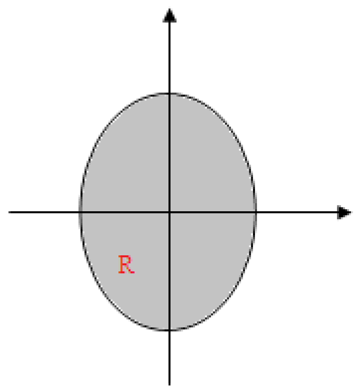
Change to Polar coordinates and evaluate

Imagine that you had to compute the double integral $\iint_R (x^2 + y^2)\mathrm{d}A$, where R is the disk of radius 6 centered at the origin.



In terms of Cartesian coordinates x and y, the disk is given by

$$-6 \leq x \leq 6 \text{ and } -\sqrt{36 - x^2} \leq y \leq \sqrt{36 - x^2}$$

Therefore, in Cartesian coordinates $\iint_R (x^2 + y^2)\mathrm{d}A$ takes the form

$$\iint_R (x^2 + y^2)\mathrm{d}A = \int_{-6}^{6} \int_{-\sqrt{36-x^2}}^{\sqrt{36-x^2}} (X^2 + y^2)\mathrm{d}y \ \mathrm{d}x$$

Here, this integral would be lot easier if we could change variables to polar coordinates. In polar coordinates, the disk is the region and is defined by $0 \leq r \leq 6$ and $0 \leq \theta \leq 2\pi$. Hence region of integration is simpler to describe using polar coordinates. Moreover, the integrand $x^2 + y^2$ in polar coordinates takes the form r^2. Thus we have to integrate r^2 over the region defined by $0 \leq r \leq 6$ and $0 \leq \theta \leq 2\pi$. Now, $\mathrm{d}A$ stands for the area of a little bit region of R. In Cartesian coordinate we replaced$\mathrm{d}A$ by either $\mathrm{d}x \ \mathrm{d}y$ or $\mathrm{d}y \ \mathrm{d}x$. In polar coordinates, $\mathrm{d}A$ does not becomes $\mathrm{d}r \ \mathrm{d}\theta$, therefore we need to determine what $\mathrm{d}A$ becomes in polar coordinates.
We know that,

$$\mathrm{d}x\mathrm{d}y = |J|\mathrm{d}r \ \mathrm{d}\theta$$

where, J is the Jacobian transformation obtained from the relation $x = r\cos\theta$ and $y = r\sin\theta$. Thus,

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Thus, $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r \ \mathrm{d}\theta$.

Now, we consider the procedure to change given Cartesian coordinate integral into polar.

step-a): First sketch region of integral using limits of given integral

step-b): Convert all boundaries i.e. limits of integration into the polar coordinates using the relations $x = r\cos\theta$ and $y = r\sin\theta$, wherever necessary

step-c): Convert integrating function in terms of r and θ using above relation.

step-d): Consider the integrating strip starting from origin and assign the limits in polar coordinates (r, θ)

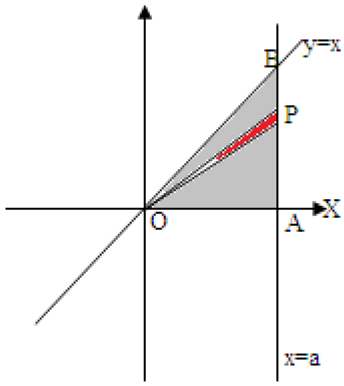
step-e): Finally replace dxdy or dydx by rdrdθ and then integrate w.r.t r first and then w.r.t. θ.

Example 1. Change the integral $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} \mathrm{d}x\, \mathrm{d}y$ into polar coordinates and hence evaluate.

Solution: Consider,

$$I = \int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} \mathrm{d}x\, \mathrm{d}y$$

Now, plot the curves $x = y$, $x = a$, $y = 0$ and $y = a$ and consider the region of integration (shown shaded) as shown in following figure.



Now, the line $x = y$ converted into $\theta = \pi/4$ in polar coordinates and $x = a$ takes the form $r\cos\theta = a$ i.e. $r = a\sec\theta$. Also, $\frac{x^2}{\sqrt{x^2+y^2}} = \frac{r^2\cos^2\theta}{\sqrt{r^2}} = r\cos^2\theta$ and $\mathrm{d}x\,\mathrm{d}y = r\mathrm{d}r\,\mathrm{d}\theta$.

To change into polar coordinates consider a integrating strip from origin as shown in above figure. The point O lies on origin i.e. $r = 0$ and P lies on $x = a$ i.e. $r = a\sec\theta$. To cover the shaded region we have to rotate integrating strip from $\theta = 0$ to $\theta = \pi/4$. Therefore, θ varies from 0 to $\pi/4$. Therefore, we get

$$I = \int_0^{\pi/4} \int_0^{a\sec\theta} r\cos^2\theta r\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi/4} \int_0^{a\sec\theta} r^2\cos^2\theta \mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi/4} \cos^2\theta \left[\frac{r^3}{3}\right]_0^{a\sec\theta} \mathrm{d}\theta$$

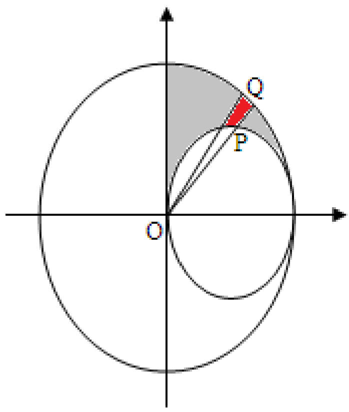
$$= \frac{1}{3}\int_0^{\pi/4} \cos^2\theta[a^3\sec^3\theta]\mathrm{d}\theta = \frac{a^3}{3}\int_0^{\pi/4} \sec\theta\mathrm{d}\theta = \frac{a^3}{3}[\log(\sec\theta + \tan\theta)]_0^{\pi/4}$$

$$= \frac{a^3}{3}\log(1+\sqrt{2})$$

Example 2. Evaluate $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{\mathrm{d}x\mathrm{d}y}{\sqrt{a^2-x^2-y^2}}$ by changing to polar coordinates.
Solution: Consider,

$$I = \int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{\mathrm{d}x\mathrm{d}y}{\sqrt{a^2-x^2-y^2}}$$

Consider the region of integration bounded by $y = \sqrt{ax-x^2}$ i.e. $(x-a/2)^2 + y^2 = a^2/4$ (circle with center at $(a/2, 0)$ and radius $a/2$ and $y = \sqrt{a^2-x^2}$ i.e. $x^2+y^2 = a^2$ (circle with center at $(0,0)$ and radius a) as shown in the following figure.



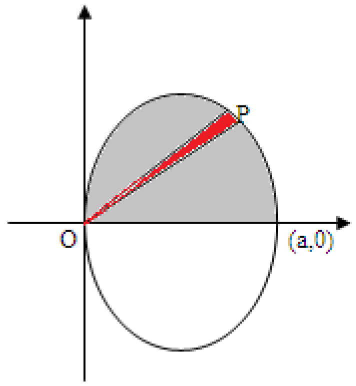
In polar coordinate system $y = \sqrt{ax-x^2}$ takes the form $r = a\cos\theta$ and $y = \sqrt{a^2-x^2}$ reduces to $r = a$. Also, $\frac{1}{\sqrt{a^2-x^2-y^2}} = \frac{1}{a^2-r^2}$ and we have $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$.
Now to change to polar coordinates consider an integrating strip starting from origin as shown in above figure. The point P lies on $r = a\cos\theta$ and Q lies on $r = a$. Therefore, r varies from $a\cos\theta$ to a. To cover the shaded region, we have to slide strip from x-axis to y-axis i.e. from $\theta = 0$ to $\theta = \pi/2$. Thus θ varies from 0 to $\pi/2$. Therefore,

$$\begin{aligned}
I &= \int_0^{\pi/2} \int_{a\cos\theta}^{a} \frac{1}{\sqrt{a^2-r^2}} r\mathrm{d}r\mathrm{d}\theta = -\frac{1}{2}\int_0^{\pi/2} \int_{a\cos\theta}^{a} \frac{-2r}{\sqrt{a^2-r^2}}\mathrm{d}r\mathrm{d}\theta \\
&= -\frac{1}{2}\int_0^{\pi/2} [2\sqrt{a^2-r^2}]_{a\cos\theta}^{a}\mathrm{d}\theta \qquad \left(\text{using } \int \frac{f'(x)}{\sqrt{f(x)}}\mathrm{d}x = 2\sqrt{f(x)}\right) \\
&= -\int_0^{\pi/2} [0 - \sqrt{a^2 - a^2\cos^2\theta}]\mathrm{d}r = -\int_0^{\pi/2} a\sin\theta\mathrm{d}\theta \\
&= a[-\cos\theta]_0^{\pi/2} = a
\end{aligned}$$

Example 3. Evaluate $\int_0^1 \int_0^{\sqrt{x-x^2}} \frac{4xy}{x^2+y^2} e^{-(x^2+y^2)}\mathrm{d}x\ \mathrm{d}y$ by changing to polar coordinates.
Solution: Consider,

$$I = \int_0^1 \int_0^{\sqrt{x-x^2}} \frac{4xy}{x^2+y^2} e^{-(x^2+y^2)}\mathrm{d}x\ \mathrm{d}y$$

Consider the region of integration bounded by $y = 0$, $y = \sqrt{x - x^2}$ i.e. $(x - a/2)^2 + y^2 = a^2/4$ (circle with center at $(a/2, 0)$ and radius $a/2$) as shown in the following figure.



In polar coordinate system $y = \sqrt{x - x^2}$ takes the form $r = \cos\theta$ and $\dfrac{4xye^{-(x^2+y^2)}}{x^2 + y^2} = 4\sin\theta\cos\theta\ e^{-r^2}$ and we have $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$.

Now to change to polar coordinates consider an integrating strip starting from origin as shown in above figure. The point O lies on origin i.e. $r = 0$ and Q lies on $r = \cos\theta$. Therefore, r varies from 0 to $\cos\theta$. To cover the shaded region, we have to slide strip from x-axis to y-axis i.e. from $\theta = 0$ to $\theta = \pi/2$. Thus θ varies from 0 to $\pi/2$. Therefore,

$$I = \int_0^{\pi/2}\int_0^{\cos\theta} 4\cos\theta\ \sin\theta\ e^{-r^2}\ r\mathrm{d}r\mathrm{d}\theta = \int_0^{\pi/2} 4\cos\theta\sin\theta\int_0^{\cos\theta} e^{-r^2} r\mathrm{d}r\mathrm{d}\theta$$

Put $r^2 = t$. This gives $r\mathrm{d}r = \dfrac{1}{2}\mathrm{d}t$ and when $r = 0$, we get $t = 0$ and when $r = \cos\theta$, we get $t = \cos^2\theta$. Thus above equation becomes,

$$\begin{aligned} I &= \frac{1}{2}\int_0^{\pi/2} 4\cos\theta\sin\theta\int_0^{\cos^2\theta} e^{-t}\mathrm{d}t\mathrm{d}\theta = 2\int_0^{\pi/2}\cos\theta\sin\theta\left[-e^{-t}\right]_0^{\cos^2\theta}\mathrm{d}\theta \\ &= 2\int_0^{\pi/2}\cos\theta\sin\theta\left[1 - e^{-\cos^2\theta}\right]\mathrm{d}\theta \end{aligned}$$

Now put $\cos^2\theta = t$, then $2\cos\theta\sin\theta\ \mathrm{d}\theta = -\mathrm{d}t$. When $\theta = 0$ we get $t = 1$ and when $\theta = \pi/2$ we get $t = 0$. Thus, above equation takes the form

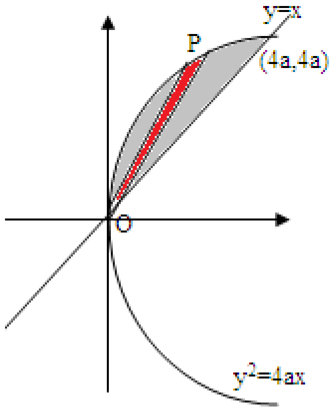
$$I = \int_1^0 (1 - e^{-t})(-\mathrm{d}t) = -[t + e^{-t}]_1^0 = -[1 - 1 - e^{-1}] = \frac{1}{e}$$

Example 4. Change to polar form and evaluate $\displaystyle\int_0^{4a}\int_{y^2/4a}^{y}\frac{x^2 - y^2}{x^2 + y^2}\mathrm{d}x\ \mathrm{d}y$

Solution: Consider,

$$\int_0^{4a}\int_{y^2/4a}^{y}\frac{x^2 - y^2}{x^2 + y^2}\mathrm{d}x\ \mathrm{d}y$$

Consider the region of integration bounded by $x = y^2/4a$ i.e. $y^2 = 4ax$, a line $x = y$ as shown in the following figure.



In polar coordinate system $y^2 = 4ax$ takes the form $r = \dfrac{4a\cos\theta}{\sin^2\theta} = r_1$ say and $\dfrac{x^2-y^2}{x^2+y^2} = \cos^2\theta - \sin^2\theta$ and we have $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$.

Now to change to polar coordinates consider an integrating strip starting from origin as shown in above figure. The point O lies on origin i.e. $r = 0$ and Q lies on $y^2 = 4ax$ i.e. $r = \dfrac{4a\cos\theta}{\sin^2\theta} = r_1$. Therefore, r varies from 0 to r_1. To cover the shaded region, we have to slide strip from the line $y = x$ to y-axis i.e. from $\theta = \pi/4$ to $\theta = \pi/2$. Thus θ varies from $\pi/4$ to $\pi/2$. Therefore,

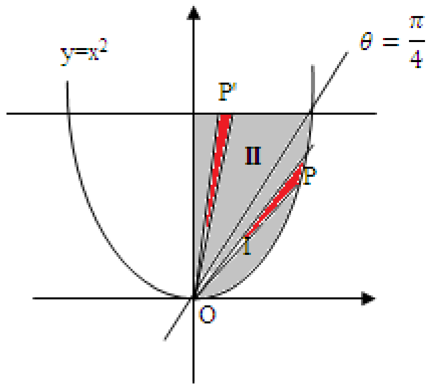
$$
\begin{aligned}
I &= \int_{\pi/4}^{\pi/2}\int_0^{r_1}(\cos^2\theta - \sin^2\theta)r\mathrm{d}r\,\mathrm{d}\theta = \int_{\pi/4}^{\pi/2}(\cos^2\theta-\sin^2\theta)\left[\frac{r^2}{2}\right]_0^{r_1}\mathrm{d}\theta \\
&= \frac{1}{2}\int_{\pi/4}^{\pi/2}(\cos^2\theta-\sin^2\theta)(r_1^2)\mathrm{d}\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2}(\cos^2\theta-\sin^2\theta)\left[\frac{16a^2\cos^2\theta}{\sin^2\theta}\right]\mathrm{d}\theta \\
&= 8a^2\int_{\pi/4}^{\pi/2}(\cot^4\theta - \cot^2\theta)\mathrm{d}\theta \\
&= 8a^2\int_{\pi/4}^{\pi/2}(\cot^2\theta(\operatorname{cosec}^2\theta - 1) - \cot^2\theta)\mathrm{d}\theta \\
&= 8a^2\int_{\pi/4}^{\pi/2}(\cot^2\theta\operatorname{cosec}^2\theta - 2\cot^2\theta)\mathrm{d}\theta \\
&= 8a^2\left\{\int_{\pi/4}^{\pi/2}(\cot^2\theta\operatorname{cosec}^2\theta)\mathrm{d}\theta - 2\int_{\pi/4}^{\pi/2}(\operatorname{cosec}^2\theta - 1)\mathrm{d}\theta\right\} \\
&= 8a^2\left\{\left[-\frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2} - 2\left[-\cot\theta - \theta\right]_{\pi/4}^{\pi/2}\right\} \\
&= 8a^2\left\{-\frac{1}{3}(0-1) - 2\left[0 - \frac{\pi}{2} + 1 + \frac{\pi}{4}\right]\right\} \\
&= 8a^2\left[\frac{1}{3} - 2 + \frac{\pi}{2}\right] = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)
\end{aligned}
$$

Example 5. Change to polar form and evaluate $\displaystyle\int_0^1\int_{x^2}^1 \frac{y}{x^2+y^2}\mathrm{d}x\,\mathrm{d}y$

Solution: Consider,

$$\int_0^1\int_{x^2}^1 \frac{y}{x^2+y^2}\mathrm{d}x\,\mathrm{d}y$$

Consider the region of integration bounded by parabola $y = x^2$, a line $y = 1$, $x = 1$ as shown in the following figure.



In polar coordinate system $y = x^2$ takes the form $r = \dfrac{\sin\theta}{\cos^2\theta}$ and $y = 1$ takes the form $r = \operatorname{cosec}\theta$. Also, $\dfrac{y}{x^2+y^2} = \dfrac{\sin\theta}{r}$ and we have $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$.
Here we have two regions as shown in fig. To change to polar coordinates consider an integrating strip in both the region starting from origin as shown in above figure. In the first region the point O lies on origin i.e. $r = 0$ and Q lies on $y = x^2$ i.e. $r = \dfrac{\sin\theta}{\cos^2\theta}$. Therefore, r varies from 0 to $\dfrac{\sin\theta}{\cos^2\theta}$. To cover the first shaded region, we have to slide strip from the initial to the line $y = x$ i.e. from $\theta = 0$ to $\theta = \pi/4$. Thus θ varies from 0 to $\pi/4$. Therefore,

$$I_1 = \int_0^{\pi/4}\int_0^{\sin\theta/\cos^2\theta} \frac{\sin\theta}{r} r\mathrm{d}r\ \mathrm{d}\theta = \int_0^{\pi/4}\int_0^{\sin\theta/\cos^2\theta} \sin\theta\mathrm{d}r\ \mathrm{d}\theta \tag{1}$$

In second region strip starts from origin i.e. $r = 0$ and the point P' lies on $y = 1$ i.e. $r = \operatorname{cosec}\theta$. To cover second region we have to slide integrating strip from $\theta = \pi/4$ to $\theta = \pi/2$. Therefore,

$$I_2 = \int_{\pi/4}^{\pi/2}\int_0^{\operatorname{cosec}\theta} \frac{\sin\theta}{r} r\mathrm{d}r\ \mathrm{d}\theta = \int_{\pi/4}^{\pi/2}\int_0^{\operatorname{cosec}\theta} \sin\theta\mathrm{d}r\ \mathrm{d}\theta \tag{2}$$

By (1) and (2), we get

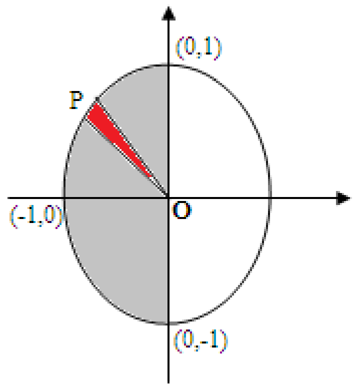
$$\begin{aligned}
I &= \int_0^{\pi/4}\int_0^{\sin\theta/\cos^2\theta} \sin\theta\mathrm{d}r\ \mathrm{d}\theta + \int_{\pi/4}^{\pi/2}\int_0^{\operatorname{cosec}\theta} \sin\theta\mathrm{d}r\ \mathrm{d}\theta \\
&= \int_0^{\pi/4} \sin\theta\,[r]_0^{\sin\theta/\cos^2\theta}\ \mathrm{d}\theta + \int_{\pi/4}^{\pi/2} \sin\theta\,[r]_0^{\operatorname{cosec}\theta}\ \mathrm{d}\theta \\
&= \int_0^{\pi/4} \sin\theta\left[\frac{\sin\theta}{\cos^2\theta}\right]\ \mathrm{d}\theta + \int_{\pi/4}^{\pi/2} \sin\theta\,[\operatorname{cosec}\theta]\ \mathrm{d}\theta \\
&= \int_0^{\pi/4} \tan^2\theta\mathrm{d}\theta + \int_{\pi/4}^{\pi/2} .\mathrm{d}\theta = \int_0^{\pi/4}(\sec^2\theta - 1)\mathrm{d}\theta + \frac{\pi}{2} - \frac{\pi}{4} \\
&= [\tan^2\theta - \theta]_0^{\pi/4} + \frac{\pi}{4} = 1 - \frac{\pi}{4} + \frac{\pi}{4} = 1
\end{aligned}$$

Example 6. Evaluate $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0}\frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}\mathrm{d}x\mathrm{d}y$

Solution: Consider,

$$I=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0}\frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}\mathrm{d}x\mathrm{d}y$$

The region of integration is bounded by $x=-\sqrt{1-y^2}$ i.e. $x^2+y^2=1$, $x=0$, $y=1$ and $y=-1$. Since $x=-\sqrt{1-y^2}$ i.e. negative root. Thus is starts from L.H.S. of symmetric line $x=0$. The region of integration is shown in the following diagram.



In polar coordinate system $x=-\sqrt{1-y^2}$ takes the form $r=1$ and $\frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}=\frac{4r}{1+r^2}$ and we have $\mathrm{d}x\mathrm{d}y=r\mathrm{d}r\mathrm{d}\theta$.

Now to change to polar coordinates consider an integrating strip starting from origin as shown in above figure. The point O lies on origin i.e. $r=0$ and Q lies on circle i.e. $r=1$. Therefore, r varies from 0 to 1. Since region is in third and fourth quadrant, to cover the shaded region, we have to slide strip from $\theta=\pi/2$ to $\theta=3\pi/2$. Therefore θ varies from $\theta=\pi/2$ to $\theta=3\pi/2$. Therefore,

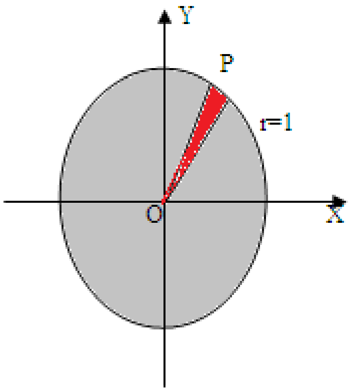
$$\begin{aligned}
I&=\int_{\pi/2}^{3\pi/2}\int_0^1\frac{4r}{1+r^2}r\mathrm{d}r\mathrm{d}\theta=4\int_{\pi/2}^{3\pi/2}\int_0^1\frac{r^2}{1+r^2}\mathrm{d}r\mathrm{d}\theta\\
&=4\int_{\pi/2}^{3\pi/2}\int_0^1\frac{(1+r^2)-1}{1+r^2}\mathrm{d}r\mathrm{d}\theta=4\int_{\pi/2}^{3\pi/2}\int_0^1\left(1-\frac{1}{1+r^2}\right)\mathrm{d}r\mathrm{d}\theta\\
&=4\int_{\pi/2}^{3\pi/2}[r-\tan^{-1}r]_0^1\mathrm{d}\theta\\
&=4\int_{\pi/2}^{3\pi/2}\left(1-\frac{\pi}{4}\right)\mathrm{d}\theta\\
&=\int_{\pi/2}^{3\pi/2}(4-\pi)\mathrm{d}\theta\\
&=(4-\pi)[\theta]_{\pi/2}^{3\pi/2}=(4-\pi)\pi
\end{aligned}$$

Example 7. Evaluate $\iint_R\sin\pi(x^2+y^2)\mathrm{d}x\,\mathrm{d}y$ over the region bounded by the circle $x^2+y^2=1$.

Solution: Consider,

$$I=\iint_R\sin\pi(x^2+y^2)\mathrm{d}x\,\mathrm{d}y$$

We will solve this problem by changing to polar coordinates. The region of integration bounded by the circle $x^2 + y^2 = 1$ is shown in the following figure.



To change into polar consider an integrating strip starting from origin as shown in above figure. At origin, we have $r = 0$ and the point P lies on a circle i.e. $r = 1$. To cover this whole circle we have to rotate this strip in 360°. Therefore θ varies from 0 to 2π. Also, $\sin \pi(x^2 + y^2) = \sin \pi r^2$ and $\mathrm{d}x\,\mathrm{d}y = r\mathrm{d}r\,\mathrm{d}\theta$. Therefore, we have

$$I = \int_0^{2\pi} \int_0^1 \sin(\pi r^2)\, r\mathrm{d}r\,\mathrm{d}\theta$$

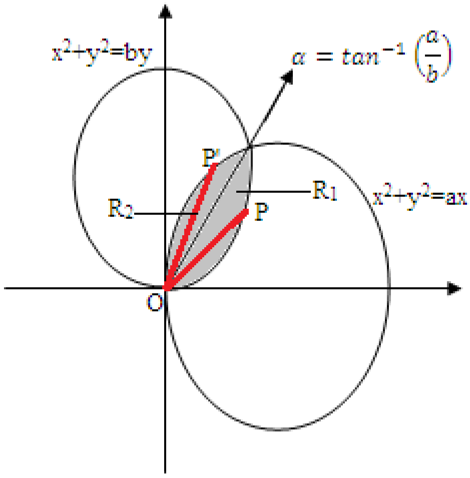
Now, put $r^2 = t \Rightarrow\ r\mathrm{d}r = \frac{1}{2}\mathrm{d}t$ and t varies from 0 to 1. Therefore above reduces to

$$I = \int_0^{2\pi} \int_0^1 \sin(\pi t)\,\frac{1}{2}\mathrm{d}t\,\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi} \left[-\frac{\cos \pi t}{\pi}\right]_0^1 \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} [-(-1)+1]\mathrm{d}\theta = \frac{1}{\pi}[\theta]_0^{2\pi}$$
$$= 2$$

Example 8. Evaluate $\iint_R \frac{(x^2+y^2)^2}{x^2y^2}\mathrm{d}x\,\mathrm{d}y$ over the area common to the circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$, $a > b > 0$.
Solution: Consider,

$$I = \iint_R \frac{(x^2+y^2)^2}{x^2y^2}\mathrm{d}x\,\mathrm{d}y$$

Here, $x^2 + y^2 = ax$ is the circle with center at $(a/2, 0)$ and radius $a/2$ and $x^2 + y^2 = by$ is the circle with center at $(0, b/2)$ and radius $b/2$. Consider the region of integration bounded by these two circles as shown in the following figure.



In polar coordinate system $x^2+y^2=ax$ takes the form $r=a\cos\theta$ and $x^2+y^2=by$ takes the form $r=a\sin\theta$. The point of intersection of these two circles is given by equating $r=a\cos\theta$ and $r=b\sin\theta$. Therefore $a\cos\theta=b\sin\theta$ gives $\theta=\tan^{-1}(b/a)=\alpha$ say. Also, $\dfrac{(x^2+y^2)^2}{x^2y^2}=\dfrac{1}{\sin^2\theta\cos^2\theta}$ and $\mathrm{d}x\,\mathrm{d}y=r\mathrm{d}r\,\mathrm{d}\theta$.

From figure, we gave two regions for change into polar coordinates. In region R_1, the point O lies on origin i.e. $r=0$ and P lies on $x^2+y^2=ax$ i.e. $r=a\cos\theta$. Therefore r varies from 0 to $a\cos\theta$ and θ varies from 0 to $alpha$. In region R_2, the point P lies on $x^2+y^2=by$ i.e. $r=a\sin\theta$. Therefore r varies from 0 to $a\sin\theta$ and θ varies from $alpha$ to $\pi/2$. Therefore, we get

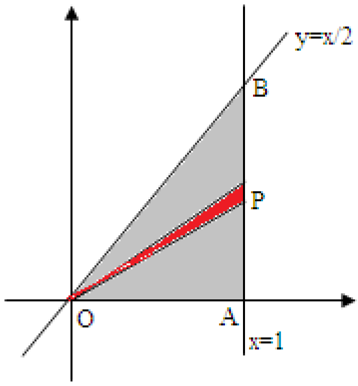
$$\begin{aligned}
I &= \int_0^{\alpha}\int_0^{b\sin\theta}\frac{1}{\sin^2\theta\,\cos^2\theta}r\mathrm{d}r\,\mathrm{d}\theta+\int_{\alpha}^{\pi/2}\int_0^{a\cos\theta}\frac{1}{\sin^2\theta\,\cos^2\theta}r\mathrm{d}r\,\mathrm{d}\theta\\
&= \int_0^{\alpha}\frac{1}{\sin^2\theta\,\cos^2\theta}\left[\frac{r^2}{2}\right]_0^{b\sin\theta}\mathrm{d}\theta+\int_{\alpha}^{\pi/2}\frac{1}{\sin^2\theta\,\cos^2\theta}\left[\frac{r^2}{2}\right]_0^{a\cos\theta}\mathrm{d}\theta\\
&= \int_0^{\alpha}\frac{1}{\sin^2\theta\,\cos^2\theta}\left[\frac{b^2\sin^2\theta}{2}\right]\mathrm{d}\theta+\int_{\alpha}^{\pi/2}\frac{1}{\sin^2\theta\,\cos^2\theta}\left[\frac{a^2\cos^2\theta}{2}\right]\mathrm{d}\theta\\
&= \frac{b^2}{2}\int_0^{\alpha}\sec^2\theta\,\mathrm{d}\theta+\frac{a^2}{2}\int_{\alpha}^{\pi/2}\mathrm{cosec}^2\,\theta\,\mathrm{d}\theta\\
&= \frac{b^2}{2}\tan\theta|_0^{\alpha}+\frac{a^2}{2}[-\cot\theta]_{\alpha}^{\pi/2}\\
&= \frac{b^2}{2}[\tan\alpha]+\frac{a^2}{2}[0+\cot\alpha]\\
&= \frac{b^2}{2}\frac{a}{b}+\frac{a^2}{2}\frac{b}{a}\\
&= \frac{ab}{2}+\frac{ab}{2}=ab
\end{aligned}$$

Example 9. Change to polar coordinate and hence evaluate $\displaystyle\int_0^1\int_0^{x/2}\frac{x}{x^2+y^2}\mathrm{d}x\,\mathrm{d}y$.

Solution: Consider,

$$\int_0^1\int_0^{x/2}\frac{x}{x^2+y^2}\mathrm{d}x\,\mathrm{d}y$$

The region of integration is bounded by $y = x/2$, $x = 1$, x-axis and y-axis is shown in the following figure.



In polar coordinate system, $y = x/2$ takes the form $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, $x = 1$ reduces to $r = \sec\theta$, $\dfrac{x}{x^2+y^2} = \dfrac{\cos\theta}{r}$ and we have $\mathrm{d}x\,\mathrm{d}y = r\mathrm{d}r\,\mathrm{d}\theta$.
To change in polar coordinates, consider an integrating strip starting from origin as shown in above figure. At origin, we have $r = 0$ and the point P lies $x = 1$ i.e. $r = \sec\theta$ and θ varies from 0 to $\tan^{-1}(1/2)$. Therefore,

$$\begin{aligned}
I &= \int_0^{\tan^{-1}(1/2)} \int_0^{\sec\theta} \frac{\cos\theta}{r}\, r\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\tan^{-1}(1/2)} \int_0^{\sec\theta} \cos\theta\, \mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{\tan^{-1}(1/2)} \cos\theta\, [r]_0^{\sec\theta} \mathrm{d}\theta = \int_0^{\tan^{-1}(1/2)} \cos\theta\, \sec\theta\, \mathrm{d}\theta = [\theta]_0^{\tan^{-1}(1/2)} \\
&= \tan^{-1}\left(\frac{1}{2}\right)
\end{aligned}$$