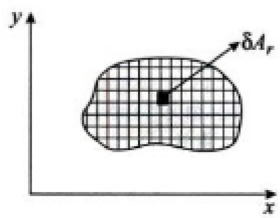
Multiple Integrals

1 Introduction

Let $f(x, y)$ be a continuous function defined in a closed and bounded region R in the $xy-$plane. Divide the region R into small elementary rectangles by drawing lines parallel to coordinate axes as shown in the following figure.



Let the total number of complete rectangles which lies inside the region R is n. Let δA_r be the area of r^{th}rectangle and (x_r, y_r) be any point in this rectangle.
Consider the sum

$$S = \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \tag{1}$$

If we increase the number of elementary rectangles i.e. n, then the area of each rectangle decreases. Hence, as $n \to \infty$ we have $\delta A_r \to 0$. The limit of the sum given by Eq.(1),if it exists, is called the double integral of $f(x, y)$ over the region R and is denoted by $\iint_R F(x, y)\mathrm{d}A$.
Hence,

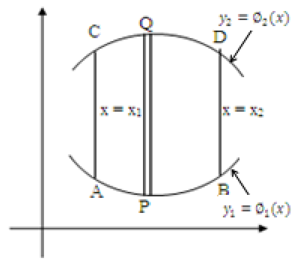
$$\iint_R f(x, y)\mathrm{d}A = \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r$$

where, $\mathrm{d}A = \mathrm{d}x\ \mathrm{d}y$ or $\mathrm{d}A = \mathrm{d}y\ \mathrm{d}x$

2 Evaluation of double integrals

The methods of evaluating the double integration depends upon the nature of the curves bounding the region R. Let the region be bounded by the curves $x = x_1$, $x = x_2$ and $y = y_1$, $y = y_2$.

a) Let R be the region bounded by the curves $x = x_1$, $x = x_2$, $y_1 = \phi_1(x)$ and $y_2 = \phi_2(x)$ as shown in the following figure.

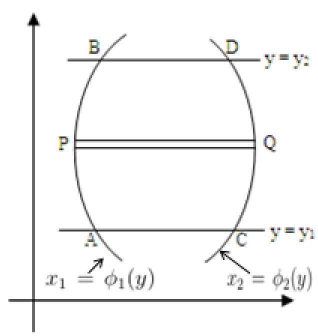


Let AB and CD be the curves $y_1 = \phi_1(x)$ and $y_2 = \phi_2(x)$. Take a vertical strip PQ of width δx. Here the double integral is evaluated first w.r.t y (treating x constant). The resulting expression which is a function of x is integrated w.r.t. x between the limits $x = x_1$ and $x = x_2$. Thus,

$$\iint_R f(x,y)\mathrm{d}x\ \mathrm{d}y = \int_{x_1}^{x_2} \left(\int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} f(x,y)\mathrm{d}y \right) \mathrm{d}x$$

Geometrically, the integral in the round bracket indicates that the integration is performed along vertical strip PQ (keeping x constant) while outer integration corresponds to the sliding of the strip PQ from AC to BD thus covering the entire region ABDC of integration.

b) Let R be the region bounded by the curves $x_1 = \phi_1(y)$, $x_2 = \phi_2(y)$, $y = y_1$ and $y = y_2$ as shown in the following figure.

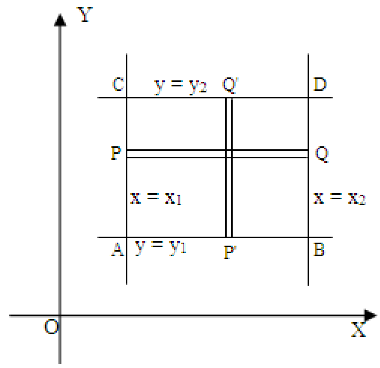


Let AB and CD be the curves $x_1 = \phi_1(y)$ and $x_2 = \phi_2(y)$. Take a horizontal strip PQ of width δx. Here the double integral is evaluated first w.r.t x (treating y constant). The resulting expression which is a function of y is integrated w.r.t. y between the limits $y = y_1$ and $y = y_2$. Thus,

$$\iint_R f(x,y)\mathrm{d}x\ \mathrm{d}y = \int_{y_1}^{y_2} \left(\int_{x_1=\phi_1(y)}^{x_2=\phi_2(y)} f(x,y)\mathrm{d}x \right) \mathrm{d}y$$

Geometrically, the integral in the round bracket indicates that the integration is performed along horizontal strip PQ (keeping y constant) while outer integration corresponds to the sliding of the strip PQ from AC to BD thus covering the entire region ABDC of integration.

c) Let R be the region bounded by the curves $x = x_1$, $x = x_2$, $y = y_1$ and $y = y_2$, where x_1, x_2, y_1 and y_2 are constants, as shown in the following figure.



Here the region of integration is rectangle ABDC. It is immaterial whether we integrate first along the horizontal strip PQ and then slide from AB to CD or we integrate first along the vertical strip $P'Q'$ and then slide it from AC to BD. Thus the order of integration is immaterial, provided the limits of integration are changed accordingly.

$$\iint_R f(x,y)\mathrm{d}x\ \mathrm{d}y = \int_{y_1}^{y_2} \left(\int_{x_1}^{x_2} f(x,y)\mathrm{d}x \right) \mathrm{d}y = \int_{x_1}^{x_2} \left(\int_{y_1}^{y_2} f(x,y)\mathrm{d}y \right) \mathrm{d}x$$

Note: From cases (a) and (b) above, we observe that if inner limits are functions of x then first we integrate w.r.t. x and then w.r.t y and if inner limits are functions of y then first we integrate w.r.t. y and then w.r.t x

3 Double Integral Over The Given Region R

First we shall consider the procedure for evaluating double integral over the given region R. Followings are the steps:

a) Sketch the region of integral using given equation(s) of curve and line. Highlight the region over which we have to integrate.

b) Draw a strip either parallel to y axis (vertical strip) or parallel to x axis (horizontal strip) in the region of integration, where both ends of the strip should be on boundary of region.

c) If strip is parallel to y-axis, then the inner limits i.e. limits of y are functions of x and outer limits i.e. limits of x are constants.

   i) The inner limits are measured or expressed from lower to upper part of the region or strip. The limits of y are obtained by solving those equations for y on which the ends of the strip lies.

   ii) To obtain outer limits, draw $x = a$ and $x = b$ parallel to y axis such that the whole region lie between these two lines. The the outer limits i.e. limits of x are a to b.

   iii) The double integration can be evaluated by integrating first w.r.t. y treating x constant and then w.r.t. x.

d) If strip is parallel to x-axis, then the inner limits i.e. limits of x are functions of y and outer limits i.e. limits of y are constants.

   i) The inner limits are measured or expressed from left to right part of the region or strip. The limits of x are obtained by solving those equations for x on which the ends of the strip lies.

   ii) To obtain outer limits, draw $y = c$ and $y = d$ parallel to x axis such that the whole region lie between these two lines. The the outer limits i.e. limits of y are c to d.

   iii) The double integration can be evaluated by integrating first w.r.t. x treating x constant and then w.r.t. y.

Example 1. Evaluate $\iint_R y\mathrm{d}x\mathrm{d}y$, where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$

Solution: Consider,

$$I = \iint_R y\mathrm{d}x\mathrm{d}y$$

Sketch the parabolas parabolas $y^2 = 4x$ and $x^2 = 4y$ and consider the shaded part as the region of integration as shown in the following Fig. a.

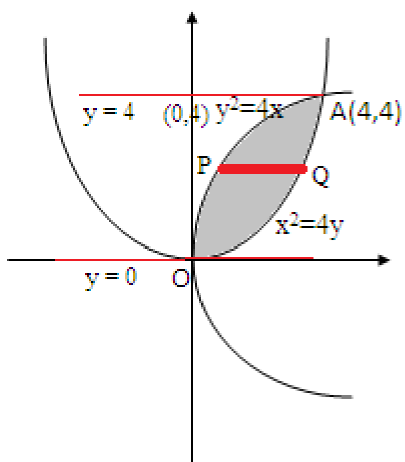


Fig. a

Solving $y^2 = 4x$ and $x^2 = 4y$, we have

$$\begin{aligned} x^2 = 4y &\Rightarrow x^4 = 16y^2 \\ &\Rightarrow x^4 = 16(4x) \\ &\Rightarrow x^4 - 64x = 0 \\ &\Rightarrow x(x^3 - 64) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 4 \end{aligned}$$

If $x = 0$, then $y^2 = 4x$ gives $y = 0$ and if $x = 0$, then $y^2 = 4x$ gives $y = 4$. Therefore, parabolas intersect at $(0, 0)$ and $(4, 4)$.

Now, Consider a strip PQ parallel to x-axis as shown in above (one can take strip parallel to

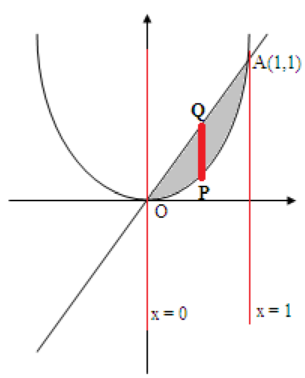
y axis also). Here the point P lies on the curve $y^2 = 4x$ i.e $x = \dfrac{y^2}{2}$ and Q lies on $x^2 = 4y$ i.e. $x = 2\sqrt{y}$. Therefore, x varies from $\dfrac{y^2}{4}$ to $2\sqrt{y}$.
Again, the whole region is bounded between $x = 0$ i.e. y axis and $x = 4$. Therefore y varies from 0 to 4. Thus, we get

$$\begin{aligned}
I = \iint_R y\mathrm{d}x\mathrm{d}y &= \int_0^4 \int_{y^2/4}^{2\sqrt{y}} y\mathrm{d}x \, \mathrm{d}y \\
&= \int_0^4 y[x]_{y^2/4}^{2\sqrt{y}} \mathrm{d}y \\
&= \int_0^4 y\left(2\sqrt{y} - \frac{y^2}{4}\right)\mathrm{d}y \\
&= \int_0^4 \left(2y^{3/2} - \frac{y^3}{4}\right)\mathrm{d}y \\
&= \left[\frac{4y^{5/2}}{5} - \frac{y^4}{16}\right]_0^4 \\
&= \frac{4(4)^{5/2}}{5} - \frac{4^4}{16} = \frac{128}{5} - 16 \\
&= \frac{48}{5}
\end{aligned}$$

Example 2. Evaluate $\iint_R xy(x+y)\mathrm{d}x\mathrm{d}y$,where R is region between $y = x^2$ and $y = x$
Solution: Consider,

$$I = \iint_R xy(x+y)\mathrm{d}x\mathrm{d}y$$

Sketch the parabola $y = x^2$ and the straight line $y = x$ and consider the shaded part as the region of integration



Solving $y = x^2$ and $y = x$, (putting $y = x$ in $y = x^2$) we get, $x = x^2 \Rightarrow x(x-1) = 0$. This gives $x = 0$ and $x = 1$.
When $x = 0$, we get $y = 0$ and when $x = 1$, we get $y = 1$. Thus the curves $y = x^2$ and $y = x$ intersects at $(0, 0)$ and $(1, 1)$.
Now, Consider a strip PQ parallel to y-axis as shown in above fig. Here the point P lies on

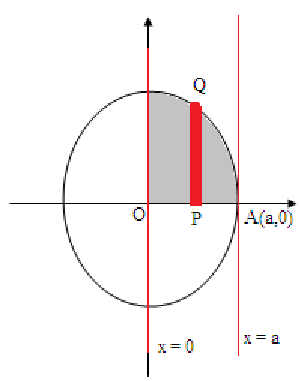
the curve $y = x^2$ and Q lies on $y = x$. Therefore, y varies from x^2 to x.
Again, the whole region is bounded between $y = 0$ i.e. x axis and $y = 1$. Therefore x varies from 0 to 1. Thus, we get

$$\begin{aligned}
I = \iint_R xy(x+y)\mathrm{d}x\mathrm{d}y &= \int_0^1 \int_{x^2}^{x} xy(x+y)\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_{x^2}^{x} (x^2y + xy^2)\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} \mathrm{d}x \\
&= \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right] \mathrm{d}x \\
&= \int_0^1 \left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right] \mathrm{d}x \\
&= \left[\frac{x^5}{6} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1 \\
&= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} \\
&= \frac{3}{56}
\end{aligned}$$

Example 3. Evaluate $\iint_R xy \; \mathrm{d}x\mathrm{d}y$, where R is the quadrant of the circle $x^2 + y^2 = a^2$ where $x \geq 0$ and $y \geq 0$.
Solution: Consider,

$$I = \iint_R xy \; \mathrm{d}x\mathrm{d}y$$

Clearly, the region of integration be the first quadrant of the circle $x^2 + y^2 = a^2$ as shown in following figure.



Consider a strip PQ parallel to y-axis as shown in above fig. Here the point P lies on the curve x-axis i.e $y = 0$ and Q lies on $x^2 + y^2 = a^2$ i.e. $y = \sqrt{a^2 - x^2}$. Therefore, y varies from 0 to $\sqrt{a^2 - x^2}$.

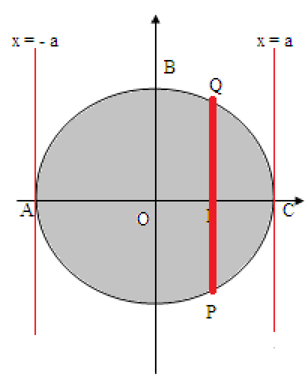
Again, the whole region is bounded between $x = 0$ i.e. y axis and $x = a$. Therefore x varies from 0 to a. Thus, we get

$$
\begin{aligned}
I = \iint_R xy \, \mathrm{d}x\mathrm{d}y &= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\mathrm{d}y\mathrm{d}x \\
&= \int_0^a \left[\frac{xy^2}{2}\right]_0^{\sqrt{a^2-x^2}} \mathrm{d}x \\
&= \int_0^a \left[\frac{x(a^2-x^2)}{2}\right] \mathrm{d}x \\
&= \int_0^a \frac{a^2x-x^3}{2} \\
&= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a \\
&= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] \\
&= \frac{a^4}{8}
\end{aligned}
$$

Example 4. Evaluate $\iint (x+y)^2 \mathrm{d}x\mathrm{d}y$ over the area bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
Solution: Consider,

$$I = \iint (x+y)^2 \mathrm{d}x\mathrm{d}y$$

Consider the upper half of the ellipse as the area of integration as shown by shaded portion in the following example.



Consider a strip PQ parallel to y-axis as shown in above fig. Here the point P lies on the curve x-axis i.e $y = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ i.e. $y = -\dfrac{b}{a}\sqrt{a^2-x^2}$ and Q lies on $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ i.e. $y = \dfrac{b}{a}\sqrt{a^2-x^2}$. Therefore, y varies from $-\dfrac{b}{a}\sqrt{a^2-x^2}$ to $\dfrac{b}{a}\sqrt{a^2-x^2}$. Again, the whole region is bounded between $x = -a$ and $x = a$. Therefore x varies from $-a$ to a. Thus, we get

$$I = \iint (x+y)^2 \mathrm{d}x\mathrm{d}y = \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x+y)^2 \mathrm{d}y\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2+2xy)\mathrm{d}y\mathrm{d}x \\
&= \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\mathrm{d}y\mathrm{d}x + \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (2xy)\mathrm{d}y\mathrm{d}x \\
&= 2\int_{-a}^{a}\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\mathrm{d}y\mathrm{d}x + 0 \text{ since } 2xy \text{ is odd and } x^2+y^2 \text{ is even} \\
&= 2\int_{-a}^{a}\left[x^2y+\frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\mathrm{d}x \\
&= 2\int_{-a}^{a}\left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]\mathrm{d}x \\
&= 4\int_{0}^{a}\left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]\mathrm{d}x
\end{aligned}
$$

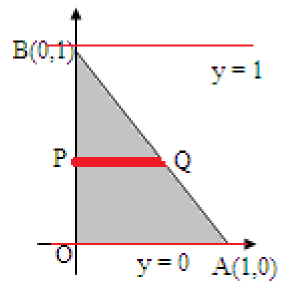
Now, put $x = a\sin\theta$. This gives $\mathrm{d}x = a\cos\theta$ and when $x = 0$ we get $\theta = 0$ and when $x = a$ we get $\theta = \frac{\pi}{2}$. Thus, we get

$$
\begin{aligned}
I &= 4\int_0^{\pi/2}\left[\frac{b}{a}a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}+\frac{b^3}{3a^3}(a^2-a^2\sin^2\theta)^{3/2}\right]a\cos\theta\mathrm{d}\theta \\
&= 4\int_0^{\pi/2}\left[a^3b\sin^2\theta\cos^2\theta+\frac{ab^3}{3}\cos^4\theta\right]\mathrm{d}\theta \\
&= 4\left[a^3b\int_0^{\pi/2}\sin^2\theta\cos^2\theta\mathrm{d}\theta+\frac{ab^3}{3}\int_0^{\pi/2}\cos^4\theta\mathrm{d}\theta\right] \\
&= 4\left[a^3b\frac{1}{2}\beta\left(\frac{3}{2},\frac{3}{2}\right)+\frac{ab^3}{3}\frac{1}{2}\beta\left(\frac{5}{2},\frac{1}{2}\right)\right] \\
&= 2\left[a^3b\frac{\Gamma 3/2\Gamma 3/2}{\Gamma 3}+\frac{ab^3}{3}\frac{\Gamma 5/2\Gamma 1/2}{\Gamma 3}\right] \\
&= 2\left[a^3b\frac{1/2\Gamma 1/2\ 1/2\Gamma 1/2}{2!}+\frac{ab^3}{3}\frac{3/2\ 1/2\Gamma 1/2\Gamma 1/2}{2!}\right] \\
&= \frac{\pi ab}{4}(a^2+b^2)
\end{aligned}
$$

Example 5. Evaluate $\iint_R xy\mathrm{d}x\mathrm{d}y$ over the region R in the positive quadrant for which $x+y \leq 1$
Solution: Consider,

$$I = \iint_R xy\mathrm{d}x\mathrm{d}y$$

The region R in the positive quadrant for which $x+y \leq 1$ i.e. region bounded by $x \geq 0$, $y \geq 0$ and $x+y = 1$ is shown by shaded portion in the following figure.



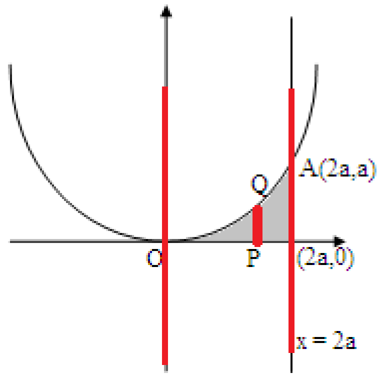
Consider a strip PQ parallel to x-axis as shown in above fig. Here the point P lies on the curve y-axis i.e $x = 0$ and Q lies on $x + y = 1$ i.e. $x = 1 - y$. Therefore, x varies from 0 to $1 - y$. Again, the whole region is bounded between x-axis i.e $y = 0$ and $y = 1$. Therefore y varies from 0 to 1. Thus, we get

$$\begin{aligned}
I = \iint_R xy\mathrm{d}x\mathrm{d}y &= \int_0^1 \int_0^{1-y} xy\mathrm{d}x\mathrm{d}y \\
&= \int_0^1 \left[y\frac{x^2}{2} \right]_0^{1-y} \mathrm{d}y \\
&= \int_0^1 \left[\frac{y(1-y)^2}{2} \right] \mathrm{d}y \\
&= \frac{1}{2} \int_0^1 \left[y - 2y^2 + y^3 \right] \mathrm{d}y \\
&= \frac{1}{2} \left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4} \right]_0^1 \\
&= \frac{1}{2} \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] \\
&= \frac{1}{24}
\end{aligned}$$

Example 6. Evaluate $\iint_R xy\ \mathrm{d}x\mathrm{d}y$,where R is the region bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.
Solution: Consider,

$$I = \iint_R xy\ \mathrm{d}x\mathrm{d}y$$

Sketch the parabolas parabolas $x^2 = 4ay$ and a line $x = 2a$ and consider the region of integration bounded by $x^2 = 4ay$, $x = 2a$ and x-axis as shown in the following Fig. (see shaded portion)



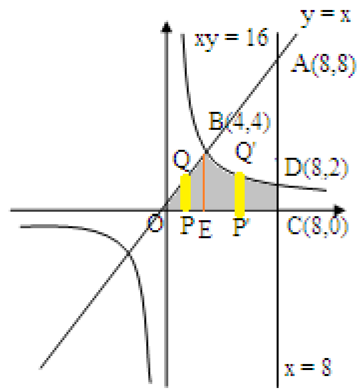
Solving $x^2 = 4ay$ and $x = 2a$ i.e. putting $x = 2a$ in $x^2 = 4ay$, we get $y = a$. Therefore the parabola $x^2 = 4ay$ and a line $x = 2a$ intersect at $(2a, a)$.
Now, consider a strip PQ parallel to y-axis as shown in above fig. Here the point P lies on the curve x-axis i.e $y = 0$ and Q lies on $x^2 = 4ay$ i.e. $y = \dfrac{x^2}{4a}$. Therefore, y varies from 0 to $\dfrac{x^2}{4a}$. Again, the whole region is bounded between $x = 0$ and $x = 2a$. Therefore x varies from 0 to $2a$. Thus, we get

$$\begin{aligned}
I = \iint_R xy \, \mathrm{d}x\mathrm{d}y &= \int_0^{2a} \int_0^{x^2/4a} xy \, \mathrm{d}y\mathrm{d}x \\
&= \int_0^{2a} \left[x\frac{y^2}{2} \right]_0^{x^2/4a} \mathrm{d}x \\
&= \int_0^{2a} \frac{x}{2} \left[\frac{x^4}{16a^2} \right] ; \mathrm{d}x \\
&= \frac{1}{32a^2} \int_0^{2a} x^5 \, \mathrm{d}x \\
&= \frac{1}{32a^2} \left[\frac{x^6}{6} \right]_0^{2a} \\
&= \frac{1}{32a^2} \left[\frac{64a^6}{6} \right] \\
&= \frac{a^4}{3}
\end{aligned}$$

Example 7. Evaluate $\iint_R x^2 \, \mathrm{d}\mathrm{d}y$, where R is the region in the first quadrant bounded by the lines $x = y$, $y = 0$, $x = 8$ and the curve $xy = 16$.
Solution: Consider,

$$I = \iint_R x^2 \, \mathrm{d}x\mathrm{d}y$$

Sketch the lines $x = y$, $y = 0$, $x = 8$ and the curve $xy = 16$ as shown in the following figure. The bounded region of integration is shown by shaded portion in the following figure.



Solving $y = x$ and $x = 8$, we get the point $A(8, 8)$. Also, solving $xy = 16$ and $y = x$, we get $x = 4$. Therefore, $y = 4$. The curve $xy = 16$ and $y = x$ intersect at $B(4, 4)$ in the first quadrant. Also, the curve $xy = 16$ and $x = 8$ intersect at $D(8, 2)$.
Now, to evaluate the given integral over the area OCDB, we divide this area into two parts OEB and ECDB. Thus,

$$I = \iint_{OEB} x^2 \, \mathrm{d}x\mathrm{d}y + \iint_{ECDB} x^2 \, \mathrm{d}x\mathrm{d}y \tag{1}$$

First consider a strip PQ parallel to y-axis in the region OEB. Here the point P lies on the curve x-axis i.e $y = 0$ and Q lies on $y = x$. Therefore, y varies from 0 to x. Again, the whole region is bounded between $x = 0$ and $x = 4$. Therefore x varies from 0 to 4.
Now, consider a strip $P'Q'$ parallel to y-axis in the region ECDB. Here the point P' lies on the curve x-axis i.e $y = 0$ and Q' lies on $xy = 16$ i.e $y = \dfrac{16}{x}$. Therefore, y varies from 0 to $16/x$. Again, the whole region is bounded between $x = 4$ and $x = 8$. Therefore x varies from 4 to 8. Therefore, by Eq-(1), we get

$$\begin{aligned}
I &= \int_0^4 \int_0^x x^2 \, \mathrm{d}y\mathrm{d}x + \int_4^8 \int_0^{16/x} x^2 \, \mathrm{d}y\mathrm{d}x \\
&= \int_0^4 x^2 [y]_0^x \, \mathrm{d}x + \int_4^8 x^2 [y]_0^{16/x} \, \mathrm{d}x \\
&= \int_0^4 x^3 \, \mathrm{d}x + \int_4^8 16x \, \mathrm{d}x \\
&= \left[\frac{x^4}{4}\right]_0^4 + 16 \left[\frac{x^2}{2}\right]_4^8 \\
&= 448
\end{aligned}$$

Exercise

1) Evaluate $\iint e^{2x+3y}\, \mathrm{d}x\mathrm{d}y$ over the triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$

2) Evaluate $\iint x^2y^2\, \mathrm{d}x\mathrm{d}y$ over the circle $x^2 + y^2 = 1$

3) Evaluate $\iint_R (4xy - y^2)\, \mathrm{d}x\mathrm{d}y$, where R is the rectangle bounded by $x = 1$, $x = 2$, $y = 0$ and $y = 3$.

4) Evaluate the integral $\iint_R x\, \mathrm{d}x\mathrm{d}y$, where R is the region bounded by the curves $y = x^3$, $x + y = 2$ and $x = 0$.

5) Find the integral $\iint_R x^2y\, \mathrm{d}x\mathrm{d}y$, where the region R is the segment of the circle. The boundaries of the segment are defined by equation $x^2 + y^2 = 4$ and $x + y - 2 = 0$

6) Find the integral $\iint_R y\, \mathrm{d}x\mathrm{d}y$, where R is bounded by the straight line $y = 2x$ and the parabola $y = 3 - x^2$.

7) Calculate the double integral $\iint_R x \sin y\, \mathrm{d}y\mathrm{d}x$, where R is bounded by $y = 0$, $y = x^2$ and $x = 1$.

8) Find the double integral $\iint_R (x + y)\, \mathrm{d}x\mathrm{d}y$, where the region R is the parallelogram with the sides $y = x$, $y = x + a$, $y = a$ and $y = 2a$, where a is constant.

Answers

1) $\frac{1}{6}(e-1)^2(2e+1)$ 2) $\frac{\pi}{24}$ 3) 18 4) $\frac{7}{15}$ 5) $\frac{8}{5}$ 6) $-\frac{64}{15}$ 7) 0.08 8) $\frac{5a^3}{2}$