# Trigonometric and Inverse Trig Problems

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Formula, $y = \sin(ax+b)$, $y^{(n)} = a^n \sin\left((ax+b) + n\frac{\pi}{2}\right)$

1) $y = (\sin 2x \sin 3x)\cos 4x = \sin 2x\,(\sin 3x \cos 4x)$

$= \sin 2x \left(\frac{1}{2}(\sin 7x - \sin x)\right)$

$= \frac{1}{4}(2\sin 2x \sin 7x - 2\sin 2x \sin x)$

$= \frac{1}{4}\left[\cos 5x - \cos 9x - (\cos x - \cos 3x)\right]$

$y = \frac{1}{4}\left[\cos 5x - \cos 9x - \cos x + \cos 3x\right]$

Apply formula, $y = \cos(ax+b)$, $y^{(n)} = a^n \cos\left(ax+b+n\frac{\pi}{2}\right)$

$y^{(n)} = \frac{1}{4}\Big[5^n \cos\left(5x+n\frac{\pi}{2}\right) - 9^n \cos\left(9x+n\frac{\pi}{2}\right)$

$- \cos\left(x+n\frac{\pi}{2}\right) + 3^n \cos\left(3x+n\frac{\pi}{2}\right)\Big]$

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2) $y = \cos^2 x \sin^3 x$ (H.W.)

$y = \frac{1}{8}\sin x - \frac{1}{16}(\sin 5x - \sin 3x)$

3) If $y = \sin^7 x$ Find $y^{(n)}$, $(y_n)$

(1 7 21 35 35 21 7 1)

Let $t = \cos x + i\sin x$, $\frac{1}{t} = \cos x - i\sin x$ / $\left(t^n - \frac{1}{t^n}\right) = 2i\sin nx$

$\left(t - \frac{1}{t}\right) = 2i\sin x \Rightarrow \left(t-\frac{1}{t}\right)^7 = 2^7 i^7 \sin^7 x$

$\sin^7 x = \frac{1}{2^7(-i)}\Big(t^7 - 7t^6\left(\frac{1}{t}\right) + 21t^5\left(\frac{1}{t}\right)^2 - 35t^4\left(\frac{1}{t}\right)^3$

$+ 35t^3\left(\frac{1}{t}\right)^4 - 21t^2\left(\frac{1}{t}\right)^5 + 7t\left(\frac{1}{t}\right)^6 - \left(\frac{1}{t}\right)^7\Big]$

$= -\frac{1}{2^7 i}\left[t^7 - 7t^5 + 21t^3 - 35t + 35\frac{1}{t} - \frac{21}{t^3} + \frac{7}{t^5} - \frac{1}{t^7}\right)$

$= -\frac{1}{2^7 i}\left[\left(t^7 - \frac{1}{t^7}\right) - 7\left(t^5 - \frac{1}{t^5}\right) + 21\left(t^3 - \frac{1}{t^3}\right) - 35\left(t - \frac{1}{t}\right)\right)$

$= -\frac{1}{2^7 i}\left[2i\sin 7x - 7(\cancel{2i}\sin 5x) + 21(\cancel{2i}\sin 3x) - 35(\cancel{2i}\sin x)\right]$

$\sin^7 x = -\frac{1}{2^7 i}\Big[\ldots + 21\sin 3x - 35\sin x\Big]$

$y = -\frac{1}{64}\left[ \sin 7x - 7\sin 5x + 21 \sin 3x \ldots \right]$

Apply formula

$$y^{(n)} = -\frac{1}{64}\left[ 7^n \sin\left(7x + n\frac{\pi}{2}\right) - 7(5)^n \sin\left(5x + n\frac{\pi}{2}\right) + 21(3)^n \sin\left(3x + n\frac{\pi}{2}\right) - 35 \sin\left(x + n\frac{\pi}{2}\right) \right]$$

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4) $y = \sin^4 x \cos^3 x$  Find $y^{(n)}$ (H.W)

$$= \left(t - \frac{1}{t}\right)^4 \left(t + \frac{1}{t}\right)^3 \rightarrow \left[\left(t - \frac{1}{t}\right)\left(t + \frac{1}{t}\right)\right]^3 \left(t - \frac{1}{t}\right) = \left(t^2 - \frac{1}{t^2}\right)^3 \left(t - \frac{1}{t}\right)$$

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5) Find $y^{(n)}$, If $y = 2^x \cos 9x = e^{(\log 2)x} \cos 9x$

By formula, $y = e^{ax} \cos(bx + c)$, $y^{(n)} = r^n e^{ax} \cos(bx + c + n\phi)$

$r = \sqrt{a^2 + b^2}$, $\phi = \tan^{-1}(b/a)$

$$y^{(n)} = r^n e^{(\log 2)x} \cos(9x + n\phi)$$

$$r = \sqrt{(\log 2)^2 + 9^2}, \quad \phi = \tan^{-1}\left(\frac{9}{\log 2}\right)$$

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6) $y = e^x(\sin x + \cos x) = e^x \sin x + e^x \cos x$ (HW)

$$= e^x \sqrt{2}\left[\sin x\left(\frac{1}{\sqrt{2}}\right) + \cos x\left(\frac{1}{\sqrt{2}}\right)\right] = \sqrt{2}e^x\left[\sin\left(x + \frac{\pi}{4}\right)\right]$$

formula

$$y^{(n)} = \sqrt{2}\, r^n e^x \sin\left(\left(x + \frac{\pi}{4}\right) + n\phi\right), \quad r = \sqrt{1+1} = \sqrt{2}, \quad \phi = \tan^{-1}(1) = \frac{\pi}{4}$$

$$y^{(n)} = \sqrt{2}(\sqrt{2})^n e^x \sin\left(\left(x + \frac{\pi}{4}\right) + n\frac{\pi}{4}\right)$$

$$= (\sqrt{2})^{n+1} e^x \sin\left(x + (n+1)\frac{\pi}{4}\right)$$

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Inverse trig

If $y = \tan^{-1} x$, Then Prove that, … where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

① $y^{(n)} = (-1)^{n-1}(n-1)!\ \sin^n\theta\ \sin n\theta$ where ...

② $y^{(n)} = \dfrac{(-1)^{n-1}(n-1)!}{(x^2+1)^{n/2}} \sin\left(n \tan^{-1}\left(\frac{1}{x}\right)\right)$

Sol$^n$: Consider 1$^{st}$ order diff.

$$y' = \frac{1}{x^2+1} = \frac{1}{(x+i)(x-i)} = \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

Now diff this $(n-1)$ times (Apply formula for $n-1$) $\left[\dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}\right]$

$$y^{(n)} = \frac{1}{2i}\left[\frac{(-1)^{(n-1)}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]$$

$$\rightarrow y^{(n)} = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right] \quad ①$$

Simplification, put $x = r\cos\theta$, $1 = r\sin\theta$

$\therefore r = \sqrt{x^2+1}$, $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

$$\therefore \frac{1}{(x-i)^n} = \frac{1}{[r(\cos\theta - i\sin\theta)]^n} = \frac{1}{r^n(\cos n\theta - i\sin n\theta)} = \frac{1}{r^n}[\cos n\theta + i\sin n\theta]$$

$$\& \ \frac{1}{(x+i)^n} = \frac{1}{[r(\cos\theta + i\sin\theta)]^n} = \frac{1}{r^n}[\cos n\theta - i\sin n\theta]$$

put in ①

$$y^{(n)} = \frac{(-1)^{n-1}(n-1)!}{\cancel{2i}}\left[\frac{1}{r^n}\left(\cancel{\cos n\theta} + \cancel{i}\sin n\theta - \cancel{\cos n\theta} + \cancel{i}\sin n\theta\right)\right]$$

$$= (-1)^{n-1}(n-1)!\ \frac{1}{r^n}\sin n\theta$$

for (A) put $\frac{1}{r} = \sin\theta$, $\boxed{y^{(n)} = (-1)^{n-1}(n-1)!\ \sin^n\theta\ \sin n\theta \text{ where } \theta = \tan^{-1}\left(\tfrac{1}{x}\right)}$

$\rightarrow$ (B) put $\frac{1}{r} = \frac{1}{(x^2+1)^{1/2}}$, $\boxed{y^{(n)} = \dfrac{(-1)^{n-1}(n-1)!}{(x^2+1)^{n/2}} \sin\left(n\tan^{-1}\left(\tfrac{1}{x}\right)\right)}$

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2) $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ Then Prove that

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$y^{(n)} = 2(-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta$

Sol$^n$: Put $x = \tan\alpha$, $\alpha = \tan^{-1}x$

$$y = \sin^{-1}\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right) = \sin^{-1}(\sin 2\alpha) = 2\alpha$$
$$= 2\tan^{-1}x$$

(H.W) (Solve)

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3) $y = x\tan^{-1}x$ Then Find $y^{(n)}$

Sol$^n$: Diff $y$ w.r.to $x$, $y' = \underbrace{\frac{x}{1+x^2}}_{A} + \underbrace{\tan^{-1}x}_{B} = A + B$ (Say)

$$y^{(n)} = A^{(n-1)} + B^{(n-1)}$$

for A, $A = \frac{x}{x^2+1} = \frac{x}{(x+i)(x-i)} = \frac{1}{2}\left[\frac{1}{x+i} + \frac{1}{x-i}\right]$

$$A^{(n)} = \frac{1}{2}\left[\frac{(-1)^n n!}{(x+i)^{n+1}} + \frac{(-1)^n n!}{(x-i)^{n+1}}\right]$$

$$A^{(n-1)} = \frac{1}{2}\left[\frac{(-1)^{n-1}(n-1)!}{(x+i)^n} + \frac{(-1)^{n-1}(n-1)!}{(x-i)^n}\right]$$

$$A^{(n-1)} = (-1)^{n-1}(n-1)!\,\frac{1}{r^n}\cos n\theta, \quad \theta = \tan^{-1}\left(\frac{1}{x}\right)$$

$$B^{(n)} = (-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta$$

$$B^{(n-1)} = (-1)^{n-2}(n-2)!\,\sin^{n-1}\theta\,\sin(n-1)\theta$$

$$\boxed{\begin{aligned} y^{(n)} &= A^{(n-1)} + B^{(n-1)} \\ &= (-1)^{n-1}(n-1)!\,\sin^n\theta\cos\theta + (-1)^{n-2}(n-2)!\,\sin^{n-1}\theta\,\sin(n-1)\theta \\ &\qquad \theta = \tan^{-1}\left(\frac{1}{x}\right)\end{aligned}}$$