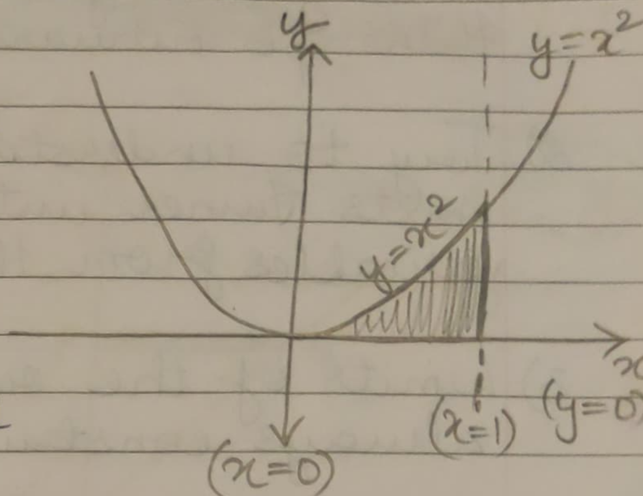
20/6/22

Double Integration

★ Evaluate

1) $\displaystyle\int_{x=0}^{x=1}\int_{y=0}^{y=x^2} (e^{y/x})\, \overset{1}{dy}\, \overset{2}{dx}$



$$\int_0^1 \left[\int_0^{x^2} e^{y/x}\, dy\right] dx$$

$$\int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx$$

$$\int_0^1 x\left[e^{x^2/x} - e^{0/x}\right] dx$$

x will be constant here because partial integration

$$\int_0^1 x\left[e^{x} - 1\right] dx$$

$$= \int_0^1 (xe^x - x)\, dx$$

Apply uv rule integration

$$= \left[xe^x - e^x - \frac{x^2}{2}\right]_0^1$$

$$= e^1 - e^1 - 1/2 + 1$$

$$= 1/2$$

★ Note:

1) Always integrate from inside and then go outwards.

2) Try to understand the functions for limits. Inner integral will have variables from the outer integral.

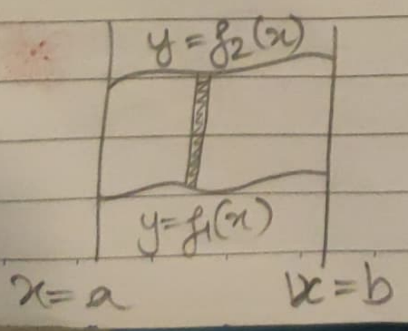
3) Limits of the outermost integral is always constant.

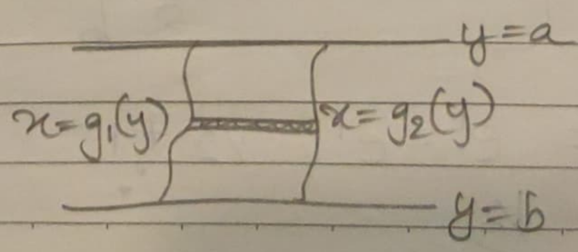
4) If limit of both integrals is constant, then order of integration can be reversed.

$$\int_c^d \int_a^b dx\, dy = \int_a^b \int_c^d dy\, dx$$

5) Any double integral can be performed in both direction by taking elementary strip parallel to y axis or x-axis.

6) x-limit will be always from left to right (→) & y-limits will be from down to up (↑).



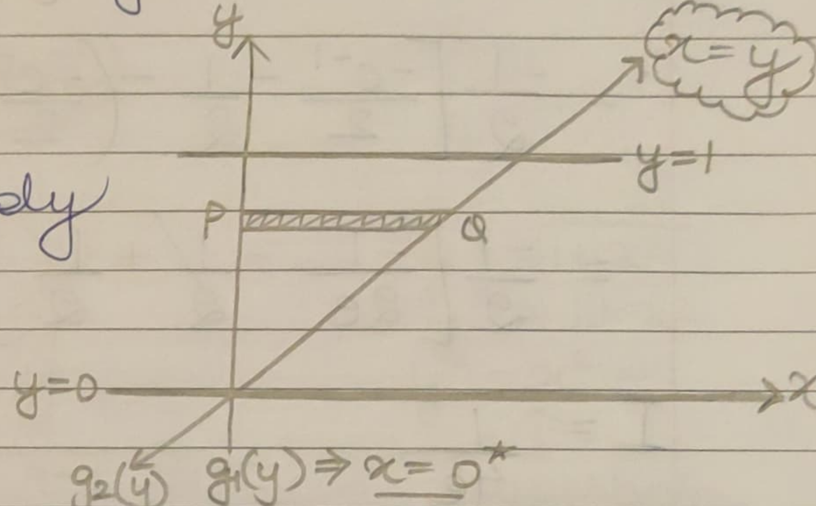


* Don't draw diagrams for evaluation as in Exam.

Q. $\int_{y=0}^{y=1}\int_{x=0}^{x=y} xy\, e^{-x^2}\, dy\, dx$ → Confusion ke liye aese ho sakta hai

$\int_0^1\int_0^y xy\, e^{-x^2}\, dx\, dy$

$\int_0^1 y\left[\int_0^y x e^{-x^2}\, dx\right] dy$



Consider

$\int x e^{-x^2}\, dx$

If we do it directly toh limits bhi change karni padengi

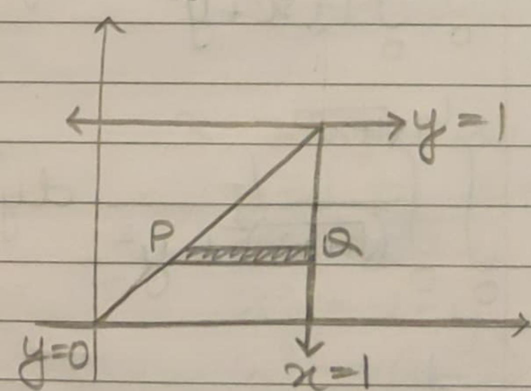
Let $x^2 = t$

$2x\, dx = dt$

$x\, dx = \frac{dt}{2}$

$\int e^{-t}\frac{dt}{2} = \frac{1}{2}\frac{e^{-t}}{(-1)} = \frac{-e^{-x^2}}{2}$

If $\int_0^1\int_{x=y}^{x=1}$



Put back

$I = \int_{y=0}^{y=1} y\left[\frac{-e^{-x^2}}{2}\right]_0^y dx$

$I = -\frac{1}{2}\int_0^1 y\cdot e^{-y^2} - y\; dy$

done a similar int peeches ~~ ~~ use the same ~~result~~

$$I = \frac{-1}{2}\int_0^1 \left[y\cdot e^{-y^2} - y\right] dy$$

$$= \frac{-1}{2}\left[\frac{-e^{-y^2}}{2} - \frac{y^2}{2}\right]_0^1$$

$$= \frac{-1}{2}\left[\frac{-e^{-1}}{2} - \frac{1}{2} - \left(\frac{-e^0}{2} - 0\right)\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{2e} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}}\right]$$

$$I = \frac{1}{4e}$$

Q. $$\int_0^1\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy\, dx$$

$$\int_0^1\int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2 + y^2}\, dy\, dx$$ ----→ of the form $1/a^2+x^2$

$$\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}}$$

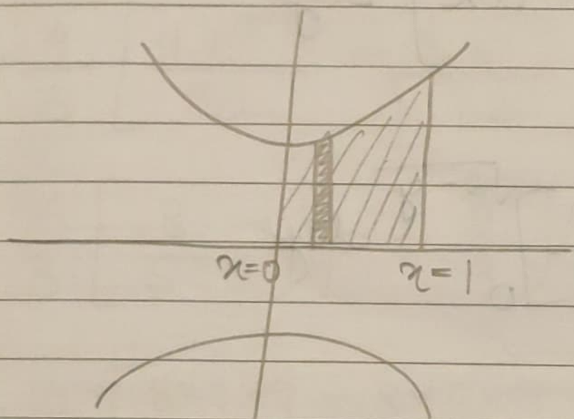
$$\int_0^1 \frac{1}{\sqrt{1+x^2}}\left(\pi/4 - 0\right) dx$$

$$\frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1+x^2}}$$

$$I = \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1$$

$$= \frac{\pi}{4}\left[\log(\sqrt{2}+1) - 0\right]$$

$$I = \frac{\pi}{4}\log(1+\sqrt{2})$$



$y = \sqrt{1+x^2}$

$y^2 = 1+x^2$

$y^2 - x^2 = 1$

↑

hyperbola (vertical

Q. $$\int_{y=1}^{y=2}\int_{x=-\sqrt{2-y}}^{x=\sqrt{2-y}} 2x^2y^2\,dx\,dy$$

$x = -\sqrt{2-y}$ & $x = \sqrt{2-y}$

$y = 1$ to $y = 2$

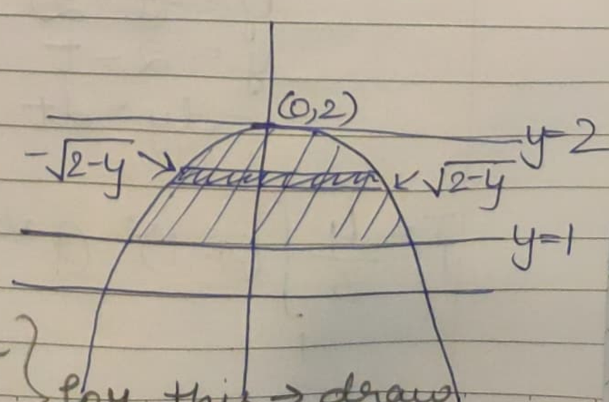
$x = \pm\sqrt{2-y}$

$x^2 = 2-y$

$x^2 = -(y-2)$

$$\int_1^2 2y^2\left[\frac{x^3}{3}\right]_{-\sqrt{2-y}}^{\sqrt{2-y}} dy$$

$$\int_1^2 \frac{2y^2}{3}\left[(2-y)^{3/2} - \left[-(2-y)\right]^{3/2}\right]$$



(very tough to solve)

Use formula

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)$ for even
$= 0$ → for odd } for this → draw diagram

Date:

The fig is symmetric about y-axis & Integrand is even f^n

$$I = 2\int_1^2 \int_0^{\sqrt{2-y}} 2x^2y^2 \, dx \, dy$$

$$I = 2\int_1^2 2y^2 \left[\int_0^{\sqrt{2-y}} x^2 \, dx\right] dy$$

$$I = 2\times 2\int_1^2 y^2 \left[\frac{x^3}{3}\right]_0^{\sqrt{2-y}} dy$$

$$= \frac{4}{3}\int_1^2 y^2 \left[(\sqrt{2-y})^3 - 0\right] dy$$

$$= \frac{4}{3}\int_1^2 y^2 \left[(2-y)^{3/2}\right] dy$$

Put $2-y = t$

$y = -(t-2)$

$y = 2-t$

$-dy = dt$

y	1	2
t	1	0

$$I = \frac{4}{3}\int_1^0 (2-t)^2 (t^{3/2}) (-dt)$$

$$= \frac{4}{3}\int_0^1 t^{3/2}(2-t)^2\,dt$$

$$= \frac{4}{3}\int_0^1 t^{3/2}(4-4t+t^2)\,dt$$

$$= \frac{4}{3}\int_0^1 4t^{3/2} - 4t^{5/2} + t^{7/2}\,dt$$

$$= \frac{4}{3}\left[\frac{4t^{5/2}}{5/2} - \frac{4t^{7/2}}{7/2} + \frac{t^{9/2}}{9/2}\right]_0^1$$

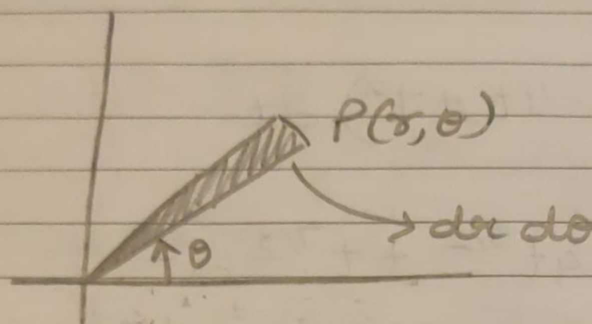
$$= \frac{4}{3}\left[\frac{8}{5} - \frac{8}{7} + \frac{2}{9}\right]$$

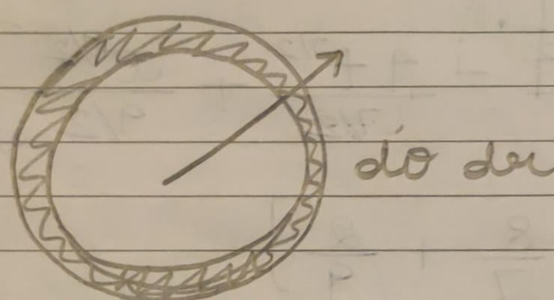
$$= \frac{4}{3}\left[\frac{8\times7\times9 - 8\times5\times9 + 2\times35}{315}\right]$$

$$= \frac{856}{945}$$

$$\begin{array}{r} 63 \\ \times 5 \\ \hline 315 \end{array} \quad \begin{array}{r} 72\times2 \\ 144 \\ 70 \\ \hline 214 \\ \times 4 \\ \hline 856 \end{array} \quad \begin{array}{r} 315 \\ \times 3 \\ \hline 945 \end{array}$$

★ Polar limits of Integration





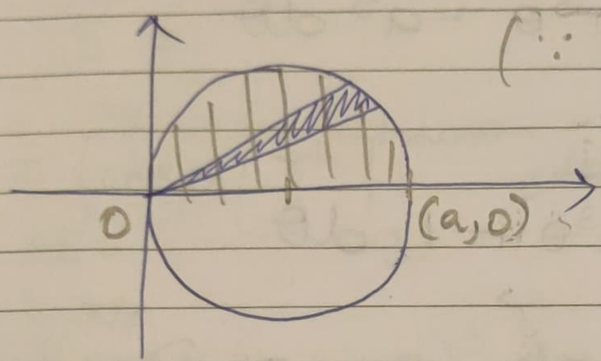
→ r mai kabhi bhi negative nahi aayega

→ $r=0 \Rightarrow$ ∴ Starting from origin
agar any other value of r
$\Rightarrow$ ∴ punctured beech mai se

1) $$\int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=a\cos\theta} r\sqrt{a^2-r^2}\, dr\, d\theta$$

Consider $r = a\cos\theta$
Circle rightward touching y-axis.

$(\because 0 \to \pi/2)$



Since it's a radial strip till circle & θ is 0 to $\pi/2$ means region of Integration is in 1st Quadrant

Consider for integration

$$\int r\sqrt{a^2-r^2}\, dr$$

put $(a^2-r^2) = t$
$-2r\,dr = dt$
$$r\,dr = \frac{-dt}{2}$$

$$\int \sqrt{t}\left(\frac{-dt}{2}\right) = \frac{-1}{2}\frac{t^{3/2}}{3/2} = \frac{-1}{3}\left(a^2-r^2\right)^{3/2}$$

Put in I

$$I = \int_0^{\pi/2} -\frac{1}{3}\left[\left(a^2-r^2\right)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{-1}{3}\int_0^{\pi/2} (a^2 - a^2\cos^2\theta)^{5/2} - (a^2)^{3/2}\, d\theta$$

$$= \frac{-1}{3}\int_0^{\pi/2}\left[(a^2)^{3/2}(\sin^2\theta)^{3/2} - a^3\right] d\theta$$

$$= \frac{-1}{3}\int_0^{\pi/2} a^3\sin^3\theta - a^3\, d\theta$$

$$= \frac{-a^3}{3}\int_0^{\pi/2}\sin^3\theta - 1\, d\theta$$

$$= \frac{-a^3}{3}\times\left[\frac{1}{2}\beta\left(2,\tfrac{1}{2}\right) - [\theta]_0^{\pi/2}\right]$$

$$= \frac{-a^3}{3\cancel{\times 2}}\left[\frac{1}{2}\times\frac{\Gamma 2\times\Gamma\frac{1}{2}}{\Gamma\frac{5}{2}} - \left(\pi/2\right)\right]$$

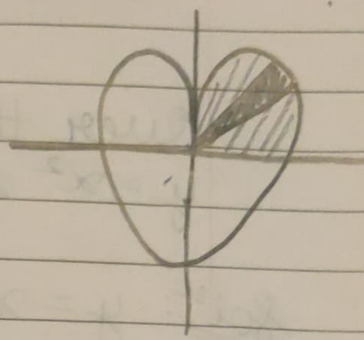
$$= \frac{-a^3}{3}\left[\frac{1}{\cancel{2}}\times\frac{\cancel{\Gamma\frac{1}{2}}}{\frac{3}{\cancel{2}}\times\frac{1}{2}\times\cancel{\Gamma\frac{1}{2}}} - \frac{\pi}{2}\right]$$

$$= \frac{-a^3}{3}\left[\frac{2}{3} - \frac{\pi}{2}\right]$$

Q. $$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=1-\sin\theta} r^2\cos\theta \, dr\, d\theta$$

$r = 1-\sin\theta$

$r = a(1-\sin\theta)$



$$I = \int_0^{\pi/2}\int_0^{1-\sin\theta} r^2\cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta\left[\frac{r^3}{3}\right]_0^{1-\sin\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}\cos\theta\,(1-\sin\theta)^3\, d\theta$$

put $\sin\theta = t$

$\cos\theta\, d\theta = dt$

$$= \frac{1}{3}\int_0^1 (1-t)^3\, dt$$

$$= \frac{1}{3}\beta(1,4)$$

$$= \frac{1}{3}\frac{\Gamma 1\,\Gamma 4}{\Gamma 5}$$

$$= \frac{1}{3}\times\frac{1\times 3!}{4!} = \frac{1}{3}\times\frac{\cancel{3\times 2}\ \cancel{3!}}{4\times \cancel{3!}} = \frac{1}{12}$$

★ Over a given region (find limits)

1) Evaluate $\iint_R xy \, dx \, dy$

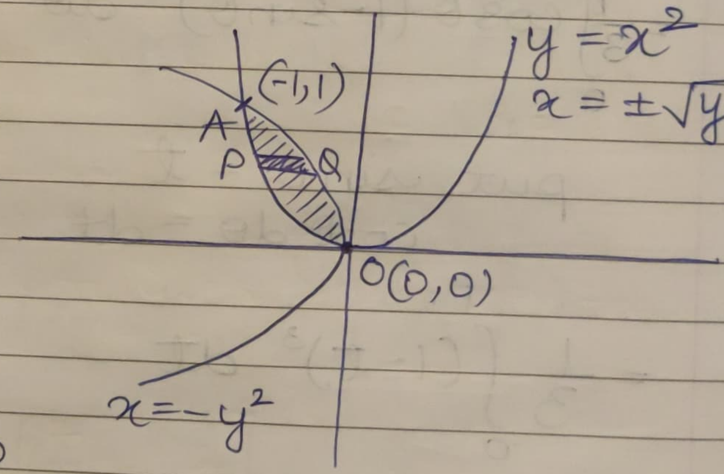
Over the area bounded by parabola $y = x^2$ & $x = -y^2$

Sol^n: $y = x^2$ is upward standard parabola

$x = -y^2$ is a leftward standard parabola as shown in fig

To find pt. of intersection
Put $y = x^2$

$x = -y^2 = -x^4$
$x^4 + x = 0$
$x(x^3+1) = 0$
$\Rightarrow x = 0 \quad \Rightarrow x^3 + 1 = 0$
$\Rightarrow (x-1)(x^2-x+1) = 0$
(real root)
$\underline{x = -1}$



Then $y = x^2$
$\Rightarrow y = 0 \,/\, y = 1$
$O(0,0)$ & $A(-1,1)$
are pts of intersection

consider Horizontal strip PQ, starting at P, i.e. $x = -\sqrt{y}$ and ending at Q i.e. $x = -y^2$.

Then y varies from $y = 0$ to $y = 1$

$$I = \int_{y=0}^{y=1} \int_{x=-\sqrt{y}}^{x=-y^2} x\cancel{+}y \; dx \, dy$$

$$I = \int_0^1 y \left[\frac{x^2}{2}\right]_{-\sqrt{y}}^{-y^2} dy$$

$$= \frac{1}{2}\int_0^1 y \left[(-y^2)^2 - (-\sqrt{y})^2\right] dy$$

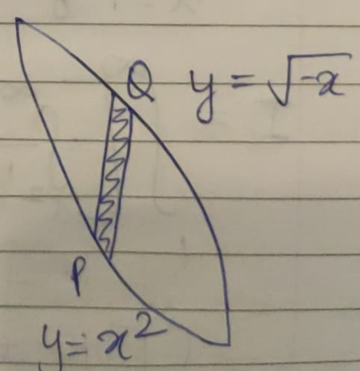
$$= \frac{1}{2}\int_0^1 y \left[y^4 - y\right] dy$$

$$= \frac{1}{2}\int_0^1 y^5 - y^2 \; dy$$

$$= \frac{1}{2}\left[\frac{y^6}{6} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{6} - \frac{1}{3}\right]$$

$$= \frac{-1}{12}$$

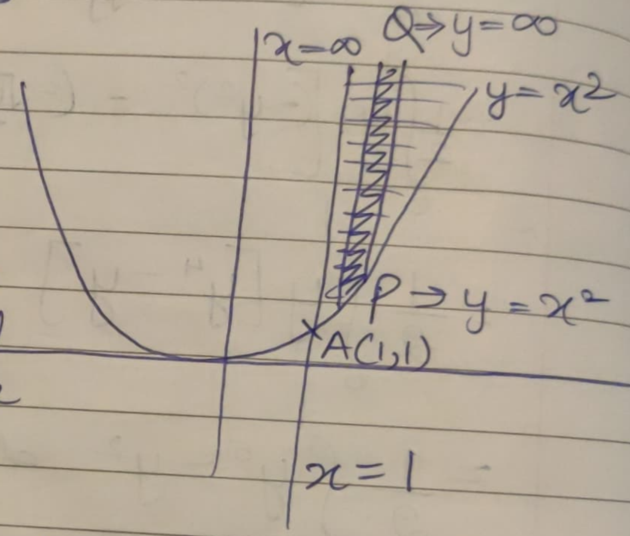


Can't do the same ques. with vertical strip

2) Evaluate $\iint_R \frac{1}{x^4+y^2}\, dx\, dy$

Where R is the region $x \geq 1$, $y \geq x^2$

Sol^n $y = x^2$ is an upward parabola (vertex at origin) & it intersects line $x=1$ at $A(1,1)$



Region of integration is unbounded shaded region in the figure

Consider vertical strip PQ starting from P i.e. $y=x^2$ to $y=\infty$ & x varies from $x=1$ to $x=\infty$

$$I = \int_{x=1}^{x=\infty} \int_{y=x^2}^{y=\infty} \frac{1}{x^4+y^2}\, dy\, dx$$

$$I = \int_1^{\infty} \left[\int_{x^2}^{\infty} \frac{1}{y^2+(x^2)^2}\, dy \right] dx$$

$$= \int_1^{\infty} \left[\frac{1}{x^2} \tan^{-1}\left(\frac{y}{x^2}\right) \right]_{x^2}^{\infty} dx$$

$$= \int_1^{\infty} \frac{1}{x^2} \left[\frac{\pi}{2} - \frac{\pi}{4} \right] dx$$

$$I = \frac{\pi}{4}\int_1^\infty \frac{1}{x^2}\,dx$$

$$= \frac{\pi}{4}\left[\frac{-1}{x}\right]_1^\infty$$

$$I = \frac{\pi}{4}$$

★ Evaluation of Double Integral over a given region (Find limits)

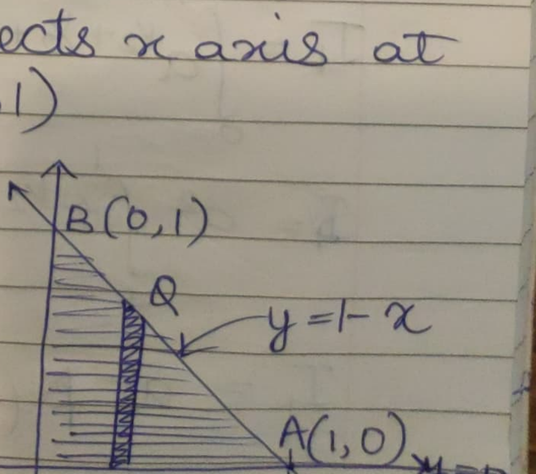
3) Evaluate $\iint_R \sqrt{xy(1-x-y)}\,dx\,dy$

over the area bounded by $x=0$, $y=0$ and $x+y=1$

4) Evaluate $\iint_R \sqrt{xy-y^2}\,dx\,dy$, where R is a triangle whose vertices are $(0,0)$ $(10,1)$, $(1,1)$

Ans: 3) Consider $x+y=1$ intersects x axis at $A(1,0)$ & y axis at $B(0,1)$

Then Region of Integration is a Triangular Region OAB as shown in fig.



We will consider vertical strip PQ

& x varies from $x=0$ to $x=1$

$$I = \int_{x=0}^{1}\int_{y=0}^{y=1-x} \sqrt{xy(1-x-y)}\, dy\, dx$$

Consider separately

$$I_1 = \int_{0}^{1-x} \sqrt{y(1-x-y)}\, dy$$

Put $1-x = a$

~~dx~~ $x = 1-a$

$$I_1 = \int_{0}^{a} \sqrt{y(a-y)}\, dy$$

Put $y = at$

$dy = a\,dt$

y	0	a
t	0	1

$$I_1 = \int_{0}^{1} (at)^{1/2} (a-at)^{1/2} (a\,dt)$$

$$I_1 = \int_{0}^{1} a^{1/2+1/2+1}\, t^{1/2} (1-t)^{1/2}\, dt$$

$$I_1 = a^2 \int_{0}^{1} t^{1/2} (1-t)^{1/2}\, dt$$

$$I_1 = a^2 \beta\left(\frac{3}{2}, \frac{3}{2}\right)$$

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$$I_1 = a^2 \frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma 3}$$

$$= a^2 \frac{\left(\frac{1}{2}\Gamma\frac{1}{2}\right)^2}{2}$$

$$I_1 = \frac{a^2\pi}{8}$$

Put in (I)

$$I = \int_0^1 \sqrt{x}\,(I_1)\,dx$$

$$= \int_0^1 \sqrt{x}\left[\frac{a^2\pi}{8}\right]dx$$

$$a = 1-x$$

$$= \frac{\pi}{8}\int_0^1 x^{1/2}(1-x)^2\,dx$$

$$= \frac{\pi}{8}\,\beta\left(\frac{3}{2}, 3\right)$$

$$= \frac{\pi}{8}\,\frac{\Gamma\frac{5}{2}\,\Gamma 4}{\Gamma 9/2} = \frac{2\pi}{105}$$

4) Evaluate $\iint_R \sqrt{xy-y^2}\,dx\,dy$

where R is a triangle whose vertices are (0,0), (10,1), (1,1) $= \underline{\underline{6}}$

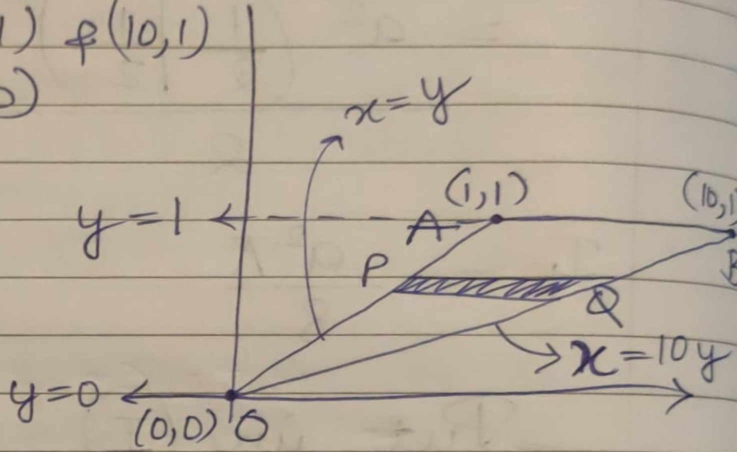
Eq^n of line w (1,1) & (10,1)

$$y-1 = \frac{1-1}{10-1}(x-10)$$

$$y-1=0$$

$AB \Rightarrow \underline{\underline{y=1}}$

$OA \Rightarrow x=y$



$OB \Rightarrow$

$$y-1 = \frac{1-0}{10-0}(x-10)$$

$$y-1 = \frac{1}{10}(x-10)$$

$$10y - \cancel{10} = x - \cancel{10}$$

$$10y = x$$

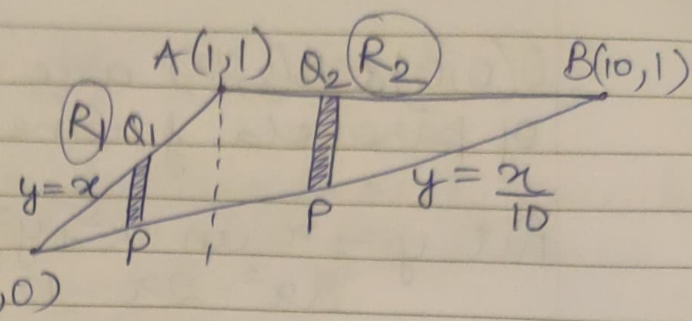
$$\underline{\underline{x = 10y}}$$

of the form $(ax+b)$

$$\int_{y=0}^{y=1}\int_{x=y}^{x=10y} \sqrt{xy-y^2}\;dx\,dy$$

$$\int_0^1\int_y^{10y} \frac{(xy-y^2)^{3/2}}{3/2\,(3)}\,dy$$

We will have to consider 2 regions for vertical strip.

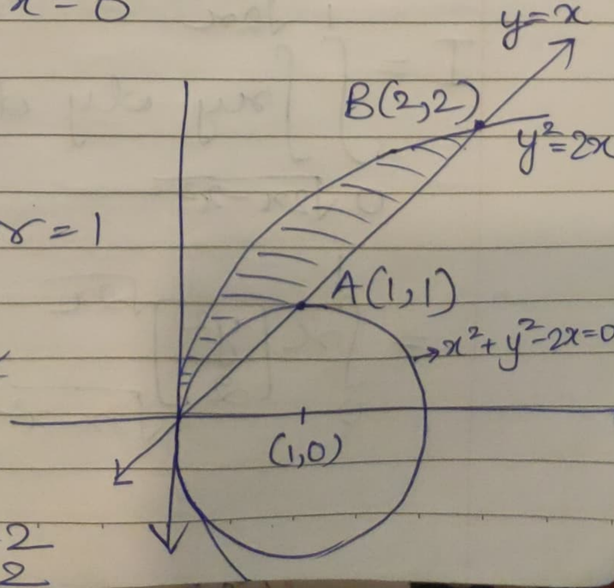


$$\int_{x=0}^{x=1}\int_{y=\frac{x}{10}}^{y=x} dy\,dx + \int_{x=1}^{x=10}\int_{y=\frac{x}{10}}^{y=1} dy\,dx$$

5) $\iint_R xy\,dx\,dy$ over the region R bounded by $x^2+y^2-2x=0$, $y^2=2x$ & $y=x$

Sol^n Region R is bounded by line $y=x$
Rightward standard parabola $y^2=2x$
& circle $x^2+y^2-2x=0$

$x^2-2x+1+y^2=1$
$(x-1)^2+y^2=1$
Centre - ~~(0,1)~~ (1,0) & $r=1$



Pt. of intersection

Put $y=x$, $y^2=2x$
$x^2=2x$
$x^2-2x=0$
$x(x-2)=0 \Rightarrow x=0, x=2$
$y=0, y=2$

$O(0,0)$ & $B(2,2)$ are pt. of intersection of parabola & line

Put $y = x$ in $x^2 + y^2 - 2x = 0$

$$2x^2 - 2x = 0$$
$$2x(x-1) = 0$$
$$x = 0, \quad x = 1$$
$$y = 0, \quad y = 1$$

Pt. of intersection of circle & line

The Region of Integration is OABCD, for evaluation we have to split it into Regions R_1(OAC) & R_2(ABC)

In R_1, we consider vertical strip PQ, where y varies from $y = \sqrt{2x - x^2}$ to $y = \sqrt{2x}$

& x varies from $x = 0$ to $x = 1$

In R_2, we consider strip P_2Q_2

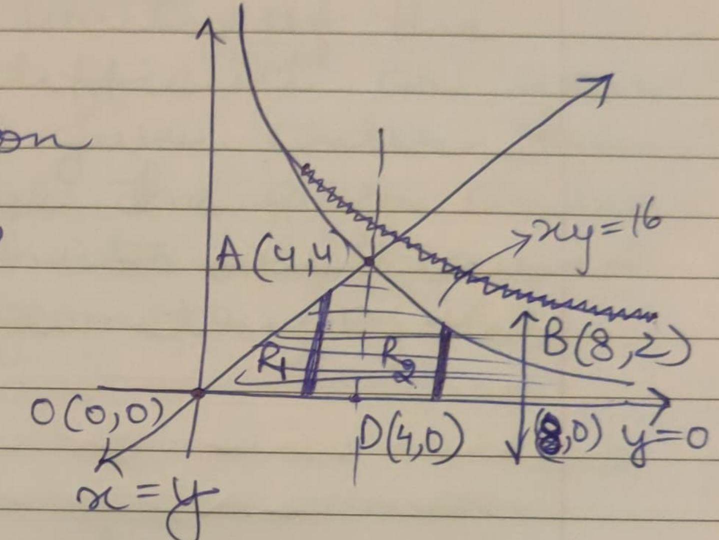
y varies from $y = x$ to $y = \sqrt{2x}$

x varies from $x = 1$ to $x = 2$

$$I = \int_0^1 \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} xy \, dy \, dx + \int_1^2 \int_x^{\sqrt{2x}} xy \, dy \, dx$$

$$= \int_0^1 x \left[\frac{y^2}{2}\right]_{\sqrt{2x-x^2}}^{\sqrt{2x}} dx + \int_1^2 x \left[\frac{y^2}{2}\right]_x^{\sqrt{2x}} dx$$

6) Evaluate $\iint_R x^2 \, dA$ whose R is region in first Quadrant bounded by hyperbola $xy=16$ & the lines $y=x$, $y=0$ & $x=8$



Pt. of intersection
$xy=16$ & $y=x$
$x^2=16 \Rightarrow x=4, y=4$
$A(4,4)$, $B(8,2)$, $C(8,0)$ & $D(4,0)$ are pts. of intersection

Region of Intersection is OABCDO as shown fig.

Then for integration, we consider the two subregion R_1(ODA) & R_2 (ABCD)

In R_1, we consider vertical

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★ Change the order & Evaluate

→ If both the x & y limits are constant, then we can change the order directly.

→ For the variable limit, if the integrand is difficult or non solvable for the given order then we usually change the given order and the new limits are obtained by considering geometry of the region.

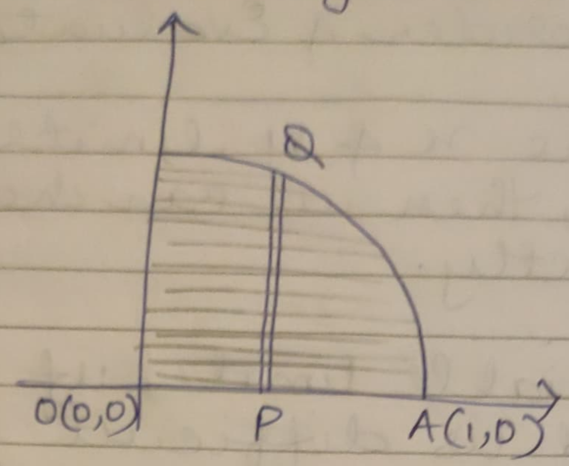
Q.1) ~~Given limits & order~~ Change the order of Integration & Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\, dy\, dx$$

① Given limits & order
First y and then x –
$y=0$ to $y=\sqrt{1-x^2}$
$\Rightarrow x^2+y^2=1$

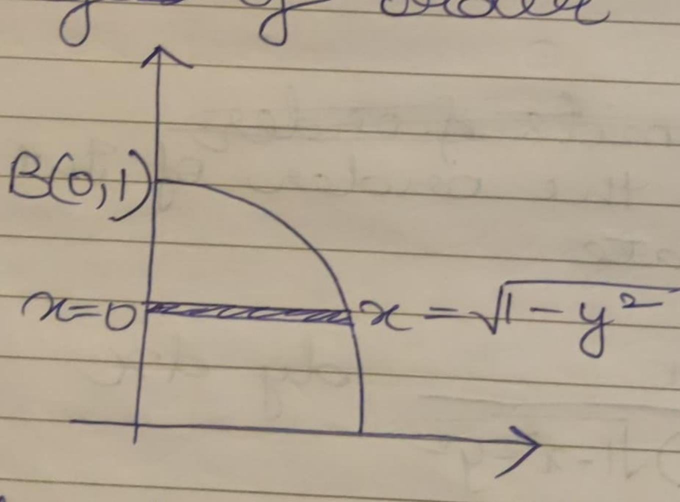
This is a standard circle with radius $=1$
Then $x=0$ to $x=1$

(2) Region of integration



Region of integration is part of a circle in I^{st} Quadrant as shown in fig

(3) Change ~~s~~ of order



Consider horizontal strip x varies from $x=0$ to $x=\sqrt{1-y^2}$ & y varies from $y=0$ to $y=1$.

$$I = \int_{y=0}^{1} \int_{x=0}^{x=\sqrt{1-y^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\, dx\, dy$$

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$$I = \int_0^{10} \frac{e^y}{(e^y+1)} \left[\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{(1-y^2)^2 - x^2}} \, dx \right] dy$$

$$I = \int_0^{10} \frac{e^y}{e^y+1} \left[\sin^{-1}\left(\frac{x}{\sqrt{1-y^2}} \right) \right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^{10} \frac{e^y}{e^y+1} \left[\frac{\pi}{2} - 0 \right] dy$$

$$= \frac{\pi}{2} \int_0^1 \left(\frac{e^y}{e^y+1} \right) dy$$

$$= \frac{\pi}{2} \left[\log(e^y+1) \right]_0^1$$

$$= \frac{\pi}{2} \left[\log(e^1+1) - \log 2 \right]$$

$$I = \frac{\pi}{2} \log\left(\frac{e^1+1}{2} \right)$$

$x^2 + 4y^2 = 1$

$4y^2 = 1 - x^2$

$y^2 = \frac{1-x^2}{4}$

$y = \frac{\sqrt{1-x^2}}{2}$

$(1-x^2) = t^2$

$-2x\,dx = 2t\,dt$

$= \int \sin\theta + \sin\theta \, d\theta$

$= \sin\theta \times -\cos\theta - \int -\cos\theta \times \cos\theta$

$= -\sin\theta\cos\theta + \int \cos^2\theta$

Q. 2) $\displaystyle\int_0^a \int_{y^2/a}^{y} \frac{y}{(a-x)\sqrt{ax-y^2}}\, dx\, dy$

① Given limits & order

First x then y

x varies from y^2/a to y

$x = y^2/a \Rightarrow y^2 = ax \rightarrow$ parabola

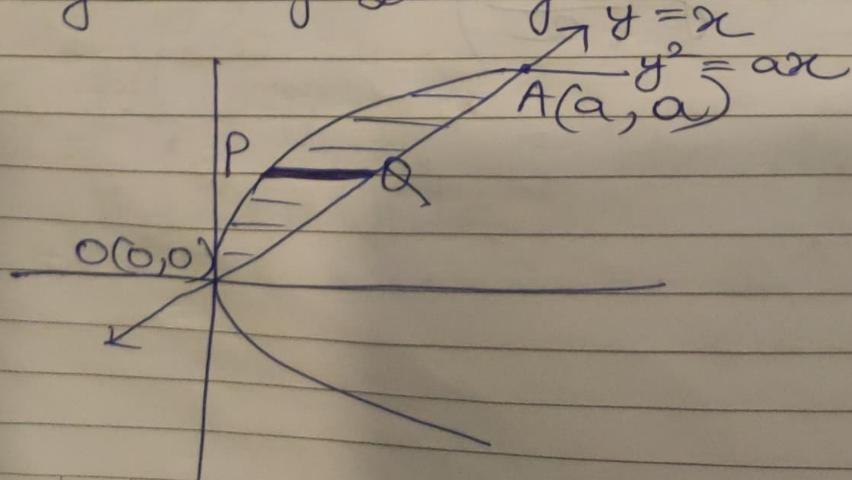
& $x = y$ is line

y varies from $y = 0$ to $y = a$

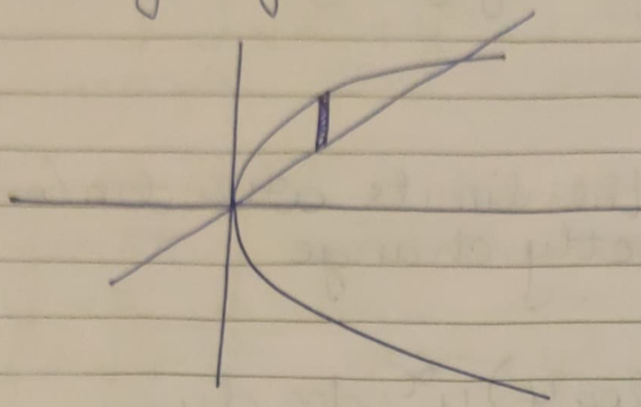
~~& x varies from x=0 to x=a~~

$\cancel{I = \int_0^a \int_{y=x}^{y=\sqrt{ax}} \frac{y}{(a-x)\sqrt{ax-y^2}}\, dy\, dx}$

② Region of Integration



(3) Change of order



H.W 5) $I = \int_0^2 \int_{\sqrt{2x}}^{2} \frac{y^2}{\sqrt{y^2-4x^2}} \, dy \, dx$

H.W 6) $I = \int_0^a \int_0^x \frac{e^y}{\sqrt{(a-x)(x-y)}} \, dy \, dx$

3) $$I = \int_{x=0}^{x=1} \int_{y=1}^{x=\infty} e^{-y} y^{x} \log y \, dy \, dx$$

Since both the limits are ~~the~~ constant we can directly change

$$I = \int_{y=1}^{y=\infty} \int_{x=0}^{1} (e^{-y} \log y) y^{x} \, dx \, dy$$

$$= \int_{1}^{\infty} e^{-y} \cancel{\log y} \left[\frac{y^{x}}{\cancel{\log y}} \right] dy$$

$$= \int_{1}^{\infty} e^{-y} \left[y^{1} - y^{0} \right] dy$$

$$= \int_{1}^{\infty} e^{-y} (y-1) \, dy$$

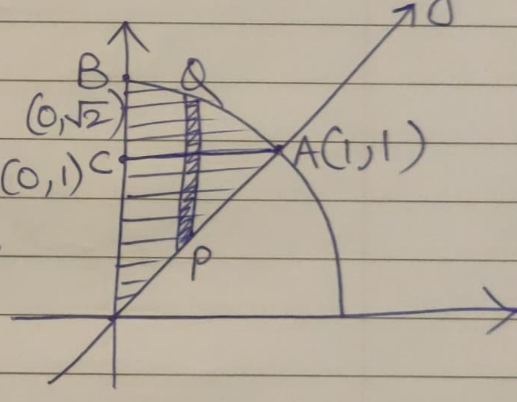
$$= \left[-e^{-y}(y-1) - e^{-y} \right]_{1}^{\infty}$$

$$= \frac{1}{e}$$

4) $I = \int_{y=0}^{1} \int_{y=x}^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} \, dy \, dx$

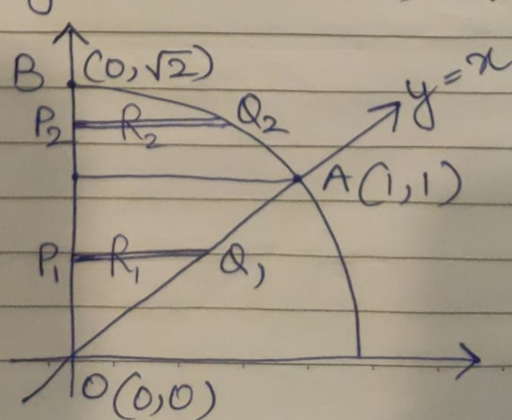
Given order of limits
first y and then x
$y = x$ is line,
$y = \sqrt{2-x^2} \Rightarrow x^2 + y^2 = 2$
is a circle w centre $(0,0)$, $r = \sqrt{2}$
x varies from 0 to 1

Region of Integration
$y = x$ at $x^2 + y^2 = 2$
$\Rightarrow 2x^2 = 2$
$\Rightarrow x^2 = 1$
$\Rightarrow x = \pm 1$



$A(1,1)$ & $(-1,-1)$
points of intersection

★ Change the order & Evaluate
When we consider horizontal strip it will be subdivided into time regions $R_1(OAC)$ & $R_2(CABC)$

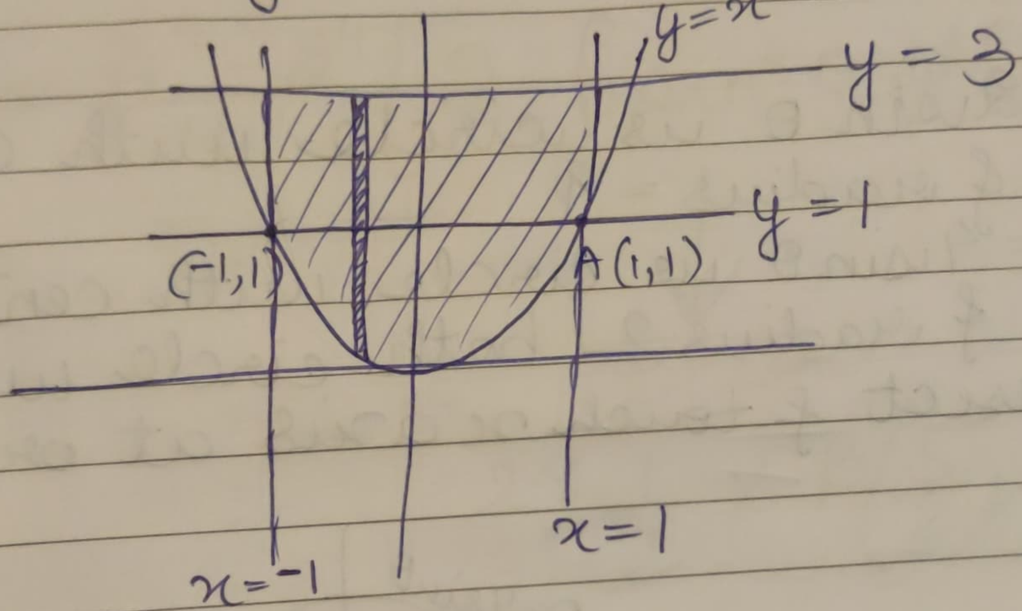


In R_1 (OAC)
x varies from 0 to y
y varies from 0 to 1

In R_2 (ABC)
x varies from 0 to $\sqrt{2-y^2}$
y varies from 1 to $\sqrt{2}$

7) $$\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} 1\,dx\,dy + \int_{y=1}^{3} dy + \int_{x=-1}^{1} dx$$

$$x=\sqrt{y}$$
$$x^2=y$$



$$I=\int_{x=-1}^{x=1}\int_{x^2}^{3} dy\,dx$$

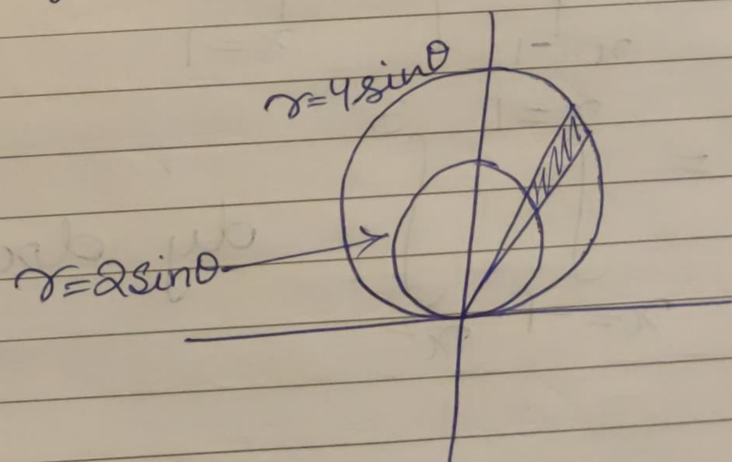
★ Evaluation over Region given in Polar coordinates

2) ~~Evaluate~~ Calculate $\iint r^3 \, dr \, d\theta$ over the area between $r = 2\sin\theta$ & $r = 4\sin\theta$

Sol^n

$r = 2\sin\theta$ is circle with centre $(0,1)$ & radius $= 1$

& $r = 4\sin\theta$ is circle with centre $(0,2)$ & radius 2 both circle with intersect & touch x axis at origin



In radial strip,

r varies from $r = 2\sin\theta$ to $r = 4\sin\theta$

θ varies from $\theta = 0$ to $\theta = \pi$

$$I = \int_{\theta=0}^{\pi} \int_{r=2\sin\theta}^{r=4\sin\theta} r^3 \, dr \, d\theta$$

$$I = \int_{0}^{\pi} \left[\frac{r^2}{4} \right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4} \int_0^{\pi} [4^4 \sin^4\theta - 2^4 \sin^4\theta] \, d\theta$$

$$= \frac{4^4 - 2^4}{4} \int_0^{\pi} \sin^4\theta \, d\theta$$

$$= \frac{4^4 - 2^4}{4} \left[2 \int_0^{\pi/2} \sin^4\theta \, d\theta \right]$$

Beta

$$2 \times \frac{1}{2} \times \beta\left(\frac{5}{2}, \frac{1}{2}\right)$$

$$I = \frac{45\pi}{2}$$

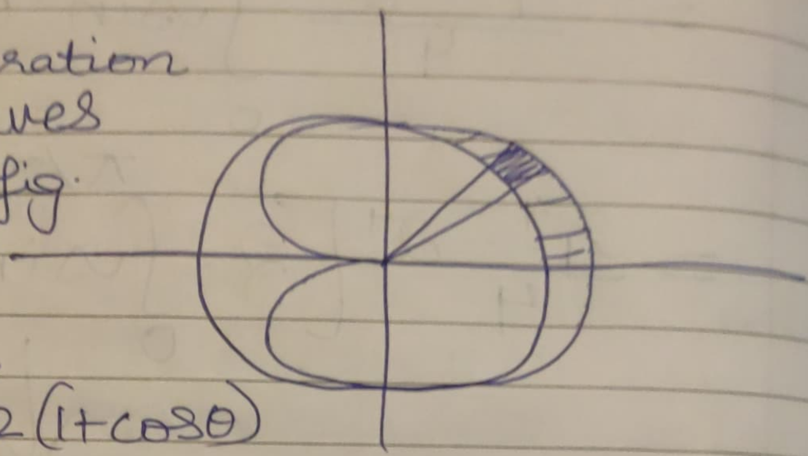
2) Evaluate $\iint_R \sin\theta \, dA$ where

R is region in first quadrant that is outside $r = 2$ & inside $r = 2(1 + \cos\theta)$

Solⁿ $r=2$ is circle w centre $(0,0)$ & radius 2.

$r=2(1+\cos\theta)$ is a cardiode rightward.

Region of integration is b/w Two curves as shown is fig.



In radial strip, r varies from $r=2$ to $r=2(1+\cos\theta)$ & θ varies from $\theta=0$ to $\theta=\frac{\pi}{2}$

$$I=\int_{0}^{\pi/2}\int_{r=2}^{r=2(1+\cos\theta)} \sin\theta\,(r\,dr\,d\theta)$$

↑ Polar coordinates

$$I=\int_{0}^{\pi/2}\sin\theta\left[\frac{r^2}{2}\right]_{2}^{2(1+\cos\theta)}d\theta$$

$$=\frac{1}{\cancel{2}}\int_{0}^{\pi/2}\sin\theta\left[\cancel{2^2}(1+\cos\theta)^2-2^2\right]d\theta$$

$$=2\int_{0}^{\pi/2}\sin\theta\left[1+2\cos\theta+\cos^2\theta-1\right]d\theta$$

$$= 2\left[\int_0^{\pi/2} a \sin\theta \cos\theta + \int_0^{\pi/2} \sin\theta \cos^2\theta \, d\theta\right]$$

$$= \frac{8}{3}$$

★ Change to Polar Coordinates & Evaluate :-

If the region is circle, cardiode or lami state or if the integrand contains functions like x^2+y^2 or $\tan^{-1}(y/x)$, etc. then it is convinient to solve the problem by converting it into polar coordinates

For circles, $x = r\cos\theta$
$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

For Ellipse, $x = a\cos\theta$
$y = b\sin\theta$

$dx\,dy = abr\,dr\,d\theta$

{This comes from Jacobian}

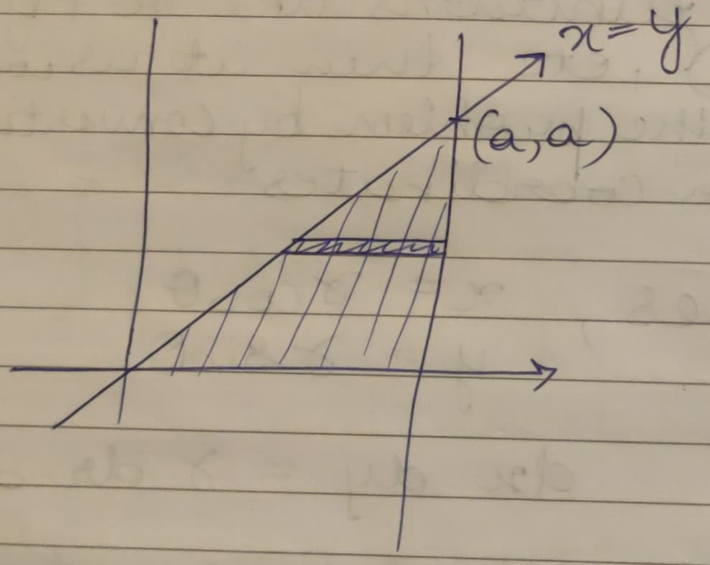
Change to Polar & Evaluate

$$I = \int_0^a \int_{x=y}^{x=a} \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy$$

Given order

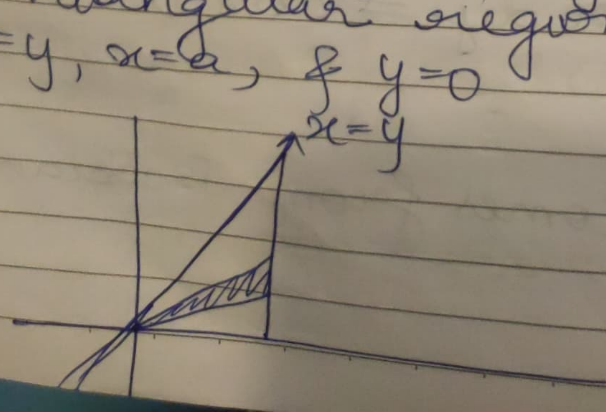
x first then y

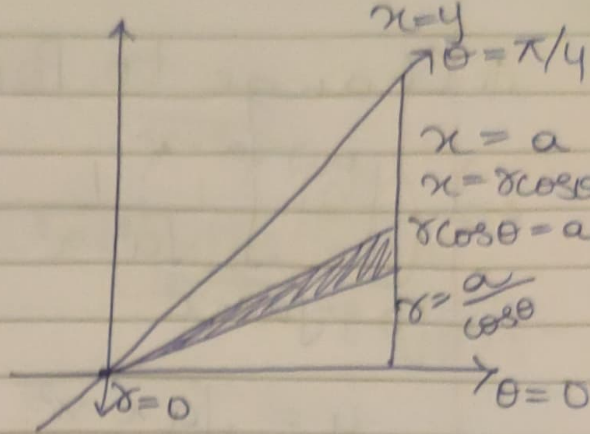
$x=y$ to $x=a$ &
$y=0$ to $y=a$



Convert to Polar & Evaluate

The region of integration is a triangular region bounded by $x=y$, $x=a$, & $y=0$





Consider radial strip

$r=0$ to $r=\dfrac{a}{\cos\theta}$
(y axis) $\quad$ $(x=a)$

& $\theta=0$ to $\theta=\pi/4$
(x axis) $\quad$ $(x=y)$

also put $x=r\cos\theta$

$y=r\sin\theta$

$dx\,dy=r\,dr\,d\theta$

$$I=\int_{\theta=0}^{\theta=\pi/4}\int_{r=0}^{r=\frac{a}{\cos\theta}}\frac{(r\cos\theta)^2}{r}\,(r\,dr\,d\theta$$

$$I=\int_0^{\pi/4}\int_0^{a/\cos\theta} r^2\cos^2\theta\,dr\,d\theta$$

$$I=\int_0^{\pi/4}\cos^2\theta\left[\frac{r^3}{3}\right]_0^{\frac{a}{\cos\theta}}d\theta$$

$$=\frac{1}{3}\int_0^{\pi/4}\cos^2\theta\left[\frac{a^3}{\cos^3\theta}\right]d\theta$$

$$=\frac{1}{3}a^3\int_0^{\pi/4}\sec\theta\,d\theta=\frac{a^3}{3}\log\left(\sqrt{2}+1\right)$$

Q. Change to Polar coordinates

2) $\int_0^a \int_{y=\sqrt{ax-x^2}}^{y=\sqrt{a^2-x^2}} \frac{1}{\sqrt{a^2-x^2-y^2}} \, dy \, dx$

$y = \sqrt{ax-x^2} \Rightarrow x^2+y^2 = ax$

$y = \sqrt{a^2-x^2} \Rightarrow x^2+y^2 = a^2$

$x^2+y^2 = ax \rightarrow$ circle with centre $\left(\frac{a}{2}, 0\right)$ & $r = \frac{a}{2}$

$x^2+y^2 = a^2 \Rightarrow$ circle w centre $(0,0)$ & $r = a$

$x = 0$ to $x = a$

Change to Polar:

$x = r\cos\theta,$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = ax \Rightarrow r = a\cos\theta$

$x^2+y^2 = a^2 \Rightarrow r = a$

r varies from $r = a\cos\theta$ to $r = a$

θ varies from $\theta = 0$ to $\theta = \pi/2$

$$I = \int_0^{\pi/2} \int_{a\cos\theta}^{a} \frac{1}{\sqrt{a^2-r^2}} \, r \, dr \, d\theta$$

$$I = \int_0^{\pi/2}$$

$a^2 - r^2 = t$

$-2r\,dr = dt$

$r\,dr = -\frac{dt}{2}$

$\int \frac{1}{\sqrt{t}}\left(\frac{dt}{-2}\right) =$

$-\sqrt{a^2-r^2}$

$$I = \int_0^{\pi/2} \left[-\sqrt{a^2-r^2}\right]_{a\cos\theta}^{a} d\theta$$

$$= \int_0^{\pi/2} \sqrt{a^2(1-\cos^2\theta)}\, d\theta$$

$$= a\int_0^{\pi/2} \sin\theta\, d\theta = a$$

3) $\iint_R \frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy$, over the area common to $x^2+y^2 = ax$ & $x^2+y^2 = by$, $\underbrace{a,b>0}$

by changing to polar coordinates.

a is +ve, ∴ circle is rightward

$\Rightarrow x^2+y^2 = ax$

$\left(x-\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$, $a>0$

Circle with centre $(a/2, 0)$, radius $= a/2$

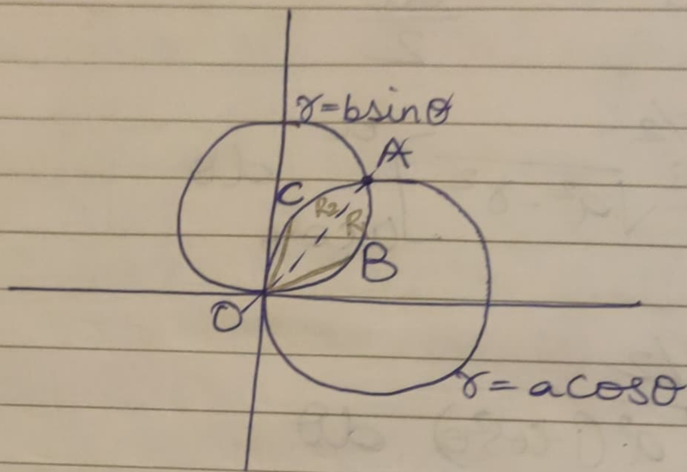
& touching y-axis

$\Rightarrow x^2+y^2 = by$

$x^2 + \left(y-\frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2$, $b>0$

Circle with centre $(0, b/2)$, $r = b/2$
& touching x-axis

The Region of Integration is common over as shown in fig



Convert into polar,

$x^2+y^2 = ax$

$r^{\cancel{2}} = a\cancel{r}\cos\theta$

$\boxed{r = a\cos\theta}$

$x^2+y^2 = by$

$r^2 = ar\sin\theta$

$\boxed{r = a\sin\theta}$

To convert to polar

$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

Point of Intersection (A)

Consider $r = r$

$a\cos\theta = b\sin\theta$

$\tan\theta = \dfrac{a}{b}$

$\theta = \tan^{-1}\left(\dfrac{a}{b}\right) = \alpha$

$$\frac{(x^2+y^2)^2}{x^2y^2} = \frac{r^4}{r^4\cos^2\theta\,\sin^2\theta} = \frac{1}{\cos^2\theta\cdot\sin^2\theta}$$

For evaluation, we have to consider 2 region $R_1(OBA)$ & $R_2(OCA)$

In $R_1(OBA)$ radial strip)
r varies from 0 to $b\sin\theta$
θ varies from 0 to $\tan^{-1}\left(\frac{a}{b}\right) = \alpha$

In $R_2(OCA)$ radial strip)
r varies from 0 to a ~~sin θ~~ $\cos\theta$
θ varies from $\tan^{-1}\left(\frac{a}{b}\right) = \alpha$ to $\frac{\pi}{2}$

$$I = \int_0^{\alpha}\int_0^{b\sin\theta} \frac{1}{\cos^2\theta\cdot\sin^2\theta}\, r\, dr\, d\theta$$

$$+ \int_{\alpha}^{\pi/2}\int_0^{a\cos\theta} \frac{1}{\cos^2\theta\cdot\sin^2\theta}\, r\, dr\, d\theta$$

$$I = I_1 + I_2$$

$$I_1 = \int_0^{\alpha} \frac{1}{\cos^2\theta\cdot\sin^2\theta}\left[\frac{r^2}{2}\right]_0^{b\sin\theta}$$

$$= \frac{1}{2}\int_0^{\alpha} \frac{1}{\cos^2\theta\cdot\cancel{\sin^2\theta}} \times b^2\cancel{\sin^2\theta}\, d\theta$$

$$= \frac{b^2}{2}\int_0^{\alpha} \sec^2\theta\, d\theta = \frac{b^2}{2}\left[\tan\theta\right]_0^{\alpha}$$

$$= \frac{b^2}{2} \tan\alpha$$

$$= \frac{b^2}{2} \tan\left(\tan^{-1} \frac{a}{b}\right)$$

$$= \frac{b^2}{2} \times \frac{a}{b}$$

$$I_1 = \frac{ab}{2}$$

$$I_2 = \int_{\alpha}^{\pi/2} \int_{0}^{a\cos\theta} \frac{1}{\sin^2\theta \cdot \cos^2\theta} \, r \, dr \, d\theta$$

$$= \int_{\alpha}^{\pi/2} \frac{1}{\sin^2\theta \cdot \cos^2\theta} \left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2} \int_{\alpha}^{\pi/2} \frac{1}{\sin^2\theta \cdot \cancel{\cos^2\theta}} \times a^2 \cancel{\cos^2\theta} \, d\theta$$

$$= \frac{a^2}{2} \int_{\alpha}^{\pi/2} \operatorname{cosec}^2\theta \, d\theta$$

$$= \frac{a^2}{2} \left[-\cot\theta\right]_{\alpha}^{\pi/2}$$

$$= \frac{a^2}{2} \left[-\cot\frac{\pi}{2} + \cot\alpha\right]$$

$$= \frac{a^2}{2}\cot\alpha$$

$$= \frac{a^2}{2}\cot\left(\tan^{-1}\frac{a}{b}\right)$$

$$= \frac{a^2}{2}\,\frac{1}{\tan\left(\tan^{-1}\frac{a}{b}\right)}$$

$$= \frac{a^2}{2}\,\frac{1}{a/b}$$

$$= \frac{a^2}{2}\times\frac{b}{a}$$

$$I_2 = \frac{ab}{2}$$

$$I = I_1 + I_2 = \frac{ab}{2} + \frac{ab}{2} = \underline{\underline{ab}}$$

Q. Solve by changing to Polar

$$\int_0^1 \int_x^{\sqrt{2x-x^2}} \underbrace{(x^2+y^2)}_{r^2}\, dy\, dx$$

$\to y^2 = 2x - x^2$

$x^2 + y^2 = 2x$

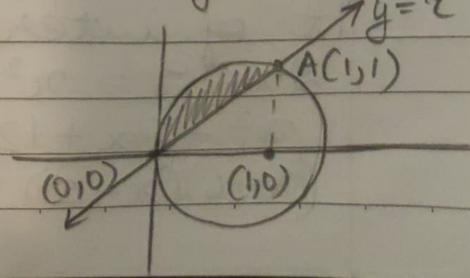
$(x-1)^2 + y^2 = 1$

$(1,0)\ ,\ r = 1$

$\to y = x$

$y = x$	$x^2+y^2 = 2x$
$\cancel{r}\sin\theta = \cancel{r}\cos\theta$	$r = 2\cos\theta$
$\theta = \pi/4$	

$$\int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{2\cos\theta} r^2 (r\, dr\, d\theta)$$



★ Application of double integral.

Type 1) Area enclosed by 2 curves $y = f_1(x)$, $y = f_2(x)$ b/w the pts. $x = x_1$ & $x = x_2$

$$A = \int_{x=x_1}^{x=x_2} \int_{f_1(x)}^{f_2(x)} 1 \; \underbrace{dy\,dx}_{\to dA}$$

Type 2) $r = f_1(\theta)$ $\quad r = f_2(\theta)$

θ_1 to θ_2

$$A = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=f_1(\theta)}^{r=f_2(\theta)} 1 \; \underbrace{r\,dr\,d\theta}_{\to dA}$$

Q.1) Find area b/w parabola $y = x^2 - 6x + 3$ & line $y = 2x - 9$

$y = x^2 - 6x + 3$

$y = (x-3)^2 - 6$

$y + 6 = (x-3)^2$

$(x-3)^2 = y - (-6)$ ← Upward parabola w vertex at $(3, -6)$

Pt. of intersection w line $y = 2x - 9$

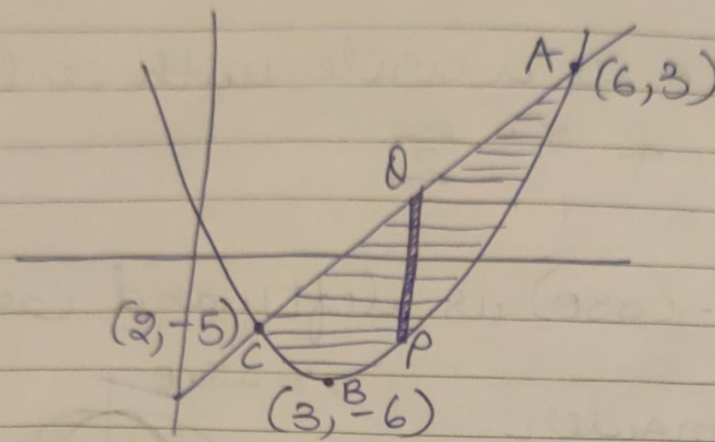
$2x - 9 = x^2 - 6x + 3$

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0$

$x = 6$	$x = 2$
$y = 3$	$y = -5$

$(6,3)$ & $(2,-5)$ are pts of intersection



Consider region of Integration as shown in fig, we consider vertical strip where

y varies from x^2-6x+3 to $2x-9$
& x varies from 2 to 6

$$A = \int_{2}^{6}\int_{x^2-6x+3}^{2x-9} 1 \, dy \, dx$$

$$= \int_{2}^{6} \left[y \right]_{x^2-6x+3}^{2x-9} dx$$

$$= \int_{2}^{6} 2x-9-x^2+6x-3$$

$$= \int_{2}^{6} -x^2+8x-12 \, dx$$

$$A = \left[-\frac{x^3}{3} + \frac{8x^2}{2} - 12x \right]_2^6 = \frac{32}{3} (\text{units})^2$$

$x^2+y^2=ay$

$x^2+\left(y-\frac{a}{2}\right)^2=\frac{a^2}{4}$

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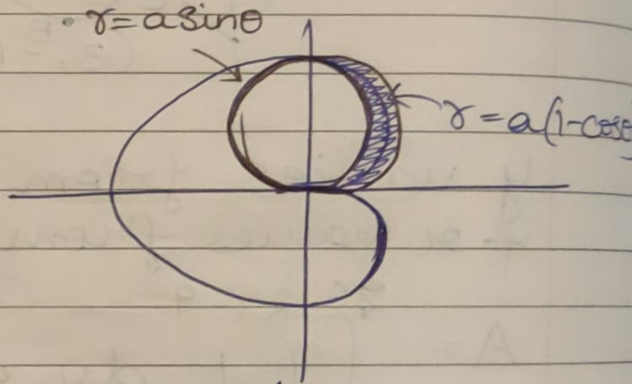
Q.2) Find the inside $r=a\sin\theta$ & outside $r=a(1-\cos\theta)$

Sol^n $r=a\sin\theta$ is a circle with centre $(0,\frac{a}{2})$ & $r=\frac{a}{2}$

$r=a(1-\cos\theta)$ is leftward cardiode

Then consider radial strip



r varies from
$r=a(1-\cos\theta)$ to
$r=a\sin\theta$
& θ varies from 0 to $\pi/2$

$$A=\int_{\theta=0}^{\theta=\pi/2}\int_{r=a(1-\cos\theta)}^{r=a\sin\theta} 1\, r\,dr\,d\theta$$

$$A=\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$A=\frac{a^2(4-\pi)}{4}$$

* Mass of Lamina

If $\rho = f(x,y) \rightarrow$ density

then mass of lamina $= \iint \rho \, dA$

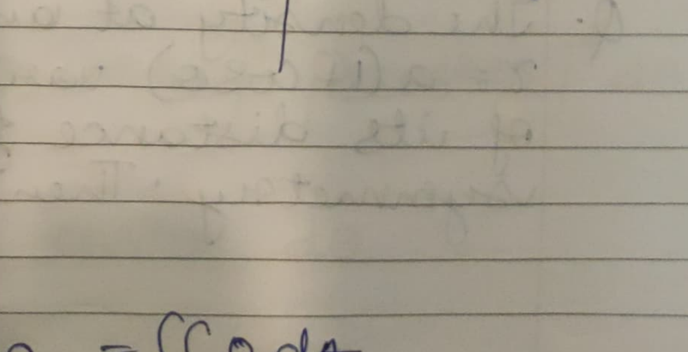
Cartesian	Polar
$\iint \rho \, dx \, dy$	$\iint \rho \, r \, dr \, d\theta$

Q. Find the total mass of the lamina in the form of ellipse $\left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$ If the density at any point ~~wherever~~ varies as the product of the distances from the axis of the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$



density $\propto xy$

$\rho = kxy$

Mass of lamina $= \iint \rho \, dA$

$$M_L = 4 \int_{x=0}^{a} \int_{y=0}^{y = b/a \sqrt{a^2 - x^2}} kxy \, dy \, dx$$

$$M_L = 4K \int_0^a x \left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 2K \int_0^a x \left(\frac{b^2}{a^2}(a^2-x^2)\right)$$

$$= 2K \frac{b^2}{a^2} \int_0^a (a^2x - x^3)\, dx$$

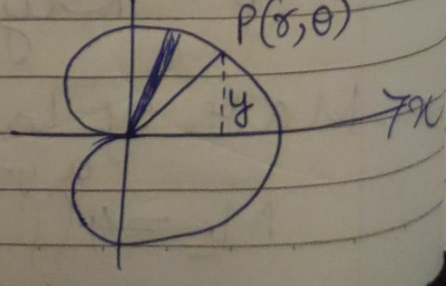
$$= 2K \frac{b^2}{a^2} \left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= 2K \frac{b^2}{a^2} \left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= 2K \cdot b^2 a^2 \left[\frac{1}{4}\right]$$

$$M_L = a^2 b^2 \frac{K}{2}$$

Q. The density at any point of a cardiode $r = a(1+\cos\theta)$ varies as the square of its distance from its axis of symmetry. Then find its mass.



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distance from x-axis

$= y = r\sin\theta$

$\rho \propto y^2$

$\rho = kr^2\sin^2\theta$

$$M_L = 2 \times \int_{\theta=0}^{\pi}\int_{r=0}^{r=a(1+\cos\theta)} \rho \, dA$$

$$= 2 \times \int_0^{\pi}\int_0^{a(1+\cos\theta)} k r^2 \sin^2\theta \; r\,dr\,d\theta$$

$$M_L = 2k \int_0^{\pi} \sin^2\theta \left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)} d\theta$$

$$M_L = \frac{k}{2}\int_0^{\pi} \sin^2\theta \cdot a^4(1+\cos\theta)^4 \, d\theta$$

$$= \frac{a^4 k}{2}\int_0^{\pi} \left[2\sin\theta/2 \cos\theta/2\right]^2 \left[2\cos^2\theta/2\right]^4 d\theta$$

$$= 2^5 a^4 k \int_0^{\pi} \sin^2\theta/2 \cdot \cos^2\theta/2 \, d\theta$$

$\theta/2 = t \quad d\theta = 2dt$

0	π
0	$\pi/2$

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$$= 2^6 a^4 k \int_0^{\pi/2} \sin^2 t \cdot \cos^{10} t \, dt$$

$$= 2^6 a^4 k \left[\frac{1}{2} \beta\left(\frac{3}{2}, \frac{11}{2}\right) \right]$$

$$= \frac{21}{35} k \pi a^4$$

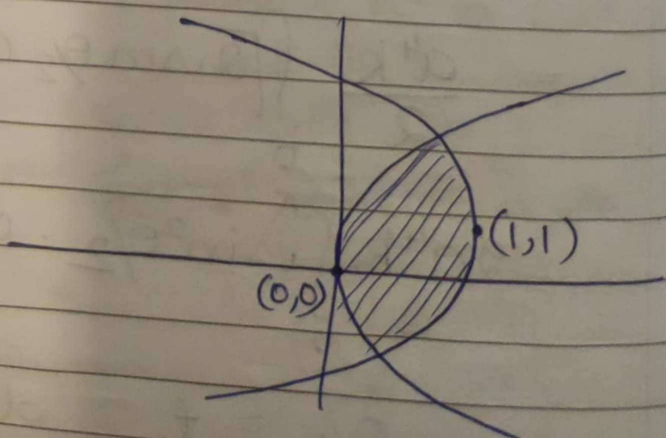
Q3) Find area bounded by parabolas $x = y^2$ & $x = 2y - y^2$

Q.4) Find area of cardiode $r = a(1+\sin\theta)$.

3) $y^2 = x$ $\qquad y^2 - 2y = -x$

$a = 1/4$ $\qquad (y-1)^2 - 1 = -x$

$(y-1)^2 = -(x-1)$



★ Triple Integration

→ Cartesian

$$\int_{z=}^{z=}\int_{x=f(z)}^{x=f(z)}\int_{y=f(x,z)}^{y=f(x,z)} dx\, dy\, dz$$

order

Q. Evaluate

1) $$\int_{z=0}^{z=2}\int_{y=0}^{y=z}\int_{x=0}^{x=yz} xyz\, dx\, dy\, dz$$

First, we have to check the order

$$\int_0^2\int_0^z \left[\frac{x^2}{2}\right]_0^{yz} \times yz\, dy\, dz$$

$$\int_0^2\int_0^z \frac{y^3z^3}{2}\, dy\, dz$$

$$\frac{1}{2}\int_0^2 \left[\frac{y^4}{4}\right]_0^z \times z^3\, dz$$

$$\frac{1}{8}\int_0^2 z^7\, dz = \frac{1}{8}\left[\frac{z^8}{8}\right]_0^2 = \frac{2^8}{64} = \frac{\cancel{64}\times 4}{\cancel{64}}$$

$$= 4$$

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2) $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, \cancel{dx}\, \cancel{dy}\, \cancel{dz}$ correct order
$dz\, dy\, dx$

$$\int_0^{\log 2}\int_0^{x} (e^{x+y}) \times \left[e^{z}\right]_0^{x+\log y} dy\, dx$$

$$\int_0^{\log 2}\int_0^{x} e^{x+y}\left[e^{x+\log y} - 1\right] dy\, dx$$

$$\int_0^{\log 2}\int_0^{x} e^{2x+y+\log y} - e^{x+y}\, dy\, dx$$

$$\int_0^{\log 2}\int_0^{x} e^{2x}\left[e^{y} \times y\right] - e^{x} \times e^{y}\, dy\, dx$$

$$\int_0^{\log} e^{2x}\left[e^{y}\cdot y - e^{y}\right]_0^{x} - e^{x}\left[e^{y}\right]_0^{x} dx$$

$$\int_0^{\log 2} e^{2x}\left[e^{x}\cdot x - e^{x} + 1\right] - e^{x}\left[e^{x} - e^{0}\right] dx$$

$$\int_0^{\log 2} e^{3x}\cdot x - e^{3x} + \cancel{e^{2x}} - \cancel{e^{2x}} + e^{x}\, dx$$

$$\left[\frac{e^{3x}}{3} \times x - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^{x}\right]_0^{\log 2}$$

$$= \frac{1}{9}\left[24 \log 2 - 19\right]$$

3) $$\int_{x=0}^{x=a}\int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}}\int_{z=0}^{z=y} z^2\,dz\,dy\,dx$$

$$\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left[\frac{z^3}{3}\right]_0^y dy\,dx$$

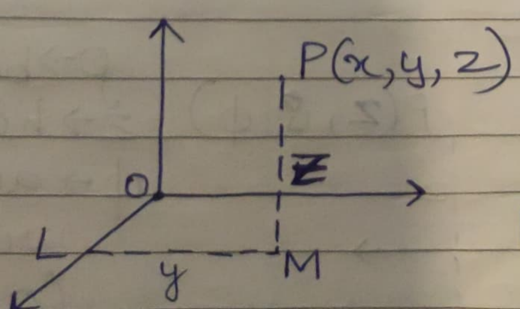
$$\frac{1}{3}\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^3\,dy\,dx$$

$$\frac{1}{3}\int_0^a\left[\frac{y^4}{4}\right]_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} dx = 0$$

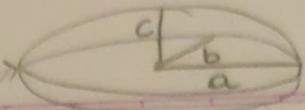
$$\cancel{\frac{1}{12}\int_0^a (a^2-x^2)^2} \quad - \quad \left[\int_{-a}^{a} f(x)\,dx = 0, \text{ odd}\right]$$
$$= 2a, \text{ even}$$

* Coordinate Systems:

1) Cartesian Coordinate System
(Planes/Intersection of planes/Cuboid)

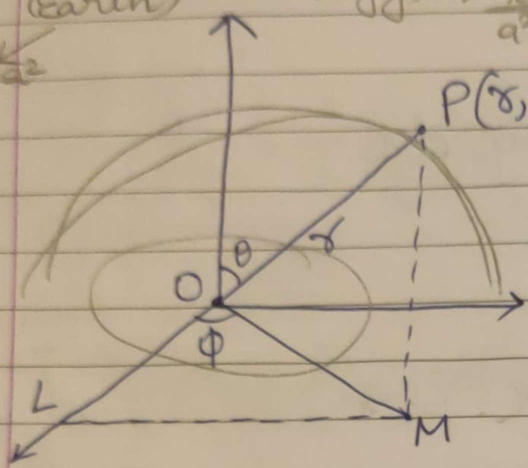


$PM = Z$
$LM = y$
$OL = x$
Perpendicular distances from plane



2) Spherical Polar Coordinates
(Spheres/Ellipsoid/their intersection)
(Earth) (egg) → $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$x^2+y^2+z^2=a^2$



θ → angle of inclined func^n
θ = 0 to π
φ → angle of rotation
φ = 0 to 2π
r → radius vector (0 to ∞)
Cone side 2 angles)
* Imagine an Earth *

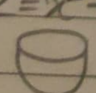
$PM = r\cos\theta = z$
$OM = r\sin\theta$
$LM = OM\sin\phi = r\sin\theta\sin\phi = y$
$OL = OM\cos\phi = r\sin\theta\cos\phi = x$

$$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$$

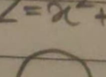
$$r^2=\sqrt{x^2+y^2+z^2},\quad \phi=\tan^{-1}\left(\frac{y}{x}\right),\quad \theta=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

3) Cylindrical Coordinate system
(paraboloid/Cone/Cylinders & intersection)
(Bowl)

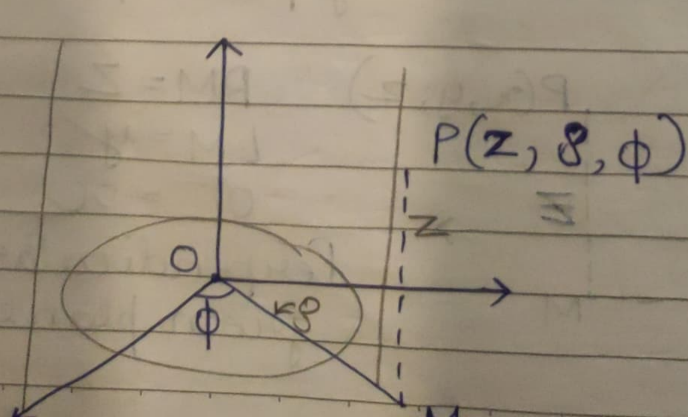
$z=x^2+y^2$



$-z=x^2+y^2$



$x=y^2+z^2$
towards me



ρ → base radius
z → height
φ → angle of rotation
(0 to 2π)

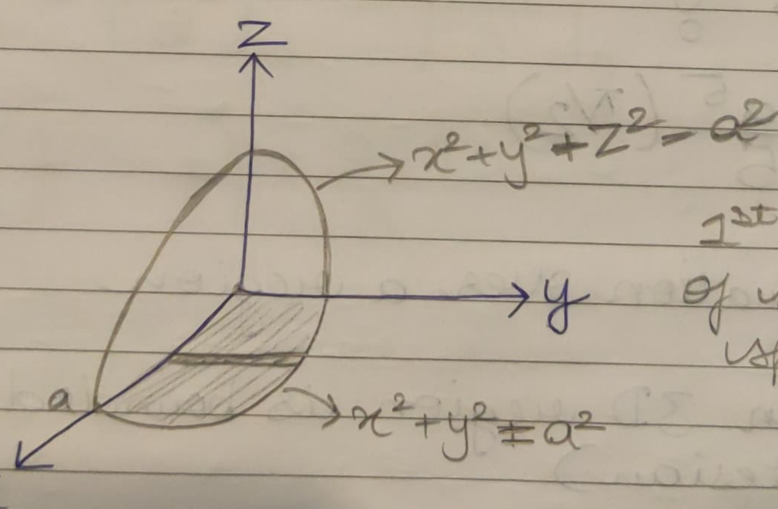
$PM = Z$
$OM = \rho$
$LM = \rho \sin\phi = y$
$OL = \rho \cos\phi = x$
$dx\, dy\, dz = \rho\, dz\, d\rho\, d\phi$

$\rho = \sqrt{x^2+y^2}$, $\phi = \tan^{-1}\left(\frac{y}{x}\right)$, $z = z$

Q. Evaluate

$$\int_{x=0}^{x=a} \underbrace{\int_{y=0}^{y=\sqrt{a^2-x^2}}}_{\text{circle}} \underbrace{\int_{z=0}^{z=\sqrt{a^2-x^2-y^2}}}_{\text{sphere}} (x^2+y^2+z^2)\, dz\, dy\, dx$$

By converting into spherical coordinates



1st octant of standard sphere

Consider, $x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$dx\, dy\, dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

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$$I = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 (r^2 \sin\theta \, dr \, d\theta \, d\phi)$$

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{r^5}{5}\right]_0^a \sin\theta \, d\theta \, d\phi$$

$$= \frac{a^5}{5} \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \, d\theta \, d\phi$$

$$= \frac{a^5}{5} \int_0^{\pi/2} (-\cos\theta)_0^{\pi/2} \, d\phi$$

$$= \frac{a^5}{5} \int_0^{\pi/2} (-0+1) \, d\phi$$

$$= \frac{a^5}{5} (\pi/2)$$

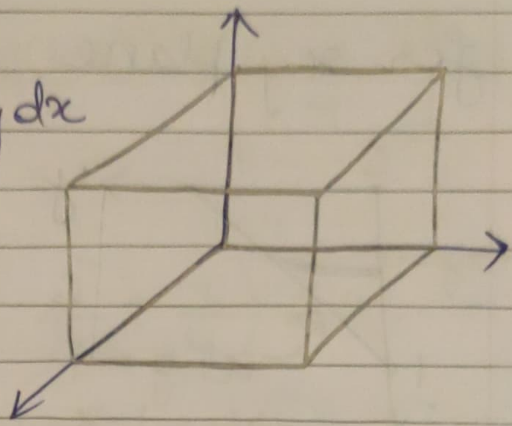
★ Evaluation over a region

A) When 3D region is bounded by planes (cartesian)

1) Evaluate $\iiint_V xyz^2 \, dV$ over the region bounded by planes $x=0, x=1, y=-1, y=2, z=0, z=3$. $= 27/4$

2) Evaluate $\iiint x^2yz \, dx \, dy \, dz$ throughout the volume bounded by planes $x=0, y=0, z=0$ & $x+y+z=1$

1) $$\int_{x=0}^{x=1}\int_{y=-1}^{y=2}\int_{z=0}^{z=3} xyz^2\,dz\,dy\,dx$$



{we can easily seperate them as we have const. limits & the inside is a product}

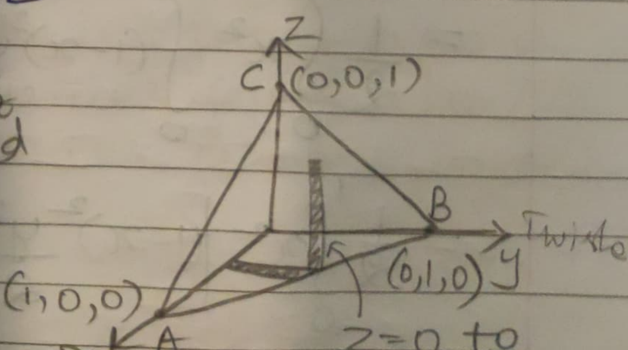
$$\int_0^1 x\,dx \times \int_{-1}^{2} y\,dy \times \int_0^3 z^2\,dz$$

$$\left[\frac{x^2}{2}\right]_0^1 * \left[\frac{y^2}{2}\right]_{-1}^{2} \times \left[\frac{z^3}{3}\right]_0^3$$

$$\frac{1}{2}\times\left(\frac{4}{2}-\frac{1}{2}\right)\times \frac{\cancel{27}\,9}{\cancel{3}}$$

$$\frac{1}{2}\times\frac{3}{2}\times 9 = \frac{27}{4}$$

2) $\iiint x^2yz\,dx\,dy\,dz$ Tetrahedron

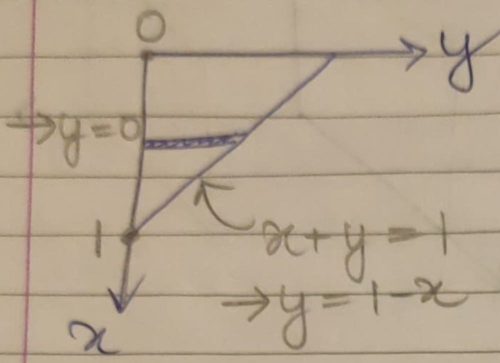


$x=0 \rightarrow yz$ plane
$y=0 \rightarrow xz$ plane
$z=0 \rightarrow yx$ plane
} I^{st} quad

$x+y+z=1 \rightarrow$ plane
(1,0,0), (0,1,0), (0,0,1)
are the intersection

$z=0$ to
$z=1-x-y$

for xy plane $(z=0)$



$$I=\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} x^2 y z \, dz\, dy\, dx$$

$$I=\int_0^1\int_0^{1-x} x^2 y\left[\frac{z^2}{2}\right]_0^{1-x-y} dy\, dx$$

$$I=\frac{1}{2}\int_0^1\int_0^{1-x} x^2 y\left[(1-x)-(y)\right]^2 dy\, dx$$

If you use $(a+b+c)^3$ formula it will get lengthy

$$=\frac{1}{2}\int_0^1\int_0^{1-x} x^2 y\left[(1-x)^2-2(1-x)y+y^2\right] dy\, dx$$

$$=\frac{1}{2}\int_0^1 x^2\int_0^{1-x}\left[(1-x)^2 y-2(1-x)y^2+y^3\right] dy\, dx$$

$$=\frac{1}{2}\int_0^1 x^2\left[(1-x)^2\frac{y^2}{2}-2(1-x)\frac{y^3}{3}+\frac{y^4}{4}\right]_0^{1-x}$$

$$I = \frac{1}{2}\int_0^1 x^2\left[\frac{(1-x)^4}{4} - \frac{2(1-x)^4}{3} + \frac{(1-x)^4}{4}\right] dx$$

$$= \frac{1}{2}\int_0^1 x^2(1-x)^4\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) dx$$

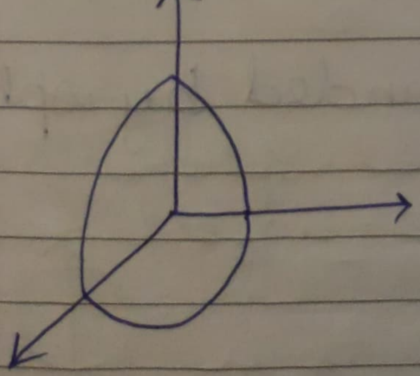
$$= \frac{1}{2}\left(\frac{1}{12}\right)\int_0^1 x^2(1-x)^4$$

$$= \frac{1}{24}\,B(3,5)$$

B) Sphere/Ellipsoid (Spherical Coordinates)

1) Evaluate $\iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2}$ throughout the volume of sphere $x^2+y^2+z^2=a^2$

Solⁿ for standard sphere of radius a we consider 1st octant



r varies from 0 to a
θ varies from 0 to π/2
φ varies from 0 to π/2

By Symmetry,

$$I = 8 \times \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} \frac{1}{r^2} \left(r^2 \sin\theta \, dr \, d\theta \, d\phi \right)$$

$$= 8 \times \int_{0}^{\pi/2} \int_{0}^{\pi/2} \sin\theta \, [r]_0^a \, d\theta \, d\phi$$

$$= 8a \int_{0}^{\pi/2} [-\cos\theta]_0^{\pi/4} \, d\phi$$

$$= 8a \int_{0}^{\pi/2} 1 \, d\phi$$

$$= 8a \left(\frac{\pi}{2} \right)$$

$$= 4a\pi$$

2) Evaluate $\iiint_V \frac{z^2}{x^2+y^2+z^2} \, dx \, dy \, dz$

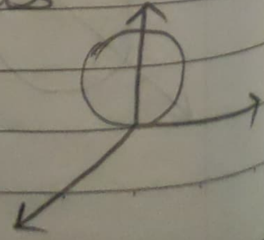
where V is volume bounded by sphere

$x^2+y^2+z^2 = z \rightarrow$ Shifted sphere on z-axis

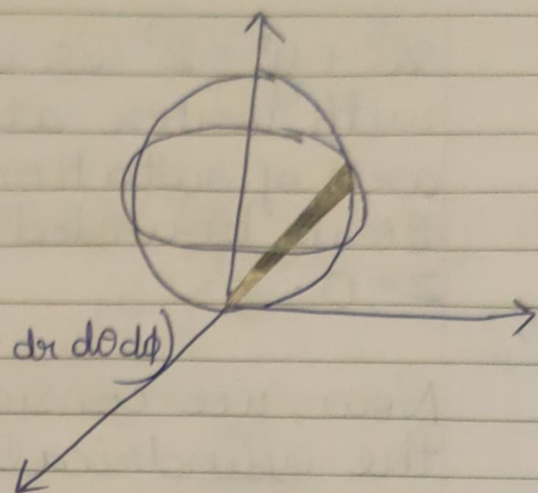
$x^2+y^2+z^2-z=0$

$x^2+y^2+(z-1/2)^2 = (1/2)^2$

Centre $(0,0,\frac{1}{2})$ & $r = \frac{1}{2}$



$$x^2+y^2+z^2=z$$
$$r^2=r\cos\theta$$
$$r=\cos\theta$$



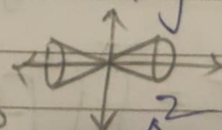
$$\int_{\phi=0}^{\phi=\pi/2}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\cos\theta}\frac{\cancel{r^2}\cos^2\theta}{\cancel{r^2}}\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right)$$

c) Cone/Cylinder/Paraboloid etc. } 3D region of int.
Use cylindrical coordinates

1) Evaluate $\iiint_V (x^2+y^2)\,dV$

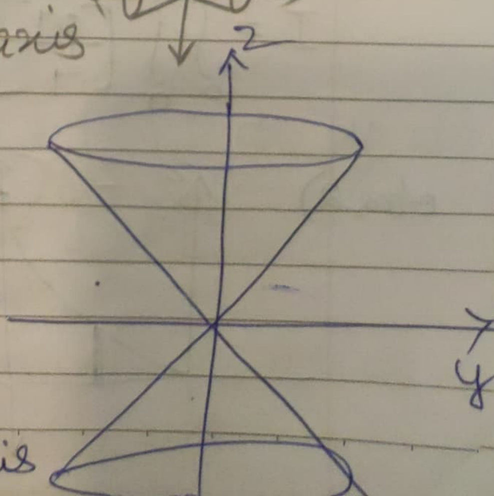
where V is solid bounded by the surface $x^2+y^2=z^2$ & the planes $z=0$ & $z=2$

HW 2) Evaluate $\iiint z^2\,dx\,dy\,dz$ over the value bounded by cylinder $x^2+y^2=a^2$ & $x^2+y^2=z$ & plane $z=0$.



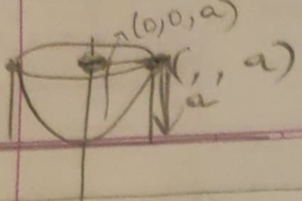
$x^2+z^2=y^2$ → Right Circular Cone about y-axis

Ans: 1) $x^2+y^2=z^2$ is a cone

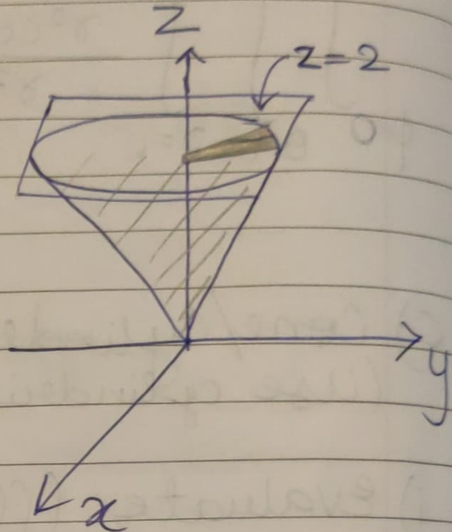


$z=0$, $x^2+y^2=0$ → circle
$y=0$, $x^2=z^2$
$x=\pm z$ → straight line

$x^2+y^2=z^2$ ← Axis about which the circular cone is



$x^2+y^2=z^2$ is a right circular cone with vertex at origin and z axis as axis of rotation
It is bounded by top $z=2$ & bottom $z=0$.



Now, we consider the cylindrical coordinate system,
$x=\rho\cos\phi$,
$y=\rho\sin\phi$,
$z=z$

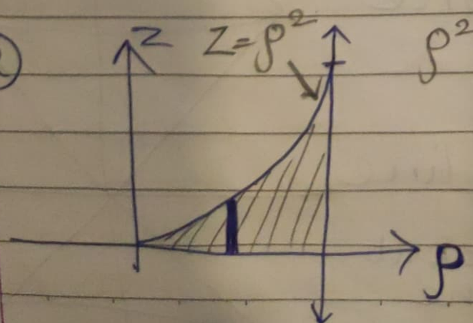
$dx\,dy\,dz=\rho\,dz\,d\rho\,d\phi$

Then for radial strip,
ϕ varies from 0 to 2π
z varies from 0 to 2
ρ varies from 0 to z

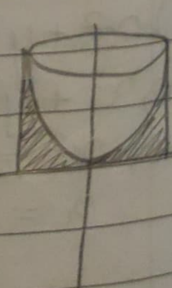
$$I=\int_{\phi=0}^{2\pi}\int_{z=0}^{2}\int_{\rho=0}^{z}\rho^2(\rho\,d\rho\,dz\,d\phi)$$

$$I=\int_0^{2\pi}\int_0^{2}\left[\frac{\rho^4}{4}\right]_0^z dz\,d\phi=\frac{16}{5}\pi$$

Ans: 2)



$\rho^2=x^2+y^2$



$z=0$ to ρ^2
$\rho=0$ to a

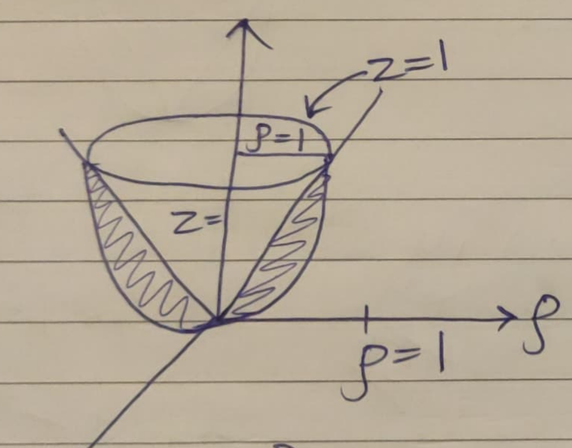
★ **Volume:** $\iiint_V 1 \, dV$

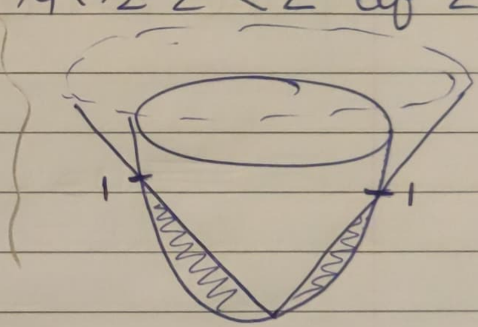
$$\int_\phi \int_\theta \int_r r^2 \sin\theta \, dr \, d\theta \, d\phi \Big/ \int_\phi \int_\rho \int_z \rho \, dz \, d\rho \, d\phi$$

Q. Find the volume bounded by $x^2+y^2=z^2$ (cone) & $x^2+y^2=z$ (Paraboloid)

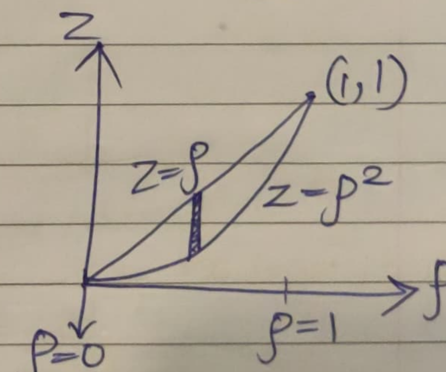
The region of integration is as shown in fig bounded by cone & paraboloid

They will intersect at 1 because –
$4>2 \quad z^2>z$ if $z>1$
$1/4 < 1/2 \quad z^2<z$ if $z<1$





$$z^2 = z$$
$$z^2 - z = 0$$
$$z=0 \,|\, z=1$$
$$\rho=0 \,|\, \rho=1$$



$x = \rho\cos\theta \qquad z = z$
$y = \rho\sin\theta$

$dx\,dy\,dz = \rho\,dz\,d\rho\,d\phi$

$$V = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{1} \int_{z=\rho^2}^{z=\rho} \rho \, dz \, d\rho \, d\phi$$

$$V = \int^{2\pi} \int^{1} \rho [z]_{\rho^2}^{\rho} \, d\rho \, d\phi$$

$$V = \int_0^{2\pi}\int_0^{1} [\rho - \rho^2]\, d\rho\, d\phi = \int_0^{2\pi} d\phi \times \left[\frac{\rho^3}{3} - \frac{\rho^4}{4}\right]_0^1$$

$$V = [\phi]_0^{2\pi} \times \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= 2\pi \times \frac{1}{12}$$

$$V = \frac{\pi}{6}\ (\text{unit})^3$$

(Q.)
If there are 2 plane $z=1$ to $z=2$