Beta function more types

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Type III
$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx \rightarrow$$ Here note that limits are not 0 to ∞. Hence substitution $\tan^2\theta$ will not work

So If limit are 0 to 1 & if integrand contain denominator of form $a+bx$ then,
$$\boxed{\frac{x}{a+bx} = \frac{t}{a+b}}$$

Problems: 1) Evaluate $I = \int_0^1 \frac{x-2x^2+x^3}{(1+x)^5}\,dx = \int_0^1 \frac{x(1-2x+x^2)}{(1+x)^5}$

$$I = \int_0^1 \frac{x(1-x)^2}{(1+x)^5}\,dx$$

$$\frac{x}{1+x} = \frac{t}{1+1} = \frac{t}{2}$$

x	0	1
t	0	1

$2x = t(1+x) = t + tx$

$x(2-t) = t \quad \therefore \boxed{x = \frac{t}{2-t}}$

$1-x = 1 - \frac{t}{2-t} = \frac{2-2t}{2-t} = \frac{2(1-t)}{2-t}$

$$\boxed{1+x = 1+\frac{t}{2-t} = \frac{2}{2-t}}$$

Then $dx = -\frac{2}{(2-t)^2}(-1)\,dt = \frac{2dt}{(2-t)^2}$

Substitute in I

$$I = \int_0^1 \frac{\left(\frac{t}{2-t}\right)\left[\frac{2(1-t)}{2-t}\right]^2}{\left(\frac{2}{2-t}\right)^5} \frac{2dt}{(2-t)^2}$$

$$= \frac{2^3}{2^5}\int_0^1 \frac{t(1-t)^2 \cancel{\frac{1}{(2-t)^5}}}{\cancel{\frac{1}{(2-t)^5}}}\,dt$$

$$= \frac{1}{2^2}\beta(2,3) = \frac{1}{4}\frac{\Gamma 2\,\Gamma 3}{\Gamma 5} = \frac{1}{48}$$

Calculate

HW 2) $I = \int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx \rightarrow$ observe here limit is 0 to 1 but denominator is not linear $(a+bx)$ form

put $x^4 = t$, $x = t^{1/4}$, $dx = \frac{1}{4}t^{-3/4}\,dt$

	0	1
	0	1

$$\therefore I = \int_0^1 \frac{(1-t)^{3/4}\,\frac{1}{4}t^{-3/4}\,dt}{(1+t)^2} = \frac{1}{4}\int_0^1 \frac{t^{-3/4}(1-t)^{3/4}}{(1+t)^2}\,dt \rightarrow \text{is required form}$$

$\rightarrow$ Now solve as above $\rightarrow \boxed{\frac{t}{1+t} = \frac{u}{2}}$

$\downarrow$ Solve

Type IV

$$\int_a^b (x-a)^m (b-x)^n \, dx$$

trick: $(x-a) = (b-a)\,t$

Problem:

$$I = \int_3^7 \sqrt[4]{(x-3)(7-x)}\, dx$$

put $(x-3) = (7-3)\,t = 4t$

$x = 4t+3$

$dx = 4\,dt$

x	3	7
t	0	1

$$I = \int_0^1 (4t)^{1/4} \left(7-(4t+3)\right)^{1/4} 4\,dt$$

$$= \int_0^1 4^{1/4}\, t^{1/4} \left[4(1-t)\right]^{1/4} 4\,dt$$

$$= 8 \int_0^1 t^{1/4} (1-t)^{1/4}\, dt$$

$$I = 8\, \beta\left(\tfrac{5}{4}, \tfrac{5}{4}\right)$$

$$= 8\, \frac{\Gamma\frac{5}{4}\, \Gamma\frac{5}{4}}{\Gamma\frac{10}{4}} = \overset{2}{\cancel{8}}\, \frac{\cancel{\frac{1}{4}}\Gamma\frac{1}{4}\;\frac{1}{4}\Gamma\frac{1}{4}}{\frac{3}{\cancel{2}}\,\frac{1}{\cancel{2}}\,\Gamma\frac{1}{2}}$$

$$= \frac{2\left(\Gamma\frac{1}{4}\right)^2}{3\sqrt{\pi}}$$

H.W.

2) $$\int_5^6 (x-5)^5 (6-x)^6 \, dx$$

$\downarrow$

Ans $\Rightarrow \beta(6,7)$

Miscellaneous Problems

1) Prove $\beta(x,x) = \dfrac{1}{2^{2x-1}}\, \beta\left(x, \tfrac{1}{2}\right)$

Sol^n:

$$\beta(x,x) = \frac{\Gamma x\, \Gamma x}{\Gamma 2x} = \left(\frac{\Gamma x}{\Gamma 2x}\right) \Gamma x \quad \text{—(1)}$$

By duplication formula

$$\Gamma m\, \Gamma\left(m+\tfrac{1}{2}\right) = \frac{1}{2^{2m-1}}\sqrt{\pi}\, \Gamma 2m$$

$$\frac{\Gamma m}{\Gamma 2m} = \frac{\sqrt{\pi}}{2^{2m-1}\, \Gamma\left(m+\frac{1}{2}\right)}$$

put $m = x$, put in (1)

$$\beta(x,x) = \left[\frac{\sqrt{\pi}}{2^{2x-1}\, \Gamma\left(x+\frac{1}{2}\right)}\right] \Gamma x$$

$$= \frac{1}{2^{2x-1}} \left(\frac{\Gamma\frac{1}{2}\; \Gamma x}{\Gamma\left(x+\frac{1}{2}\right)}\right)$$

$$\beta(x,x) = \frac{1}{2^{2x-1}}\, \beta\left(x, \tfrac{1}{2}\right)$$

Short ~~ans~~:

$$\left(\Gamma 1\; \Gamma\tfrac{3}{4}\right) \Gamma\tfrac{2}{4}\; \Gamma 1 = \left(\frac{\pi}{\ldots}\right)\left(\sqrt{\pi}\right)(1)$$

~~Short ans:~~

2) evaluate $\Gamma\frac{1}{4}\,\Gamma\frac{2}{4}\,\Gamma\frac{3}{4}\,\Gamma\frac{4}{4} = \left(\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}\right)\Gamma\frac{2}{4}\,\Gamma 1 = \left(\frac{\pi}{\sin\frac{1}{4}\pi}\right)(\sqrt{\pi})\,(1)$

$= \pi\,\sqrt{2}\sqrt{\pi} = \pi\sqrt{2\pi}$

3) $\int_0^{\pi/2} \cos^{14} x\,dx \cdot \int_0^{\pi/2} \cos^{15} x\,dx =$ (n, $n+1$)

$\left[\frac{\cancel{13}}{\cancel{14}}\,\frac{11}{\cancel{12}}\,\frac{9}{10}\,\frac{7}{8}\,\frac{5}{6}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}\,\frac{\cancel{14}}{15}\,\frac{\cancel{12}}{\cancel{13}} \times \frac{2}{3} \times 1\right] = \frac{1}{15}\,\frac{\pi}{2}$

$= \frac{1}{n+1}\,\frac{\pi}{2}$

4) Given $\int_0^\infty \frac{x^{p-1}}{1+x} = \frac{\pi}{\sin p\pi}$ Then prove that $\Gamma p\,\Gamma 1-p = \frac{\pi}{\sin p\pi}$

$0<p<1$

Sol^n: Let $I = \int_0^\infty \frac{x^{p-1}}{1+x}dx$

put $x = \tan^2\theta$

$dx = 2\tan\theta\sec^2\theta\,d\theta$

x	0	∞
θ	0	$\pi/2$

$I = \int_0^{\pi/2} \frac{(\tan^2\theta)^{p-1}}{(\cancel{1+\tan^2\theta})}\,2\tan\theta\,\cancel{\sec^2\theta}\,d\theta$

$= 2\int_0^{\pi/2} \tan^{2p-1}\theta\,d\theta$

$I = 2\int_0^{\pi/2} \sin^{2p-1}\theta\,\cos^{1-2p}\theta\,d\theta$

$I = \cancel{2}\,\frac{1}{\cancel{2}}\,\beta\left(\frac{2p\cancel{-1}\cancel{+1}}{2}, \frac{1-2p+1}{2}\right)$

$= \beta\left(p, \frac{\cancel{2}(1-p)}{\cancel{2}}\right)$

$I = \frac{\Gamma p\,\Gamma 1-p}{\Gamma 1} = \Gamma p\,\Gamma 1-p$

But Given $I = \frac{\pi}{\sin p\pi}$ $\therefore$ Comparing $\Gamma p\,\Gamma 1-p = \frac{\pi}{\sin p\pi}$

5) Prove that $\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4}\beta\left(\frac{n}{2}, \frac{n}{2}\right)$ & Hence evaluate $\int_0^\infty \operatorname{sech}^8 x\,dx$

Sol^n: Consider $I = \int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(e^x+e^{-x})^n}$ (even f^n)

$f(-x) = f(x)$ even

Sol. $\int_0^{\infty} \frac{dx}{(e^x+e^{-x})^n} \rightarrow 2\int_{-\infty}^{\infty} \frac{dx}{(e^x+e^{-x})^n} \rightarrow$ (even fn)

$e^x = \tan\theta$

x	$-\infty$	∞
θ	0	$\pi/2$

$e^x\,dx = \sec^2\theta\, d\theta$

$dx = \frac{\sec^2\theta}{\tan\theta}\, d\theta$

$$I = \frac{1}{2}\int_0^{\pi/2} \frac{1}{(\tan\theta+\cot\theta)^n}\,\frac{\sec^2\theta}{\tan\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)^n}\ \boxed{\frac{\frac{1}{\cos^2\theta}\,d\theta}{\frac{\sin\theta}{\cancel{\cos\theta}}}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\cos^n\theta\,\sin^n\theta}{\cos\theta\,\sin\theta}\, d\theta$$

$$I = \frac{1}{2}\int_0^{\pi/2} \sin^{n-1}\theta\,\cos^{n-1}\theta\, d\theta$$

$$= \frac{1}{2}\,\frac{1}{2}\,\beta\left(\frac{n}{2},\frac{n}{2}\right) = \frac{1}{4}\beta\left(\frac{n}{2},\frac{n}{2}\right)$$

Consider $e^x+e^{-x} = 2\cosh x$

& put $n = 8$ in integral

$$I = \int_0^{\infty} \frac{dx}{(2\cosh x)^8} = \frac{1}{4}\beta\left(\frac{8}{2},\frac{8}{2}\right)$$

$$I = \frac{1}{2^8}\int_0^{\infty} \operatorname{sech}^8 x\, dx = \frac{1}{4}\,\frac{\Gamma 4\,\Gamma 4}{\Gamma 8}$$

$$\therefore \int_0^{\infty} \operatorname{sech}^8 x\, dx = \frac{\cancel{2^8}^{\,16}}{\cancel{4}}\,\frac{3!\,3!}{7\times\cancel{6}\times 5\times\cancel{4}\times\cancel{3!}} = \frac{16}{35}$$

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6) Prove that $\displaystyle\int_0^{\pi} \frac{\sqrt{\sin x}}{(5+3\cos x)^{3/2}}\, dx = \frac{\left(\Gamma\frac{3}{4}\right)^2}{2\sqrt{2\pi}}$

Sol^n: Here we put $t = \tan\frac{x}{2}$

x	0	π
t	0	∞

Then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2\,dt}{1+t^2}$

$$I = \int_0^{\infty} \frac{\sqrt{2t/\cancel{1+t^2}}\left[\frac{2\,dt}{\cancel{1+t^2}}\right]}{\left[5+3\left(\frac{1-t^2}{\cancel{1+t^2}}\right)\right]^{3/2}}$$

$$= \int_0^{\infty} \frac{2\sqrt{2}\,\sqrt{t}\, dt}{\left[5(1+t^2)+3(1-t^2)\right]^{3/2}}$$

$$I = \int_0^{\infty} \frac{\cancel{2\sqrt{2}}\,\sqrt{t}\, dt}{\cancel{2^{3/2}}\,(4+t^2)^{3/2}} = \int_0^{\infty} \frac{t^{1/2}}{(4+t^2)^{3/2}}\, dt$$

$t = 2\tan\theta$

	0	∞
	0	$\pi/2$

$dt = 2\sec^2\theta\, d\theta$

$$I = \int_0^{\pi/2} \frac{(2\tan\theta)^{1/2}\,(2\sec^2\theta\, d\theta)}{(4+4\tan^2\theta)^{3/2}}$$

$$= \frac{2^{3/2}}{2^3}\int_0^{\pi/2} \frac{\sin^{1/2}\theta}{\cos^{1/2}\theta}\,\cos\theta\, d\theta$$

$$\frac{1}{\ }\int_0^{\pi/2} \sin^{1/2}\theta\,\cos^{1/2}\theta\, d\theta$$

$$= \int_0 \left[5(1+t^2) + 3(1-t^2)\right]^{1/2}$$

$$= \int_0^\infty \frac{2\sqrt{2}\,\sqrt{t}\;dt}{\left[8+2t^2\right]^{3/2}}$$

$$= \frac{1}{2^{3/2}} \int_0^{\pi/2} \sin^{1/2}\theta \cos^{1/2}\theta \, d\theta$$

$$= \frac{1}{2\sqrt{2}} \frac{1}{2} \beta\left(\frac{3}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{4\sqrt{2}} \frac{\Gamma\frac{3}{4}\,\Gamma\frac{3}{4}}{\Gamma\frac{6}{4}} = \frac{\left(\Gamma\frac{3}{4}\right)^2}{\overset{2}{\cancel{4}}\sqrt{2}\,\frac{1}{\cancel{2}}\Gamma\frac{1}{2}}$$

$$= \frac{\left(\Gamma\frac{3}{4}\right)^2}{2\sqrt{2}\sqrt{\pi}}$$