Beta function properties

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Def^n: The function of m & n $(m, n > 0)$ defined by the integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is called Beta function.

It is denoted by $\beta(m,n)$, $\therefore$ $\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx}$

$$\boxed{\int_0^1 x^m (1-x)^n\,dx = \beta(m+1, n+1)}$$

Properties of Beta function

1) $\beta(m,n) = \beta(n,m)$

2) The relation bet^n Beta & Gamma functions.

$$\boxed{\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}}$$

3) Second form of Beta f^n

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta\,d\theta$$

Rearranging,

$$\boxed{\int_0^{\pi/2} \sin^p\theta \cos^q\theta\,d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$

$(2m-1 = p)$ ↳ def^n of Beta f^n

4) $$\boxed{\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)}$$

↳ def^n (form) of Beta f^n

$$\left(\right) \int_0^\infty \frac{x^m}{(1+x)^n}\,dx = \beta(m+1, n-m-1)$$

Short explanat^n:

1) $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$t = 1-x \quad \therefore dt = -dx$

x	0	1
t	1	0

$= \int_1^0 (1-t)^{m-1} t^{n-1}(-dt) = \beta(n,m)$

3) $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$x = \sin^2\theta$, $1-x = 1-\sin^2\theta = \cos^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

x	0	1
θ	0	$\pi/2$

$I = \int_0^{\pi/2} \sin^{2(m-1)}\theta \cos^{2n-2}\theta\; 2\sin\theta\cos\theta\,d\theta$

$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$

$x = \tan^2\theta$

	0	∞
	θ	$\pi/2$

$dx = 2\tan\theta\sec^2\theta\,d\theta$

$I = \int_0^{\pi/2} \frac{(\tan^2\theta)^m}{(\sec^2\theta)^n}\, 2\tan\theta\sec^2\theta\,dx$

$n-1$

$n+1$ | x t

$\overset{\circ}{0}\ (1+x)$ ✓

5) $$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx$$

put $\boxed{\dfrac{x}{a+bx} = \dfrac{t}{a+b}}$

→ evaluation purpose:

6) $\Gamma p\,\Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$, $0<p<1$ ✓

(proof afterwards)

examples: 1) $\Gamma\frac{3}{4}\,\Gamma\frac{1}{4} = \Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right) = \dfrac{\pi}{\sin\left(\frac{1}{4}\right)\pi} = \dfrac{\pi}{\frac{1}{\sqrt{2}}} = \boxed{\sqrt{2}\,\pi}$

2) $\Gamma\frac{1}{3}\,\Gamma\frac{2}{3} = \Gamma\frac{1}{3}\,\Gamma\left(1-\frac{1}{3}\right) = \dfrac{\pi}{\sin\left(\frac{1}{3}\right)\pi} = \dfrac{\pi}{\frac{\sqrt{3}}{2}} = \dfrac{2\pi}{\sqrt{3}}$

3) $\Gamma\frac{1}{2}\,\Gamma\left(1-\frac{1}{2}\right) = \dfrac{\pi}{\sin\left(\frac{1}{2}\right)\pi} = \dfrac{\pi}{1} \Rightarrow \Gamma\frac{1}{2}\,\Gamma\frac{1}{2} = \pi$

$\Rightarrow \boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$

7) Duplication formula of Gamma function

$$\boxed{2^{2m-1}\,\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma 2m}$$

examples: 1) $\Gamma\frac{3}{4}\,\Gamma\frac{5}{4}$, if $m=\frac{3}{4}$, $m+\frac{1}{2} = \frac{3}{4}+\frac{1}{2} = \frac{5}{4}$

$2m = \frac{3}{2}$, $2m-1 = \frac{1}{2}$

$\therefore \Gamma\frac{3}{4}\,\Gamma\frac{5}{4} = \dfrac{1}{2^{2m-1}}\sqrt{\pi}\,\Gamma 2m = \dfrac{1}{\sqrt{2}}\sqrt{\pi}\,\Gamma\frac{3}{2}$

$= \sqrt{\dfrac{\pi}{2}}\,\frac{1}{2}\,\Gamma\frac{1}{2} = \boxed{\dfrac{\pi}{2\sqrt{2}}}$

2) $\Gamma\frac{5}{4}\,\Gamma\frac{7}{4} = \Gamma\frac{5}{4}\,\Gamma\left(\frac{5}{4}+\frac{1}{2}\right)$ $\quad m=\frac{5}{4}$, $2m=\frac{5}{2}$, $2m-1=\frac{3}{2}$

$\Gamma\frac{5}{4}\,\Gamma\frac{7}{4} = \dfrac{1}{2^{2m-1}}\sqrt{\pi}\,\Gamma 2m = \dfrac{1}{2^{3/2}}\sqrt{\pi}\,\Gamma\frac{5}{2} = \dfrac{\sqrt{\pi}}{2\sqrt{2}}\,\frac{3}{2}\,\frac{1}{2}\,\Gamma\frac{1}{2}$

$\boxed{\ 3\pi\ }$

$$\Gamma\tfrac{5}{4}\,\Gamma\tfrac{7}{4} = \frac{1}{2^{2m-1}}\sqrt{\pi}\,\Gamma 2m = \frac{}{2\sqrt{2}}$$

$$= \boxed{\frac{3\pi}{8\sqrt{2}}}$$