SUCCESSIVE DIFFERENTIATION

Standard Formulae

Function of x	n th derivative w.r.t. x
y=$(ax+b)^m$	$y_n = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$ if m>n $=n!\,a^n$ if m=n =0 if n>m
y=$\frac{1}{(ax+b)^m}$	$y_n = \frac{(-1)^n\ (m+n-1)!\ a^n}{(m-1)!\,(ax+b)^{m+n}}$
y=$\frac{1}{(ax+b)}$	$y_n = \frac{(-1)^n\ n!\ a^n}{(ax+b)^{n+1}}$
y=log (ax+ b)	$y_n = \frac{(-1)^{n-1}\ (n-1)!\ a^n}{(ax+b)^n}$
y=a^{mx}	$y_n = m^n a^{mx}(\log a)^n$
y=e^{mx}	$y_n = m^n e^{mx}$
y= cos (ax+ b)	$y_n = a^n \cos(ax+b+\frac{n\pi}{2})$
y=sin(ax+ b)	$y_n = a^n \sin(ax+b+\frac{n\pi}{2})$
y=$e^{ax}\cos(bx+c)$	$y_n = r^n e^{ax}\cos(bx+c+n\theta)$ where r=$\sqrt{a^2+b^2}$ & $\theta = \tan^{-1}(\frac{b}{a})$
y=$e^{ax}\sin(bx+c)$	$y_n = r^n e^{ax}\sin(bx+c+n\theta)$ where r=$\sqrt{a^2+b^2}$ & $\theta = \tan^{-1}(\frac{b}{a})$

Problems

01. Find the nth derivative w. r. t. x of the following

(a) $\frac{x}{(x-1)(x-2)(x-3)}$ **(M−16)** (b) $\frac{8x}{x^3-2x^2-4x+8}$ $(\mathbf{D-07})$

(c) $y=\frac{x^3}{(x+1)(x-2)}$ **(D-17)** (d) $\frac{x^2+4}{(x-1)^2(2x+3)}$ **(D-12)**

02. Find the nth order derivative of the following

(a) $\frac{1}{x^2+a^2}$ (b) $\tan^{-1}x$ (c) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ (d) $\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$

03. (a) If $y=(x-1)^n$ show that $y+\frac{y_1}{1!}+\frac{y_2}{2!}+L\quad+\frac{y_n}{n!}=x^n$

(b) If $y=x\log\left(\frac{x-1}{x+1}\right)$ S.T. $y_n=(-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n}-\frac{x+n}{(x+1)^n}\right]$

04. Find the nth derivative of

$(a)\cos^4 x$ $(b)\ \sin^5 x$ $(c)\ \sin x\sin x\sin 3x$ **(D−14)**

$(d)\,e^x\cos x\cos 2x$ **(M−16)** $(d)\,e^{2x}\sin\frac{x}{2}\cos\frac{x}{2}\sin 3x$ **(D−17)**

Homework

05. Find the nth derivative w. r. t. x of (a) $\frac{x}{x^2+a^2}$ **(D-07)** (b) $\frac{1}{x^3+x^2+x+1}$ **(M-09)** (c) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ $(d)\cos^3 x\sin^2 x$ **(M-19)**

06. If $I_n=\frac{d^n}{dx^n}\left(x^n\log x\right)$ show that $I_n=(n-1)!+n\,I_{n-1}$ and hence

$I_n=n!\left(\log x+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$

07. Find the nth derivative of

(a) $\cos 5x\cos 3x\cos x$ **(M-17,M−13)** $(b)\,2^x\cos x\sin^2 x$ **(D-16,D-13)**

(c) $2^x\cos^9 x$ (M−09) (d) $\sin px+\cos px$ as $p^n[1+(-1)^n\sin 2px]^{1/2}$ **(D-15,M-14)**

Leibnitz's Theorem

$$(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_1 u_{n-2} v_2 + \cdots + {}^nC_n uv_n$$

08. Find the nth derivative w. r. t. x of the following

(a) $x^3 e^{2x}$ (b) $x^3 \cos^2 x$

09. If $y = x\log(x+1)$ show that $y_n = \dfrac{(-1)^{n-2}(n-2)!(x+n)}{(x+1)^n}$

10.If $y = e^{m\sin^{-1}x}$ show that **(M-08)**

(a) $(1-x^2)y_2 - xy_1 - m^2 y = 0$

(b) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

11. If $y = \log(x+\sqrt{x^2+1})$ then show that

(a) $(1+x^2)y_2 + xy_1 = 0$

(b) $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$

(c) $y_n(0) = 0$, and $y_{2n+1}(0) = (-1)^n[1^2.3^2.\ 5^2.\ .\ .\ .\ (2n-1)^2]$

12. If y= $\cos(m\sin^{-1}x)$ S T.$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$ (**D – 13** **(D-17)**

13. If y= $\sin[\log(x^2+2x+1)]$ then show that

$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$ **(M-15, M-12)**

14. If $y = \frac{\sin h^{-1}x}{\sqrt{1+x^2}} = a_0 + a_1 x + a_2 x^2 + \cdots$ then show that

(a) $(1+x^2)y_1 + xy = 1$

(b) $(1+x^2)y_{n+2} + (2n+3)xy_{n+1} + (n+1)^2 y_n = 0$

(c) $y_{2n}(0) = 0$ and $y_{2n+1}(0) = (-1)^n 2^2.4^2.\ .\ .\ .\ (2n)^2$

(d) $(n+2)a_{n+2} + (n+1)a_n = 0$

15. If $y = (\sin^{-1}x)^2$ obtain $y_n(0)$ **(D-12)**

Homework

15. (a) $e^{x\cos\alpha}\cos(x\sin\alpha)$ (b) $x^3\cosh^2 x$ (c) $x^2e^{3x}\cos^3 x$

16. By forming the nth derivative of x^{2n} in 2 ways show that

$$1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2.2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2}+\cdots=\frac{(2n)!}{(n!)^2}$$

17. (a) If $y=\frac{\log x}{x}$ show that $y_5=\frac{5!}{x^6}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+-\log x\right)$ **(M-10)**

(b) If $y=x^n\log x$ show that $y_{n+1}=\frac{n!}{x}$ **(D-08)**

6. *If* $x=\cos\left(\frac{1}{m}\log y\right)$ *s.t.* $(x^2-1)\ y_{n+2}+(2n+1)xy_{n+1}+(n^2+m^2)y_n=0$**(D – 15)**

17. If $y=(x+\sqrt{x^2-1})$ then show that

$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m)y_n=0$ **(D-16)**

18. If $y^{\frac{1}{m}}-y^{\frac{-1}{m}}=2x$ show that

(a) $(1+x^2)y_2+xy_1-m^2y=0$

(b) $(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$ **(M-13, D-10)**

(c) $y=1+\frac{mx}{1!}+\frac{m^2x^2}{2!}+\frac{m(m^2-1^2)x^3}{3!}+\cdots$

19. If $y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$ then s.t $(1-x^2)y_{n+1}-(2n+1)xy_n-n^2y_{n-1}=0$

and find $y_n(0)$ **(D-14, M-14)**

20. If $y=e^{\tan^{-1}x}=a_0+a_1x+a_2x^2+\cdots$ show that

(a) $(1+x^2)y_1=y$

(b) $(1+x^2)y_{n+2}+[(2(n+1)x-1]y_{n+1}-n(n+1)\ \ y_n=0$ **(M-17,M-16)**

(c) $(n+2)a_{n+2}+na_n=a_{n+1}$

21. If $y=a\cos(\log x)+b\sin(\log x)$ then show that

$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$ **(M-19)**

Expansion of Functions in Series

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots \qquad \textbf{(Maclaurin Series)}$$

01. Derive the following standard expansions

(a) $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$ (b) $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$

(c) $\tan x = x+\frac{1}{3}x^3+\frac{2}{15}x^5\cdots$ (d) $e^x = 1+x+\frac{x^2}{2!}+\cdots$

(e) $\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots \qquad |x|<1$

(f) $(1+x)^n = 1+\frac{nx}{1!}+\frac{n(n-1)x^2}{2!}+\cdots \qquad |x|<1$

02. Show that $\log\sec\left(\frac{\pi}{4}+x\right)=\frac{1}{2}\log 2+x+\frac{x^2}{1}+\frac{2x^3}{3}+\cdots$ **(D-11)**

03. Show that

(a) $\tanh x = x-\frac{x^3}{3}+\frac{2}{15}x^5-\cdots$ (b) $x\,\mathrm{cosec}\,x = 1+\frac{x^2}{2}+\frac{7}{360}x^4-\cdots$ **(D-17)**

(c) $\frac{x}{e^x-1}=1-\frac{1}{2}x+\frac{1}{12}x^2-\frac{1}{720}x^4-\cdots$ and hence deduce

$\frac{x}{2}\left(\frac{e^x+1}{e^x-1}\right)=1+\frac{1}{12}x^2-\frac{1}{720}x^4-\cdots$ **(M-14)**

04. Expand in ascending powers of x upto the term x^4 the following functions

(a)$(1+x)^x$ (b)$e^{x\sin x}$ (c)$\sin(e^x-1)$ **(D-16)**(d)$\sec^2 x$ (M-16) (e) $e^x\log(1+x)$

05. Show that

(a) $\log\left[\log(1+x)^{\frac{1}{x}}\right]=-\frac{x}{2}+\frac{5x^2}{24}-\frac{x^3}{8}+\frac{251}{2880}x^4-\cdots$

(b) $\log(1+\tan x) = x-\frac{x^2}{2}+\frac{2x^3}{3}-\cdots$

(c) $\log\left(\frac{\sinh x}{x}\right)=\frac{x^2}{6}-\frac{x^4}{180}+\frac{x^6}{2835}-\cdots$

06. Show that

(a) $\sin^{-1}x = x+\frac{1}{2}\frac{x^3}{3}+\frac{1}{2}\frac{3}{4}\frac{x^5}{5}+\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7}+\cdots$ and hence

(b)$(\sin^{-1}x)^2=x^2+\frac{1}{3}x^4+\frac{8}{45}x^6\cdots$ (c) $\frac{\sin^{-1}x}{\sqrt{1-x^2}}=x+\frac{2}{3}x^3+\frac{8}{15}x^5\cdots$

(d) $\sin^{-1}(3x-4x^3)=3\sin^{-1}x$ (e) $\sec^{-1}\left(\frac{1}{1-2x^2}\right)=2\sin^{-1}x$

(f) $e^\theta = 1+\sin\theta+\frac{1}{2!}\sin^2\theta+\cdots$ (g) $e^{\cos^{-1}x}=e^{\frac{\pi}{2}}\left(1-x+\frac{x^2}{2}-\frac{x^3}{3}\cdots\right)$**(D-08)**

07. S.T. (a) $\log\sec x=\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}\cdots$ $(\mathbf{D-15, D-14, M-09})$

(b) $\log\left(x+\sqrt{1+x^2}\right)=x-\frac{x^3}{6}+\frac{3x^5}{40}-\cdots$

Taylor Series

$$f(x+h)=f(h)+x\frac{f'(h)}{1!}+x^2\frac{f''(h)}{2!}+x^3\frac{f'''(h)}{3!}+\cdots \text{ in ascending powers of x.}$$

08. Using Taylor series

(a)Find the first 4 terms of $\tan\left(\frac{\pi}{4}+x\right)$ in ascending powers of x and hence find the value of $(\tan 46^\circ 30')$ correct to 4 decimal places.

(b)Arrange $7+(x+2)+3(x+2)^3+(x+2)^4-(x+2)^5$ in ascending powers of x.**(D-13)**

$$f(x)=f(a)+(x-a)\frac{f'(a)}{1!}+(x-a)^2\frac{f''(a)}{2!}+(x-a)^3\frac{f'''(a)}{3!}+\cdots \text{ in ascending powers of(x-a).}$$

09. (a)Expand $2x^3+7x^2+x-6$ in ascending powers of (x-2) **(M-17,D-16).**

(b)Expand $\tan^{-1}x$ in powers of (x-1) and hence find $\tan^{-1}(1.003)$ **(M-15)**

(c) Expand $\tan^{-1}x$ in powers of $(x-\frac{\pi}{4})$ **(M-19)**

Homework

10. Show that (a)$\log\tan\left(\frac{\pi}{4}+x\right)=2x+\frac{4x^3}{3}+\frac{4x^5}{3}+\cdots$

(b)$\log(1+e^x)=\log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\cdots$ (c) $\sec^2 x=1+x^2+\frac{2x^4}{3}+\cdots$**(D-08)**

11. Derive the Maclaurin's series expansion for log (1+x) **(M-14)**;

hence find the series expansion of $\log\left(\frac{1+x}{1-x}\right)$ and the value of $\log\sec\left(\frac{11}{9}\right)$;

also find the expansion of $\tanh^{-1}x$ and $\log(1+x+x^2+x^3)$ **(M-13)**

and $\log(1-x+x^2)$ **(D-13)**

12. Show that $\sinh x=x+\frac{x^3}{3!}+\frac{1}{5!}x^5-\cdots$ **(M-15)**

13. Expand in ascending powers of x upto the term x^4 the following functions

(a) e^{e^x} (b) $e^{x\cos x}$ **(M-10)** (c) log (1+cos x) (d)log (1+sin x) $(\mathbf{D-12})$

14. Show that $\log\left(\frac{xe^x}{e^x-1}\right)=\frac{x}{2}-\frac{x^2}{24}+\frac{x^4}{2880}-\cdots$

15. Show that $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$ **(M-17)** and hence find expansions of

(a) $\log(1+x^2)$ **(D-10)** (b)$\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(c) $\cos^{-1}[\tanh(\log x)]$ **(D-17)** (d) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ **(M-10)**

(e) $e^\theta=1+\tan\theta+\frac{1}{2!}\tan^2\theta-\frac{1}{3!}\tan^3\theta+\frac{7}{4!}\tan^4\theta\cdots$

16.Show that $\log\cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \cdots$

Indeterminate Forms

L'Hospitals Rule for indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

Problems

01. Evaluate the following $\frac{0}{0}$ indeterminate forms

(a) $\lim_{x\to 1}\frac{x^x - x}{x - 1 - \log x}$ **(M-13)** (b) $\lim_{x\to 0}\frac{\tan x - \sin x}{\sin^3 x}$

02. Find the values of the constants a, b, c if (a) $\lim_{x\to 0}\frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$ **(M-08)**

(b) $\lim_{x\to 0}\frac{x(1+a\cos x) - b\sin x}{x^3} = 1$ **(M-16)** (c) $\lim_{x\to 0}\frac{ae^x - be^{-x} - cx}{x - \sin x} = 4$ **(D-15,D-10)**

03. Show that

(a) $\lim_{x\to 0}\frac{\tan x\tan^{-1}x - x^2}{x^6} = \frac{2}{9}$ (b) $\lim_{x\to 0}\frac{e^{x\sin x} - \cosh(\sqrt{2}x)}{x^4} = \frac{1}{6}$

(c) $\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}} - e + \frac{1}{2}ex}{x^2} = \frac{11}{24}e$ (d) $\lim_{x\to 0}\frac{e^{2x} - (1+x)^2}{x\log(1+x)}$ **(D-16)**

04. Evaluate the indeterminate forms

(a) $\lim_{x\to 0}\log_{\tan x}(\tan 2x)$ **(D-08)** (b) $\lim_{x\to\infty}\frac{\sinh^{-1}x}{\cosh^{-1}x}$ (c) $\lim_{x\to 0}\frac{\log_{\sin x}(\cos x)}{\log_{\sin\frac{x}{2}}\left(\cos\frac{x}{2}\right)}$

05. Evaluate the following $\infty - \infty$ indeterminate forms

(a) $\lim_{x\to 0}\left[\frac{a}{x}-\cot\frac{x}{a}\right]$ **(M-11)** (b) $\lim_{x\to 3}\left[\frac{1}{x-3}-\frac{1}{\log(x-2)}\right]$ (M−19)

06. Evaluate the following $0.\infty$ indeterminate forms

(a) $\lim_{x\to 0}\log(1-x)\cot\frac{\pi x}{2}$ (b) $\lim_{x\to 1}\log(1-x)\cot\frac{\pi x}{2}$

07. Evaluate the following $0^0,\infty^0,1^\infty$ indeterminate forms

(a) $\lim_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}}$ (b) $\lim_{x\to 0}(1+\tan x)^{\cot x}$ **(M-14)** (c) $\lim_{x\to 0}(\cot x)^{\sin x}$ **(D-17)**

(c) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ (d) $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x+4^x}{4}\right)^{\frac{1}{x}}$ **(M-12)** (e) $\lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}+c^{\frac{1}{x}}}{3}\right)^{3x}$

Homework

08. Evaluate

(a) $\lim_{x\to 0}\frac{\sin x - x+\frac{1}{6}x^3}{x^5}$ (b) $\lim_{x\to 0}\frac{\log_{\sec\frac{x}{2}}(\cos x)}{\log_{\sec x}\left(\cos\frac{x}{2}\right)}$ (c) $\lim_{x\to 0}\frac{\sin x\sin^{-1}x-x^2}{x^6}$ **(M-15)**

(d) $\lim_{x\to 0}\left[\frac{1}{x^2}-\cot^2 x\right]$ **(D-12)** (e) $\lim_{x\to 0}\frac{\log_{\sin x}(\cos x)}{\log_{\sin\frac{x}{2}}\left(\cos\frac{x}{2}\right)}$ **(f)** $\lim_{x\to\infty}\frac{\sinh^{-1}x}{\cosh^{-1}x}$

(g) $\lim_{x\to 0}\frac{e^x\sin x-x-x^2}{x^2+x\log(1-x)}$ **(D-11)** (h) $\lim_{x\to 0}\frac{x\sin(\sin x)-\sin^2 x}{x^6}$ (i) $\lim_{x\to 0}(\mathrm{cosec}\,x)^{\tan^2 x}$

(j) $\lim_{x\to a}\sin^{-1}\sqrt{\frac{a-x}{a+x}}\,\mathrm{cosec}\sqrt{a^2-x^2}$ **(D-07)** (k) $\lim_{x\to 0}\frac{1-x^{\sin x}}{x\log x}$

09. Find the values of a and b if (a) $\lim_{x\to 0}\frac{a\sinh x+b\sin x}{x^3}=\frac{5}{3}$ **(D-14, M-09)**

(b) $\lim_{x\to 0}\frac{a\sin^2 x+b\log\cos x}{x^4}=\frac{1}{2}$ **(D-09)**

10. Find the expansion of $\sin^{-1}x$ and hence obtain $\lim_{x\to 0}\frac{\sin^{-1}x-x}{x^3}$

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