Problems based on trigonometric formula

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| Type II |

$$\boxed{\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$

1) Using Trigonometric formula convert given problem into powers of sin & cos θ, substitute accordingly also check for limit 0 to $\pi/2$

2) algebraic f^n : They can be reduced in above form
e.g. integral of type $\left(\sqrt{a^2-x^2}\right) \to x = a\sin\theta / a\cos\theta$

$a^2+x^2 \to x = a\tan\theta$

Problems 1) Evaluate $I = \int_0^{\pi} (1-\cos\theta)^5 \, d\theta$

Sol^n: $I = \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)^5 d\theta$

$= 2^5 \int_0^{\pi} \sin^{10}(\theta/2) \, d\theta$

put $\frac{\theta}{2} = t$, $\theta = 2t$, $d\theta = 2dt$

θ	0	π
t	0	$\pi/2$

$I = 2^5 \int_0^{\pi/2} \sin^{10}t \cos^0 t \, (2dt)$

Using formula $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$ ✓

$I = 2^5 \frac{1}{\cancel{2}}\beta\left(\frac{11}{2}, \frac{1}{2}\right) \cancel{2}$

$= 2^5 \frac{\Gamma\frac{11}{2}\,\Gamma\frac{1}{2}}{\Gamma 6}$ $\left(\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}\right)$ $\Gamma n+1 = n!$ post

$= \cancel{2^5} \frac{\frac{9}{\cancel{2}}\,\frac{7}{\cancel{2}}\,\frac{\cancel{5}}{\cancel{2}}\,\frac{\cancel{3}}{\cancel{2}}\,\frac{1}{\cancel{2}}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\cancel{8}\times 4\times\cancel{3}\times 2\times 1}$

$= \frac{9\times 7}{8}\pi = \frac{63}{8}\pi$

2) $I = \int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)} \, d\theta$

$I = \int_0^{\pi} \frac{\left(2\sin\theta/2\cos\theta/2\right)^4}{2\cos^2\theta/2} \, d\theta$

$= \int_0^{\pi} 2^3 \sin^4\theta/2 \cos^2\theta/2 \, d\theta$

$\theta = 2t$, $d\theta = 2dt$

$I = 2^3 \int_0^{\pi/2} \sin^4 t \cos^2 t \, (2dt)$

$= 2^3 \frac{1}{\cancel{2}}\beta\left(\frac{5}{2}, \frac{3}{2}\right)\cancel{2}$

$I =$

~~$\int_0^{\pi/2} \sin^7\theta = \frac{6}{7}\,\frac{4}{5}\,\frac{2}{3}\,1$~~ 8

$\frac{\alpha}{2} = t, \quad \alpha = 2t, \quad d\alpha = 2dt$

~~$\int_0^{\pi/2} \sin^7 \alpha = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1$~~ $\frac{8}{105}$

Don't Use this

3) $I = \int_0^1 x^5 \sqrt{\frac{1-x^2}{1+x^2}}\, dx$

$I = \int_0^1 x^4 x \sqrt{\frac{(1-x^2)(1-x^2)}{(1+x^2)(1-x^2)}}\, dx$

$= \int_0^1 x^4 \frac{(1-x^2)}{\sqrt{1-x^4}}\, x\,dx$

$x^2 = \sin\alpha, \quad 2x\,dx = \cos\alpha\, d\alpha$

x	0	1
α	0	$\pi/2$

$I = \frac{1}{2}\int_0^{\pi/2} \sin^2\alpha \frac{(1-\sin\alpha)}{\cancel{\sqrt{1-\sin^2\alpha}}} \cancel{\cos\alpha}\, d\alpha$

$I = \int_0^{\pi/2} \sin^2\alpha - \sin^3\alpha\, d\alpha$

$= \int_0^{\pi/2} \sin^2\alpha \cos^0\alpha\, d\alpha - \int_0^{\pi/2} \sin^3\alpha \cos^0\alpha\, d\alpha$

$= \frac{1}{2}\beta\left(\frac{3}{2}, \frac{1}{2}\right) - \frac{1}{2}\beta\left(2, \frac{1}{2}\right)$

$= \frac{1}{2} \frac{\frac{1}{2}\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 2} - \frac{1}{2}\frac{\Gamma 2\,\Gamma\frac{1}{2}}{\Gamma\frac{5}{2}}$

$= \frac{1}{2}\left\{\frac{\pi}{2} - \frac{\Gamma\frac{1}{2}}{\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}\right\}$

$= \frac{1}{2}\frac{1}{2}\left\{\frac{\pi}{2} - \frac{4}{3}\right\} \quad = \frac{\pi}{8} - \frac{1}{3}$

4) $I = \int_0^1 x^4 \cos^{-1}x\, dx$

put $\cos^{-1}x = t, \quad x = \cos t, \quad dx = -\sin t\, dt$

	0	1
	$\pi/2$	0

$I = \int_{\pi/2}^{0} \cos^4 t\; t\,(-\sin t\, dt)$

$= \int_0^{\pi/2} \underset{u}{t} \underset{v}{\left(\sin t \cos^4 t\right)} dt$

Integration by parts.

$\left[t\left(-\frac{\cos^5 t}{5}\right)\right]_0^{\pi/2}$

$I = 0 + \frac{1}{5}\int_0^{\pi/2} \cos^5 t\, dt$

$= \frac{1}{5}\frac{1}{2}\beta\left(\frac{1}{2}, 3\right)$

$= \frac{8}{75}$

Integra

$$= \left[t\left(-\frac{\cos^5 t}{5}\right)\right]_0 - \int_0^{\pi/2} (1)\left(-\frac{\cos^5 t}{5}\right) dt$$

$x^6 = \tan^2\alpha$

5) $I = \int_0^{\infty} \frac{x^2 \, dx}{(1+x^6)^3}$

Sol^n: put $x^3 = \tan\alpha$

x	0	∞
α	0	$\pi/2$

$3x^2 dx = \sec^2\alpha \, d\alpha$

$$I = \int_0^{\pi/2} \frac{\frac{1}{3}\sec^2\alpha \, d\alpha}{(1+\tan^2\alpha)^3} d\alpha$$

$$= \frac{1}{3}\int_0^{\pi/2} \frac{\sec^2\alpha}{(\sec^2\alpha)^3} d\alpha = \frac{1}{3}\int_0^{\pi/2} \frac{1}{\sec^4\alpha} d\alpha$$

$$I = \frac{1}{3}\int_0^{\pi/2} \cos^4\alpha \, d\alpha$$

$$= \frac{1}{3}\left[\frac{1}{2}\beta\left(\frac{1}{2}, \frac{5}{2}\right)\right]$$

$I =$

6) $I = \int_0^{\pi} x \sin^5 x \cos^6 x \, dx$

Sol^n:

$$I = \int_0^{\pi} (\pi - x) \sin^5(\pi - x)\cos^6(\pi - x) dx$$

$$\left[\int_0^a f(x) dx = \int_0^a f(a-x) dx\right]$$

$\sin(\pi - x) = \sin x, \cos(\pi - x) = -\cos x$

$$I = \int_0^{\pi} (\pi - x) \sin^5 x \cos^6 x \, dx$$

$$I = \pi\int_0^{\pi} \sin^5 x\cos^6 x \, dx - \int_0^{\pi} x \sin^5 x \cos^6 x \, dx$$

$$I = \pi \int_0^{\pi} \sin^5 x \cos^6 x \, dx - I$$

$$2I = \pi \int_0^{\pi} \sin^5 x \cos^6 x \, dx$$

$$\left[\because \int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx\right.$$

$$= 2\int_0^a f(x) dx - \left[\begin{array}{l} \text{if} \\ f(2a-x) \\ = f(x) \end{array}\right]$$

$$2I = \pi\left\{\int_0^{\pi/2} \sin^5 x \cos^6 x \, dx + \int_0^{\pi/2} \sin^5(\pi - x)\cos^6(\pi - x) dx\right\}$$

$$\cancel{2}I = \pi\left(\cancel{2}\int_0^{\pi/2} \sin^5 x \cos^6 x \, dx\right.$$

$$I = \pi \, \frac{1}{2}\beta\left(3, \frac{7}{2}\right)$$

$$I = \frac{8\pi}{693}$$

$$2I = \pi \int_0^{\pi} \sin^5 x \cos^6 x \, dx \qquad | \qquad I = \checkmark \qquad \frac{8\pi}{693}$$

HW ~~d~~) $\int_{-\pi}^{\pi} \sin^4 x \cos^2 x \, dx$ $\quad \left(\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx , \ f(x) \text{ even} \right.$

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8) Prove that $\underbrace{\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta}_{I_1} \cdot \underbrace{\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta}_{I_1} = \frac{\pi^2}{2}$ $\rightarrow \frac{\pi}{\sqrt{2}}$

Let $I = I_1 \times I_2$

$$I_1 = \int_0^{\pi/2} \cos^{1/2}\theta \sin^{-1/2}\theta \, d\theta = \frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right) = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$I_2 =$

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9) Prove that $\int_{-\pi/6}^{\pi/3} (\sqrt{3}\sin\theta + \cos\theta)^{1/4} \, d\theta = 2^{-3/4}\beta\left(\frac{1}{2}, \frac{5}{8}\right)$

Sol^n: $I = \int_{-\pi/6}^{\pi/3} \left[2\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)\right]^{1/4} d\theta$

$$I = 2^{1/4} \int_{-\pi/6}^{\pi/3} \left[\cos\frac{\pi}{6}\sin\theta + \sin\frac{\pi}{6}\cos\theta\right]^{1/4} d\theta$$

$$= 2^{1/4} \int_{-\pi/6}^{\pi/3} \left[\sin\left(\theta + \frac{\pi}{6}\right)\right]^{1/4} d\theta$$

θ	$-\pi/6$	$\pi/3$
t	0	$\pi/2$

put $\theta + \frac{\pi}{6} = t$

$d\theta = dt$

$$I = 2^{1/4}\int_0^{\pi/2} \sin^{1/4} t \, dt$$

$$= 2^{1/4}\,\frac{1}{2}\,\beta\left(\frac{\frac{1}{4}+1}{2}, \frac{1}{2}\right)$$

$$= 2^{-3/4}\beta\left(\frac{5}{8}, \frac{1}{2}\right)$$

$$= 2^{-3/4}\beta\left(\frac{1}{2}, \frac{5}{8}\right) \quad \left(\because \beta(m,n) = \beta(n,m)\right)$$

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10) $I = \int_0^{\infty} \frac{x^5}{(2+3x)^{15}} dx = \int_0^{\infty} \frac{x^5}{\left[2\left(1+\frac{3}{2}x\right)\right]^{15}} dx$ ✓

put $\frac{3}{2}x = \tan^2\theta$

x	0	∞
t	0	$\pi/2$

$\frac{3}{2}\,dx = 2\tan\theta\sec^2\theta\, d\theta$ $\quad | \quad I = \frac{1}{\ldots}\int_0^{\pi/2} \sin^{11}\theta \cos^{17}\theta \, d\theta$

put $\frac{3}{2}x = \tan^2\theta$

x	0	∞
t	0	$\pi/2$

$x = \frac{2}{3}\tan^2\theta, \quad dx = \frac{2}{3} 2\tan\theta\sec^2\theta\, d\theta$

$$I = \int_0^{\pi/2} \frac{\left(\frac{2}{3}\tan^2\theta\right)^5}{2^{15}(1+\tan^2\theta)^{15}} \frac{4}{3}\tan\theta\sec^2\theta\, d\theta$$

$$I = \frac{1}{2^8 3^6}\int_0^{\pi/2} \frac{\tan^{11}\theta}{\sec^{28}\theta}\, d\theta$$

$$I = \frac{1}{2^8 3^6}\int_0^{\pi/2} \sin^{11}\theta\cos^{17}\theta\, d\theta$$

$$= \frac{1}{2^8 3^6}\,\frac{1}{2}\,\beta\left(\frac{12}{2}, \frac{18}{2}\right)$$

$$= \frac{1}{2^9 3^6}\beta(6,9) = \frac{1}{2^9 3^6}\frac{\Gamma 6\,\Gamma 9}{\Gamma 15}$$

$$= \frac{1}{2^9 3^6}\frac{8\times\cancel{4}\times\cancel{3}\times\cancel{2}\ \ \cancel{8!}}{14\times 13\times\cancel{12}\times 11\times\cancel{10}\times 9\times\cancel{8!}} =$$

11)
$$I = \int_0^\infty \frac{x^6(1-x^{10})}{(1+x)^{24}} = \int_0^\infty \frac{x^6}{(1+x)^{24}}dx - \int_0^\infty \frac{x^{16}}{(1+x)^{24}}dx$$

H.W.

$x = \tan^2\theta$