Gamma Function

Tuesday, March 30, 2021 11:37 AM

Improper Integrals : $\int_a^{\infty}$ $\int_{-\infty}^{t}$ $\int_a^b f(x)\, dx$ $f(x) \to \infty$

Definition :

function of n $(n>0)$

$$\Gamma n = \phi(n) = \int_0^{\infty} e^{-x} x^{n-1}\, dx$$

$\to$ integral is finite

$\left(\frac{1}{e^{\infty}}\right)(\infty^n)$, $\frac{1}{e^{\infty}} \to 0$, $\infty^n \to \infty$

$$\Gamma 5 = \int_0^{\infty} e^{-x} x^4\, dx$$

$$\int_0^{\infty} e^{-x} x^3 = \Gamma 4$$

$$\boxed{\int_0^{\infty} e^{-x} x^n\, dx = \Gamma n+1}$$

- Properties of Gamma function

1) $\Gamma n+1 = n \Gamma n$ 2) $\Gamma n+1 = n!$ if n is positive intger

3) $\Gamma \frac{1}{2} = \sqrt{\pi}$ (proof afterwards) | 4) $\Gamma \frac{3}{4} \Gamma \frac{1}{4} = \sqrt{2}\pi$

exp : 1) $\Gamma n+1 = \int_0^{\infty} e^{-x} x^n\, dx$

$$= \left[x^n\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty} - \int_0^{\infty} n x^{n-1}\left(\frac{e^{-x}}{-1}\right) dx$$

$$\lim_{x\to\infty} \frac{x^n}{e^x} \left(\frac{\infty}{\infty}\right)$$

$$+ n \int_0^{\infty} e^{-x} x^{n-1}\, dx$$

$$+ n \Gamma n$$

L'Hospital Rule

$$= \lim_{x\to\infty} \left(\frac{n x^{n-1}}{e^x}\right) \left(\frac{\infty}{\infty}\right) \text{form}$$

$$= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} \left(\frac{\infty}{\infty}\right)$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to\infty} \frac{n(n-1)x}{e^x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{n!}{e^x} \quad \left(\frac{n!}{\infty}\right) = 0$$

$$\Gamma(n+1) = 0 + n\Gamma(n) = n\Gamma(n)$$

$$\Gamma(5) = 4\Gamma(4) = 4(3\Gamma(3)) = 4\cdot 3(2\Gamma(2))$$

$$= 4\cdot 3\cdot 2\, 1$$

$$\Gamma(5) = 4!$$

$$\Gamma(n+1) = n! \quad (n \text{ is positive integer})$$

$$n! = \int_0^\infty e^{-x} x^n \, dx$$

Evaluation of gamma f^n (Γn)

1) If n is positive integer Then

$$\Gamma(n+1) = n!$$

2) $\Gamma(1) = 0! = 1$, $\Gamma(n+1) = n\Gamma(n) \Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n}$

$$\Gamma(0) = \frac{\Gamma(1)}{0} = \frac{1}{0} = \infty \text{ (not defined)}$$

negative integer $\Rightarrow \Gamma(n)$ is not defined.

$0<n<1$

3) positive fraction: $\Gamma(n+1) = n\Gamma(n)$

$$\rightarrow \Gamma\left(\frac{5}{3}\right) = \frac{2}{3}\Gamma\left(\frac{2}{3}\right) = \frac{2}{3}\left[-\frac{1}{3}\right]\Gamma\left(-\frac{1}{3}\right) = \frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\Gamma\left(\frac{4}{3}\right)$$

$$\left(\frac{5}{3}-1\right)$$

$$\rightarrow \Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right) = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{5\cdot 3\sqrt{\pi}}{2^3}$$

4) negative fractions, $\boxed{\Gamma(n) = \frac{\Gamma(n+1)}{n}}$ $\quad 0<n<1$ Stoping Criteria

$$\Gamma(-\tfrac{5}{2}) \quad \Gamma\left(-\frac{3}{2}\right) \quad \Gamma\left(-\frac{1}{2}\right) \quad \Gamma\left(\frac{1}{2}\right) \quad -\frac{2^3\sqrt{\pi}}{}$$

$$\Gamma\left(-\tfrac{5}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}\right)}{\left(-\frac{5}{2}\right)} = \frac{\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)} = \frac{\Gamma\frac{1}{2}}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = \frac{-2^3\sqrt{\pi}}{5\cdot 3}$$

Problems: Evaluate

(Type I) Substitute function which is in power of exp

1) If $I = \int_0^\infty x^2 e^{-x^8}\,dx$

$\underline{\text{Sol}^n}$: put $t = x^8$, $x = t^{1/8}$

$dx = \frac{1}{8} t^{-7/8}\,dt$

x	0	∞
t	0	∞

$$I = \int_0^\infty \left(t^{1/8}\right)^2 e^{-t}\left(\frac{1}{8} t^{-7/8}\,dt\right)$$

$$= \frac{1}{8}\int_0^\infty e^{-t}\, t^{\left(\frac{2}{8}-\frac{7}{8}\right)}\,dt = \frac{1}{8}\int_0^\infty e^{-t}\, t^{-5/8}\,dt$$

$-\frac{5}{8}+1 = \frac{3}{8}$

$\int_0^\infty e^{-t} t^n\,dt = \Gamma n+1$

$$I = \frac{1}{8}\Gamma\frac{3}{8}$$

$\int x^m e^{-\boxed{ax^k}}\,dx$

$\int e^{-\boxed{ax^k}}\,dx$

2) Evaluate $I = \int_0^\infty \sqrt{x}\,\boxed{e^{-x^2}}\,dx \cdot \int_0^\infty \frac{\boxed{e^{-x^2}}}{\sqrt{x}}\,dx$

$\underline{\text{Sol}^n}$: Let $I_1 = \int_0^\infty \sqrt{x}\, e^{-x^2}\,dx$ & $I_2 = \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx$

& $I = I_1 \times I_2$

By putting $x^2 = t$, $x = t^{1/2}$

$\therefore dx = \frac{1}{2} t^{-1/2}\,dt$

x	0	∞
t	0	∞

Consider $I_1 = \int_0^\infty x^{1/2} e^{-x^2}\,dx$

$$= \int_0^\infty t^{1/4}\, e^{-t}\left(\frac{1}{2} t^{-1/2}\,dt\right)$$

$\infty \quad 1 \quad (1/_4 - 1/_2)$

Consider $I_2 = \int_0^\infty e^{-x^2} x^{-1/2}\,dx$

$$I_2 = \int_0^\infty e^{-t}\, t^{-1/4}\left(\frac{1}{2} t^{-1/2}\,dt\right)$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{-3/4}\,dt$$

$$I_2 = \frac{1}{2}\Gamma\left(-\tfrac{3}{4}+1\right) = \frac{1}{2}\Gamma\frac{1}{4}$$

Since $I = I_1 \times I_2$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{\left(\frac{1}{4}-\frac{1}{2}\right)}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{-1/4}\, dt = \frac{1}{2}\Gamma\left(-\frac{1}{4}+1\right) = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$$

Since $I = I_1 \times I_2$

$$I = \frac{1}{2}\Gamma\left(\frac{3}{4}\right) \times \frac{1}{2}\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{4}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{4}\left(\sqrt{2}\,\pi\right)$$

$$I = \frac{\pi}{2\sqrt{2}}$$

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3) Evaluate $\displaystyle\int_0^\infty (x^2+4)\, e^{-2x^2}\, dx$

Sol^n: Let $I = \displaystyle\int_0^\infty \underline{(x^2+4)}\, e^{-\underline{2x^2}}\, dx = \int_0^\infty x^2 e^{-2x^2} dx + 4\int_0^\infty e^{-2x^2} dx$

$= I_1 + I_2$ (Assume) ✓

put $2x^2 = t$, $x = \sqrt{\frac{t}{2}}$

$x^2 = \frac{t}{2}$ $\quad dx = \frac{1}{\sqrt{2}}\,\frac{1}{2}\, t^{-1/2}\, dt$

x	0	∞
t	0	∞

$$I = \int_0^\infty \left(\frac{t}{2}\right) e^{-t}\left(\frac{1}{2\sqrt{2}}\,\boxed{t^{-1/2}}\right) dt + 4\int_0^\infty e^{-t}\left(\frac{1}{2\sqrt{2}}\, t^{-1/2}\right) dt$$

$$= \frac{1}{2\sqrt{2}}\left\{\frac{1}{2}\int_0^\infty e^{-t}\, t^{1/2}\, dt + 4\int_0^\infty e^{-t}\, t^{-1/2}\, dt\right\}$$

$$= \frac{1}{2\sqrt{2}}\left\{\frac{1}{2}\Gamma\left(\frac{3}{2}\right) + 4\,\Gamma\left(\frac{1}{2}\right)\right\} \qquad \left\{\Gamma(n+1) = n\Gamma(n)\right\}$$

$$= \frac{1}{2\sqrt{2}}\left\{\frac{1}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) + 4\,\Gamma\left(\frac{1}{2}\right)\right\} = \frac{\sqrt{\pi}}{2\sqrt{2}}\left\{\frac{1}{4}+4\right\}$$

$$I = \frac{17\sqrt{\pi}}{8\sqrt{2}} = \frac{17}{8}\sqrt{\frac{\pi}{2}}$$

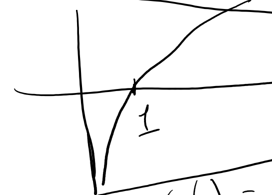
Some problems of Gamma function

Wednesday, March 31, 2021 12:04 PM

Type II $\int_0^1 x^m (\boxed{\log x})^n \, dx \longrightarrow$ identifier

$\int_0^1 x^m (1-x)^n \, dx = \beta(m,n)$

$\boxed{\log x = -t}$, $x = e^{-t}$ $dx = -e^{-t} dt$



x	0	1
t	∞	0

$\log(0) = -\infty = -t$

$\log\left(\frac{1}{x}\right) = \log 1 - \log x = -\log x$

$\log x = -t$, $-\log x = t$ | $\log\left(\frac{1}{x}\right) = t$

$I = \int_\infty^0 (e^{-t})^m (-t)^n (-e^{-t} dt) = \int_0^{} dx$

Problem 1) Evaluate $\int_0^1 x^3 \left[\log\left(\frac{1}{x}\right)\right]^4 dx$

Sol^n: Let $I = \int_0^1 x^3 \left[\log\left(\frac{1}{x}\right)\right]^4 dx$

put $\log x = -t$ | $x = e^{-t}$

$-\log x = t$ | $dx = -e^{-t} dt$

$\log\left(\frac{1}{x}\right) = t$

x	0	1
t	∞	0

$I = \int_\infty^0 (e^{-t})^3 [t]^4 (-e^{-t} dt)$

$= -\int_\infty^0 e^{-4t} t^4 dt$

$= \int_0^\infty e^{-4t} t^4 dt$

put $4t = u$ | $dt = \frac{du}{4}$

$t = \frac{u}{4}$

t	0	∞
u	0	∞

$I = \int_0^\infty e^{-u} \left(\frac{u}{4}\right)^4 \frac{du}{4}$

$= \frac{1}{4^5} \int_0^\infty e^{-u} u^4 du$

$= \frac{1}{4^5} \Gamma 5 = \frac{4!}{4^5} = \frac{3}{128}$

$[\Gamma n+1 = n!]$

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2) $I = \int_0^1 \frac{dx}{\sqrt{x \log\left(\frac{1}{x}\right)}} dx = \int_0^1 x^{-1/2} \left(\log\left(\frac{1}{x}\right)\right)^{-1/2} dx$

Type III $I = \int_0^\infty x^k \boxed{a^{-mx^n}} dx \longrightarrow e^{-t}$

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1) $I = \int_0^\infty x^2 5^{-4x^2} dx$

Sol^n: putting $5^{-4x^2} = e^{-t}$

$x^2 \log 5 = -t (\log e)$

Substituting,

$I = \int_0^\infty \left(\frac{t}{4\log 5}\right)(e^{-t})\left(\frac{1}{4\sqrt{\log 5}} \frac{dt}{\sqrt{t}}\right)$

Sol^n: putting, $s^{\ldots} = e^{\ldots}$

Apply log, $-4x^2 \log s = -t (\log e)$

$$x^2 = \frac{t}{4\log s} \quad \Big| \quad x = \frac{\sqrt{t}}{2\sqrt{\log s}}$$

$$\boxed{dx = \frac{1}{4\sqrt{\log s}} \frac{1}{\sqrt{t}}\, dt}$$

x	0	∞
t	0	∞

$$I = \int_0^{\ldots} \left(\frac{\ldots}{4\log s}\right)^{\ldots} \ldots$$

$$= \frac{1}{16(\log s)^{3/2}} \int_0^\infty e^{-t}\, t^{1/2}\, dt$$

$$= \frac{1}{16(\log s)^{3/2}} \Gamma\tfrac{3}{2}$$

$$= \frac{1}{16(\log s)^{3/2}} \frac{1}{2}\Gamma\tfrac{1}{2}$$

$$= \frac{\sqrt{\pi}}{32(\log s)^{3/2}}$$

Type IV] If integral contains $\sin bx / \cos bx$ then use IP / RP of e^{-ibx}

problem 1) $I = \int_0^\infty x^2 e^{-\sqrt{3}x} \sin x \, dx$ $\quad$ {consider $\sin x = IP$ of e^{ix}}

$$= \text{Img Part of} \left\{ \int_0^\infty x^2 e^{-\sqrt{3}x} e^{ix}\, dx \right\}$$

$$= IP \left\{ \int_0^\infty x^2 e^{-(\sqrt{3}-i)x}\, dx \right\}$$

Now put $(\sqrt{3}-i)x = t \quad \therefore x = \frac{t}{(\sqrt{3}-i)}$

$$dx = \frac{dt}{(\sqrt{3}-i)}$$

x	0	∞
t	0	∞

$$\therefore I = IP \left\{ \int_0^\infty \left(\frac{t}{\sqrt{3}-i}\right)^2 e^{-t} \left(\frac{dt}{\sqrt{3}-i}\right) \right\}$$

$$= IP \left\{ \frac{1}{(\sqrt{3}-i)^3} \int_0^\infty t^2 e^{-t}\, dt \right\}$$

$$I = IP \left\{ \frac{1}{(\sqrt{3}-i)^3} \Gamma 3 \right\} = IP \left\{ \frac{2!}{(\sqrt{3}-i)^3} \right\}$$

Consider

$$(\sqrt{3}-i)^3 = (\sqrt{3})^3 - 3(\sqrt{3})^2(i) + 3\sqrt{3}\, i^2 - i^3$$

$$= \cancel{(\sqrt{3})^3} - 9i - \cancel{(\sqrt{3})^3} + i$$

$$= -8i$$

$$\therefore I = IP \left\{ \frac{2}{-8i} \right\}$$

$$I = IP \left\{ \frac{1}{4} i \right\} =$$

$$\boxed{I = \frac{1}{4}}$$

H.W. 2) $I = \int^\infty x e^{-\sqrt{3}x} \sin x \, dx$ ✓ $\quad$ RP of

H.W.

2) $I = \int_0^\infty x e^{-\sqrt{3}x} \sin x \, dx$ ✓

3) Evaluate $I = \int_0^\infty \cos(ax^{1/n}) \, dx$

put $ax^{1/n} = t$ $\therefore x = \left(\frac{t}{a}\right)^n$

$\therefore dx = \frac{1}{a^n} n t^{n-1} \, dt$

x	0	∞
t	0	∞

$$I = \int_0^\infty \cos t \frac{1}{a^n} n t^{n-1} \, dt$$

$$= \frac{n}{a^n} \int_0^\infty t^{n-1} \cos t \, dt$$

$[\cos t = \text{R.P. of } e^{-it}]$

$$I = \text{RP of} \left\{ \frac{n}{a^n} \int_0^\infty t^{n-1} e^{-\boxed{it}} \, dt \right\}$$

put $it = v$

$dt = \frac{dv}{i}$

	0	∞
	0	∞

$$I = \text{RP} \left\{ \frac{n}{a^n} \int_0^\infty \frac{v^{n-1}}{i^{n-1}} e^{-v} \frac{dv}{i} \right\}$$

$$= \text{RP} \left\{ \frac{n}{a^n i^n} \int_0^\infty v^{n-1} e^{-v} \, dv \right\}$$

$$I = \text{RP} \left\{ \frac{n}{a^n i^n} \Gamma n \right\} \quad n > 0$$

$$I = \text{RP} \left\{ \frac{\Gamma(n+1)}{a^n i^n} \right\}$$ ✓