MATRICES: BASIC CONCEPTS

A matrix, in general sense, represents a collection of information stored or arranged in an orderly fashion. The mathematical concept of a matrix refers to a set of numbers, variables or functions ordered in rows and columns. Such a set then can be defined as a distinct entity, the matrix, and it can be manipulated as a whole according to some basic mathematical rules.

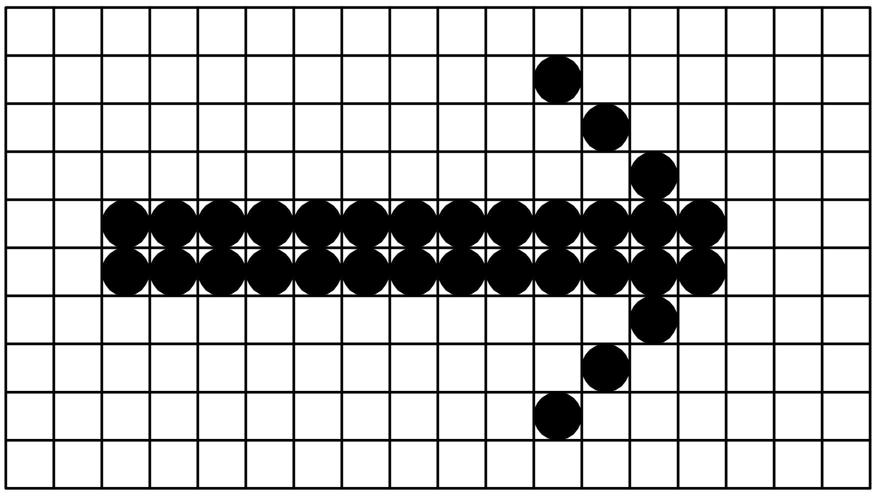
A matrix with 9 elements is shown below.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 2 \\ -7 & 4 & 6 \\ 9 & 1 & 8 \end{bmatrix}$$

Matrix [A] has 3 rows and 3 columns. Each element of matrix [A] can be referred to by its row and column number. For example,

$$a_{23} = 6$$

A computer monitor with 800 horizontal pixels and 600 vertical pixels can be viewed as a matrix of 600 rows and 800 columns.



In order to create an image, each pixel is filled with an appropriate colour.

ORDER OF A MATRIX

The order of a matrix is defined in terms of its number of rows and columns.

Order of a matrix = No. of rows $\times$ No. of columns

Matrix [A], therefore, is a matrix of order 3 $\times$ 3.

COLUMN MATRIX

A matrix with only one column is called a column matrix or column vector.

$$\begin{bmatrix} 4 \\ 6 \\ -3 \end{bmatrix}$$

ROW MATRIX

A matrix with only one row is called a row matrix or row vector.

$$\begin{bmatrix} 3 & 5 & -6 \end{bmatrix}$$

SQUARE MATRIX

A matrix having the same number of rows and columns is called a square matrix.

$$\begin{bmatrix} 2 & 4 & 7 \\ -5 & 3 & 4 \\ 2 & -4 & 9 \end{bmatrix}$$

RECTANGULAR MATRIX

A matrix having unequal number of rows and columns is called a rectangular matrix.

$$\begin{bmatrix} 5 & -3 & 7 & 1 \\ -2 & 9 & 2 & 8 \\ 5 & 4 & 1 & 13 \end{bmatrix}$$

REAL MATRIX

A matrix with all real elements is called a real matrix

PRINCIPAL DIAGONAL and TRACE OF A MATRIX

In a square matrix, the diagonal containing the elements a_{11}, a_{22}, a_{33}, a_{44}, ……, a_{nn} is called the principal or main diagonal.

The sum of all elements in the principal diagonal is called the trace of the matrix.

The principal diagonal of the matrix

$$\begin{bmatrix} 2 & 4 & 7 \\ -5 & 3 & 4 \\ 2 & -4 & 9 \end{bmatrix}$$

is indicated by the dashed box. The trace of the matrix is 2 + 3 + 9 = 14.

UNIT MATRIX

A square matrix in which all elements of the principal diagonal are equal to 1 while all other elements are zero is called the unit matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ZERO or NULL MATRIX

A matrix whose elements are all equal to zero is called the null or zero matrix.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

DIAGONAL MATRIX

If all elements except the elements of the principal diagonal of a square matrix are zero, the matrix is called a diagonal matrix.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

RANK OF A MATRIX

The maximum number of linearly independent rows of a matrix [A] is called the rank of [A] and is denoted by

Rank [A].

For a system of linear equations, a unique solution exists if the number of independent equations is at least equal to the number of unknowns.

In the following system of linear equations

$$2x - 4y + 5z = 36 \quad \ldots \ldots (1)$$

$$-3x + 5y + 7z = 7 \quad \ldots \ldots (2)$$

$$5x + 3y - 8z = -31 \quad \ldots \ldots (3)$$

all three equations are linearly independent. Therefor, if we form the augmented matrix [A] for the system where

$$[A] = \begin{bmatrix} 2 & -4 & 5 & 36 \\ -3 & 5 & 7 & 7 \\ 5 & 3 & -8 & -31 \end{bmatrix}$$

the rank of [A] will be 3.

Consider the following linear systems with 2 independent equations.

$$2x - 4y + 5z = 36 \qquad \ldots \ldots (1)$$

$$-3x + 5y + 7z = 7 \qquad \ldots \ldots (2)$$

$$-x + y + 12z = 43 \qquad \ldots \ldots (3)$$

In the above set, Eqn. (3) can be generated by adding Eqn. (1) to Eqn. (2). Therefore, Eqn. (3) is a dependent equation.

Therefor, if we form the augmented matrix [A] for the system where

$$[A] = \begin{bmatrix} 2 & -4 & 5 & 36 \\ -3 & 5 & 7 & 7 \\ -1 & 1 & 12 & 43 \end{bmatrix}$$

the rank of [A] will be 2.

MATRIX OPERATIONS

Equality of Matrices

Two matrices are equal if all corresponding elements are equal.

[A] = [B] if $a_{ij} = b_{ij}$ **for all *i* and *j***

$$[A] = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 5 & 1 \\ 3 & 7 & 8 \end{bmatrix} \qquad [B] = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 5 & 1 \\ 3 & 7 & 8 \end{bmatrix}$$

Addition and Subtraction

Two matrices can be added (subtracted) by adding (subtracting) the corresponding elements of the two matrices.

$$[C] = [A] + [B] = [B] + [A]$$

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices [A], [B] and [C] must have the same order.

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$[\mathbf{B}] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$[\mathbf{C}] = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix}$$

<u>Multiplication by a scalar</u>

If a matrix is multiplied by a scalar *k*, each element of the matrix is multiplied by *k*.

$$k[\mathrm{A}] = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$

<u>Matrix multiplication</u>

Two matrices can be multiplied together provided they are compatible with respect to their orders. The number of columns in the first matrix [A] must be equal to the number of rows in the second matrix [B]. The resulting matrix [C] will have the same number of rows as [A] and the same number of columns as [B].

$$[\mathrm{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad [B] = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$[C]=[A][B]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$[C]=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} \end{bmatrix}$$

$$c_{ij}=\sum_{k=1}^{m} a_{ik}b_{kj}$$

where m is the number of columns in [A] and also the number of rows in [B].

Example:

$$[A]=\begin{bmatrix} 2 & 3 & 1 \\ 5 & 7 & 4 \end{bmatrix} \qquad [B]=\begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}$$

$$[C]=\begin{bmatrix} 2\times2+3\times1+1\times5 & 2\times3+3\times4+1\times6 \\ 5\times2+7\times1+4\times5 & 5\times3+7\times4+4\times6 \end{bmatrix}$$

$$[C]=\begin{bmatrix} 12 & 24 \\ 37 & 67 \end{bmatrix}$$

Try the following multiplication:

$$[\mathbf{A}]=\begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & 2 \\ 4 & -2 & 5 \end{bmatrix} \qquad [\mathbf{B}]=\begin{bmatrix} 4 & 3 \\ 1 & 2 \\ 5 & 1 \end{bmatrix}$$

$$[\mathbf{C}]=[\mathbf{A}][\mathbf{B}]=\begin{bmatrix} 29 & 12 \\ 17 & 11 \\ 39 & 13 \end{bmatrix}$$

Transpose of a Matrix

The transpose $[\mathrm{A}]^{\mathrm{T}}$ of an $m \times n$ matrix $[\mathrm{A}]$ is the $n \times m$ matrix obtained by interchanging the rows and columns of $[\mathrm{A}]$.

$$[\mathrm{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 4 & 5 & 2 \\ -3 & 1 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

$$[\mathrm{A}]^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} 4 & -3 & 2 \\ 5 & 1 & 9 \\ 2 & 7 & 6 \end{bmatrix}$$

Transpose of a sum

$$([\mathrm{A}] + [\mathrm{B}])^{\mathrm{T}} = [\mathrm{A}]^{\mathrm{T}} + [\mathrm{B}]^{\mathrm{T}}$$

Transpose of a product

$$([\mathrm{A}][\mathrm{B}])^{\mathrm{T}} = [\mathrm{B}]^{\mathrm{T}}[\mathrm{A}]^{\mathrm{T}}$$

Numerical example of the product rule

$$[\mathbf{A}] = \begin{bmatrix} 2 & 3 \\ 0 & 4 \\ 5 & 1 \end{bmatrix} \quad [B] = \begin{bmatrix} 4 & 3 & 0 & 1 \\ 2 & 1 & 5 & 3 \end{bmatrix}$$

$$\left([\mathbf{A}][\mathbf{B}]\right)^{\mathrm{T}} = \begin{bmatrix} 14 & 8 & 22 \\ 9 & 4 & 16 \\ 15 & 20 & 5 \\ 11 & 12 & 8 \end{bmatrix}$$

$$[\mathbf{B}]^{\mathrm{T}}[\mathbf{A}]^{\mathrm{T}} = ?$$

Symmetric Matrices

A matrix $[\mathbf{A}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for all *i* and *j*.

$$[\mathbf{A}] = [\mathbf{A}]^{\mathrm{T}}$$

Example:

$$[\mathbf{A}] = \begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{2} \\ \mathbf{3} & \mathbf{5} & \mathbf{7} \\ \mathbf{2} & \mathbf{7} & \mathbf{0} \end{bmatrix}$$

DETERMINANT OF A MATRIX

Why determinants?

In some forms of solutions for systems of linear equations, determinants appear as denominators in a routine manner.

In a system with 3 unknowns, the determinant may appear in the solution in the following way.

$$x = \frac{D_x}{D} \quad y = \frac{D_y}{D} \quad z = \frac{D_z}{D}$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

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$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{21}\left(a_{12}a_{33} - a_{13}a_{32}\right)$$
$$+ a_{31}\left(a_{12}a_{23} - a_{13}a_{22}\right)$$

$$\mathbf{D} = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 4 & -2 \\ 3 & 5 & 6 \end{vmatrix}$$

$$= 2\begin{vmatrix} 4 & -2 \\ 5 & 6 \end{vmatrix} - 1\begin{vmatrix} -3 & 4 \\ 5 & 6 \end{vmatrix} + 3\begin{vmatrix} -3 & 4 \\ 4 & -2 \end{vmatrix}$$

$$= 2(4\times6+2\times5)-1(-3\times6-4\times5)+3(3\times2-4\times4)$$

$$= 68+38-30=76$$

Find the determinant:

$$D = \begin{vmatrix} 3 & 1 & 4 \\ 6 & 2 & 1 \\ 7 & 0 & -5 \end{vmatrix}$$

Important Properties of Determinants

1. The value of a determinant is not altered if its rows are written as columns in the same order.

$$\begin{vmatrix} 3 & 1 & 4 \\ 6 & 2 & 1 \\ 7 & 0 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 6 & 7 \\ 1 & 2 & 0 \\ 4 & 1 & -5 \end{vmatrix}$$

2. If any two rows (or two columns) of a determinant are interchanged, the value of the determinant is multiplied by –1.

$$\begin{vmatrix} 3 & 1 & 4 \\ 6 & 2 & 1 \\ 7 & 0 & -5 \end{vmatrix} = -\begin{vmatrix} 6 & 2 & 1 \\ 3 & 1 & 4 \\ 7 & 0 & -5 \end{vmatrix}$$

3. A common factor of all elements of any row (or column) can be placed before the determinant.

$$\begin{vmatrix} 3 & 8 & 1 \\ 5 & 4 & 2 \\ 1 & 12 & -3 \end{vmatrix} = \begin{vmatrix} 3 & 4\times 2 & 1 \\ 5 & 4\times 1 & 2 \\ 1 & 4\times 3 & -3 \end{vmatrix} = 4\begin{vmatrix} 3 & 2 & 1 \\ 5 & 1 & 2 \\ 1 & 3 & -3 \end{vmatrix}$$

4. If the corresponding elements of two rows (or columns) of a determinant are proportional, the value of the determinant is zero.

$$\begin{vmatrix} 3 & 2 & 5 \\ 6 & 4 & 10 \\ 2 & 7 & 8 \end{vmatrix} = 0$$

Meaning: Row 2 (Row 1) is linearly dependent on Row 1 (Row 2). Therefore, the linear system with three unknowns does not have a unique solution.

5. The value of a determinant remains unaltered if the elements of one row (or column) are altered by adding to them any

constant multiple of the corresponding elements in any other row (or column).

$$\begin{vmatrix} 3 & 1 & 4 \\ 6 & 2 & 1 \\ 7 & 0 & -5 \end{vmatrix} = \begin{vmatrix} 3+2\times 6 & 1+2\times 2 & 4+2\times 1 \\ 6 & 2 & 1 \\ 7 & 0 & -5 \end{vmatrix}$$

6. If each element of a row (or a column) of a determinant can be expressed as a sum of two, the determinant can be written as the sum of two determinants.

$$\begin{vmatrix} 3 & 1 & 4 \\ 6 & 2 & 1 \\ 7 & 0 & -5 \end{vmatrix} = \begin{vmatrix} -1+4 & 1 & 4 \\ 3+3 & 2 & 1 \\ 5+2 & 0 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1 & 4 \\ 3 & 2 & 1 \\ 5 & 0 & -5 \end{vmatrix} + \begin{vmatrix} 4 & 1 & 4 \\ 3 & 2 & 1 \\ 2 & 0 & -5 \end{vmatrix}$$

= - 49

7. Determinant of a product of matrices

$$D\left([\mathbf{A}][\mathbf{B}]\right)=D[\mathbf{A}]\,D[\mathbf{B}]$$

$$[\mathbf{A}]=\begin{bmatrix}2 & 3 & 4\\ 1 & -1 & 3\\ 4 & 2 & 1\end{bmatrix}\qquad [\mathbf{B}]=\begin{bmatrix}1 & 2 & 3\\ 4 & -6 & 5\\ 3 & 1 & 4\end{bmatrix}$$

$$[\mathbf{C}]=[\mathbf{A}][\mathbf{B}]\qquad [\mathbf{C}]=\begin{bmatrix}26 & -10 & 37\\ 6 & 11 & 10\\ 15 & -3 & 26\end{bmatrix}$$

$$D[\mathbf{C}]=D\left([\mathbf{A}][\mathbf{B}]\right)=1505$$

$$D[\mathbf{A}]=43 \quad \text{and} \quad D[\mathbf{B}]=35$$

$$D[\mathbf{A}]\,D[\mathbf{B}]=43\times 35=1505$$