Leibnitz's Theorem(rule) for product (nth order product)

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Steps (Procedure)

1) Determine function whose nth derivative known, call it 'u'
2) If derivative of some function vanishes, we take it as v (e.g. x^m, polynomials)
3) Apply the rule.

Explanation:

$y = uv$

$y' = (u'v + uv')$

$y'' = u''v + u'v' + u'v' + uv''$

$= (u''v + 2u'v' + uv'')$

$y''' = u'''v + 3u''v' + 3u'v'' + uv'''$

$$\boxed{y^{(n)} = {}^nC_0 u^{(n)} v + {}^nC_1 u^{(n-1)} v' + \dots + {}^nC_r u^{(n-r)} v^{r} + \dots + {}^nC_n u v^{(n)}}$$

This is Leibnitz's rule for product

Problems:

1) $y = x^2 \sin x$ Find $y^{(n)}$

Let $u = \sin x$

$u^{(n)} = \sin\left(x + n\frac{\pi}{2}\right)$

$v = x^2$

By Leibnitz's Product rule

$y^{(n)} = u^{(n)} v + (n) u^{(n-1)} v' + \frac{n(n-1)}{2} u^{(n-2)} v'' + \dots$

$$\therefore \boxed{y^{(n)} = \sin\left(x + n\frac{\pi}{2}\right)(x^2) + n\left(\sin\left[x + (n-1)\frac{\pi}{2}\right]\right)(2x) + \frac{n(n-1)}{\cancel{2}}\left(\sin\left[x + (n-2)\frac{\pi}{2}\right](\cancel{2})\right) + 0 + 0}$$

2) $y = x^3 a^x$ Find $y^{(5)}$

Let $u = a^x$

$u^{(n)} = (\log a)^n a^x$

$v = x^3$

$\longrightarrow$ Leibnitz $\Rightarrow u^{(n)} v + n u^{(n-1)} v' + \dots$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$y^{(5)} = a^x (\log a)^5 (x^3) + 5 a^x (\log a)^4 (3x^2) +$
$10 a^x (\log a)^3 (6x) + 10 a^x (\log a)^2 (6) + 0$

$y^{(5)} = a^x \left[x^3 (\log a)^5 + 15x^2 (\log a)^4 + 60x (\log a)^3 \right.$

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H.W. 3) If $y = \frac{\log x}{x}$, Then Prove that $(\log x)\left(\frac{1}{x}\right)$

$$y^{(5)} = \frac{5!}{x^6}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} - \log x\right]$$

H.W. 4) If $y = x^n \log x$ Then prove that $y_{n+1} = \frac{n!}{x}$

★ **Problems with recurrence Relations :**

1) If $y = \tan^{-1}x$, Then prove that

$$\boxed{(1+x^2)\,y_{n+2} + 2(n+1)\,x\,y_{n+1} + n(n+1)\,y_n = 0}$$

Hence deduce that $y_n(0) = 0$ if n is even

$y_n(0) = (-1)^{\frac{n-1}{2}}(n-1)!$ if n is odd

Sol^n: Consider $y = \tan^{-1}x$ — (A)

diff. w.r.to x, $y' = \frac{1}{1+x^2}$

multiply $(1+x^2)$ b.s., $(1+x^2)\cdot y' = 1$ — (1)

diff. w.r.to x

$$\boxed{(1+x^2)\,y''} + \left[y'(2x)\right] = 0 \quad — (2)$$

Now apply nth order derivative & Leibnitz's rule.

$$(uv)^{(n)} = u^{(n)}v + n\,u^{(n-1)}v' + \frac{n(n-1)}{2}u^{(n-2)}v'' + \cdots$$

$$\Rightarrow \left[y_{n+2}(1+x^2) + n\,y_{n+1}(2x) + \frac{n(n-1)}{\cancel{2}}y_n(\cancel{2}) + 0\right]$$

$$+\left[y_{n+1}(2x) + n\,y_n(2) + 0\right] = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + (2xn+2x)\,y_{n+1} + \left[n^2 - n + 2n\right]y_n = 0$$

$n^2 + n$

$$\boxed{(1+x^2)\,y_{n+2} + 2(n+1)\,x\,y_{n+1} + n(n+1)\,y_n = 0} \quad (3)$$

put $x = 0$ in (A) (1) (2) & (3)

$y(0) = \tan^{-1}(0) = 0$

$y_1(0) = \frac{1}{1+x^2} = 1$,

$(1+x^2)y'' + y'(2x) = 0$

$y''(0) + 0 = 0 \Rightarrow y''(0) = 0$

put in (3)

$(1+0)\,y_{n+2}(0) + 0 + n(n+1)\,y_n(0) = 0$

$$\boxed{y_{n+2}(0) = -n(n+1)\,y_n(0)} \longrightarrow \text{recurrance relation.}$$

If n is even, put $n = 2, 4, 6, 8$

$y_4(0) = -(2)(3)\,y_2(0) = 0$

$y_6(0) = -(4)(5)\,y_4(0) = 0$ & so on

$y_n(0) = -n(n+1)\,y_n(0) = 0$ if n is even

If n is odd, put $n = 1, 3, 5$

$y_3(0) = -(1)(2)\,y_1(0) = -(1)(2) = -2!$

$y_5(0) = -(3)(4)\,y_3(0) = -(3)(4)[-(1)(2)]$

$= 4!$

$y_n(0) = -n(n+1)\,y_{n-2}(0) = (-1)^{\frac{n-1}{2}}(n-1)!$

if n is odd.

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2) If $m\sin^{-1}x = \sin^{-1}y$ Then prove that

$\rightarrow (1-x^2)\,y_{n+2} - (2n+1)x\,y_{n+1} + (m^2-n^2)\,y_n = 0$

Hence deduce that $y_n(0) = 0$ if n even &

$y_n(0) = ((n-2)^2 - m^2)\ldots(3^2-m^2)(1^2-m^2)\,m$ if n is odd

Sol^n: Consider $y = \sin[m\sin^{-1}x]$ — (A)

diff. w.r.to x, $y' = \cos[m\sin^{-1}x]\left[\frac{m}{\sqrt{1-x^2}}\right]$

multiply by $\sqrt{1-x^2}$, $\sqrt{1-x^2}\,y' = m\cos[m\sin^{-1}x]$ — (1)

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diff w.r.to x, $\sqrt{1-x^2}\,y'' + y'\dfrac{(-\cancel{2}x)}{\cancel{2}\sqrt{1-x^2}} = -m\sin[m\sin^{-1}x]\left[\dfrac{m}{\sqrt{1-x^2}}\right]$

multiply by $\sqrt{1-x^2}$, $(1-x^2)y'' - xy' = -m^2 y$

$$\boxed{(\underbrace{1-x^2}_{v})\underbrace{y''}_{u} - \underbrace{x}_{v}\underbrace{y'}_{u} + m^2 y = 0} \quad — (2)$$

Apply nth order derivative and Leibnitz's Rule.

$\rightarrow (uv)^{(n)} = u^{(n)}v + n\,u^{(n-1)}v' + \dfrac{n(n-1)}{2}u^{(n-2)}v'' + \cdots\cdots$

$\therefore \left[y_{n+2}(1-x^2) + n\,y_{n+1}(-2x) + \dfrac{n(n-1)}{\cancel{2}}y_n(-\cancel{2}) + 0\right]$

$- \left[y_{n+1}(x) + n\,y_n(1) + 0\right] + m^2 y_n = 0$

$\therefore (1-x^2)y_{n+2} + (-2xn - x)y_{n+1} + (-n^2 + \cancel{n} - \cancel{n} + m^2)y_n = 0$

$$\therefore \boxed{(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} + (m^2-n^2)y_n = 0} \quad (3)$$

put $x=0$ in (A) (1) (2) & (3)

$y(0) = \sin[m\sin^{-1}(0)] = 0$

$y_1(0) = \cos[m\sin^{-1}(0)]\left(\dfrac{m}{\sqrt{1-0}}\right) = m(1) = m$

put in (2)

$(1-x^2)y'' - \cancel{x}y' + m^2 y = 0$

$$\boxed{y''(0) = -m^2 y(0)} = 0$$

put in (3)

$(1-0)y_{n+2}(0) - 0 + (m^2-n^2)y_n(0) = 0$

$$\boxed{y_{n+2}(0) = (n^2-m^2)y_n(0)}$$ required recurrence relation.

for n is even, put $n = 2, 4, 6$

$$y_4(0) = (2^2 - m^2)\, y_2(0) = 0$$

Similarly $y_6(0) = (4^2 - m^2)\, y_4(0) = 0$

$$y_n(0) = 0 \quad \text{if } n \text{ is even.}$$

for n is odd, put $n = 1, 3, 5$

$$y_3(0) = (1^2 - m^2)\, y_1(0) = (1^2 - m^2)\, m$$

$$y_5(0) = (3^2 - m^2)\, y_3(0) = (3^2 - m^2)(1^2 - m^2)\, m$$

$$\vdots$$

$$y_n(0) = ((n-2)^2 - m^2)\, y_{n-2}(0) = ((n-2)^2 - m^2) \cdots (3^2 - m^2)(1^2 - m^2)\, m$$

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3) $I_n = \dfrac{d^n}{dx^n}(x^n \log x)$ Then Prove that $I_n = n I_{n-1} + (n-1)!$ & Hence show that

$$I_n = n!\left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right]$$

Sol^n: Consider

$$I_n = \frac{d^n}{dx^n}(x^n \log x)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}(x^n \log x)\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[n x^{n-1} \log x + x^n \frac{1}{x}\right]$$

$$I_n = n \frac{d^{n-1}}{dx^{n-1}}(x^{n-1} \log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$\boxed{I_n = n I_{n-1} + (n-1)!}$$

$$I_0 = \frac{d^0}{dx^0}(x^0 \log x]$$

$$I_1 = \frac{d}{dx}[x \log x]$$

$$I_2 = \frac{d^2}{dx^2}[x^2 \log x]$$

Now, to get req expression, divide b.s. by $n!$

$$\frac{I_n}{n!} = \frac{n}{n!} I_{n-1} + \frac{(n-1)!}{n!}$$

$$\frac{I_n}{n!} = \frac{n}{n!} I_{n-1} \qquad \frac{\cdot}{n!}$$

$$\boxed{\frac{I_n}{n!} = \frac{1}{(n-1)!} I_{n-1} + \frac{1}{n}}$$

putting $n = 2, 3, 4, \dots, n$

$n=2 \Rightarrow \frac{I_2}{2!} = \frac{1}{1!} I_1 + \frac{1}{2}$

$n=3 \Rightarrow \frac{I_3}{3!} = \frac{1}{2!} I_2 + \frac{1}{3} = \underline{I_1 + \frac{1}{2} + \frac{1}{3}}$

$n=4 \Rightarrow \frac{I_4}{4!} = \frac{1}{3!} I_3 + \frac{1}{4} = I_1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$\frac{I_n}{n!} = \frac{1}{(n-1)!} I_{n-1} + \frac{1}{n} = \underline{I_1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n}}$

$I_n = n! \left[I_1 + \qquad \right]$

$\underline{I_1} = \frac{d}{dx}\left[x^1 \log x\right] = x\left(\frac{1}{x}\right) + \log x = \underline{\log x + 1}$

$$\boxed{I_n = n! \left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots \frac{1}{n} \right]}$$

H.W. 4) If $y^{1/m} + y^{-1/m} = 2x$ then prove that

$(x^2 - 1) y_{n+2} + (2n+1) x y_{n+1} + (n^2 - m^2) y_n = 0$

HW 5) If $\sin^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$ Then prove that

$x^2 y_{n+2} + (2n+1) x y_{n+1} + 2n^2 y_n = 0$

HW 6) If $x = e^t$ & $y = \cos mt$ Then prove that

$x^2 y_{n+2} + (2n+1) x y_{n+1} + (m^2 + n^2) y_n = 0$

HW

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + (m^2+n^2) y_n = 0$$

4) we have to express y explicitely as function of x

$$y^{1/m} + y^{-1/m} = 2x$$

$$\therefore y^{1/m} + \frac{1}{y^{1/m}} = 2x \Rightarrow y^{2/m} + 1 = 2xy^{1/m} \qquad (t = y^{1/m})$$

$$t^2 - 2xt + 1 = 0$$

$$y^{1/m} = t = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = \frac{\cancel{2}x \pm \cancel{2}\sqrt{x^2-1}}{\cancel{2}} = x \pm \sqrt{x^2-1}$$

take m^{th} power both sides

$$\rightarrow \boxed{y = (x \pm \sqrt{x^2-1})^m} \longrightarrow A$$

$$y' = m(x \pm \sqrt{x^2-1})^{m-1}\left(1 \pm \frac{(\cancel{2}x)}{\cancel{2}\sqrt{x^2-1}}\right)$$

$$y' = m(x \pm \sqrt{x^2-1})^{m-1}\left[\frac{\sqrt{x^2-1} \pm x}{\sqrt{x^2-1}}\right]$$

$$(\sqrt{x^2-1})y' = m(x \pm \sqrt{x^2-1})^m = my$$

$$(\sqrt{x^2-1})y' - my = 0 \quad \text{— } (1)$$

diff again w.r.to x

$$\sqrt{x^2-1}\, y'' + y' \frac{(\cancel{2}x)}{\cancel{2}\sqrt{x^2-1}} - my' = 0$$

multiply by $\sqrt{x^2-1}$

$$(x^2-1)y'' + xy' - my' = 0$$

(Complete remaining Part)

5) $$\sin^{-1}\left(\frac{y}{b}\right) = \boxed{\log\left(\frac{x}{n}\right)^n} = n\log x - n\log n$$

$$\rightarrow \boxed{y = b\sin[n\log x - n\log n]}$$

complete remaining

Hint $\rightarrow$ $x^2 y'' + xy' + n^2 y = 0$

remaining

Hint $\Rightarrow$ $x^2y'' + xy' + n^2y = 0$

6) $x = e^t$ & $y = \cos mt$

$t = \log x \Rightarrow \boxed{y = \cos[m \log x]}$

$y' = -\sin(m \log x)\left(\frac{m}{x}\right)$

(Complete remaining part.)

$\boxed{x^2y'' + xy' + m^2y = 0}$