SOMAIYA VIDYAVIHAR UNIVERSITY

K J Somaiya College of Engineering

Somaiya TRUST

PARTIAL DIFFERENTIATION

FYBTECH SEM-I

MODULE-4

PARTIAL DIFFERENTIATION OF AN IMPLICIT FUNCTION

- If $f(x,y)=0$, be an implicit relation between x and y which defines y as a differentiable function of x, then $\frac{dy}{dx}=-\frac{f_x}{f_y}$
- **(i)** The result $\frac{dy}{dx}=-\frac{f_x}{f_y}$ i.e. $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$ is sometimes referred to as a rule of term by term differentiation of an implicit function.
- It states that, for an implicit function differentiate each term as usual and multiply the derivative of a term by dy/dx if you are differentiating a function of y.
- **(ii)** If $f(x,y,z)=0$ then z is an implicit function of x,y and if we want to find partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ we differentiate each term separately treating one variable constant.
- If we differentiate z partially w.r.t. x treating y constant we multiply the result by $\partial z/\partial x$.
- Similar is the case of partial differentiation w.r.t y

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EXAMPLE-19

- If $(\cos x)^y = (\sin y)^x, find\ \frac{dy}{dx}$
- **Solution:** Taking logarithms of both sides
- $y \log \cos x = x \log \sin y$
- Differentiating term by term,
- $\log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot (-\sin x) = \log \sin y + x \cdot \frac{1}{\sin y} \cdot \cos y \frac{dy}{dx}$
- $\therefore \frac{dy}{dx} = \frac{\log \sin y + y \tan x}{\log \cos x - x \cot x}$
- Alternatively
- Taking logarithms of both sides
- $f(x, y) \equiv y \log \cos x - x \log \sin y = 0$
- $\therefore \frac{\partial f}{\partial x} = -y \cdot \frac{1}{\cos x} \cdot \sin x - \log \sin y$
- and $\frac{\partial f}{\partial y} = \log \cos x - x \cdot \frac{1}{\sin y} \cdot \cos y$
- $\therefore \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{y \tan x + \log \sin y}{\log \cos x - x \cot y}$

EXAMPLE-20

- If $u = x \log xy \; and \; x^3 + y^3 + 3axy = 0, \; find \; \frac{du}{dx}$
- **Solution:** We have $\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$
- $= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$ .... (i)
- Now, $\frac{\partial u}{\partial x} = x \cdot \left(\frac{1}{xy} \cdot y\right) + \log xy = 1 + \log xy$
- and $\frac{\partial u}{\partial y} = x \cdot \frac{1}{xy} \cdot x = \frac{x}{y}$
- Now if $f(x, y) = x^3 + y^3 - 3axy = 0$
- $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2+3ay}{3y^2+3ax} = -\frac{(x^2+ay)}{(y^2+ax)}$
- $\therefore$ From (i), we get,
- $\frac{du}{dx} = 1 + \log xy - \frac{x}{y} \cdot \left(\frac{x^2+ay}{y^2+ax}\right)$

EXAMPLE-21

- If $z^3 - xz - y = 0$, prove that $\frac{\partial^2 z}{\partial x \partial y} = -\frac{3z^2+x}{(3z^2-x)^3}$
- **Solution:** We treat x, y as independent variables, z as a dependent variable and z as an implicit function of x, y
- Differentiating the given relation partially w.r.t. x, we get,
- $3z^2 \frac{\partial z}{\partial x} - z - x \frac{\partial z}{\partial x} - 0 = 0$ .... (i)
- $\therefore \frac{\partial z}{\partial x} = \frac{z}{3z^2 - x}$
- Differentiating the given relation partially w.r.t. y, we get,
- $3z^2 \frac{\partial z}{\partial y} - x \frac{\partial z}{\partial y} - 1 = 0$
- $\therefore \frac{\partial z}{\partial y} = \frac{1}{3z^2 - x}$ .... (ii)
- Differentiating (ii) w.r.t. x, we get,
- $\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{(3z^2-x)^2}\left[6z \cdot \frac{\partial z}{\partial x} - 1\right]$
- $= -\frac{1}{(3z^2-x)^2}\left[6z \cdot \frac{z}{(3z^2-x)} - 1\right]$
- $= -\frac{1}{(3z^2-x)^2} \cdot \frac{(3z^2+x)}{(3z^2-x)} = -\frac{3z^2+x}{(3z^2-x)^3}$

EXAMPLE-22

- If $x^x y^y z^z = C$, and $x = y = z$, prove that **(i)** $\frac{\partial^2 z}{\partial x \partial y} = -[x \log ex]^{-1}$
- **(ii)** $\frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \frac{2(x^2-2)}{x(1+\log\ x)}$
- **Solution:** **(i)** Taking logarithms of both sides
- $x \log x + y \log y + z \log z = \log C$
- Differentiating this implicit function partially w.r.t. x treating y constant
- $x \cdot \frac{1}{x} + \log x + \left(z \cdot \frac{1}{z} + \log z\right) \frac{\partial z}{\partial x} = 0$
- $\therefore (1 + \log x) + (1 + \log z) \frac{\partial z}{\partial x} = 0$ .... (1)
- $\therefore \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}$
- Similarly, $\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)}$
- Differentiating (1) partially w.r.t. y treating x constant,
- $(1 + \log z) \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} \left[\frac{1}{z} \cdot \frac{\partial z}{\partial y}\right] = 0$
- $\therefore (1 + \log z) \frac{\partial^2 z}{\partial x \partial y} + \left[-\frac{1+\log x}{1+\log z}\right] \cdot \frac{1}{z} \cdot \left[-\frac{1+\log y}{1+\log z}\right] = 0$
- Putting $x = y = z$,
- $(1 + \log x) \frac{\partial^2 z}{\partial x \partial y} + \frac{1}{x} = 0$ $\quad \therefore \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1+\log x)}$ .... (2)
- $\therefore \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(\log e+\log x)} = -\frac{1}{x \log ex} = -(x \log ex)^{-1}$

EXAMPLE-22

- **(ii)** Differentiating the result (1) of the above example w.r.t. x, partially
- $\frac{1}{x}+\frac{1}{z}\left(\frac{\partial z}{\partial x}\right)^2+(1+\log z)\frac{\partial^2 z}{\partial x^2}=0$
- From (1) when $x=y=z, \frac{\partial z}{\partial x}=-1$ and then from above
- $\therefore \frac{1}{x}+\frac{1}{x}+(1+\log x)\frac{\partial^2 z}{\partial x^2}=0$
- $\therefore \frac{\partial^2 z}{\partial x^2}=-\frac{2}{x(1+\log x)}$ .... (3)
- By symmetry at $x=y=z$,
- $\frac{\partial^2 z}{\partial y^2}=-\frac{2}{x(1+\log x)}$ .... (4)
- From (2), $\frac{\partial^2 z}{\partial x \partial y}=-\frac{1}{x(1+\log x)}$ .... (5)
- Hence at $x=y=z$,
- LHS $=-\frac{2}{x(1+\log x)}+\frac{2x^2}{x(1+\log x)}-\frac{2}{x(1+\log x)}=\frac{2(x^2-2)}{x(1+\log\ x)}=$ RHS

EXAMPLE-23

- If $a^2x^2 + b^2y^2 = c^2z^2$, prove that $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2 z}$
- **Solution:** Differentiating the given function partially w.r.t. x, we get,
- $a^2 \cdot 2x = c^2 \cdot 2z \cdot \frac{\partial z}{\partial x}$ $\quad \therefore \frac{\partial z}{\partial x} = \frac{a^2 x}{c^2 z}$
- $\therefore \frac{1}{a^2}\frac{\partial z}{\partial x} = \frac{1}{c^2}\left(\frac{x}{z}\right)$
- Differentiating this again, partially w.r.t. x
- $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\left[\frac{z \cdot 1 - x(\partial z/\partial x)}{z^2}\right] = \frac{1}{c^2}\left[\frac{z - x\left(a^2 x/c^2 z\right)}{z^2}\right] = \frac{1}{c^2}\left[\frac{c^2z^2 - a^2x^2}{c^2z^3}\right]$ .... (i)
- Differentiating the given function partially w.r.t. y, now,
- $b^2 y = c^2 z \frac{\partial z}{\partial y}$ $\quad \therefore \frac{\partial z}{\partial y} = \frac{b^2 y}{c^2 z}$
- $\therefore \frac{1}{b^2} \cdot \frac{\partial z}{\partial y} = \frac{1}{c^2}\left(\frac{y}{z}\right)$
- By symmetry, $\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\left[\frac{c^2z^2 - b^2y^2}{c^2z^3}\right]$ .... (ii)
- Adding (i) and (ii), we get,
- $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{2c^2z^2 - \left(a^2x^2 + b^2y^2\right)}{c^4z^3}$
- $= \frac{2c^2z^2 - c^2z^2}{c^4z^3} = \frac{c^2z^2}{c^4z^3}$ [By data]
- $= \frac{1}{c^2 z}$

EXAMPLE-24

- If $\Phi\left(\frac{z}{x^3}, \frac{y}{x}\right) = 0$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$
- **Solution:** Let $u = \frac{z}{x^3}, v = \frac{y}{x}$ $\quad \therefore \Phi(u, v) = 0$
- $\therefore \frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \Phi}{\partial v} \cdot \frac{\partial v}{\partial x} = 0$ .... (i)
- and $\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \Phi}{\partial v} \cdot \frac{\partial v}{\partial y} = 0$ .... (ii)
- Now $u = \frac{z}{x^3}$
- $\therefore \frac{\partial u}{\partial x} = \frac{1}{x^3}\frac{\partial z}{\partial x} - \frac{3}{x^4} \cdot z;$ $\quad \frac{\partial u}{\partial y} = \frac{1}{x^3}\frac{\partial z}{\partial y}$
- $v = \frac{y}{x}$
- $\therefore \frac{\partial v}{\partial x} = -\frac{y}{x^2},$ $\quad \frac{\partial v}{\partial y} = \frac{1}{x}$

EXAMPLE-24

- Hence, from (i) we get,
- $\frac{\partial \Phi}{\partial u} \cdot \left(\frac{1}{x^3}\frac{\partial z}{\partial x} - \frac{3z}{x^4}\right) + \frac{\partial \Phi}{\partial v}\left(-\frac{y}{x^2}\right) = 0$
- Multiplying by x^4, we get,
- $\therefore \frac{\partial \Phi}{\partial u}\left(x\frac{\partial z}{\partial x} - 3z\right) + \frac{\partial \Phi}{\partial v}(-x^2y) = 0$ ………………… (iii)
- And from (ii), we get,
- $\frac{\partial \Phi}{\partial u}\left(\frac{1}{x^3}\frac{\partial z}{\partial y}\right) + \frac{\partial \Phi}{\partial v}\left(\frac{1}{x}\right) = 0$
- Multiplying by x^3y
- $\frac{\partial \Phi}{\partial u}\left(y\frac{\partial z}{\partial y}\right) + \frac{\partial \Phi}{\partial v}(x^2y) = 0$ ………………… (iv)
- From (iii) and (iv), we get,
- $\frac{\partial \Phi/\partial u}{\partial \Phi/\partial v} = \frac{x^2y}{x(\partial z/\partial x)-3z}$
- and $\frac{\partial \Phi/\partial u}{\partial \Phi/\partial v} = -\frac{x^2y}{y(\partial z/\partial y)}$
- Equating the two $\quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$