CURVE TRACING

Tracing of Cartesian Curves:

We give below some of the main rules which definitely help in determining the shape of the curves from their equations.

1. Symmetry:

(a) If powers of y in the equation of the curve, are all even, then the curve is symmetrical about X −axis, i.e., whatever is the shape of the curve on one side of X −axis, same should be on the other side of it. Because for one value of X we get two values of y which differ in sign only. E.g. the curve $y^2 = 4ax$ is symmetrical about X −axis. Similarly, if all powers of x in the equation are even, the curve is symmetric about y −axis. E.g. the curve $x^2 = 4ay$ is symmetric about y −axis

(b) If on interchanging x and y, the equation of the curve remains unchanged, then the curve is symmetrical about the line $y = x$. E.g. the curve $xy = c^2$ and $x^3 + y^3 = 3axy$ both are symmetrical about the line $y = x$

(c) If x is changed to $-x$ and y to y to $-y$, the equation of the curve remains unchanged, then the curve is symmetrical in the opposite quadrants, i.e. whatever is the shape of the curve in the first quadrant same is in the third quadrant and similarly, for the second and fourth quadrants. E.g. the curves $xy = c^2$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are symmetrical in opposite quadrants.

2. The Origin:

(a) If there is no constant term in the equation, the curve passes through the origin

(b) If the curve is passing through the origin, we can find the equation of the tangent to the curve at the origin, by equating to zero the lowest degree term in the equation. E.g. the curve $y^2 = 4ax$ has y −axis as tangent at the origin and the curve $y = x^3$ has x −axis as tangent to the curve at the origin

(c) If there are or more tangents to the curve at the origin, it is called a multiple point. Further, the origin is called a node, a cusp or isolated point according as the tangents there are real and distinct real and coincident or imaginary respectively. E.g. in the curve $y^2(a^2+x^2)+x^2(a^2-x^2)=0$, the origin is an isolated point and for the curve $y^2(a-x)=x^3$ the origin is a cusp

3. Intersection with the Coordinate Axes:

(a) Find the points where the curve intersects the x −axis and the y −axis separetly.

(b) Find the tangent to the curve at its point of intersection with the coordinate axes by first shifting the origin to this point and then equating to zero the lowest degree term

4. Region of Absence of Curve:

If possible, express y in terms of x **or** x in terms of y explicitly, and examine the nature, i.e. how y varies as x varies and how x varies as y varies. On solving for y, say, in terms of x, suppose we find that between $x=a$ and $x=b, y$ is imaginary (i.e. y^2 is negative), then the curve does not exist in the region between the lines $x=a$ and $x=b$. E.g. in the curve $y^2(1-x)=x^3, y$ is imaginary when $x<0$ and $x>1$ hence no curve lies on the right of the line $x=1$ and on the left of y −axis

5. Maxima, Minima and Points of Inflexion:

Find $\frac{dy}{dx}$ and equate it to zero to get the critical points at which y may be maximum or minimum. Also find $\frac{d^2y}{dx^2}$ and equate it is zero to get the possible points of inflexion

6. Asymptotes:

Find the asymptotes parallel to x −axis (or y −axis) by equating to zero the coefficnet of the highest degree term in x (or y). E.g. in the curve $y^2(1-x)=x^3$ the asymptote parallel to y −axis is the line $x=1$. To find oblique asymptotes put $y=mx+c$ in the equation of the curve and then equate to zero the coefficient of highest and the next heights degree terms in x, this will give equations to find m and c of the equations of asymptotes

7. Conversion of Polar Form:

In needed, transform the given equation into polar coordinates r and θ by putting $x = r\cos\theta$, $y = r\sin\theta$ and hence $x^2 + y^2 = r^2, \frac{y}{x} = \tan\theta$. Convenient when there is $(x^2 + y^2)^{n/2}$ in the equation. The polar form of some curves is more convenient to handle than Cartesian form.

Curves in Parametric Form:

Following few points will help in tracing the curves whose equations are given in parametric form or can be expressed in parametric form: $x = f_1(t), y = f_2(t)$

1. If $x = f_1(t)$ is an even function of t and $y = f_2(t)$ is an odd function of t, the curve is symmetrical about x −axis and vice versa

2. If for some value of t both x and y become zero then the curve passes through the origin

3. **(a)** Find the value of t for which $f_2(t) = 0$ and for this value of t find $x = f_1(t)$ and get the point where the curve meets the x −axis.

(b) Similarly, find the value of t for which $f_1(t) = 0$ and for this value of t find $y = f_2(t)$ and get the point where the curve meets the y −axis.

4. If possible find the greatest and the least values of x and y which give the region or regions in which the curve lies.

5. Find $\frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$ and find the points on the curve where the tangent to the curve is parallel to either coordinate axes

6. Find the possible asymptotes of the curve, parallel to the coordinate axes

Tracing of Polar Curves:

Curves in polar coordinates r and θ are traced basically in the same manner as the Cartesian curves. The hints given below would definitely help in sketching the polar curves.

1. **Symmetry:**

(a) If θ is changed to $-\theta$ and the equation remains unchanged, the curve is symmetrical about the initial line (x −axis)

(b) If θ is changed to $\pi - \theta$ and the equation remains unchanged, the curve is symmetrical about the line $\theta = \frac{\pi}{2}$ (i.e. y −axis)

(c) If θ is changed to $\pi + \theta$ and the equation remains unchanged, the curve is symmetrical about the pole (i.e. same as symmetry in opposite quadrants). In other words we can say that if r is changed to $-r$ and the equation remains unchanged, the curve is symmetrical about the pole

(d) If θ is changed to $\left(\frac{\pi}{2} - \theta\right)$ and the equation remains unaltered, the curve is symmetrical about the line $\theta = \frac{\pi}{4}$. (Same as symmetry about the line $y = x$)

2. **Pole:**

We find whether the pole lies on the curve or not. If $r = 0$ for $\theta = \alpha$, then the curve passes through the pole and the tangent to the curve at the pole is the line $\theta = \alpha$

3. **Direction of the Tangent:**

Differentiate the equation of the curve with respect to θ and using $\tan \emptyset = r\frac{d\theta}{dr}$ we obtain $\emptyset$ which is the angle between the tangent to the curve at a point and the radius vector of this point. This angle $\emptyset$ gives the direction of the tangent to the curve at the point

4. **Region:**

Solve the given equation of the curve for r in terms of θ and if we can find the range of θ, say $\alpha < \theta < \beta$ in which r becomes imaginary (i.e. r^2 is negative) then we conclude that no portion of the curve lies in the triangular strip $\theta = \alpha$ and $\theta = \beta$

5. **Asymptotes:**

If r becomes infinite for $\theta = \theta_1$, say, then there may exist asymptote of the curve

6. **Intersection with axes:**

Determine where the curve meets the initial line, the lines $\theta = \frac{\pi}{2}, \theta = \pi, \theta = \frac{3\pi}{2}$ etc. and the tangents, at these points.

7. Conversion to Cartesian Form:

Transform into Cartesian equation, if necessary, by putting $r^2 = x^2 + y^2$,

$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \tan\theta = \frac{y}{x}$

Preliminaries and Definitions:

Singular Points:

An unusual point on a curve is called **singular point** such as, a point of inflexion, a double point, a multiple point, cusp, node or a conjugate point

1. **Point of Inflexion:** A point where the curve unusually crosses its tangent is called a **point of inflexion**. In Fig 1, P is the point of inflexion

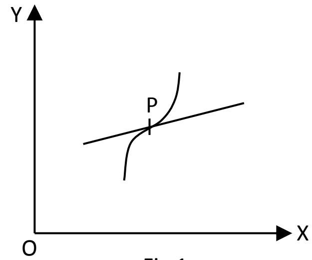


Fig 1

2. **Multiple Point:** A point through which is more than one branches of a curve pass is called a multiple point of the curve

3. **A Double Point:** A point on curve is called a **double point**, if two branches of the curve pass through it. A **Triple Point** if three branches pass through it

4. **Node:** A double point is called node if the branches of curve passing through it are real and the tangents at the common point of intersection are real and distinct. (not coincident)

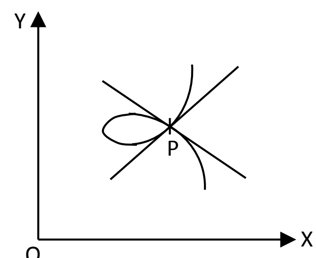


Fig 2

5. **Cusp:** A double point is called a **cusp** if the tangents at it to the two branches of the curve are coincident

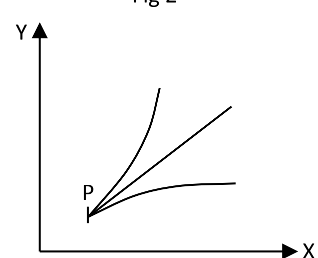


Fig 3

Standard Curves:

1. Straight Lines:

(i) $x = 0$ is the equation of y −axis, $y = 0$ is the equation of x −axis, $x = \pm a$, line parallel to y −axis, $y = \pm b$, line is parallel to x −axis

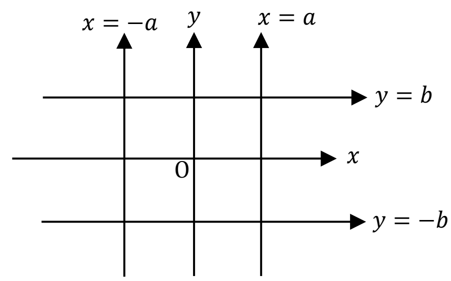


Fig 4

(ii) $y = \pm mx$, a line passing through origin having slope $m, m = \tan\theta$

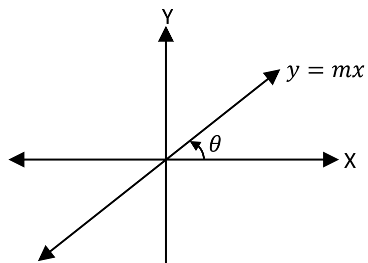


Fig 5

θ is acute angle when m is positive

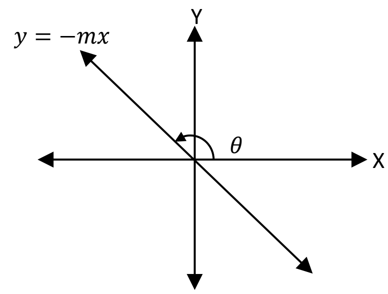


Fig 6

θ is obtuse angle, m is negative

(iii) Slope Intercept Form: $y = mx + c$, where m is the slope and c is the y −intercept

(a) $y = mx + c$

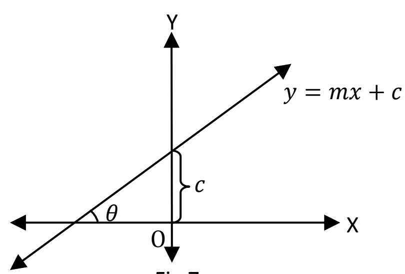


Fig 7

θ is acute, c is positive intercept

(b) $y = -mx + c$

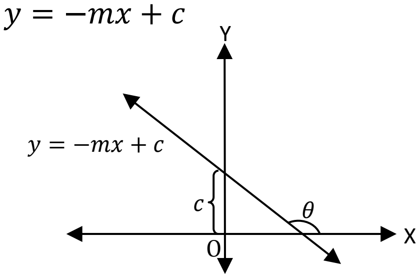


Fig 8

θ is obtuse, c is positive intercept

(c) $y = mx - c$

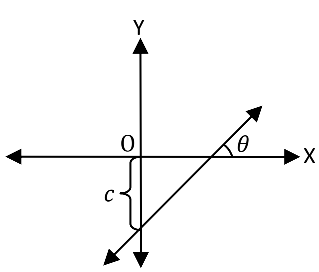


Fig 9

θ is acute, c is negative intercept

(d) $y = -mx - c$

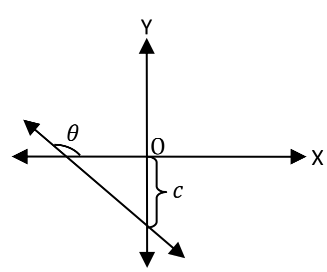


Fig 10

θ is obtuse, c is negative intercept

Note: For positive slope m, θ is acute and for negative slope m, θ is obtuse

(iv) Two-Intercept Form:

$\frac{x}{a}+\frac{y}{b}=1$

a is x −intercept b is y −intercept

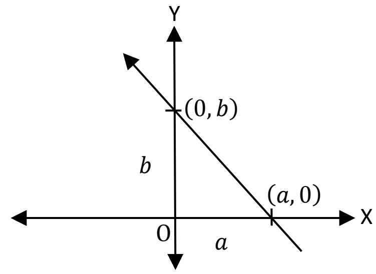


Fig 11

2. Circles:

(i) Standard circle with centre at $(0,0)$ and radius $r=a$

Cartesian equation: $x^2+y^2=a^2$

Parametric equation: $x=a\cos\theta\,, y=a\sin\theta$

Polar equation: $r=a$

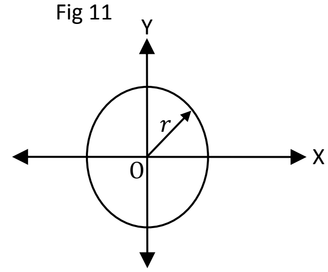


Fig 12

(ii) Centre Radius Form:

Cartesian equation: $(x-a)^2+(y-b)^2=r^2$

Centre $\equiv (a,b)$ Radius $=r$

Parametric equation: $x=a+r\cos\theta\,, y=b+r\sin\theta$

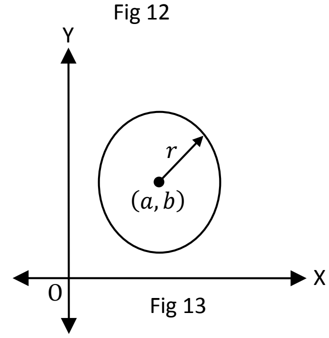


Fig 13

(iii) General Form:

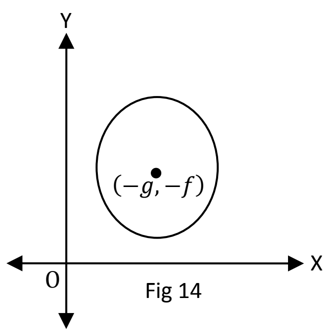


Fig 14

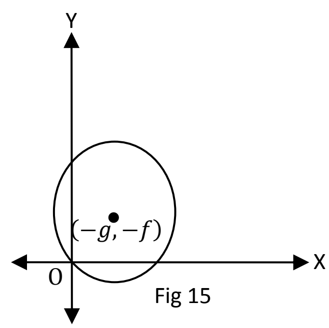


Fig 15

$x^2+y^2+2gx+2fy+c=0$

Centre $\equiv(-g,-f)$ Radius $=\sqrt{g^2+f^2-c}$

$x^2+y^2+2gx+2fy+c=0$

Centre $\equiv(-g,-f)$ Radius $=\sqrt{g^2+f^2}$

(iv) Circle with Centre on Coordinate Axes:

Cartesian equation: $(x-a)^2+y^2=a^2$ or $x^2+y^2=2ax$

Centre $\equiv(a,0)$ Radius $=a$

Parametric equation: $x=a(1+\cos\theta), y=a\sin\theta$

Polar equation: $r=2a\cos\theta$

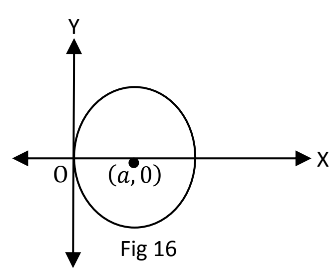


Fig 16

(v) Cartesian equation: $x^2 + (y-a)^2 = a^2$ or $x^2 + y^2 = 2ay$

Centre $\equiv (0, a)$ Radius $= a$

Parametric equation: $x = a\cos\theta\,, y = a(1+\sin\theta)$

Polar equation: $r = 2a\sin\theta$

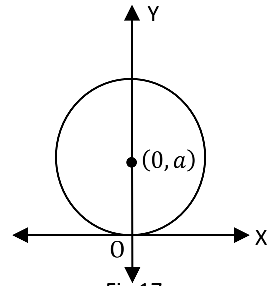


Fig 17

3. Parabola:

(i) Right-handed Parabola:

Cartesian equation: $y^2 = 4ax$

Parametric equation: $x = at^2, y = 2at$

Vertex at $O(0,0)$, Directrix: $x = -a$

Focus $S \equiv (a, 0)$ $LL' =$ Latus rectum $= 4a$ Eccentricity $= e = \frac{SP}{PM} = 1$

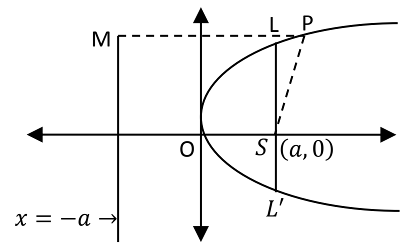


Fig 18

(ii) Left-handed Parabola:

Cartesian equation: $y^2 = -4ax$

Parametric equation: $x = -at^2, y = 2at$

Vertex at $O(0,0)$, Directrix: $x = a$

Focus $S \equiv (-a, 0)$ $LL' =$ Latus rectum $= 4a$

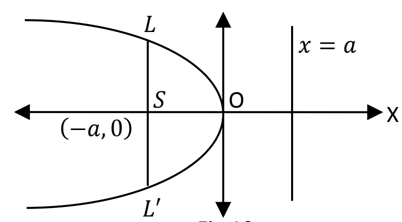


Fig 19

(iii) Upward Parabola:

Cartesian equation: $x^2 = 4ay$

Parametric equation: $x = 2at, y = at^2$

Vertex at $O(0,0)$, Directrix: $y = -a$

Focus $S \equiv (0, a)$ $LL' =$ Latus rectum $= 4a$

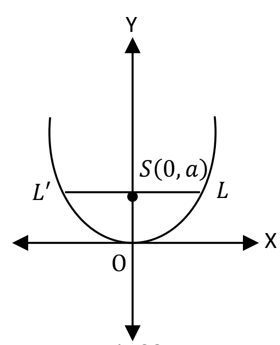


Fig 20

(iv) Downward Parabola:

Cartesian equation: $x^2 = -4ay$

Parametric equation: $x = 2at, y = -at^2$

Vertex at $O(0,0)$, Directrix: $y = a$

Focus $S \equiv (0, -a)$ $LL' =$ Latus rectum $= 4a$

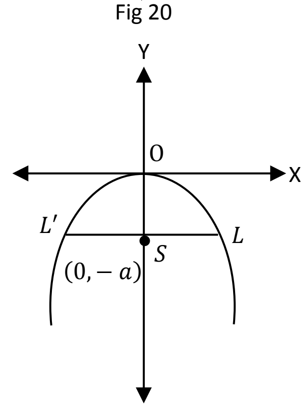


Fig 21

(v) General Form of Parabola:

$(x-h)^2 = 4a(y-k)$

Vertex at $A = (h,k)$

The most general equation of parabola is

$y = ax^2 + bx + c$ where a, b, c are constants

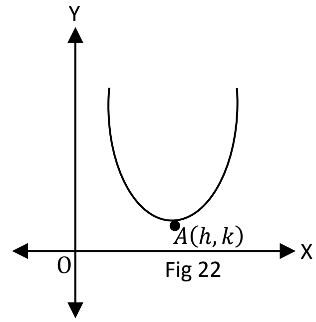


Fig 22

(vi) General Form of Parabola:

$(y-k)^2 = 4a(x-h)$

Vertex at $A = (h,k)$

The most general equation of parabola is

$x = ay^2 + by + c$ where a, b, c are constants

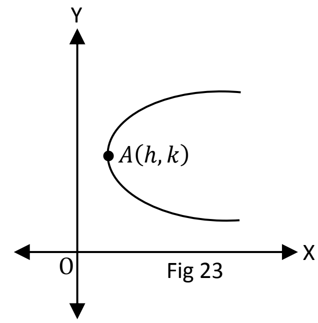


Fig 23

4. Ellipse:

(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b) \ \ b^2 = a^2(1-e^2), 0 < e < 1$

Focus $S \equiv (ae, 0)$

Major axis $= 2a$ Minor axis $= 2b$

Parametric equations: $x = a\cos\theta\,, y = b\sin\theta$

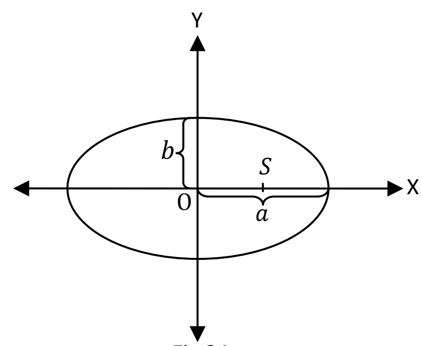


Fig 24

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (b > a) \ \ a^2 = b^2(1-e^2), 0 < e < 1$

Focus $S \equiv (0, be)$ $e \equiv$ eccenticity

Major axis $= 2b$ Minor axis $= 2a$

Parametric equations: $x = a\sin\theta\,, y = b\cos\theta$

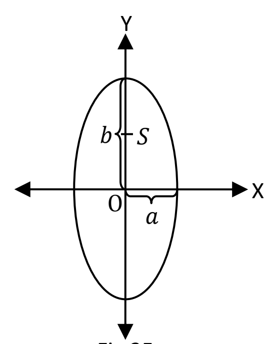


Fig 25

(iii) $\frac{(x-l)^2}{a^2} + \frac{(y-m)^2}{b^2} = 1 \quad (a > b)$

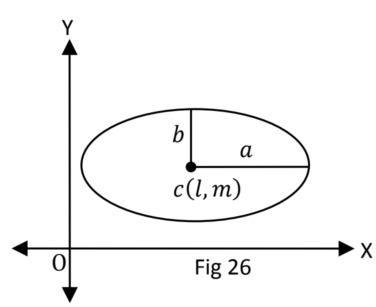


Fig 26

5. Hyperbola:

(i) A Double Curve:

Cartesian equation: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad b^2=a^2(e^2-1)$

$e \equiv$ eccentricity $e>1$ Focus $S \equiv (ae,0)$

Transverse axis $=2a$ Conjugate axis $=2b$

Parametric equations: $x=a\sec\theta\,, y=b\tan\theta$

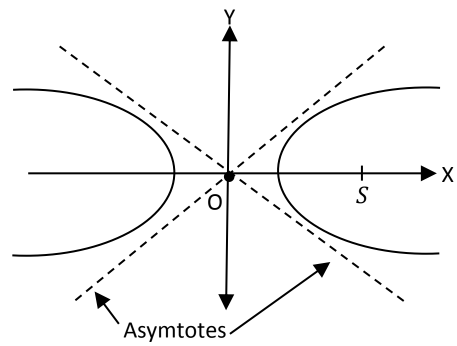


Fig 27

(ii) Rectangular Hyperbolas:

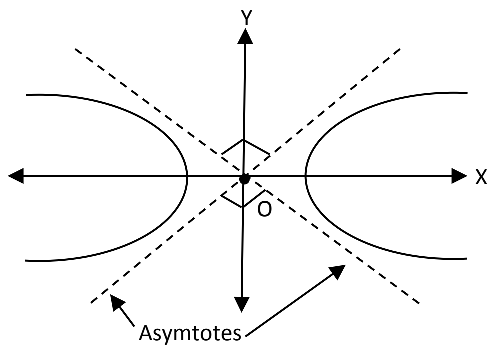


Fig 28

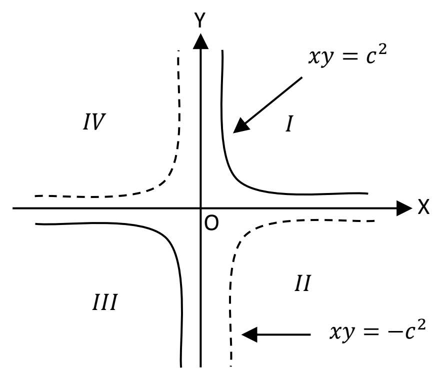


Fig 29

Here, $a=b$

$x^2+y^2=a^2 \quad e=\sqrt{2}$

Asymptotes are perpendicular to each other

$I: xy=c^2, \quad II: xy=-c^2$

Coordinate axes are asymptotes

(which are perpendicular to each other)

6. Cycloids:

(i) $x=a(\theta+\sin\theta), y=a(1+\cos\theta)$

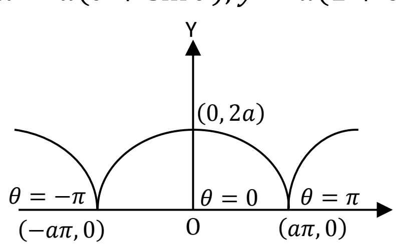


Fig 30

(ii) $x=a(\theta-\sin\theta), y=a(1-\cos\theta)$

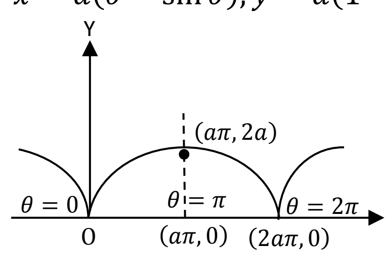


Fig 31

(iii) $x=a(\theta+\sin\theta), y=a(1-\cos\theta)$

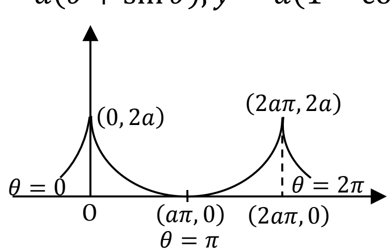


Fig 32

(iv) $x=a(\theta-\sin\theta), y=a(1+\cos\theta)$

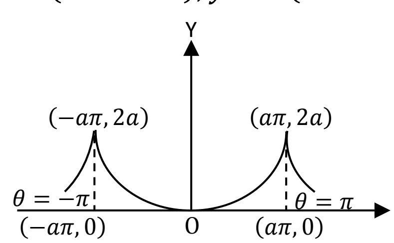


Fig 33

7. Lemniscate of Bernoulli:

Cartesian equation: $(x^2+y^2)^2 = a^2(x^2-y^2)$

Polar equation: $r^2 = a^2\cos 2\theta$

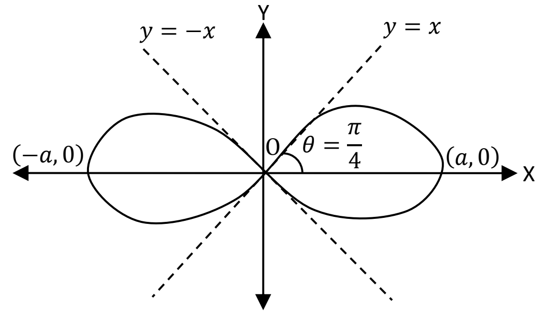


Fig 34

8. Astroid: (Hypocycloid)

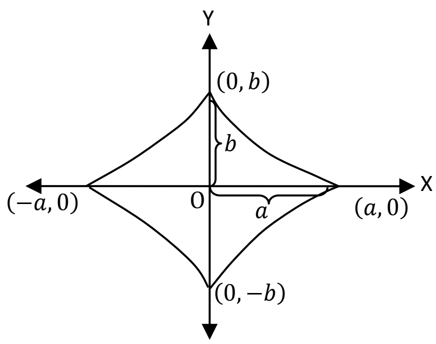


Fig 35

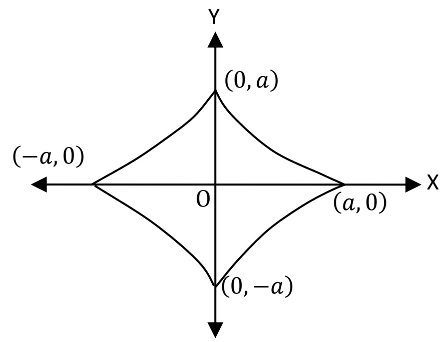


Fig 36

Cartesian Form: $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$

Parametric Form: $x = a\cos^3\theta\,, y = b\sin^3\theta$

Cartesian Form: $x^{2/3} + y^{2/3} = a^{2/3}$

Parametric Form: $x = a\cos^3\theta\,, y = a\sin^3\theta$

9. Catenary:

$y = c\cosh\left(\frac{x}{c}\right)$

$A(0,c)$ is the vertex

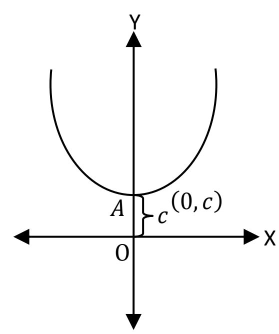


Fig 37

10. Cardioid:

(i) $r = a(1+\cos\theta)$

(ii) $r = a(1-\cos\theta)$

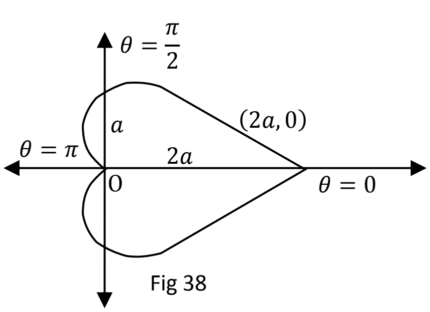


Fig 38

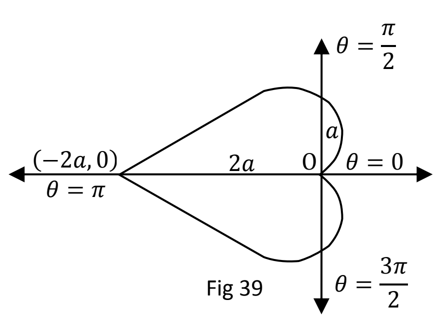


Fig 39

(iii) $r = a(1+\sin\theta)$

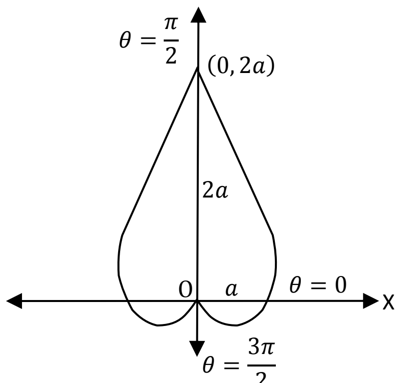


Fig 40

(iv) $r = a(1-\sin\theta)$

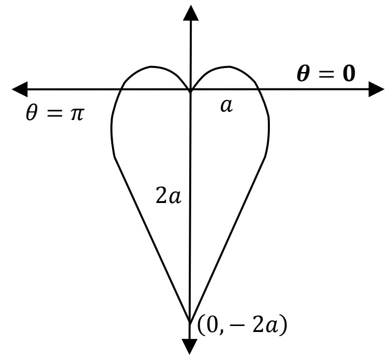


Fig 41

11. Cubic (or Cubical) Parabola and Semi- Cubic Parabola

A polynomial of 3rd degree is called cubic parabola

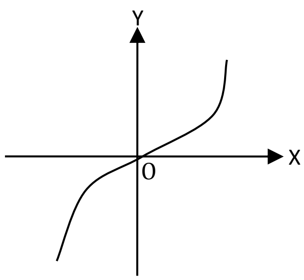


Fig 42

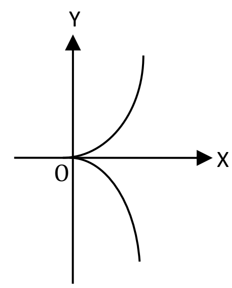


Fig 43

Cubic Parabola: $ay = x^3$

Semi-Cubic Parabola $ay^2 = x^3$

The most general form $y = ax^3 + bx^2 + cx + d$

12. Cissoid:

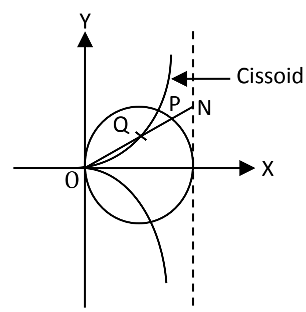


Fig 44

Polar form of the equation of cissoid is $r = 2a(\sec\theta - \cos\theta)$

or $r\cos\theta = 2a\sin^2\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Cartesian form: $(2a - x)y^2 = x^3, x \in [0, 2a]$

and Parametric form: $x = 2a\sin^2\theta$, $y = 2a\left(\tan\theta - \frac{1}{2}\sin 2\theta\right)$

13. Folium of Descartes:

Folium of Descartes forms a loop in the first quadrant with a double point at the origin and $x + y + a = 0$ is an asymptote. It is symmetrical about $y = x$

Cartesian equation: $x^3 + y^3 - 3axy = 0$

Polar equation: $r = \frac{3a \sin\theta \cdot \cos\theta}{\sin^3\theta + \cos^2\theta}$ or $r = \frac{3a \sec\theta \cdot \tan\theta}{1+\tan^3\theta}$

Parametric equation: $x = \frac{3at}{1+t^3}, y = \frac{3at^2}{1+t^3}$

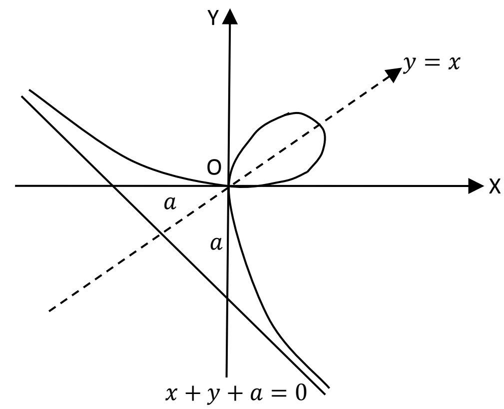


Fig 45

14. Loop:

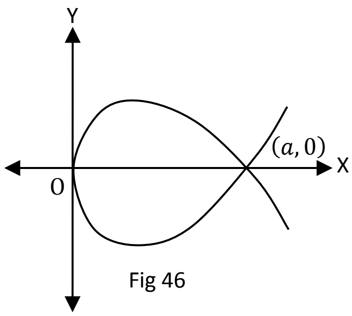


Fig 46

$ay^2 = x(x-a)^2$

Symmetric about x −axis

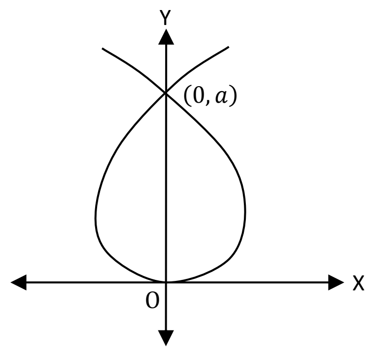


Fig 47

$ax^2 = y(y-a)^2$

Symmetric about y −axis

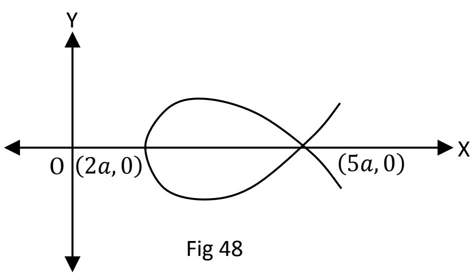


Fig 48

$9ay^2 = (x-2a)(x-5a)^2$

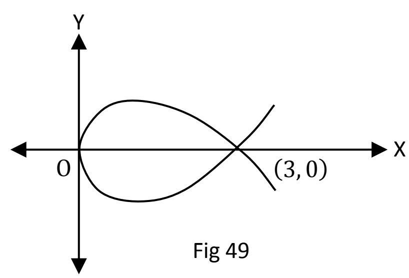


Fig 49

Cartesian equation: $y^2 = x\left(1 - \frac{1}{3}x\right)^2$

Parametric Equation: $x = t^2, y = t\left(1 - \frac{t^2}{3}\right)$

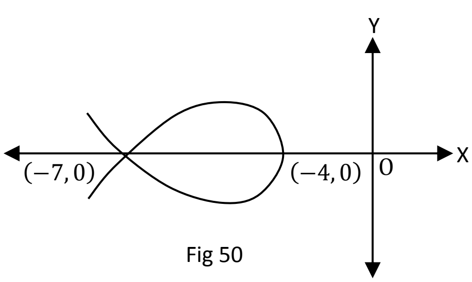


Fig 50

$9y^2 = (x+7)^2(x+4)$

Three Dimensional Objects:

1. Planes:

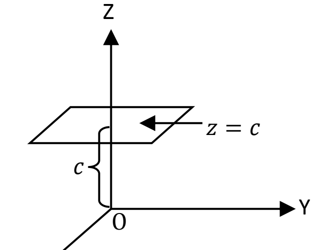
(i) Equation of the coordinate planes are: $z = 0$ is the xy −plane, $x = 0$ is the yz −plane and $y = 0$ is the zx −plane

(ii) Plane-parallel to the coordinate planes:

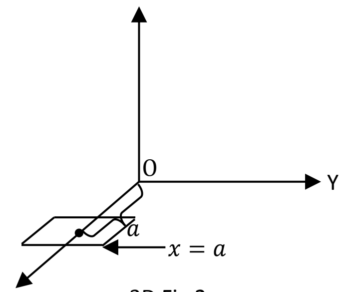
$z = c$ is the plane parallel to xy −plane,

$x = a$ is the plane parallel to yz −plane and

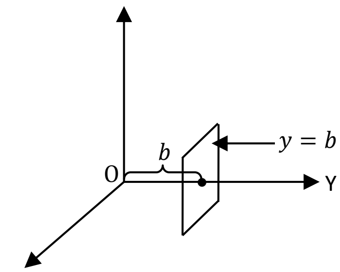
$y = b$ is the plane parallel to zx −plane



3D Fig 1



3D Fig 2



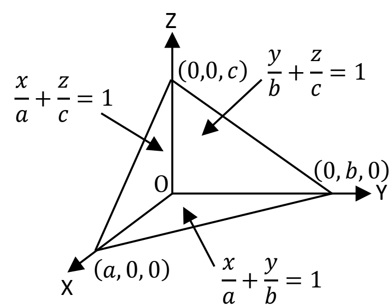
3D Fig 3

2. Tetrahedron:

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is a tetrahedron

When $z = 0, \frac{x}{a} + \frac{y}{b} = 1$ is a plane which is parallel to z −axis.

Similarly, the planes $\frac{y}{b} + \frac{z}{c} = 1$ is parallel to x −axis and $\frac{x}{a} + \frac{z}{c} = 1$ is parallel to y −axis



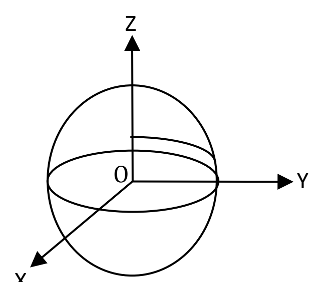
3D Fig 4

3. Sphere:

Cartesian Form: $x^2 + y^2 + z^2 = a^2$

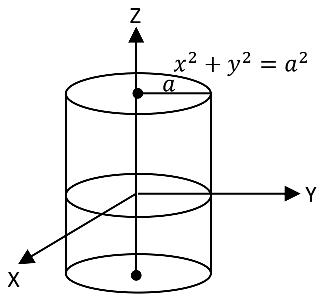
Polar form: $r = a$

Centre at origin and radius a

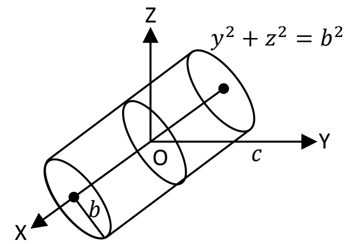


3D Fig 5

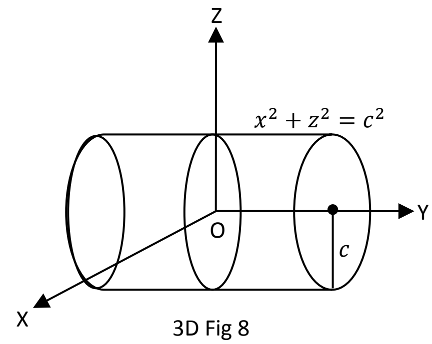
4. Right Circular Cylinders:



3D Fig 6

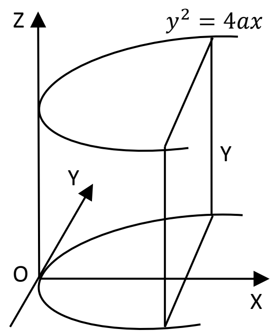


3D Fig 7

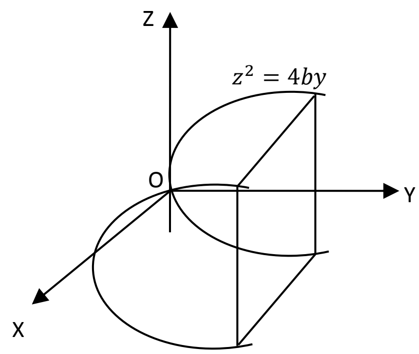


3D Fig 8

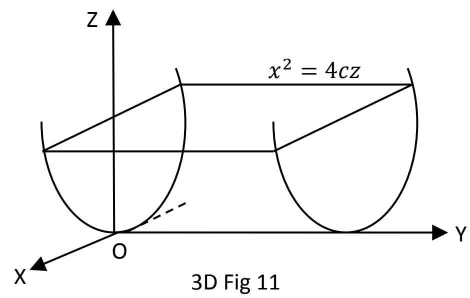
5. Parabolic Cylinders:



3D Fig 9



3D Fig 10



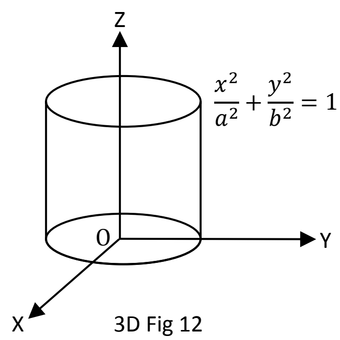
3D Fig 11

6. Elliptic Cylinders:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ \text{axis} = z - \text{axis}$$

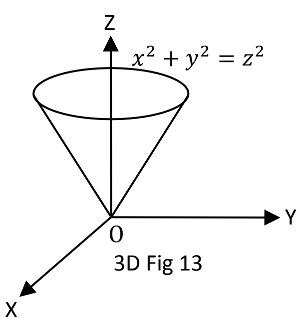
$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \ \text{axis} = x - \text{axis}$$

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \ \text{axis} = y - \text{axis}$$

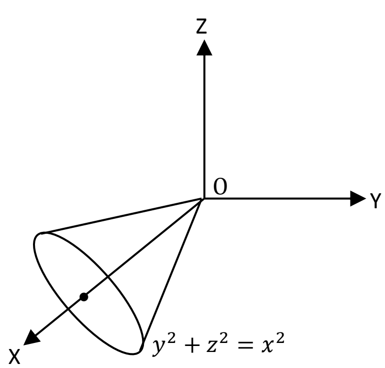


3D Fig 12

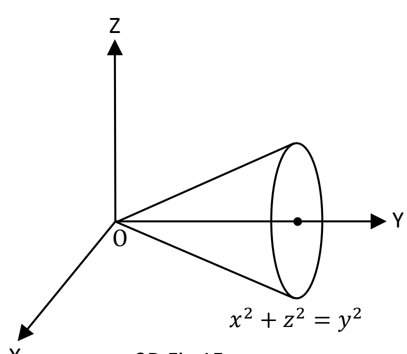
7. Cone:



3D Fig 13

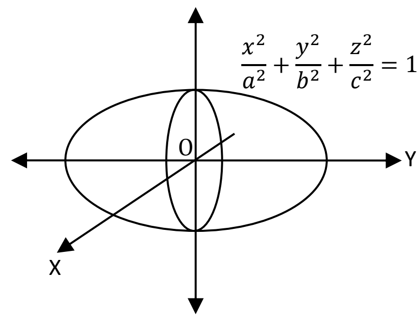


3D Fig 14



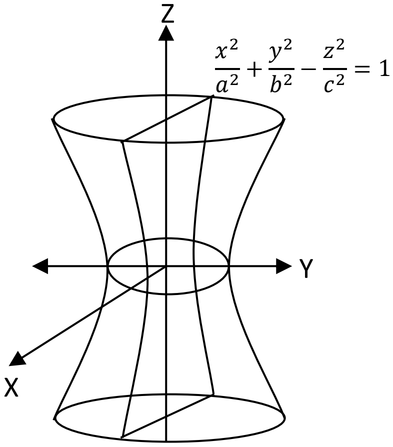
3D Fig 15

8. Ellipsoid:



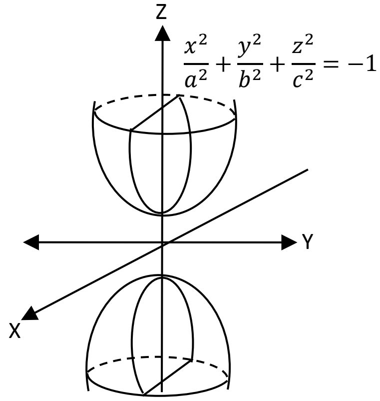
3D Fig 16

9. Hyperboloids:



3D Fig 17

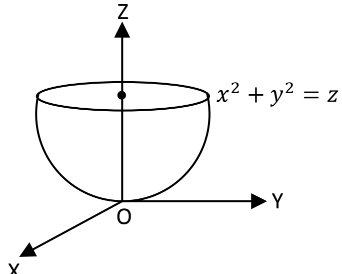
Hyperboloid of one sheet



3D Fig 18

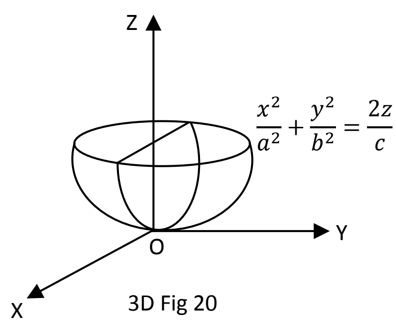
Hyperboloid of two sheet

10. Paraboloids:



3D Fig 19

Circular paraboloid



3D Fig 20

Elliptic paraboloid