Differentiation under integral sign (DUIS)

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Consider $I = \int_a^b f(x)\,dx$, this integrand converted into f^n of $\underline{x \,\&\, \alpha}$,

where α is a parameter,

1) $f(x,\alpha)$ is continuous function & $\frac{\partial f}{\partial \alpha}$ is also cont. in both variable x & α throughout $[a,b]$

If $\underline{a(\alpha)}$ & $\underline{b(\alpha)}$ are limits $\Rightarrow I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx$ Then its

derivative w.r. to $\alpha \Rightarrow$ $\boxed{\frac{d}{d\alpha}(I\alpha)) = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x,\alpha)\,dx + \left\{ f(b,\alpha)\frac{\partial b}{\partial \alpha} - f(a,\alpha)\frac{\partial a}{\partial \alpha} \right\}}$

Now, if a & b are constants OR independent $\alpha \Rightarrow \frac{\partial b}{\partial \alpha} = \frac{\partial a}{\partial \alpha} = 0 \Rightarrow$ $\boxed{\frac{d}{d\alpha}(I(\alpha)) = \int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)\,dx}$

$I = \int_0^1 \left(\frac{x^3 - 1}{\log x}\right) dx$ $\longrightarrow$ $I = \int_0^1 \frac{x^\alpha - 1}{\log x}\,dx$ Hence find $\int_0^1 \frac{x-1}{\log x}\,dx$

$f(x) \longrightarrow \boxed{f(x,\alpha)}$

$\boxed{I(\alpha) = \int_0^1 \left(\frac{x^\alpha - 1}{\log x}\right) dx}$ — (1)

$\frac{dI}{d\alpha} = \frac{\partial}{\partial \alpha}\int_0^1 \left[f(x,\alpha) \right] dx$

$= \int_0^1 \frac{\partial}{\partial \alpha} f(x,\alpha)\,dx$

diff. w.r. to α

$\frac{dI}{d\alpha} = \int_0^1 \frac{\partial}{\partial \alpha}\left[\frac{x^\alpha - 1}{\log x}\right] dx$

$= \int_0^1 \left[\frac{1}{\cancel{\log x}}\right]\left(x^\alpha \cancel{\log x}\right) dx$

$\frac{dI}{d\alpha} = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$

Integrating w.r. to α

$\underset{\text{L.H.S.}}{I(\alpha)} = \int \frac{1}{\alpha+1}\,d\alpha = \underset{\text{R.H.S.}}{\log(\alpha+1) + C}$

put $\boxed{\alpha = 0}$,

R.H.S. $= \log(0+1) + C = 0 + C$

L.H.S. $= I(0) = \int_0^1 \frac{x^0 - 1}{\log x}\,dx = \int_0^1 \frac{1-1}{\log x}\,dx$

$$\Rightarrow \boxed{C = 0} \qquad = 0$$

$$\boxed{I(\alpha) = \log(\alpha+1)} \qquad \alpha = 3$$

$$I(\alpha) = \int_0^1 \frac{x^{\alpha}-1}{\log x}\,dx = \int_0^1 \frac{x^3-1}{\log x} = \log(3+1) = \log 4$$

$$\int_0^1 \frac{x^{10}-1}{\log x} = \log(10+1) = \boxed{\log 11}$$

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2) Show that $\int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx = \cot^{-1} a$

& Hence Deduce that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

Sol^n: Let $I(a) = \int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx$

Then By rule of DUIS

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\,dx = \int_0^\infty \frac{\sin x}{\cancel{x}} e^{-ax}(-\cancel{x})$$

$$= -\int_0^\infty e^{-ax}\sin x\,dx$$

$$= -\left\{\frac{e^{-ax}}{a^2+1}\left[-a\sin x - \cos x\right]\right\}_0^\infty$$

$$\frac{dF}{da} = -\frac{1}{a^2+1}$$

To find I, Integrating b.s. w.r.to a

$$I(a) = \int -\frac{1}{a^2+1}\,da = \cot^{-1}(a) + C$$

To find C, put $a = \infty$

$$L.H.S. \Rightarrow I(\infty) = \int_0^\infty \frac{e^{-\infty}\sin x}{x}\,dx = 0 \qquad \to 0$$

$$R.H.S. \Rightarrow \cot^{-1}(\infty) + C = 0 + C$$

$$\Rightarrow \boxed{C = 0}$$

$$\therefore \boxed{I(a) = \cot^{-1}(a)}$$

for deduction, $a = 0$

$$I(0) = \int_0^\infty \frac{e^{-0x}\sin x}{x}\,dx = \int_0^\infty \frac{\sin x}{x}\,dx$$

$$\boxed{\int_0^\infty \frac{\sin x}{x}\,dx = I(0)} = \cot^{-1}(0)$$

$I(a) = \int \quad a^2+1$

$$\boxed{\int_0^\infty \frac{\sin x}{x}\,dx = I(0) = \frac{\pi}{2}} = \cot^{-1}(0)$$

H.W.

1) Show that $\displaystyle\int_0^\infty \left[\frac{\log(1+ax^2)}{x^2}\right] dx = \pi\sqrt{a} \quad (a>0)$

Hence evaluate $\displaystyle\int_0^\infty \frac{\log(1+x^2)}{x^2}\,dx$

Sol^n: $I(a) =$

$$\frac{dF}{da} = \int_0^\infty \frac{1}{\cancel{x^2}} \frac{1}{(1+ax^2)} (\cancel{x^2})\,dx$$

$$= \frac{1}{a}\int_0^\infty \frac{1}{\left(\frac{1}{a}+x^2\right)}\,dx$$

$$= \frac{1}{a}\left[\frac{1}{1/\sqrt{a}}\tan^{-1}\left(\frac{x}{1/\sqrt{a}}\right)\right]_0^\infty$$

$$\boxed{\frac{dF}{da} = \frac{\pi}{2}\frac{1}{\sqrt{a}}}$$

$$\boxed{I = \pi\sqrt{a} + C}$$

$a=0$

$$\boxed{C=0}$$

$$\boxed{I(a) = \pi\sqrt{a}}$$

4) Let $I(a) = \int_0^\infty$ —— (1)

Sol^n: By rule of DUIS

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}(f)\, dx$$

$$= \int_0^\infty \frac{e^{-x}}{x}\left(1 + \frac{1}{\cancel{x}} e^{-ax}(-\cancel{x})\right) dx$$

$$\left(\frac{dI}{da}\right) = \int_0^\infty \frac{e^{-x}}{x}\left(1 - \boxed{e^{-ax}}\right) dx \quad —— (2)$$

Still we have x in the D'r

∴ Applying rule of DUIS again,

$$\frac{d^2I}{da^2} = \int_0^\infty \frac{e^{-x}}{\cancel{x}}\left(- e^{-ax}(-\cancel{x})\right) dx$$

$$= \int_0^\infty e^{-x(1+a)}\, dx$$

$$\frac{d^2I}{da^2} = \left[\frac{e^{-x(1+a)}}{-(1+a)}\right]_0^\infty = \frac{1}{1+a} \quad —— (3)$$

To find I,

Integrating (3) w.r.to a

$$\frac{dI}{da} = \log(1+a) + C_1$$

To find C_1, put $a = 0$

$$L.H.S. = \frac{dI}{da}(0) = \int_0^\infty \frac{e^{-x}}{x}(1 - e^0)\, dx \quad — \text{from (2)}$$

$$= 0$$

$$R.H.S. = \log(1+0) + C_1 = C_1$$

$$\Rightarrow \boxed{C_1 = 0}$$

$$\frac{dI}{da} = \log(1+a)$$

Integrating w.r.to a

$$I(a) = \int \log(1+a)\, da$$

$$= \log(1+a)(a) - \int \frac{a}{1+a}\, da + C_2$$

$$I(a) = a\log(1+a) - \int \frac{a+1-1}{a+1}\, da + C_2$$

$$= a\log(1+a) - a + \log(1+a) + C_2$$

$$I(a) = (a+1)\log(a+1) - a + C_2$$

put $a = 0$

$$L.H.S. = I(0) = \int_0^\infty \frac{e^{-x}}{x}\left(-\frac{1}{x} + \frac{1}{x}\right) dx \quad \boxed{\text{from (1)}} = 0$$

$$R.H.S. = 1\log(1) - 0 + C_2 = C_2$$

$$\Rightarrow \boxed{C_2 = 0}$$

R.H.S. $= 1 \log(1) - 0 + C_2 = C_2$

$\Rightarrow \boxed{C_2 = 0}$

$$\boxed{I(a) = (a+1)\log(a+1) - a}$$

DUIS (Two Parameter)

1) Prove that $\int_0^\infty \left(\frac{\tan^{-1}ax - \tan^{-1}bx}{x}\right)dx = \frac{\pi}{2}\log\left(\frac{a}{b}\right)$

Sol^n: Let $I(a) = \int_0^\infty \left(\frac{\tan^{-1}(ax) - \tan^{-1}bx}{x}\right)dx$

By Rule of DUIS

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\,dx = \int_0^\infty \frac{1}{\cancel{x}}\left(\frac{1}{1+a^2x^2}\right)(\cancel{x})\,dx$$

$$= \frac{1}{a^2}\int_0^\infty \frac{1}{(1/a)^2 + x^2}\,dx$$

$$= \frac{1}{a^{\cancel{2}}}\left[\frac{1}{1/\cancel{a}}\tan^{-1}\left(\frac{x}{1/a}\right)\right]_0^\infty$$

$$\frac{dI}{da} = \frac{1}{a}\left[\frac{\pi}{2} - 0\right] = \boxed{\frac{\pi}{2a}}$$

Integrating w.r. to a

$$I(a) = \frac{\pi}{2}\int \frac{1}{a}\,da = \frac{\pi}{2}\log a + C \quad \text{--- (1)}$$

put $\boxed{a = b}$

$$\text{L.H.S.} = I(b) = \int_0^\infty \frac{\tan^{-1}bx - \tan^{-1}bx}{x}\,dx$$

$$= 0$$

$$\text{R.H.S.} = \frac{\pi}{2}\log b + C$$

$$\Rightarrow \boxed{C = -\frac{\pi}{2}\log b}$$

$$I(a) = \frac{\pi}{2}\log a - \frac{\pi}{2}\log b$$

$$\boxed{I = \frac{\pi}{2}\log\left(\frac{a}{b}\right)}$$

Miscan:

1) Show that $\int_0^\infty e^{-\left(x^2 + \frac{a^2}{x^2}\right)}dx = \boxed{\frac{\sqrt{\pi}}{2}e^{-2a}}$, $a > 0$

Given $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ & Hence evaluate $\int_0^\infty e^{-\left(x^2 + \frac{1}{x^2}\right)}dx$

Sol^n: Let $I(a) = \int_0^\infty e^{-\left(x^2 + \frac{a^2}{x^2}\right)}dx$

$\rightarrow$ (D.E.)

Sol: Let $I(a) = \int_0^{\infty} e^{-\left(x^2+\frac{a^2}{x^2}\right)} dx$

By Rule of DUIS

$$\frac{dI}{da} = \int_0^{\infty} \frac{\partial f}{\partial a} dx = \int_0^{\infty} e^{-\left(x^2+\frac{a^2}{x^2}\right)}\left(-\frac{2a}{x^2}\right)dx$$

Put $\frac{a}{x} = y$, $-\frac{a}{x^2}dx = dy$

x	0	∞
y	∞	0

$$\frac{dI}{da} = 2\int_{\infty}^{0} e^{-\left(\frac{a^2}{y^2}+y^2\right)} dy$$

$$\frac{dI}{da} = -2\int_0^{\infty} e^{-\left(y^2+\frac{a^2}{y^2}\right)} dy = -2I$$

$\Rightarrow \frac{dI}{da} = -2I \rightarrow$ (D.E.) V.S./linear

$\Rightarrow \frac{dI}{I} = -2\,da$

$\Rightarrow \log I = \underline{-2a} + \log C$

$\log\left(\frac{I}{C}\right) = -2a$

$\Rightarrow \boxed{I(a) = Ce^{-2a}} \rightarrow (2)$

To find C, put $a = 0$

R.H.S. $= Ce^{-0} = C$

L.H.S. $= I(0) = \int_0^{\infty} e^{-(x^2+0)} dx$

$= \frac{\sqrt{\pi}}{2}$

$\Rightarrow C = \frac{\sqrt{\pi}}{2}$

$\Rightarrow \boxed{I(a) = \frac{\sqrt{\pi}}{2} e^{-2a}}$

To evaluate $\int e^{-\left(x^2+\frac{1}{x^2}\right)} dx$ put $a = 1$

$\Rightarrow I(1) = \boxed{\frac{\sqrt{\pi}}{2} e^{-2}} = \frac{\sqrt{\pi}}{2e^2}$

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2) S.T. $\int_0^{\infty} e^{-bx^2} \cos 2ax \, dx = \frac{1}{2}\sqrt{\frac{\pi}{b}}\, e^{-a^2/b}$,

Assume $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Sol^n: Let $I(a) = \int_0^{\infty} e^{-bx^2} \cos \underline{2ax}\, dx$

By Rule of DUIS

$$\frac{dI}{da} = \int_0^{\infty} \frac{\partial f}{\partial a} dx = \int_0^{\infty} \boxed{e^{-bx^2}}\,\boxed{(-\sin 2ax)}\,(2x)\, dx$$

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\,dx = \int_0^\infty e^{-bx^2}(-\sin 2ax)(2x)\,dx$$

To obtain the integral, (Integration by parts)

Put $-bx^2 = t \Rightarrow -2bx\,dx = dt$ ∴

$$\therefore \int e^{-bx^2}(-2x)\,dx = \int e^t\left(\frac{dt}{b}\right) = \frac{e^t}{b} = \frac{e^{-bx^2}}{b}$$

$$\frac{dI}{da} = \left[\sin 2ax\left(\frac{e^{-bx^2}}{b}\right)\right]_0^\infty - \int_0^\infty \cos 2ax\,(2a)\left(\frac{e^{-bx^2}}{b}\right)dx$$

$$= 0 - \frac{2a}{b}\underbrace{\boxed{\int_0^\infty e^{-bx^2}\cos 2ax\,dx}}_{I}$$

$$\boxed{\frac{dI}{da} = -\frac{2a}{b}I} \longrightarrow \text{D.E. (v.s./linear)}$$