Chapter 1

Linear Differential Equation With Constant Coefficients

1.1 Introduction

One thing that will never change is the fact that the world is constantly changing. Mathematically, rates of change are described by derivatives. If you try and use maths to describe the world around you say the growth of a plant, the fluctuations of the stock market, the spread of diseases, or physical forces acting on an object you soon find yourself dealing with derivatives of functions. The way they inter-relate and depend on other mathematical parameters is described by differential equations. These equations are at the heart of nearly all modern applications of mathematics to natural phenomena. The applications are almost unlimited, and they play a vital role in much of modern technology.

1.2 Definitions

Definition. An equation involving derivatives or differential coefficients of depedent variable with respect to one or more independent variables is called **differential equations**.

Thus,

$$e^x \mathrm{d}x + e^y \mathrm{d}y = 0 \tag{1}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + n^2 y = 0 \tag{2}$$

$$y = x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{\mathrm{d}y/\mathrm{d}x} \tag{3}$$

$$\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{3/2} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \tag{4}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \tag{5}$$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{6}$$

all are examples of differential equations.

Definition. A differential equation is called an **ordinary differential equation** if it contains differential coefficients of dependent variable w.r.t single ndependent variable.

Thus the equations (1)-(4) are all ordinary differential equations.

Definition. A differential equation is called an **partial differential equation** if it contains differential coefficients of dependent variable w.r.t. more than one independent variables.

Thus the equations (5) and (6) are partial differential equations.

Definition. The **order** of a differential equation is the highest order derivative occurring in the differential equation.

The the equation (1), (3) and (5) are of first order and the equations (2), (4) and (6) are of order 2.

Definition. The **degree** of a differential equation is the degree of the highest order derivative occurring in the differential equation provided that the equation has been made free from radicals and fractions as far as derivative are concerned.

The equation (1), (2), (5) and (6) are of degree 1. By removing fraction term from equation (3) it will takes the form $y\frac{\mathrm{d}y}{\mathrm{d}x} = x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + x$. Thus the degree of equation (3) is 2. The equation (4) contain radical term, by removing it we get $\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^3 = \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2$. Thus the degree of equation (4) is 2.

Definition. A differential equation is said to be **linear** if i) all the differential coefficients and dependent variable occur in the first degree only and ii) there is no product of differential coefficients and/or dependent variable.

Thus the equations (1), (2), (5) and (6) are linear.

Definition. A differential equation is said to be **non-linear** if it is not linear. i.e. at leat one differential coefficient is of degree more than one or there is product of differential coefficients and/or dependent variable.

Thus the equations (3) and (4) are non-linear

Definition. The relation between dependent and independent variable when substituted in the differential equation reduces to an identity is called **solution** of the differential equation.

For example the relation $y = c_1 \cos x + c_2 \sin x$ is the solution of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0$.

Definition. If the order of the ordinary differential equation is equal to the number of arbitrary constants in the solution of the differential equation, then the solution is called the **complete solution** or **general solution** and if we assign the particular values to the arbitrary constants in the solution then that solution is called **particular solution**.

Thus the solution $y = c_1 \cos x + c_2 \sin x$ is complete solution of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0$. If we assign $c_1 = 1$ and $c_2 = -1$ then above complete solution becomes $y = \cos x - \sin x$ is the particular solution of the equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0$.

1.3 Linear differential equations with constant coefficients

A linear differential equation with constant coefficient of nth order is of the form

$$a_0 \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1 \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + a_2 \frac{\mathrm{d}^{n-2} y}{\mathrm{d}x^{n-2}} + \cdots + a_{n-1}\frac{\mathrm{d}y}{\mathrm{d}x} + a_n y = X,\ a_0 \neq 0 \tag{7}$$

where a_0, a_1, a_2, …, a_n are constants and X is either function of x or constant.

In above equation if $X = 0$ for all x, then it is called homogeneous equation, where as if $X \neq 0$ for some x then it is called non-homogeneous equation.

1.3.1 Differential Operator D

Let us introduce the differential operator $D \equiv \dfrac{\mathrm{d}}{\mathrm{d}x}$ so that $\dfrac{\mathrm{d}y}{\mathrm{d}x} \equiv Dy$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} \equiv D^2y$, …, $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n} \equiv D^n y$. Then equation (7) becomes

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_{n-1} D y + a_n y = X$$

$$\Rightarrow \quad (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n) y = X$$

$$\Rightarrow \quad f(D)y = X$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n$. i.e $f(D)$ is a polynomial in D. The equation $f(D)y = X$ is called symbolic form of the equation (7).

1.3.2 Solution of $f(D)y = X$

The complete solution of equation (7) is complementary function + particular integral. The complete solution of $f(D)y = 0$ is called complementary solution and the particular solution of $f(D)y = X$ is called particular integral.

Rules to find the complementary function or Complete solution of $f(D)(y) = 0$

Consider the equation

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n) y = 0 \tag{8}$$

where a_0, a_1, a_2,..., a_n are constants. Note that $a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n$ is a polynomial in D of degree n. Therefore by fundamental theorem of algebra the equation

$$a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_{n-1}m + a_n = 0 \tag{9}$$
$$\text{i.e. } f(m) = 0$$

has exactly n roots. Let m_1, m_2,..., m_n be the n roots of the equation (9). The equation (9) is called the **auxiliary equation**.
The solution of equation (8) depends on the nature of the roots of the auxiliary equation. Following are the various cases:

Case-I: If all the roots of auxiliary equation are real and distinct
Let m_1, m_2,..., m_n be the n roots of the equation (9) and suppose that all the roots are the distinct and real. Then the equation (8) can be written as

$$(D - m_1)(D - m_2)\ldots(D - m_n)y = 0 \tag{10}$$

Here the equation (11) will be satisfied by the solution of the equations

$$(D - m_1)y = 0, (D - m_2)y = 0, (D - m_3)y = 0, \ldots, (D - m_n)y = 0$$

Now consider

$$(D - m_1)y = 0 \quad \text{i.e } \frac{dy}{dx} - m_1y = 0$$

This is linear in y. Therefore, the integrating factor is $e^{\int -m_1 dx} = e^{-m_1x}$ and the complete solution of $(D - m_1)y = 0$ is

$$ye^{-m_1x} = \int 0.e^{-m_1x}dx + c_1$$

$$\Rightarrow \qquad y = c_1e^{m_1x}$$

Similarly, the solution of $(D - m_2)y = 0$ is $y = c_2e^{m_2x}$, the solution of $(D - m_3)y = 0$ is $y = c_3e^{m_3x}$ and so on we get, $(D - m_n)y = 0$ is $y = c_ne^{m_nx}$.
Hence the complete solution of equation (8) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \cdots + c_ne^{m_nx}$$

Case-II: If two roots of the auxiliary equation are real and equal
Let $m_1 = m_2$, m_3, m_4, ..., m_n are the roots of equation (9). Then the equation (8) can be written as

$$(D - m_1)(D - m_1)(D - m_2)\ldots(D - m_n)y = 0 \tag{11}$$

The equation (11) will be satisfied by the solution of the equations

$$(D - m_1)(D - m_1)y = 0, (D - m_2)y = 0, (D - m_3)y = 0, \ldots, (D - m_n)y = 0$$

To find solution of $(D - m_1)(D - m_1)y = 0$
Let $(D - m_1)y = v$. Then above equation becomes,

$$(D - m_1)v = 0$$

This is linear in v. Its solution is $v = c_1 e^{m_1 x}$. Putting this value in $(D - m_1)y = v$, we get

$$(D - m_1)y = c_1 e^{m_1 x}$$

Which can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} - m_1 y = c_1 e^{m_1 x}$$

Above equation is linear in y. Therefore, the integration factor I.F. $= e^{\int m_1 \mathrm{d}x} = e^{-m_1 x}$. Thus the general solution of above equation is

$$\begin{aligned} y.(I.F) &= \int (I.F) c_1 e^{m_1 x} \mathrm{d}x + c_2 \\ y.e^{-m_1 x} &= \int e^{-m_1 x} c_1 e^{m_1 x} \mathrm{d}x + c_2 \\ &= \int c_1 \mathrm{d}x + c_2 \\ &= c_1 x + c_2 \end{aligned}$$

Thus, the solution of $(D - m_1)(D - m_1)y = 0$ is $y = (c_1 x + c_2)e^{m_1 x}$.
Hence the complete solution of equation (8) is

$$y = (c_1 x + c_2)e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

Note: Let $m_1 = m_2 = m_3$ be the three equal roots of the auxiliary equation of the equation (8). Then the complete solution of equation (8) is

$$y = (c_1 x^2 + c_2 x + c_3)e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

Case-III: If two roots of auxiliary equation are imaginary

Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ be two imaginary roots and m_3, m_4,..., m_n are real and distinct roots of the auxiliary equation (9). Then as in case (I) the complete solution of equation (8) is

$$\begin{aligned} y &= c_1 e^{\alpha + i\beta} x + c_2 e^{\alpha - i\beta} x + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \\ &= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \\ &= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \\ &= e^{\alpha x}[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \\ &= e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \end{aligned}$$

Thus complete solution of equation (8) in this case is

$$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

Case-IV: If two pairs of imaginary roots are equal

Let $m_1 = m_2 = \alpha + i\beta$ and $m_3 = m_4 = \alpha - i\beta$. Then by Case-(II) and Case-(III) the complete solution of equation (8) is

$$y = e^{\alpha x}[(C_1 x + C_2)\cos\beta x + (C_3 x + C_4)\sin\beta x] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}$$

Summary

Let m_1, m_2, $\ldots$, m_n be the roots of the equation $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n = 0$. Depending upon the nature of roots, the complete solution of $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n)y = 0$ is given in the following table.

Types of roots	**Complete solution**
All the roots are real and distinct	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
Two roots m_1 and m_2 are repeated and remaining all are real and distinct	$y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
Three roots m_1, m_2 and m_3 are repeated and remaining all are real and distinct	$y = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$
Two roots are imaginary i.e. $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$	$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
Pair of imaginary roots are repeated i.e. $m_1 = m_2 = \alpha + i\beta$ and $m_3 = m_4 = \alpha - i\beta$	$y = e^{\alpha x}[(C_1 x + C_2)\cos\beta x + (C_3 x + C_4)\sin\beta x] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}$

Remember

To solve the equation of the type $f(D)(y) = 0$, we have to find the solution of the equation $f(D) = 0$. Followings are the formulae and methods to solve $f(D) = 0$.

(i) $D^2 - a^2 = (D - a)(D + a)$

(ii) $D^2 + a^2 = (D + ia)(D - ia)$

(iii) If $aD^2 + bD + c = 0$, then $D = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(iv) $D^3 - a^3 = (D - a)(D^2 + aD + a^2)$

(v) $D^3 + a^3 = (D + a)(D^2 - aD + a^2)$

(vi) For the equation of the type $aD^3 + bD^2 + cD + d = 0$, where a, b, c and d are constants. Find one root by trial and error method and use synthetic division method to find the remaining roots.

(vii) $D^4 - a^4 = (D^2 - a^2)(D^2 + a^2) = (D - a)(D + a)(D - ia)(D + ia)$

(viii) $D^4 + a^4 = 0$
Adding and subtracting middle term $2a^2D^2$ to make perfect square, we get

$$D^4 + 2a^2D^2 + a^4 - 2a^2D^2 = 0$$

$$\Rightarrow (D^2 + a^2)^2 - 2a^2D^2 = 0$$
$$\Rightarrow (D^2 + \sqrt{2}aD + a^2)(D^2 - \sqrt{2}aD + a^2) = 0$$

To find the roots of $D^2 + \sqrt{2}aD + a^2 = 0$ and $D^2 - \sqrt{2}aD + a^2 = 0$ use the formula given in (iii).

(ix) For equation $f(D) = 0$ of degree $n \geq 3$, use synthetic division method.

Working rule to find the complementary function

Step:1 If the given equation is in the form

$$a_0\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + a_2\frac{\mathrm{d}^{n-2} y}{\mathrm{d}x^{n-2}} + \cdots + a_{n-1}\frac{\mathrm{d}y}{\mathrm{d}x} + a_n y = 0$$

Then write $\dfrac{\mathrm{d}}{\mathrm{d}x} = D$ and write the equation in symbolic form as

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n)y = X$$

Step:2 Write the auxiliary equations as

$$a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_{n-1}m + a_n = 0$$

Step:3 Find the roots of the equation $a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_{n-1}m + a_n = 0$. Let $m = m_1, m_2, \ldots, m_n$ be the n roots of the auxiliary equation.

Step:4 Depending upon the nature of roots write the complementary function by using the rules of finding the complementary function.

Examples of the type $f(D)y = 0$

Example 1. Solve $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 7\dfrac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0$

Sol. Consider the given equation as

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 7\frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0 \tag{1}$$

In symbolic form, we get

$$(D^3 - 7D - 6)y = 0$$

The auxiliary equations is

$$m^3 - 7m - 6 = 0$$

Here $m = -1$ is the root of the above equation. Therefore by synthetic division method, we get

$$
\begin{array}{r|rrrr}
 & 1 & 0 & -7 & -6 \\
-1 & & -1 & 1 & 6 \\
\hline
 & 1 & -1 & -6 & 0
\end{array}
$$

Therefore

$$m^3 - 7m - 6 = (m+1)(m^2 - m - 6) = 0$$

$$\Rightarrow (m+1)(m-3)(m+2) = 0$$
$$\Rightarrow m = -1, 3, -2$$

The complete solution of equation (1) is

$$y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{-2x}$$

Example 2. Solve $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 8y = 0$

Sol. Consider the given equation as

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 8y = 0$$

In symbolic form, the given equation can be written as

$$(D^3 + 8)y = 0$$

The auxiliary equation is

$$m^3 + 8 = 0 \Rightarrow m^3 + 2^3 = 0 \Rightarrow (m+2)(m^2 - 2m + 4) = 0$$
$$\Rightarrow m = -2, \frac{2 \pm \sqrt{4 - 16}}{2}$$
$$\Rightarrow m = -2, \frac{2 \pm 2\sqrt{3}i}{2}$$
$$\Rightarrow m = -2, 1 \pm \sqrt{3}i$$

$\therefore$ The complete solution is

$$y = c_1 e^{-2x} + e^{x}[c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x]$$

Example 3. Solve $\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4} + 4y = 0$

Sol. The given equation as

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + 4y = 0 \tag{1}$$

The auxiliary equation is

$$m^4 + 4 = 0$$

$$\begin{aligned}
&\Rightarrow (m^2)^2 + (\sqrt{2})^4 = 0\\
&\Rightarrow (m^2)^2 + 2(m^2)(\sqrt{2})^2 + (\sqrt{2})^4 - 2(m^2)(\sqrt{2})^2 = 0\\
&\Rightarrow (m^2)^2 + 4m^2 + 2^2 - 4m^2 = 0\\
&\Rightarrow \left(m^2+2\right)^2 - (2m)^2 = 0\\
&\Rightarrow (m^2+2m+2)(m^2-2m+2) = 0\\
&\Rightarrow m = \frac{-2 \pm \sqrt{4-8}}{2},\ \frac{2 \pm \sqrt{4-8}}{2}\\
&\Rightarrow m = -1 \pm i,\ 1 \pm i
\end{aligned}$$

Therefore, the complete solution of given equation is

$$y = e^{-x}(c_1 \cos x + c_3 \sin x) + e^{x}(c_3 \cos x + c_4 \sin x)$$

Example 4. Solve $(D^2 - D - 6)y = 0$ given that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 0$

Sol. The given equation as

$$(D^2 - D - 6)y = 0 \tag{1}$$

The auxiliary equation is

$$m^2 - m - 6 = 0$$

$$\begin{aligned}
&\Rightarrow (m-3)(m+2) = 0\\
&\Rightarrow m = 3, -2
\end{aligned}$$

Therefore, the complete solution is

$$y = c_1 e^{3x} + c_2 e^{-2x} \tag{2}$$

To find c_1 and c_2:

Differentiate equation (2) w.r.t x, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3c_1 e^{3x} - 2c_2 e^{-2x} \tag{3}$$

Given that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 0$. Therefore, by equation (2) and (3) we get

$$c_1 + c_2 = 0 \tag{4}$$

and

$$3c_1 - 2c_2 = 4 \tag{5}$$

Solving above two equations we get, $c_1 = \frac{4}{5}$ and $c_2 = -\frac{4}{5}$. Putting these values in equation (2), we get

$$y = \frac{4}{5}\left(e^{3x} - e^{-2x}\right)$$

This is the required solution of given equation.

Example 5. Solve $(D^6 - 64)y = 0$

Sol. The given equation as

$$(D^6 - 64)y = 0 \tag{1}$$

The auxiliary equation is

$$m^6 - 64 = 0$$

$$\begin{aligned}
&\Rightarrow (m^3)^2 - (8)^2 = 0\\
&\Rightarrow (m^3 - 8)(m^3 + 8) = 0\\
&\Rightarrow (m^3 - 2^3)(m^3 + 2^3) = 0\\
&\Rightarrow (m - 2)(m^2 - 2m + 4)(m + 2)(m^2 + 2m + 4) = 0\\
&\Rightarrow m = 2,\ \frac{2 \pm \sqrt{4 - 16}}{2},\ -2,\ \frac{-2 \pm \sqrt{4 - 16}}{2}\\
&\Rightarrow m = 2, -2, 1 \pm \sqrt{3}i, -1 \pm \sqrt{3}i
\end{aligned}$$

The complete solution of given equation is

$$y = c_1e^{2x} + c_2e^{-2x} + e^x[c_3 \cos \sqrt{3}x + c_4 \sin \sqrt{3}x] + e^{-x}[c_5 \cos \sqrt{3}x + c_6 \sin \sqrt{3}x]$$

Example 6. Solve $\{(D^2 + 1)^3(D^2 + D + 1)^2\}y = 0$

Sol. The given equation as

$$\{(D^2 + 1)^3(D^2 + D + 1)^2\}y = 0 \tag{1}$$

The auxiliary equation is

$$(m^2 + 1)^3(m^2 + D + 1)^2 = 0$$

$$\begin{aligned}
&\Rightarrow m^2 + 1 = 0 \text{ or } m^2 + m + 1 = 0\\
&\Rightarrow m = \pm i \text{ or } m = \frac{-1 \pm \sqrt{1 - 4}}{2}\\
&\Rightarrow m = \pm i \text{ or } m = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}
\end{aligned}$$

The factor $m^2 + 1$ is repeated three times and $m^2 + m + 1$ repeated two times. Thus the roots are

$$m = \pm\, i, \pm\, i, \pm\, i, -\frac{1}{2} \pm \frac{\sqrt{3}i}{2},\ -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

The complete solution of equation (1) is

$$\begin{aligned}
y =& (c_1 + c_2x + c_3x^2) \cos x + (c_4 + c_5x + c_6x^2) \sin x +\\
& e^{-\frac{1}{2}x} \left((c_7 + c_8x) \cos \frac{\sqrt{3}}{2}x + (c_9 + c_{10}x) \sin \frac{\sqrt{3}}{2}x \right)
\end{aligned}$$

Exercise

Solve the following differential equations

1) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\dfrac{\mathrm{d}y}{\mathrm{d}x}+4y=0$

2) $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}-5\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+8\dfrac{\mathrm{d}y}{\mathrm{d}x}-4y=0$

3) $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+6\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+11\dfrac{\mathrm{d}y}{\mathrm{d}x}+6y=0$

4) $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\dfrac{\mathrm{d}y}{\mathrm{d}x}+12y=0$

5) $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}-3\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\dfrac{\mathrm{d}y}{\mathrm{d}x}-y=0$

6) $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+y=0$

7) $(D^3+D^2+4D+4)y=0$

8) $(D^4-4D+4)y=0$

9) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\dfrac{\mathrm{d}y}{\mathrm{d}x}+29y=0$, given that $y=0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=15$ when $x=0$

10) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+5\dfrac{\mathrm{d}y}{\mathrm{d}x}+6y=0$, given that $y=0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=15$ when $x=0$

Answers

1) $y=(c_1+c_2x)e^{2x}$

2) $y=(c_1+c_2x)e^{2x}+c_3e^{x}$

3) $y=c_1e^{-x}+c_2e^{-2x}+c_3e^{-3x}$

4) $y=c_1e^{-3x}+e^{x}[c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x]$

5) $y=(c_1+c_2x+c_3x^2)e^{x}$

6) $y=c_1e^{-x}+e^{x/2}\left(c_2\cos\dfrac{\sqrt{3}x}{2}+c_3\sin\dfrac{\sqrt{3}x}{2}\right)$

7) $y=c_1e^{-x}+c_2\cos 2x+c_3\sin 2x$

8) $y=(c_1+c_2x)e^{\sqrt{2}x}+(c_3+c_4x)e^{-\sqrt{2}x}$

9) $y=3e^{-2x}\sin 5x$

10) $y=15(e^{-2x}-e^{-3x})$

1.4 The operator $\dfrac{1}{f(D)}$

Definition. $\dfrac{1}{f(D)}X$ is that function of x, *free from arbitrary constants*, which when operated upon by $f(D)$ gives X.
Thus,

$$f(D)\left\{\frac{1}{f(D)}\,X\right\}=X$$

$\Rightarrow \quad f(D)$ and $\dfrac{1}{f(D)}$ are inverse operators.

1.4.1 Particular integral of $f(D)y = X$

Consider the liner differential equation as

$$a_0\frac{\mathrm{d}^n y}{\mathrm{d}x^n}+a_1\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}+a_2\frac{\mathrm{d}^{n-2} y}{\mathrm{d}x^{n-2}}+\cdots+a_{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}+a_n y=X \tag{2}$$

The symbolic form of above equation is

$$f(D)y=X \tag{3}$$

Putting $y=\dfrac{1}{f(D)}X$ in equation (3) we get

$$f(D)\left\{\frac{1}{f(D)}\,X\right\}=X\ \Rightarrow\ X=X$$

$\Rightarrow \quad y=\dfrac{1}{f(D)}X$ is solution of equation (3). Since it contain no arbitrary constants, it is the particular integral of $f(D)y=X$. Therefore,

$$\boxed{\text{P. I.}=\frac{1}{f(D)}X}$$

Note

1) The symbol D stands for the differential operator. Therefore, $\dfrac{1}{D}$ stands for the operation of integration. Thus,

$$\frac{1}{D}X=\int X\mathrm{d}x,\ \frac{1}{D^2}X=\int\int X(\mathrm{d}x)^2$$

2) $\dfrac{1}{D-a}X = e^{ax}\displaystyle\int Xe^{-ax}\mathrm{d}x$

Let us consider

$$\frac{1}{D-a}X = y$$

Operating on both side by $D-a$, we get

$$\begin{aligned}
& (D-a)\left(\frac{1}{D-a}X\right) = (D-a)y \\
\Rightarrow \qquad & X = \frac{\mathrm{d}y}{\mathrm{d}x} - ay \\
\Rightarrow \qquad & \frac{\mathrm{d}y}{\mathrm{d}x} - ay = X
\end{aligned}$$

Which is linear equation. Therefore, $I.F. = e^{\int a\mathrm{d}x} = e^{ax}$. Therefore, its solution is

$$\begin{aligned}
& ye^{ax} = \int X\, e^{ax}\mathrm{d}x \\
\Rightarrow \qquad & y = e^{-ax}\int X\, e^{ax}\mathrm{d}x \\
\Rightarrow \qquad & \frac{1}{D-a}X = e^{-ax}\int X\, e^{ax}\mathrm{d}x
\end{aligned}$$

Hence

$$\frac{1}{D-a}X = e^{-ax}\int X\, e^{ax}\mathrm{d}x$$

Similarly, one can prove

$$\frac{1}{D+a}X = e^{ax}\int X\, e^{-ax}\mathrm{d}x$$

Rules for finding the particular integral (P.I.)

The methods of finding the particular integrals for the linear differential equation $f(D)y = X$, where $f(D) = a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n$, depends on the special forms of the function X on R.H.S. These special forms are

i) $X = e^{ax}$, where a is a is constant

ii) $X = \sin ax$ or $\cos ax$, where a is constant.

iii) $X = x^m$ or polynomial in x of degree m

iv) $X = e^{ax}V$, where V is a function of x

v) $X = x^mV$, where V is function of x.

1.5 P.I. of linear differential equation

Consider the liner differential equation as

$$a_0\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + a_2\frac{\mathrm{d}^{n-2} y}{\mathrm{d}x^{n-2}} + \cdots + a_{n-1}\frac{\mathrm{d}y}{\mathrm{d}x} + a_n y = X \tag{4}$$

In symbolic form, above equation can be written as

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n)y = X \tag{5}$$

It can be written as

$$f(D)y = X$$

where, $a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n = f(D)$. Then the particular integral of equation (4) is

$$P.I. = \frac{1}{f(D)}X$$

1.6 Particular integral when $X = e^{ax}$

We have $De^{ax} = ae^{ax}$, $D^2e^{ax} = a^2e^{ax}$, $\ldots$, $D^ne^{ax} = a^ne^{ax}$. Then

$$\begin{aligned} f(D)e^{ax} &= (D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n)e^{ax} \\ &= D^ne^{ax} + a_1D^{n-1}e^{ax} + a_2D^{n-2}e^{ax} + \cdots + a_{n-1}De^{ax} + a_ne^{ax} \\ &= a^ne^{ax} + a_1a^{n-1}e^{ax} + a_2a^{n-2}e^{ax} + \cdots + a_{n-1}ae^{ax} + a_ne^{ax} \\ &= (a^n + a_1a^{n-1} + a_2a^{n-2} + \cdots + a_{n-1}a + a_n)e^{ax} \\ &= f(a)e^{ax} \end{aligned}$$

Therefore,

$$f(D)\left\{\frac{1}{f(D)}\,e^{ax}\right\} = \frac{1}{f(D)}[e^{ax}f(a)]$$

This implies

$$e^{ax} = \frac{1}{f(D)}[e^{ax}f(a)]$$

$$\Rightarrow\ \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$$

Therefore the particular integral is given by

$$\boxed{\text{P. I.} = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}, \text{ provided } f(a) \neq 0}$$

Case of failure: If $f(a) = 0$, then above methods. Since $f(a) = 0$, $D = a$ is a root of the auxiliary equation $f(D) = 0$. Therefore, $D - a$ is factor of $f(D)$. Thus one can write

$$f(D) = (D - a)\phi(D) \tag{6}$$

Now consider $\phi(a) \neq 0$, then

$$\begin{aligned}\frac{1}{f(D)}e^{ax} &= \frac{1}{(D-a)\phi(D)} = \frac{1}{D-a}\,\frac{1}{\phi(D)}e^{ax}\\ &= \frac{1}{\phi(a)}\,\frac{1}{D-a}e^{ax} = \frac{1}{\phi(a)}e^{ax}\int e^{ax}\,e^{-ax}\,\mathrm{d}x\\ &= \frac{1}{\phi(a)}e^{ax}\int 1\,\mathrm{d}x = x\,\frac{1}{\phi(a)}e^{ax}\end{aligned}$$

Now, differentiate equation (6) w.r.t. x, we get

$$f'(D) = (D-a)\phi'(D) + \phi(D)$$

$\Rightarrow$ $$f'(a) = \phi(a)$$

Therefore,

$$\frac{1}{f(D)}e^{ax} = x\,\frac{1}{f'(a)}e^{ax} \quad \text{provided } f'(a) \neq 0$$

Again if $f'(a) = 0$, then $\dfrac{1}{f(D)}e^{ax} = x^2\,\dfrac{1}{f''(a)}e^{ax}$ provided $f''(a) \neq 0$

and so on.

Remember: The P.I. of the differential equation $f(D)y = e^{ax}$ is given by

1) $P.I. = \dfrac{1}{f(D)}e^{ax} = \dfrac{1}{f(a)}e^{ax}$, provided $f(a) \neq 0$

2) If $f(a) = 0$, then $P.I. = x\dfrac{1}{f'(D)}e^{ax} = x\dfrac{1}{f'(a)}e^{ax}$, provided $f'(a) \neq 0$

3) If $f'(a) = 0$, then $P.I. = x^2\dfrac{1}{f''(D)}e^{ax} = x^2\dfrac{1}{f''(a)}e^{ax}$, provided $f''(a) \neq 0$

4) If $f^{n-1}(a) = 0$, then $P.I. = x^n\dfrac{1}{f^n(D)}e^{ax} = x^n\dfrac{1}{f^n(a)}e^{ax}$, provided $f^n(a) \neq 0$

Examples

Example 7. Solve $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} - 3\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = e^{3x}$

Sol. The given equation is

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = e^{3x} \tag{1}$$

In symbolic form, above equation can be written as

$$(D^3 - 3D^2 + 4)y = e^{3x}$$

The auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$

Here, $m = -1$ is root of the above equation. Therefore by synthetic division method we get

$$\begin{array}{c|cccc} & 1 & -3 & 0 & 4 \\ 2 & & 2 & -2 & -4 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

Therefore,

$$\begin{aligned} & m^3 - 3m^2 + 4 = (m-2)(m^2 - m - 2) = 0 \\ \Rightarrow & (m-2)(m-2)(m+1) = 0 \\ \Rightarrow & m = -1, 2, 2 \end{aligned}$$

Therefore, the complementary function is

$$C.F. = c_1 e^{-x} + (c_2 + c_3 x)e^{2x}$$

Now,

$$P.I. = \frac{1}{D^3 - 3D^2 + 4} e^{3x} = \frac{1}{3^3 - 3 \times 3^2 + 4} e^{3x} = \frac{1}{4} e^{3x}$$

The complete solution of equation (1.7) is

$$y = c_1 e^{-x} + (c_2 + c_3 x)e^{2x} + \frac{1}{4} e^{3x}$$

Example 8. Solve $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = -2\cos hx$

Sol. The given equation is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = -2\cos hx \tag{1}$$

The symbolic form of equation is

$$(D^2 + 4D + 5)y = -\cosh x$$

The auxiliary equation is

$$m^2 + 4m + 5 = 0 \ \Rightarrow m = \frac{-4 \pm \sqrt{16 - 4 \times 5}}{2} = -2 \pm i$$

The complementary function is

$$C.F. = e^{-2x}[c_1 \cos x + c_2 \sin x]$$

Now, P.I. corresponding $-2\cos hx$ is given by

$$P.I. = \frac{1}{D^2 + 4D + 5}(-2\cos hx) = -2\frac{1}{D^2 + 4D + 5}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= -\frac{1}{D^2+4D+5}e^x - \frac{1}{D^2+4D+5}e^{-x} = -\frac{1}{1+4+5}e^x - \frac{1}{1-4+5}e^{-x}$$
$$= -\frac{1}{10}e^x - \frac{1}{2}e^{-x}$$

The complete solution of equation (13) is

$$y = e^{-2x}[c_1\cos x + c_2\sin x] - \frac{1}{10}e^x - \frac{1}{2}e^{-x}$$

Example 9. Solve $\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 2e^x\cos h2x$

Sol. Consider the given equation as

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 2e^x\cos h2x \tag{1}$$

The symbolic form of equation is

$$(D^4 + D^2 + 1)y = 2e^x\cos h2x$$

The auxiliary equation is

$$m^4 + m^2 + 1 = 0$$

$$\begin{aligned} m^4 + m^2 + 1 = 0 &\Rightarrow m^4 + 2m^2 + 1 - m^2 = 0 \\ &\Rightarrow (m^2+1)^2 - m^2 = 0 \\ &\Rightarrow (m^2+m+1)(m^2-m+1) = 0 \\ &\Rightarrow m = \frac{-1\pm\sqrt{-3}}{2},\ \frac{1\pm\sqrt{-3}}{2} \\ &\Rightarrow m = \frac{-1}{2}\pm\frac{\sqrt{3}i}{2},\ \frac{1}{2}\pm\frac{\sqrt{3}i}{2} \end{aligned}$$

The complementary function is

$$y = e^{\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2} + c_2\sin\frac{\sqrt{3}}{2}\right) + e^{-\frac{1}{2}x}\left(c_3\cos\frac{\sqrt{3}}{2} + c_4\sin\frac{\sqrt{3}}{2}\right)$$

Now, P.I. corresponding to $2e^x\cos h2x$ is given by

$$\begin{aligned} P.I. &= \frac{1}{D^4+D^2+1}2e^x\cos h2x = \frac{1}{D^4+D^2+1}2e^x\left(\frac{e^{2x}+e^{-2x}}{2}\right) \\ &= \frac{1}{D^4+D^2+1}(e^{3x}+e^{-x}) = \frac{1}{D^4+D^2+1}e^{3x} + \frac{1}{D^4+D^2+1}e^{-x} \\ &= \frac{1}{3^4+3^2+1}e^{3x} + \frac{1}{(-1)^4+(-1)^2+1}e^{-x} = \frac{1}{91}e^{3x} + \frac{1}{3}e^{-x} \end{aligned}$$

The complete solution of equation (14) is

$$y = e^{\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}x}{2} + c_2\sin\frac{\sqrt{3}x}{2}\right) + e^{-\frac{1}{2}x}\left(c_3\cos\frac{\sqrt{3}x}{2} + c_4\sin\frac{\sqrt{3}x}{2}\right) + \frac{1}{91}e^{3x} + \frac{1}{3}e^{-x}$$

Example 10. Solve $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4y = (1+e^x)^2$

Sol. The given equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4y = (1+e^x)^2 \tag{1}$$

The symbolic form of equation is

$$(D^2-4)y = (1+e^x)^2$$

The auxiliary equation is

$$m^2 - 4 = 0 \ \Rightarrow m = 2, -2$$

The complementary function is

$$y = c_1e^{2x} + c_2e^{-2x}$$

Now, P.I. corresponding to $(1+e^x)^2$ is given by

$$\begin{aligned} P.I. &= \frac{1}{D^2-4}(1+e^x)^2 = \frac{1}{D^2-4}(1+2e^x+e^{2x}) = \frac{1}{D^2-4}(1) + 2\frac{1}{D^2-4}e^x + \frac{1}{D^2-4}e^{2x} \\ &= \frac{1}{D^2-4}(e^{0x}) + 2\frac{1}{D^2-4}e^x + \frac{1}{D^2-4}e^{2x} = -\frac{1}{4} - \frac{2}{3}e^x + \frac{1}{0}e^{2x} \qquad \text{(case of failure)} \\ &= -\frac{1}{4} - \frac{2}{3}e^x + x\frac{1}{2D}e^{2x} = -\frac{1}{4} - \frac{2}{3}e^x + x\frac{1}{4}e^{2x} \end{aligned}$$

The complete solution of equation (15) is

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}$$

Example 11. Solve $(D^3+3D^2+3D+1)y = e^{-x}$

Sol. The given equation as

$$(D^3+3D^2+3D+1)y = e^{-x} \tag{1}$$

The auxiliary equation is

$$m^3 + 3m^2 + 3m + 1 = 0$$

Here $m = -1$ is root of the above equation. Therefore by synthetic division method we get

$$\begin{array}{r|rrrr} & 1 & 3 & 3 & 1 \\ -1 & & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

Thus

$$m^3 + 3m^2 + 3m + 1 = (m+1)(m^2+2m+1) = (m+1)^3$$

Therefore,

$$m^3 + 3m^2 + 3m + 1 = 0 \ \Rightarrow (m+1)^3 = 0 \ \Rightarrow m = -1, -1, -1$$

The complementary function is

$$C.F. = (c_1 + c_2x + c_3x^2)e^{-x}$$

Now, P.I. corresponding to e^{-x} is given by

$$\begin{aligned}
P.I. &= \frac{1}{D^3+3D^2+3D+1}e^{-x} = \frac{1}{(-1)^3+3(-1)^2+3(-1)+1}e^{-x} && \text{(case of failure)}\\
&= x\frac{1}{3D^2+6D+3}e^{-x} = x\frac{1}{3(-1)^2+6(-1)+3}e^{-x} && \text{(case of failure)}\\
&= x^2\frac{1}{6D+6}e^{-x} = x^2\frac{1}{6(-1)+6}e^{-x} && \text{(case of failure)}\\
&= x^3\frac{1}{6}
\end{aligned}$$

The complete solution of equation (16) is

$$y = (c_1 + c_2x + c_3x^2)e^{-x} + \frac{1}{6}x^3$$

Exercise

Solve the following differential equations

1) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\dfrac{\mathrm{d}y}{\mathrm{d}x} + 6y = e^x$

2) $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} + y = 3 + 5e^x$

3) $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} - 2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\dfrac{\mathrm{d}y}{\mathrm{d}x} + 6y = e^{4x}$

4) $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$

5) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \dfrac{\mathrm{d}y}{\mathrm{d}x} + y = (1 - e^x)^2$

6) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\dfrac{\mathrm{d}y}{\mathrm{d}x} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$

Answers

1) $y = c_1e^{-2x} + c_2e^{-3x} + \dfrac{e^x}{12}$

2) $y = c_1e^x + e^{\frac{1}{2}x}\left(c_2\cos\dfrac{\sqrt{3}}{2} + c_3\sin\dfrac{\sqrt{3}}{2}\right)$

3) $y = c_1e^x + c_2e^{-2x} + c_3e^{3x} + \dfrac{1}{18}e^{4x}$

4) $y = (c_1 + c_2x)e^x + c_3e^{3x} + \dfrac{1}{8}(xe^{3x} - x^2e^x)$

5) $y = e^{-\frac{1}{2}x}\left(c_1\cos\dfrac{\sqrt{3}}{2} + c_2\sin\dfrac{\sqrt{3}}{2}\right) + 1 - \dfrac{2}{3}e^x + \dfrac{1}{7}e^{2x}$

6) $y = (c_1 + c_2 x)e^{3x} + 3x^2 e^{3x} + \frac{7}{25}e^{-2x} - \frac{1}{9}\log 2$

1.7 Particular integral when $X = \sin ax$ or $\cos ax$

We have

$$D\sin ax = a\cos ax,\ D^2\sin ax = -a^2\sin ax,$$
$$D^3\sin ax = -a^3\cos ax,\ D^4\sin ax = (-a^2)^2\sin ax$$
$$D^5\sin ax = -a^5\cos ax,\ D^6\sin ax = (-a^2)^3\sin ax$$

and so on. In general one can write $(D^2)^r\sin ax = (-a^2)^r\sin ax$.

Now suppose $f(D) = \phi(D^2)$ and $\phi(D^2) = p_0(D^2)^n + p_1(D^2)^{n-1} + p_2(D^2)^{n-2} + \cdots + p_{n-1}(D^2) + p_n$. Then

$$\begin{aligned}\phi(D^2)\sin ax &= [p_0(D^2)^n + p_1(D^2)^{n-1} + p_2(D^2)^{n-2} + \cdots + p_{n-1}(D^2) + p_n]\sin ax\\ &= [p_0(-a^2)^n + p_1(-a^2)^{n-1} + p_2(-a^2)^{n-2} + \cdots + p_{n-1}(-a^2) + p_n]\sin ax\\ &= \phi(-a^2)\sin ax\end{aligned}$$

Operating on both side by $\dfrac{1}{\phi(D^2)}$, we get

$$\frac{1}{\phi(D^2)}\big(\phi(D^2)\sin ax\big) = \frac{1}{\phi(D^2)}\big(\phi(-a^2)\sin ax\big)$$

This implies

$$\sin ax = \frac{1}{\phi(D^2)}\big(\phi(-a^2)\sin ax\big)$$

Dividing by $\dfrac{1}{\phi(-a^2)}$, we get

$$\frac{1}{\phi(-a^2)}\sin ax = \frac{1}{\phi(D^2)}\sin ax$$

Therefore,

$$\boxed{\text{P.I.} = \frac{1}{f(D)}\sin ax = \frac{1}{\phi(D^2)}\sin ax = \frac{1}{\phi(-a^2)}\sin ax,\ \text{provided } \phi(-a^2) \neq 0}$$

Case of failure: If $\phi(-a^2) = 0$, then above methods. Then the particular integral is given by

$$\begin{aligned}P.I. &= \frac{1}{f(D)}\sin ax = \frac{1}{\phi(D^2)}\sin ax\\ &= \frac{1}{\phi(D^2)}\text{Im part of } e^{iax}\\ &= \text{Im part of}\frac{1}{\phi(D^2)}e^{iax}\\ &= \text{Im part of}\frac{1}{\phi(-a^2)}e^{iax} \qquad \text{case of failure}\end{aligned}$$

$$= \text{Im part of } \ x\,\frac{1}{\phi'(D^2)}e^{iax} = x\,\frac{1}{\phi'(D^2)}\sin ax$$

Therefore,

$$\boxed{\text{P.I.} = \frac{1}{f(D)}\sin ax = \frac{1}{\phi(D^2)}\sin ax = x\frac{1}{\phi'(D^2)}\sin ax = x\frac{1}{\phi'(-a^2)}\sin ax, \text{ provided } \phi'(-a^2)\neq 0}$$

Again if $\phi'(-a^2) = 0$, then P.I corresponding $\sin ax$ is given by

$$\boxed{\text{P.I.} = \frac{1}{f(D)}\sin ax = \frac{1}{\phi(D^2)}\sin ax = x^2\frac{1}{\phi''(D^2)}\sin ax = x\frac{1}{\phi''(-a^2)}\sin ax, \text{ provided } \phi''(-a^2)\neq 0}$$

Similarly P.I corresponding to $\cos ax$ is

$$\boxed{\text{P.I.} = \frac{1}{f(D)}\cos ax = \frac{1}{\phi(D^2)}\cos ax = \frac{1}{\phi(-a^2)}\cos ax, \text{ provided } \phi(-a^2)\neq 0}$$

If $\phi(-a^2) = 0$, then the particular integral is given by

$$\boxed{\text{P.I.} = \frac{1}{f(D)}\cos ax = \frac{1}{\phi(D^2)}\cos ax = x\frac{1}{\phi'(D^2)}\cos ax = x\frac{1}{\phi'(-a^2)}\cos ax, \text{ provided } \phi'(-a^2)\neq 0}$$

Again if $\phi'(-a^2) = 0$, then P.I corresponding $\sin ax$ is given by

$$\boxed{\text{P.I.} = \frac{1}{f(D)}\sin ax = \frac{1}{\phi(D^2)}\cos ax = x^2\frac{1}{\phi''(D^2)}\cos ax = x\frac{1}{\phi''(-a^2)}\cos ax, \text{ provided } \phi''(-a^2)\neq 0}$$

Examples

Example 12. Solve $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = \sin 3x$

Sol. Consider the given equation as

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin 3x$$

The symbolic form of equation is

$$(D^2 - 5D + 6)y = \sin 3x$$

The auxiliary equation is

$$\begin{aligned} m^2 - 5m + 6 = 0 \ &\Rightarrow (m-2)(m-3) = 0 \\ &\Rightarrow m = 2, 3 \end{aligned}$$

$\therefore$ The complementary function is

$$C.F. = c_1 e^{2x} + c_2 e^{3x}$$

Now, P.I. corresponding to $\sin 3x$ is

$$\begin{aligned}
P.I. &= \frac{1}{D^2 - 5D + 6}\sin 3x = \frac{1}{-9 - 5D + 6}\sin 3x \\
&= -\frac{1}{3 + 5D}\sin 3x = -\frac{1}{3 + 5D}\frac{3 - 5D}{3 - 5D}\sin 3x \\
&= -\frac{3 - 5D}{9 - 25D^2}\sin 3x = -\frac{3 - 5D}{9 - 25(-9)}\sin 3x \\
&= -\frac{3 - 5D}{234}\sin 3x = -\frac{1}{234}[3\sin 3x - 5D\sin 3x] \\
&= -\frac{1}{234}[3\sin 3x - 15\cos 3x] = \frac{1}{78}[5\cos 3x - \sin 3x]
\end{aligned}$$

$\therefore$ The complete solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{78}[5\cos 3x - \sin 3x]$$

Example 13. Solve $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \dfrac{\mathrm{d}y}{\mathrm{d}x} + y = \cos 2x$

Sol. The given equation is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} + y = \cos 2x$$

The symbolic form of equation is

$$(D^2 - D + 1)y = \cos 2x$$

The auxiliary equation is

$$\begin{aligned}
m^2 - m + 1 = 0 \ \Rightarrow m &= \frac{1 \pm \sqrt{1 - 4}}{2} \\
\Rightarrow m &= \frac{1}{2} \pm \frac{\sqrt{3}i}{2}
\end{aligned}$$

$\therefore$ The complementary function is

$$C.F. = e^{\frac{1}{2}x}[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x]$$

Now, the P.I. corresponding to $\cos 2x$ is given by

$$P.I. = \frac{1}{D^2 - D + 1}\cos 2x = \frac{1}{-4 - D + 1}\cos 2x$$

$$
\begin{aligned}
&= \frac{1}{-3-D}\cos 2x = -\frac{1}{3+D}\cos 2x \\
&= -\frac{1}{3+D}\,\frac{3-D}{3-D}\cos 2x = -\frac{3-D}{9-D^2}\cos 2x \\
&= -\frac{3-D}{9-(-4)}\cos 2x = -\frac{1}{13}[3\cos 2x - D\cos 2x] \\
&= -\frac{1}{13}[3\cos 2x + 2\sin 2x]
\end{aligned}
$$

$\therefore$ The complete solution is

$$y = e^{\frac{1}{2}x}[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x] - \frac{1}{13}[3\cos 2x + 2\sin 2x]$$

Example 14. Solve $(D^2 - 3D + 2)y = 6e^{-3x} + \sin 2x$

Sol. The given equation is

$$(D^2 - 3D + 2)y = 6e^{-3x} + \sin 2x$$

The auxiliary equation is

$$
\begin{aligned}
m^2 - 3m + 2 = 0 &\Rightarrow (m-1)(m-2) = 0 \\
&\Rightarrow m = 1, 2
\end{aligned}
$$

Therefore, the complementary function is

$$C.F = c_1e^x + c_2e^{2x}$$

Now, P.I. corresponding to $6e^{-3x} + \sin 2x$ is

$$
\begin{aligned}
P.I. &= \frac{1}{D^2-3D+2}[6e^{-3x} + \sin 2x] = \frac{6}{D^2-3D+2}e^{-3x} + \frac{1}{D^2-3D+2}\sin 2x \\
&= \frac{6}{9-3(-3)+2}e^{-3x} + \frac{1}{-4-3D+2}\sin 2x = \frac{6}{20}e^{-3x} - \frac{1}{2+3D}\sin 2x \\
&= \frac{6}{20}e^{-3x} - \frac{1}{2+3D}\,\frac{2-3D}{2-3D}\sin 2x = \frac{6}{20}e^{-3x} - \frac{2-3D}{4-9D^2}\sin 2x \\
&= \frac{6}{20}e^{-3x} - \frac{2-3D}{4-9(-4)}\sin 2x = \frac{6}{20}e^{-3x} - \frac{2\sin 2x - 3D\sin 2x}{40} \\
&= \frac{6}{20}e^{-3x} - \frac{2\sin 2x - 6\cos 2x}{40} = \frac{3}{10}e^{-3x} - \frac{1}{20}[\sin 2x - 3\cos 2x]
\end{aligned}
$$

$\therefore$ The complete solution is

$$y = c_1e^x + c_2e^{2x} + \frac{3}{10}e^{-3x} - \frac{1}{20}[\sin 2x - 3\cos 2x]$$

Example 15. Solve $\dfrac{d^3y}{dx^3} + 9\dfrac{dy}{dx} = \cos 3x$

Sol. The given equation is

$$\frac{d^3y}{dx^3} + 9\frac{dy}{dx} = \cos 3x$$

The symbolic form of equation is

$$(D^3 + D)y = \cos 3x$$

The auxiliary equation is

$$\begin{aligned} m^3 + m = 0 &\Rightarrow m(m^2 + 9) = 0 \\ &\Rightarrow m = 0, \pm 3i \end{aligned}$$

$\therefore$ The complementary function is

$$C.F. = c_1 + c_2 \cos 3x + c_3 \sin 3x$$

Now, P.I. corresponding to $\cos 3x$ is

$$\begin{aligned} P.I. &= \frac{1}{D^3 + 9D} \cos 3x = \frac{1}{D(D^2 + 9)} \cos 3x \\ &= \frac{1}{D(-9 + 9)} \cos 3x \qquad\qquad \text{case of failure} \\ &= x\frac{1}{3D^2 + 9} \cos 3x = x\frac{1}{3(-9) + 9} \cos 3x \\ &= -x\frac{1}{18} \cos 3x \end{aligned}$$

$\therefore$ The complete solution is

$$y = c_1 + c_2 \cos 3x + c_3 \sin 3x - \frac{x}{18} \cos 3x$$

Example 16. Solve $(D^2 + 1)y = \sin x \ \sin 2x$

Sol. The given equation is

$$(D^2 + 1)y = \sin x \ \sin 2x$$

The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$\therefore$ The complementary function is

$$C.F. = c_1 \cos x + c_2 \sin x$$

Now, P.I. corresponding to $\sin x \ \sin 2x$ is given by

$$P.I. = \frac{1}{D^2 + 1}[\sin x \ \sin 2x] = \frac{1}{2}\,\frac{1}{D^2 + 1}[2 \sin x \ \sin 2x]$$

$$= \frac{1}{2}\;\frac{1}{D^2+1}[\cos x - \cos 3x] = \frac{1}{2}\frac{1}{D^2+1}\cos x - \frac{1}{2}\frac{1}{D^2+1}\cos 3x$$
$$= \frac{1}{2}\frac{1}{-1+1}\cos x - \frac{1}{2}\frac{1}{(-9+1)}\cos 3x \qquad \text{case of failure}$$
$$= \frac{x}{2}\frac{1}{2D}\cos x + \frac{1}{2}\frac{1}{8}\cos 3x = \frac{x}{4}\int \cos x \mathrm{d}x + \frac{1}{16}\cos 3x$$
$$= \frac{x}{4}\sin x \mathrm{d}x + \frac{1}{16}\cos 3x$$

$\therefore$ The complete solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{x}{4}\sin x \mathrm{d}x + \frac{1}{16}\cos 3x$$

Example 17. Solve $(D^2+4)y = \cos x\ \cos 2x\ \cos 3x$

Sol. The given equation is

$$(D^2+4)y = \cos x\ \cos 2x\ \cos 3x$$

The auxiliary equation is

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$\therefore$ The complementary function is

$$C.F. = c_1 \cos 2x + c_2 \sin 2x$$

Now, P.I. corresponding to $\cos x \cos 2x \cos 3x$ is given by

$$P.I. = \frac{1}{D^2+4}[\cos x \cos 2x \cos 3x] \tag{1}$$

Here first we simplify the term $\cos x \cos 2x \cos 3x$.

$$\begin{aligned}
\cos x \cos 2x \cos 3x &= \frac{1}{2}[(2\cos x \cos 2x)\cos 3x] \\
&= \frac{1}{2}[(\cos 3x + \cos(-x))\cos 3x] \\
&= \frac{1}{2}[(\cos 3x + \cos x)\cos 3x] \\
&= \frac{1}{2}[\cos^2 3x + \cos 3x \cos x] \\
&= \frac{1}{4}[2\cos^2 3x + 2\cos 3x \cos x] \\
&= \frac{1}{4}[2\frac{1+\cos 6x}{2} + \cos 4x + \cos 2x] \\
&= \frac{1}{4}[1 + \cos 6x + \cos 4x + \cos 2x]
\end{aligned}$$

Then by (1), we get

$$\begin{aligned}
P.I. &= \frac{1}{D^2+4}[\cos x \cos 2x \cos 3x] = \frac{1}{4} \; \frac{1}{D^2+4}[1+\cos 6x+\cos 4x+\cos 2x] \\
&= \frac{1}{4}\left[\frac{1}{D^2+4}e^{0x} + \frac{1}{D^2+4}\cos 6x + \frac{1}{D^2+4}\cos 4x + \frac{1}{D^2+4}\cos 2x\right] \\
&= \frac{1}{4}\left[\frac{1}{0+4}e^{0x} + \frac{1}{-36+4}\cos 6x + \frac{1}{-16+4}\cos 4x + \frac{1}{-4+4}\cos 2x\right] \\
&= \frac{1}{4}\left[\frac{1}{4} - \frac{1}{32}\cos 6x - \frac{1}{12}\cos 4x + \frac{x}{2D}\cos 2x\right] \\
&\qquad\qquad [\because D^2 = -4 \Rightarrow D^2+4=0, \text{ hence case fails for last term}] \\
&= \frac{1}{4}\left[\frac{1}{4} - \frac{1}{32}\cos 6x - \frac{1}{12}\cos 4x + \frac{x}{2}\int \cos 2x \mathrm{d}x\right] \\
&= \frac{1}{4}\left[\frac{1}{4} - \frac{1}{32}\cos 6x - \frac{1}{12}\cos 4x + \frac{x}{4}\sin 2x\right]
\end{aligned}$$

$\therefore$ The complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}\left[\frac{1}{4} - \frac{1}{32}\cos 6x - \frac{1}{12}\cos 4x + \frac{x}{4}\sin 2x\right]$$

Example 18. Solve $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = e^x + \sin 2x$

Sol. Consider the given equation as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = e^x + \sin 2x$$

The symbolic form of equation is

$$(D^2+4)y = e^x + \sin 2x$$

The auxiliary equation is

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

The complementary function is

$$C.F. = c_1 \cos 2x + c_2 \sin 2x$$

Now, P.I. corresponding to $e^x + \sin 2x$ is given by

$$P.I. = \frac{1}{D^2+4}[e^x + \sin 2x] = \frac{1}{D^2+4}e^x + \frac{1}{D^2+4}\sin 2x$$

$$= \frac{1}{1+4}e^x + \frac{1}{-4+4}\sin 2x$$
$$= \frac{1}{5}e^x + x\frac{1}{2D}\sin 2x$$
$$[\because D^2 = -4 \Rightarrow D^2 + 4 = 0, \text{ hence case fails for last term}]$$
$$= \frac{1}{5}e^x + \frac{x}{2}\int \sin 2x \mathrm{d}x$$
$$= \frac{1}{5}e^x - \frac{x}{4}\cos 2x$$

$\therefore$ The complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^x - \frac{x}{4}\cos 2x$$

Exercise

1) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} + 5y = \sin 3x$

2) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = \sin 3x \cos 2x$

3) $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + y = \sin 2x$

4) $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 9\frac{\mathrm{d}y}{\mathrm{d}x} - 27y = \cos 3x$

5) $(D^4 + 5D^2 + 4)y = \cos\frac{x}{2}\ \cos\frac{3x}{2}$

6) $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + y = \sin 3x - \cos^2\left(\frac{x}{2}\right)$

7) $(D^2 + 4)(D^+1)y = \cos 2x$

8) $(D^4 + 8D^2 + 16)y = \sin^2 x$

Answers

1) $y = e^x(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{20}(3\cos 3x - 2\sin 3x)$

2) $y = c_1 e^x + c_2 e^{3x} + \frac{1}{884}[10\cos 5x - 11\sin 5x] + \frac{1}{20}[\sin x + 2\cos x]$

3) $y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + \frac{1}{15}[2\cos 2x - \sin 2x]$

4) $y = c_1 e^{3x} + (c_2 \cos 3x + c_3 \sin 3x) - \frac{x}{36}[\cos 3x + \sin 3x]$

5) $y = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x + \frac{x}{12}\sin x - \frac{x}{24}\sin 2x$

6) $y = c_1 e^{-x} + e^{\frac{1}{2}x}[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x] + \frac{1}{730}[\sin 3x + 27\cos 3x] - \frac{1}{2} - \frac{1}{4}[\cos x - \sin x]$

7) $y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos x + c_4 \sin x - \dfrac{x}{12} \sin 2x$

8) $y = (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x + \dfrac{1}{32} + \dfrac{x^2}{64} \cos 2x$

1.8 Particular integral when $X = x^m$

When $X = x^m$, m being positive integer, the particular integral is

$$P.I. = \frac{1}{f(D)}x^m$$

To find particular integral we follow the following steps

✠ Take out the lowest degree term with sign from $f(D)$ to make the first term unity. The remaining factor will be of the form $1+\phi(D)$ or $1-\phi(D)$

✠ Take this factor $1+\phi(D)$ or $1-\phi(D)$ in the numerator so it takes the form $[1+\phi(D)]^{-1}$ or $[1-\phi(D)]^{-1}$.

✠ Expand $[1+\phi(D)]^{-1}$ or $[1-\phi(D)]^{-1}$ in ascending powers of D as far as the term containing D^m. Because $D^{m+1}x^m = 0$, $D^{m+2}x^m = 0$ and so on.

✠ Operate on x^m term by term

To find the P.I. of this type one should remember the following expansions

i) $(1+D)^{-1} = 1 - D + D^2 - D^3 - D^4 + D^5 - \dots$

ii) $(1-D)^{-1} = 1 + D + D^2 + D^3 + D^4 + D^5 + \dots$

iii) $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 - 5D^4 + 6D^5 - \dots$

iv) $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + 5D^4 + 6D^5 + \dots$

Examples

Example 19. Solve $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4y = x^2$

Sol. Consider the given equation as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4y = x^2$$

The symbolic form of equation is

$$(D^2 - 4)y = x^2$$

The auxiliary equation is

$$m^2 - 4 = 0 \;\Rightarrow m = 2, -2$$

∴ The complementary function is

$$C.F. = c_1e^{2x} + c_2e^{-2x}$$

Now, P.I. corresponding to x^2 is given by

$$
\begin{aligned}
P.I. = \frac{1}{D^2-4}x^2 &= \frac{1}{-4\left(1-\frac{D^2}{4}\right)}x^2 = -\frac{1}{4}\left(1-\frac{D^2}{4}\right)^{-1}x^2 \\
&= -\frac{1}{4}\left[1+\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2+\dots\right]x^2 \\
&= -\frac{1}{4}\left[x^2+\frac{D^2}{4}x^2+\left(\frac{D^2}{4}\right)^2x^2+\dots\right] \\
&= -\frac{1}{4}\left[x^2+\frac{2}{4}+0+\dots\right] \\
&= -\frac{1}{4}\left[x^2+\frac{1}{2}\right]
\end{aligned}
$$

$\therefore$ The complete solution is

$$y = c_1e^{2x}+c_2e^{-2x}-\frac{1}{4}\left[x^2+\frac{1}{2}\right]$$

Example 20. Solve $(D^2+2D+1)y = x^2$

Sol. The given equation is

$$(D^2+2D+1)y = x^2$$

The auxiliary equation is

$$
\begin{aligned}
m^2+2m+1=0 &\Rightarrow (m+1)^2=0 \\
&\Rightarrow m=-1,-1
\end{aligned}
$$

$\therefore$ The complementary function is

$$C.F. = (c_1+c_2x)e^{-x}$$

Now, P.I corresponding to x^2 is

$$
\begin{aligned}
P.I. &= \frac{1}{D^2+2D+1}x^2 = \frac{1}{(1+D)^2}x^2 \\
&= (1+D)^{-2}x^2 \\
&= (1-2D^2+3(D^2)^2-4(D^2)^3+\dots)x^2 \\
&= x^2-2D^2x^2+(D^2)^2x^2-(D^2)^3x^2+\dots \\
&= x^2-4
\end{aligned}
$$

$\therefore$ The complete solution is

$$y = (c_1+c_2x)e^{-x}+x^2-4$$

Example 21. Solve $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = 1 + x^2$

Sol. The given equation is

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1 + x^2$$

The symbolic form of equation is

$$(D^3 - D^2 - 6D)y = 1 + x^2$$

The auxiliary equation is

$$\begin{aligned} m^3 - m^2 - 6m = 0 &\Rightarrow m(m^2 - m - 6) = 0 \\ &\Rightarrow m(m-3)(m+2) = 0 \\ &\Rightarrow m = 0, 3, -2 \end{aligned}$$

$\therefore$ The complementary function is

$$C.F. = c_1 + c_2e^{3x} + c_3e^{-2x}$$

Now, P.I. corresponding to $1 + x^2$ is given by

$$\begin{aligned} P.I. &= \frac{1}{D^3 - D^2 - 6D}(1 + x^2) = \frac{1}{-6D\left(1 + \dfrac{D^2 - D^3}{6D}\right)}(1 + x^2) \\ &= \frac{1}{-6D\left(1 + \dfrac{D - D^2}{6}\right)}(1 + x^2) = -\frac{1}{6D}\left(1 + \frac{D - D^2}{6}\right)^{-1}(1 + x^2) \\ &= -\frac{1}{6D}\left[1 - \frac{D - D^2}{6} + \left(\frac{D - D^2}{6}\right)^2 - \left(\frac{D - D^2}{6}\right)^3 + \dots\right](1 + x^2) \\ &= -\frac{1}{6D}\left[1 - \frac{1}{6}(D - D^2) + \frac{1}{36}(D^2 - 2D^3 + D^4) - \frac{1}{218}(D^3 - 3D^4 + 3D^5 - D^6) + \dots\right](1 + x^2) \\ &= -\frac{1}{6D}\left[1 + x^2 - \frac{1}{6}(D - D^2)(1 + x^2) + \frac{1}{36}(D^2 - 2D^3 + D^4)(1 + x^2)\right. \\ &\qquad \left. - \frac{1}{218}(D^3 - 3D^4 + 3D^5 - D^6)(1 + x^2) + \dots\right] \\ &= -\frac{1}{6D}\left[1 + x^2 - \frac{1}{6}(2x - 2) + \frac{1}{36}(2) - 0 + \dots\right] \\ &= -\frac{1}{6D}\left[1 + x^2 - \frac{x}{3} + \frac{1}{3} + \frac{1}{18}\right] = -\frac{1}{6D}\left[x^2 - \frac{x}{3} + \frac{25}{18}\right] \\ &= -\frac{1}{6}\int\left[x^2 - \frac{x}{3} + \frac{25}{18}\right]dx \\ &= -\frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18}\right] \end{aligned}$$

$\therefore$ The complete solution is

$$y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18}\right]$$

Example 22. Solve $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \dfrac{\mathrm{d}y}{\mathrm{d}x} = x + x^2$

Sol. The given equation is

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = x + x^2$$

The symbolic form of equation is

$$(D^3 + 2D^2 + D)y = x + x^2$$

The auxiliary equation is

$$\begin{aligned} m^3 + 2m^2 + m = 0 &\Rightarrow m(m^2 + 2m + 1) = 0 \\ &\Rightarrow m(m+1)(m+1) = 0 \\ &\Rightarrow m = 0, -1, -1 \end{aligned}$$

$\therefore$ The complementary function is

$$C.F. = c_1 + (c_2 + c_3 x)e^{-x}$$

Now, P.I. corresponding to $x + x^2$ is given by

$$\begin{aligned} P.I. &= \frac{1}{D^3 + 2D^2 + D}(x + x^2) = \frac{1}{D(D^2 + 2D + 1)}(x + x^2) = \frac{1}{D}\,\frac{1}{(1+D)^2}(x + x^2) \\ &= \frac{1}{D}\,(1+D)^{-2}(x + x^2) = \frac{1}{D}\,[1 - 2D + 3D^2 - 4D^3 + \ldots](x + x^2) \\ &= \frac{1}{D}[x + x^2 - 2(1 + 2x) + 3(2) - 4(0) + \ldots] = \frac{1}{D}[x^2 - 3x + 4] \\ &= \int [x^2 - 3x + 4]\mathrm{d}x = \frac{x^3}{3} - \frac{3x^2}{2} + 4x \end{aligned}$$

$\therefore$ The complete solution is

$$y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$$

Example 23. Solve $(D^2 - 4D + 4)y = 8(e^{2x} + \sin 2x + x^2)$

Sol. The given equation is

$$(D^2 - 4D + 4)y = 8(e^{2x} + \sin 2x + x^2)$$

The auxiliary equation is

$$m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$$

$\therefore$ The complementary function is

$$C.F. = (c_1 + c_2 x)e^{2x}$$

Now, P.I. corresponding to $8(e^{2x} + \sin 2x + x^2)$ is

$$\begin{aligned} P.I. &= \frac{1}{D^2 - 4D + 4} 8(e^{2x} + \sin 2x + x^2) \\ &= 8\left[\frac{1}{D^2 - 4D + 4}e^{2x} + \frac{1}{D^2 - 4D + 4}\sin 2x + \frac{1}{D^2 - 4D + 4}x^2\right] \qquad (1) \end{aligned}$$

Now consider,

$$\begin{aligned} \frac{1}{D^2 - 4D + 4}e^{2x} &= \frac{1}{2^2 - 4(2) + 4}e^{2x} && \text{(case of failure)} \\ &= x\frac{1}{2D - 4}e^{2x} \\ &= x\frac{1}{4 - 4}e^{2x} && \text{(case of failure)} \\ &= x^2\frac{1}{2}e^{2x} \end{aligned}$$

Again,

$$\frac{1}{D^2 - 4D + 4}\sin 2x = \frac{1}{-4 - 4D + 4}\sin 2x = \frac{1}{-4D}\sin 2x = \frac{1}{8}\cos 2x$$

and

$$\begin{aligned} \frac{1}{D^2 - 4D + 4}x^2 = \frac{1}{(D-2)^2}x^2 = \frac{1}{4\left(1 - \frac{D}{2}\right)^2}x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2}x^2 \\ = \frac{1}{4}\left[1 + 2\frac{D}{2} + 3\frac{D^2}{4} + 4\frac{D^3}{8} + \dots\right]x^2 = \frac{1}{4}\left[x^2 + 2\frac{D}{2}x^2 + 3\frac{D^2}{4}x^2 + 4\frac{D^3}{8} + \dots\right] \\ = \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right] \end{aligned}$$

Therefore by (1) the P.I. is

$$\begin{aligned} P.I. &= 8\left(x^2\frac{1}{2}e^{2x} + \frac{1}{8}\cos 2x + \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right]\right) \\ &= 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3 \end{aligned}$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x)e^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

Exercise

Solve the following:

1) $(D^3 - 3D + 2)y = x$

2) $(D^3 - D^2 - 6D)y = x^2 + \sin x$

3) $(D^2 + D)y = x^2 + 2x + 4$

4) $(D^4 - 2D^3 + D^2)y = x^3$

Answers

1) $y = (c_1 + c_2 x)e^x + \dfrac{1}{2}\left[x + \dfrac{3}{2}\right]$

2) $y = c_1 + c_2 e^{-2x} + c_3 e^{3x} - \dfrac{1}{6}\left[\dfrac{x^3}{3} - \dfrac{x^2}{6} + \dfrac{7x}{18}\right] + \dfrac{1}{50}(\sin x + 7\cos x)$

3) $y = c_1 + c_2 e^{-x} + \dfrac{x^3}{3} + 4x$

4) $y = (c_1 + c_2 x) + (c_3 + c_4 x)e^x + \dfrac{x^5}{20} + \dfrac{x^4}{2} + 3x^3 + 12x^2$

1.9 Particular integral when $X = e^{ax}V$, V is a function of x

Let u be a function of x. Then by successive differentiation, We have

$$De^{ax}u = e^{ax}Du + ae^{ax}u = e^{ax}(D+a)u$$

$D^2e^{ax}u = De^{ax}(D+a)u = e^{ax}(D^2+aD)u+ae^{ax}(D+a)u = e^{ax}(D^2+2aD+a^2)u = e^{ax}(D+a)^2u$

Similarly, one can obtain

$$D^3e^{ax}u = e^{ax}(D+a)^3u, D^4e^{ax}u = e^{ax}(D+a)^4u$$

In general we have,

$$D^ne^{ax}u = e^{ax}(D+a)^nu$$

Therefore,

$$\begin{aligned}
f(D)e^{ax}u &= \left(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n\right)e^{ax}u \\
&= a_0D^ne^{ax}u + a_1D^{n-1}e^{ax}u + a_2D^{n-2}e^{ax}u + \cdots + a_{n-1}De^{ax}u + a_ne^{ax}u \\
&= e^{ax}a_0(D+a)^nu + a_1e^{ax}(D+a)^{n-1}u + \cdots + a_{n-1}e^{ax}(D+a)u + a_ne^{ax}u \\
&= e^{ax}[a_0(D+a)^n + a_1(D+a)^{n-1} + \cdots + a_{n-1}(D+a) + a_n]u \\
&= e^{ax}f(D+a)u
\end{aligned}$$

Operating on both sides by $\dfrac{1}{f(D)}$, we get

$$\begin{aligned}
\frac{1}{f(D)}\left[f(D)e^{ax}u\right] &= \frac{1}{f(D)}\left[e^{ax}f(D+a)u\right] \\
\Rightarrow \qquad e^{ax}u &= \frac{1}{f(D)}\left[e^{ax}f(D+a)u\right]
\end{aligned}$$

Take $f(D+a)u = V \Rightarrow u = \dfrac{V}{f(D+a)}$. Then above equality reduces to

$$e^{ax}\frac{1}{f(D+a)} = \frac{1}{f(D)}e^{ax}V \text{ or } \frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}$$

Therefore,

$$P.I. = \frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}$$

It means that take out e^{ax} to the left of $\dfrac{1}{f(D)}$ and at denominator in $f(D)$ write $D+a$ for every D so that $f(D)$ becomes $f(D+a)$ and operate $\dfrac{1}{f(D+a)}$ with V alone by previous methods.

Examples

Example 24. Solve $(D^2-4D+3)y=e^x\cos 2x$

Sol. The given equation is

$$(D^2-4D+3)y=e^x\cos 2x$$

The auxiliary equation is

$$\begin{aligned} m^2-4m+3=0 &\Rightarrow (m-1)(m-3)=0 \\ &\Rightarrow m=1,3 \end{aligned}$$

$\therefore$ The complementary function is

$$C.F.=c_1e^x+c_2e^{3x}$$

Now, P.I. corresponding to $e^x\cos 2x$ is

$$\begin{aligned} P.I. &= \frac{1}{D^2-4D+3}e^x\cos 2x = e^x\frac{1}{(D+1)^2-4(D+1)+3}\cos 2x \\ &= e^x\frac{1}{D^2+2D+1-4D-4+3}\cos 2x = e^x\frac{1}{D^2-2D} \\ &= e^x\frac{1}{-2^2-2D}\cos 2x = -\frac{1}{2}e^x\frac{1}{2+D}\cos 2x \\ &= -\frac{1}{2}e^x\frac{1}{2+D}\,\frac{2-D}{2-D}\cos 2x = -\frac{1}{2}e^x\frac{2-D}{4-D^2}\cos 2x \\ &= -\frac{1}{2}e^x\frac{2-D}{4-(-2^2)}\cos 2x = -\frac{1}{16}e^x[2\cos 2x - D\cos 2x] \\ &= -\frac{1}{16}e^x[2\cos 2x+2\sin 2x] = -\frac{1}{8}e^x[\cos 2x+\sin 2x] \end{aligned}$$

$\therefore$ The complete solution is

$$y=c_1e^x+c_2e^{3x}-\frac{1}{8}e^x[\cos 2x+\sin 2x]$$

Example 25. Solve $(D^2-3D+2)y=e^x\sin x$

Sol. The given equation is

$$(D^2-3D+2)y=e^x\sin x$$

The auxiliary equation is

$$\begin{aligned} m^2-3m+2=0 &\Rightarrow (m-1)(m-2)=0 \\ &\Rightarrow m=1,2 \end{aligned}$$

$\therefore$ The complementary function is

$$C.F.=c_1e^x+c_2e^{2x}$$

Now, P.I. corresponding to $e^x \sin x$ is

$$
\begin{aligned}
P.I. &= \frac{1}{D^2 - 3D + 2} e^x \sin x = e^x \frac{1}{(D+1)^2 - 3(D+1) + 2} \sin x \\
&= e^x \frac{1}{D^2 + 2D + 1 - 3D - 3 + 2} \sin x \\
&= e^x \frac{1}{D^2 - D} \sin x = e^x \frac{1}{-1 - D} \sin x \\
&= -e^x \frac{1}{1 + D} \frac{1 - D}{1 - D} \sin x = -e^x \frac{1 - D}{1 - D^2} \sin x \\
&- e^x \frac{\sin x - D \sin x}{1 - (-1)} = -e^x \frac{1}{2} [\sin x - \cos x]
\end{aligned}
$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{2x} - \frac{e^x}{2} [\sin x - \cos x]$$

Example 26. Solve $(D^2 - 2D + 4)y = e^x \cos^2 x$

Sol. The given equation is

$$(D^2 - 2D + 4)y = e^x \cos^2 x$$

The auxiliary equation is

$$
\begin{aligned}
m^2 - 2m + 4 = 0 \Rightarrow m &= \frac{2 \pm \sqrt{4 - 16}}{2} \Rightarrow m = \frac{2 \pm \sqrt{-12}}{2} \\
\Rightarrow m &= \frac{2 \pm 2\sqrt{3}i}{2} \Rightarrow m = 1 \pm \sqrt{3}i
\end{aligned}
$$

$\therefore$ The complementary function is

$$C.F. = e^x (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$$

Now, P.I corresponding to $e^x \cos^2 x$ is given by

$$
\begin{aligned}
P.I. &= \frac{1}{D^2 - 2D + 4} e^x \cos^2 x = \frac{1}{D^2 - 2D + 4} e^x \left(\frac{1 + \cos 2x}{2} \right) \\
&= \frac{1}{2} \left[\frac{1}{D^2 - 2D + 4} e^x + \frac{1}{D^2 - 2D + 4} e^x \cos 2x \right] \qquad (1)
\end{aligned}
$$

Now consider

$$\frac{1}{D^2 - 2D + 4} e^x = \frac{1}{1^2 - 2(1) + 4} e^x = \frac{1}{3} e^x$$

and

$$\frac{1}{D^2 - 2D + 4} e^x \cos 2x = e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos 2x$$

$$
\begin{aligned}
&= e^x \frac{1}{D^2+2D+1-2D-2+4}\cos 2x \\
&= e^x \frac{1}{D^2+3}\cos 2x \\
&= e^x \frac{1}{-2^2+3}\cos 2x \\
&= -e^x \cos 2x
\end{aligned}
$$

By (1), P.I corresponding to $e^x \cos^2 x$ is

$$P.I. = \frac{1}{6}e^x - \frac{1}{2}e^x \cos 2x$$

$\therefore$ The complete solution is

$$y = e^x(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + \frac{1}{6}e^x - \frac{1}{2}e^x \cos 2x$$

Example 27. Solve $(D^2-2D+2)y = x^2e^{3x}$

Sol. The given equation is

$$(D^2-2D+2)y = x^2e^{3x}$$

The auxiliary equation is

$$
\begin{aligned}
m^2-2m+2=0 \Rightarrow m &= \frac{2\pm\sqrt{4-8}}{2} \\
\Rightarrow m &= \frac{2\pm 2i}{2} = 1 \pm i
\end{aligned}
$$

$\therefore$ The complementary function is

$$C.F. = e^x(c_1\cos x + c_2 \sin x)$$

The P.I. corresponding to x^2e^{3x} is given by

$$
\begin{aligned}
P.I. &= \frac{1}{D^2-2D+2}x^2e^{3x} = e^{3x}\frac{1}{(D+3)^2-2(D+3)+2}x^2 \\
&= e^{3x}\frac{1}{D^2+6D+9-2D-6+2}x^2 = e^{3x}\frac{1}{D^2+4D+5}x^2 \\
&= e^{3x}\frac{1}{5\left[1+\dfrac{D^2+4D}{5}\right]}x^2 = e^{3x}\frac{1}{5}\left[1+\frac{D^2+4D}{5}\right]^{-1}x^2 \\
&= e^{3x}\frac{1}{5}\left[1-\frac{D^2+4D}{5}+\left(\frac{D^2+4D}{5}\right)^2-\left(\frac{D^2+4D}{5}\right)^3+\dots\right]x^2 \\
&= e^{3x}\frac{1}{5}\left[1-\frac{1}{5}(D^2+4D)+\frac{1}{25}(D^4+8D^3+16D^2)-\frac{1}{125}(D^6+12D^5+48D^4+64D^3)+\dots\right]x^2
\end{aligned}
$$

$$= e^{3x}\frac{1}{5}\left[x^2 - \frac{1}{5}(2+8x) + \frac{1}{25}(0+0+32) - \frac{1}{125}(0+0+0+0) + \dots\right]$$
$$= e^{3x}\frac{1}{5}\left[x^2 - \frac{2}{5} + \frac{8x}{5} + \frac{32}{25}\right]$$
$$= \frac{1}{5}e^{3x}\left[x^2 - \frac{8x}{5} + \frac{22}{25}\right]$$

$\therefore$ The complete solution is

$$y = e^x(c_1\cos x + c_2\sin x) + \frac{1}{5}e^{3x}\left[x^2 - \frac{8x}{5} + \frac{22}{25}\right]$$

Example 28. Solve $(D^2 - 6D + 9)y = \dfrac{e^{3x}}{x^2}$

Sol. The given equation is

$$(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$$

The auxiliary equation is

$$m^2 - 6m + 9 = 0 \Rightarrow (m-3)(m-3) = 0$$
$$\Rightarrow m = 3, 3$$

$\therefore$ The complementary function is

$$C.F. = (c_1 + c_2x)e^{3x}$$

Now, P.I corresponding to $\dfrac{e^{3x}}{x^2}$ is

$$\begin{aligned}
P.I. &= \frac{1}{D^2 - 6D + 9}\frac{e^{3x}}{x^2} = \frac{1}{(D-3)^2}\frac{e^{3x}}{x^2} \\
&= e^{3x}\frac{1}{[(D+3)-3]^2}\frac{1}{x^2} = e^{3x}\frac{1}{D^2}\frac{1}{x^2} \\
&= e^{3x}\frac{1}{D}\,\frac{1}{D}x^{-2} = e^{3x}\frac{1}{D}\int x^{-2}\mathrm{d}x \\
&= e^{3x}\frac{1}{D}(-x^{-1}) = -e^{3x}\int x^{-1}\mathrm{d}x \\
&= -e^{3x}\log x
\end{aligned}$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2x)e^{3x} - e^{3x}\log x$$

Example 29. Solve $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4y = x\sinh x$

Sol. The given equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4y = x\sinh x$$

The symbolic form of equation is

$$(D^2-4)t = x\sinh x$$

The auxiliary equation is

$$m^2 - 4 = 0 \Rightarrow m = 2, -2$$

$\therefore$ The complementary function is

$$C.F. = c_1e^{2x} + c_2e^{-2x}$$

Now, P.I corresponding to $x\sinh x$ is given by

$$\begin{aligned} P.I. &= \frac{1}{D^2-4}x\sinh x = \frac{1}{D^2-4}x\left(\frac{e^x-e^{-x}}{2}\right) \\ &= \frac{1}{2}\left[\frac{1}{D^2-4}xe^x - \frac{1}{D^2-4}xe^{-x}\right] \end{aligned} \tag{1}$$

Now consider

$$\begin{aligned} \frac{1}{D^2-4}xe^x &= e^x\frac{1}{(D+1)^2-4}x = e^x\frac{1}{D^2+2D+1-4}x \\ &= e^x\frac{1}{D^2+2D-3}x = e^x\frac{1}{-3\left(1-\dfrac{D^2+2D}{3}\right)}x \\ &= -e^x\frac{1}{3}\left(1-\frac{D^2+2D}{3}\right)^{-1}x \\ &= -e^x\frac{1}{3}\left[1+\frac{D^2+2D}{3}+\left(\frac{D^2+2D}{3}\right)^2+\dots\right]x \\ &= -e^x\frac{1}{3}\left[1+\frac{1}{3}(D^2+2D)+\frac{1}{9}(D^4+2D^3+4D^2)+\dots\right]x \\ &= -e^x\frac{1}{3}\left[x+\frac{1}{3}(D^2x+2Dx)+\frac{1}{9}(D^4x+2D^3x+4D^2x)+\dots\right] \\ &= -e^x\frac{1}{3}\left[x+\frac{2}{3}\right] \end{aligned}$$

Also

$$\frac{1}{D^2-4}xe^{-x} = e^{-x}\frac{1}{(D-1)^2-4}x = e^{-x}\frac{1}{D^2-2D+1-4}-x$$

$$= e^{-x}\frac{1}{D^2-2D-3}x = e^{-x}\frac{1}{-3\left(1-\dfrac{D^2-2D}{3}\right)}x$$
$$= -e^{-x}\frac{1}{3}\left(1-\frac{D^2-2D}{3}\right)^{-1}x$$
$$= -e^{-x}\frac{1}{3}\left[1+\frac{D^2-2D}{3}+\left(\frac{D^2-2D}{3}\right)^2+\dots\right]x$$
$$= -e^{-x}\frac{1}{3}\left[1+\frac{1}{3}(D^2-2D)+\frac{1}{9}(D^4-2D^3+4D^2)+\dots\right]x$$
$$= -e^{-x}\frac{1}{3}\left[x+\frac{1}{3}(D^2x-2Dx)+\frac{1}{9}(D^4x-2D^3x+4D^2x)+\dots\right]$$
$$= -e^{-x}\frac{1}{3}\left[x-\frac{2}{3}\right]$$

By (1) the P.I. is

$$P.I. = \frac{1}{2}\left[-e^{x}\frac{1}{3}\left(x+\frac{2}{3}\right)-e^{-x}\frac{1}{3}\left(x-\frac{2}{3}\right)\right]$$
$$= -\frac{1}{3}\left(\frac{e^x+e^{-x}}{2}\right)-\frac{2}{9}\left(\frac{e^x-e^{-x}}{2}\right)$$
$$= -\frac{1}{3}\cosh x-\frac{2}{9}\sinh x$$

$\therefore$ The complete solution is

$$y = c_1e^{2x}+c_2e^{-2x}-\frac{1}{3}\cosh x-\frac{2}{9}\sinh x$$

Exercise

Solve the following:

1) $\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}+6y=e^{-2x}\sin 2x$

2) $(D^3-D^2+D-1)y=e^x\cos x$

3) $\dfrac{d^2y}{dx^2}+2y=x^2e^{3x}+e^x\cos 3x$

4) $\dfrac{d^4y}{dx^4}-y=\cos x\cosh x$

5) $(D^2+4D+3)y=e^{-x}\sin x+xe^{3x}$

6) $(D^2-3D+2)y=2e^x\cos\dfrac{x}{2}$

Answers

1) $y=c_1e^{-2x}+c_2e^{-3x}-\dfrac{e^{-2x}}{10}(\cos 2x+2\sin 2x)$

2) $y = c_1 e^x + c_2 \cos x + c_3 \sin x - \frac{e^x}{5}(2\cos x - \sin x)$

3) $y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + e^{3x}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right] + \frac{e^x}{6}(\sin 3x - \cos 3x)$

4) $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}\cos x \cosh x$

5) $y = c_1 e^{-x} + c_2 e^{-3x} - \frac{e^{-x}}{5}(\sin x + 2\cos x) + \frac{e^{3x}}{22}\left(x - \frac{5}{11}\right)$

6) $y = (c_1 + c_2 x)e^x + c_3 e^{-2x} - \frac{1}{37}\left(\sin\frac{x}{2} + 6\cos\frac{x}{2}\right)$

1.10 Particular integral when $X = x^m V$, V is a function of x

Consider a linear differential equation with constant coefficient as

$$f(D)y = x^m V$$

Where, V is a function of x.

✠ When $V = e^{ax}$. Then $x^m V = x^m e^{ax}$ i.e. $x^m V = e^{ax} x^m$. In this case P.I. can be obtained by the method of finding P.I. when $X = e^{ax} V$, where V is a function of x

✠ When $V = x^n$. Then $x^m V = x^m x^n = x^{m+n}$. In this case P.I. can be obtained by the method of finding P.I. when $X = x^m$

✠ When $V = \cos ax$ or $V = \sin ax$ Then use the following.

i) When $V = \cos ax$. Then P.I. is given by

$$\begin{aligned} P.I. &= \frac{1}{f(D)} x^m \cos ax = \frac{1}{f(D)} x^m \text{Real part of } e^{iax} \\ &= \text{Real Part of } \frac{1}{f(D)} x^m e^{iax} \\ &= \text{Real Part of } e^{iax} \frac{1}{f(D+ia)} x^m \end{aligned}$$

ii) When $V = \sin ax$. Then P.I. is given by

$$\begin{aligned} P.I. &= \frac{1}{f(D)} x^m \sin ax = \frac{1}{f(D)} x^m \text{Imaginary part of } e^{iax} \\ &= \text{Imaginary Part of } \frac{1}{f(D)} x^m e^{iax} \\ &= \text{Imaginary Part of } e^{iax} \frac{1}{f(D+ia)} x^m \end{aligned}$$

Examples

Example 30. Solve $(D^2+4)y = x \sin x$

Sol. The given equation is

$$(D^2+4)y = x \sin x$$

The auxiliary equation is

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$\therefore$ The complementary function is

$$C.F. = c_1 \cos 2x + c_2 \sin 2x$$

Now, P.I. corresponding to $x\sin x$ is given by

$$\begin{aligned}
P.I. &= \frac{1}{D^2+4}x\sin x = \frac{1}{D^2+4}x \text{ Imaginary Part of } \ e^{ix} \\
&= \text{Im. Part of } \ \frac{1}{D^2+4}xe^{ix} = \text{Im. Part of } \ e^{ix}\frac{1}{(D+i)^2+4} \\
&= \text{Im. Part of } \ e^{ix}\frac{1}{D^2+2iD-1+4} = \text{Im. Part of } \ e^{ix}\frac{1}{D^2+2iD+3}x \\
&= \text{Im. Part of } \ e^{ix}\frac{1}{3\left(1+\dfrac{D^2+2iD}{3}\right)}x = \text{Im. Part of } \ e^{ix}\frac{1}{3}\left(1+\frac{D^2+2iD}{3}\right)^{-1}x \\
&= \text{Im. Part of } \ e^{ix}\frac{1}{3}\left[1-\frac{D^2+2iD}{3}+\left(\frac{D^2+2iD}{3}\right)^2-\dots\right]x \\
&= \text{Im. Part of } \ e^{ix}\frac{1}{3}\left[1-\frac{1}{3}(D^2+2iD)+\frac{1}{9}(D^4+4iD^3-4D^2-\dots\right]x \\
&= \text{Im. Part of } \ e^{ix}\frac{1}{3}\left[x-\frac{1}{3}(D^2x+2iDx)+\frac{1}{9}(D^4x+4iD^3x-4D^2x-\dots\right] \\
&= \text{Im. Part of } \ e^{ix}\frac{1}{3}\left[x-\frac{2ix}{3}+0-\dots\right] \\
&= \text{Im. Part of } \ (\cos x+i\sin x)\ \frac{1}{3}\left[x-\frac{2ix}{3}\right] \\
&= \frac{x}{3}\sin x-\frac{2}{9}\cos x
\end{aligned}$$

$\therefore$ The complete solution is

$$y = c_1\cos 2x + c_2\sin 2x + \frac{x}{3}\sin x - \frac{2}{9}\cos x$$

Example 31. Solve $(D^2+2D+1)y = x\cos x$

Sol. The given equation is

$$(D^2+2D+1)y = x\cos x$$

The auxiliary equation is

$$m^2+2m+1=0 \Rightarrow (m+1)^2=0 \Rightarrow m=-1,-1$$

$\therefore$ The complementary function is

$$C.F. = (c_1+c_2x)e^{-x}$$

Now, P.I. corresponding to $x\cos x$ is given by

$$P.I. = \frac{1}{D^2+2D+1}x\cos x = \frac{1}{D^2+2D+1}x \text{ Real part of } e^{ix}$$

$$
\begin{aligned}
&= \text{Real part of } \frac{1}{D^2+2D+1} x e^{ix} = \text{Real part of } e^{ix} \frac{1}{(D+i)^2+2(D+i)+1} x \\
&= \text{Real part of } e^{ix} \frac{1}{D^2+2iD-1+2D+2i+1} x = \text{Real part of } e^{ix} \frac{1}{D^2+2(i+1)D+2i} x \\
&= \text{Real part of } e^{ix} \frac{1}{2i\left(1+\dfrac{D^2+2(i+1)D}{2i}\right)} x \\
&= \text{Real part of } e^{ix} \frac{1}{2i}\left(1+\frac{D^2+2(i+1)D}{2i}\right)^{-1} x \\
&= \text{Real part of } e^{ix} \frac{1}{2i}\left[1-\frac{D^2+2(i+1)D}{2i}+\dots\right] x \\
&= \text{Real part of } e^{ix} \frac{1}{2i}\left[x-\frac{1}{2i}(D^2+2(i+1)D)x+\dots\right] \\
&= \text{Real part of } e^{ix} \frac{1}{2i}\left[x-\frac{1}{2i}(0+2(i+1))\right] = \text{Real part of } e^{ix} \frac{1}{2i}\left[x+i(i+1)\right] \\
&= -\text{Real part of } e^{ix} \frac{i}{2}\left[x+i(i+1)\right] = -\text{Real part of } \frac{e^{ix}}{2}\left[ix-(i+1)\right] \\
&= -\text{Real part of } \frac{e^{ix}}{2}\left[ix-i-1\right] = -\text{Real part of } \frac{1}{2}(\cos x+i\sin x)\left[i(x-1)-1\right] \\
&= -\frac{1}{2}[-\cos x-(x-1)\sin x] = \frac{1}{2}[\cos x+(x-1)\sin x]
\end{aligned}
$$

∴ The complete solution is

$$y = (c_1+c_2x)e^{-x} + \frac{1}{2}[\cos x+(x-1)\sin x]$$

Example 32. Solve $(D^2+1)y = x^2\sin 2x$

Sol. The given equation is

$$(D^2+1)y = x^2\sin 2x$$

The auxiliary equation is

$$m^2+1=0 \Rightarrow m=\pm i$$

∴ The complementary function is

$$C.F. = c_1\cos x + c_2\sin x$$

Now, P.I. corresponding to $x^2\sin 2x$ is given by

$$
\begin{aligned}
P.I. &= \frac{1}{D^2+1}x^2\sin 2x = \frac{1}{D^2+1}x^2 \text{ Im. part of } e^{2ix} \\
&= \text{Im part of } \frac{1}{D^2+1}x^2e^{2ix} = \text{Im part of } e^{2ix}\frac{1}{(D+2i)^2+1}x^2
\end{aligned}
$$

$$= \text{Im part of } e^{2ix}\frac{1}{D^2+4iD-4+1}x^2 = \text{Im part of } e^{2ix}\frac{1}{D^2+4iD-3}x^2$$
$$= \text{Im part of } e^{2ix}\frac{1}{-3\left(1-\dfrac{D^2+4iD}{3}\right)}x^2 = -\text{Im part of } e^{2ix}\frac{1}{3}\left(1-\frac{D^2+4iD}{3}\right)^{-1}x^2$$
$$= -\text{Im part of } e^{2ix}\frac{1}{3}\left[1+\frac{D^2+4iD}{3}+\left(\frac{D^2+4iD}{3}\right)^2+\dots\right]x^2$$
$$= -\text{Im part of } e^{2ix}\frac{1}{3}\left[1+\frac{1}{3}(D^2+4iD)+\frac{1}{9}(D^4+8iD^3-16D^2)+\dots\right]x^2$$
$$= -\text{Im part of } e^{2ix}\frac{1}{3}\left[x^2+\frac{1}{3}(D^2x^2+4iDx^2)+\frac{1}{9}(D^4x^2+8iD^3x^2-16D^2x^2)+\dots\right]$$
$$= -\text{Im part of } e^{2ix}\frac{1}{3}\left[x^2+\frac{1}{3}(2+8ix)+\frac{1}{9}(0+0-32)+\dots\right]$$
$$= -\text{Im part of } e^{2ix}\frac{1}{3}\left[x^2+\frac{1}{3}(2+8ix)-\frac{32}{9}\right]$$
$$= -\text{Im part of } (\cos 2x+i\sin 2x)\frac{1}{3}\left[\left(x^2-\frac{26}{9}\right)+\frac{8x}{3}i\right]$$
$$= -\frac{1}{3}\left[\frac{8x}{3}\cos 2x+\left(x^2-\frac{26}{9}\right)\sin 2x\right]$$
$$= -\frac{1}{27}\left[24x\cos 2x+\left(9x^2-26\right)\sin 2x\right]$$

$\therefore$ The complete solution is given by

$$y = c_1\cos x+c_2\sin x-\frac{1}{27}\left[24x\cos 2x+\left(9x^2-26\right)\sin 2x\right]$$

Example 33. Solve $(D^4+2D^2+1)y = x^2\cos x$

Sol. The given equation is

$$(D^4+2D^2+1)y = x^2\cos x$$

The auxiliary equation is

$$\begin{aligned} m^4+2m^2+1=0 &\Rightarrow (m^2+1)^2=0 \\ &\Rightarrow (m^2+1)(m^2+1)=0 \\ &\Rightarrow m=\pm i,\pm i \end{aligned}$$

$\therefore$ The complementary function is

$$C.F. = (c_1+c_2x)\cos x+(c_3+c_4x)\sin x$$

Now, P.I. corresponding to $x^2\cos x$ is given by

$$P.I. = \frac{1}{D^4+2D^2+1}x^2\cos x = \frac{1}{D^4+2D^2+1}\text{ Real Part of } x^2e^{ix}$$

$$= \text{Real Part of } \frac{1}{(D^2+1)^2} x^2 e^{ix} = \text{Real Part of } e^{ix} \frac{1}{[(D+i)^2+1)]^2} x^2$$
$$= \text{Real Part of } e^{ix} \frac{1}{[D^2+2iD-1+1]^2} x^2 = \text{Real Part of } e^{ix} \frac{1}{[D^2+2iD]^2} x^2$$
$$= \text{Real Part of } e^{ix} \frac{1}{-4D^2\left[1+\frac{D^2}{2iD}\right]^2} x^2 = \text{Real Part of } e^{ix} \frac{1}{-4D^2\left[1+\frac{D}{2i}\right]^2} x^2$$
$$= -\text{Real Part of } e^{ix} \frac{1}{4D^2}\left[1+\frac{D}{2i}\right]^{-2} x^2$$
$$= -\text{Real Part of } e^{ix} \frac{1}{4D^2}\left[1-2\frac{D}{2i}+3\left(\frac{D}{2i}\right)^2-4\left(\frac{D}{2i}\right)^3+\dots\right] x^2$$
$$= -\text{Real Part of } e^{ix} \frac{1}{4D^2}\left[x^2-\frac{1}{i}Dx^2-\frac{3}{4}D^2x^2+\frac{4}{8i}D^3x^2+\dots\right]$$
$$= -\text{Real Part of } e^{ix} \frac{1}{4D^2}\left[x^2-\frac{2x}{i}-\frac{6}{4}+0+\dots\right]$$
$$= -\text{Real Part of } e^{ix} \frac{1}{4D^2}\left[x^2+2ix-\frac{3}{2}\right] \qquad \left[\because \frac{1}{i}=-i\right]$$
$$= -\text{Real Part of } e^{ix} \frac{1}{4D}\frac{1}{D}\left[x^2+2ix-\frac{3}{2}\right]$$
$$= -\text{Real Part of } e^{ix} \frac{1}{4D}\int\left[x^2+2ix-\frac{3}{2}\right] \mathrm{d}x$$
$$= -\text{Real Part of } e^{ix} \frac{1}{4D}\left[\frac{x^3}{3}+ix^2-\frac{3x}{2}\right]$$
$$= -\text{Real Part of } e^{ix} \frac{1}{4}\int\left[\frac{x^3}{3}+ix^2-\frac{3x}{2}\right] \mathrm{d}x$$
$$= -\text{Real Part of } e^{ix} \frac{1}{4}\left[\frac{x^4}{12}+\frac{x^3}{3}i-\frac{3x^2}{4}\right]$$
$$= -\text{Real Part of } (\cos x+i\sin x)\frac{1}{4}\left[\left(\frac{x^4}{12}-\frac{3x^2}{4}\right)+\frac{x^3}{3}i\right]$$
$$= -\frac{1}{4}\left[\left(\frac{x^4}{12}-\frac{3x^2}{4}\right)\cos x-\frac{x^3}{3}\sin x\right]$$
$$= -\frac{1}{48}\left[\left(x^4-9x^2\right)\cos x-4x^3\sin x\right]$$

$\therefore$ The complete solution is

$$y = (c_1+c_2x)\cos x+(c_3+c_4x)\sin x-\frac{1}{48}\left[\left(x^4-9x^2\right)\cos x-4x^3\sin x\right]$$

Example 34. Solve $(D^2-1)y = xe^x\sin x$

Sol. The given equation is

$$(D^2-1)y = xe^x\sin x$$

The auxiliary equation is

$$m^2 - 1 = 0 \Rightarrow m = \pm i$$

$\therefore$ The complementary function is

$$C.F. = c_1 e^x + c_2 e^{-x}$$

Now, P.I corresponding to $xe^x \sin x$ is given by

$$\begin{aligned}
P.I. &= \frac{1}{D^2-1} xe^x \sin x = e^x \frac{1}{(D+1)^2-1} x \sin x \\
&= e^x \frac{1}{D^2+2D+1-1} x \sin x = e^x \frac{1}{D^2+2D} x \sin x \\
&= e^x \frac{1}{D^2+2D} x \text{ Im part of } e^{ix} = \text{Im part of } e^x \frac{1}{D^2+2D} xe^{ix} \\
&= \text{Im part of } e^x e^{ix} \frac{1}{(D+i)^2+2(D+i)} x = \text{Im part of } e^x e^{ix} \frac{1}{D^2+2iD-1+2D+2i} x \\
&= \text{Im part of } e^x e^{ix} \frac{1}{D^2+2(i+1)D+(2i-1)} x \\
&= \text{Im part of } e^x e^{ix} \; \frac{1}{(2i-1)\left(1+\dfrac{D^2+2(i+1)D}{2i-1}\right)} x \\
&= \text{Im part of } e^x e^{ix} \; \frac{1}{2i-1} \left(1+\frac{D^2+2(i+1)D}{2i-1}\right)^{-1} x \\
&= \text{Im part of } e^x e^{ix} \; \frac{1}{2i-1} \left[1-\frac{D^2+2(i+1)D}{2i-1}+\left(\frac{D^2+2(i+1)D}{2i-1}\right)^2-\dots\right] x \\
&= \text{Im part of } e^x e^{ix} \; \frac{1}{2i-1} \left[x-\frac{D^2+2(i+1)D}{2i-1}x+\left(\frac{D^2+2(i+1)D}{2i-1}\right)^2 x-\dots\right] \\
&= \text{Im part of } e^x e^{ix} \; \frac{1}{2i-1} \left[x-\frac{0+2(i+1)}{2i-1}+0-\dots\right] \\
&= \text{Im part of } e^x e^{ix} \; \frac{1}{2i-1} \left[x-\frac{2(i+1)}{2i-1}\right] \\
&= \text{Im part of } e^x e^{ix} \; \frac{1}{2i-1} \frac{2i+1}{2i+1} \left[x-\frac{2(i+1)(2i+1)}{(2i-1)(2i+1)}\right] \\
&= \text{Im part of } e^x e^{ix} \; \frac{2i+1}{-5} \left[x-\frac{2(-1+3i)}{-5}\right] \\
&= \text{Im part of } e^x e^{ix} \; \frac{1}{-5}(1+2i) \left[x+\frac{2}{5}(-1+3i)\right] \\
&= \text{Im part of } e^x e^{ix} \; \frac{1}{-5}(1+2i) \left[\left(x-\frac{2}{5}\right)+\frac{6}{5}i\right]
\end{aligned}$$

$$= -\text{Im part of } \frac{e^x}{5}(\cos x + i\sin x)\left\{\left[\left(x-\frac{2}{5}\right)-\frac{12}{5}\right]+i\left[2\left(x-\frac{2}{5}\right)+\frac{6}{5}\right]\right\}$$
$$= -\text{Im part of } \frac{e^x}{5}(\cos x + i\sin x)\left[\left(x-\frac{14}{5}\right)+i\left(2x+\frac{2}{5}\right)\right]$$
$$= -\frac{e^x}{5}\left[\left(x-\frac{14}{5}\right)\sin x+\left(2x+\frac{2}{5}\right)\cos x\right]$$
$$= -\frac{e^x}{5}\left[x\left(\sin x+2\cos x\right)+\frac{2}{5}\left(\cos x-7\sin x\right)\right]$$

$\therefore$ The complete solution is

$$y = c_1e^x + c_2e^{-x} - \frac{e^x}{5}\left[x\left(\sin x+2\cos x\right)+\frac{2}{5}\left(\cos x-7\sin x\right)\right]$$

Exercise

Solve the following:

1) $(D^2+a^2)y = x\sin ax$
2) $(D^2+3D+2)y = x\sin 2x$
3) $(D^2-1)y = x^2\sin 3x$
4) $(D^2+4)y = x\sin^x$
5) $(D^2-2D+1)y = x\sin x$
6) $(D^2-1)y = x\sin x + (1+x^2)e^x$

Answers

1) $y = c_1\cos ax + c_2\sin ax - \dfrac{x^2}{4a}\cos ax + \dfrac{x}{4a^2}\sin ax$
2) $y = c_1e^{-x} + c_2e^{-2x} + \dfrac{7-30x}{200}\cos 2x + \dfrac{12-5x}{100}\sin 2x$
3) $y = c_1e^{-x} + c_2e^{-x} - \dfrac{1}{10}\left[x^2\sin 3x + \dfrac{6}{5}x\cos 3x - \dfrac{13}{25}\sin 3x\right]$
4) $y = c_1\cos 2x + c_2\sin 2x + \dfrac{x}{8} - \dfrac{x}{32}\cos 2x - \dfrac{x^2}{16}\sin 2x$
5) $y = (c_1+c_2x)e^x\dfrac{1}{2}[x\cos x + \cos x - \sin x]$
6) $y = c_1e^x + c_2e^{-x} - \dfrac{1}{2}[x\sin x + \cos x] + \dfrac{1}{12}xe^x(2x^2-3x+9)$

1.11 Particular integral when X is any other function of x

Consider the linear differential equation with constant coefficient as

$$f(D)y = X$$

Where $f(D) = a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n$ and X does not belongs partially or completely to any one of the previously discussed forms i.e. e^{ax}, $\sin ax$ or $\cos ax$, x^m, $e^{ax}V(x)$ and $x^mV(x)$. Then

$$\begin{aligned} P.I. &= \frac{1}{f(D)}X \\ &= \frac{1}{a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n}X \qquad (1) \end{aligned}$$

Suppose $D = m_1, m_2, \ldots m_n$ are the roots of $D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n = 0$. Then one can write

$$a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n = a_0(D - m_1)(D - m_2)\ldots(D - m_n)$$

Therefore, resolving into partial factors (1) reduces to

$$\begin{aligned} P.I. &= \frac{1}{a_0(D - m_1)(D - m_2)\ldots(D - m_n)}X = \left[\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \cdots + \frac{A_n}{D - m_n}\right]X \\ &= \frac{A_1}{D - m_1}X + \frac{A_2}{D - m_2}X + \cdots + \frac{A_3}{D - m_3}X \\ &= A_1e^{m_1x}\int Xe^{-m_1x}\mathrm{d}x + A_2e^{m_2x}\int Xe^{-m_2x}\mathrm{d}x + \cdots + A_ne^{m_nx}\int Xe^{-m_nx}\mathrm{d}x \\ &\qquad \left[\because \frac{1}{D - a}X = e^{ax}\int Xe^{-ax}\mathrm{d}x\right] \end{aligned}$$

Remember

1. When X does not belong to partially or completely to any one of the previously discussed form use this general method.
2. To solve examples using this general method we use following to formulae

   i) $\dfrac{1}{D - a}X = e^{ax}\displaystyle\int Xe^{-ax}\mathrm{d}x$ ii) $\dfrac{1}{D + a}X = e^{-ax}\displaystyle\int Xe^{ax}\mathrm{d}x$

Examples

Example 35. Solve $(D^2 + 3D + 2)y = \sin e^x$

Sol. The given equation is

$$(D^2 + 3D + 2)y = \sin e^x$$

The auxiliary equation is

$$m^2 + 3m + 2 = 0 \Rightarrow (m + 1)(m + 2) = 0$$
$$m = -1, -2$$

$\therefore$ The complementary function is

$$C.F. = c_1 e^{-x} + c_2 e^{-2x}$$

Now, P.I. corresponding to $\sin e^x$ is given by

$$\begin{aligned}
P.I. &= \frac{1}{D^2 + 3D + 2} \sin e^x = \frac{1}{(D + 1)(D + 2)} \sin e^x \\
&= \left(\frac{1}{D + 1} - \frac{1}{D + 2} \right) \sin e^x \\
&= e^{-x} \int e^x \sin e^x \mathrm{d}x - e^{-2x} \int e^{2x} \sin e^x \mathrm{d}x
\end{aligned}$$

Put $e^x = p$. Therefore, $e^x \mathrm{d}x = \mathrm{d}p$. Then above equality becomes

$$\begin{aligned}
P.I. &= e^{-x} \int \sin p \mathrm{d}p - e^{-2x} \int p \sin p \mathrm{d}p \\
&= e^{-x}(-\cos p) - e^{-2x}[p(-\cos p) - (1)(-\sin p)] \\
&= -e^{-x} \cos p - e^{-2x}[-p \cos p + \sin p] \\
&= -e^{-x} \cos e^x - e^{-2x}[-e^x \cos e^x + \sin e^x] \\
&= -e^{-x} \cos e^x - e^{-x} \cos e^x + e^{-2x} \sin e^x \\
&= -e^{-2x} \sin e^x
\end{aligned}$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin e^x$$

Example 36. Solve $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{e^x}$

Sol. The given equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{e^x}$$

The symbolic form of equation is

$$(D^2 + 3D + 2)y = e^{e^x}$$

The auxiliary equation is

$$m^2 + 3m + 2 = 0 \Rightarrow (m + 1)(m + 2) = 0$$

$$\Rightarrow m = -1, -2$$

$\therefore$ The complementary function is

$$C.F. = c_1 e^{-x} + c_2 e^{-2x}$$

Now, P.I. corresponding to e^{e^x} is given by

$$\begin{aligned}
P.I. &= \frac{1}{D^2 + 3D + 2} e^{e^x} = \frac{1}{(D+1)(D+2)} e^{e^x} \\
&= \left(\frac{1}{D+1} - \frac{1}{D+2} \right) e^{e^x} \\
&= e^{-x} \int e^x e^{e^x} \mathrm{d}x - e^{-2x} \int e^{2x} e^{e^x} \mathrm{d}x
\end{aligned}$$

Put $e^x = p$. Therefore, $e^x \mathrm{d}x = \mathrm{d}p$. Then above equality becomes

$$\begin{aligned}
P.I. &= e^{-x} \int e^p \mathrm{d}p - e^{-2x} \int p e^p \mathrm{d}p \\
&= e^{-x} e^p - e^{-2x} [p(e^p) - (1)(e^p)] \\
&= e^{-x} e^{e^x} - e^{-2x} [e^x e^{e^x} - e^{e^x}] \\
&= e^{-x} e^{e^x} - e^{-x} e^{e^x} + e^{-2x} e^{e^x} \\
&= e^{-2x} e^{e^x}
\end{aligned}$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \sin e^x$$

Example 37. Solve $(D^2 + a^2)y = \operatorname{cosec} ax$

Sol. The given equation is

$$D^2 + a^2 = 0$$

The auxiliary equation is

$$m^2 + a^2 = 0 \Rightarrow m = \pm ai$$

$\therefore$ The auxiliary equation is

$$C.F. = c_1 \cos ax + c_2 \sin ax$$

Now, P.I. corresponding to $\operatorname{cosec} ax$ is

$$\begin{aligned}
P.I. &= \frac{1}{D^2 + a^2} \operatorname{cosec} ax = \frac{1}{(D+ia)(D-ia)} \operatorname{cosec} ax \\
&= \frac{1}{2ia} \left[\frac{1}{D-ia} - \frac{1}{D+ia} \right] \operatorname{cosec} ax = \frac{1}{2ia} \left[\frac{1}{D-ia} \operatorname{cosec} ax - \frac{1}{D+ia} \operatorname{cosec} ax \right] \\
&= \frac{1}{2ia} \left[e^{iax} \int e^{-iax} \frac{1}{\sin ax} \mathrm{d}x - e^{-iax} \int e^{iax} \frac{1}{\sin ax} \mathrm{d}x \right]
\end{aligned}$$

$$= \frac{1}{2ia}\left[e^{iax}\int(\cos ax - i\sin ax)\frac{1}{\sin ax}\mathrm{d}x - e^{-iax}\int(\cos ax + i\sin ax)\frac{1}{\sin ax}\mathrm{d}x\right]$$
$$= \frac{1}{2ia}\left[e^{iax}\int(\cot ax - i)\mathrm{d}x - e^{-iax}\int(\cot ax + i)\mathrm{d}x\right]$$
$$= \frac{1}{2ia}\left[e^{iax}\left(\frac{\log(\sin ax)}{a} - ix\right) - e^{-iax}\left(\frac{\log(\sin ax)}{a} + ix\right)\right]$$
$$= \frac{1}{2ia}\left[\frac{\log(\sin ax)}{a}(e^{iax} - e^{-iax}) - ix(e^{iax} + e^{iax})\right]$$
$$= \frac{\log(\sin ax)}{a^2}\left(\frac{e^{iax} - e^{-iax}}{2i}\right) - \frac{x}{a}\left(\frac{e^{iax} + e^{iax}}{2}\right)$$
$$= \frac{\sin ax}{a^2}\log\sin ax - \frac{x}{a}\cos ax$$

$\therefore$ The complete solution is

$$y = c_1\cos ax + c_2\sin ax + \frac{\sin ax}{a^2}\log\sin ax - \frac{x}{a}\cos ax$$

Example 38. Solve $(D^2 + a^2)y = \sec ax$

Sol. The given equation is

$$D^2 + a^2 = 0$$

The auxiliary equation is

$$m^2 + a^2 = 0 \Rightarrow m = \pm ai$$

$\therefore$ The auxiliary equation is

$$C.F. = c_1\cos ax + c_2\sin ax$$

Now, P.I. corresponding to $\sec ax$ is

$$P.I. = \frac{1}{D^2 + a^2}\sec ax = \frac{1}{(D + ia)(D - ia)}\sec ax$$
$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right]\sec ax = \frac{1}{2ia}\left[\frac{1}{D - ia}\sec ax - \frac{1}{D + ia}\sec ax\right]$$
$$= \frac{1}{2ia}\left[e^{iax}\int e^{-iax}\frac{1}{\cos ax}\mathrm{d}x - e^{-iax}\int e^{iax}\frac{1}{\cos ax}\mathrm{d}x\right]$$
$$= \frac{1}{2ia}\left[e^{iax}\int(\cos ax - i\sin ax)\frac{1}{\cos ax}\mathrm{d}x - e^{-iax}\int(\cos ax + i\sin ax)\frac{1}{\cos ax}\mathrm{d}x\right]$$
$$= \frac{1}{2ia}\left[e^{iax}\int(1 - i\tan ax)\mathrm{d}x - e^{-iax}\int(1 + i\tan ax)\mathrm{d}x\right]$$
$$= \frac{1}{2ia}\left[e^{iax}\left(x - i\frac{\log(\sec ax)}{a}\right) - e^{-iax}\left(x + \frac{i\log(\sec ax)}{a}\right)\right]$$
$$= \frac{1}{2ia}\left[x(e^{iax} - e^{-iax}) - i\frac{\log\sec ax}{a}(e^{iax} + e^{iax})\right]$$

$$
\begin{aligned}
&= \frac{x}{a}\left(\frac{e^{iax} - e^{-iax}}{2i}\right) - \frac{\log \sec ax}{a^2}\left(\frac{e^{iax} + e^{iax}}{2}\right) \\
&= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log \cos ax
\end{aligned}
$$

$\therefore$ The complete solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log \cos ax$$

Example 39. Solve $(D^2 - 1)y = \dfrac{2}{1 + e^x}$

Sol. The given equation is

$$(D^2 - 1)y = \frac{2}{1 + e^x}$$

The auxiliary equation is

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

$\therefore$ The auxiliary equation is

$$C.F. = c_1 e^x + c_2 e^{-x}$$

Now, P.I. corresponding to $\dfrac{2}{1 + e^x}$ is

$$
\begin{aligned}
P.I. &= \frac{1}{D^2 - 1}\,\frac{2}{1 + e^x} = \frac{2}{(D-1)(D+1)}\,\frac{1}{1 + e^x} = \left[\frac{1}{D-1} - \frac{1}{D+1}\right]\,\frac{1}{1 + e^x} \\
&= \frac{1}{D-1}\,\frac{1}{1 + e^x} - \frac{1}{D+1}\,\frac{1}{1 + e^x} = e^x \int e^{-x}\frac{1}{1 + e^x}\mathrm{d}x - e^{-x}\int e^x \frac{1}{1 + e^x}\mathrm{d}x \\
&= e^x \int \frac{1}{e^x(1 + e^x)}\mathrm{d}x - e^{-x}\int e^x \frac{1}{1 + e^x}\mathrm{d}x = e^x \int \left[\frac{1}{e^x} - \frac{1}{1 + e^x}\right]\mathrm{d}x - e^{-x}\int e^x \frac{1}{1 + e^x}\mathrm{d}x \\
&= e^x \int \left[\frac{1}{e^x} - \frac{(1 + e^x) - e^x}{1 + e^x}\right]\mathrm{d}x - e^{-x}\int e^x \frac{1}{1 + e^x}\mathrm{d}x \\
&= e^x \int \left[\frac{1}{e^x} - 1 + \frac{e^x}{1 + e^x}\right]\mathrm{d}x - e^{-x}\int e^x \frac{1}{1 + e^x}\mathrm{d}x \\
&= e^x[-e^{-x} - x + \log(1 + e^x)] - e^{-x}\log(1 + e^x) \\
&= -1 - xe^x - e^x \log(1 + e^x) - e^{-x}\log(1 + e^x)
\end{aligned}
$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-x} - 1 - xe^x - e^x \log(1 + e^x) - e^{-x}\log(1 + e^x)$$

Exercise

Solve the following:

1) $(D^2 - 2D + y)y = xe^x \sin x$

2) $(D^2 - D - 2)y = 2\log x + \dfrac{1}{x} + \dfrac{1}{x^2}$

3) $(D^2 + 2D + 1)y = 4e^{-x} \log x$

4) $(D^2 + a^2)y = 2a \tan ax$

Answers

1) $y = (c_1 + c_2x)e^x - e^x(x \sin x + 2\cos x)$

2) $y = c_1e^{-x} + c_2e^x - \log x$

3) $y = (c_1 + c_2x)e^{+}e^{-}x^2(2\log x - 3)$

4) $y = c_1 \cos ax + c_2 \sin ax - \dfrac{2}{a} \cos ax \ \log\tan\left(\dfrac{\pi}{4} + \dfrac{ax}{2}\right)$

1.12 Homogeneous Linear Differential Equations

Definition. An equation of the form

$$x^n a_0 \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1 x^{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + a_2 x^{n-2} \frac{\mathrm{d}^{n-2} y}{\mathrm{d}x^{n-2}} + \cdots + a_{n-1} x \frac{\mathrm{d}y}{\mathrm{d}x} + a_n y = X$$

Where $0 \neq a_0, a_1, a_2, \ldots, a_n$ are constants and X is either function of x or constant, is called homogeneous linear differential equation. It is also called **Cauchy's homogeneous linear differential equation.**

The following is the method to reduce the given homogeneous linear differential equation to the form linear differential equation with constant coefficient.

Method of solution

Consider a homogeneous linear differential equation of order n as

$$x^n a_0 \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1 x^{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + a_2 x^{n-2} \frac{\mathrm{d}^{n-2} y}{\mathrm{d}x^{n-2}} + \cdots + a_{n-1} x \frac{\mathrm{d}y}{\mathrm{d}x} + a_n y = X \tag{1}$$

Where $a_0, a_1, a_2, \ldots, a_n$ are constants and X is either function of x or constant. To reduce to the linear differential equation with constant coefficient we use the substitution

$$x = e^z \therefore z = \log x \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{x}$$

Take $D = \frac{\mathrm{d}}{\mathrm{d}z}$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z}\frac{1}{x} \Rightarrow x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z} = Dy$$

$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}z}\right) = -\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right) \\ &= -\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{1}{x}\frac{\mathrm{d}^2 y}{\mathrm{d}z^2}\frac{1}{x} \\ &= -\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{1}{x^2}\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = \frac{1}{x^2}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}\right)\end{aligned}$$

Therefore

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{x^2}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}\right) \Rightarrow x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z} = D(D-1)y$$

Similarly,

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = D(D-1)(D-2)y, \ \frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = D(D-1)(D-2)(D-3)y$$

and so on.

Since R.H.S. is a either function of x or constant, the substitution $x = e^z$ gives that R.H.S is either a function of z or constant. Putting all above values in the L.H.S of equation (1), L.H.S becomes function of D. Therefore equation (1) reduces to the form $f(D)y = Z$, which is linear differential equation with constant coefficient. Which can be solved by previously discussed method. The complete solution of (1) is obtained by putting $z = \log x$ in the complete solution of $f(D)y = Z$.

Note. If the degree of x and order of differential coefficient in each term of the equation is not same then multiply the given equation by suitable power of x so that degree of x and order of differential coefficient become same and then apply the method to solve the equation.

Examples

Example 40. Solve $x^3\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$

Sol. The given equation is

$$x^3\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2y = 10\left(x + \frac{1}{x}\right) \tag{1}$$

Put $x = e^z \Rightarrow z = \log x$. Take, $D = \dfrac{\mathrm{d}}{\mathrm{d}z}$. Then

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = Dy,\ x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = D(D-1)y \text{ and } x^3\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = D(D-1)(D-2)y$$

Then the (1) becomes

$$\begin{aligned} & [D(D-1)(D-2) + 2D(D-1) + 2]y = 10\left(e^z + \frac{1}{e^z}\right) \\ \Rightarrow \quad & (D^3 - 3D^2 + 2D + 2D^2 - 2D + 2)y = 10\left(e^z + e^{-z}\right) \\ \Rightarrow \quad & (D^3 - D^2 + 2) = 10\left(e^z + e^{-z}\right) \end{aligned}$$

Therefore

$$(D^3 - D^2 + 2)y = 10\left(e^z + e^{-z}\right) \tag{2}$$

The auxiliary equation of equation (1) is

$$m^3 - m^2 + 2 = 0 \tag{3}$$

Here $m = -1$ is the one root of the above equation. Therefore, by synthetic division method we get

$$\begin{array}{r|rrrr} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

Therefore

$$m^3 - m^2 + 2 = (m+1)(m^2 - 2m + 2) = 0$$

$$\begin{aligned} &\Rightarrow (m+1)(m^2 - 2m + 2) = 0 \\ &\Rightarrow m = -1, \frac{2 \pm \sqrt{4-8}}{2} \\ &\Rightarrow m = -1, 1 \pm i \end{aligned}$$

$\therefore$ The complementary function is

$$y = c_1 e^{-z} + e^z(c_2 \cos z + c_3 \sin z)$$

Now, P.I corresponding to $10\left(e^z + e^{-z}\right)$ is

$$\begin{aligned} P.I. &= \frac{1}{D^3 - D^2 + 2} 10\left(e^z + e^{-z}\right) = 10\left[\frac{1}{D^3 - D^2 + 2} e^z + \frac{1}{D^3 - D^2 + 2} e^{-z}\right] \\ &= 10\left[\frac{1}{1^3 - 1^2 + 2} e^z + \frac{1}{(-1)^3 - (-1)^2 + 2} e^{-z}\right] \text{case of failure in second part} \\ &= 10\left[\frac{1}{2} e^z + z\frac{1}{3D^2 - 2D} e^{-z}\right] = 10\left[\frac{1}{2} e^z + z\frac{1}{3(-1)^2 - 2(-1)} e^{-z}\right] \\ &= 10\left[\frac{1}{2} e^z + z\frac{1}{5} e^{-z}\right] = 5e^z + 2ze^{-z} \end{aligned}$$

$\therefore$ The complete solution of equation (2) is

$$y = c_1 e^{-z} + e^z(c_2 \cos z + c_3 \sin z) + 5e^z + 2ze^{-z}$$

Putting $z = \log x$ in above equation. The complete solution of equation (1) is

$$y = \frac{c_1}{x} + x(c_2 \cos\log x + c_3 \sin\log x) + 5x + \frac{2}{x}\log x$$

Example 41. Solve $x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \log x$

Sol. The given equation is

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \log x \tag{1}$$

Put $x = e^z \Rightarrow z = \log x$. Take, $D = \frac{\mathrm{d}}{\mathrm{d}z}$. Then

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = Dy,\ x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = D(D-1)y$$

Then the (1) becomes

$$[D(D-1)-D+1]y = z \Rightarrow (D^2-2D+1)y = z$$

Therefore

$$(D^2-2D+1)y = z \tag{2}$$

The auxiliary equation of equation (2) is

$$m^2-2m+1=0 \Rightarrow (m-1)^2=0 \Rightarrow m=1,1$$

$\therefore$ The complementary function is

$$y=(c_1+c_2z)e^z$$

Now, P.I corresponding to z is

$$\begin{aligned} P.I. &= \frac{1}{D^2-2D+1}z = \frac{1}{(D-1)^2}z = \frac{1}{(1-D)^2} = (1-D)^{-2}z \\ &= (1+2D+3D^2+\dots)z \\ &= z+2Dz+3D^2z+\dots \\ &= z+2 \end{aligned}$$

$\therefore$ The complete solution of equation (2) is

$$y=(c_1+c_2z)e^z+z+z$$

Putting $z=\log x$, the complete solution of equation (1) is

$$y=(c_1+c_2\log x)x+\log x+2$$

Example 42. Solve $x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{2y}{x}=\dfrac{\sin x}{x}$

Sol. The given equation is

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2y}{x}=\frac{\sin x}{x} \tag{1}$$

Multiplying both side by x, we get

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4x\frac{\mathrm{d}y}{\mathrm{d}x}+2y=\sin x \tag{2}$$

Put $x=e^z \Rightarrow z=\log x$. Take, $D=\dfrac{\mathrm{d}}{\mathrm{d}z}$. Then

$$x\frac{\mathrm{d}y}{\mathrm{d}x}=Dy,\ x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=D(D-1)y$$

Then the (2) becomes

$$[D(D-1)+4D+2]y=\sin e^z \ \Rightarrow (D^2+3D+2)y=\sin e^z$$

Therefore

$$(D^2 + 3D + 2)y = \sin e^z \tag{3}$$

The auxiliary equation of equation (3) is

$$m^2 + 3m + 2 = 0 \Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$$

$\therefore$ The complementary function is

$$y = c_1 e^{-z} + c_2 e^{-2z}$$

Now, P.I corresponding to $\sin e^z$ is

$$\begin{aligned}
P.I. &= \frac{1}{D^2 + 3D + 2} \sin e^z = \frac{1}{(D+1)(D+2)} \sin e^z \\
&= \left(\frac{1}{D+1} - \frac{1}{D+2} \right) \sin e^z \\
&= e^{-z} \int e^z \sin e^z \mathrm{d}z - e^{-2z} \int e^{2z} \sin e^z \mathrm{d}z
\end{aligned}$$

Put $e^z = p$. Therefore, $e^z \mathrm{d}z = \mathrm{d}p$. Then above equality becomes

$$\begin{aligned}
P.I. &= e^{-z} \int \sin p \mathrm{d}p - e^{-2z} \int p \sin p \mathrm{d}p \\
&= e^{-z}(-\cos p) - e^{-2z}[p(-\cos p) - (1)(-\sin p)] \\
&= -e^{-z} \cos p - e^{-2z}[-p \cos p + \sin p] \\
&= -e^{-z} \cos e^z - e^{-2z}[-e^z \cos e^z + \sin e^z] \\
&= -e^{-z} \cos e^z - e^{-z} \cos e^z + e^{-2z} \sin e^z \\
&= -e^{-2z} \sin e^z
\end{aligned}$$

$\therefore$ The complete solution of equation (3) is

$$y = c_1 e^{-z} + c_2 e^{-2z} - e^{-2z} \sin e^z$$

Putting $z = \log x$, the complete solution of (1) is

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} - \frac{\sin x}{x^2}$$

Example 43. Solve $x^2 \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3x \dfrac{\mathrm{d}y}{\mathrm{d}x} + 5y = x^2 \sin \log x$

Sol. The given equation is

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3x \frac{\mathrm{d}y}{\mathrm{d}x} + 5y = x^2 \sin \log x \tag{1}$$

Put $x = e^z \Rightarrow z = \log x$. Take, $D = \dfrac{\mathrm{d}}{\mathrm{d}z}$. Then

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = Dy,\ x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = D(D-1)y$$

Then the (1) becomes

$$[D(D-1) - 3D + 5]y = e^{2z}\sin z \ \Rightarrow (D^2 - 4D + 5)y = e^{2z}\sin z$$

Therefore

$$(D^2 - 4D + 5)y = e^{2z}\sin z \tag{2}$$

The auxiliary equation of equation (2) is

$$m^2 - 4m + 5 = 0 \Rightarrow m = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$\therefore$ The complementary function is

$$C.F. = e^{2z}(c_1\cos z + c_2\sin z)$$

Now, P.I. corresponding to $e^{2z}\sin z$ is given by

$$\begin{aligned}
P.I. &= \frac{1}{D^2-4D+5}e^{2z}\sin z = e^{2z}\frac{1}{(D+2)^2-4(D+2)+5}\sin z = e^{2z}\frac{1}{D^2+4D+4-4D-8+5}\sin z \\
&= e^{2z}\frac{1}{D^2+1}\sin z = e^{2z}\frac{1}{-1^2+1}\sin z \qquad\qquad \textbf{case of failure} \\
&= e^{2z}z\frac{1}{2D}\sin z = -\frac{1}{2}ze^{2z}\cos z
\end{aligned}$$

$\therefore$ The complete solution of equation (2) is

$$y = e^{2z}(c_1\cos z + c_2\sin z) - \frac{1}{2}ze^{2z}\cos z$$

Putting $z = \log x$, the complete solution of equation (1) is

$$y = x^2[c_1\cos\log x + c_2\sin\log x] - \frac{1}{2}x^2\log x\cos\log x$$

Example 44. Solve $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 3y = x^2\log x$

Sol. The given equation is

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = x^2\log x \tag{1}$$

Put $x = e^z \Rightarrow z = \log x$. Take, $D = \dfrac{\mathrm{d}}{\mathrm{d}z}$. Then

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = Dy,\ x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = D(D-1)y$$

Then the (1) becomes

$$[D(D-1)-D-3]y = e^{2z}z \Rightarrow (D^2-2D-3)y = e^{2z}z$$

Therefore

$$(D^2-2D-3)y = e^{2z}z \tag{2}$$

The auxiliary equation of equation (2) is

$$\begin{aligned} m^2-2m-3=0 &\Rightarrow (m+1)(m-3)=0 \\ &\Rightarrow m=-1,3 \end{aligned}$$

$\therefore$ The complementary function is

$$C.F. = c_1e^{-z}+c_2e^{3z}$$

Now, P.I. corresponding to $e^{2z}z$ is given by

$$\begin{aligned} P.I. &= \frac{1}{D^2-2D-3}e^{2z}z = e^{2z}\frac{1}{(D+2)^2-2(D+2)-3}z \\ &= e^{2z}\frac{1}{D^2+4D+4-2D-4-3}z = e^{2z}\frac{1}{D^2+2D-3}z \\ &= e^{2z}\frac{1}{-3\left(1-\dfrac{D^2+2D}{3}\right)}z = -\frac{e^{2z}}{3}\left(1-\frac{D^2+2D}{3}\right)^{-1}z \\ &= -\frac{e^{2z}}{3}\left[1+\frac{D^2+2D}{3}+\left(\frac{D^2+2D}{3}\right)^2+\dots\right]z \\ &= -\frac{e^{2z}}{3}\left[1+\frac{1}{3}(D^2+2D)+\frac{1}{9}\left(D^4+4D^3+4D^2\right)+\dots\right]z \\ &= -\frac{e^{2z}}{3}\left[z+\frac{1}{3}(D^2z+2Dz)+\frac{1}{9}\left(D^4z+4D^3z+4D^2z\right)+\dots\right] \\ &= -\frac{e^{2z}}{3}\left[z+\frac{2}{3}+\frac{1}{9}(0)+\dots\right] = -\frac{e^{2z}}{3}\left[z+\frac{2}{3}\right] \end{aligned}$$

$\therefore$ The complete solution of equation (2) is

$$y = c_1e^{-z}+c_2e^{3z}-\frac{e^{2z}}{3}\left[z+\frac{2}{3}\right]$$

Putting $z=\log x$ in above. The complete solution of equation (1) is

$$y = \frac{c_1}{x}+c_2x^3-\frac{x^2}{3}\left(\log x+\frac{2}{3}\right)$$

Example 45. The radial displacement u in a rotating disc at a distance r from the axis is given by the $\dfrac{\mathrm{d}^2u}{\mathrm{d}r^2}+\dfrac{1}{r}\dfrac{\mathrm{d}u}{\mathrm{d}r}-\dfrac{u}{r^2}+kr=0$. Find the displacement if $u=0$ when $r=0$ and $r=a$.

Sol. Consider the given equation as

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2} + kr = 0$$

Multiplying both side by r^2, we get

$$r^2\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + r\frac{\mathrm{d}u}{\mathrm{d}r} - u = -kr^3 \tag{1}$$

Put $r = e^z \Rightarrow z = \log r$. Take, $D = \dfrac{\mathrm{d}}{\mathrm{d}z}$. Then

$$r\frac{\mathrm{d}y}{\mathrm{d}r} = Dy,\ r^2\frac{\mathrm{d}^2y}{\mathrm{d}r^2} = D(D-1)y$$

Then the (1) becomes

$$[D(D-1) + D - 1]y = -ke^{3z}$$

Therefore,

$$(D^2 - 1)y = -ke^{3z} \tag{2}$$

The auxiliary equation of (2) is

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

$\therefore$ The complementary function is

$$C.F. = c_1e^z + c_2e^{-z}$$

Now, P.I. corresponding to $-ke^{3z}$ is

$$P.I. = \frac{1}{D^2-1}(-ke^{3z}) = -k\frac{1}{D^2-1}e^{3z} = -k\frac{1}{3^2-1}e^{3z} = -\frac{k}{8}e^{3z}$$

$\therefore$ The complete solution of equation (2) is

$$u = c_1e^z + c_2e^{-z} - \frac{k}{8}e^{3z}$$

Thus, the complete solution of equation (1) is

$$u = c_1r + \frac{c_2}{r} - \frac{k}{8}r^3 \tag{3}$$

To find c_1 and c_2

By (3) one can write

$$ur = c_1r^2 + c_2 - \frac{k}{8}r^4 \tag{4}$$

Given that $u = 0$ when $r = 0$ and $r = a$. Using these conditions in above equations we get, $c_2 = 0$ and $c_1 = \dfrac{k}{8}a^2$. Therefore by (3), the complete solution is

$$u = r\frac{k}{8}a^2 - \frac{k}{8}r^3 = \frac{k}{8}r(a^2 - r^2)$$

Exercise

Solve the following differential equations

1) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 2\log x$

2) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\dfrac{\mathrm{d}y}{\mathrm{d}x}y + 4y = \cos\log x + x\sin\log x$

3) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 2y = x^{-1}$

4) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3x\dfrac{\mathrm{d}y}{\mathrm{d}x} + 4y = 2x$

5) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = (\log x)^2 + x\sin(\log x)$

6) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^x$

7) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 6y = x^3\log x$

8) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 2\sin\log x$

Answers

1) $y = (c_1 + c_2\log x)x + 2(\log x + 2)$

2) $y = x[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)] + \dfrac{1}{13}[3\cos\log x - 2\sin\log x] + \dfrac{x}{2}\sin(\log x)$

3) $y = c_1x + c_2x^2 + \dfrac{1}{6x}$

4) $y = (c_1 + c_2\log x)x^2 + 2x$

5) $y = c_1\cos\log x + c_2\sin\log x + (\log x)^2 - \dfrac{1}{5}x[2\cos\log x - \sin\log x] - 2$

6) $y = \dfrac{c_1}{x} + \dfrac{c_2}{x^2} + \dfrac{e^x}{x^2}$

7) $y = c_1x^2 + \dfrac{c_2}{x^3} + \dfrac{x^3}{6}\left[\log x - \dfrac{7}{6}\right]$

8) $y = c_1\cos\log x + c_2\sin\log x - \log x\cos\log x$

Objective Type Questions For GATE

1) The order of the differential equation $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 + y^4 = e^{-x}$ is

   a) 1 b) 2 c) 3 d) 4

2) The differential equation $\dfrac{d^2y}{dx^2} + \sin x\dfrac{dy}{dx} + ye^x = \sinh x$ is

   a) first order and linear
   b) first order and non-linear
   c) second order and linear
   d) second order and non-linear

3) The solution of the ordinary differential equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$ is

   a) $y = c_1e^{3x} + c_2e^{-2x}$ b) $y = c_1e^{3x} + c_2e^{2x}$ c) $y = c_1e^{-3x} + c_2e^{2x}$ d) $y = c_1e^{-3x} + c_2e^{-2x}$

4) The solution of the differential equation $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$ is

   a) $y = e^{2x} + e^{-3x}$ b) $y = e^{2x} + e^{3x}$ c) $y = e^{-2x} + e^{3x}$ d) $y = e^{-2x} + e^{-3x}$

5) Which of the following is the solution of the ordinary differential equation $\dfrac{d^2y}{dx^2} + p\dfrac{dy}{dx} + (q-1)y = 0$? for $p = 4$, $q = 5$

   a) e^{-3x} b) xe^{-x} c) xe^{-2x} d) x^2e^{-2x}

6) The general solution of $\dfrac{d^2y}{dx^2} + y = 0$ is

   a) $y = p\cos x + q\sin x$ b) $y = p\cos x$ c) $y = p\sin x$ d) $y = p\sin^2 x$

7) What condition is to be satisfied so that the solution of the differential equation $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = 0$ is of the form $y = (c_1 + c_2x)e^{mx}$, where c_1 and c_2 are constants.

   a) $a^2 = b$ b) $b^2 = a$ c) $a^2 = 4b$ d) $b^2 = 4a$

8) For the differential equation $x''(t) + 3x'(t) + 2x(t) = 5$, the solution $x(t)$ approaches which of the following values as $t \to \infty$?

   a) zero b) $\dfrac{5}{2}$ c) 5 d) 10

9) For the differential equation $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 8x = 0$ with initial conditions $x(0) = 1$ and $\left(\dfrac{dx}{dt}\right)_{t=0} = 0$, the solution is

a) $x(t) = 2e^{-6t} - e^{-2t}$ b) $x(t) = 2e^{-6t} - e^{-4t}$ c) $x(t) = -e^{-6t} + 2e^{-4t}$ d) $x(t) = e^{-2t} + 2e^{-4t}$

10) The solution of the differential equation $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 2y = 0$ with initial conditions $y(0) = 0$ and $\left(\dfrac{dy}{dt}\right)_{t=0} = -1$, the solution is

a) $-t\sin t$ b) $-e^{-t}(1 - \cos t)$ c) $-(t + \sin t)/2$ d) $-e^{-t}\sin t$

11) The function $n(x)$ satisfies the differential equation $\dfrac{d^2n(x)}{dx^2} - \dfrac{n(x)}{L^2} = 0$, where L is constant. The boundary conditions are $n(0) = k$ and $n(\infty) = 0$. The solution of this equation is

a) $n(x) = ke^{x/L}$ b) $n(x) = ke^{-x/\sqrt{L}}$ c) $n(x) = k^2e^{-x/L}$ d) $n(x) = ke^{-x/L}$

12) For $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 3e^{2x}$, the particular integral is

a) $\dfrac{1}{15}e^{2x}$ b) $\dfrac{1}{5}e^{2x}$ c) $3e^{2x}$ d) $c_1e^{-x} + c_2e^{-3x}$

13) Which one of the following is not a solution of the differential equation $\dfrac{d^2y}{dx^2} + y = 1$?

a) $y = 1$ b) $y = 1 + \cos x$ c) $y = 1 + \sin x$ d) $y = 2 + \sin x + \cos x$

14) With $y = e^{ax}$ if the sum $S = \dfrac{dy}{dx} + \dfrac{d^2y}{dx^2} + \cdots + \dfrac{d^ny}{dx^n}$ approaches $2y$ as $n \to \infty$, then the value of a is

a) $\dfrac{1}{3}$ b) $\dfrac{1}{2}$ c) $\dfrac{2}{3}$ d) 2

15) The solution of the differential equation $x^2\dfrac{d^3y}{dx^3} + 2x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 0$ is given by

a) $y = c_1x + c_2x^{-2} + c_3$ c) $y = c_1x^2 + c_2x^{-1} + c_3$

b) $y = c_1x^2 + c_2x^{-2} + c_3$ d) $y = c_1x + c_2x^{-1} + c_3$

16) For $f(D)(y) = e^{ax}$, which of the following is false

a) $P.I. = \dfrac{1}{f(a)}e^{ax}$, if $f(a) \neq 0$

b) $P.I. = \dfrac{1}{f'(a)}e^{ax}$, if $f(a) = 0$ and $f'(a) \neq 0$

c) $P.I. = x\dfrac{1}{f'(a)}e^{ax}$, if $f(a) = 0$ and $f'(a) \neq 0$

d) $P.I. = x^2\dfrac{1}{f''(a)}e^{ax}$, if $f(a) = f'(a) = 0$ and $f''(a) \neq 0$

17) The complementary function of the differential equation $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4y = e^{3x}$ is

a) $ae^{-x} + be^{2x} + cxe^{2x}$

b) $ae^{-x} + be^{-2x} + cxe^{-2x}$

c) $ae^{-x} + be^{-2x} + ce^{2x}$

d) $\frac{1}{4}e^{3x}$

18) The particular solution of $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{4} = \frac{1}{\sqrt{x}}$ is

a) $\frac{1}{2\sqrt{x}}$

b) $\frac{\log x}{2\sqrt{x}}$

c) $\frac{(\log x)^2}{2\sqrt{x}}$

d) $\frac{\sqrt{x}\log x}{2}$

19) Transformation to LDE with constant coefficient by substituting $x = e^z$ of the equation $x^3\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0$ is

a) $\frac{\mathrm{d}^3y}{\mathrm{d}z^3} - 4\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + 5\frac{\mathrm{d}y}{\mathrm{d}z} - 2y = 0$

b) $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0$

c) $\frac{\mathrm{d}^3y}{\mathrm{d}z^3} - 2\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + 5\frac{\mathrm{d}y}{\mathrm{d}z} - 2y = 0$

d) $\frac{\mathrm{d}^3y}{\mathrm{d}z^3} + 4\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + 5\frac{\mathrm{d}y}{\mathrm{d}z} + 2y = 0$

20) Particular Integral of differential equation $(D^2 - 4D + 3)y = x^3e^{2x}$ is

a) $e^{2x}(x^3 + 6x)$

b) $e^{2x}(x^3 - 6x)$

c) $-e^{2x}(x^3 + 6x)$

d) $-e^{2x}(x^3 - 6x)$

21) For $y'' - y' + 2y = x$, which of the following is wrong

a) $C.F. = e^x(c_1 \cos x + c_2 \sin x)$

b) $P.I. = \frac{1}{2}(x + 1)$

c) $y = e^x(c_1 \cos x + c_2 \sin x) + \frac{1}{2}(x + 1)$

d) None of these