Triple Integration

(A) Evaluate if limits are given:

(I) If all limits are constant (not a single integral limit is variable), then given order i.e. $dx\,dy\,dz$ is very important.

e.g. (1) If $\int_0^1 \int_0^2 \int_{-1}^3 xyz \, dx\,dy\,dz$,

in this we have to integrate w.r.to 'x' first, then w.r.to 'y' and finally w.r.to 'z' since order is $dx\,dy\,dz$

$$\therefore I = \int_0^1 \left[\int_0^2 \left(\int_{-1}^3 xyz \, dx \right) dy \right] dz$$

$$= \int_0^1 \left[\int_0^2 \left[yz \frac{x^2}{2} \right]_{-1}^3 dy \right] dz$$

$$= \int_0^1 \left(\int_0^2 yz \left[\frac{9}{2} - \frac{1}{2} \right] dy \right) dz$$

$$= 4 \int_0^1 \left(\int_0^2 yz \, dy \right) dz = 4 \int_0^1 \left[z \frac{y^2}{2} \right]_0^2 dz$$

$$= 4 \int_0^1 2z \, dz = 4 \left[\frac{2z^2}{2} \right]_0^1 = \underline{\underline{4}}$$

(2) $\int_0^1 \int_0^2 \int_0^3 (x+y+z) \, dz\,dy\,dx$

[Here order is w.r.to 'z' first, then w.r.to y & finally w.r.to x]

$$I = \int_0^1 \left[\int_0^2 \left(\int_0^3 (x+y+z) \, dz \right) dy \right] dx$$

$$= \int_0^1 \left[\int_0^2 \left(xz + yz + \frac{z^2}{2} \right)_0^3 dy \right]$$

$$= \int_0^1 \left[\int_0^2 \left(3x + 3y + \frac{9}{2} \right) dy \right] dx$$

$$I = \int_0^1 \left[3xy + \frac{3y^2}{2} + \frac{9}{2}y \right]_0^2 dx$$

$$= \int_0^1 [6x + 6 + 9]\, dx = \int_0^1 (6x+15)\, dx$$

$$= \left[\frac{6x^2}{2} + 15x \right]_0^1 = \underline{18}$$

(II) If one of the integration is a variable then also order given in the problem is ~~ati~~ important. (i.e. follow the same order)

e.g. - (1) $\displaystyle\int_0^4 \int_0^{2\sqrt{2}} \int_0^{\sqrt{4x-x^2}} y \cdot dy\, dx\, dz$

$= \int_0^4$ [Here w.r.to 'y' first then w.r.to 'x' & finally w.r.to 'z']

$$I = \int_0^4 \left[\int_0^{2\sqrt{2}} \left(\int_0^{\sqrt{4x-x^2}} y\, dy \right) dx \right] dz$$

$$= \int_0^4 \left[\int_0^{2\sqrt{2}} \left[\frac{y^2}{2} \right]_0^{\sqrt{4x-x^2}} dx \right] dz$$

$$= \frac{1}{2} \int_0^4 \left[\int_0^{2\sqrt{2}} (4x - x^2)\, dx \right] dz$$

$$= \frac{1}{2} \int_0^4 \left[\frac{4x^2}{2} - \frac{x^3}{3} \right]_0^{2\sqrt{2}} dz$$

$$= \frac{1}{2} \int_0^4 \left[2(2\sqrt{2})^2 - \frac{(2\sqrt{2})^3}{3} - 0 \right] dz$$

$$= \frac{1}{2} \int_0^4 \left(16 - \frac{16\sqrt{2}}{3} \right) dz$$

$$= \frac{1}{2} \left[\left(16 - \frac{16\sqrt{2}}{3} \right) z \right]_0^4$$

$$= \underline{2\left[16 - \frac{16\sqrt{2}}{3} \right]}$$

(III) If integrating limits are variable, then we can select order from ~~right~~ to ~~left~~ Left to right of the integral. In this from left first integrating limits are always constant then see the second integrating limits (i.e. middle one) if these are in 'x' then take first integration 'dx' then see the last (third) integrating limits if that are in y then take middle as dy and finally ~~w~~ dz.

e.g. (1) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz \, dx\, dy\, dz$

[Here from left to right, first integration limits are constant but middle one in x thus write dx after first integral. Next in third integration limits are in x & y i.e. $(1-x-y)$ thus take middle as dy (already dx selected) & finally w.r.to 'z'.]

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} xyz \, dz$$

$$= \int_0^1 dx \int_0^{1-x} dy \left[xy \frac{z^2}{2} \right]_0^{1-x-y} = \frac{1}{2} \int_0^1 dx \int_0^{1-x} xy (1-x-y)^2 \, dy$$

$$= \frac{1}{2} \int_0^1 dx \int_0^{1-x} xy \left[(1-x) - y \right]^2 dy$$

$$= \frac{1}{2} \int_0^1 dx \int_0^{1-x} xy \left[(1-x)^2 - 2(1-x)y + y^2 \right] dy$$

$$= \frac{1}{2} \int_0^1 dx \int_0^{1-x} x \left[(1-x)^2 \cdot y - 2(1-x) y^2 + y^3 \right] dy$$

$$= \frac{1}{2} \int_0^1 dx \left[x \left[(1-x)^2 \cdot \frac{y^2}{2} - 2(1-x)\frac{y^3}{3} + \frac{y^4}{4} \right] \right]_0^{1-x}$$

$$= \frac{1}{2} \int_0^1 dx \; x \left[\frac{(1-x)^4}{2} - \frac{2}{3}(1-x)^4 + \frac{(1-x)^4}{4} \right] - 0$$

$$= \frac{1}{2} \int_0^1 dx \; x \left[\frac{(1-x)^4}{12} \right] dx \quad \text{(Apply by parts}$$

$$= \frac{1}{24} \left[x \frac{(1-x)^5}{-5} - \int \frac{(1-x)^5}{-5} dx \right]_0^1 = \frac{1}{720}$$

Q2) $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dx\, dy\, dz$ (March)

[Here, middle integration limits are in y, thus we take 'dy' for final integral, the third integral is in 'x' then take middle as 'dx' & finally 'dz']
From left to write

$$I = \int_0^1 dy \int_{y^2}^1 dx \int_0^{1-x} x\, dz$$

$$= \int_0^1 dy \int_{y^2}^1 dx \left[x z \right]_0^{1-x} = \int_0^1 dy \int_{y^2}^1 dx \left[x(1-x) - 0 \right]$$

$$= \int_0^1 dy \int_{y^2}^1 (x - x^2)\, dx = \int_0^1 dy \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{y^2}^1$$

$$= \int_0^1 \left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{y^4}{2} - \frac{y^6}{3} \right) \right] dy$$

$$= \int_0^1 \left[\frac{1}{6} - \frac{y^4}{2} - \frac{y^6}{3} \right] dy = \left[\frac{y}{6} - \frac{y^5}{10} - \frac{y^7}{21} \right]_0^1$$

$$= \left[\frac{1}{6} - \frac{1}{10} + \frac{1}{21} \right] = \frac{4}{35} \,//.$$

(3) $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z - x^2}} dx\, dy\, dz$

$$I = \int_0^4 dz \int_0^{2\sqrt{z}} dx \int_0^{\sqrt{4z-x^2}} dy$$ [See the limits from left to write of integration]

$$= \int_0^4 dz \int_0^{2\sqrt{z}} dx \left[y \right]_0^{\sqrt{4z-x^2}} = \int_0^4 dz \int_0^{2\sqrt{z}} \sqrt{4z - x^2}\, dx$$

$$I = \int_0^4 dz \left[\frac{x}{2}\sqrt{4z - x^2} + \frac{4z}{2} \sin^{-1}\left(\frac{x}{\sqrt{4z}}\right) \right]_0^{2\sqrt{z}}$$

[using formula $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a}$]

$$I = \int_0^4 dz \left[0 + 2z \sin^{-1}(1) - 0 \right]$$

$$= \int_0^4 2 \frac{\pi}{2} z\, dz = \pi \left[\frac{z^2}{2} \right]_0^4 = 8\pi$$

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4) Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

[Here already given the order as dy dx dz i.e. First integrate w.r.to 'y', then 'x' & finally w.r.to 'z']

$$I=\int_{-1}^{1}dz\int_{0}^{z}dx\int_{x-z}^{x+z}(x+y+z)\,dy$$

$$=\int_{-1}^{1}dz\int_{0}^{z}dx\left[(x+z)y+\frac{y^2}{2}\right]_{x-z}^{x+z}$$

$$=\int_{-1}^{1}dz\int_{0}^{z}dx\left[(x+z)^2+\frac{(x+z)^2}{2}-(x+z)(x-z)-\frac{(x-z)^2}{2}\right]$$

$$=\int_{-1}^{1}dz\int_{0}^{z}\left[\frac{3}{2}(x+z)^2-(x^2-z^2)-\frac{(x-z)^2}{2}\right]dx$$

$$=\int_{-1}^{1}dz\left[\frac{3}{2}\frac{(x+z)^3}{3}-\frac{x^3}{3}-xz^2-\frac{(x-z)^3}{6}\right]_0^z$$

$$=\int_{-1}^{1}dz\left[\left(\frac{1}{2}(2z)^3-\frac{z^3}{3}-z^3-\frac{0}{6}\right)-\left(\frac{1}{2}z^3-0-0-\frac{(-z)^3}{6}\right)\right]$$

$$=\int_{-1}^{1}\left(4z^3-\frac{4}{3}z^3-\frac{1}{2}z^3+\frac{z^3}{6}\right)dz$$

$=0$ [$\because$ the integrating function is odd]

5) $\int_{0}^{\frac{\pi}{2}}\int_{0}^{a\cos\theta}\int_{0}^{\sqrt{a^2-r^2}} r\,dr\,d\theta\,dz$

[Here from left to right middle integral limit is in 'θ', thus first integral take $d\theta$ then third integral is in 'r' take second integral 'dr' finally last integral 'dz']

$$I=\int_{0}^{\frac{\pi}{2}}d\theta\int_{0}^{a\cos\theta}dr\int_{0}^{\sqrt{a^2-r^2}} r\,dz$$

$$=\int_{0}^{\frac{\pi}{2}}d\theta\int_{0}^{a\cos\theta}dr\,\Big[r\cdot z\Big]_{0}^{\sqrt{a^2-r^2}}$$

$$\therefore I = \int_0^{\frac{\pi}{2}} d\theta \int_0^{a\cos\theta} x\sqrt{a^2-x^2}\, dx$$

Using $\int f'(x)\, f(x)^n dx = \frac{f(x)^{n+1}}{n+1}$

$$= \int_0^{\frac{\pi}{2}} d\theta \left(\frac{-1}{2}\right) \int_0^{a\cos\theta} -2x\,(a^2-x^2)^{\frac{1}{2}} dx$$

$$= \frac{-1}{2} \int_0^{\frac{\pi}{2}} d\theta \left[\frac{2}{3}(a^2-x^2)^{3/2}\right]_0^{a\cos\theta}$$

$$= \frac{-1}{3} \int_0^{\frac{\pi}{2}} \left[(a^2-a^2\cos^2\theta)^{\frac{3}{2}} - (a^2)^{\frac{3}{2}}\right] d\theta$$

$$= -\frac{a^3}{3} \int_0^{\frac{\pi}{2}} (\sin^3\theta - 1)\, d\theta$$

$$= -\frac{a^3}{3} \left[\int_0^{\frac{\pi}{2}} \sin^3\theta\, (\cos\theta)^0 d\theta - \int_0^{\frac{\pi}{2}} d\theta \right]$$

$$= -\frac{a^3}{3} \left[\frac{1}{2} B\left(2, \frac{1}{2}\right) - [\theta]_0^{\frac{\pi}{2}} \right]$$

$$= -\frac{a^3}{3} \left[\frac{1}{2} \frac{\Gamma 2\, \Gamma\frac{1}{2}}{\Gamma\frac{5}{2}} - \frac{\pi}{2} \right] = -\frac{a^3}{3} \left[\frac{1}{2} \frac{\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} - \frac{\pi}{2} \right]$$

$$= -\frac{a^3}{3} \left[\frac{2}{3} - \frac{\pi}{2} \right]$$

$$= \frac{a^3}{18} [3\pi - 4]$$

Page (7)

$$\therefore I = \int_0^{\frac{\pi}{2}} d\theta \int_0^{a\cos\theta} x\sqrt{a^2-x^2}\, dx$$

$$= \int_0^{\frac{\pi}{2}} d\theta \left(\frac{-1}{2}\right) \int_0^{a\cos\theta} -2x\,(a^2-x^2)^{\frac{1}{2}}\, dx$$

using $\int f'(x)\, f(x)^n\, dx = \frac{f(x)^{n+1}}{n+1}$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta \left[\frac{2}{3}(a^2-x^2)^{3/2}\right]_0^{a\cos\theta}$$

$$= \frac{-1}{3} \int_0^{\frac{\pi}{2}} \left[(a^2-a^2\cos^2\theta)^{\frac{3}{2}} - (a^2)^{\frac{3}{2}}\right] d\theta$$

$$= -\frac{a^3}{3} \int_0^{\frac{\pi}{2}} (\sin^3\theta - 1)\, d\theta$$

$$= -\frac{a^3}{3} \left[\int_0^{\frac{\pi}{2}} \sin^3\theta\, (\cos\theta)^0\, d\theta - \int_0^{\frac{\pi}{2}} d\theta\right]$$

$$= -\frac{a^3}{3} \left[\frac{1}{2} B\left(2, \frac{1}{2}\right) - [\theta]_0^{\frac{\pi}{2}}\right]$$

$$= -\frac{a^3}{3} \left[\frac{1}{2} \frac{\Gamma 2\, \Gamma\frac{1}{2}}{\Gamma\frac{5}{2}} - \frac{\pi}{2}\right] = -\frac{a^3}{3}\left[\frac{1}{2} \frac{\sqrt{\pi}}{\frac{3}{2}\frac{1}{2}\sqrt{\pi}} - \frac{\pi}{2}\right]$$

$$= -\frac{a^3}{3}\left[\frac{2}{3} - \frac{\pi}{2}\right]$$

$$= \frac{a^3}{18}\left[3\pi - 4\right]$$

(B) Triple Integration over a given volume 'V'

(1) Evaluate $\iiint_V (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.

$\Rightarrow$ Let V be the volume enclosed by the plane $x+y+z=1$ & it meets the co-ordinate axes in $A(1,0,0)$, $B(0,1,0)$ & $C(0,0,1)$ respectively.

Here PQ be strip which is parallel to Z-axis

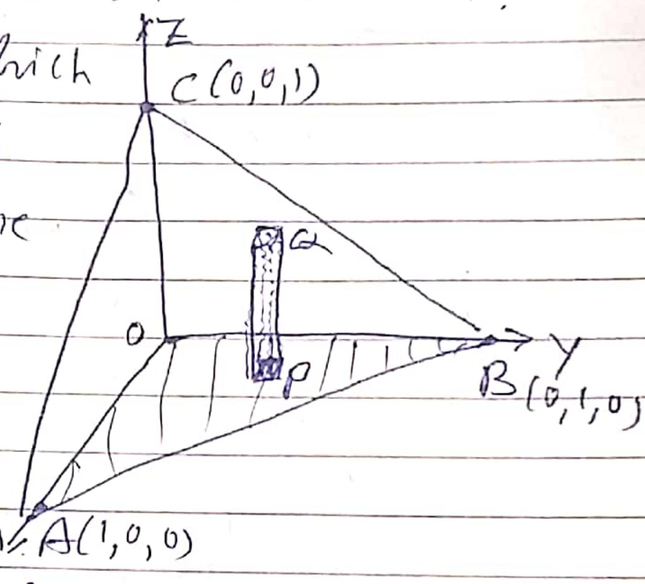
Point P is on xy-plane

i.e. $z=0$

& pt. Q is on plane

$x+y+z=1$

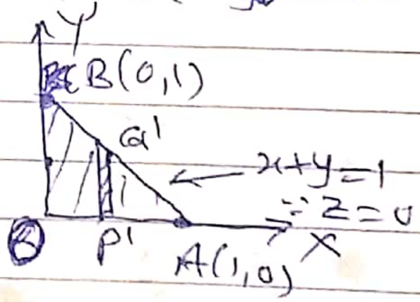
i.e. $z=1-x-y$



Thus we have limits for z as $z \to 0$ to $1-x-y$

Taking projection on xy-plane, we will get a ΔAOB:

Taking P'Q' integrating strip parallel to y-axis



$P \to y=0$, $Q \to y=1-x$.

$\therefore y \to 0$ to $(1-x)$

& $x \to 0$ to 1.

$$\therefore I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} (x+y+z)\,dz.$$

$$= \int_0^1 dx \int_0^{1-x} dy \left[\frac{[x+y+z]^2}{2} \right]_0^{1-x-y}$$

$$\therefore I = \frac{1}{2}\int_0^1 dx \int_0^{1-x} \left[1-(x+y)^2\right] dy$$

$$= \frac{1}{2}\int_0^1 dx \left[y - \frac{(x+y)^3}{3}\right]_0^{1-x}$$

$$= \frac{1}{2}\int_0^1 dx \left[(1-x) - \frac{1}{3} - 0 + \frac{x^3}{3}\right]$$

$$= \frac{1}{2}\left[x - \frac{x^2}{2} - \frac{x}{3} + \frac{x^4}{12}\right]_0^1$$

$$= \frac{1}{2}\left[1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{12}\right]$$

$$I = \frac{1}{8} //.$$

(2) Evaluate $\iiint_V \frac{dx\,dy\,dz}{(1+x+y+z)^3}$ where V is the volume bounded by the planes $x=0, y=0, z=0, x+y+z=1$.

(Here figure & limits same as problem (1) above)

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3}$$

$$= \int_0^1 dx \int_0^{1-x} dy \left[\frac{(1+x+y+z)^{-2}}{-2}\right]_0^{1-x-y}$$

$$= \int_0^1 dx \int_0^{1-x} dy \left[\frac{-1}{2(1+x+y+z)^2}\right]_0^{1-x-y}$$

$$= \int_0^1 dx \int_0^{1-x} \left(\frac{-1}{2(2)^2} + \frac{1}{2(1+x+y)^2}\right] dy$$

$$= \int_0^1 dx \left[-\frac{y}{8} + \frac{1}{2}\left[\frac{-1}{(1+x+y)}\right]\right]_0^{1-x}$$

$$= \int_0^1 dx \left[-\frac{(1-x)}{8} + \frac{1}{2}\left(\frac{-1}{2} + \frac{1}{1+x}\right)\right]$$

$$= \int_0^1 \left(-\frac{1}{8} + \frac{x}{8} - \frac{1}{4} + \frac{1}{2}\left(\frac{1}{1+x}\right)\right) dx$$

$$= \left[-\frac{x}{8} + \frac{x^2}{16} - \frac{x}{4} + \frac{1}{2}\log(1+x)\right]_0^1$$

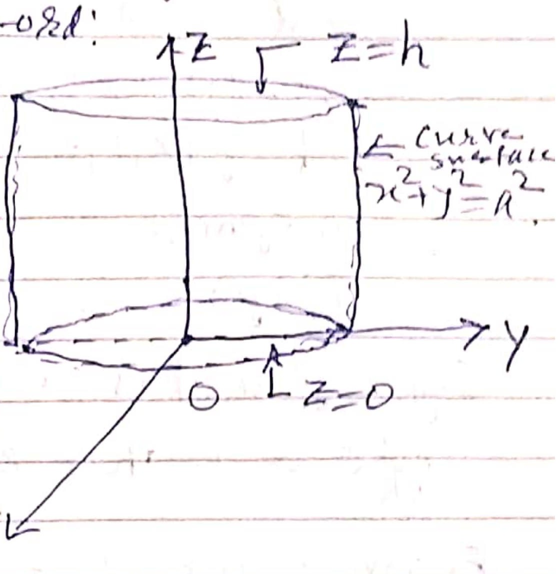
$$= \left[-\frac{1}{8} + \frac{1}{16} - \frac{1}{4} + \frac{1}{2}\log 2 - 0\right] = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right] //$$

(C) Cylindrical and Spherical polar co-ord:

(I) Cylindrical co-ord: (r, θ, z)

Consider a right circular cylinder, bounded by surfaces $x^2+y^2=a^2$, $z=0$ to $z=h$

Using cylindrical polar co-ord:



$x = r\cos\theta,\ y = r\sin\theta,\ z = z.$

$dx\,dy\,dz = |J|\,dr\,d\theta\,dz.$

$$J = J\left(\frac{x,y,z}{r,\theta,z}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

$\therefore dx\,dy\,dz = r\,dr\,d\theta\,dz$

(I) For whole cylinder limits are as

$r \to 0$ to a (radius)

$\theta \to 0$ to 2π (Take projection on xy-plane it gives a circle)

$z \to 0$ to h (height of cylinder)

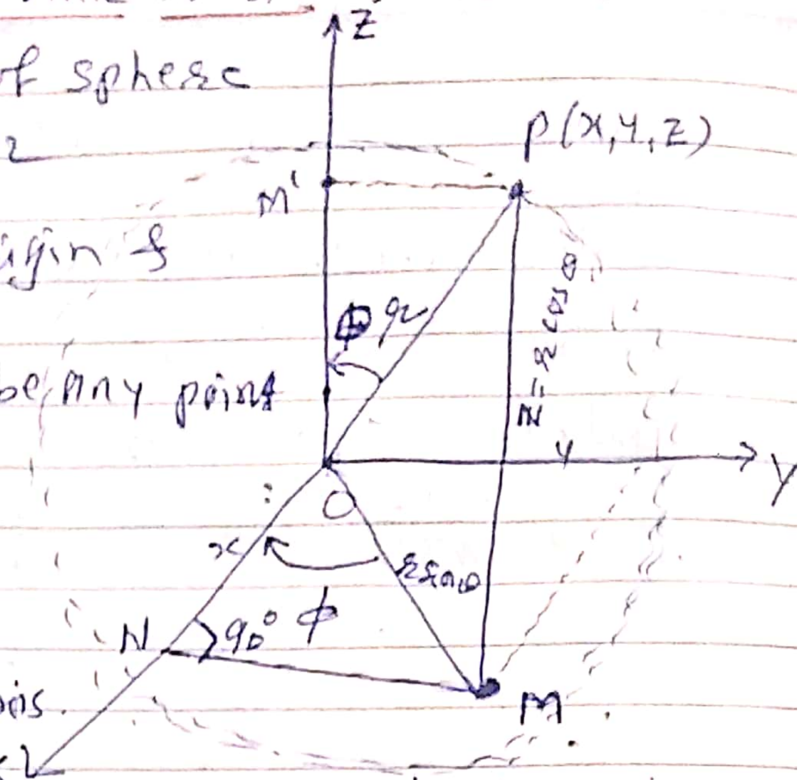
(II) Cylinder lies in first octant (it is also called positive octant. As the whole space is divided into eight parts by three-dimensional co-ord. system.)

$r \to 0$ to a

$\theta \to 0$ to $\frac{\pi}{2}$ (Proj. on xy-plane is a quadrant of a circle)

$z \to 0$ to h. $\therefore \iiint_V f(x,y,z)\,dx\,dy\,dz = \int_r \int_\theta \int_z F(r,\theta,z)\,r\,dr\,d\theta\,dz$

(II) Spherical polar co-ord : (r, θ, ϕ).



Consider eqn of sphere

$x^2+y^2+z^2 = a^2$

centre at Origin &

radius 'a'.

let $p(x,y,z)$ be any point

on sphere.

$l(OP) = r$

Seg. OP makes an

angle θ with z-axis.

From point P take PM perpendicular on xy-plane

∴ As OM' is same as PM. But from $\triangle OPM'$

we have $OM' = r\cos\theta$.

∴ $PM = r\cos\theta$.

Now consider $\triangle MON$ in xy-plane

Also $OM = r\sin\theta$ (i.e. PM')

~~OM = x~~ $\cos\phi = \frac{ON}{OM} = \frac{x}{r\sin\theta}$

∴ $x = r\sin\theta\cos\phi$.

Also $\sin\phi = \frac{MN}{OM} = \frac{y}{r\sin\theta}$

∴ $y = r\sin\theta\sin\phi$.

& $z = r\cos\theta$

The relation between cartesian &

spherical polar co-ord. are

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

& $z = r\cos\theta$.

$dx\,dy\,dz = |J|\, dr\, d\theta\, d\phi$.

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = r^2 \sin\theta$$

$\therefore dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Limits:

(I) For whole sphere (complete sphere)

$x^2+y^2+z^2=a^2$,

$r \to 0$ to a

$\theta \to 0$ to π [circle ~~rotated~~ rotation along z-axis is half rotation)

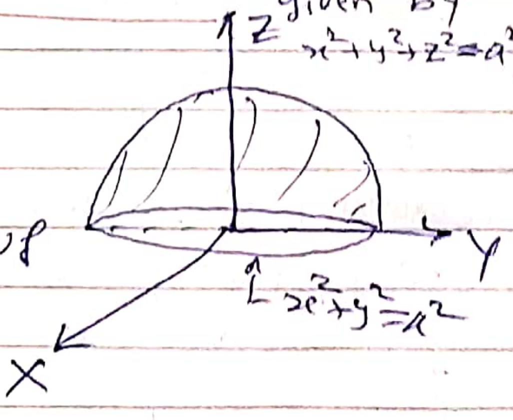
$\phi \to 0$ to 2π (This is in xy-plane & projection of sphere on xy-plane is circle)

(II) Sphere lies above xy-plane: Hemi-sphere given by $x^2+y^2+z^2=a^2, z\geq 0$

$r \to 0$ to a

$\theta \to 0$ to $\frac{\pi}{2}$

& $\phi \to 0$ to 2π (proj. of hemi-sphere on xy-plane is a circle $x^2+y^2=a^2$)



(III) Sphere lies in first octant (positive octant)

$r \to 0$ to a

$\theta \to 0$ to $\frac{\pi}{2}$

$\phi \to 0$ to $\frac{\pi}{2}$ [projection on xy-plane is a quadrant of a circle].

Type I: change to cylindrical polar co-ord: (r, θ, z)

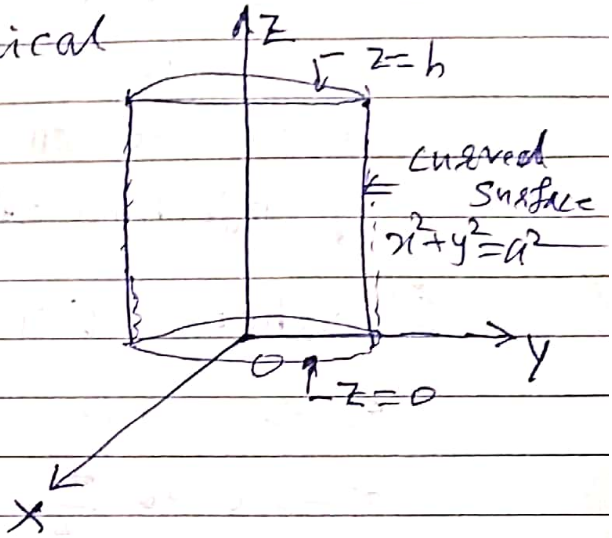
Put $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ into integrating function & convert into r, θ, z.

Use. $dx\,dy\,dz = r\,dr\,d\theta\,dz$.

Relatively ~~wri~~ assign limits for r, θ & z.

e.g. (1) Evaluate $\iiint z(x^2+y^2+z^2)\,dx\,dy\,dz$ over the volume 'v' of the cylinder $x^2+y^2=a^2$ between the planes $z=0$ & $z=h$ (positive)

$\Rightarrow$ Converting into cylindrical co-ord.



Put $x = r\cos\theta$, $y = r\sin\theta$

$z = z$.

Limits:

$r \to 0$ to a

$\theta \to 0$ to 2π

$z \to 0$ to h

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$x^2+y^2+z^2 = r^2+z^2$

$$I = \int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{h} z(r^2+z^2)\cdot r\,dz\,d\theta\,dr$$

$$= \int_0^a dr \int_0^{2\pi} d\theta \int_0^h r[r^2 z + z^3]\,dz$$

$$= \int_0^a dr \int_0^{2\pi} d\theta\; r\left[\frac{r^2z^2}{2} + \frac{z^4}{4}\right]_0^h$$

$$= \int_0^a dr \int_0^{2\pi} r\left[\frac{h^2r^2}{2} + \frac{h^4}{4}\right] = \int_0^a \left(\frac{h^2}{2}r^3 + \frac{h^4}{4}\cdot r\right)dr\,[\theta]_0^{2\pi}$$

$$= 2\pi \int_0^a \left(\frac{h^2r^3}{2} + \frac{h^4}{4}\cdot r\right)dr = \boxed{\frac{\pi a^2h^2}{4}(a^2+h^2)}$$

e.g. ② Evaluate $\iiint_V (x^2+y^2)\,dx\,dy\,dz$, V is the volume bounded by the surface $x^2+y^2=2z$ and the plane $z=2$.

$\Rightarrow$ Here the surface $x^2+y^2=2z$ is parabolic cylinder

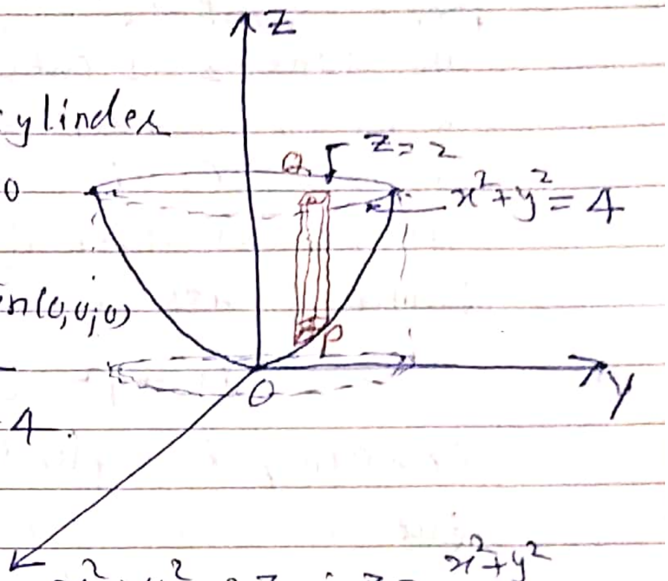
At $z=0$ (xy-plane), $x^2+y^2=0$

$\Rightarrow x=0,\ y=0$

$\therefore$ The vertex is at origin(0,0,0)

The plane $z=2$ cuts the surface in circle $x^2+y^2=4$.

Consider PQ be strip parallel to z-axis.



pt. P lies on surface $x^2+y^2=2z \therefore z=\frac{x^2+y^2}{2}$

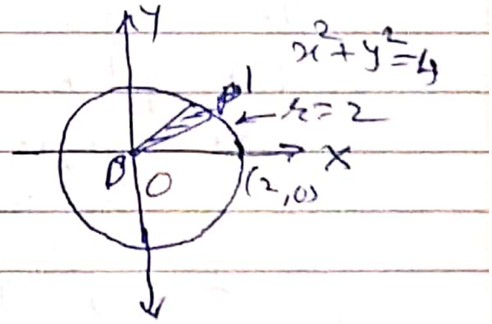
changing to cylindrical polar co-ord., we have $z=\frac{r^2}{2}$

~~point Q lies on plane z = 2~~

$\therefore z \to \frac{r^2}{2}$ to 2.

Taking projection on xy-plane, the region is circle $x^2+y^2=4$, $z=0$

OP' be integrating strip for polar co-ord.



$\therefore r \to 0$ to 2

& $\theta \to 0$ to 2π

$dx\,dy\,dz = r\,dr\,d\theta\,dz$, $x^2+y^2=r^2$ in polar.

$$\therefore I = \int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=\frac{r^2}{2}}^{2} r^2\,(r\,dz\,d\theta\,dr)$$

$$= \int_0^2 r^3\,dr \int_0^{2\pi} d\theta \int_{\frac{r^2}{2}}^{2} dz = \int_0^2 r^3 dr \int_0^{2\pi} d\theta\,[z]_{\frac{r^2}{2}}^{2}$$

$$= \int_0^2 r^3 dr \int_0^{2\pi} d\theta \cdot \left(2-\frac{r^2}{2}\right) = \int_0^2 r^3\left(2-\frac{r^2}{2}\right)dr\,[\theta]_0^{2\pi}$$

$$= \int_0^2 2\pi\left[2r^3-\frac{r^5}{2}\right]dr = 2\pi\left[\frac{2r^4}{4}-\frac{r^6}{12}\right]_0^2 = \boxed{\frac{16\pi}{3}}$$

e.g (3) Evaluate $\iiint_V \sqrt{x^2+y^2}\,dx\,dy\,dz$, where V is the volume of solid bounded by $x^2+y^2=z^2$, $z=0$ & $z=1$.

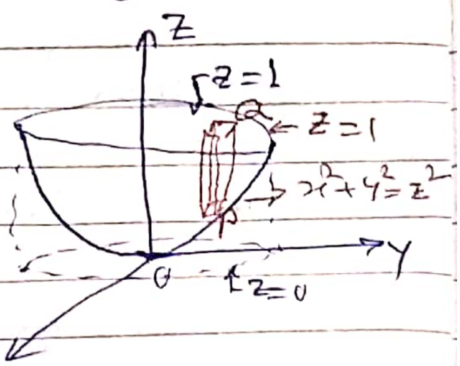
Here surface is $x^2+y^2=z^2$ e

The plane $z=1$ cuts a circle $x^2+y^2=1$, on xy-plane i.e. $z=0$, it meets at origin

Let PQ be strip ∥ to z-axis.

Point P lies on $x^2+y^2=z^2$

$\therefore z=\sqrt{x^2+y^2}$ (above xy-plane)



changing to cylindrical polar co-ord.

put $x=r\cos\theta$, $y=r\sin\theta$, $z=z$.

$dx\,dy\,dz = r\,dr\,d\theta\,dz$.

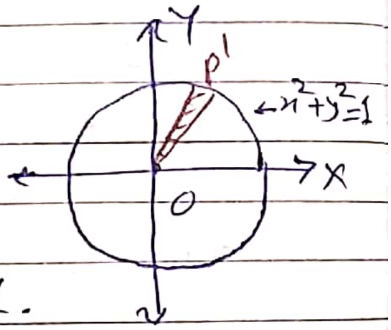
At point P, $z=\sqrt{r^2}=r$

pt. Q gives $z=1$

$\therefore z \rightarrow r$ to 1

Taking projection on xy-plane.

The region R is circle $x^2+y^2=1$



Let OP' be strip.

At origin $r=0$, at pt. P, $r=1$.

$\therefore r \rightarrow 0$ to 1 & $\theta \rightarrow 0$ to 2π

$$\therefore I = \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=r}^{1}\sqrt{r^2}\,[r\,dz\,d\theta\,dr]$$

$$=\int_0^1 r^2\,dr\int_0^{2\pi}d\theta\,.\int_r^1 dz = \int_0^1 r^2\,dr\int_0^{2\pi}d\theta\,[z]_r^1$$

$$=\int_0^1 r^2\,dr\int_0^{2\pi}d\theta\,[1-r] = \int_0^1 r^2(1-r)\,dr\,[\theta]_0^{2\pi}$$

$$=2\pi\int_0^1(r^2-r^3)\,dr = 2\pi\left[\frac{r^3}{3}-\frac{r^4}{4}\right]_0^1$$

$$=2\pi\left[\frac{1}{12}\right]=\boxed{\frac{\pi}{6}}$$

Type II : Change to spherical polar co-ord. :-

~~eg~~ ~~Here~~ use $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$,

$z = r\cos\theta$, & $x^2+y^2+z^2 = r^2$

& $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$.

e.g. ① Evaluate $\iiint_V z^2\,dx\,dy\,dz$, over the volume of sphere $x^2+y^2+z^2=1$.

$\Rightarrow$ Here evaluate over ~~compl~~ whole sphere $x^2+y^2+z^2=1$ ∴ radius is 1.

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$.

Limits for whole sphere : see page 11

$r \to 0$ to 1, $\theta \to 0$ to π & $\phi = 0$ to 2π

Also $z = r\cos\theta$.

$$\therefore I = \int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(r\cos\theta)^2 \cdot r^2\sin\theta\,d\phi\,d\theta\,dr$$

$$= \int_0^1 r^4\,dr \int_0^{\pi}\cos^2\theta\sin\theta\,d\theta \int_0^{2\pi} d\phi$$

$$= \int_0^1 r^4\,dr \int_0^{\pi}\cos^2\theta\sin\theta\,d\theta\,[\phi]_0^{2\pi}$$

$$= 2\pi\int_0^1 r^4\,dr\,(-1)\int_0^{\pi}(\cos\theta)^2(-\sin\theta)\,d\theta$$

using $\int f'(\theta)\,f(\theta)^n\,d\theta = \frac{f(\theta)^{n+1}}{n+1}$

$$= 2\pi\int_0^1 r^4\,dr\,(-1)\left[\frac{\cos^3\theta}{3}\right]_0^{\pi}$$

$$= 2\pi\int_0^1 r^4\,dr\,(-1)\left[\frac{\cos^3\pi - \cos 0}{3}\right]$$

$$= 2\pi\int_0^1 r^4\,dr\,(-1)\left[\frac{-2}{3}\right]$$

$$= \frac{4\pi}{3}\left[\frac{r^5}{5}\right]_0^1 = \frac{4\pi}{3}\left[\frac{1}{5}\right] = \boxed{\frac{4\pi}{15}}$$

eg (2) Evaluate $\iiint_V (x^2+y^2+z^2)^m \, dx\,dy\,dz$, where V is the volume of sphere $x^2+y^2+z^2=1$.

$\Rightarrow$ changing to spherical polar co-ord.

$x^2+y^2+z^2=r^2$, & $dx\,dy\,dz = r^2 \sin\theta \, dr\,d\theta\,d\phi$.

Limits for whole sphere with radius '1'.

$r \to 0$ to 1, $\theta \to 0$ to π, $\phi \to 0$ to 2π (*) see page (11)

$$I = \int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} (r^2)^m \cdot r^2 \sin\theta \, d\phi\,d\theta\,dr$$

$$= \int_0^1 r^{2m+2}\,dr \int_0^{\pi} \sin\theta\,d\theta \int_0^{2\pi} d\phi$$

$$\Rightarrow \int_0^1 r^{2m+2}\,dr \int_0^{\pi} \sin\theta\,d\theta \, [\phi]_0^{2\pi}$$

$$= 2\pi \int_0^1 r^{2m+2}\,dr \, [-\cos\theta]_0^{\pi}$$

$$= 2\pi \int_0^1 r^{2m+2}\,dr \, [-(-1)-1]$$

$$= 4\pi \int_0^1 r^{2m+2}\,dr = 4\pi \left[\frac{r^{2m+2+1}}{2m+2+1}\right]_0^1$$

$$\boxed{= \frac{4\pi}{2m+3}}$$

(3) Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$, by changing to spherical polar co-ord.

Here the solid is bounded by

$z=0$ to $z=\sqrt{1-x^2-y^2}$ (i.e. $x^2+y^2+z^2=1$)

$y=0$ to $y=\sqrt{1-x^2}$ (i.e. $x^2+y^2=1$)

& $x=0$ to $x=1$

(Since all ~~sort~~ limits starts from 0 to positive square root. Thus sphere lies in positive octant ~~quadrant~~)

Changing to spherical polar co-ord,
put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
so that $x^2+y^2+z^2 = r^2$
& $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$.
Since sphere $x^2+y^2+z^2=1$ lies in positive octant.
Limits: $r \to 0$ to 1
$\theta \to 0$ to $\frac{\pi}{2}$
$\phi \to 0$ to $\frac{\pi}{2}$ (*)

(*) see Page (11)

$$\therefore I = \int_{r=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{\frac{\pi}{2}} \frac{r^2\sin\theta}{\sqrt{1-r^2}}\,d\phi\,d\theta\,dr.$$

$$= \int_0^1 \frac{r^2}{\sqrt{1-r^2}}dr \int_0^{\frac{\pi}{2}} \sin\theta\,d\theta \int_0^{\frac{\pi}{2}} d\phi$$

$$= \int_0^1 \frac{r^2}{\sqrt{1-r^2}}dr \cdot \int_0^{\frac{\pi}{2}} \sin\theta\,d\theta\,[\phi]_0^{\frac{\pi}{2}} \quad (=\tfrac{\pi}{2}-0)$$

$$= \frac{\pi}{2}\int_0^1 \frac{r^2}{\sqrt{1-r^2}}dr\,[-\cos\theta]_0^{\frac{\pi}{2}} \longrightarrow [0+1]$$

$$= \frac{\pi}{2}\int_0^1 \frac{r^2}{\sqrt{1-r^2}}dr\,[0+1]$$

Put $r = \sin t$
$dr = \cos t\,dt$

r	0	1
t	0	$\pi/2$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{\sqrt{1-\sin^2 t}}\cos t\,dt = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sin^2 t\,dt$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{1-\cos 2t}{2}dt = \frac{\pi}{4}\left[t - \frac{\sin 2t}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}\left[\frac{\pi}{2}-0\right] = \boxed{\frac{\pi^2}{8}}$$

e.g. (4) Evaluate $\displaystyle\int_0^\infty\int_0^\infty\int_0^\infty \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$.

convert into spherical polar co-ord.
As the volume is the complete first (positive) octant. Let us consider a sphere with centre

at origin & radius R' lies in first octant.

To get the complete first octant we take $R \to \infty$ ⊛

∴ Using $x = r \sin\theta \cos\phi$, $y = r\sin\theta \sin\phi$, $z = r\cos\theta$

$$\therefore 1+x^2+y^2+z^2 = 1+r^2$$

& $dx\,dy\,dz = r^2 \sin\theta \, dr\, d\theta\, d\phi$.

Limits: sphere lies in first octant &

$r \to 0$ to ∞ ⊛

$\theta \to 0$ to $\frac{\pi}{2}$

$\phi \to 0$ to $\frac{\pi}{2}$

see page 11

$$\therefore I = \int_{r=0}^{\infty}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{\frac{\pi}{2}} \frac{r^2 \sin\theta \, d\phi\, d\theta\, dr}{(1+r^2)^2}$$

$$= \int_0^{\infty} \frac{r^2}{(1+r^2)^2}dr \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \; [\phi]_0^{\frac{\pi}{2}}$$

$$= \int_0^{\infty} \frac{r^2}{(1+r^2)^2}dr \cdot [-\cos\theta]_0^{\frac{\pi}{2}} \; \left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{2} \int_0^{\infty} \frac{r^2}{(1+r^2)^2}dr \; \left[-\cos\frac{\pi}{2} + \cos 0\right]$$

put $r = \tan t$

$dr = \sec^2 t\, dt$

r	0	∞
t	0	$\pi/2$

$$I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\tan^2 t}{(1+\tan^2 t)^2} \times \sec^2 t\, dt$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\tan^2 t}{\sec^4 t} \cdot \sec^2 t\, dt = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{\cos^2 t} \cdot \cos^2 t\, dt$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2t}{2}\right) dt = \frac{\pi}{4}\left[t - \frac{\sin 2t}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}\left[\frac{\pi}{2} - 0\right] = \boxed{\frac{\pi^2}{8}}$$

Test Your Knowlege :-

Triple Integral.

(A)

1) $\int_0^{2a}\int_0^{x}\int_y^{x} xyz \, dx\,dy\,dz \quad \left(=\frac{4}{3}a^6\right)$

(2) $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z} \, dx\,dy\,z \quad \left(=\frac{5}{8}\right)$

(3) $\int_0^{1}\int_{y^2}^{1}\int_0^{1-x} x \, dx\,dy\,dz \quad \left(=\frac{4}{35}\right)$

(4) $\int_1^{e}\int_1^{\log y}\int_1^{e^x} \log z \, dx\,dy\,dz \quad \left(=\frac{e^2-8e+13}{4}\right)$

(5) $\int_1^{3}\int_{\frac{1}{x}}^{1}\int_0^{\sqrt{xy}} xyz \, dx\,dy\,dz. \quad \left(=\frac{13}{9}-\frac{1}{6}\log 3\right)$

(6) $\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\left(\frac{a^2-r^2}{a}\right)} r \, dr\,d\theta\,dz \quad \left(=\frac{5a^3\pi}{64}\right)$

(7) $\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r \, dr\,d\theta\,dz \quad \left(=\frac{a^3}{18}(3\pi-4)\right)$

(B) Evaluate (I) $\iiint_V \frac{dx\,dy\,dz}{(1+x+y+z)^3} \quad \left(=\frac{1}{2}\left(\log 2-\frac{5}{8}\right)\right)$

(II) $\iiint_V \frac{dx\,dy\,dz}{(x+y+2z+1)^3}, \quad \left(=\frac{1}{4}(\log 3-1)\right)$

where V is the volume bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.

(C) Change to spherical polar co-ord. & evaluate

[$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$|J| = r^2\sin\theta$]

(I) $\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} (x^2+y^2+z^2) \, dx\,dy\,dz \quad \left(=\frac{\pi a^5}{10}\right)$

Page ② Evaluate $\iiint_V (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by the planes $x=0, y=0, z=0$, $x+y+z=1$.

(2) $$\int_0^\infty\int_0^\infty\int_0^\infty \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2} \quad \left(=\frac{\pi^2}{8}\right)$$

(3) $$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} \quad \left(=\frac{\pi^2}{8}\right)$$

(4) Evaluate $\iiint_V z(x^2+y^2+z^2)\,dx\,dy\,dz$ throughout the volume of the cylinder $x^2+y^2=a^2$ intercepted by the planes $z=0$ and $z=h$. $\left(=\frac{\pi}{4}a^2h^2(a^2+h^2)\right)$

(5) Evaluate $\iiint_V z(x^2+y^2)\,dx\,dy\,dz$ over the volume of cylinder $x^2+y^2=1$ between $z=2$ & $z=3$ $\left(=\frac{5\pi}{4}\right)$

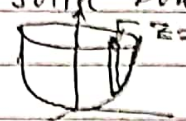
(6) $\iiint_V z^2\,dx\,dy\,dz$ over $x^2+y^2+z^2=1$ $\quad\left(=\frac{4\pi}{15}\right)$

(7) $\iiint_V (y^2z^2+z^2x^2+x^2y^2)\,dx\,dy\,dz$, V is the volume of $x^2+y^2+z^2=a^2$ $\quad\left(=\frac{4\pi}{35}a^7\right)$

(8) $\iiint_V (x^2+y^2)\,dx\,dy\,dz$, V is the volume bounded by the surfaces $x^2+y^2=2z$ and the plane $z=2$ $\left(=\frac{16\pi}{3}\right)$

Hint $$I=\int_0^2\int_0^{2\pi}\int_{\frac{r^2}{2}}^{2} r^2(r\,dr\,d\theta\,dz)$$

(9) $\iiint_V \sqrt{x^2+y^2}\,dx\,dy\,dz$ $\quad V$ is the volume of solid bounded by $x^2+y^2=z^2$, $z=0$ & $z=1$.



$$=\int_{-1}^{1}dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\int_{\sqrt{x^2+y^2}}^{1}\sqrt{x^2+y^2}\,dz \quad \left(=\frac{\pi}{6}\right)$$

Evaluate ① $$\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dz \quad \left(=\frac{8}{3}\log 2-\frac{19}{9}\right)$$

② $$\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz=\frac{1}{8}e^{4a}-\frac{3}{4}e^{2a}+e^{a}-\frac{3}{8}$$

③ $$\int_0^a\int_0^a\int_0^a (xy+yz+xz)\,dx\,dy\,dz \quad \left(=\frac{3}{4}a^5\right)$$

④ $$\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz \quad (=0)$$

⑤ $$\int_0^1\int_0^{x^2}\int_0^{x+y}(2x-y-z)\,dx\,dy\,dz$$

⑥ $$\int_{-1}^{1}\int_0^{2\pi}\int_0^{1+\cos\theta} 4r\,dz\,d\theta\,dr$$

⑦ $$\int_0^{2\pi}\int_0^{3}\int_0^{z/3} r^3\,dr\,dz\,d\theta \quad \left(=\frac{3\pi}{10}\right)$$