Polar Form (r, θ)

Wednesday, May 05, 2021 3:54 PM

1) length of arc of the curve given by $r = f(\theta)$ is

$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad ✓$$

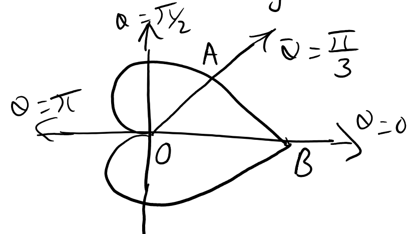
2) length of arc of the curve given by $\theta = g(r)$ is

$$S = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr \quad ✓$$

Problems:

1) (a) Find the length of the perimeter for $r = a(1+\cos\theta)$, (b) also prove that the upper half of the cardiode is bisected by the line $\theta = \frac{\pi}{3}$

Sol^n: Consider the cardiode as depicted in figure



Let length of upper half of given cardiode is S

$$l[\text{Arc } OB] = S = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, dr$$

$$S = \int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + [-a\sin\theta]^2}\, dr$$

$$= \int_0^{\pi} a\sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, dr$$

$$= a\int_0^{\pi} \sqrt{2(2\cos^2\theta/2)}\, dr$$

$$= 2a\int_0^{\pi} \cos(\theta/2)\, dr \quad \text{—— (1)}$$

$$S = 2a\left[2\sin(\theta/2)\right]_0^{\pi}$$

$$S = 4a[+1] = 4a$$

Now perimeter of cardiode is $= 2 \times S$
$= 8a$

& arc where the line $\theta = \frac{\pi}{3}$ intersect the cardiode is given by

$$\text{arc}(AB) = \int_0^{\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, dr$$

from (1) $= 2a\int_0^{\pi/3} \cos(\theta/2)\, dr$

$$\text{arc}(AB) = 2a\left[2\sin(\theta/2)\right]_0^{\pi/3}$$

$$= 2a\left[\cancel{2}\,\frac{1}{\cancel{2}}\right) = 2a$$

$$\text{arc}(AB) = \frac{1}{2}\,\text{arc}(OB)$$

Hence we proved that line $\theta = \frac{\pi}{3}$ bisects the upper half of given cardiode

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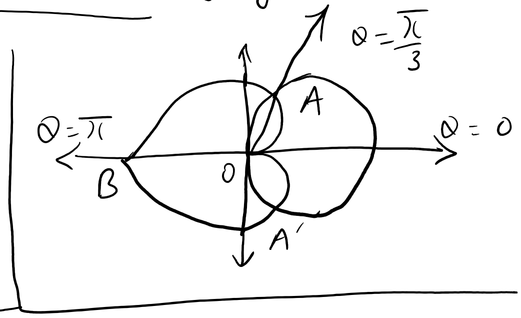
2) Find the length of the cardiode $r = a(1-\cos\theta)$ lying outside the circle $r = a\cos\theta$



Sol^n: The given curves are as shown in the figure

Now, they intersect where,

$\cancel{a}(1-\cos\theta) = \cancel{a}\cos\theta$

$1 - 2\cos\theta =$

$(1-2\cos\theta) = 0$

$\cos\theta = \frac{1}{2} \Rightarrow \boxed{\theta = \frac{\pi}{3}}$

This is point of intersection

Then length of cardiode outside circle is $= 2\, l[\text{Arc } AB]$

& at A, $\theta = \frac{\pi}{3}$ & at B, $\theta = \pi$

Required length $(s) = 2\int_{\pi/3}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, dr$

$$S = 2\int_{\pi/3}^{\pi} \sqrt{a^2(1-\cos\theta)^2 + (a\sin\theta)^2}\, d\theta$$

$$= 2a\int_{\pi/3}^{\pi} \sqrt{1 - 2\cos\theta + \underline{\cos^2\theta + \sin^2\theta}}\, d\theta$$

$$= 2a\int_{\pi/3}^{\pi} \sqrt{2(2\sin^2\theta/2)}\, d\theta$$

$$= 4a\int_{\pi/3}^{\pi} \sin(\theta/2)\, d\theta$$

$$= 4a\left[-2\cos(\theta/2)\right]_{\pi/3}^{\pi}$$

$$S = 4a\left[+\cancel{2}\frac{\sqrt{3}}{\cancel{2}}\right] = \underline{4\sqrt{3}a}$$

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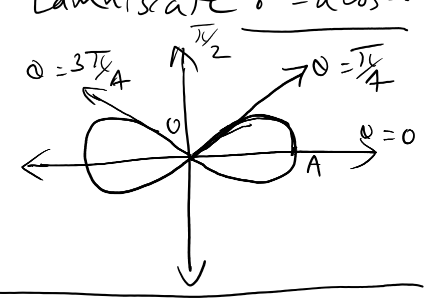
3) Find the length of upper arc of one loop of Lamniscate $r^2 = a^2\cos 2\theta$



Sol^n: The curve is as shown in figure for upper half of one loop

θ varies from $\theta = 0$ to $\theta = \frac{\pi}{4}$

$$\therefore S = \int_0^{\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$r^2 = a^2\cos 2\theta$, diff w.r. to θ

$\therefore \cancel{2}r\frac{dr}{d\theta} = a^2(-\cancel{2}\sin 2\theta)$

$= a^2 \sin 2\theta$

$$S = a\int_0^{\pi/4} \frac{1}{\sqrt{\cos 2\theta}}\, d\theta$$

put $t = 2\theta$, $dt = 2d\theta$

θ	0	$\pi/4$
t	0	$\pi/2$

$\pi/2$ ... $\pi/2$ ... $-1/2$

$\therefore \cancel{2} r\frac{dr}{d\theta} = a^2(-\cancel{2}\sin 2\theta)$

$$\frac{dr}{d\theta} = \frac{-a^{\cancel{2}}\sin 2\theta}{\cancel{a}\sqrt{\cos 2\theta}}$$

$$\therefore S = \int_0^{\pi/4}\sqrt{a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta}}\,d\theta$$

$$= \int_0^{\pi/4}\frac{\cancel{\sqrt{a^2(\cos^2 2\theta + \sin^2 2\theta)}}}{\sqrt{\cos 2\theta}}\,d\theta$$

put $t = 2\theta$, $dt = 2\,d\theta$ ...

$$S = \frac{a}{2}\int_0^{\pi/2}\frac{1}{\sqrt{\cos t}}\,dt = \frac{a}{2}\int_0^{\pi/2}\sin^0 t\cos^{-1/2} t\,dt$$

$\boxed{\frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)}$

$$= \frac{a}{2}\cdot\frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{4}\right)$$

$\boxed{\Gamma\tfrac{1}{4}\,\Gamma\tfrac{3}{4} = \sqrt{2}\pi}$

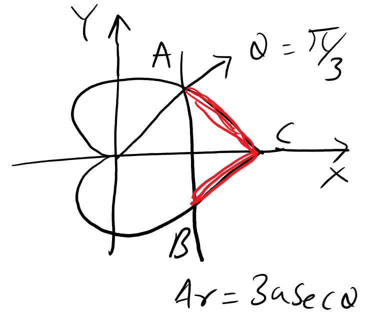
$$S = \frac{a}{4}\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}} = \frac{a}{2}\frac{\sqrt{\pi}\,\Gamma\frac{1}{4}}{\left(\frac{\sqrt{2}\pi}{\Gamma\frac{1}{4}}\right)}$$

$$\boxed{S = \frac{a}{4\sqrt{2\pi}}\left(\Gamma\tfrac{1}{4}\right)^2}$$

4) Find length of arc of parabola $r = \frac{6}{1+\cos\theta}$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$

5) Show that the length of arc of that part of cardiode $r = a(1+\cos\theta)$ which lies on the side of the line $4r = 3a\sec\theta$, away from pole is $4a$

Sol^n: The cardiode is as shown in fig



Now consider $4r = 3a\sec\theta$

$4r\cos\theta = 3a$ $\quad (\because x = r\cos\theta)$

$\therefore 4x = 3a \Rightarrow x = \frac{3a}{4}$, This is line parallel to y axis

Now at the point of intersection A

$$a(1+\cos\theta) = \frac{3a}{4}\sec\theta$$

i.e. $4\cancel{a}(1+\cos\theta)\cos\theta = 3\cancel{a}$

$\therefore 4\cos^2\theta + 4\cos\theta - 3 = 0$

[Quadratic eq^n in $\cos\theta$]

solving, we get $\cos\theta = -\frac{3}{2}, \frac{1}{2}$

Since, Point of intersection A is in first quadrant $\cos\theta = -\frac{3}{2}$ (negative)

we want to find length away from pole (origin)

as highlighted in figure arc ACB

$\therefore$ req length $(S) = 2[\text{Arc } AC]$

$$= 2\int_0^{\pi/3}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$\therefore S = 2\int_0^{\pi/3}\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

(as solved before)

$$= 2a\int_0^{\pi/3} 2\cos(\theta/2)\,d\theta$$

$$= 4a\left[2\sin\theta/2\right]_0^{\pi/3}$$

Since, for ...
first quadrant $\cos\theta = -\frac{3}{2}$ (negative)
is impossible
$\therefore \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$$= 4a \left[2 \sin \theta/2 \right]_0^{\pi/3}$$

$$S = 4a \left[\cancel{2}\left(\frac{1}{\cancel{2}}\right) - 0 \right] = \boxed{4a}$$

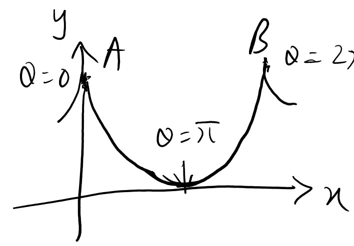
Parametric form

Friday, May 07, 2021 4:47 PM

If length of arc of a curve is given in parametric form as $x = f_1(t)$ & $y = f_2(t)$ Then

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Problems: 1) Find the length of one arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1+\cos\theta)$ (from one cusp to another cusp)



Sol^n: As shown in fig, the curve is at A for $\theta = 0$ & it is at B for $\theta = 2\pi$

Now $\frac{dx}{d\theta} = a(1-\cos\theta)$, $\frac{dy}{d\theta} = -a\sin\theta$

Then required length of curve is

$$S = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{a^2(1-\cos\theta)^2 + (a\sin\theta)^2}\, d\theta$$

$$= a\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$S = a\int_0^{2\pi} \sqrt{2(2\sin^2\theta/2)}\, d\theta$$

$$S = 2a\int_0^{2\pi} \sin(\theta/2)\, d\theta$$

$$= 2a\left[-2\cos(\theta/2)\right]_0^{2\pi}$$

$$= 2a[2+2]$$

$$\boxed{S = 8a}$$

H.W. 2) Prove that the length of the arc of the curve $x = a\sin 2\theta(1+\cos 2\theta)$, $y = a\cos 2\theta(1-\cos 2\theta)$ measured from origin to (x, y) is $\frac{4}{3}a\sin 3\theta$

Check $\frac{dx}{d\theta} = 2a\cos 2\theta + 2a\cos 4\theta$ | $\frac{dy}{d\theta} = -2a\sin 2\theta + 2a\sin 4\theta$

$$S = \int_0^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

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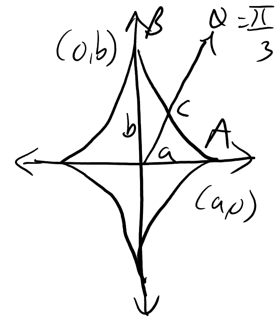
(3) ... length of curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$, $b > a$

3) (a) Find the total length of curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$, $b>a$

(b) Hence deduce the total length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ ✓

(c) Also show that the line $\theta = \frac{\pi}{3}$ divides the length of astroid $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant in the ratio 1 : 3



Sol^n: The curve is astroid as shown in figure

a) for this curve we can use the parametric equations, $x = a\cos^3\theta$, $y = b\sin^3\theta$

For total length, $s = 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$$\therefore s = 4\int_0^{\pi/2} \sqrt{[-3a\cos^2\theta \sin\theta]^2 + [3b\sin^2\theta\cos\theta]^2}\, d\theta$$

$$s = 4\int_0^{\pi/2} 3\sqrt{[a^2\cos^2\theta(\cos^2\theta\sin^2\theta)] + [b^2\sin^2\theta(\cos^2\theta\sin^2\theta)]}\, d\theta$$

$$\rightarrow s = 4\int_0^{\pi/2} 3\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,(\cos\theta\sin\theta)\, d\theta \quad \text{--- (1)}$$

To find this integral we put

$\rightarrow a^2\cos^2\theta + b^2\sin^2\theta = t^2$

$[-a^2\cancel{2}\cos\theta\sin\theta + b^2\cancel{2}\sin\theta\cos\theta]d\theta = \cancel{2}t\,dt$

$(b^2 - a^2)\cos\theta\sin\theta\, d\theta = t\,dt$

$$\therefore \boxed{\cos\theta\sin\theta\, d\theta = \frac{t}{(b^2-a^2)}\, dt}$$

θ	0	$\pi/2$
t	a	b

$$s = 12\int_a^b t\left[\frac{t}{b^2-a^2}\right] dt$$

$$= \frac{\cancel{12}\,4}{(b^2-a^2)}\left[\frac{t^3}{\cancel{3}}\right]_a^b$$

$$= \frac{4}{(b^2-a^2)}(b^3 - a^3)$$

$$\boxed{s = 4\frac{(b^2+ab+a^2)}{(b+a)}} \quad \text{--- (2)}$$

(b) for deduction put b = a

Hence total length of astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is

$$S_1 = \cancel{4}\,2\left[\frac{3a^2}{\cancel{2a}}\right] = 6a$$

c) Since length of the given astroid in First quadrant is

$6a - 3a = l[\text{arc } AB]$

c) Since length of ... $\frac{6a}{4} = \frac{3}{2}a = \underline{l[\text{arc } AB]}$

To find the length cut off by $\theta = \frac{\pi}{3}$

$$l[\text{arc } AC] = \int_0^{\pi/3} 3a \sin\theta \cos\theta \, d\theta \quad - [\text{from } (1) \text{ putting } b = a]$$

$$l[\text{arc } AC] = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/3} = \frac{3a}{2}\left[\frac{3}{4}\right] = \frac{9a}{8} ✓$$

$$l[\text{arc } BC] = \frac{3a}{2} - \frac{9a}{8} = \frac{3a}{8} ✓$$

& observe that $\dfrac{l[\text{arc } BC]}{l(\text{arc } AC]} = \dfrac{3a/8}{9a/8} = \dfrac{1}{3}$

$\rightarrow$ line $\theta = \frac{\pi}{3}$ ~~~~~ in ratio 1 : 3

HW 4) Show that <u>length</u> of tactrix, $x = a[\cos t + \log \tan(t/2)]$, $y = a \sin t$ from $t = \frac{\pi}{2}$ to any point t is, $\underline{a \log(\sin t)}$

Explanation for figure(curves)

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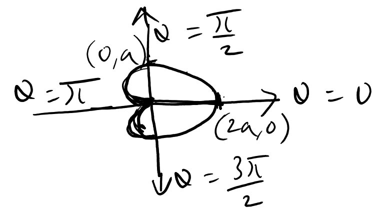
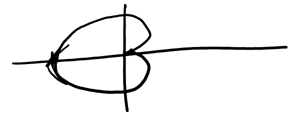
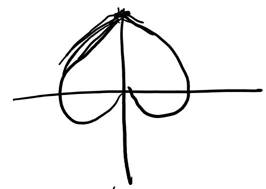
1) $r = a(1+\cos\theta)$ This is cardiode



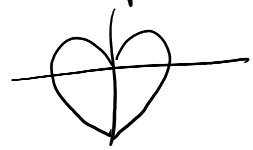
$r = a(1-\cos\theta)$



$r = a(1+\sin\theta)$



$r = a(1-\sin\theta)$



2) $x = r\cos\theta$

$\boxed{\frac{x}{r} = \cos\theta}$

$\boxed{r = a\cos\theta}$

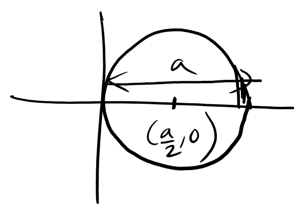
$\frac{r}{\cos\theta} = a$

$r\left(\frac{r}{x}\right) = a \quad \therefore r^2 = ax$

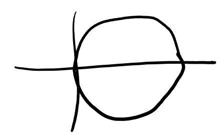
$x^2+y^2 = ax$

$(x-\frac{a}{2})^2 \quad (x^2 - ax + \frac{a^2}{4}) + y^2 = \frac{a^2}{4}$

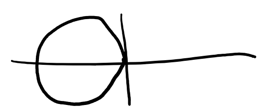
Center $\left(\frac{a}{2}, 0\right)$, radius $\frac{a}{2}$



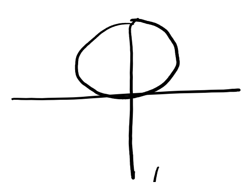
$r = a\cos\theta$

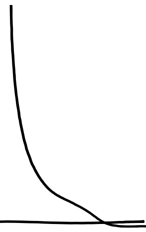


$r = -a\cos\theta$

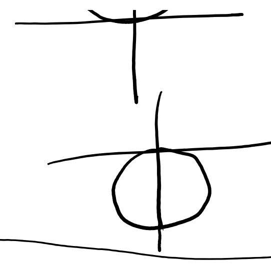


$r = a\sin\theta$

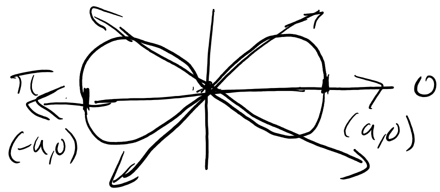




$r = -a\sin\theta$



3) $r^2 = a^2 \cos 2\theta$

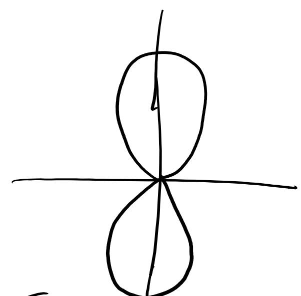


at $\theta = 0$, $r^2 = a^2 \Rightarrow r = \pm a$

at $\theta = \pi$, $r^2 = a^2 \Rightarrow \underline{r = a}$

at $\theta = \frac{\pi}{2}$, $r^2 = -a^2 \Rightarrow \underline{r \text{ is img}}$

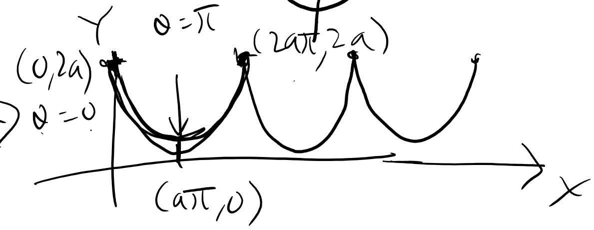
at $\theta = \frac{\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4} \Rightarrow r^2 = 0 \Rightarrow \underline{r = 0}$

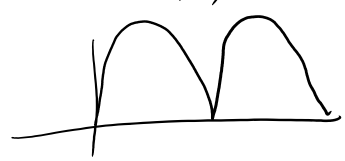


$r^2 = a^2 \sin 2\theta$

4) $x = a(\theta - \sin\theta)$

$y = \underline{a(1+\cos\theta)}$





at $\underline{\theta = 0}$, $x = 0$

$y = 2a$

at $\underline{\theta = \pi}$, $x = \underline{a\pi}$

$y = 0$

$\underline{\theta = 2\pi}$, $x = 2\pi a$

$y = 2a$

$$\left[\begin{array}{l} x = a(\theta + \sin\theta) \\ y = a(1 - \cos\theta) \end{array}\right]$$