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Experiment / assignment / tutorial No. ______
Grade: AA/ AB / BB / BC / CC / CD / DD

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Q1) $I = \int_0^{a\sqrt{3}} \int_0^{\sqrt{x^2+a^2}} \frac{x}{y^2+(x^2+a^2)} \, dy \, dx$

Integrating with x as constant

$I = \int_0^{a\sqrt{3}} x \left[\frac{1}{\sqrt{x^2+a^2}} \tan^{-1}\left(\frac{y}{\sqrt{x^2+a^2}}\right)\right]_{y=0}^{y=\sqrt{x^2+a^2}} dx$

$= \int_0^{a\sqrt{3}} \frac{x}{\sqrt{x^2+a^2}} \left(\tan^{-1}(1) - \tan^{-1}(0)\right) dx$

$= \int_0^{a\sqrt{3}} \frac{\pi}{4} \times \frac{x}{\sqrt{x^2+a^2}} \, dx$

$x^2+a^2 = u$

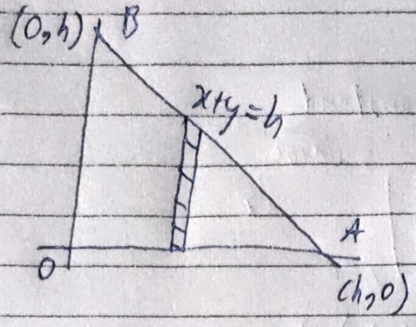
$2x\,dx = du$

$= \int_{a^2}^{4a^2} \frac{\pi}{8} \frac{1}{\sqrt{u}}$

$= \frac{\pi}{4}\left[\sqrt{x^2+a^2}\right]_0^{a\sqrt{3}}$

$= \frac{\pi}{4} a$

Q2) $\iint x^{m-1} y^{n-1} \, dx\, dy$ over $x+y=h, x=0, y=0$



y varies from 0 to h
strip moves $x=0$ to $x=h$

$$I = \int_0^h \int_0^{h-x} x^{m-1} y^{n-1} \, dy \, dx$$

$$= \int_0^h x^{m-1} \left[\frac{y^n}{n} \right]_0^{h-x} dx$$

$$= \int_0^h x^{m-1} \frac{(h-x)^n}{n} \, dx$$

$x = ht$

$$\cancel{= \int_0^1 x^{m-1} \frac{t^n}{n}}$$

$$= \int_0^1 \frac{h^{m-1}}{n} \cdot t^{m-1} \times h^n (1-t)^n \cdot h\,dt$$

$$= \frac{h^{m+n}}{n} \int_0^1 t^{m-1} (1-t)^n \, dt$$

$$= \frac{h^{m+n} \Gamma m \, \Gamma n}{(m+n) \Gamma m+n}$$

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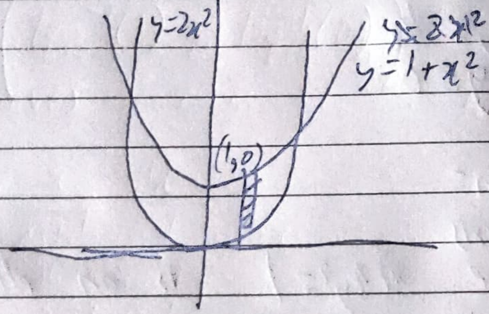
Q3) $\iint (x+2y)\,dA$ over $y=2x^2$ & $y=1+x^2$

$y=1+x^2$ is a parabola which opens upwords with vertex at $(0,1)$.
$y=2x^2$ is standerd upward parabola.
They both intersect at $x=\pm 1$



$$\iint_R (x+2y)\,dy\,dx = 2\int_0^1\int_{2x^2}^{x^2+1} (x+2y)\,dy\,dx$$

$$=2\int_0^1 \left[xy+y^2\right]_{2x^2}^{x^2+1} dx$$

$$=2\int_0^1 \left[\left[x(x^2+1)+(x^2+1)^2\right]-(2x^3+4x^4)\right]dx$$

$$\iint (x+2y)\,dA = \int_{-1}^{1} (-3x^4-x^3+2x^2+x+1)\,dx$$

$$=\left[\frac{-3x^5}{5}-\frac{x^4}{4}+\frac{2x^3}{3}+\frac{x^2}{2}+x\right]_{-1}^{1}$$

$$=\left(\frac{-3}{5}-\frac{1}{4}+\frac{2}{3}+\frac{1}{2}+1\right)-\left(\frac{3}{5}+\frac{1}{4}-\frac{2}{3}+\frac{1}{2}-1\right)$$

$$=\frac{32}{15}$$

Q4) $\displaystyle\int_0^{1/2}\int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}$

Given limits

Internal limits :

$x = \sqrt{1-4y^2}$

$x = 0$

External limits

$y = 1/2$

$y = 0$

Region of integration -

$x = 0$ is y axis

$x = \sqrt{1-4y^2}$ i.e $x^2 + 4y^2 = 1$ is an ellipse with centre at origin

with major axis and minor axis 1 & $1/2$.

Thus ~~are~~ region of integration is area of ellipse in 1st quadrant

Change order of integration:

Consider a strip parallel to y axis on strip

Internal limits :

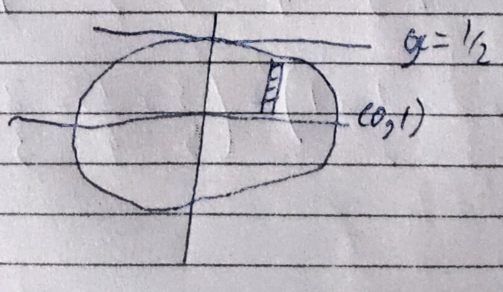
$y = \dfrac{\sqrt{1-x^2}}{2}$

$y = 0$

External Limits :

$x = 1$

$x = 0$



$$I = \int_0^1 \int_0^{\frac{\sqrt{1-x^2}}{2}} \frac{1+x^2}{\sqrt{1-x^2}} \times \frac{dy}{\sqrt{(1-x^2)-y^2}}\, dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{y}{\sqrt{1-x^2}}\right)\right]_0^{\frac{\sqrt{1-x^2}}{2}} dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0\right] dx$$

$$= \frac{\pi}{6} \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}$$

put $x = \sin\theta$ $\quad dx = \cos\theta\, d\theta$

$$I = \frac{\pi}{6} \int_0^{\pi/2} \frac{1+\sin^2\theta}{\cos\theta} \times \cos\theta\, d\theta$$

$$= \frac{\pi}{6} \int_0^{\pi/2} (1+\sin^2\theta)\, d\theta$$

$$= \frac{\pi}{6} \left[(\theta)_0^{\pi/2} + \frac{1}{2} \times \frac{\pi}{4} \right]$$

$$= \frac{\pi}{6} \left[\frac{3\pi}{4}\right]$$

$$= \frac{\pi^2}{8}$$

Q5) $$\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy$$

$x = r\cos\theta \qquad y = r\sin\theta \qquad dx\,dy = r\,dr\,d\theta$

$$\int_0^{\pi/4} \int_0^{a/\cos\theta} \frac{r^2\cos^2\theta}{r} \times r\,dr\, d\theta$$

$$= \int_0^{\pi/4} \left[\frac{r^3}{3}\right]_0^{a/\cos\theta} \cos^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \left[\frac{a^3}{3} \times \frac{1}{\cos^3\theta} \times \cos^2\theta \, d\theta\right]$$

$$= \int_0^{\pi/4} \frac{a^3}{3} \times \sec\theta \, d\theta$$

$$= \frac{a^3}{3}\left[\log(\sec\theta + \tan\theta)\right]_0^{\pi/4}$$

$$= \frac{a^3}{3}\left[\log(\sqrt{2}+1) - \log 1\right]$$

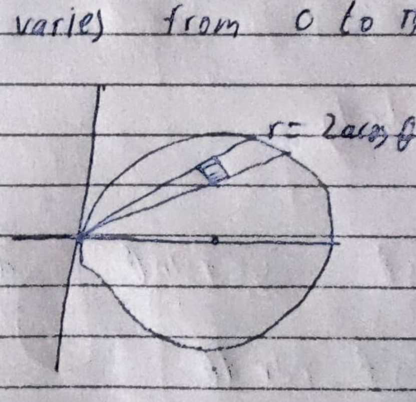
$$= \frac{a^3}{3} \log(1+\sqrt{2})$$

Q6) $\iint r e^{-r^2/a^2} \cos\theta \sin\theta \, d\theta \, dr$ over upper half of circle $r = 2a\cos\theta$

$r = 2a\cos\theta$ i.e $r^2 = 2ar\cos\theta$ i.e $x^2+y^2 = 2ax$

i.e $(x-a)^2 + y^2 = a^2$ has centre at $(a, 0)$

Consider a radical strip on this strip, r varies from 0 to $2a\cos\theta$ & θ varies from 0 to $\pi/2$



$$I = \int_0^{\pi/2}\int_0^{2a\cos\theta} r e^{-r^2/a^2} \sin\theta\cos\theta \, d\theta \, dr$$

$$= \int_0^{\pi/2} \left[\frac{-a^2}{2} e^{-r^2/a^2}\right]_0^{2a\cos\theta} \sin\theta\cos\theta \, d\theta$$

$$= \frac{-a^2}{2}\left[\frac{1}{8} e^{-4\cos^2\theta} - \frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$$

$$= \frac{-a^2}{2}\left[\frac{1}{8} - \frac{1}{2} - \frac{1}{8}e^{-4}\right]$$

$$= \frac{a^2(3+e^{-4})}{16}$$

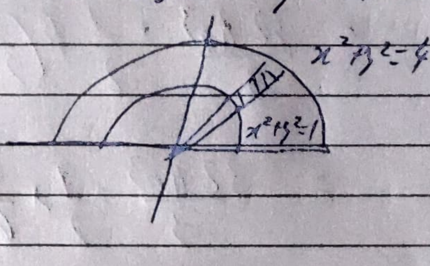
Q7) $\iint_R (3x+4y^2)\, dy\, dx$ bounded by $x^2+y^2=1$, $x^2+y^2=4$

Region of integration- between two semicircles

Region in polar coordinates- $x = r\cos\theta$ $y = r\sin\theta$

$r = 1$ to $r = 2$

$\theta = 0$ to $\theta = \pi$



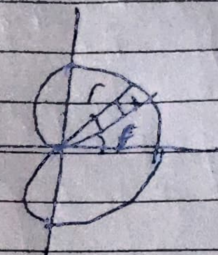
$$I = \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$I = \int_0^{\pi} \left[r^3\cos\theta + r^4\sin^2\theta\right]_1^2 d\theta$$

$$= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\, d\theta = \int_0^{\pi}\left[7\cos\theta + \frac{15}{2}(1-\cos 2\theta)\right] d\theta$$

$$= \left[7\sin\theta + \frac{15}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{\pi} = \frac{15\pi}{2}$$

98) The density at any point $P(r, \theta)$ is $\rho = kr$



Consider radial strip in first quadrant. On this strip r varies from $r=0$ to $r = a(1+\cos\theta)$ & then θ varies from $\theta = 0$ to $\theta = \pi$

$$\text{Mass} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} (kr)\, r\, dr\, d\theta$$

$$= \frac{2k}{3}\int_0^{\pi} a^3 (1+\cos\theta)^3 d\theta$$

$$= \frac{16}{3} ka^3 \int_0^{\pi} \cos^6\frac{\theta}{2}\, d\theta$$

$\theta/2 = t$

$$= \frac{16}{3} ka^3 \int_0^{\pi/2} \cos^6 t \cdot 2dt$$

$$= \frac{32ka^3}{3} \cdot \frac{1}{2} \beta\left(\frac{1}{2}, \frac{7}{2}\right)$$

$$= \frac{32ka^3}{3} \times \frac{1}{2} \times \frac{5}{6} \times \frac{3}{4} \times \frac{\pi}{2}$$

$$= \frac{5}{3} ka^3 \pi$$

Q9) $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$

$I = \int_0^{\log 2}\int_0^{x} e^{x+y}\left[e^{z}\right]_0^{x+y} dy\, dx$

$= \int_0^{\log 2}\int_0^{x} \left(e^{2(x+y)} - e^{(x+y)}\right) dy\, dx$

$= \int_0^{\log 2} \left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x}\right] dx$

$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^{x}\right]_0^{\log 2}$

$= \left(\frac{16}{8} - \frac{4}{2} - \frac{4}{4} + 2\right) - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1\right)$

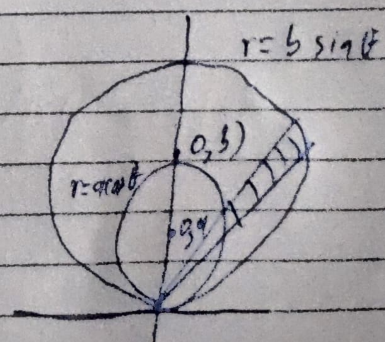
$= \frac{5}{8}$

Q10) $r = 2a\sin\theta$, $r = 2b\sin\theta$ $\quad b > a$

$\sqrt{x^2+y^2} = 2a\, \frac{y}{\sqrt{x^2+y^2}}$

i.e. $x^2+y^2 = 2ay$

$x^2 + (y-a)^2 = a^2$



similarly $x + (y-b)^2 = b^2$

r varies from $r = 2a\sin\theta$ to $2b\sin\theta$

θ varies from $\theta = 0$ to $\theta = \pi/2$

$$A = 2\int_0^{\pi/2}\int_{2a\sin\theta}^{2b\sin\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{2a\sin\theta}^{2b\sin\theta} d\theta$$

$$= 4\int_0^{\pi/2}(b^2\sin^2\theta - a^2\sin^2\theta)\,d\theta$$

$$= 4(b^2-a^2)\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= 4(b^2-a^2)\times\frac{1}{2}\times\frac{\pi}{2}$$

$$= (b^2-a^2)\pi$$