{2}$

$$LHS = \tanh\frac{x}{2} = \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} = \frac{e^x - 1}{e^x + 1}$$

$(e^x = \sec\theta + \tan\theta)$

$$\tanh\frac{x}{2} = \frac{\sec\theta + \tan\theta - 1}{\sec\theta + \tan\theta + 1}$$

$$= \frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta}$$

$$= \frac{(1 - \cos\theta) + \sin\theta}{(1 + \cos\theta) + \sin\theta}$$

$$= \frac{2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{2\sin\theta/2\,(\sin\theta/2 + \cos\theta/2)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)}$$

$\tanh\frac{x}{2} = \tan\frac{\theta}{2}$ Hence proved.

SEPARATION OF REAL AND IMAGINARY PARTS

Monday, October 11, 2021 11:39 AM

Many a time we are required to separate real and imaginary parts of a given complex function.
For this, we have to use identities of circular and hyperbolic functions.

In problem where we are given $\tan(\alpha + i\beta) = x + i\,y$, we proceed as shown below

$2\beta = (\alpha + i\beta) + (\alpha - i\beta)$

$\tan(2\beta) = \tan(\quad)$

Since $\tan(\alpha + i\beta) = x + i\,y$, we get $\tan(\alpha - i\beta) = x - i\,y$.

$\therefore\ \tan 2\alpha = \tan[(\underbrace{\alpha + i\beta}_{A}) + (\underbrace{\alpha - i\beta}_{B})]$

$$= \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta).\tan(\alpha-i\beta)}$$

$$= \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \frac{2x}{1-x^2-y^2}$$

$$\tan 2\alpha = \frac{2x}{1-x^2-y^2}$$

$\therefore 1 - x^2 - y^2 = 2x \cot 2\alpha$

$\therefore x^2 + y^2 + 2x \cot 2\alpha - 1 = 0$

Further, $\tan(2i\beta) = \tan[(\alpha + i\beta) - (\alpha - i\beta)]$

$$= \frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\ \tan(\alpha-i\beta)}$$

$$i \tanh 2\beta = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = \frac{2i\,y}{1+x^2+y^2}$$

$$\therefore \tanh 2\beta = \frac{2y}{1+x^2+y^2}$$

$\therefore 1 + x^2 + y^2 = 2y \coth 2\beta \qquad$ i.e., $x^2 + y^2 - 2y \coth 2\beta + 1 = 0$

$$\tan(2i\beta) = \tan[(\alpha+i\beta) - (\alpha - i\beta)]$$

$$= \frac{\tan(\alpha+i\beta) - \tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$$

$$= \frac{(x+iy) - (x-iy)}{1+(x+iy)(x-iy)}$$

SOME SOLVED EXAMPLES:

2. If $\sin(\alpha - i\,\beta) = x + i\,y$ then prove that $\frac{x^2}{cosh^2\beta} + \frac{y^2}{sinh^2\beta} = 1$ and $\frac{x^2}{sin^2\alpha} - \frac{y^2}{cos^2\alpha} = 1$

Soln, $\sin(\alpha - i\beta) = x + iy$

$(\sin(A - B) = \sin A \cos B - \cos A \sin B)$

$\sin\alpha \cos(i\beta) - \cos\alpha \sin(i\beta) = x + iy$

$(\cos(i\beta) = \cosh\beta \quad \& \quad \sin(i\beta) = i \sinh\beta)$

$\sin\alpha \cosh\beta - i\cos\alpha \sinh\beta = x + iy$

$\therefore x = \sin\alpha\cosh\beta \quad \& \quad y = -\cos\alpha\sinh\beta$ —(i)

(i) $\frac{x}{\ } \quad \sin\alpha \qquad \frac{y}{\ } = -\cos\alpha$

$\therefore x = \sin\alpha\cosh\beta$

(i) $\dfrac{x}{\cosh\beta} = \sin\alpha \quad , \quad \dfrac{y}{\sinh\beta} = -\cos\alpha$

squaring & adding

$$\frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta} = \sin^2\alpha + \cos^2\alpha = 1$$

(ii) $\dfrac{x}{\sin\alpha} = \cosh\beta \quad , \quad \dfrac{y}{\cos\alpha} = -\sinh\beta \qquad$ (using (i))

square & subtract

$$\frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha} = \cosh^2\beta - \sinh^2\beta = 1$$

4. If $x + i\,y = \tan(\pi/6 + i\,\alpha)$, prove that $x^2 + y^2 + 2x/\sqrt{3} = 1$

Soln: we are given $x + iy = \tan\left(\frac{\pi}{6} + i\alpha\right)$

$\therefore \quad x - iy = \tan\left(\frac{\pi}{6} - i\alpha\right)$

$$\tan\left[\left(\frac{\pi}{6} + i\alpha\right) + \left(\frac{\pi}{6} - i\alpha\right)\right] = \frac{\tan\left(\frac{\pi}{6}+i\alpha\right) + \tan\left(\frac{\pi}{6}-i\alpha\right)}{1 - \tan\left(\frac{\pi}{6}+i\alpha\right)\tan\left(\frac{\pi}{6}-i\alpha\right)}$$

$$= \frac{(x+iy) + (x-iy)}{1-(x+iy)(x-iy)}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{2x}{1-x^2-y^2}$$

$$\sqrt{3} = \frac{2x}{\quad}$$

$$\sqrt{3} = \frac{2x}{1-x^2-y^2}$$

$$1-x^2-y^2 = \left(\frac{2}{\sqrt{3}}\right)x$$

$$x^2+y^2+\left(\frac{2}{\sqrt{3}}\right)x = 1$$

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1. Separate into real and imaginary parts $tan^{-1}(e^{i\theta})$

Soln:- Let $\tan^{-1}(e^{i\theta}) = x+iy$

$$\therefore \quad e^{i\theta} = \tan(x+iy)$$

$$\therefore \tan(x+iy) = \cos\theta + i\sin\theta$$

$$\Rightarrow \quad \tan(x-iy) = \cos\theta - i\sin\theta$$

$$\therefore \quad \tan(2x) = \tan\left[(x+iy)+(x-iy)\right]$$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)}{1-(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)}$$

$$= \frac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)} = \frac{2\cos\theta}{1-1}$$

$$= \frac{2\cos\theta}{0}$$

$$\therefore \tan 2x = \infty$$

$\therefore 2x = \pi$ …

$\because \tan 2x = \infty$

$\therefore 2x = \frac{\pi}{2}$ $\qquad \therefore \boxed{x = \frac{\pi}{4}}$

Similarly, $\tan(2iy) = \tan[(x+iy)-(x-iy)]$

$$= \frac{\tan(x+iy) - \tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{1+(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{2i\sin\theta}{1+(\cos^2\theta + \sin^2\theta)} = \frac{2i\sin\theta}{2}$$

$\therefore \tan(2iy) = i\sin\theta$

$\therefore i\tanh 2y = i\sin\theta$ $\qquad$ $\left(\tan(ix) = i\tanh x\right)$

$\therefore \tanh 2y = \sin\theta$

$\therefore 2y = \tanh^{-1}(\sin\theta)$

$\therefore y = \frac{1}{2}\tanh^{-1}(\sin\theta)$

$\therefore \tan^{-1}(e^{i\theta}) = x+iy = \frac{\pi}{4} + i\frac{1}{2}\tanh^{-1}(\sin\theta)$

3. If $\cos(x+iy) = \cos\alpha + i\sin\alpha$, *prove that*
(i) $\sin\alpha = \pm\sin^2 x = \pm\sinh^2 y$ **(ii)** $\cos 2x + \cosh 2y = 2$

Soln:- $\cos(x+iy) = \cos\alpha + i\sin\alpha$

$\left(\cos(A+B) = \cos A\cos B - \sin A\sin B\right)$

$\cos x \cos iy - \sin x \sin(iy) = \cos\alpha + i\sin\alpha$

$(\cos iy = \cosh y \;\; \& \;\; \sin(iy) = i\sinh y)$

$\therefore \cos x \cosh y - i \sin x \sinh y = \cos\alpha + i\sin\alpha$

comparing real & imaginary parts

$\cos\alpha = \cos x \cosh y \;\; \& \;\; \sin\alpha = \underbrace{-\sin x \sinh y}_{(A)}$

(i) TPT. $\sin\alpha = \pm \sin^2 x = \pm \sinh^2 y$

we know $\sin^2\alpha + \cos^2\alpha = 1$

$\sin^2 x \sinh^2 y + \cos^2 x \cosh^2 y = 1$

$\sin^2 x \sinh^2 y + (1-\sin^2 x)(1+\sinh^2 y) = 1$

$\sin^2 x \sinh^2 y + 1 + \sinh^2 y - \sin^2 x - \sin^2 x \sinh^2 y = 1$

$\therefore \sinh^2 y - \sin^2 x = 0$

$\Rightarrow \boxed{\sin^2 x = \sinh^2 y}$ ✓

$\Rightarrow \sin x = \pm \sinh y$ or $\sinh y = \pm \sin x$

Now $\sin\alpha = -\sin x \sinh y$ (using (A))

$= -\sin x(\pm \sin x) = \pm \sin^2 x$

or $= (\pm \sinh y)\sinh y = \pm \sinh^2 y$

(ii) TPT $\cos 2x + \cosh 2y = 2$

LHS $= \cos 2x + \cosh 2y$

$= 1 - 2\sin^2 x + 1 + 2\sinh^2 y$

$= 2 - 2\sin^2 x + 2\sinh^2 y$

but in (i) we proved that
$\sin^2 x = \sinh^2 y$

$\therefore$ LHS $= 2 - 0 = 2$.

5. If $x + i\,y = c\cot(u + i\,v)$, show that $\frac{x}{\sin 2u} = -\frac{y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}$.

Soln:- $x + iy = c\cot(u+iv)$

$\therefore x - iy = c\cot(u - iv)$

$$2x = (x+iy) + (x-iy)$$

$$= c\left(\cot(u+iv) + \cot(u-iv)\right)$$

$$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)} + \frac{\cos(u-iv)}{\sin(u-iv)}\right]$$

$$= c\left[\frac{\cos(u+iv)\sin(u-iv) + \sin(u+iv)\cos(u-iv)}{\sin(u+iv)\sin(u-iv)}\right]$$

$$= c\left[\frac{\sin\left[(u+iv)+(u-iv)\right]}{\frac{1}{2}\left[\cos(u+iv-u+iv) - \cos(u+iv+u-iv)\right]}\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\therefore 2x = \frac{c\sin 2u}{\frac{1}{2}\left[\cos(2iv) - \cos(2u)\right]}$$

$$\frac{1}{2}\left[\cos(2iv) - \cos(2u)\right]$$

$$\therefore \frac{x}{\sin 2u} = \frac{c}{\cosh 2v - \cos 2u}$$

For other result, we take

$$2iy = (x+iy) - (x-iy)$$

$$= c\left[\cot(u+iv) - \cot(u-iv)\right]$$

H.W

H.W

$$\therefore \frac{-y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}$$

6. If $u + i\,v = cosec\left(\frac{\pi}{4} + i\,x\right)$, *prove that* $\left(u^2 + v^2\right)^2 = 2\left(u^2 - v^2\right)$

Soln: $u + iv = \operatorname{cosec}\left(\frac{\pi}{4} + ix\right)$

$$\Rightarrow \frac{1}{\sin\left(\frac{\pi}{4} + ix\right)} = u + iv$$

$$\Rightarrow \sin\left(\frac{\pi}{4} + ix\right) = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow \sin\frac{\pi}{4}\cos ix + \cos\frac{\pi}{4}\sin ix = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

$$\left(\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},\ \cos ix = \cosh x,\ \sin ix = i\sinh x\right)$$

$$\therefore \frac{1}{\sqrt{2}}\cosh x + i\frac{1}{\sqrt{2}}\sinh x = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

comparing real & imaginary parts

$$\cosh x = \frac{\sqrt{2}u}{u^2+v^2} \quad \& \quad \sinh x = \frac{-\sqrt{2}v}{u^2+v^2}$$

Now $\cosh^2 x - \sinh^2 x = 1$

$$\frac{2u^2}{(u^2+v^2)^2} - \frac{2v^2}{(u^2+v^2)^2} = 1$$

$$2(u^2-v^2) = (u^2+v^2)^2$$ Hence proved

7. If $x + iy = \cos(\alpha + i\beta)$ or if $\cos^{-1}(x + iy) = \alpha + i\beta$ express x and y in terms of α and β.
Hence show that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation $\lambda^2 - (x^2 + y^2 + 1)\lambda + x^2 = 0$

Soln :- $\cos(\alpha + i\beta) = x + iy$

$\cos\alpha \cos i\beta - \sin\alpha \sin i\beta = x + iy$

$\cos\alpha \cosh\beta - i\sin\alpha \sinh\beta = x + iy$

$\cos i\beta = \cosh\beta$
$\sin i\beta = i\sinh\beta$

$\therefore x = \cos\alpha \cosh\beta, \quad y = -\sin\alpha \sinh\beta$ ——— (1)

we know that, in terms of the roots, the quadratic equation is given by

$$\lambda^2 - (\text{sum of roots})\lambda + (\text{product of roots}) = 0$$

Hence to prove that given equation has roots as $\cos^2\alpha$ & $\cosh^2\beta$ we have to show that

$x^2+y^2+1 = \cos^2\alpha + \cosh^2\beta$ & $\underline{x^2 = \cos^2\alpha \cosh^2\beta}$

from (1) $x = \cos\alpha \cosh\beta \Rightarrow x^2 = \cos^2\alpha \cosh^2\beta$

Now $x^2+y^2+1 = (\cos\alpha \cosh\beta)^2 + (-\sin\alpha \sinh\beta)^2 + 1$

$= \cos^2\alpha \cosh^2\beta + \sin^2\alpha \sinh^2\beta + 1$

$= \cos^2\alpha \cosh^2\beta + (1-\cos^2\alpha)(\cosh^2\beta - 1) + 1$

$= \cos^2\alpha + \cosh^2\beta =$ sum of the roots.

$\therefore \cos^2\alpha$ & $\cosh^2\beta$ are roots of given equation.

INVERSE HYPERBOLIC FUNCTIONS

Thursday, October 28, 2021 2:09 PM

If $x = \sinh u$ then $u = \sinh^{-1} x$ is called sine hyperbolic inverse of x, where x is real.

Similarly we can define $\cosh^{-1}x$, $\tanh^{-1}x$, $\coth^{-1}x$, $\operatorname{sech}^{-1}x$, $\operatorname{cosech}^{-1}x$.

Theorem: If x is real.

(i) $\sinh^{-1}x = \log\left(x + \sqrt{x^2+1}\right)$

(ii) $\cosh^{-1}x = \log\left(x + \sqrt{x^2-1}\right)$

(iii) $\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Soln:- (i) Let $\sinh^{-1} x = y$

$$\therefore x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$\therefore 2x = e^y - e^{-y} = e^y - \frac{1}{e^y} = \frac{e^{2y}-1}{e^y}$$

$$\therefore e^{2y} - 1 = 2x e^y$$

$$\therefore e^{2y} - 2x e^y - 1 = 0$$

This is a quadratic in e^y

$$\therefore e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$\therefore y = \log\left(x \pm \sqrt{x^2+1}\right)$$

but $x - \sqrt{x^2+1} < 0$ $\therefore \log\left(x - \sqrt{x^2+1}\right)$ is not defined.

but $x - \sqrt{x^2+1} < 0$ $\therefore \log(x - \sqrt{x^2+1})$ is not defined.

$\therefore y = \log(x + \sqrt{x^2+1})$

$$\therefore \boxed{\sinh^{-1} x = \log(x + \sqrt{x^2+1})}$$

(ii) TPT $\cosh^{-1} x = \log(x + \sqrt{x^2-1})$

let $\cosh^{-1} x = y$

$$x = \cosh y = \frac{e^y + e^{-y}}{2} \Rightarrow 2x = e^y + \frac{1}{e^y} = \frac{e^{2y}+1}{e^y}$$

$$\therefore e^{2y} - 2x e^y + 1 = 0$$

This is a quadratic in e^y

$$\therefore e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)}$$

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$\therefore e^y = x \pm \sqrt{x^2-1}$$

$$\therefore y = \log(x \pm \sqrt{x^2-1}) \quad \text{——— } 1$$

consider, $y = \log(x - \sqrt{x^2-1})$ ——— 2

$$e^y = x - \sqrt{x^2-1}$$

$$e^{-y} = \frac{1}{x - \sqrt{x^2-1}}$$

$$e^{-y} = \frac{1}{x - \sqrt{x^2 - 1}}$$

$$= \frac{1}{x - \sqrt{x^2-1}} \times \frac{x + \sqrt{x^2-1}}{x + \sqrt{x^2-1}}$$

$$e^{-y} = \frac{x + \sqrt{x^2-1}}{(x)^2 - (\sqrt{x^2-1})^2} = \frac{x + \sqrt{x^2-1}}{x^2 - x^2 + 1}$$

$$e^{-y} = x + \sqrt{x^2-1}$$

$y = \log(x - \sqrt{x^2-1})$

$$-y = \log(x + \sqrt{x^2-1})$$

$$\therefore y = -\log(x + \sqrt{x^2-1}) \quad \text{--- } (3)$$

from (2) & (3) $\Rightarrow \log(x - \sqrt{x^2-1}) = -\log(x + \sqrt{x^2-1})$

using in (1) $\quad y = \pm \log(x + \sqrt{x^2-1})$

$$\therefore \cosh^{-1} x = \pm \log(x + \sqrt{x^2-1})$$

$$\therefore x = \cosh\left[\pm \log(x + \sqrt{x^2-1})\right]$$

$[\cosh(-z) = \cosh z]$

$$\therefore x = \cosh[\log(x + \sqrt{x^2-1})]$$

$$\therefore \cosh^{-1} x = \log(x + \sqrt{x^2-1})$$

(iii) $\quad \tanh^{-1} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

let $\tanh^{-1} x = y$

$$x = \tanh y$$

$$\frac{x}{1} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$\frac{1+x}{1-x} = \frac{(e^y + e^{-y}) + (e^y - e^{-y})}{(e^y + e^{-y}) - (e^y - e^{-y})}$$

$$\frac{1+x}{1-x} = \frac{2e^y}{2e^{-y}} = e^{2y}$$

$$\therefore \quad e^{2y} = \frac{1+x}{1-x}$$

$$2y = \log\left(\frac{1+x}{1-x}\right) \qquad \therefore \ y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\therefore \tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

SOME SOLVED EXAMPLES:

1. Prove that tanh $log\ \sqrt{x} = \frac{x-1}{x+1}$ Hence deduce that tanh log$\sqrt{5/3}$ + tanh log$\sqrt{7}$ = 1

Soln: we know that $\tanh y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$

$$\therefore \tanh(\log\sqrt{x}) = \frac{e^{\log(\sqrt{x})} - e^{-\log(\sqrt{x})}}{e^{\log(\sqrt{x})} + e^{-(\log(\sqrt{x}))}}$$

$$\tanh(\log\sqrt{x}) = \frac{\sqrt{x} - \frac{1}{\sqrt{x}}}{\sqrt{x} + \frac{1}{\sqrt{x}}} = \frac{x-1}{x+1}$$

$$\frac{\quad}{\sqrt{x}+\frac{1}{\sqrt{x}}} = x+1$$

Now $\tanh\log\left(\sqrt{\frac{5}{3}}\right) = \dfrac{\frac{5}{3}-1}{\frac{5}{3}+1} = \dfrac{\frac{2}{3}}{8/3} = \frac{1}{4}$

$\tanh\log(\sqrt{7}) = \dfrac{7-1}{7+1} = \dfrac{6}{8} = \dfrac{3}{4}$

$\therefore \tanh\log\left(\sqrt{\frac{5}{3}}\right) + \tanh\log(\sqrt{7}) = \frac{1}{4}+\frac{3}{4} = 1$

2. (i) Prove that $cosh^{-1}\sqrt{1+x^2} = sinh^{-1}x$

(ii) Prove that $tanh^{-1}x = sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

(iii) Prove that $cosh^{-1}\left(\sqrt{1+x^2}\right) = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

(iv) Prove that $cot\,h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$

(v) Prove that $sech^{-1}(\sin\theta) = \log cot\frac{\theta}{2}$

Soln.:- (i) Tpt $\cosh^{-1}\left(\sqrt{1+x^2}\right) = \sinh^{-1}x$

Soln:- Let $\cosh^{-1}\sqrt{1+x^2} = y$

$\sqrt{1+x^2} = \cosh y$

squaring both sides

$1+x^2 = \cosh^2 y$

$\therefore x^2 = \cosh^2 y - 1 = \sinh^2 y$

$\therefore x = \sinh y$

$\therefore y = \sinh^{-1}(x)$

$\therefore \cosh^{-1}\left(\sqrt{1+x^2}\right) = \sinh^{-1}(x)$

(ii) Tpt. $\tanh^{-1}x = \sinh^{-1}\dfrac{x}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}$

Soln :- Let $\tanh^{-1} x = y$ $\quad \therefore x = \tanh y$

$$\therefore \frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\sqrt{1-\tanh^2 y}} = \frac{\tanh y}{\sqrt{\operatorname{sech}^2 y}} = \frac{\tanh y}{\operatorname{sech} y}$$

$$= \sinh y$$

$$\therefore y = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

(iii) T.P.T. $\cosh^{-1}\left(\sqrt{1+x^2}\right) = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ H.W

(iv) T.P.T. $\coth^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}\log\left(\frac{x+a}{x-a}\right)$

Soln :- Let $\coth^{-1}\left(\frac{x}{a}\right) = y$

$$\therefore \frac{x}{a} = \coth y \longrightarrow$$

$$\therefore \tanh y = \frac{a}{x}$$

$$\therefore y = \tanh^{-1}\left(\frac{a}{x}\right)$$

$$\left[\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\right]$$

$$= \frac{1}{2}\log\left[\frac{1+\frac{a}{x}}{1-\frac{a}{x}}\right]$$

$$y = \frac{1}{2}\log\left[\frac{x+a}{x-a}\right]$$

$$\left[\frac{x+a}{x-a}\right]$$

$$\therefore \coth^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}\log\left[\frac{x+a}{x-a}\right]$$

<u>Note</u> :- (i) $a=1$

(V) TPT. $\operatorname{sech}^{-1}(\sin\theta) = \log\cot\frac{\theta}{2}$

<u>Soln</u>:- Let $\operatorname{sech}^{-1}(\sin\theta) = x$

$$\therefore \sin\theta = \operatorname{sech} x$$

$$\therefore \sin\theta = \frac{2}{e^{x}+e^{-x}} \qquad \left[\operatorname{sech}x = \frac{1}{\cosh x} = \frac{1}{\frac{e^{x}+e^{-x}}{2}}\right.$$

$$\therefore \sin\theta = \frac{2e^{x}}{e^{2x}+1}$$

$$(\sin\theta)e^{2x} - 2e^{x} + \sin\theta = 0$$

This is a quadratic in e^{x}

$$\therefore e^{x} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(\sin\theta)(\sin\theta)}}{2\sin\theta}$$

$$= \frac{2 \pm \sqrt{4-4\sin^2\theta}}{2\sin\theta}$$

$$e^{x} = \frac{1 \pm \sqrt{1-\sin^2\theta}}{\sin\theta} = \frac{1\pm\cos\theta}{\sin\theta}$$

$$\therefore e^x = \frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2\theta/2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\theta/2}{\sin\theta/2}$$

$$\therefore e^x = \cot\frac{\theta}{2} \Rightarrow x = \log\left[\cot\frac{\theta}{2}\right]$$

3. Separate into real and imaginary parts $cos^{-1}e^{i\theta}$ **or** $cos^{-1}(\cos\theta + i\sin\theta)$

$x = ?$, $y = ?$

Soln:- let $\cos^{-1}(e^{i\theta}) = x+iy$

$$\therefore \cos(x+iy) = e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos x \cos iy - \sin x \sin iy = \cos\theta + i\sin\theta$$

$$\cos x \cosh y - i\sin x \sinh y = \cos\theta + i\sin\theta$$

$$\cos x \cosh y = \cos\theta \quad \& \quad -\sin x \sinh y = \sin\theta \quad \text{--- (1)}$$

Since $\cosh^2 y - \sinh^2 y = 1$

$$\left(\frac{\cos\theta}{\cos x}\right)^2 - \left(\frac{\sin\theta}{-\sin x}\right)^2 = 1$$

$$\frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$$

$$\frac{1-\sin^2\theta}{1-\sin^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$$

$$\sin^2 x - \sin^2\theta\sin^2 x - \sin^2\theta + \sin^2\theta\sin^2 x = \sin^2 x - \sin^4 x$$

$$\sin^2 x - \sin^2\theta = \sin^2 x - \sin^4 x$$

$$\therefore \sin^2\theta = \sin^4 x$$

$$\therefore \sin x = \sqrt{\sin\theta} \quad \text{--- (2)}$$

$$\therefore x = \sin^{-1}\left(\sqrt{\sin\theta}\right)$$

Using ① $\quad \sin\theta = -\sin x \sinh y$

$$= -\sqrt{\sin\theta}\,\sinh y \quad (\text{using ②})$$

$$-\sqrt{\sin\theta} = \sinh y$$

$$\therefore y = \sinh^{-1}\left(-\sqrt{\sin\theta}\right) \checkmark$$

$$\left[\sinh^{-1}x = \log\left(x + \sqrt{x^2+1}\right)\right]$$

$$\therefore y = \log\left[-\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}\right)$$

$$\therefore y = \log\left(\sqrt{\sin\theta+1} - \sqrt{\sin\theta}\right)$$

$$\therefore \cos^{-1}(e^{i\theta}) = x + iy = \sin^{-1}\left(\sqrt{\sin\theta}\right) + i\log\left(\sqrt{\sin\theta+1} - \sqrt{\sin\theta}\right)$$

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Q.4 separate into real and imaginary parts

$$\sinh^{-1}(ix)$$

Solⁿ:- Let $\sinh^{-1}(ix) = \alpha + i\beta$

$$ix = \sinh(\alpha + i\beta)$$

$$= \sinh\alpha \cosh(i\beta) + \cosh\alpha \sinh(i\beta)$$

$$\cosh(i\beta) = \cos\beta$$

$$\sinh(i\beta) = i\sin\beta$$

$$ix = \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$$

comparing real & imaginary parts

$\sinh\alpha\cos\beta = 0 \Rightarrow \cos\beta = 0 \Rightarrow \beta = \frac{\pi}{2}$

also $\cosh\alpha\sin\beta = x \Rightarrow \cosh\alpha\sin\frac{\pi}{2} = x$

$$\Rightarrow \cosh\alpha = x$$
$$\Rightarrow \alpha = \cosh^{-1}(x)$$

$$\therefore \sinh^{-1}(ix) = \alpha + i\beta = \cosh^{-1}(x) + i\frac{\pi}{2}$$

Q5:- If $\tan z = \frac{i}{2}(1-i)$, prove that

$$z = \frac{1}{2}\tan^{-1}(2) + \frac{i}{4}\log(5)$$

Soln:- $\tan z = \frac{i}{2}(1-i) = \frac{1}{2}(i - i^2) = \frac{1}{2} + \frac{i}{2}$

let $z = x + iy$

$\tan(x+iy) = \frac{1}{2} + \frac{i}{2} \Rightarrow \tan(x - iy) = \frac{1}{2} - \frac{i}{2}$

$$\tan(2x) = \tan[(x+iy) + (x-iy)]$$

$$= \frac{\tan(x+iy) + \tan(x-iy)}{1 - \tan(x+iy)\tan(x-iy)}$$

$$= \frac{\frac{1}{2} + \frac{i}{2} + \frac{1}{2} - \frac{i}{2}}{1 - \left(\frac{1}{2}+\frac{i}{2}\right)\left(\frac{1}{2}-\frac{i}{2}\right)} = \frac{1}{1 - \left(\frac{1}{4}+\frac{1}{4}\right)} = \frac{1}{\frac{1}{2}}$$

$$\therefore \tan(2x) = 2$$

$$\therefore \boxed{x = \frac{1}{2}\tan^{-1}2}$$

Also $\tan(2iy) = \tan[(x+iy)-(x-iy)]$

$$= \frac{\tan(x+iy) - \tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$\tan(2iy) = \frac{\left(\frac{1}{2}+\frac{i}{2}\right) - \left(\frac{1}{2}-\frac{i}{2}\right)}{1+\left(\frac{1}{2}+\frac{i}{2}\right)\left(\frac{1}{2}-\frac{i}{2}\right)}$$

$$\tan(ix) = i\tanh(x)$$

$$i\tanh(2y) = \frac{i}{1+\frac{1}{2}}$$

$$\tanh(2y) = \frac{2}{3} \Rightarrow 2y = \tanh^{-1}\left(\frac{2}{3}\right)$$

$$\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\therefore 2y = \frac{1}{2}\log\left(\frac{1+2/3}{1-2/3}\right) = \frac{1}{2}\log 5$$

$$\therefore y = \frac{1}{4}\log 5$$

$$\therefore z = x+iy = \frac{1}{2}\tan^{-1}(2) + \frac{i}{4}\log 5$$

Q.6 Show that $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \frac{i}{2}\log\frac{x}{a}$.

Soln:- Let $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \theta$

$$\therefore \tan\theta = i\left(\frac{x-a}{x+a}\right)$$

$\left(\; x+a \right)$

but $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\therefore \tan\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$

$$\therefore \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} = i\left(\frac{x-a}{x+a}\right)$$

$$\therefore \frac{x-a}{x+a} = \frac{e^{i\theta} - e^{-i\theta}}{i^2(e^{i\theta} + e^{-i\theta})} = \frac{e^{-i\theta} - e^{i\theta}}{(e^{i\theta} + e^{-i\theta})}$$

componendo- dividendo.

$$\frac{(x-a)+(x+a)}{(x-a)-(x+a)} = \frac{(e^{-i\theta} - e^{i\theta}) + (e^{i\theta} + e^{-i\theta})}{(e^{-i\theta} - e^{i\theta}) - (e^{i\theta} + e^{-i\theta})}$$

$$\frac{2x}{-2a} = \frac{2e^{-i\theta}}{-2e^{i\theta}}$$

$$\therefore \frac{x}{a} = e^{-2i\theta}$$

$$\therefore -2i\theta = \log\left(\frac{x}{a}\right)$$

$$\therefore \theta = \frac{-1}{2i} \log\left(\frac{x}{a}\right)$$

$$= -\frac{i}{2i^2} \log\left(\frac{x}{a}\right)$$

$\theta = i \; \log\left(\frac{x}{\;}\right)$

$\frac{}{2} \log\left(\frac{}{a}\right)$

LOGARITHMS OF COMPLEX NUMBERS

Monday, November 8, 2021 2:44 PM

Let $z = x + iy$ and also let $x = r\cos\theta\,, y = r\sin\theta$ so that $r = \sqrt{x^2+y^2}$ and $\theta = tan^{-1}(y/x)$.

Hence, $\log z = \log\big(r(\cos\theta + i\sin\theta)\big) = \log(\,r.e^{i\theta})$

$= \log r + \log e^{i\theta} = \log r + i\theta$

$\therefore\ \log(x+iy) = \log r + i\theta$

$\therefore\ \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\,tan^{-1}\frac{y}{x}$ .... (1)

This is called **principal value** of log (x + iy)

The general value of log (x + iy) is denoted by Log (x + iy) and is given by

$\therefore \text{Log}\,(\mathrm{x}+\mathrm{iy}) = 2\mathrm{n}\pi\mathrm{i} + \log(\mathrm{x}+\mathrm{iy})$

$\therefore \text{Log}\,(\mathrm{x}+\mathrm{iy}) = 2\mathrm{n}\pi\mathrm{i} + \frac{1}{2}\log(x^2+y^2) + i\,tan^{-1}\frac{y}{x}$

$\text{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i(2n\pi + tan^{-1}\frac{y}{x})$ .... (2)

Caution: $\theta = tan^{-1}y/x$ only when x and y are both positive.

In any other case θ is to be determined from $x = r\cos\theta\,, y = r\sin\theta\,, -\pi \le \theta \le \pi$.

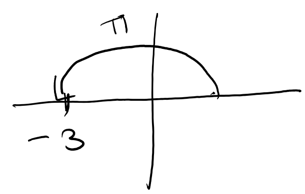
SOME SOLVED EXAMPLES:

1. Considering the principal value only prove that $\log_2(-3) = \frac{\log 3 + i\,\pi}{\log 2}$

Soln :- $\log_2(-3) = \dfrac{\log(-3)}{\log(2)}$

$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

$\therefore \log(-3) = \frac{1}{2}\log(9) + i\tan^{-1}\left(\frac{0}{-3}\right)$



$\log(-3) = \log(3) + i(\pi)$

$\therefore \log_2(-3) = \dfrac{\log(3) + i\pi}{\log 2}.$

2. Find the general value of $\text{Log}(1+\text{i}) + \text{Log}\,(1-\text{i})$

Soln :- $\text{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\left(2n\pi + \tan^{-1}\left(\frac{y}{x}\right)\right)$

$\text{Log}(1+i) = \frac{1}{2}\log(1^2+1^2) + i\left(2n\pi + \tan^{-1}\left(\frac{1}{1}\right)\right)$

$$\text{Log}(1+i) = \frac{1}{2}\log(2) + i\left(2n\pi + \frac{\pi}{4}\right)$$

$$\text{Log}(1-i) = \frac{1}{2}\log(2) - i\left(2n\pi + \frac{\pi}{4}\right)$$

$$\text{Log}(1+i) + \text{Log}(1-i) = \frac{1}{2}\log(2) + \frac{1}{2}\log(2)$$
$$= \log(2)$$

3. Prove that $\log(1 + e^{2i\theta}) = \log(2\cos\theta) + i\theta$

Soln :- $\log(1+e^{2i\theta}) = \log(1+\cos 2\theta + i\sin 2\theta)$

$$= \log(2\cos^2\theta + i\,2\cos\theta\sin\theta)$$
$$= \log\left[2\cos\theta(\cos\theta + i\sin\theta)\right]$$
$$= \log(2\cos\theta) + \log(\cos\theta + i\sin\theta)$$
$$= \log(2\cos\theta) + \log(e^{i\theta})$$
$$= \log(2\cos\theta) + i\theta$$

Other way.

$x = 1 + \cos 2\theta$

$y = \sin 2\theta$

$$\log(1+e^{2i\theta}) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

4. Find the value of $\log[\sin(x + iy)]$

Soln :- $\sin(x+iy) = \sin x\cos iy + \cos x\sin iy$

$\cos iy = \cosh y$

$\sin iy = i\sinh y$

$\therefore \sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$

$$\log(a+ib) = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)$$

$$\log[\sin(x+iy)] = \frac{1}{2}\log[\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y]$$

$$+ i\tan^{-1}\left(\frac{\cos x\sinh y}{\sin x\cosh y}\right)$$

Now $\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y$

$$= (1-\cos^2 x)\cosh^2 y + \cos^2 x(\cosh^2 y - 1)$$

$$= \cosh^2 y - \cos^2 x\cosh^2 y + \cos^2 x\cosh^2 y - \cos^2 x$$

$$= \cosh^2 y - \cos^2 x$$

$$\therefore \log[\sin(x+iy)] = \frac{1}{2}\log(\cosh^2 y - \cos^2 x)$$

$$+ i\tan^{-1}(\cot x\tanh y)$$

5. Show that $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$

Soln:- $\log(a+ib) = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)$

$$\log(a-ib) = \frac{1}{2}\log(a^2+b^2) - i\tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore \log\left(\frac{a-ib}{a+ib}\right) = \log(a-ib) - \log(a+ib)$$

$$= -2i\tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore i\log\left(\frac{a-ib}{a+ib}\right) = -2i^2\tan^{-1}\left(\frac{b}{a}\right)$$

$$i \log\left(\frac{a-ib}{a+ib}\right) = 2\tan^{-1}\left(\frac{b}{a}\right)$$

$$\tan\left[i \log\left(\frac{a-ib}{a+ib}\right)\right] = \tan\left(2\tan^{-1}\left(\frac{b}{a}\right)\right)$$

$$= \tan 2\theta \qquad \text{where } \theta = \tan^{-1}\frac{b}{a}$$

$$= \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{b/a}{1-(b/a)^2}$$

$$\tan\left[i \log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$$

6. Prove that $\cos\left[i \log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{a^2-b^2}{a^2+b^2}$ H.W.

7. Separate into real and imaginary parts $\sqrt{i}^{\sqrt{i}}$

Soln:- we have $\sqrt{i} = i^{1/2} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2}$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}$$

$$\sqrt{i} = i^{1/2} = \left(e^{i\pi/2}\right)^{1/2} = e^{i\pi/4}$$

Now $\left(\sqrt{i}\right)^{\sqrt{i}} = \left(e^{i\pi/4}\right)^{\left(\frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}\right)} = e^{i\frac{\pi}{4\sqrt{2}} - \frac{\pi}{4\sqrt{2}}}$

$$= e^{-\frac{\pi}{4\sqrt{2}}}\left(e^{i\frac{\pi}{4\sqrt{2}}}\right)$$

$$\left(\sqrt{i}\right)^{\sqrt{i}} = e^{-\frac{\pi}{4\sqrt{2}}}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$$

$$(\sqrt{i})^{\sqrt{i}} = e^{-\frac{\pi}{4\sqrt{2}}}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$$

$\therefore$ Real part $e^{-\frac{\pi}{4\sqrt{2}}}\cos\frac{\pi}{4\sqrt{2}}$

Imaginary part $e^{-\frac{\pi}{\sqrt{2}}}\sin\frac{\pi}{4\sqrt{2}}$

8. Find the principal value of $(1+i)^{1-i}$

Soln:- $Z = (1+i)^{1-i}$

$$\log Z = (1-i)\log(1+i)$$

$$= (1-i)\left[\frac{1}{2}\log(1^2+1^2) + i\tan^{-1}\left(\frac{1}{1}\right)\right]$$

$$= (1-i)\left[\frac{1}{2}\log 2 + i\frac{\pi}{4}\right]$$

$$\log Z = \left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)$$

$$= x + iy \quad (\text{say})$$

$$\therefore Z = e^{x+iy} = e^{x}\cdot e^{iy}$$

$$= e^{x}[\cos y + i\sin y]$$

$$Z = e^{\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right)}\left[\cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)\right]$$

$$= \sqrt{2}\, e^{\pi/4}\left[\cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)\right]$$

$$\left(e^{\frac{1}{2}\log 2} = e^{\log\sqrt{2}} = \sqrt{2}\right)$$

9. Prove that the general value of $(1 + i\tan\alpha)^{-i}$ is $e^{2m\pi+\alpha}[\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)]$

9. Prove that the general value of $(1+i\tan\alpha)^{-i}$ is $e^{2m\pi+\alpha}[\cos(\log\cos\alpha)+i\sin(\log\cos\alpha)]$

Soln:

$$Z = (1+i\tan\alpha)^{-i}$$

$$\operatorname{Log} Z = -i \operatorname{Log}(1+i\tan\alpha)$$

$$= -i\left[\frac{1}{2}\log(1^2+\tan^2\alpha)+i\left[\tan^{-1}\left(\frac{\tan\alpha}{1}\right)+2m\pi\right]\right]$$

$$= -i\left[\frac{1}{2}\log\sec^2\alpha+i(2m\pi+\alpha)\right]$$

$$= -i\left[\log(\sec\alpha)+i(2m\pi+\alpha)\right]$$

$$= (2m\pi+\alpha)-i\log(\sec\alpha)$$

$$\operatorname{Log} Z = (2m\pi+\alpha)+i\log(\cos\alpha)$$

$$\therefore Z = e^{(2m\pi+\alpha)+i\log(\cos\alpha)}$$

$$= e^{(2m\pi+\alpha)}\cdot e^{i\log(\cos\alpha)}$$

$$= e^{(2m\pi+\alpha)}\left[\cos(\log(\cos\alpha))+i\sin(\log(\cos\alpha))\right]$$

10. Considering only principal value, if $(1+i\tan\alpha)^{1+i\tan\beta}$ is real, prove that its value is $(\sec\alpha)^{sec^2\beta}$

Solution:- Let $Z = (1+i\tan\alpha)^{(1+i\tan\beta)}$

Taking logarithm on both sides

$$\log Z = (1+i\tan\beta)\cdot\log(1+i\tan\alpha)$$

$$=(1+i\tan\beta)\left[\frac{1}{2}\log(1+\tan^2\alpha)+i\tan^{-1}\left(\frac{\tan\alpha}{1}\right)\right]$$

$$= (1+i\tan\beta)\left[\tfrac{1}{2}\log\sec^2\alpha + i\alpha\right]$$

$$= (1+i\tan\beta)\left[\log\sec\alpha + i\alpha\right]$$

$$\log z = \left(\log\sec\alpha - \alpha\tan\beta\right) + i\left(\alpha + \tan\beta\log\sec\alpha\right)$$

$$= x+iy \qquad (\text{say})$$

$$x = \log\sec\alpha - \alpha\tan\beta, \quad y = \alpha + \tan\beta\log\sec\alpha.$$

Now $z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

$$z = e^x\cos y + i e^x \sin y$$

Now since z is real,

$$e^x \sin y = 0$$

$$\Rightarrow \sin y = 0 \Rightarrow \underline{\underline{y=0}}.$$

$$\therefore \quad z = e^x\cos y = e^x\cos(0) = e^x$$

$$\therefore z = e^{\log\sec\alpha - \alpha\tan\beta}$$

$$= e^{\log\sec\alpha}\cdot e^{-\alpha\tan\beta} = \sec\alpha \cdot e^{-\alpha\tan\beta} \quad \text{(1)}$$

Since $y=0 \Rightarrow \alpha + \tan\beta\log\sec\alpha = 0$

$$\Rightarrow -\alpha = \tan\beta \cdot \log\sec\alpha.$$

$$\Rightarrow -\alpha\tan\beta = \tan^2\beta\log\sec\alpha$$

$$= \log(\sec\alpha)^{\tan^2\beta}$$

$$\Rightarrow e^{-\alpha\tan\beta} = (\sec\alpha)^{\tan^2\beta}$$

$\therefore$ substituting in ①

$$z = \sec\alpha \cdot e^{-\alpha \tan\beta} = \sec\alpha \cdot (\sec\alpha)^{\tan^2\beta}$$

$$= (\sec\alpha)^{1+\tan^2\beta}$$

$$z = (\sec\alpha)^{\sec^2\beta}$$

11. If $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = \alpha + i\,\beta,$ find α and β

<u>Soln</u>:

$$\alpha + i\beta = \frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}}$$

Taking log on both sides

$$\log(\alpha+i\beta) = (x+iy)\log(a+ib) - (x-iy)\log(a-ib)$$

$$= (x+iy)\left[\frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)\right]$$

$$- (x-iy)\left[\frac{1}{2}\log(a^2+b^2) - i\tan^{-1}\left(\frac{b}{a}\right)\right]$$

$$= \frac{x}{2}\log(a^2+b^2) + ix\tan^{-1}\left(\frac{b}{a}\right) + i\frac{y}{2}\log(a^2+b^2) - y\tan^{-1}\frac{b}{a}$$

$$- \left[\frac{x}{2}\log(a^2+b^2) - ix\tan^{-1}\left(\frac{b}{a}\right) - i\frac{y}{2}\log(a^2+b^2) - y\tan^{-1}\frac{b}{a}\right]$$

$$= 2i\left[x\tan^{-1}\left(\frac{b}{a}\right) + \frac{y}{2}\log(a^2+b^2)\right]$$

$$\log(\alpha+i\beta) = 2ik \qquad \text{where } k = x\tan^{-1}\left(\frac{b}{a}\right) + \frac{y}{2}\log(a^2+b^2)$$

$$\alpha + i\beta = e^{2ik}$$

$$\alpha + i\beta = \cos(2k) + i\sin 2k$$

$$\therefore \alpha = \cos 2k = \cos 2\left[x\tan^{-1}\left(\frac{b}{a}\right) + \frac{y}{2}\log(a^2+b^2)\right]$$

$$\beta = \sin 2k = \sin 2\left[x\tan^{-1}\left(\frac{b}{a}\right) + \frac{y}{2}\log(a^2+b^2)\right]$$

12. If $i^{\alpha+i\beta} = \alpha + i\beta$ $\left(\text{or } i^{i^{i\ldots\ldots\infty}} = \alpha + i\beta\right)$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$ Where n is any positive integer

Soln:

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = \cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)$$

we have $i^{\alpha+i\beta} = \alpha + i\beta$

$$\left[\cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)\right]^{\alpha+i\beta} = \alpha + i\beta$$

$$\left[e^{i\left(2n\pi + \frac{\pi}{2}\right)}\right]^{\alpha+i\beta} = \alpha + i\beta$$

$$\Rightarrow e^{-\beta\left(2n\pi + \frac{\pi}{2}\right)} \cdot e^{i\alpha\left(2n\pi + \frac{\pi}{2}\right)} = \alpha + i\beta$$

$$\Rightarrow e^{-\left(2n\pi + \frac{\pi}{2}\right)\beta}\left[\cos\left(2n\pi + \frac{\pi}{2}\right)\alpha + i\sin\left(2n\pi + \frac{\pi}{2}\right)\alpha\right] = \alpha + i\beta$$

equating real and imaginary parts

$$\alpha = e^{-\left(2n\pi + \frac{\pi}{2}\right)\beta}\cos\left(2n\pi + \frac{\pi}{2}\right)\alpha$$

$$\beta = e^{-\left(2n\pi + \frac{\pi}{2}\right)\beta}\sin\left(2n\pi + \frac{\pi}{2}\right)\alpha$$

$$-2\left(2n\pi + \frac{\pi}{2}\right)\beta$$

$$\therefore \alpha^2+\beta^2 = e^{-2\left(2n\pi+\frac{\pi}{2}\right)\beta}\left[\cos^2\left(2n\pi+\frac{\pi}{2}\right)\alpha + \sin^2\left(2n\pi+\frac{\pi}{2}\right)\alpha\right]$$

$$= e^{-(4n\pi+\pi)\beta}$$

$$\therefore \alpha^2+\beta^2 = e^{-(4n+1)\pi\beta}$$

13. Prove that $\log tan\left(\frac{\pi}{4}+i\frac{x}{2}\right) = i\, tan^{-1}(\sinh x)$. (H.W).

$$\log \tan\left(\frac{\pi}{4}+i\frac{x}{2}\right) = \log\left[\frac{1+\tan(ix/2)}{1-\tan(ix/2)}\right]$$

$$= \log\left[\frac{1+i\tanh(x/2)}{1-i\tanh(x/2)}\right]$$