Type I

$$\int_0^1 x^m (1-x^n)^p \, dx$$

$$\int_0^a x^m (a^n - x^n)^p \, dx$$

$$\int_0^a x^m (a-x)^p \, dx$$

$$\int_0^{1/a} x^m (1-(ax)^n)^p \, dx$$

Try a substitution t such that inside bracket we will get $(1-t)$ term & limit will be converted to 0 to 1

explanatⁿ: $$I = \int_0^a x^m (a^n - x^n)^p \, dx$$

x	0	a
t	0	1

put $x^n = a^n t$

$x = a t^{1/n}$, $dx = a \frac{1}{n} t^{\frac{1}{n}-1} dt$

$$I = \int_0^1 (a t^{1/n})^m (a^n - a^n t)^p \frac{a}{n} t^{\frac{1}{n}-1} dt$$

$$= \int_0^1 a^m t^{\frac{m}{n}} a^{np} (1-t)^p \frac{a}{n} t^{\frac{1-n}{n}} dt$$

$$= \frac{a^{m+np+1}}{n} \int_0^1 t^{\frac{m+1-n}{n}} (1-t)^p dt$$

$$\frac{a^{m+np+1}}{n} \beta\left(\frac{m-n+1}{n}+1, p+1\right)$$

Problems: 1) Evaluate $\int_0^9 x^{3/2}(9-x)^{1/2} dx$

Solⁿ: Let $I = \int_0^9 x^{3/2}(9-x)^{1/2} dx$

x	0	9
t	0	1

put $x = 9t$

$dx = 9\,dt$

substitute, $I = \int_0^1 (9t)^{3/2} (9-9t)^{1/2}\, 9\, dt$

$$I = \int_0^1 3^3 t^{3/2} 9^{1/2} (1-t)^{1/2}\, 9\, dt$$

$$= 9^3 \int_0^1 t^{3/2} (1-t)^{1/2} dt$$

$$I = 9^3 \beta\left(\frac{3}{2}+1, \frac{1}{2}+1\right)$$

(Using $\int_0^1 x^m (1-x)^n dx = \beta(m+1, n+1)$)

$$= 9^3 \beta\left(\frac{5}{2}, \frac{3}{2}\right)$$

(Using formula $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$)

$$I = 9^3 \frac{\Gamma\frac{5}{2}\,\Gamma\frac{3}{2}}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)}$$

($\Gamma n+1 = n\Gamma n$; $\Gamma n+1 = n!$, n positive integer; $\Gamma\frac{1}{2} = \sqrt{\pi}$)

$$= 9^3 \frac{\left(\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}\right)\left(\frac{1}{2}\Gamma\frac{1}{2}\right)}{\Gamma 4}$$

$$= 9^3 \left(\frac{\cancel{3}}{2}\frac{1}{2}\frac{1}{2}\pi\right)$$

$$= \frac{9^3\left(\frac{\cancel{3}}{2}\frac{1}{2}\frac{1}{2}\pi\right)}{\cancel{3}\times 2\times 1}$$

$$I = \frac{729}{16}\pi$$

2) Evaluate $\int_0^2 x\sqrt[3]{8-x^3}\,dx$

Let $I = \int_0^2 \underline{x}(8-\underline{x^3})^{1/3}\,dx$

put $x^3 = 8t$

x	0	2
t	0	1

$x = \underline{2t^{1/3}}$, $dx = 2\frac{1}{3}t^{-2/3}\,dt$

Substitute $I = \int_0^1 (2t^{1/3})(8-8t)^{1/3}\cdot \frac{2}{3}t^{-2/3}\,dt$

$$I = \int_0^1 2\,t^{1/3}\left(8(1-t)\right)^{1/3}\frac{2}{3}t^{-2/3}\,dt$$

$$= \frac{8}{3}\int_0^1 t^{-1/3}(1-t)^{1/3}\,dt$$

$$I = \frac{8}{3}\beta\left(\frac{2}{3},\frac{4}{3}\right)$$

(write Formula)

$$= \frac{8}{3}\frac{\Gamma\frac{2}{3}\,\Gamma\frac{4}{3}}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)} \rightarrow \Gamma\frac{6}{3} \quad \Gamma 2 = 1!$$

$$\boxed{I = \frac{8}{3}\Gamma\frac{2}{3}\,\Gamma\frac{4}{3}}$$

3) Let $I = \int_0^1 (1-\sqrt[5]{x})^{3/2}\,dx$

put $x^{1/5} = t$

x	0	1
t	0	1

$x = t^5$, $dx = 5t^4\,dt$

$$I = \int_0^1 (1-t)^{3/2}\,5t^4\,dt$$

$$= 5\int_0^1 t^4(1-t)^{3/2}\,dt$$

$$I = 5\,\beta\left(5,\frac{5}{2}\right)$$

$$I = 5\frac{\Gamma 5\,\Gamma\frac{5}{2}}{\Gamma\left(5+\frac{5}{2}\right)}$$

$$I = 5\frac{4\times 3\times 2\times 1\;\Gamma\frac{5}{2}}{\Gamma\frac{15}{2}}$$

$$= \frac{\cancel{5}\times 4\times \cancel{3}\times 2\times 1\;\cancel{\Gamma\frac{5}{2}}}{\left(\frac{13}{2}\right)\left(\frac{11}{2}\right)\left(\frac{\cancel{9}^{3}}{2}\right)\left(\frac{7}{2}\right)\left(\frac{\cancel{5}}{2}\right)\cancel{\Gamma\frac{5}{2}}}$$

$$= \frac{2^8}{13\times 11\times 3\times 7} = \frac{256}{3003}$$

4) $I = \int_0^1 \frac{x^2(4-x^2)}{\sqrt{1-x^2}}\,dx = 4\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx - \int_0^1 \frac{x^4}{\sqrt{1-x^2}}\,dx$

4) $I = \int_0^1 \frac{x^2(4-x^2)}{\sqrt{1-x^2}}\,dx = 4\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx - \int_0^1 \frac{x^4}{\sqrt{1-x^2}}\,dx$

$= 4\int_0^1 x^2(1-x^2)^{-1/2}\,dx - \int_0^1 x^4(1-x^2)^{-1/2}\,dx$

$\quad I_1 - I_2$

Put $x^2 = t$, $dx = \frac{1}{2\sqrt{t}}\,dt$

x	0	1
t	0	1

$\therefore I_1 = \cancel{4}^{2}\int_0^1 t(1-t)^{-1/2}\,\frac{1}{\cancel{2}}t^{-1/2}\,dt$

$= 2\int_0^1 t^{1/2}(1-t)^{-1/2}\,dt$

$I_1 = 2\beta\left(\frac{3}{2},\frac{1}{2}\right)$

$I_2 = \int_0^1 t^2(1-t)^{-1/2}\left(\frac{1}{2}t^{-1/2}\,dt\right) = \frac{1}{2}\int_0^1 t^{3/2}(1-t)^{-1/2}\,dt$

$I_2 = \frac{1}{2}\beta\left(\frac{5}{2},\frac{1}{2}\right)$

$I = I_1 - I_2 = 2\beta\left(\frac{3}{2},\frac{1}{2}\right) - \frac{1}{2}\beta\left(\frac{5}{2},\frac{1}{2}\right)$

$= 2\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma 2} - \frac{1}{2}\frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma 3} = \cancel{2}\,\frac{1}{\cancel{2}}\left(\Gamma\frac{1}{2}\right)^2 - \frac{1}{2}\,\frac{\frac{3}{\cancel{2}}\,\frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2}{2}$

$= \pi - \frac{3}{16}\pi = \frac{13}{16}\pi$

Dell at 4/12/2021 12:28 PM

H.W 5) $\int_0^{1/2} x^3\sqrt{1-4x^2}\,dx \cdot \int_0^{1/2}\sqrt{2y-4y^2}\,dy = \frac{\pi}{1920}$

Let $I = I_1 \times I_2$

$I = \int_0^{1/2} x^3(1-4x^2)^{1/2}\,dx \cdot \int_0^{1/2}\sqrt{2y(1-2y)}\,dy$

In $I_1 \Rightarrow 4x^2 = t$

$x = \left(\frac{t}{4}\right)^{1/3}$

x	0	1/2
t	0	1

In $I_2 \Rightarrow 2y = t$

x	0	1/2
t	0	1