APPLICATIONS OF DE MOIVRE'S THEOREM:

1) Expansion of $\sin n\theta, \cos n\theta$ in powers of $\sin\theta, \cos\theta$:

By De Moivre's theorem $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

$= \cos^n\theta + {}^nC_1\cos^{n-1}\theta . i\sin\theta + {}^nC_2\cos^{n-2}\theta.(i\sin\theta)^2 + {}^nC_3\cos^{n-3}\theta(i\sin\theta)^3 + \ldots\ldots$

$= (\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots\ldots) + i({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots\ldots)$

Comparing real & imaginary parts on both sides

$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots\ldots\ldots\ldots$

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-2}\sin^3\theta + \ldots\ldots\ldots\ldots$

SOME SOLVED EXAMPLES:

1. Using De Moivre's Theorem, prove that, $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$

Solution: By De Moivre's theorem,

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

$= (\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

$= \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

Equating real and imaginary parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

2. Using De Moivre's Theorem express $\sin 3\theta, \cos 3\theta, \tan 3\theta$ in terms of powers of $\sin\theta, \cos\theta, \tan\theta$ respectively.

Solution: continue as example (1) and obtain

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta - 3\sin^2\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \qquad = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^2\theta$

$= 4\cos^3\theta - 3\cos\theta$

$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$

Dividing the numerator and denominator by $\cos^3\theta$

$\tan 3\theta = \frac{(3\tan\theta - \tan^3\theta)}{(1-3\tan^2\theta)}$

3. Show that, **(i)** $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$

(ii) $\cos 5\theta = 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta$

Solution: By De Moivre's Theorem, $(\cos 5\theta + i\sin 5\theta) = (\cos\theta + i\sin\theta)^5$

$$= cos^5\theta + 5\,cos^4\theta(i\sin\theta) + 10cos^3\theta(i\sin\theta)^2 + 10cos^2\theta(i\sin\theta)^3$$
$$+ 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5 \quad \ldots \text{Using Binomial Theorem}$$
$$= cos^5\theta + i\,5cos^4\theta\,\sin\theta - 10cos^3\theta sin^2\theta + i.10cos^2\theta sin^3\theta + 5\cos\theta sin^4\theta + i\,sin^5\theta$$
$$= \left(cos^5\theta - 10\,cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta\right) + i(5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta)$$

Equating real and imaginary parts

$$\cos 5\,\theta = cos^5\theta - 10\,cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta$$
$$\sin 5\,\theta = 5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta$$

We have $\sin 5\theta = 5\,cos^4\,\theta\,\sin\theta - 10cos^2\theta sin^3\theta + sin^5\theta$

$$= 5(1 - sin^2\theta)^2\sin\theta - 10(1 - sin^2\theta)\,sin^3\,\theta + sin^5\theta$$
$$= 5(1 - 2\,sin^2\theta + sin^4\theta)\sin\theta - 10(1 - sin^2\theta)sin^3\,\theta + sin^5\theta$$
$$= 5\sin\theta - 20sin^3\,\theta + 16\,sin^5\theta$$

And $\cos 5\theta = cos^5\theta - 10\,cos^3\theta\,sin^2\theta + 5\cos\theta\,sin^4\theta$

$$= cos^5\theta - 10cos^3\theta(1 - cos^2\theta) + 5\cos\theta(1 - cos^2\theta)^2$$
$$= cos^5\theta - 10cos^3\theta(1 - cos^2\theta) + 5\cos\theta\,(1 - 2cos^2\theta + cos^4\theta)$$
$$= 5\,cos\theta - 20cos^3\theta + 16\,cos^5\theta$$

4. Show that, $\frac{\sin 5\theta}{\sin\theta} = 16\,cos^4\theta - 12\,cos^2\theta + 1$

Solution: From above example (3)

$$\sin 5\theta = 5cos^4\theta\,\sin\theta - 10cos^2\theta sin^3\theta + sin^5\theta$$
$$\therefore \frac{\sin 5\theta}{\sin\theta} = 5cos^4\theta - 10cos^2\theta sin^2\theta + sin^4\theta$$
$$= 5cos^4\theta - 10cos^2\theta(1 - cos^2\theta) + (1 - cos^2\theta)^2$$
$$= 5cos^4\theta - 10cos^2\theta + 10cos^4\theta + 1 - 2cos^2\theta + cos^4\theta$$
$$= 16\,cos^4\theta - 12\,cos^2\theta + 1$$

5. Use De Moiver's Theorem to show that $tan5\theta = \frac{5\tan\theta - 10\,tan^3\theta + tan^5\theta}{1 - 10tan^2\theta + 5tan^4\theta}$ and hence deduce that $5tan^4\frac{\pi}{10} - 10tan^2\frac{\pi}{10} + 1 = 0$

Solution: From above example (3)

$$\cos 5\,\theta = cos^5\theta - 10\,cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta$$
$$\sin 5\,\theta = 5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta$$
$$\therefore \tan 5\theta = \frac{\sin 5\,\theta}{\cos 5\,\theta} = \frac{5\,cos^4\theta\sin\theta - 10cos^2\theta sin^3\theta + sin^5\theta}{cos^5\theta - 10\,cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta}$$

Dividing the numerator and denominator by $cos^5\theta$

$$\tan 5\theta = \frac{\tan\theta - 10tan^3\theta + tan^5\theta}{1 - 10tan^2\theta + 5tan^4\theta} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Now, Put $\theta = \frac{\pi}{10}$.

Then $\tan 5\theta = \tan\frac{\pi}{2} = \infty$ and hence the denominator in (1) must be zero.

$$\therefore 5\, tan^4\frac{\pi}{10} - 10tan^2\frac{\pi}{10} + 1 = 0\,.$$

6. If $\sin 6\theta = a\, cos^5\theta \sin\theta + b\, cos^3\theta\, sin^3\theta + c\cos\theta\, sin^5\theta$,

find the values of a, b, c.

Solution: By De Moivre's Theorem $\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$

$$= (\cos\theta)^6 + 6(\cos\theta)^5(i\sin\theta) + 15(\cos\theta)^4(i\sin\theta)^2 + 20(\cos\theta)^3(i\sin\theta)^3$$
$$+15(\cos\theta)^2(i\sin\theta)^4 + 6(\cos\theta)^1(i\sin\theta)^5 + (i\sin\theta)^6$$
$$= \cos^6\theta + i6\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - i20\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta$$
$$+i6\cos\theta\sin^5\theta - \sin^6\theta$$
$$= (\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta)$$
$$+i(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta)$$

Equating imaginary parts, $\sin 6\theta = 6\cos^6\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

Comparing with $\sin 6\theta = a\cos^5\theta\sin\theta + b\cos^3\theta\sin^3\theta + c\cos\theta\sin^5\theta$

we get, $a = 6, b = -20, c = 6$

7. Prove that,

$$\cos 8\theta = cos^8\theta - 28\, cos^6\theta\, sin^2\theta + 70\, cos^4\theta\, sin^4\theta - 28\, cos^2\theta\, sin^6\theta + sin^8\theta$$
$$\sin 8\theta = 8\, cos^7\theta\sin\theta - 56\, cos^5\theta\, sin^3\theta + 56cos^3\theta sin^5\theta - 8\cos\theta\, sin^7\theta.$$

Solution: By De Moivre's Theorem $\cos 8\theta + i\sin 8\theta = (\cos\theta + i\sin\theta)^8$

$$= (\cos\theta)^8 + 8(\cos\theta)^7(i\sin\theta) + 28(\cos\theta)^6(i\sin\theta)^2 + 56(\cos\theta)^5(i\sin\theta)^3$$
$$+70(\cos\theta)^4(i\sin\theta)^4 + 56(\cos\theta)^3(i\sin\theta)^5 + 28(\cos\theta)^2(i\sin\theta)^6$$
$$+8(\cos\theta)(i\sin\theta)^7 + (i\sin\theta)^8$$
$$= \cos^8\theta + i\cos^7\theta\sin\theta - 28\cos^6\theta\sin^2\theta - i56\cos^5\theta\sin^3\theta + 28\cos^4\theta\sin^4\theta$$
$$+i56\cos^3\theta\sin^5\theta - 28\cos^2\theta\sin^6\theta - i8\cos\theta\sin^7\theta + \sin^8\theta$$
$$= (\cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta)$$
$$+i(8\cos^7\theta\sin\theta - 56\cos^5\theta\sin^3\theta + 56\cos^3\theta\sin^5\theta - 8\cos\theta\sin^7\theta)$$

Equating real and imaginary parts

$$\cos 8\theta = cos^8\theta - 28\, cos^6\theta\, sin^2\theta + 70\, cos^4\theta\, sin^4\theta - 28\, cos^2\theta\, sin^6\theta + sin^8\theta$$
$$\sin 8\theta = 8\, cos^7\theta\sin\theta - 56\, cos^5\theta\, sin^3\theta + 56cos^3\theta sin^5\theta - 8\cos\theta\, sin^7\theta.$$

8. Using De Moivre's theorem prove that,

$2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$ *where* $x = 2\cos\theta$.

Solution: $2(1 + \cos 8\theta) = 2(2cos^2 4\,\theta) = (2\cos 4\theta)^2$ ....(1)

To find $\cos 4\theta$ in powers of $\cos\theta$,

Consider $(\cos 4\theta + i\sin 4\theta) = (\cos\theta + i\sin\theta)^4$

$= cos^4\theta + 4cos^3\theta i \sin\theta + 6cos^2\theta i^2 sin^2\theta + 4\,cos\theta i^3 sin^3\theta + i^4 sin^4\theta$

$= (cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta) + i(4cos^3\theta \sin\theta - 4\,cos\theta sin^3\theta)$

Equating real Parts. $\cos 4\theta = cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta$

$= cos^4\theta - 6cos^2\theta(1 - cos^2\theta) + (1 - cos^2\theta)^2$

$= cos^4\theta - 6cos^2\theta + 6cos^4\theta + 1 - 2cos^2\theta + cos^4\theta$

$= 8cos^4\theta - 8cos^2\theta + 1$

$\therefore 2\cos 4\theta = 16cos^4\theta - 16cos^2\theta + 2$ Putting this value in (1)

$2(1 + \cos 8\theta) = (16cos^4\theta - 16cos^2\theta + 2)^2$

$= [(2\cos\theta)^4 - 4(2\cos\theta)^2 + 2]^2$

$= (x^4 - 4x^2 + 2)^2$ where $x = 2\cos\theta$

9. Prove that $\frac{1+\cos 9A}{1+\cos A} = [16cos^4A - 8\,cos^3A - 12\,cos^2A + 4\cos A + 1]^2$

Solution: $\frac{1+\cos 9A}{1+\cos A} = \frac{2\cos^2\left(\frac{9A}{2}\right)}{2\cos^2\left(\frac{A}{2}\right)} = \left[\frac{\cos\left(\frac{9A}{2}\right)}{\cos\left(\frac{A}{2}\right)}\right]^2$

$= \left(\frac{2\cos\left(\frac{9A}{2}\right)\sin\left(\frac{A}{2}\right)}{2\cos\left(\frac{A}{2}\right)\sin\left(\frac{A}{2}\right)}\right)^2 = \left[\frac{\sin\left(\frac{9A}{2}+\frac{A}{2}\right)-\sin\left(\frac{9A}{2}-\frac{A}{2}\right)}{\sin A}\right]^2$

$= \left(\frac{\sin(5A)-\sin(4A)}{\sin A}\right)^2$ .... (1)

By De Moivre's Theorem, $(cos\,5\theta + i\,sin\,5\theta) = (cos\,\theta + i\,sin\,\theta)^5$

$= cos^5\theta + 5\,cos^4\theta(i\sin\theta) + 10cos^3\theta(i\sin\theta)^2 + 10cos^2\theta(i\sin\theta)^3$
$+ 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$ .... Using Binomial Theorem

$= cos^5\theta + i\,5cos^4\theta\,\sin\theta - 10cos^3\theta sin^2\theta - i.10cos^2\theta sin^3\theta + 5\cos\theta sin^4\theta + i\,sin^5\theta$

$= (cos^5\theta - 10\,cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta) + i(5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta)$

Equating imaginary parts

$\sin 5\,\theta = 5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta$ .... (2)

Consider $(\cos 4\theta + i\sin 4\theta) = (\cos\theta + i\sin\theta)^4$

$= cos^4\theta + 4cos^3\theta i\sin\theta + 6cos^2\theta i^2 sin^2\theta + 4\,cos\theta i^3 sin^3\theta + i^4 sin^4\theta$

$= (cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta) + i(4cos^3\theta\sin\theta - 4\,cos\theta sin^3\theta)$

Equating imaginary parts

$\sin 4\theta = 4\,cos^3\theta\sin\theta - 4cos\theta\,sin^3\theta$ .... (3)

Put (2) & (3) in (1) we get

$\frac{1+\cos 9A}{1+\cos A} = \left[\frac{(5\,cos^4A\sin A - 10cos^2A sin^3A + sin^5A) - (4\,cos^3A\sin A - 4cosA sin^3A)}{\sin A}\right]^2$

$= (5\cos^2 A - 10\cos^2 A\sin^2 A + \sin^4 A - 4\cos^2 A + 4\cos A\sin^2 A)^2$

$= [5\cos^2 A - 10\cos^2 A(1 - \cos^2 A) + (1 - \cos^2 A)^2 - 4\cos^3 A + 4\cos A(1 - \cos^2 A)]^2$

$= [5\cos^2 A - 10\cos^2 A + 10\cos^4 A + 1 - 2\cos^2 A + \cos^4 A - 4\cos^3 A + 4\cos A -$
$4\cos^3 A]^2$

$$= (16\cos^4 A - 8\cos^3 A - 12\cos^2 A + 4\cos A + 1)^2$$

10. Prove that $\frac{1-\cos 9A}{1-\cos A} = [16\,cos^4A + 8\,cos^3A - 12\,cos^2A - 4\cos A + 1]^2$

Solution: $\frac{1-\cos 9A}{1-\cos A} = \frac{2\sin^2\left(\frac{9A}{2}\right)}{2\sin^2\left(\frac{A}{2}\right)} = \left(\frac{\sin\left(\frac{9A}{2}\right)}{\sin\left(\frac{A}{2}\right)}\right)^2 = \left(\frac{2\sin\left(\frac{9A}{2}\right)\cos\left(\frac{A}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}\right)^2 = \left[\frac{\sin\left(\frac{9A}{2}+\frac{A}{2}\right)+\sin\left(\frac{9A}{2}-\frac{A}{2}\right)}{\sin A}\right]^2$

$= \left(\frac{\sin(5A)+\sin(4A)}{\sin A}\right)^2$ Continue as above example

SOME PRACTICE PROBLEMS

1. Using De Moivre's Theorem prove that,

   $\cos 4\theta = cos^4\theta - 6\,cos^2\theta sin^2\theta + sin^4\theta$ and

   $\sin 4\theta = 4cos^3\theta sin\theta - 4\cos\theta\, sin^3\theta$

2. Prove that. $\frac{\sin 6\theta}{\sin 2\theta} = 16\,cos^4\theta - 16\,cos^2\theta + 3$

3. If $\cos 6\theta = a\,cos^6\theta + b\,cos^4\theta\,sin^2\theta + c cos^2\theta\,sin^4\theta + d\,sin^6\theta$. find a. b. c. d.

4. Express $\sin 7\theta$ *and* $\cos 7\theta$ in terms of powers of $\sin\theta$ *and* $\cos\theta$.

5. Prove that, $\frac{\sin 7\theta}{\sin\theta} = 7 - 56\,sin^2\theta + 112\,sin^4\theta - 64\,sin^6\theta$

6. Show that $\tan 7\theta = \frac{7\tan\theta - 35tan^3\theta + 21tan^5\theta - tan^7\theta}{1 - 21tan^2\theta + 35tan^4\theta - 7tan^6\theta}$.

7. Express $\tan 7\theta$ in terms of powers of $\tan\theta$

Hence deduce $7\,tan^6\frac{\pi}{14} - 35\,tan^4\frac{\pi}{14} + 21\,tan^2\frac{\pi}{14} - 1 = 0$

8. Prove that $\frac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2$ *where* $x = 2\cos\theta$

9. Prove that $\frac{1-\cos 7\theta}{1-\cos\theta} = (x^3 + x^2 - 2x - 1)^2$ *where* $x = 2\cos\theta$

Answers

3. $a = 1, b = -15, c = 15, d = -1$

4. $\sin 7\theta = 7cos^6\theta\sin\theta - 35\,cos^4\theta\,sin^3\theta + 21\,cos^2\theta\ \,sin^5\theta - sin^7\theta$;

   $\cos 7\theta = cos^7\theta - 21\,cos^5\theta\,sin^2\theta + 35\,cos^3\theta\,sin^4\theta - 7\cos\theta\ sin^6\theta$.

2) Expansion of $sin^n\theta, cos^n\theta$ in term of $\sin n\,\theta, \cos n\theta$ (n is a + ve integer):

Let $x = \cos\theta + i\sin\theta = e^{i\theta}$ $\therefore \frac{1}{x} = \cos\theta - i\sin\theta = e^{-i\theta}$

Hence $x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

Again, $x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = e^{in\theta}$

$\frac{1}{x^n} = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta = e^{-in\theta}$

$x^n + \frac{1}{x^n} = 2\cos n\theta$ and $x^n - \frac{1}{x^n} = 2i\sin n\theta$

To expand $cos^n\theta$, write $cos^n\theta = \frac{1}{2^n}\left(x+\frac{1}{x}\right)^n$

To expand $sin^n\theta$, write $sin^n\theta = \frac{1}{(2i)^n}\left(x-\frac{1}{x}\right)^n$ and expand R.H.S. using binomial expansion

$$(x+a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \dots\dots\dots + a^n$$

SOME SOLVED EXAMPLES:

1. Show that $sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Solution: Let $x = \cos\theta + i\sin\theta, then\ \frac{1}{x} = cos\theta - i\sin\theta$

$\therefore x+\frac{1}{x} = 2\cos\theta \quad and\ x-\frac{1}{x} = 2\,\mathrm{i}\sin\theta$ ....(1)

$x^n = \cos n\theta + i\sin n\theta \ and\ \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n+\frac{1}{x^n} = 2\cos n\theta \quad and\ x^n-\frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta$ ....(2)

$\therefore (2\,i\sin\theta)^5 = \left(x-\frac{1}{x}\right)^5$ from (1)

$= x^5 - 5x^4.\frac{1}{x} + 10x^3.\frac{1}{x^2} - 10x^2.\frac{1}{x^3} + 5x.\frac{1}{x^4} - \frac{1}{x^5}$

$= x^5 - 5x^3 + 10x - 10\frac{1}{x} + 5\frac{1}{x^3} - \frac{1}{x^5}$

$32\,i^5sin^5\theta = \left(x^5-\frac{1}{x^5}\right) - 5\left(x^3-\frac{1}{x^3}\right) + 10\left(x-\frac{1}{x}\right)$

$\therefore 32\,i\,sin^5\theta = 2\,i\sin 5\,\theta - 5(2i\sin 3\theta) + 10\,(2i\,sin\theta)$ from (2)

$\therefore sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

2. Using De Moivre's Theorem prove that. $cos^6\theta + sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$

Solution: Let $x = \cos\theta + i\sin\theta, then\ \frac{1}{x} = \cos\theta - i\,sin\theta$

$\therefore x+\frac{1}{x} = 2\cos\theta \quad and\ x-\frac{1}{x} = 2\,\mathrm{i}\sin\theta$ ....(1)

$x^n = \cos n\theta + i\sin n\theta \ and\ \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n+\frac{1}{x^n} = 2\cos n\theta \quad and\ x^n-\frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta$ ....(2)

$(2\cos\theta)^6 = \left(x+\frac{1}{x}\right)^6$ from (1)

$= x^6 + 6x^5.\frac{1}{x} + 15x^4.\frac{1}{x^2} + 20x^3.\frac{1}{x^3} + 15x^2.\frac{1}{x^4} + 6x.\frac{1}{x^5} + \frac{1}{x^6}$

$2^6cos^6\theta = x^6 + 6x^4 + 15x^2 + 20 + 15.\frac{1}{x^2} + 6.\frac{1}{x^4} + \frac{1}{x^6}$ ....(3)

$(2\,i\sin\theta)^6 = \left(x-\frac{1}{x}\right)^6$ from (1)

$= x^6 - 6x^5.\frac{1}{x} + 15x^4.\frac{1}{x^2} - 20x^3.\frac{1}{x^3} + 15x^2.\frac{1}{x^4} - 6x.\frac{1}{x^5} + \frac{1}{x^6}$

$-2^6sin^6\theta = x^6 - 6x^4 + 15x^2 - 20 + 15.\frac{1}{x^2} - 6.\frac{1}{x^4} + \frac{1}{x^6}$

$\therefore 2^6 sin^6\theta = -x^6 + 6x^4 - 15x^2 + 20 - 15.\frac{1}{x^2} + 6.\frac{1}{x^4} - \frac{1}{x^6}$ ....(4)

Adding (3) and (4) $2^6(cos^6\theta + sin^6\theta) = 12x^4 + 40 + 12.\frac{1}{x^4}$

$2^6(cos^6\theta + sin^6\theta) = 4\left[3\left(x^4 + \frac{1}{x^4}\right) + 10\right]$

$\therefore cos^6\theta + sin^6\theta = \frac{1}{16}\left[3\left(x^4 + \frac{1}{x^4}\right) + 10\right]$

$\therefore cos^6\theta + sin^6\theta = \frac{1}{16}[6\cos 4\theta + 10]$ from (2)

$= \frac{1}{8}[3 \cos 4\theta + 5]$

3. Expand $sin^7\theta$ in a series of sines of multiples of θ

Solution: Let $x = \cos\theta + i\sin\theta$ $\therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore x + \frac{1}{x} = 2\cos\theta$ *and* $x - \frac{1}{x} = 2\,\text{i}\sin\theta$ ....(1)

$x^n = \cos n\theta + i\sin n\theta$ *and* $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta$ *and* $x^n - \frac{1}{x^n} = 2\,\text{i}\sin n\theta$ ....(2)

$(2\,i\sin\theta)^7 = \left(x - \frac{1}{x}\right)^7$ from (1)

$= x^7 - 7x^6.\frac{1}{x} + 21x^5.\frac{1}{x^2} - 35x^4.\frac{1}{x^3} + 35x^3.\frac{1}{x^4} - 21x^2.\frac{1}{x^5} + 7x.\frac{1}{x^6} - \frac{1}{x^7}$

$= x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}$

$= \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)$

$-2^7 i\, sin^7\theta = 2i\sin 7\theta - 7.(2i\sin 5\theta) + 21.(2i\sin 3\theta) - 35.(2i\sin\theta)$ from (2)

$\therefore -2^6 sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$

$\therefore sin^7\theta = -\frac{1}{2^6}(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta)$

4. Expand $cos^7\theta$ in a series of cosines of multiples of θ

Solution: Let $x = \cos\theta + i\sin\theta$ $\therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore x + \frac{1}{x} = 2\cos\theta$ *and* $x - \frac{1}{x} = 2\,\text{i}\sin\theta$ ....(1)

$x^n = \cos n\theta + i\sin n\theta$ *and* $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta$ *and* $x^n - \frac{1}{x^n} = 2\,\text{i}\sin n\theta$ ....(2)

$(2\cos\theta)^7 = \left(x + \frac{1}{x}\right)^7$ .... from (1)

$= x^7 + 7x^6\frac{1}{x} + 21x^5\frac{1}{x^2} + 35x^4\frac{1}{x^3} + 35x^3\frac{1}{x^4} + 21x^2\frac{1}{x^5} + 7x\frac{1}{x^6} + \frac{1}{x^7}$

$= x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$

$= \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$

$\therefore 2^7\cos^7\theta = 2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(\cos\theta)$ From (2)

$$\cos^7\theta = \frac{1}{2^6}[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta]$$

5. Show that $2^5 sin^4\theta cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$.

Solution: Let $x = \cos\theta + i\sin\theta \quad \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\mathrm{i}\sin\theta$ ....(1)

$x^n = \cos n\theta + i\sin n\theta \quad and \quad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta$ ....(2)

$(2i\sin\theta)^4(2\cos\theta)^4 = \left(x-\frac{1}{x}\right)^4\left(x+\frac{1}{x}\right)^2$ From (1)

$\therefore 2^6\sin^4\theta\cos^2\theta = \left(x-\frac{1}{x}\right)^2\left(x-\frac{1}{x}\right)^2\left(x+\frac{1}{x}\right)^2 = \left(x-\frac{1}{x}\right)^2\left(x^2-\frac{1}{x^2}\right)^2$

$= \left(x^2-2+\frac{1}{x^2}\right)\left(x^4-2+\frac{1}{x^4}\right)$

$= x^6 - 2x^2 + \frac{1}{x^2} - 2x^4 + 4 - \frac{2}{x^4} + x^2 - \frac{2}{x^2} + \frac{1}{x^6}$

$= \left(x^6+\frac{1}{x^6}\right) - 2\left(x^4+\frac{1}{x^4}\right) - \left(x^2+\frac{1}{x^2}\right) + 4$

$= 2\cos 6\theta - 2(2\cos 4\theta) - (2\cos 2\theta) + 4$ From (2)

$\therefore 2^5\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

6. Prove that $cos^5\theta sin^3\theta = -\frac{1}{2^7}[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta]$

Solution: Let $x = \cos\theta + i\sin\theta \quad \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\mathrm{i}\sin\theta$ ....(1)

$x^n = \cos n\theta + i\sin n\theta \quad and \quad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta$ ....(2)

$(2\cos\theta)^5(2i\sin\theta)^3 = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^3$

$2^8 i^3\cos^5\theta\sin^3\theta = \left(x+\frac{1}{x}\right)^2\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^3$

$-2^8 i\cos^5\theta\sin^3\theta = \left(x+\frac{1}{x}\right)^2\left(x^2-\frac{1}{x^2}\right)^3$

$= \left(x^2-2+\frac{1}{x^2}\right)\left(x^6-3x^2+\frac{3}{x^2}-\frac{1}{x^6}\right)$

$= x^8 - 3x^4 + 3 - \frac{1}{x^4} + 2x^6 - 6x^2 + \frac{6}{x^2} - \frac{2}{x^6} + x^4 - 3 + \frac{3}{x^4} - \frac{1}{x^8}$

$= \left(x^8-\frac{1}{x^8}\right) + 2\left(x^6-\frac{1}{x^6}\right) - 2\left(x^4-\frac{1}{x^4}\right) - 6\left(x^2-\frac{1}{x^2}\right)$

$= (2i\sin 8\theta) + 2(2i\sin 6\theta) - 2(2i\sin 4\theta) - 6(2i\sin 2\theta)$ From (2)

$$\therefore -2^7 \cos^5\theta \sin^3\theta = \sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta$$

$$\therefore \cos^5\theta \sin^3\theta = -\frac{1}{2^7}\left[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right]$$

SOME PRACTICE PROBLEMS:

1. Show that $cos^6\theta = \frac{1}{32}[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10]$
2. Prove that $cos^6\theta - sin^6\theta = \frac{1}{16}[\cos 6\theta + 15\cos 2\theta]$
3. Express $sin^8\theta$ in a series of cosines of multiples of θ.
4. Prove that, $cos^8\theta = \frac{1}{2^7}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$
5. Prove that $cos^8\theta + sin^8\theta = \frac{1}{64}[\cos 8\theta + 28\cos 4\theta + 35]$.
6. Show that $2^6 sin^4\theta cos^3\theta = \cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta$.
7. Prove that $sin^7\theta cos^3\theta = -\frac{1}{512}[\sin 10\theta - 4\sin 8\theta + 3\sin 6\theta + 8\sin 4\theta - 14\sin 2\theta]$
8. If $sin^4\theta\, cos^3\theta = a_1\cos\theta + a_3\cos 3\theta + a_5\cos 5\theta + a_7\cos 7\theta$,
   Prove that $a_1 + 9a_3 + 25a_5 + 49a_7 = 0$.

Answers:

3. $sin^8\theta = \frac{1}{2^7}[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35]$

3) Minor, Cofactor of an Element And Adjoint of a Matrix:

Minor of an element:

The minor of an element a_{ij} of a square matrix $A = [a_{ij}]_{m\times m}$ is the value of the determinant obtained by deleting the i[th] row and j[th] column of the matrix A. The minor of a_{ij} is denoted by M_{ij}. Note that, if A is matrix of order $m \times m$ then the minor of any element of A is the value of the Determinant of order $(m-1)\times(m-1)$

For example: Consider the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 5 \\ 6 & -2 & 1 \end{bmatrix}$

Here $a_{11} = 2$ (element in 1[st] row and 1[st] column). The minor of a_{11} is the value of the determinant
obtained from the matrix A by deleting 1[st] row and 1[st] column.

Hence, the minor of $a_{11} = M_{11} = \begin{vmatrix} 4 & 5 \\ -2 & 1 \end{vmatrix} = 4 + 10 = 14$

Similarly $a_{23} = 5$ and the minor of a_{23} is $M_{23} = \begin{vmatrix} 2 & -1 \\ 6 & -2 \end{vmatrix} = -4 + 6 = 2$

Cofactor of an element:

The cofactor of an element a_{ij} of a square matrix $A = [a_{ij}]$ is given by $(-1)^{i+j}M_{ij}$, where M_{ij} is the minor of a_{ij} The cofactor of a_{ij} is denoted by A_{ij} i.e $A_{ij} = (-1)^{i+j} M_{ij}$

For example: For the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 5 \\ 6 & -2 & 1 \end{bmatrix}$

$$\text{Cofactor of } a_{11} = (-1)^{1+1}M_{11} = (-1)^2 \begin{vmatrix} 4 & 5 \\ -2 & 1 \end{vmatrix} = 4 + 10 = 14$$

$$\text{Cofactor of } a_{23} = (-1)^{2+3}M_{23} = (-1)^5 \begin{vmatrix} 2 & -1 \\ 6 & -2 \end{vmatrix} = -(-4 + 6) = -2$$

Cofactor Matrix:

The cofactor matrix of the square matrix $A = [a_{ij}]_{m\times m}$ is a matrix of order $m \times m$ where each element a_{ij} of the matrix A is replaced by its cofactor A_{ij}.

i.e If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then Cofactor matrix $= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Adjoint of a Matrix:

The adjoint of a matrix $A = [a_{ij}]$ is the transpose of the cofactor matrix. It is denoted by 'adj A'.

i.e if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

For example: Consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 6 & 8 \end{bmatrix}$

Here, $a_{11} = 3,\ M_{11} = 8$ and $A_{11} = (-1)^{1+1}(8) = 8$

$a_{12} = -2,\ M_{12} = 6$ and $A_{12} = (-1)^{1+2}(6) = -6$

$a_{21} = 6,\ M_{21} = -2$ and $A_{21} = (-1)^{2+1}(-2) = 2$

$a_{22} = 8,\ M_{22} = 3$ and $A_{22} = (-1)^{2+2}(3) = 3$

$\therefore$ cofactor matrix $= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 8 & -6 \\ 2 & 3 \end{bmatrix}$ $\qquad \therefore adj\ A = \begin{vmatrix} 8 & 2 \\ -6 & 3 \end{vmatrix}$

INVERSE BY ADJOINT METHOD:

If $A = [a_{ij}]_{m\times m}$ is a non – singular square matrix, i.e $|A| \neq 0$, then its inverse exits and it is given as $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

1. Find the adjoint of the matrix $\begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix}$

$a_{11} = -1, \quad M_{11} = 4 \qquad \therefore A_{11} = (-1)^{1+1}(4) = 4$

$a_{12} = 2, \quad M_{12} = -1 \qquad \therefore A_{12} = (-1)^{1+2}(-1) = 1$

$a_{21} = -3, \quad M_{21} = 2 \qquad \therefore A_{21} = (-1)^{2+1}(2) = -2$

$a_{22} = 4, \quad M_{22} = -1 \qquad \therefore A_{22} = (-1)^{2+2}(-1) = -1$

Hence the cofactor matrix $= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -2 & -1 \end{bmatrix}$

$\therefore adjA = \begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}$

2. Find the adjoint of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 5 \\ -2 & 0 & -1 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 5 \\ -2 & 0 & -1 \end{bmatrix}$

First we have to find the cofactor matrix $= [A_{ij}]_{3\times3}$ where $A_{ij} = (-1)^{i+j}.M_{ij}$

Now, $M_{11} = \begin{vmatrix} 3 & 5 \\ 0 & -1 \end{vmatrix} = -3 - 0 = -3 \qquad \therefore A_{11} = (-1)^{1+1}(-3) = -3$

$M_{12} = \begin{vmatrix} -2 & 5 \\ -2 & -1 \end{vmatrix} = 2 + 10 = 12 \qquad \therefore A_{12} = (-1)^{1+2}(12) = -12$

$M_{13} = \begin{vmatrix} -2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6 \qquad \therefore A_{13} = (-1)^{1+3}(6) = 6$

$M_{21} = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1 - 0 = 1 \qquad \therefore A_{21} = (-1)^{2+1}(1) = -1$

$M_{22} = \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = -1 + 4 = 3 \qquad \therefore A_{22} = (-1)^{2+2}(3) = 3$

$M_{23} = \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 0 - 2 = -2 \qquad \therefore A_{23} = (-1)^{2+3}(-2) = 2$

$M_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5 - 6 = -11 \qquad \therefore A_{31} = (-1)^{3+1}(-11) = -11$

$M_{32} = \begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} = 5 + 4 = 9 \qquad \therefore A_{32} = (-1)^{3+2}(9) = -9$

$M_{33} = \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} = 3 - 2 = 1 \qquad \therefore A_{33} = (-1)^{3+3}(1) = 1$

Hence the cofactor matrix $= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -3 & -12 & 6 \\ -1 & 3 & 2 \\ -11 & -9 & 1 \end{bmatrix}$

$\therefore \; adj\,A = \begin{bmatrix} -3 & -1 & -11 \\ -12 & 3 & -9 \\ 6 & 2 & 1 \end{bmatrix}$

3. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$, verify that $A(adj\ A) = (adj\ A)A = |A|.I.$

Solution: $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix} = 1(0+0) + 1(9+2) + 2(0-0) = 0 + 11 + 0 = 11$$

First we have to find the cofactor matrix $= [A_{ij}]_{3\times3}$, where $A_{ij} = (-1)^{i+j}M_{ij}$

$$A_{11} = (-1)^{1+1}M_{11} = \begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} = 0 + 0 = 0$$

$$A_{12} = (-1)^{1+2}M_{12} = -\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -(9+2) = -11$$

$$A_{13} = (-1)^{1+3}M_{13} = \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$$

$$A_{21} = (-1)^{2+1}M_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = -(-3-0) = 3$$

$$A_{22} = (-1)^{2+2}M_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = 1$$

$$A_{23} = (-1)^{2+3}M_{23} = -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -(0+1) = -1$$

$$A_{31} = (-1)^{3+1}M_{31} = \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = 2 - 0 = 2$$

$$A_{32} = (-1)^{3+2}M_{32} = -\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -(-2-6) = 8$$

$$A_{33} = (-1)^{3+3}M_{33} = \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 0 + 3 = 3$$

Hence the cofactor matrix $= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix}$

$$\therefore adj\ A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\therefore \text{A(adj A)} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0-0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \quad \text{..................(1)}$$

$$\therefore \text{(adj A) A} = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0-0+0 & 0+2+9 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \quad \text{..........(2)}$$

$$|A|I = 11\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}11 & 0 & 0\\0 & 11 & 0\\0 & 0 & 11\end{bmatrix} \qquad \text{.......................(3)}$$

from (1), (2) and (3), we get $A(\text{adj } A) = (\text{adj } A)A = |A|.I$

Note: This relation is valid for any non – singular matrix A.

4. Find the inverse of the matrix $\begin{bmatrix}1 & 0 & 0\\0 & cos\theta & sin\theta\\0 & sin\theta & -cos\theta\end{bmatrix}$ by the adjoint method

Solution: Let $A = \begin{bmatrix}1 & 0 & 0\\0 & cos\theta & sin\theta\\0 & sin\theta & -cos\theta\end{bmatrix}$

$$\therefore |A| = \begin{vmatrix}1 & 0 & 0\\0 & cos\theta & sin\theta\\0 & sin\theta & -cos\theta\end{vmatrix} = 1(-cos^2\theta - sin^2\theta) - 0 + 0$$

$$= -(cos^2\theta + sin^2\theta) = -1 \neq 0$$

$\therefore A^{-1} exists$

First we have to find the cofactor matrix $= [A_{ij}]_{3\times3}$, where $A_{ij} = (-1)^{i+j}M_{ij}$

Now, $A_{11} = (-1)^{1+1}M_{11} = \begin{vmatrix}cos\theta & sin\theta\\sin\theta & -cos\theta\end{vmatrix} = -cos^2\theta - sin^2\theta = -1$

$$A_{12} = (-1)^{1+2}M_{12} = -\begin{vmatrix}0 & sin\theta\\0 & -cos\theta\end{vmatrix} = -(0-0) = 0$$

$$A_{13} = (-1)^{1+3}M_{13} = -\begin{vmatrix}0 & cos\theta\\0 & sin\theta\end{vmatrix} = 0 - 0 = 0$$

$$A_{21} = (-1)^{2+1}M_{21} = -\begin{vmatrix}0 & 0\\sin\theta & -cos\theta\end{vmatrix} = -(-0-0) = 0$$

$$A_{22} = (-1)^{2+2}M_{22} = \begin{vmatrix}1 & 0\\0 & -cos\theta\end{vmatrix} = -cos\theta - 0 = -cos\theta$$

$$A_{23} = (-1)^{2+3}M_{23} = -\begin{vmatrix}1 & 0\\0 & sin\theta\end{vmatrix} = -(sin\theta - 0) = -sin\theta$$

$$A_{31} = (-1)^{3+1}M_{31} = -\begin{vmatrix}0 & 0\\cos\theta & sin\theta\end{vmatrix} = 0 - 0 = 0$$

$$A_{32} = (-1)^{3+2}M_{32} = -\begin{vmatrix}1 & 0\\0 & sin\theta\end{vmatrix} = -(sin\theta - 0) = -sin\theta$$

$$A_{33} = (-1)^{3+3}M_{33} = \begin{vmatrix}1 & 0\\0 & cos\theta\end{vmatrix} = cos\theta - 0 = cos\theta$$

Hence the cofactor matrix $= \begin{bmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\0 & -cos\theta & -sin\theta\\0 & -sin\theta & cos\theta\end{bmatrix}$

$$\therefore adjA = \begin{bmatrix}-1 & 0 & 0\\0 & -cos\theta & -sin\theta\\0 & -sin\theta & cos\theta\end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adj\ A) = \frac{1}{-1}\begin{bmatrix}-1 & 0 & 0\\0 & -cos\theta & -sin\theta\\0 & -sin\theta & cos\theta\end{bmatrix} \qquad \therefore A^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & cos\theta & sin\theta\\0 & sin\theta & -cos\theta\end{bmatrix}$$

5. Find the inverse of the matrix $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ by the adjoint method

Solution: Let $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} -1 & 5 \\ -3 & 2 \end{vmatrix} = -2 + 15 = 13 \neq 0$

$\therefore A^{-1} exists$

First we have to find the cofactor matrix $= [A_{ij}]_{2\times 2}$, where $A_{ij} = (-1)^{i+j} M_{ij}$

Now, $A_{11} = (-1)^{1+1} M_{11} = 2$ $\qquad A_{12} = (-1)^{1+2} M_{12} = -(-3) = 3$

$A_{21} = (-1)^{2+1} M_{21} = -5$ $\qquad A_{22} = (-1)^{2+2} M_{22} = -1$

Hence, the cofactor matrix $= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

$\therefore adjA = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|}(adj\ A) = \frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

SOME MORE RESULTS:

Theorem: If A be any n – rowed matrix, then prove that $A(adj\ A) = (adj\ A)A = |A|\ I_n$

where I_n is the n – rowed unit matrix.

[The theorem states that the matrices A and $(adj\ A)$ are commutative and that their product is a scalar matrix, every diagonal element of which is $|A|$.]

Theorem: If A is any n – rowed matrix, then prove that $|adj\ A| = |A|^{n-1}$

Theorem: If A is any n – rowed matrix, then prove that $adj(adj\ A) = |A|^{n-2}.A$

Corollary: Prove that $|adj\ (adj\ A)| = |A|^{(n-1)^2}$

Invertible matrices: Inverse or Reciprocal of a Matrix.

Definition: Let A be any n – rowed matrix. If there exists a matrix B such that $AB = BA = I_n$ then B is called an inverse of A and is denoted by A^{-1}. The matrix A is called an invertible matrix

THEOREMS ON INVERSE OF MATRICES

Theorem: Existence of the Inverse: The necessary and sufficient condition for a square matrix A to possess an inverse is that A is non – singular.

Note: If A is invertible matrix, then the inverse of A is $\frac{1}{|A|}.(adj\ A)$

Theorem : Every invertible matrix possesses a unique inverse.

Theorem: Reversal law for the inverse of a product.

If A and B are two n – rowed non – singular matrices, then prove that

(i) AB is non – singular and

(ii) $(AB)^{-1} = B^{-1}A^{-1}$ i.e The inverse of a product is the product of the inverses taken in the reverse order.

Theorem: If A is an n – rowed non – singular matrix, then prove that $(A^T)^{-1} = (A^{-1})^T$

i.e The inverse of the transpose of any non – singular matrix is equal to the transpose of the inverse of that matrix.

Inverse of an Adjoint: We have proved that $A\ adj\ A = |A|\ I$

$$\therefore \left(\frac{A}{|A|}\right).(adjA) = I$$

Since the product of $\frac{A}{|A|}$ and adj A is a unit matrix

by definition of the inverse each is the inverse of the other.

$$\therefore (adjA)^{-1} = \frac{A}{|A|} \quad \text{and} \quad \left(\frac{A}{|A|}\right)^{-1} = adjA$$

NOTE:

(1) If A and B are two non – singular square matrices of the same order, adj (AB) = (adjB) (adjA)

(2) If A is a square matrix, then $(adj\ A)^T = adj(A^T)$.

(3) If A is symmetric matrix,then adj A is also symmetric.

(4) If P and Q are the matrices such that $|P| = |Q| = 1$ *and* $adj\ B = A$,

then PAQ is the adjoint of $Q^{-1}BP^{-1}$

(5) If A is a square matrix of order n and $|A| \neq 0$, then

(i) $|A^{-1}| = \frac{1}{|A|}$ (ii) $\text{adj}(A^{-1}) = (\text{ adj}A)^{-1}$

(6) A is a skew – symmetric matrix of odd order then, A is singular i.e $|A| = 0$

EXERCISE

1. Show that the $adj(adjA)$ of $A = \frac{1}{9}\begin{bmatrix} -1 & -8 & 4 \\ -4 & 4 & 7 \\ -8 & -1 & -4 \end{bmatrix}$ is A itself

2. If A = $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, find $adjA$ and hence , find $|A|$ without evaluating it.

3. Find $adjA$ for A where A = $\begin{bmatrix} -2 & 1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & 5 \end{bmatrix}$. What is $adj(adjA)$?

4. If A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, find **(i)** $adjA$ and **(ii)** $adj(adjA)$

5. Verify that $(adjA)' = (adjA')$ for **(i)** $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

6. If A = $\begin{bmatrix} 1 & 2 & 1 \\ a & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ and $adj(adjA) = A$, find a

7. Find the adjoint of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ and verify the theorem

$A(adj\ A) = (adj\ A)A = |A|I.$

8. Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ and verify that

$A(adj\ A) = (adj\ A)A = |A|I_3.$

9. Let I be the unit matrix of order n and $adj(2I) = 2^k.I$. Find the value of k.

10. If $A(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Prove that $[A(\alpha)]^{-1} = A(-\alpha)$

11. Find the inverse of the matrix $A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ if $a^2 + b^2 + c^2 + d^2 = 1$

12. If A = $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$, find $adjA$, A^{-1}. Also find B such that $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$

13. Find the inverse of $\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & b & 1 \end{bmatrix}$, hence , find the inverse of $\begin{bmatrix} 1+ab & a & 0 \\ b & 1+ab & a \\ 0 & b & 1 \end{bmatrix}$

14. If A = $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 2 & 3 \end{bmatrix}$, find A^{-1} if it exists . Hence, find the inverse of B = $\begin{bmatrix} 3 & 6 & -3 \\ 0 & 3 & -3 \\ 6 & 6 & 9 \end{bmatrix}$

15. If $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 2 & 2 & 0 \end{bmatrix}$ verify that $A(adj\ A) = |A|I$. Hence find the inverse of $(adj\ A)$.

16. Find the inverse of $(adj\ A)$ if it exists, where, $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

17. Verify that $A(adjA) = |A|I$ and $adj(adjA) = A|A|$ also find $(adjA)^{-1}$ for

(i) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ 4 & -5 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 3 \end{bmatrix}$

18. Find the matrix A , if $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

19. Find the inverse of A if $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

20. Prove that $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -tan\left(\frac{\theta}{2}\right) \\ tan\left(\frac{\theta}{2}\right) & 1 \end{bmatrix} \begin{bmatrix} 1 & tan\left(\frac{\theta}{2}\right) \\ -tan\left(\frac{\theta}{2}\right) & 1 \end{bmatrix}^{-1}$

21. Prove that an inverse of a skew – symmetric matrix of odd order does not exists.

22. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ $and\ B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ verify the result $(AB)^{-1} = B^{-1}A^{-1}$

23. If A = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $and\ B = \begin{bmatrix} \frac{q+r}{2} & \frac{r-p}{2} & \frac{q-p}{2} \\ \frac{r-q}{2} & \frac{r+p}{2} & \frac{p-q}{2} \\ \frac{q-r}{2} & \frac{p-r}{2} & \frac{p+q}{2} \end{bmatrix}$, prove that , ABA^{-1} is a diagonal matrix.

24. Find the inverse of the matrix $S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and show that SAS^{-1} is a diagonal matrix,

where $A = \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$

25. Find the inverse of the matrix $S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and prove that SAS^{-1} is a diagonal matrix,

where $A = \frac{1}{2}\begin{bmatrix} 4 & -1 & 1 \\ -2 & 3 & -1 \\ 2 & 1 & 5 \end{bmatrix}$

ANSWERS

2. $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, $|A| = 1$ 3. $\begin{bmatrix} 4 & 8 & 20 \\ 12 & 4 & 16 \\ 4 & 4 & 8 \end{bmatrix}$, $16A$ 4. (i) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ -4 & 3 & -1 \end{bmatrix}$

(ii) A itself

6. $a = 3$ 7. $\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$ 8. $\begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$

9. $k = (n-1)$

11. $\begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$

12. $adjA = \begin{bmatrix} 9 & -2 & -4 \\ 1 & 2 & -1 \\ -12 & 1 & 7 \end{bmatrix}, A^{-1} = \frac{1}{5}\begin{bmatrix} 9 & -2 & -4 \\ 1 & 2 & -1 \\ -12 & 1 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

13. $A^{-1} = \begin{bmatrix} 1 & -a & a^2 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ b^2 & -b & 1 \end{bmatrix}, (AB)^{-1} = \begin{bmatrix} 1 & -a & a^2 \\ -b & ab+1 & -a^2b-a \\ b^2 & -ab^2-b & a^2b^2+ab+1 \end{bmatrix}$

14. $A^{-1} = \frac{1}{3}\begin{bmatrix} 5 & -8 & -1 \\ -2 & 5 & 1 \\ -2 & 2 & 1 \end{bmatrix}, B^{-1} = \frac{1}{3}A^{-1} = \frac{1}{9}\begin{bmatrix} 5 & -8 & -1 \\ -2 & 5 & 1 \\ -2 & 2 & 1 \end{bmatrix}$

15. $\frac{1}{2}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 2 & 2 & 0 \end{bmatrix}$

16. $\frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

18. $\begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}$

19. $\begin{bmatrix} -21 & 11 & 9 \\ 14 & -7 & -6 \\ -2 & 1 & 1 \end{bmatrix}$

24. $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

25. $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$