Expansion of Functions

Tuesday, March 23, 2021 9:59 AM

Under certain conditions, functions can be expressed as 'power series' (polynomials)

There are two theorem which enable us to write series

1) Taylor's Th^m 2) Maclaurin's Th^m

1) Taylor's Th^m (Series)

Assume function $f(x)$ has continuous derivatives of all order in an interval containing 'a' Then $f(x)$ can be expanded as power series around point 'a'

$$\rightarrow f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

we take $x = a+h$, $h = (x-a)$

$$\rightarrow \boxed{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots}$$

2) Maclaurin's Series $a=0$

with same assumption, $f(x)$ can be expressed around origin

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots}$$

alternate notation, $y = f(x) \Rightarrow y_0 = f(0)$, $y_1(0) = f'(0)$, $y_2(0) = f''(0)$

$$y = y_0 + x y_1(0) + \frac{x^2}{2!} y_2(0) + \cdots \infty$$

Expansion of Standard function (Using Maclaurin's)

1) $f(x) = e^x$

Then, $f'(x) = f''(x) = f'''(x) = \text{so on} = e^x$

$f'(0) = f''(0) = \cdots = e^0 = 1$

$$\therefore f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$e^x = f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

2) $f(x) = e^{-x}$

$f'(x) = -e^{-x}$, $f''(x) = e^{-x}$, $f'''(x) = -e^{-x}$

$x = 0$

$\therefore$ $f(0) = e^{-0} = 1$, $f'(0) = -e^{-0} = -1$, $f''(0) = \underline{1}$, $f'''(0) = -1$

formula $\to$

$$e^{-x} = f(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots\cdots$$

3) $f(x) = \sin x$

4) $f(x) = \cos x$

---

3) $f(x) = \sin x$

$f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{iv}(x) = \sin x$, $f^{v}(x) = \cos x$

Then $f(0) = 0$, $f'(0) = \underline{1}$, $f''(0) = 0$, $f'''(0) = -1$, $f^{iv}(0) = 0$, $f^{v}(0) = 1$

$f^{vi}(0) = 0$, $f^{vii}(0) = -1$

put in formula $\to$ [ ]

$$\sin x = f(x) = 0 + x + 0 + \frac{x^3}{3!}(-1) + 0 + \frac{x^5}{5!}(1) + \cdots$$

$$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = \underline{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

---

5) $f(x) = \underline{a^x} = e^{x(\log a)}$

$f'(x) = (\log a)\,\underline{e}^{x(\log a)}$, $f''(x) = (\log a)^2\,\underline{e^{x(\log a)}}$

put $x = 0$

$f(0) = 1$, $f'(0) = \log a$, $f''(0) = (\log a)^2$

formula $\rightarrow$

$$f(x) = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + \frac{x^3 (\log a)^3}{3!} + \cdots$$

---

6) $f(x) = (1+x)^m$ (binomial)

$$= 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots$$

7) $\frac{1}{1-x} = \underline{1 - x + x^2 - x^3 + \cdots}$

$f(x) = \frac{1}{1-x}$, $f'(x) = -\frac{1}{(1-x)}(-1)$, $f''(x) = -2\frac{1}{(1-x)^2}$

$x = 0$ $\qquad f'''(x) = (3.2)$

8) $\frac{1}{1+x} = 1 + x + x^2 + x^3 + \cdots$

Problem: 1) $f(x) = \tan x$ write Maclaurin's Series for $f(x)$

$y = \tan x$

Sol^n:

$y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2$

$y'' = 2\sec x(\sec x \tan x)$

$y'' = 2yy'$

$y''' = 2yy'' + 2(y')^2$

$y^{iv} = 2(yy''' + y''y') + 4y'y''$

$= 6y'y'' + 2yy'''$

$y^{v} = 6y'y''' + 6y''y'' + 2yy^{iv} + 2y'''y'$

$y(0) = \tan 0 = 0$

$y'(0) = \sec^2(0) = 1$

$y''(0) = 2y(0)y'(0) = 0$

$y'''(0) = 2(0)(0) + 2(1)^2$

$= 2$

$y^{iv}(0) = 6(1)(0) + 2(0)(2)$

$= 0$

$y^{v}(0) = 6(1)(2) + 6(0) + 2(0)(0) + 2(2)(1)$

$y''''' = \ldots$ | $y'''''(0) = \ldots$

$2yy^{iv} + 2y'''y'$ | $+ 2(0)(0) + 2(2)(1)$

$= 16$

Then formula is given by,

$$y = y(0) + x\,y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \cdots$$

put $$y = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(\cancel{16})^2$$

$\cancel{120}$ 15

$$\tan x = y = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$$

---

another Method (Use prev exp)

$$\tan x = \frac{\sin x}{\cos x} = \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right]\left[\frac{1}{1-\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)}\right]$$

$$\tan x = \underset{A}{\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right]}\left[\underset{B}{1} + \underset{C}{\left[\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right]} + \left[\frac{x^2}{2!} - \frac{x^4}{4!} + \right]^2 \cdots\right]$$

$\left[\frac{x^4}{4} \; D \; -\right.$

$$= \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right] + \left[\frac{x^3}{2!} - \frac{x^5}{4!} - \frac{x^5}{12} \cdots\right]$$

$$+ \left[\frac{x^5}{4} \cdots\cdots\right]$$

$$= x + x^3\left[\frac{1}{2} - \frac{1}{6}\right] + x^5\left[\frac{1}{120} - \frac{1}{24} - \frac{1}{12} + \frac{1}{4}\right]$$

$$\underline{\tan x} = x + \frac{1}{3}x^3 + x^5\left[\frac{1 - 5 - 10 + 30}{120}\right]$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

→ 2) Expand $e^x \sec x$ by Maclaurin's Series

Sol^n: Let $y(x) = e^x \sec x$

$y' = e^x \sec x + e^x \sec x \tan x$

put $x = 0$

$y(0) = e^0 \sec(0) = \underline{1}$

$y'(0) = y(0) + y(0)\tan(0)$

Sol^n: Let $y(x) = e^x \sec x$

$$y' = e^x \sec x + e^x \sec x \tan x$$
$$= (\underline{y} + \underline{y \tan x})$$
$$y'' = y' + \underline{y(\sec^2 x)} + y' \tan x$$

$$y''' = y'' + y\, 2\sec x(\sec x \boxed{\tan x}) + y' \sec^2 x$$
$$+ y' \sec^2 x + y'' \boxed{\tan x}$$

$y(0) \ldots$

$$y'(0) = \underline{y(0)} + \underline{y(0)\tan(0)}$$
$$= 1 + 0 = 1$$
$$y''(0) = y'(0) + y(0)\sec^2(0) + y'(0)\cancel{\tan(0)}$$
$$= 1 + 1(1) = 2$$
$$y'''(0) = y''(0) + 0 + 2y'(0)\sec^2(0)$$
$$= 2 + 2(1)(1) = 4$$

Then formula for Maclaurin's Series,

$$y = y(0) + x y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \cdots$$

Put, $y = 1 + x + \frac{x^2}{\cancel{2!}}(\cancel{2}) + \frac{x^3}{\cancel{3!}_{\,3}}(\cancel{4})^{2}$

$$\boxed{e^x \sec x = y = 1 + x + x^2 + \frac{2}{3}x^3 + \cdots\cdots}$$

You can also solve by 2nd method (Prev. Expansion)

3) Show that $e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11}{24}x^4 + \cdots\cdots$

Sol^n: Assume $\underline{x\cos x} = y$, $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots$

$$e^{x\cos x} = 1 + (x\cos x) + \frac{(x\cos x)^2}{2!} + \frac{(x\cos x)^3}{3!} + \frac{(x\cos x)^4}{4!}$$

Since $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$e^{x\cos x} = 1 + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\cdots\right) + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^2$$
$$+ \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^3 + \frac{x^4}{4!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^4$$

$x\cos x \qquad x^3 \qquad (\qquad x^4 \qquad 2x \quad + \cdots)$

$$e^{x\cos x} = 1 + x - \frac{x^3}{2!} + \cdots + \frac{x^2}{2!}\left(1 + \frac{x^4}{4} - \cancel{2}\frac{x^2}{\cancel{2!}} + \cdots\right)$$

$$+ \frac{x^3}{3!}\left(1 + \cdots\right) + \frac{x^4}{4!}\left(1 + \cdots\right)$$

$$\boxed{= 1 + x - \frac{x^3}{2} + \cdots + \frac{x^2}{2} - \frac{x^4}{2} + \cdots + \frac{x^3}{6} + \cdots + \frac{x^4}{24} + \cdots}$$

$$= 1 + x + \frac{x^2}{2} + x^3\left(\frac{1}{6} - \frac{1}{2}\right) + x^4\left(\frac{1}{24} - \frac{1}{2}\right)$$

$$\boxed{e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{1}{3}x^3 - \frac{11}{24}x^4}$$

Problems on Taylor's Th^m

4) Expand $\log(\cos x)$ about point $\pi/3$

$f(x) = y = \log(\cos x)$

$f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$

$f''(x) = -\sec^2 x$

$f'''(x) = -2\sec^2 x \tan x$

Put $a = \pi/3$, $x = \pi/3$

$f(a) = \log\left(\cos\frac{\pi}{3}\right) = \log\left(\frac{1}{2}\right) = -\log 2$

$f'(a) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$

$f''(a) = -\sec^2\left(\frac{\pi}{3}\right) = -4$

$f'''(a) = -2\sec^2\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right)$

$= -2(4)(\sqrt{3}) = -8\sqrt{3}$

Taylor's Series formula $\Rightarrow$ $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$

$$\log(\cos x) = (-\log 2) + (x-a)(-\sqrt{3}) + \frac{(x-a)^2}{2!}(-4) + \frac{(x-a)^3}{3!}(-8\sqrt{3})$$

$$\boxed{\log(\cos x) = -\log 2 - \sqrt{3}\left(x - \frac{\pi}{3}\right) - 2\left(x - \frac{\pi}{3}\right)^2 - \frac{4}{\sqrt{3}}\left(x - \frac{\pi}{3}\right)^3}$$

... x^3 ... $x^2 + 4x + 3$ in powers of $(x-2)$

4) Expand x^3-3x^2+4x+3 in powers of $(x-2)$

Sol^n: $(x-2)$ compare with $(x-a)$, $a = 2$

$f(x) = x^3-3x^2+4x+3$

$f'(x) = 3x^2-6x+4$

$f''(x) = 6x-6$

$f'''(x) = 6$

$f(a) = 2^3-3(2)^2+4(2)+3 = 7$

$f'(a) = 4$

$f''(a) = 6$

$f'''(a) = 6$

formula $\left[f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)\right]$

Then $f(x) = 7+4(x-2)+6\frac{(x-2)^2}{2!}+\frac{\cancel{6}}{\cancel{3!}}(x-2)^3$

$$\boxed{f(x) = 7+4(x-2)+3(x-2)^2+(x-2)^3}$$

5) Using Taylor's Th^m, Find $\sqrt{9.12}$ correct upto five decimals.

H.W. 6) Prove that $\log(\sin(x+h)) = \log(\sin x)+h\cot x-\frac{h^2}{2!}\operatorname{cosec}^2 x+\frac{h^3}{3!}\left(\frac{\cos x}{\sin^3 x}\right)$

5) Let $f(x) = \sqrt{x}$, $a = 9$, $h = 0.12$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f''(x) = -\frac{1}{4}x^{-3/2}$

$f'''(x) = \frac{3}{8}x^{-5/2}$

$f(a) = \sqrt{9} = 3$

$f'(a) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

$f''(a) = -\frac{1}{4}(9)^{-3/2} = -\frac{1}{108}$

$f'''(a) = \frac{3}{8}(9)^{-5/2} = \frac{1}{648}$

By taylor's Th^m $\left[f(a+h) = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)\right]$

$\ldots\ (\ldots)^2\ (\ldots)\ \ldots\ \frac{(0.12)^3}{\ldots}\left(\frac{1}{\ldots}\right)$

$2! \quad 3!$

$$\sqrt{9.12} = 3 + (0.12)\frac{1}{6} + \frac{(0.12)^2}{2}\left(-\frac{1}{108}\right) + \frac{(0.12)^3}{6}\left(\frac{1}{648}\right)$$

$$= 3.01992$$