CHAPTER 1

Differential Equations of the First Order and of the First Degree

1. Introduction

Last year you have learnt how to form a differential equation from a given primitive and also to solve some simple differential equations.

In this chapter we shall learn some more methods of solving the differential equations of the first order and first degree *i.e.* the differential equation of the type $dy/dx = f(x, y)$. The method of solving a differential equation of the first order and first degree depends upon the type to which it belongs. These types are :

1. **Variables Separable Type.**
2. **Homogeneous Equation.**
3. **Non - homogeneous Equation.**
4. **Exact Differential Equation.**
5. **Linear Differential Equation.**
6. **Bernoulli's Differential Equation**

You have already studied how to solve the equations of the first three types. We shall briefly **review** them again for the sake or completeness and shall discuss in details the remaining three types.

Note ....

Students may omit the articlas 2, 3 and 4 **after reading** them. But they are advised to learn **carefully** the remaining three articals.

Example 1 : Solve $x - y\dfrac{dx}{dy} = a\left(x^2 + \dfrac{dx}{dy}\right)$.

Sol. : The equation can be written as,

$$x - ax^2 = (a + y)\frac{dx}{dy} \quad \therefore \frac{dy}{a+y} = \frac{dx}{x - ax^2}$$

$$\therefore \frac{dy}{a+y} = \left(\frac{1}{x} + \frac{a}{1-ax}\right)dx. \qquad \text{[By partial fractions]}$$

Integrating, we get,

$$\log(a + y) = \log x - \log(1 - ax) + \log c$$

$$\therefore \log(a + y) = \log\left(\frac{cx}{1-ax}\right) \quad \therefore a + y = \frac{cx}{1-ax}$$

$\therefore$ The solution is $(a + y)(1 - ax) = cx$.

3. Equations Homogeneous in x and y (Review)

An expression in x and y is said to be **homogeneous** (homo = same) if the degree of every term is the same *e.g.* $x^2y + x^3 + xy^2 - y^3$ is homogeneous of the third degree. $x^3y + x^2y^2 + xy^3 - x^4$ is homogeneous of the fourth degree. A homogeneous expression of degree n can be expressed, by taking x^n *common*, in the form $x^n f(y/x)$. For example, the homogeneous expressions given above can be written as

$$x^3\left\{\frac{y}{x} + 1 + \left(\frac{y}{x}\right)^2 - \left(\frac{y}{x}\right)^3\right\} \text{ and } x^4\left\{\frac{y}{x} + \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3 - 1\right\}.$$

Further, if $f_1(x, y)$ and $f_2(x, y)$ are functions of the same degree in x and y then $f_1(x, y) / f_2(x, y)$ can be put in the form $f(y/x)$ or $F(x/y)$.

Consider two homogeneous functions $f_1(x, y)$ and $f_2(x, y)$ where $f_1(x, y) = 2x^2y + 3xy^2$ and $f_2(x, y) = x^2y - xy^2$. Since both are homogeneous functions of degree three we can write their ratio as

$$\frac{f_1(x,y)}{f_2(x,y)} = \frac{x^3[2(y/x) + 3(y/x)^2]}{x^3[(y/x) - (y/x)^2]} = \frac{2(y/x) + 3(y/x)^2}{(y/x) - (y/x)^2} = F\left(\frac{y}{x}\right).$$

An equation of the form $\dfrac{dy}{dx} = \dfrac{f_1(x,y)}{f_2(x,y)}$.

where f_1 and f_2 are homogeneous expressions of the same degree in x and y can be reduced to variable separable type by putting $y = vx$. And then mere integration gives the solution of the equation.

If we put $v = y/x$ in the above illustration, we get

$$\frac{f_1(x,y)}{f_2(x,y)} = \frac{2v + 3v^2}{v - v^2} = F(v)$$

This is a function of v only.

To solve homogeneous equation in x and y

Let the given equation be $\dfrac{dy}{dx} = \dfrac{f_1(x,y)}{f_2(x,y)}$ .... (1)

where f_1 and f_2 are homogeneous functions in x and y of the same degree.

Now, we put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

Since f_1 and f_2 are homogeneous functions of the same degree, say n, in x and y we can take x^n common from both numerator and denominator. Cancelling it out the right hand side of (1), then becomes a function of v only, say $F(v)$. Hence, the equation (1) reduces to

$$v + x\frac{dv}{dx} = F(v) \quad \therefore\ x\frac{dv}{dx} = F(v) - v$$

$$\therefore\ \frac{dv}{F(v) - v} = \frac{dx}{x}$$ which is of variable separable type.

Integrating both sides, we get,

$$\int \frac{dv}{F(v - v)} = \log x + c$$

Resubstituting $v = y/x$, we get the required solution.

Example 1 : Solve $(x^3 + y^3)\dfrac{dy}{dx} = x^2 y$.

Sol. : Since the equation is homogeneous of the third degree in x and y.

Putting $y = vx$, $\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

$\therefore$ The equation reduces to

$$(x^3 + v^3 x^3)\left(v + x\frac{dv}{dx}\right) = x^3 v \quad \therefore\ (1 + v^3)\left(v + x\frac{dv}{dx}\right) = v$$

$$\therefore\ v + v^4 + x(1 + v^3)\frac{dv}{dx} = v \quad \therefore\ x(1 + v^3)\frac{dv}{dx} + v^4 = 0.$$

$$\therefore\ \frac{1 + v^3}{v^4}\,dv + \frac{dx}{x} = 0 \quad \therefore\ \frac{dv}{v^4} + \frac{dv}{v} + \frac{dx}{x} = 0.$$

Integrating, $-\dfrac{1}{3v^3} + \log v + \log x = -\log c$

$$\therefore\ \log v + \log x + \log c = \frac{1}{3v^3} \quad \therefore\ \log vxc = \frac{1}{3v^3}.$$

Putting $v = \dfrac{y}{x}$, $\log cy = \dfrac{x^3}{3y^3}$. $\quad \therefore$ The solution is $cy = e^{x^3/3y^3}$.

4. Non-homogeneous Linear Equations (Review)

An equation of the form $\boxed{\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}}$ .... (1)

is non-homogeneous linear in x and y.

or $(ax + by + c)\,dx - (a'x + b'y + c')\,dy = 0$

To solve the equation

To solve the above equation we put $x = x' + h$ and $y = y' + k$ and find the values of the constants h and k such that the equation becomes homogeneous. Then it can be solved by the previous method.

Putting $x = x' + h$, $y = y' + k$ since $dx = dx'$ and $dy = dy'$, the equation (1) reduces to

$$\frac{dy'}{dx'} = \frac{a(x'+h) + b(y'+k) + c}{a'(x'+h) + b'(y'+k) + c'} \quad \text{.................... (2)}$$

We choose h and k such that the equation (2) is homogeneous of the first degree. This will be so if the constants on the right hand side of (2) in numerator and denominator are zero *i.e.* if

$$ah + bk + c = 0 \qquad \text{and} \qquad a'h + b'k + c' = 0 \quad \text{.................... (3)}$$

For these value of h and k the equation (2) reduces to

$$\frac{dy'}{dx'} = \frac{ax'+by'}{a'x'+b'y'} \quad \text{.................... (4)}$$

But this is homogeneous and hence, by putting $y' = vx'$ and $\frac{dy'}{dx'} = v + x'\frac{dv}{dx'}$, the equation (4) becomes

$$v + x'\frac{dv}{dx'} = \frac{a+bv}{a'+b'v} \qquad i.e. \qquad x'\frac{dv}{dx'} = \left(\frac{a+bv}{a'+b'v}\right) - v$$

where the variables are separated and the solution can be obtained by integration.

The solution of the equation (1) is then obtained by resubstituting $v = y'/x'$ and $x' = x - h$, $y' = y - k$ in terms of x and y.

The method, however, fails if $\frac{a}{a'} = \frac{b}{b'}$

because the roots of the equation (3) are then indeterminate.

Suppose $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$ then the equation (1) reduces to

$$\frac{dy}{dx} = \frac{ax+by+c}{m(ax+by)+c'} \quad \text{.................... (5)}$$

On putting $ax + by = v$ and $a + b\frac{dy}{dx} = \frac{dv}{dx}$ the equation (5) reduces to

$$\frac{1}{b}\left\{\frac{dv}{dx} - a\right\} = \frac{v+c}{mv+c'} \qquad \therefore \quad \frac{dv}{dx} = b\left\{\frac{v+c}{mv+c'}\right\} + a$$

where the variables are separated and the solution can be obtained by mere integration. Resubstituting $v = ax + by$, we get the solution of the equation (1).

Type I : $\frac{a}{a'} \neq \frac{b}{b'}$

Example 1 : Solve $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$.

Sol. : We have $\frac{dy}{dx} = -\frac{4x+3y+1}{3x+2y+1}$.

Since $\frac{a}{a'} \neq \frac{b}{b'}$, we solve the equations

$4x + 3y + 1 = 0$ and $3x + 2y + 1 = 0$ and get $x = -1$ and $y = 1$.

Hence, putting $x = x' - 1$ and $y = y' + 1$, we get, $\frac{dy'}{dx'} = -\frac{4x'+3y'}{3x'+2y'}$

Now, put $y' = vx'$ and $\frac{dy'}{dx'} = v + x'\frac{dv}{dx'}$

$$\therefore v + x'\frac{dv}{dx'} = -\frac{4+3v}{3+2v} \quad \therefore x'\frac{dv}{dx'} = -\frac{2v^2+6v+4}{3+2v}$$

$$\frac{2v+3}{2(v^2+3v+2)}dv = -\frac{dx'}{x'}$$

Integrating $\frac{1}{2}\log(v^2+3v+2) = -\log x' + \log c'$

$$\log(v^2+3v+2) + 2\log x' = \log c.$$

$$\therefore \log(v^2+3v+2)x'^2 = \log c \quad \therefore y'^2 + 3x'y' + 2x'^2 = c$$

$$\therefore (y-1)^2 + 3(x+1)(y-1) + 2(x+1)^2 = c.$$

$\therefore$ The solution is $2x^2 + 3xy + y^2 + x + y = c'$. $[\, c' = c - 1 \,]$

Type II : $\frac{a}{a'} = \frac{b}{b'}$

Example 1 : Solve $(2x+4y+3)\frac{dy}{dx} = (2y+x+1)$.

Sol. : This is a non-homogeneous equation of the first degree in x and y.

Since, $\frac{a}{a'} = \frac{b}{b'}$, we put $x + 2y = v$ $\therefore 1 + 2\frac{dy}{dx} = \frac{dv}{dx}$

The given equation reduces to

$$(2v+3)\frac{1}{2}\left\{\frac{dv}{dx} - 1\right\} = v + 1 \quad \therefore (2v+3)\left\{\frac{dv}{dx} - 1\right\} = 2v + 2$$

$$\therefore \frac{dv}{dx} - 1 = \frac{2v+2}{2v+3} \quad \therefore \frac{dv}{dx} = 1 + \frac{2v+2}{2v+3} = \frac{2v+3+2v+2}{2v+3}$$

$$\therefore \frac{dv}{dx} = \frac{4v+5}{2v+3} \quad \therefore \frac{2v+3}{4v+5}dv = dx$$

$$\therefore \frac{1}{2}\left(1 + \frac{1}{4v+5}\right)dv = dx \qquad \text{[By actual division]}$$

Integrating, $\frac{1}{2}v + \frac{1}{8}\log(4v+5) = x + c$.

Putting $v = x + 2y$ $\therefore \frac{1}{2}(x+2y) + \frac{1}{8}\log(4x+8y+5) = x + c.$

$\therefore$ The solution is $\log(4x+8y+5) = 4x - 8y + c'$.

The articles 2, 3, 4 are meant for the purpose of **review** only.

5. Exact Differential Equations

Definition : A differential equation which is obtained from its primitive by differentiation only and without any operation of elimination or reduction is called an **exact differential equation.**

If $u = c$ where u is a function of x and y, is a primitive then

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0$$

is an exact differential equation. Thus, an exact differential equation is obtained from its primitive by equating its total differential to zero. For example,

If $u = x^2 + y^2 = c$ then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 2x\,dx + 2y\,dy$.

Equating $du = 0$, we get the equation $x\,dx + y\,dy = 0$ which is exact.

We shall now prove that if the equation $M\,dx + N\,dy = 0$ is exact, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and conversely if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then $M\,dx + N\,dy = 0$ is exact. In other words we prove that :

The *Necessary and Sufficient Condition for M dx + N dy = 0 to be exact is*

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Proof : (a) The Condition is necessary

Given $M\,dx + N\,dy = 0$ is exact, to prove that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Since, the given equation is exact, by definition, it is obtained by putting total differential du of some function u of x and y to zero.

Hence, the l.h.s. of the equation is du i.e. $du = M\,dx + N\,dy$.

But $du \equiv \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$\therefore\ M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$ $\qquad \therefore\ \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x\,\partial y}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y\,\partial x}$

But $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$ hence, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

(b) The condition is sufficient

Given $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ *to prove that $M\,dx + N\,dy = 0$ is an exact differential equation.*

Let $\int M\,dx = V$ then $\frac{\partial V}{\partial x} = M$ and $\frac{\partial^2 V}{\partial x\,\partial y} = \frac{\partial M}{\partial y}$.

But $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (given) $\quad \therefore\ \frac{\partial^2 V}{\partial x\,\partial y} = \frac{\partial N}{\partial x} \qquad \therefore\ \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right)$

$\therefore$ Integrating, we get

$$N = \frac{\partial V}{\partial y} + f(y) = \frac{\partial V}{\partial y} + \phi'(y) \qquad [\text{where } \phi'(y) = f(y)]$$

where, $\Phi'(y)$ is a function of y alone. (a constant with respect to partial integrating)

Hence, putting the values of M and N

$$M\,dx + N\,dy = \frac{\partial V}{\partial x}dx + \left\{\frac{\partial V}{\partial y} + \Phi'(y)\right\}dy$$

$$= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \Phi'(y)\,dy$$

$$\therefore \quad M\,dx + N\,dy = d[V + \Phi(y)]$$

$\therefore$ $M\,dx + N\,dy$ is an exact differential equation.

(c) Rule for finding the solution

If $M\,dx + N\,dy = 0$ is an exact differential equation, it must have been obtained by equating to zero, total differential du of some function u of x and y.

Since, $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ and $M = \frac{\partial u}{\partial x}$, $N = \frac{\partial u}{\partial y}$ by integrating $M\,dx$ w.r.t. x, we get part of u. Its partial derivative w.r.t. y is in N. The terms in u which are free from x do not appear in M but they appear in N. Hence, the remaining part of u is obtained by integrating w.r.t. y those terms in N which are free from x. Thus, we get,

The Rule 1

Integrate $M\,dx$ w.r.t. x treating y constant. Integrate only those terms in N which are free from x w.r.t. y. Equate the sum to a constant. This is the solution.

In symbols, $\boxed{\int M\,dx \text{ (treating } y \text{ constant)} + \int (\text{Terms in } N \text{ from } x)\,dy = c}$

Alternatively we may first integrate N with respect to y treating x constant.

The Rule 2

Integrate $N\,dy$ w.r.t. y treating x constant. Integrate only those terms in M which are from y w.r.t. x. Equate the sum to a constant. This is the solution . (See Ex. 1 below)

In symbols, $\int N\,dy \text{ (treating } x \text{ constant)} + \int (\text{Terms in } M \text{ free from } y)\,dx = c.$

(See solved Ex. 7, page 1-19.)

Solved Examples : Class (a) : 4 Marks

Example 1 (a) : Solve $(\tan y + x)\,dx + (x \sec^2 y - 3y)\,dy = 0.$

Sol. : Here, $M = \tan y + x$, $N = x \sec^2 y - 3y$

Now $\frac{\partial M}{\partial y} = \sec^2 y = \frac{\partial N}{\partial x}$ $\quad\therefore$ The equation is exact.

Now, $\int M\,dx = \int (\tan y + x)\,dx = x \tan y + \frac{x^2}{2}$

And $\int (\text{terms in } N \text{ free from } x)\,dy = \int -3y\,dy = -\frac{3y^2}{2}$

$\therefore$ The solution is $x \tan y + \frac{x^2}{2} - \frac{3y^2}{2} = c'$ $\quad$ *i.e.* $\quad 2x \tan y + x^2 - 3y^2 = c.$

Alternatively :

$$\int N\,dy \text{ (treating } x \text{ constant)} = \int (x\sec^2 y - 3y)\,dy = x\tan y - \frac{3y^2}{2}$$

$$\int (\text{terms in } M \text{ free from } y)\,dx = \int x\,dx = \frac{x^2}{2}.$$

$\therefore$ The solution is $x\tan y - \dfrac{3y^2}{2} + \dfrac{x^2}{2} = c.$

Example 2 (a) : Solve $x\,dx + y\,dy = \dfrac{a(x\,dy - y\,dx)}{x^2+y^2}$. **(M.U. 1997, 99)**

Sol. : The equation can be written as

$$\left\{x + \frac{ay}{x^2+y^2}\right\}dx + \left\{y - \frac{ax}{x^2+y^2}\right\}dy = 0$$

$$\therefore\ M = x + \frac{ay}{x^2+y^2} \qquad \therefore\ \frac{\partial M}{\partial y} = \frac{a}{x^2+y^2} - \frac{2ay^2}{(x^2+y^2)^2} = \frac{ax^2 - ay^2}{(x^2+y^2)}$$

$$N = y - \frac{ax}{x^2+y^2} \qquad \therefore\ \frac{\partial N}{\partial x} = -\frac{a}{x^2+y^2} + \frac{2ax^2}{(x^2+y^2)^2} = \frac{ax^2 - ay^2}{(x^2+y^2)^2}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

Now, $\displaystyle\int M\,dx = \int x\,dx + ay\int \frac{dx}{x^2+y^2} = \frac{x^2}{2} + ay\cdot\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) = \frac{x^2}{2} + a\cdot\tan^{-1}\left(\frac{x}{y}\right)$

And $\displaystyle\int (\text{terms in } N \text{ free from } x)\,dy = \int y\,dy = \frac{y^2}{2}$

$\therefore$ The solution is $\dfrac{x^2}{2} + \dfrac{y^2}{2} + a\tan^{-1}\left(\dfrac{x}{y}\right) = c'$ i.e. $x^2 + y^2 + 2a\tan^{-1}\left(\dfrac{x}{y}\right) = c.$

Example 3 (a) : Solve $[1 + \log(xy)]\,dx + \left(1 + \dfrac{x}{y}\right)dy = 0.$

Sol. : Here, $M = 1 + \log(xy)$; $N = 1 + \dfrac{x}{y}.$

$$\therefore\ \frac{\partial M}{\partial y} = \frac{1}{xy}\cdot x = \frac{1}{y},\quad \frac{\partial N}{\partial x} = \frac{1}{y} \qquad \therefore\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The equation is exact.

$$\therefore\ \int M\,dx = \int (1 + \log xy)\,dx = x + \int \log(xy)\,dx$$

Now, $\displaystyle\int \log(xy)\cdot 1\cdot dx = \log(xy)\cdot x - \int x\cdot\frac{1}{xy}\cdot y\,dx$ [Integrating by parts]

$$\therefore\ \int \log(xy)\cdot 1\cdot dx = x\log(xy) - \int dx = x\log(xy) - x$$

And $\displaystyle\int (\text{terms in } N \text{ free from } x)\,dy = \int 1\cdot dy = y$

$\therefore$ The solution is $x + x\log(xy) - x + y = c$ i.e. $y + x\log(xy) = c.$

Example 4 (a) : Solve $\dfrac{dy}{dx}+\dfrac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0.$ **(M.U. 1989, 92, 2003, 12)**

Sol. : We have, $(y\cos x+\sin y+y)\,dx+(\sin x+x\cos y+x)\,dy=0$

$\therefore\ M=y\cos x+\sin y+y\ ;\ N=\sin x+x\cos y+x$

$$\frac{\partial M}{\partial y}=\cos x+\cos y+1\ ;\ \frac{\partial N}{\partial x}=\cos x+\cos y+1$$

$\therefore\ \dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$. The equation is exact.

Now, $\int M\,dx=\int(y\cos x+\sin y+y)\,dx$

$\qquad = y\sin x+x\sin y+xy$

And $\int(\text{terms in } N \text{ free from } x)\,dy=0$

$\therefore$ The solution is $y\sin x+x\ \sin y+xy=c.$

Example 5 (a) : Solve $2\left(1+x^2\sqrt{y}\right)y\,dx+\left(x^2\sqrt{y}+2\right)x\,dy=0.$ **(M.U. 1991)**

Sol. : Here, $M=2y+2x^2y^{3/2}\ ;\ N=x^3\sqrt{y}+2x.$

$\therefore\ \dfrac{\partial M}{\partial y}=2+3x^2y^{1/2},\quad \dfrac{\partial N}{\partial x}=3x^2\sqrt{y}+2$

$\because\ \dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$. The equation is exact.

Now, $\int M\,dx=\int(2y+2x^2y^{3/2})\,dx=2xy+\dfrac{2}{3}x^3y^{3/2}$

And $\int(\text{terms in } N \text{ free from } x)\,dy=\int 0\cdot dy=0$

$\therefore$ The solution is $2xy+\dfrac{2}{3}x^3y^{3/2}=c.$

Example 6 (a) : Solve $\dfrac{y}{x^2}\cos\left(\dfrac{y}{x}\right)dx-\dfrac{1}{x}\cos\left(\dfrac{y}{x}\right)dy+2x\,dx=0.$ **(M.U. 2000, 04)**

Sol. : We have $\left[2x+\dfrac{y}{x^2}\cos\left(\dfrac{y}{x}\right)\right]dx+\left[-\dfrac{1}{x}\cos\dfrac{y}{x}\right]dy=0$

$\therefore\ M=2x+\dfrac{y}{x^2}\cos\dfrac{y}{x}$ and $N=-\dfrac{1}{x}\cos\dfrac{y}{x}$

$$\therefore\ \frac{\partial M}{\partial y}=\frac{1}{x^2}\cos\left(\frac{y}{x}\right)-\frac{y}{x^2}\sin\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

$$\frac{\partial N}{\partial x}=\frac{1}{x^2}\cos\left(\frac{y}{x}\right)-\frac{1}{x}\sin\left(\frac{y}{x}\right)\cdot\frac{y}{x^2}$$

Since $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, the equation is exact.

$\therefore\ \int M\,dx=\int\left(2x+\dfrac{y}{x^2}\cos\dfrac{y}{x}\right)dx$

$$\therefore \int M\,dx = \int 2x\,dx + \int \cos\left(\frac{y}{x}\right)\cdot\frac{y}{x^2}\,dx = x^2 + I_2$$

For I_2, put $\frac{y}{x} = t$, $-\frac{y}{x^2}\,dx = dt$

$$\therefore \int M\,dx = x^2 - \int \cos t\,dt = x^2 - \sin t = x^2 - \sin\left(\frac{y}{x}\right)$$

$$\int (\text{Term in } N \text{ free from } x)\,dy = 0$$

$\therefore$ The solution is $x^2 - \sin\frac{y}{x} = c$.

Example 7 (a) : Solve $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$, given $y(0) = 4$. **(M.U. 2000, 15)**

Sol. : Here $M = 1 + e^{x/y}$, $N = e^{x/y}\left(1 - \frac{x}{y}\right)$

$$\therefore \frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right)$$

$$\frac{\partial N}{\partial x} = e^{x/y}\cdot\frac{1}{y}\left(1 - \frac{x}{y}\right) - e^{x/y}\left(\frac{1}{y}\right) = e^{x/y}\left(-\frac{x}{y^2}\right)$$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

$$\therefore \int M\,dx = \int \left(1 + e^{x/y}\right)dx = x + ye^{x/y}$$

$$\int (\text{Term is } N \text{ free from } x)\,dy = \int 0\,dy = 0$$

$\therefore$ The solution is $x + ye^{x/y} = c$.

By data when $x = 0$, $y = 4$ $\quad\therefore\ 4 = c$.

The particular solution is $x + ye^{x/y} = 4$.

Remark ....

The above differential equation can also be solved by putting $x = vy$ and then by separating the variables as in § 3, page 1-2.

Example 8 (a) : Solve $(x - 2e^y)\,dy + (y + x\sin x)\,dx = 0$. **(M.U. 2013)**

Sol. : We have $M = y + x\sin x$ and $N = x - 2e^y$

$$\therefore \frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$$

$$\therefore \int M\,dx = \int (y + x\sin x)\,dx$$

$$= yx + x(-\cos x) - \int (-\cos x)\cdot 1\cdot dx \qquad [\text{ Integration by parts }]$$

$$= xy - x\cos x + \sin x$$

$$\int (\text{Term is } N \text{ free from } x)\,dy = \int -2e^y\,dy = -2e^y$$

$\therefore$ The solution is $xy - x\cos x + \sin x - 2e^y = c$.

Example 9 (a) : Solve $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+(x+\log x-x\sin y)\,dy=0.$

(M.U. 1998, 2006, 15)

Sol. : We have $M = y\left(1+\frac{1}{x}\right)+\cos y$ and $N = x+\log x - x\sin y$

$$\therefore \quad \frac{\partial M}{\partial y} = x+\frac{1}{x}-\sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = 1+\frac{1}{x}-\sin y$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence, the given equation is exact.

Now, $\int M\,dx = \int\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dy = y(x+\log x)+x\cos y$

And $\int(\text{Term is } N \text{ free from } x)\,dy = \int 0\cdot dy = 0$

$\therefore$ The solution is $y(x+\log x)+x\cos y = 0.$

Example 10 (a) : Solve $(x^2-4xy-2y^2)\,dx+(y^2-4xy-2x^2)\,dy=0.$ **(M.U. 2014)**

Sol. : We have $M = x^2-4xy-2y^2$ and $N = y^2-4xy-2x^2$

$$\therefore \quad \frac{\partial M}{\partial y} = -4x-4y \quad \text{and} \quad \frac{\partial N}{\partial x} = -4y-4x$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence, the given equation is exact.

Now, $\int M\,dx = \int(x^2-4xy-2y^2)\,dx = \frac{x^3}{3}-2x^2y-2y^2x$

And $\int(\text{Term is } N \text{ free from } x)\,dy = \int y^2\cdot dy = \frac{y^3}{3}$

$\therefore$ The solution is

$$\frac{x^3}{3}-2x^2y-2y^2x+\frac{y^3}{3} = c' \qquad \therefore \quad x^3+y^3-6x^2y-6y^2x = c.$$

Remark ....

The above differential equation can also be solved by putting $y = vx$ and then by separating the variables as in § 3, page 1-2.

Example 11 : Solve $\left(x+y\sqrt{1-x^2y^2}\right)dx+\left(x\sqrt{1-x^2y^2}-y\right)dy=0.$

Sol. : We have $M = x+y\sqrt{1-x^2y^2}$ and $N = x\sqrt{1-x^2y^2}-y$

$$\therefore \quad \frac{\partial M}{\partial y} = 1\cdot\sqrt{1-x^2y^2}+y\left[\frac{-x^2y}{\sqrt{1-x^2y^2}}\right]$$

$$\therefore \quad \frac{\partial N}{\partial x} = 1\cdot\sqrt{1-x^2y^2}+x\left[\frac{-xy}{\sqrt{1-x^2y^2}}\right]$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence, the given equation is exact.

Now, $\int M\,dx = \int\left(x + y\sqrt{1 - x^2y^2}\right)dx = \frac{x^2}{2} + y^2\int\sqrt{\frac{1}{y^2} - x^2}\,dx$

$$\left[\text{By } \int\sqrt{a^2 - x^2}\cdot dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$\therefore \quad \int M\,dx = \frac{x^2}{2} + y^2\left[\frac{x}{2}\sqrt{\frac{1}{y^2} - x^2} + \frac{1}{y^2}\cdot\frac{1}{2}\sin^{-1}\left(\frac{x}{1/y}\right)\right]$$

$$= \frac{x^2}{2} + \frac{xy}{2}\sqrt{1 - x^2y^2} + \frac{1}{2}\sin^{-1}(xy)$$

And $\int(\text{Term is } N \text{ free from } x)\,dy = \int -y\cdot dy = -\frac{y^2}{2}$

$\therefore$ The solution is $\frac{x^2 - y^2}{2} + \frac{xy}{2}\sqrt{1 - x^2y^2} + \frac{1}{2}\sin^{-1}(xy) = c.$

Example 12 : Solve $\left[x\sqrt{x^2 + y^2} - y\right]dx + \left[y\sqrt{x^2 + y^2} - x\right]dy = 0.$ **(M.U. 2014)**

Sol. : We have $M = x\sqrt{x^2 + y^2} - y$ and $N = y\sqrt{x^2 + y^2} - x$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{x\cdot y}{\sqrt{x^2 + y^2}} - 1 \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{y\cdot x}{\sqrt{x^2 + y^2}} - 1$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence, the given equation is exact.

Now, $\int M\,dx = \int\left(x\sqrt{x^2 + y^2} - y\right)dx$

$\therefore$ To find $\int x\sqrt{x^2 + y^2}\,dx$, we put $x^2 + y^2 = t$. $\quad \therefore \; 2x\,dx = dt.$

$$\therefore \quad \int x\sqrt{x^2 + y^2}\cdot dx = \int\frac{1}{2}t^{1/2}\,dt = \frac{1}{2}\cdot\frac{t^{3/2}}{3/2} = \frac{1}{3}t^{3/2} = \frac{1}{3}(x^2 + y^2)^{3/2}$$

Hence, $\int M\,dx = \frac{1}{3}(x^2 + y^2)^{3/2} - xy$

And $\int(\text{Term is } N \text{ free from } x)\,dy = \int 0\cdot dy = 0$

$\therefore$ The complete solution is

$$\frac{1}{3}(x^2 + y^2)^{3/2} - xy = c' \qquad \therefore \quad (x^2 + y^2)^{3/2} - 3xy = c.$$

Note ....

Some **non-homogeneous linear** and **some homogeneous** differential equations happen to be exact and can be solved by using the method of § 5. For instance Ex. 5 below is non-homogeneous as well as exact and Ex. 10, page 1-11 is homogeneous as well as exact.

EXERCISE - I

Solve the following equations : Class (a) : 4 Marks

1. $(\sin x \cos y + e^{2x})\, dx + (\cos x \sin y + \tan y)\, dy = 0$
2. $(2x^2 + 3y^2 - 7)\, x\, dx + (3x^2 + 2y^2 - 8)\, y\, dy = 0$ **(M.U. 1991)**
3. $\dfrac{dy}{dx} = \dfrac{y+1}{(y+2)\, e^y - x}$ **(M.U. 2010)**
4. $(y^2 e^{xy^2} + 4x^3)\, dx + (2xy\, e^{xy^2} - 3y^2)\, dy = 0$ **(M.U. 1997, 99)**
5. $(4x + 3y - 4)\, dx + (3x - 7y - 3)\, dy = 0.$
6. $\dfrac{dy}{dx} = \dfrac{y}{2y \log y + y - x}$
7. $\dfrac{dy}{dx} = \dfrac{a^2 - 2xy - y^2}{(x+y)^2}$
8. $\dfrac{y\, dx - x\, dy}{(x+y)^2} = \dfrac{dx}{2\sqrt{1-x^2}}$ **(M.U. 1989)**
9. $(ye^{xy} - 2y^3)\, dx + (xe^{xy} - 6xy^2 - 2y)\, dy = 0$
10. $(x^2 - x \tan^2 y + \sec^2 y)\, dy = (\tan y - 2xy - y)\, dx$ **(M.U. 2001)**
11. $\left(\dfrac{y^2}{1+x^2} - 2y\right) dx + (2y \tan^{-1} x - 3x + \sin hy)\, dy = 0$
12. $[\, y \sin(xy) + xy^2 \cos(xy)\,]\, dx + [\, x \sin(xy) + x^2 y \cos(xy)\,]\, dy = 0$ **(M.U. 1998, 99, 2003)**
13. $(2x^2 y + 4x^3 - 12\, xy^2 + 3y^2 + xe^y + e^{2x})\, dy + (12\, x^2 y + 2\, xy^2 + 4\, x^3 - 4y^3 + 2\, ye^{2x} + e^y)\, dx = 0$
14. $\left[\dfrac{y^2}{(y-x)^2} - \dfrac{1}{x}\right] dx + \left[\dfrac{1}{y} - \dfrac{x^2}{(x-y)^2}\right] dy = 0$
15. $(2xy \cos x^2 - 2xy + 1)\, dx + (\sin x^2 - x^2)\, dy = 0.$

[**Ans :** (1) $-\cos x \cos y + \dfrac{1}{2} e^{2x} + \log \sec y = c$ (2) $2x^4 + 2y^4 + 3x^2y^2 - (7x^2 + 8y^2) = c$

(3) $(y+1)(e^y - x) = c$ (4) $e^{xy^2} + x^4 - y^3 = c$

(5) $4x^2 + 6xy - 8x - 7y^2 - 6y = c$ (6) $y^2 \log y - xy = c$

(7) $x^2 y + xy^2 + \dfrac{y^3}{3} - a^2 x = c$ (8) $\dfrac{y}{x-y} + \dfrac{1}{2} \sin^{-1} x = c$

(9) $e^{xy} - 2xy^3 - y^2 = c$ (10) $x \tan y - x^2 y - xy - \tan y = c$

(11) $y^2 \tan^{-1} x - 2xy + \cos hy = c$ (12) $xy \sin(xy) = c$

(13) $4x^3 y + x^2 y^2 + x^4 - 4xy^3 + ye^{2x} + xe^y + y^3 = c$

(14) $\dfrac{y^2}{y-x} + \log\left(\dfrac{y}{x}\right) = c$ (18) $y \sin x^2 - x^2 y + x = c.$]

6. Equations Reducible to Exact (Integrating Factor)

Sometimes a given equation is not exact but is rendered exact if it is multiplied by a suitable factor. Such a factor is called an **integrating factor**.

For instance, the equation $y\,dx - x\,dy = 0$ is not exact. If we multiply it by $1/y^2$, then it becomes

$$\frac{1}{y}dx - \frac{x}{y^2}dy = 0 \qquad \text{.............. (1)}$$

Since, $\frac{\partial M}{\partial y} = -\frac{1}{y^2} = \frac{\partial N}{\partial x}$, it is exact and $M\,dx = \int \frac{1}{y}dx = \frac{x}{y}$, $\int N\,dy = 0$.

Hence, its solution is $\frac{x}{y} = c$.

Again if we multiply (1) by $\frac{1}{x^2}$, then it becomes $\frac{y}{x^2}dx - \frac{dy}{x} = 0$.

Since, $\frac{\partial M}{\partial y} = \frac{1}{x^2} = \frac{\partial N}{\partial x}$, it is exact and its solution is $-\frac{y}{x} = c'$. *i.e.* $\frac{x}{y} = -\frac{1}{c'} = c$ as before.

Also, if we multiply (1) by $\frac{1}{xy}$, then it becomes $\frac{dx}{x} - \frac{dy}{y} = 0$.

Since, $\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$, it is exact and its solution is

$$\log x - \log y = \log c. \quad i.e. \quad \log\frac{x}{y} = \log c \quad i.e. \quad \frac{x}{y} = c \text{ as before.}$$

Thus, $\frac{1}{y}, \frac{1}{y^2}, \frac{1}{xy}$ are the integrating factors of the equation (1). *i.e.*, there can be move than one integrating factors.

Integrating factor by inspection

Sometimes integrating factor can be obtained by inspection. Careful study of the following results will help you in this direction.

(1) If $u = xy$, $du = x\,dy + y\,dx$

(2) If $u = x^2y^2$, $du = 2xy^2\,dx + 2x^2y\,dy$

(3) If $u = \frac{x}{y}$, $du = \frac{y\,dx - x\,dy}{y^2}$

(4) If $u = \frac{y}{x}$, $du = \frac{x\,dy - y\,dx}{x^2}$

(5) If $u = x^2y$, $du = 2xy\,dx + x^2\,dy$

(6) If $u = xy^2$, $du = y^2\,dx + 2xy\,dy$

(7) If $u = \tan^{-1}\frac{x}{y}$, $du = \frac{y\,dx - x\,dy}{x^2+y^2}$

(8) If $u = \tan^{-1}\frac{y}{x}$, $du = \frac{x\,dy - y\,dx}{x^2+y^2}$

(9) If $u = \log(x-y)$, $du = \frac{dx - dy}{x-y}$

(10) If $u = \log(x+y)$, $du = \frac{dx + dy}{x+y}$

(11) If $u = \frac{y^2}{x}$, $du = \frac{2xy\,dy - y^2dx}{x^2}$

(12) If $u = \frac{x^2}{y}$, $du = \frac{2xy\,dx - x^2dy}{y^2}$

(13) If $u = \frac{1}{xy}$, $du = -\frac{(x\,dy + y\,dx)}{x^2y^2}$.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $(x+y)(dx-dy) = dx+dy$.

Sol. : From the result (10) we see that if we divide by $(x+y)$, the equation will be exact.

Dividing by $(x+y)$,

$$dx - dy = \frac{dx+dy}{x+y} \quad i.e. \quad d(x) - d(y) = d[\log(x+y)]$$

$\therefore$ Its solution is $x - y = \log(x+y) + c$.

Example 2 (b) : Solve $y\,dx - x\,dy + (1+x^2)\,dx + x^2 \sin y\,dy = 0$

Sol. : From the last term we see that we must get rid of the term x^2. Hence, dividing by x^2

$$\frac{y\,dx - x\,dy}{x^2} + \left(\frac{1}{x^2}+1\right)dx + \sin y\,dy = 0$$

From the result (4) we see that, the equation is exact.

We have, $d\left(-\dfrac{y}{x}\right) + d\left[-\dfrac{1}{x}+x\right] + d(-\cos y) = 0$

$\therefore$ The solution is $-\dfrac{y}{x} - \dfrac{1}{x} + x - \cos y = c$.

i.e. $y + 1 - x^2 + x\cos y = cx$.

Example 3 (b) : Solve $y(2xy + e^x)\,dx - e^x\,dy = 0$. **(M.U. 2014)**

Sol. : In order to render the term $2xy^2\,dx$ exact we must divide by y^2.

Hence, $2x\,dx + \dfrac{y \cdot e^x dx - e^x dy}{y^2} = 0 \quad \therefore \; d(x^2) + d\left(\dfrac{e^x}{y}\right) = 0$

This is exact and its solution is

$$x^2 + \frac{e^x}{y} = c \quad i.e. \quad x^2 y + e^x = cy.$$

Example 4 (b) : Solve $y - x\dfrac{dy}{dx} = x + y\dfrac{dy}{dx}$.

Sol. : The equation can be written as $y\,dx - x\,dy = x\,dx + y\,dy$.

Dividing throughout by $x^2 + y^2$, we get,

$$\frac{y\,dx - x\,dy}{x^2+y^2} = \frac{x\,dx + y\,dy}{x^2+y^2} \qquad \therefore \; d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = d\left[\log\sqrt{x^2+y^2}\right]$$

Hence, its solution is $\tan^{-1}\left(\dfrac{y}{x}\right) = \log\sqrt{x^2+y^2} + c$.

Example 5 (b) : Solve $(y\log x - 1)\,y\,dx = x\,dy$.

Sol. : The equation can be written as $y^2 \log x\,dx = x\,dy + y\,dx$

Dividing by x^2y^2, we get, $\dfrac{\log x}{x^2}\,dx = \dfrac{x\,dy + y\,dx}{x^2y^2}$ which is exact.

Now, integrating by parts

$$\int \frac{\log x}{x^2}\,dx = \log x \cdot \left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right) \cdot \frac{1}{x}\,dx$$

$$= -\frac{1}{x}\log x - \int -\frac{1}{x^2}\,dx = -\frac{1}{x}\log x - \frac{1}{x}$$

$$\therefore\ d\left[-\frac{1}{x}\log x - \frac{1}{x}\right] = d\left[-\frac{1}{xy}\right]$$

Hence, by integration,

$$-\frac{1}{x} \cdot \log x - \frac{1}{x} = -\frac{1}{xy} + c$$

$\therefore\ y(\log x + 1) = 1 - cxy$ is the solution.

Example 6 (b) : Solve $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0$.

Sol. : The equation can be written as

$$y\,dx + x\,dy + xy^2\,dx - x^2y\,dy = 0$$

Dividing by x^2y^2, we get,

$$\frac{x\,dy + y\,dx}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0 \qquad \therefore\ d\left[-\frac{1}{xy}\right] + d[\log x] - d[\log y] = 0$$

Integrating, $-\dfrac{1}{xy} + \log x - \log y = \log c$

$\therefore\ \log\dfrac{x}{cy} = \dfrac{1}{xy} \qquad \therefore\ x = c\,ye^{1/(xy)}$ is the comlete solution.

EXERCISE - II

Solve the following equations : Class (b) : 6 Marks

1. $x\,dy - y\,dx + 2x^3dx = 0$
2. $y(1+x)\,dx + x(1-y)\,dy = 0$
3. $x\,dy - y\,dx = (x^2+y^2)(dx+dy)$
4. $(1+2xy)\,y\,dx + (1-2xy)\,x\,dy = 0$
5. $(1-xy+x^2y^2)\,dx + (x^3y - x^2)\,dy = 0$
6. $x\,dy - y\,dx = a(x^2+y^2)\,dy$
7. $a(x\,dy + 2y\,dx) = xy\,dy$.

[**Ans. :** (1) I.F. $\dfrac{1}{x^2}$; $y + x^3 = cx$ (2) I.F. $\dfrac{1}{xy}$; $(\log xy) + x - y = c$

(3) I.F. $\dfrac{1}{x^2+y^2}$; $\tan^{-1}\left(\dfrac{y}{x}\right) = x + y + c$ (4) I.F. $\dfrac{1}{x^2y^2}$; $\log\left(\dfrac{x}{y}\right)^2 = c + \dfrac{1}{xy}$

(5) I.F. $\dfrac{1}{x}$; $2\log x - 2xy + x^2y^2 = c$ (6) I.F. $\dfrac{1}{x^2+y^2}$; $\tan^{-1}\left(\dfrac{y}{x}\right) = ay + c$

(7) I.F. $\dfrac{1}{xy}$; $a\log(x^2y) = y + c$.]

Now, integrating by parts

$$\int \frac{\log x}{x^2}\,dx = \log x \cdot \left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right) \cdot \frac{1}{x}\,dx$$

$$= -\frac{1}{x}\log x - \int -\frac{1}{x^2}\,dx = -\frac{1}{x}\log x - \frac{1}{x}$$

$$\therefore\ d\left[-\frac{1}{x}\log x - \frac{1}{x}\right] = d\left[-\frac{1}{xy}\right]$$

Hence, by integration,

$$-\frac{1}{x}\cdot\log x - \frac{1}{x} = -\frac{1}{xy} + c$$

$\therefore\ y(\log x + 1) = 1 - cxy$ is the solution.

Example 6 (b) : Solve $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$.

Sol. : The equation can be written as

$$y\,dx + x\,dy + xy^2\,dx - x^2y\,dy = 0$$

Dividing by x^2y^2, we get,

$$\frac{x\,dy + y\,dx}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0 \qquad \therefore\ d\left[-\frac{1}{xy}\right] + d[\log x] - d[\log y] = 0$$

Integrating, $-\dfrac{1}{xy} + \log x - \log y = \log c$

$\therefore\ \log\dfrac{x}{cy} = \dfrac{1}{xy} \qquad \therefore\ x = c\,y e^{1/(xy)}$ is the comlete solution.

EXERCISE - II

Solve the following equations : Class (b) : 6 Marks

1. $x\,dy - y\,dx + 2x^3dx = 0$
2. $y(1 + x)\,dx + x(1 - y)\,dy = 0$
3. $x\,dy - y\,dx = (x^2 + y^2)(dx + dy)$
4. $(1 + 2xy)\,y\,dx + (1 - 2xy)\,x\,dy = 0$
5. $(1 - xy + x^2y^2)\,dx + (x^3y - x^2)\,dy = 0$
6. $x\,dy - y\,dx = a(x^2 + y^2)\,dy$
7. $a(x\,dy + 2y\,dx) = xy\,dy.$

[**Ans. :** (1) I.F. $\dfrac{1}{x^2}$; $y + x^3 = cx$ (2) I.F. $\dfrac{1}{xy}$; $(\log xy) + x - y = c$

(3) I.F. $\dfrac{1}{x^2 + y^2}$; $\tan^{-1}\left(\dfrac{y}{x}\right) = x + y + c$ (4) I.F. $\dfrac{1}{x^2y^2}$; $\log\left(\dfrac{x}{y}\right)^2 = c + \dfrac{1}{xy}$

(5) I.F. $\dfrac{1}{x}$; $2\log x - 2xy + x^2y^2 = c$ (6) I.F. $\dfrac{1}{x^2 + y^2}$; $\tan^{-1}\left(\dfrac{y}{x}\right) = ay + c$

(7) I.F. $\dfrac{1}{xy}$; $a\log(x^2y) = y + c.$]

7. Equations Reducible To Exact By Integrating Factors (Rules of Finding Integrating Factors)

We shall now learn some standard rules of obtaining integrating factors.

Rule 1

If $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) \Big/ N$ is a function of x only, say $f(x)$, then $e^{\int f(x)\,dx}$ is an integrating factor.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0$. **(M.U. 2007)**

Sol. : We have, $M = x^2 + y^2 + 1$ and $N = -2xy$.

$$\therefore \quad \frac{\partial M}{\partial y} = 2y, \ \frac{\partial N}{\partial x} = -2y \qquad \therefore \quad \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ N = \frac{4y}{-2xy} = -\frac{2}{x} = f(x)$$

$$\therefore \quad \text{I.F.} = e^{\int -(2/x)\,dx} = e^{-2\log x} = e^{\log(1/x^2)} = \frac{1}{x^2}$$

Multiplying by the I.F., we get $\left(1 + \dfrac{y^2}{x^2} + \dfrac{1}{x^2}\right) dx + \left(-\dfrac{2y}{x}\right) dy = 0$, which is exact.

Now, $\displaystyle\int M\,dx = \int \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx = x - \frac{y^2}{x} - \frac{1}{x}$

and $\displaystyle\int N\,dy = \int (\text{Terms in } N \text{ free from } x)\,dy = 0$

$\therefore$ The solution is $x - \dfrac{y^2}{x} - \dfrac{1}{x} = c$ i.e. $x^2 - y^2 - 1 = cx$.

Example 2 (b) : Solve $(y - 2x^3)\,dx - x(1 - xy)\,dy = 0$. **(M.U. 1995)**

Sol. : We have $M = y - 2x^2$ and $N = -x + x^2 y$.

$$\therefore \quad \frac{\partial M}{\partial y} = 1 \text{ and } \frac{\partial N}{\partial x} = -1 + 2xy$$

$$\therefore \quad \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ N = \frac{1 + 1 - 2xy}{-x(1 - xy)} = \frac{2(1 - xy)}{-x(1 - xy)} = -\frac{2}{x} = f(x)$$

$$\therefore \quad \text{I.F.} = e^{\int f(x)\,dx} = e^{-\int (2/x)\,dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying by $\dfrac{1}{x^2}$, we get $\left(\dfrac{y}{x^2} - 2x\right) dx - \left(\dfrac{1}{x} - y\right) dy = 0$, which is exact.

$$\therefore \quad \int M\,dx = \int \left(\frac{y}{x^2} - 2x\right) dx = -\frac{y}{x} - x^2$$

and $\displaystyle\int N\,dy = \int (\text{Terms in } N \text{ free from } x)\,dy = \int y\,dy = \frac{y^2}{2}$

$\therefore$ The solution is $-\dfrac{y}{x} - x^2 + \dfrac{y^2}{2} = c$.

Example 3 (b) : Solve $(2x\log x - xy)\,dy + 2y\,dx = 0.$ **(M.U. 1995, 2003)**

Sol. : Here, $M = 2y,\quad N = 2x\log x - xy$

$$\therefore \quad \frac{\partial M}{\partial y} = 2,\ \frac{\partial N}{\partial x} = 2(1+\log x) - y$$

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\Big/ N = \frac{-2\log x + y}{x(2\log x - y)} = -\frac{1}{x} = f(x)$$

$$\therefore \quad \text{I.F.} = e^{\int f(x)\,dx} = e^{\int(-1/x)\,dx} = e^{-\log x} = e^{\log(1/x)} = \frac{1}{x}$$

Multiply the given equation by $1/x$ and rearrange the terms,

$$\frac{2y}{x}\,dx + (2\log x - y)\,dy = 0, \text{ which is exact.}$$

$$\therefore \quad \int M\,dx = 2y\int \frac{dx}{x} = 2y\log x$$

$$\int (\text{Terms in } N \text{ free from } x)\,dy = \int -y\,dy = -\frac{y^2}{2}$$

$$\therefore \text{ The solution is } 2y\log x - \frac{y^2}{2} = c.$$

Example 4 (b) : Solve $(4xy + 3y^2 - x)\,dx + x(x+2y)\,dy = 0.$

(M.U. 2006, 12)

Sol. : We have, $M = 4xy + 3y^2 - x,\ N = x^2 + 2xy.$

$$\therefore \quad \frac{\partial M}{\partial y} = 4x + 6y,\ \frac{\partial N}{\partial x} = 2x + 2y$$

$$\therefore \quad \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\Big/ N = \frac{4x + 6y - 2x - 2y}{x(x+2y)} = \frac{2x+4y}{x(x+2y)} = \frac{2}{x} = f(x)$$

$$\therefore \quad \text{I.F.} = e^{\int f(x)\,dx} = e^{\int(2/x)\,dx} = e^{2\log x} = e^{\log x^2} = x^2$$

Multiplying the given equation by x^2,

$$(4x^3y + 3x^2y^2 - x^3)\,dx + (x^4 + 2x^3y)\,dy = 0, \text{ which is exact.}$$

$$\therefore \quad \int M\,dx = \int (4x^3y + 3x^2y^2 - x^3)\,dx = x^4y + x^3y^2 - \frac{x^4}{4}.$$

$$\int (\text{Terms in } N \text{ free from } x)\,dy = 0$$

$$\therefore \text{ The solution is } x^4y + x^3y^2 - \frac{x^4}{4} = c' \quad \therefore \quad 4x^4y + 4x^3y^2 - x^4 = c.$$

Example 5 (b) : Solve $(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0.$ **(M.U. 2004, 07)**

Sol. : We have $\dfrac{\partial M}{\partial y} = 2xy,\ \dfrac{\partial N}{\partial x} = -2xy$

$$\therefore \quad \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\Big/ N = \frac{2xy - (-2xy)}{-x^2y} = \frac{4xy}{-x^2y} = -\frac{4}{x} = f(x)$$

$$\therefore \quad \text{I.F.} = e^{\int f(x)\,dx} = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = e^{\log(1/x^4)} = \frac{1}{x^4}$$

Multiplying the given equation by $\frac{1}{x^4}$, we get $\left(\frac{xy^2 - e^{1/x^3}}{x^4}\right)dx - \frac{x^2y}{x^4}dy = 0$, which is exact.

$$\therefore \int M\,dx = \int\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx = -\frac{y^2}{2x^2} + \frac{1}{3}e^{1/x^3}. \quad \left[\text{Put } \frac{1}{x^3} = t\right]$$

$$\int (\text{Terms in } N \text{ free from } x)\,dy = 0$$

$$\therefore \text{ The solution is } \frac{1}{3}e^{1/x^3} - \frac{y^2}{2x^2} = c.$$

Note ....

See Ex. 6, page 1-54 for another method.

Example 6 (b) : Solve $(x^4 + y^4)\,dx - xy^3 dy = 0$. **(M.U. 1996, 2002)**

Sol. : We have $\frac{\partial M}{\partial y} = 4y^3,\ \frac{\partial N}{\partial x} = -y^3$

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\Big/ N = \frac{4y^3 + y^3}{-xy^3} = \frac{5y^3}{-xy^3} = -\frac{5}{x} = f(x)$$

$$\therefore \text{ I.F.} = e^{\int f(x)\,dx} = e^{\int -(5/x)\,dx} = e^{-5\log x} = e^{\log(1/x^5)} = \frac{1}{x^5}$$

Multiplying the given equation by $\frac{1}{x^5}$, we get $\left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx - \frac{y^3}{x^4}\,dy = 0$, which is exact.

$$\therefore \int M\,dx = \int\left(\frac{1}{x} + y^4 \cdot x^{-5}\right)dx = \log x - \frac{y^4 \cdot x^{-4}}{4}$$

$$\int (\text{Terms in } N \text{ free from } x)\,dy = 0$$

$$\therefore \text{ The solution is } \log x - \frac{y^4}{4x^4} = c' \quad i.e.\ \ 4x^4 \log x - y^4 = cx^4.$$

Example 7 (b) : Solve $x \sin x\,dy + [\,y\,(x \cos x - \sin x) - 2\,]\,dx = 0$. **(M.U. 2015)**

Sol. : We have $M = xy\cos x - y\sin x$ and $N = x\sin x$.

$$\therefore \frac{\partial M}{\partial y} = x\cos x - \sin x \quad \text{and} \quad \frac{\partial N}{\partial x} = \sin x + x\cos x$$

$$\therefore \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\Big/ N = \frac{(x\cos x - \sin x) - (\sin x + x\cos x)}{x\sin x}$$

$$= -\frac{2\sin x}{x\sin x} = -\frac{2}{x} = f(x)$$

$$\therefore \text{ I.F.} = e^{\int f(x)\,dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log\left(\frac{1}{x^2}\right)} = \frac{1}{x^2}$$

Multiplying the given equation by $\frac{1}{x^2}$, we get

$$\frac{\sin x}{x}\,dy + \left(\frac{y}{x}\cos x - \frac{y}{x^2}\sin x - \frac{2}{x^2}\right)dx = 0$$

Since, $\int M\,dy$ is difficult, we apply the rule 2, page 1-7.

$$\therefore \quad \int N\,dy \text{ (treating } x \text{ constant)} = \int \frac{\sin x}{x}\,dy = \frac{\sin x}{x}\cdot y$$

$$\int (\text{Terms in } N \text{ free from } y)\,dy = \int -\frac{2}{x^2}\,dx = \frac{2}{x}$$

$\therefore$ The solution is $\frac{y}{x}\sin x + \frac{2}{x} = c.$

EXERCISE - III

Solve the following equations : Class (b) : 6 Marks

1. $(x^2 + y^2)\,dx - 2xy\,dy = 0$
2. $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}(x + xy^2)\,dy = 0$ **(M.U. 2000, 08**
3. $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$
4. $(x^3e^x - my^2)\,dx + m\,xy\,dy = 0$
5. $(y - 2x^2)\,dx - x(1 - xy)\,dy = 0$
6. $(x^4e^x - 2m\,xy^2)\,dx + 2\,mx^2y\,dy = 0$ **(M.U. 2003**

[**Ans. :** **(1)** $x^2 - y^2 = cx$, **(2)** Multiply by 12. $x^6 + 3x^4y + x^4y^3 = c$,
(3) $e^x(x^2 + y^2) = c,$ **(4)** $2x^2e^x + my^2 = cx^2,$
(5) $xy^2 - 4x^2 - 2y = cx,$ **(6)** $x^2e^x + my^2 = cx^2$.]

Rule 2

If $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Big/ M$ **is a function of** y **only, say** $f(y)$**, then** $e^{\int f(y)\,dy}$ **is an integrating facto**

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $(x + 2y^3)\frac{dy}{dx} = y.$

Sol. : The equation can be written as $y\,dx - (x + 2y^3)\,dy = 0.$

Here $M = y$ and $N = -(x + 2y^3)$.

Now, $\frac{\partial M}{\partial y} = 1,\ \frac{\partial N}{\partial x} = -1 \quad \therefore \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Big/ M = \frac{-2}{y} = f(y)$

$$\therefore \quad \text{I.F.} = e^{\int -(2/y)\,dy} = e^{-2\log y} = \frac{1}{y^2}$$

Multiplying by the I.F., we get $\frac{1}{y}\,dx - \left(\frac{x}{y^2} + 2y\right)dy = 0$, which is exact.

$$\therefore \quad \int M\,dx = \int \frac{1}{y}\,dx = \frac{x}{y} \quad \text{and} \quad \int N\,dy = \int -2y\,dy = -y^2$$

$\therefore$ The solution is $\frac{x}{y} - y^2 = c$ i.e. $x - y^3 = cy.$

Example 2 (b) : Solve $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0.$

(M.U. 1996, 2005, 10)

Sol. : We have, $M = 2x\,y^4\,e^y + 2xy^3 + y$

and $N = x^2\,y^4\,e^y - x^2\,y^2 - 3x.$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x\,(y^4e^y + 4y^3e^y) + 6xy^2 + 1$$

and $$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\therefore \quad \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \Big/ M = \frac{-8xy^2 - 4 - 8xy^3e^y}{y\,(2xy^3e^y + 2xy^2 + 1)} = -\frac{4}{y} \cdot \frac{(2xy^3e^y + 2xy^2 + 1)}{(2xy^3e^y + 2xy^2 + 1)}$$

$$= -\frac{4}{y} = f(y)$$

$$\therefore \quad \text{I.F.} = e^{\int -(4/y)\,dy} = e^{-4\log y} = e^{\log(1/y^4)} = \frac{1}{y^4}.$$

Multiplying by the I.F., we get,

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right) dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right) dy = 0 \text{ which is exact.}$$

$$\therefore \quad \int M\,dx = \int \left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right) dx = x^2e^y + \frac{x^2}{y} + \frac{x}{y^3}$$

$$\int (\text{Terms in } N \text{ free from } x)\,dy = 0$$

$$\therefore \quad \text{The solution is } x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c.$$

(M.U. 1999)

Example 3 (b) : Solve $y\,(xy + e^x)\,dx - e^x dy = 0.$

Sol. : We have, $M = y\,(xy + e^x)$ and $N = -\,e^x.$

$$\therefore \quad \frac{\partial M}{\partial y} = 2xy + e^x \text{ and } \frac{\partial N}{\partial x} = -\,e^x$$

$$\therefore \quad \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \Big/ M = \frac{-\,e^x - 2xy - e^x}{y(xy + e^x)} = \frac{-2(xy + e^x)}{y(xy + e^x)} = -\frac{2}{y} = f(y)$$

$$\therefore \quad \text{I.F.} = e^{\int (-2/y)\,dy} = e^{-2\log y} = e^{\log(1/y^2)} = \frac{1}{y^2}.$$

Multiplying by the I.F., we get, $\left(x + \dfrac{e^x}{y}\right) dx - \dfrac{e^x}{y^2}\,dy = 0$, which is exact.

$$\therefore \quad \int M\,dx = \int \left(x + \frac{e^x}{y}\right) dx = \frac{x^2}{2} + \frac{e^x}{y}$$

$$\int (\text{Terms in } N \text{ free from } x)\,dy = 0$$

$$\therefore \quad \text{The solution is } \frac{x^2}{2} + \frac{e^x}{y} = c.$$

Example 4 (b) : Solve $\left(\frac{y}{x}\sec y-\tan y\right)dx+(\sec y\log x-x)\,dy=0.$

Sol. : We have $M=\frac{y}{x}\sec y-\tan y$ and $N=\sec y\log x-x.$

$$\therefore\quad \frac{\partial M}{\partial y}=\frac{1}{x}\sec y+\frac{y}{x}\sec y\tan y-\sec^2 y$$

$$\therefore\quad \frac{\partial N}{\partial x}=\frac{\sec y}{x}-1$$

$$\therefore\quad \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\Big/ M=\frac{\left(\frac{\sec y}{x}-1-\frac{\sec y}{x}-\frac{y}{x}\sec y\tan y+\sec^2 y\right)}{\left(\frac{y}{x}\sec y-\tan y\right)}$$

$$=\frac{-\frac{y}{x}\sec y\tan y-1+\sec^2 y}{\frac{y}{x}\sec y-\tan y}=\frac{-\frac{y}{x}\sec y\tan y+\tan^2 y}{\frac{y}{x}\sec y-\tan y}$$

$$=\frac{-\tan y\left(\frac{y}{x}\sec y-\tan y\right)}{\frac{y}{x}\sec y-\tan y}=-\tan y=f(y)$$

$$\therefore\quad \text{I.F.}=e^{\int-\tan y\,dy}=e^{\int\frac{-\sin y}{\cos y}dy}=e^{\log\cos y}=\cos y$$

Multiplying the equation by $\cos y$, we get

$$\left(\frac{y}{x}-\sin y\right)dx+(\log x-x\cos y)\,dy=0$$

$$\therefore\quad \int M\,dx=\int\left(\frac{y}{x}-\sin y\right)dx=y\log x-x\sin y$$

$$\int(\text{Terms in } N \text{ free from } x)\,dy=\int 0\cdot dy=0$$

$\therefore$ The solution is $y\log x-x\sin y=c.$

Example 5 (b) : Solve $(xy^3+y)\,dx+2\,(x^2y^2+x+y^4)\,dy=0.$ **(M.U. 2009, 12)**

Sol. : We have $M=xy^3+y$ and $N=2x^2y^2+2x+2y^4$

$$\therefore\quad \frac{\partial M}{\partial y}=3xy^2+1\quad\text{and}\quad\frac{\partial N}{\partial x}=4xy^2+2$$

$$\therefore\quad \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\Big/ M=\frac{4xy^2+2-3xy^2-1}{y(xy^2+1)}=\frac{xy^2+1}{y(xy^2+1)}=\frac{1}{y}=f(y)$$

$$\therefore\quad \text{I.F.}=e^{\int(1/y)\,dy}=e^{\log y}=y$$

Multiplying the equation by y, we get,

$$(xy^4+y^2)\,dx+2\,(x^2y^3+xy+y^5)\,dy=0$$

$$\therefore\quad \int M\,dx=\int(xy^4+y^2)\,dx=\frac{x^2}{2}\cdot y^4+xy^2$$

$$\int(\text{Terms in } N \text{ free from } x)\,dy=\int y^5\cdot dy=\frac{y^6}{6}$$

$\therefore$ The solution is

$$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{6} = c' \quad \therefore \ 3x^2y^4 + 6xy^2 + y^6 = c.$$

EXERCISE - IV

Solve the following equations : Class (b) : 6 Marks

1. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$ **(M.U. 1997, 2003)**
2. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$ **(M.U. 2010)**
3. $(2xy^2 - y)\,dx + x\,dy = 0$ **(M.U. 1994)**
4. $y(x^2y + e^x)\,dx - e^x\,dy = 0$
5. $(2x^2y + e^x)\,y\,dx - (e^x + y^3)\,dy = 0$
6. $xe^x\,(dx - dy) + e^x\,dx + ye^y\,dy = 0$

[**Ans. :** (1) $x\left(y + \frac{2}{y^2}\right) + y^2 = c$ (2) $x^3y^3 + x^2 = cy$ (3) $x^2y - x = cy$

(4) $\frac{x^3}{3} + \frac{e^x}{y} = c$ (6) $4x^3y - 3y^3 + 6e^x = cy$ (7) $2xe^{x-y} + y^2 = c.$]

Rule 3

If the equation is of the form $f_1\,(xy)\;y\;dx + f_2\,(xy)\;x\;dy = 0$ and $Mx - Ny \neq 0$ then $1 / (Mx - Ny)$ is an integrating factor.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0.$ **(M.U. 2012)**

Sol. : The equation is of the above form and

$$Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3$$

Dividing by x^3y^3, we get $\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$, which is exact.

(If you divide by $3x^3y^3$, the term 3 gets cancelled because l.h.s. is zero.)

$$\therefore \int M\,dx = \int \left(\frac{1}{x^2y} + \frac{2}{x}\right)dx = -\frac{1}{xy} + 2\log x$$

$$\int (\text{Terms in } N \text{ free from } x)\,dy = \int -\frac{1}{y}\,dy = -\log y$$

$\therefore$ The solution is $2\log x - \log y = \frac{1}{xy} + c$ i.e. $\log\left(\frac{x^2}{y}\right) = \frac{1}{xy} + c.$

Example 2 (b) : Solve $y\,(1 + xy)\,dx + x\,(1 + xy + x^2y^2)\,dy = 0.$ **(M.U. 1989, 95)**

Sol. : The equation is of the above form and

$$Mx - Ny = xy + x^2y^2 - xy - x^2y^2 - x^3y^3 = -x^3y^3$$

Dividing by $-x^3y^3$, we get,

$$\left(-\frac{1}{x^3y^2} - \frac{1}{x^2y}\right)dx + \left(-\frac{1}{x^2y^3} - \frac{1}{xy^2} - \frac{1}{y}\right)dy = 0, \text{ which is exact.}$$

$$\therefore \int M\,dx = \int\left(-\frac{1}{x^3y^2}-\frac{1}{x^2y}\right)dx = \frac{1}{2x^2y^2}+\frac{1}{xy}$$

$$\int(\text{Terms in } N \text{ free from } x)\,dy = \int -\frac{1}{y}\,dy = -\log y$$

$$\therefore \text{ The solution is } \frac{1}{2x^2y^2}+\frac{1}{xy}-\log y = c.$$

Example 3 (b) : Solve $y\,(xy+2x^2y^2)\,dx + x\,(xy - x^2y^2)\,dy = 0$.

Sol. : Cancelling out the common factor xy, the equation can be written as

$$y\,(1+2xy)\,dx + x\,(1-xy)\,dy = 0$$

The equation is of the above form and

$$Mx - Ny = xy\,(1+2xy) - xy\,(1-xy) = 3x^2y^2$$

Dividing (1) by $3x^2y^2$, we get, (You may divide by x^2y^2 as well.)

$$\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy = 0$$

This is an exact differential equation.

$$\therefore \int M\,dx = \int\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx = -\frac{1}{3xy}+\frac{2}{3}\log x$$

$$\int(\text{Terms in } N \text{ free from } x)\,dy = \int -\frac{1}{3y}\,dy = -\frac{1}{3}\log y$$

$$\therefore \text{ The solution is } -\frac{1}{3xy}+\frac{2}{3}\log x - \frac{1}{3}\log y = c' \quad i.e. \quad 2\log x - \frac{1}{xy} - \log y = c.$$

[**Note** that the term 3 is elimintated.]

Example 4 (b) : Solve $(xy\sin xy + \cos xy)\,y\,dx + (xy\sin xy - \cos xy)\,x\,dy = 0$.

(M.U. 1997, 98, 2002, 16)

Sol. : The equation is of the above form and

$$Mx - Ny = x^2y^2\sin xy + xy\cos xy - x^2y^2\sin xy + xy\cos xy$$
$$= 2xy\cos xy$$

$\therefore$ Dividing by $2xy\cos xy$, we get,

$$\frac{1}{2}\left(y\tan xy + \frac{1}{x}\right)dx + \frac{1}{2}\left(x\tan xy - \frac{1}{y}\right)dy = 0$$

$$\therefore \left(y\tan xy + \frac{1}{x}\right)dx + \left(x\tan xy - \frac{1}{y}\right)dy = 0, \text{ which is exact.}$$

$$\therefore \int M\,dx = \int\left(y\tan xy + \frac{1}{x}\right)dx = \log\sec xy + \log x$$

$$\int(\text{Terms in } N \text{ free from } x)\,dy = \int -\frac{1}{y}\,dy = -\log y$$

$\therefore$ The solution is $\log\sec xy + \log x = \log y + \log c$ $\quad i.e. \quad x\sec xy = cy$.

Example 5 (b) : Solve $\frac{dy}{dx} = -\frac{x^2y^3 + 2y}{2x - 2x^3y^2}$. **(M.U. 1998)**

Sol. : The equation can be written as

$$(x^2y^3 + 2y)\,dx + (2x - 2x^3y^2)\,dy = 0$$

i.e. $y(2 + x^2y^2)\,dx + x(2 - 2x^2y^2)\,dy = 0$

The equation is of the above form and

$$Mx - Ny = 2xy + x^3y^3 - 2xy + 2x^3y^3 = 3x^3y^3$$

Dividing the equation by $3x^3y^3$, we get

$$\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \left(\frac{2}{3x^2y^3} - \frac{2}{3y}\right)dy = 0$$

Now, $\int M\,dx = \int\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx = \frac{1}{3}\log x - \frac{1}{3x^2y^2}$

And $\int(\text{Term in } N \text{ free from } x)\,dy = \int -\frac{2}{3}y\,dy = -\frac{2}{3}\log y$

$\therefore$ The solution is $\frac{1}{3}\log x - \frac{1}{3x^2y^2} - \frac{2}{3}\log y = c$. $\quad\therefore\ \frac{1}{3}\log\frac{x}{y^2} - \frac{1}{3x^2y^2} = c.$

$$\therefore\ \log\frac{x}{y^2} - \frac{1}{x^2y^2} = 3c = c'$$

Example 6 (b) : Solve $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$. **(M.U. 2006, 15)**

Sol. : The equation can be written as $y(1 - xy)\,dx - x(1 + xy)\,dy = 0$

$\therefore$ The equation is of the above form and $Mx - Ny = xy - x^2y^2 + xy + x^2y^2 = 2xy$

Dividing by $2xy$,

$$\frac{1}{2}\left(\frac{1}{x} - y\right)dx - \frac{1}{2}\left(\frac{1}{y} + x\right)dy = 0 \quad \therefore\ \frac{dx}{x} - \frac{dy}{y} - (y\,dx + x\,dy) = 0$$

Integrating, we get $\log x - \log y - xy = c \quad \therefore\ \log\frac{x}{y} = c + xy.$

(Or find $\int M\,dx$ and $\int N\,dy$ if you please.)

Example 7 (b) : Solve $(x^3y^4 + x^2y^3 + xy^2 + y)\,dx + (x^4y^3 - x^3y^2 - x^2y + x)\,dy = 0.$ **(M.U. 2013)**

Sol. : The equation is of the form

$$f_1(x, y)\,y\,dx + f_2(xy)\,x\,dy = 0.$$

Now, $Mx - Ny = x^4y^4 + x^3y^3 + x^2y^2 + xy - x^4y^4 + x^3y^3 + x^2y^2 - xy$

$$= 2x^3y^3 + 2x^2y^2 = 2x^2y^2(xy + 1)$$

Dividing the equation by $2x^2y^2(xy + 1)$, we get

$$\left[\frac{x^2y^3(xy + 1) + y(xy + 1)}{2x^2y^2 \cdot (xy + 1)}\right]dx + \left[\frac{x^3y^2(xy - 1) - x(xy - 1)}{2x^2y^2 \cdot (xy + 1)}\right]dy = 0$$

$$\therefore \quad \frac{(xy+1)(x^3y^3+y)}{2x^2y^2\cdot(xy+1)}dx+\frac{(xy-1)(x^3y^2-x)}{2x^2y^2\cdot(xy+1)}dy=0$$

$$\therefore \quad \frac{y(xy+1)(x^2y^2+1)}{2x^2y^2\cdot(xy+1)}dx+\frac{x(xy-1)(x^2y^2-1)}{2x^2y^2\cdot(xy+1)}dy=0$$

Now, $x^2y^2-1=(xy-1)(xy+1)$.

$\therefore$ The above equation becomes

$$\frac{(x^2y^2+1)}{2x^2y}dx+\frac{(xy-1)^2}{2xy^2}dy=0$$

This is of the form $M\,dx+N\,dy=0$ and is now exact.

$$\therefore \quad \int M\,dx=\int\frac{x^2y^2+1}{2x^2y}dx=\int\left(\frac{y}{2}+\frac{1}{2x^2y}\right)dx=\frac{xy}{2}-\frac{1}{2xy}$$

Now, $N=\dfrac{x^2y^2-2xy+1}{2xy^2}=\dfrac{x}{2}-\dfrac{1}{y}+\dfrac{1}{2xy^2}$

$$\therefore \quad \int(\text{Terms in } N \text{ free from } x)\,dy=\int-\frac{1}{y}dy=-\log y$$

$\therefore$ The solution is

$$\frac{xy}{2}-\frac{1}{2xy}-\log y=c' \qquad \therefore \quad xy-\frac{1}{xy}-2\log y=c$$

i.e., $\quad xy-\dfrac{1}{xy}-(\log y^2)=c.$

EXERCISE - V

Solve the following equations : Class (b) : 6 Marks

1. $y(1+xy+x^2y^2)\,dx+x(1-xy+x^2y^2)\,dy=0$
2. $y(1+xy)\,dx+x\,(1-xy)\,dy=0$ **(M.U. 2005)**
3. $y(2xy+1)\,dx+x(1+2xy-x^3y^3)\,dy=0$
4. $y(xy+2x^2y^2)\,dx+x(xy+x^2y^2)\,dy=0$
5. $y(\sin xy+xy\cos xy)\,dx+x\,(xy\cos xy-\sin xy)\,dy=0$

[**Ans. : (1)** $xy+\log\left(\dfrac{x}{y}\right)-\dfrac{1}{xy}=c$ **(2)** $xy-\log\left(\dfrac{x}{y}\right)=c$ **(3)** $\dfrac{1}{x^2y^2}+\dfrac{1}{3x^3y^3}+\log y=c$

(4) $\log(x^2y)=\dfrac{1}{xy}+c$ **(5)** $x\sin(xy)=cy$]

Rule 4

If the equation $M\,dx+N\,dy=0$ is homogeneous and $Mx+Ny\neq 0$ then $1/(Mx+Ny)$ is an integrating factor.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $(x^2y-2xy^2)\,dx-(x^3-3x^2y)\,dy=0.$ **(M.U. 1995, 98, 2001, 16)**

Sol. : The equation is homogeneous and $Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2$.

Hence, $\dfrac{1}{Mx+Ny} = \dfrac{1}{x^2y^2}$ is an integrating factor.

Dividing by x^2y^2, the equation becomes $\left(\dfrac{1}{y} - \dfrac{2}{x}\right)dx + \left(-\dfrac{x}{y^2} + \dfrac{3}{y}\right)dy = 0$.

Now, $\int M\,dx = \int \left(\dfrac{1}{y} - \dfrac{2}{x}\right)dx = \dfrac{x}{y} - 2\log x$

And $\int$ (Terms in N free from x) $dy = \int \dfrac{3}{y}\,dy = 3\log y$

$\therefore$ The solution is $\dfrac{x}{y} - 2\log x + 3\log y = -\log c$

i.e. $\dfrac{x}{y} + \log\dfrac{cy^3}{x^2} = 0$ i.e. $\log\dfrac{cy^3}{x^2} = -\dfrac{x}{y}$ i.e. $\dfrac{cy^3}{x^2} = e^{-x/y}$.

Alternatively : A homogeneous equation can also be solved by putting $y = vx$ and then separating the variables.

Now, put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ The equation $\dfrac{dy}{dx} = \dfrac{x^2y - 2xy^2}{x^3 - 3x^2y}$ reduces to $v + x\dfrac{dv}{dx} = \dfrac{v - 2v^2}{1 - 3v}$.

$\therefore\ x\dfrac{dv}{dx} = \dfrac{v - 2v^2}{1 - 3v} - v \quad \therefore\ x\dfrac{dv}{dx} = \dfrac{v^2}{1 - 3v} \quad \therefore\ \left(\dfrac{1 - 3v}{v^2}\right)dv = \dfrac{dx}{x}$

$\therefore\ \int\left(\dfrac{1}{v^2} - \dfrac{3}{v}\right)dv = \int\dfrac{dx}{x} \quad \therefore\ -\dfrac{1}{v} - 3\log v = \log x + \log c \quad \therefore\ -\dfrac{1}{v} = \log x + \log v^3 + \log c$

$\therefore\ -\dfrac{x}{y} = \log x + \log\dfrac{y^3}{x^3} + \log c = \log\left(\dfrac{cy^3}{x^2}\right) \quad \therefore\ \dfrac{cy^3}{x^2} = e^{-x/y}$

$\therefore\ cy^3 = x^2e^{-x/y}$ which is the same as above.

Example 2 (b) : Solve $y(x + y)\,dx - x(y - x)\,dy = 0$. **(M.U. 1987, 2002)**

Sol. : The equation is homogeneous and $Mx + Ny = x^2y + xy^2 - xy^2 + x^2y = 2x^2y$.

Hence, $\dfrac{1}{Mx+Ny} = \dfrac{1}{2x^2y}$ is an integrating factor.

Dividing by $2x^2y$, we get $\dfrac{y(x+y)}{2x^2y}\,dx - \dfrac{x(y-x)}{2x^2y}\,dy = 0$.

$\therefore\ \left(\dfrac{1}{2x} + \dfrac{y}{2x^2}\right)dx - \left(\dfrac{1}{2x} - \dfrac{1}{2y}\right)dy = 0$

$\therefore\ \int M\,dx = \int\left(\dfrac{1}{2x} + \dfrac{y}{2x^2}\right)dx = \dfrac{1}{2}\log x - \dfrac{y}{2x}$

$\int$ (Terms in N free from x) $dy = \int\dfrac{1}{2y}\,dy = \dfrac{1}{2}\log y$

$\therefore$ The solution is $\dfrac{1}{2}\log x + \dfrac{1}{2}\log y - \dfrac{y}{2x} = c.$

$\therefore \quad \log x + \log y - \dfrac{y}{x} = 2c = c' \qquad \therefore \quad \log xy - \dfrac{y}{x} = c'$

Example 3 (b) : Solve $(x^2 + y^2)\,dx - (x^2 + xy)\,dy = 0.$ **(M.U. 1993)**

Sol. : The equation is homogeneous and

$$Mx + Ny = x(x^2 + y^2) - y(x^2 + xy) = x^3 - x^2 y = x^2 (x - y).$$

Hence, $\dfrac{1}{Mx + Ny} = \dfrac{1}{x^2(x - y)}$ is an integrating factor.

Dividing by $x^2 (x - y)$, we get $\dfrac{x^2 + y^2}{x^2(x - y)}\,dx - \dfrac{x^2 + xy}{x^2(x - y)}\,dy = 0$

Now, this is an exact differential equation.

$$\therefore \quad \int M\,dx = \int \frac{x^2 + y^2}{x^2(x - y)}\,dx = \int \frac{dx}{x - y} + y^2 \int \frac{dx}{x^2(x - y)}$$

$$= \int \frac{dx}{x - y} + y^2 \left[\frac{1}{y^2}\int \frac{dx}{x - y} - \frac{1}{y^2}\int \frac{dx}{x} - \frac{1}{y}\int \frac{dx}{x^2}\right] \quad \text{.............. (A)}$$

(By partial fraction treating y constant. Let $\dfrac{1}{x^2(x - y)} = \dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x - y}$)

$$\therefore \quad \int M\,dx = 2\int \frac{dx}{x - y} - \int \frac{dx}{x} - y\int \frac{dx}{x^2} = 2\log(x - y) - \log x + \frac{y}{x}$$

$$\int (\text{Terms in } N \text{ free from } x)\,dy = 0$$

$\therefore$ The solution is $\dfrac{y}{x} = \log x - 2\log(x - y) + c = \log \dfrac{x}{(x - y)^2} + c.$

Example 4 (b) : Solve $(x^2 - xy + y^2)\,dx - xy\,dy = 0.$

Sol. : The equation is homogeneous and $Mx + Ny = x^3 - x^2 y + xy^2 - xy^2 = x^2(x - y).$

Hence, $\dfrac{1}{Mx + Ny} = \dfrac{1}{x^2(x - y)}$ is an integrating factor.

Dividing by $x^2 (x - y)$, we get $\dfrac{x^2 - xy + y^2}{x^2(x - y)}\,dx - \dfrac{xy}{x^2(x - y)}\,dy = 0$

$$\therefore \quad \frac{x^2 - xy}{x^2(x - y)}\,dx + \frac{y^2}{x^2(x - y)}\,dx - \frac{y}{x(x - y)}\,dy = 0$$

$$\therefore \quad \left[\left(\frac{1}{x}\right) + \left(\frac{1}{x - y} - \frac{1}{x} - \frac{y}{x^2}\right)\right]dx + \left[\frac{1}{x} - \frac{1}{x - y}\right]dy = 0$$

(By partial fraction as in (A) of Ex. 3 above)

$$\therefore \quad \int M\,dx = \int \frac{dx}{x - y} - \int \frac{y}{x^2}\,dx = \log(x - y) + \frac{y}{x}.$$

$$\int (\text{Terms in } N \text{ free from } x)\,dy = 0$$

$\therefore$ The solution is $\log(x - y) + \dfrac{y}{x} = c.$

Note ....

A homogeneous equation can also be solved by putting $y = vx$ and then separating the variables as seen earlier in Ex. 1.

EXERCISE - VI

Solve the following equations : Class (b) : 6 Marks

1. $x^2 y\, dx - (x^3 + y^3)\, dy = 0$
2. $(3xy^2 - y^3)\, dx - (2x^2 y - xy^2)\, dy = 0$
3. $(3xy - 2ay^2)\, dx + (x^2 - 2axy)\, dy = 0$
4. $(x^3 + y^3)\, dx - xy^2\, dy = 0$
5. $x(x - y)\, dy + y^2\, dx = 0$
6. $(x^4 + y^4)\, dx - xy^3 dy = 0$

(M.U. 1996, 2002, 04)

[**Ans. :** **(1)** $cy = e^{x^3/3y^3}$ **(2)** $cx^3 = y^2 e^{-(y/x)}$ **(3)** $x^2 y(x - ay) = c$
(4) $cx = e^{y^3/3x^3}$ **(5)** $cy = e^{y/x}$ **(6)** $4x^4 \log x - y^4 = cx^4$.]

8. Linear Equations

(a) Definition

A differential equation is said to be **linear** if the dependent variable and its derivatives appear only in the first degree. The form of the linear equation of the first order is,

$$\boxed{\frac{dy}{dx} + Py = Q}$$

where P and Q are functions of x or constants only.

For example, $\dfrac{dy}{dx} + 3xy = x^2$, $\dfrac{dy}{dx} + y = e^x$ are linear equations.

A linear equation of the form $\dfrac{dy}{dx} + Py = Q$ is not solvable as it is.

However, if we multiply it by factor $e^{\int P\,dx}$ it becomes exact and hence can be solved by mere integration.

(b) To show that $e^{\int P\,dx}$ is an I.F.

Consider $\dfrac{dy}{dx} + Py = Q$ i.e. $(Py - Q)\, dx + dy = 0$. .... (1)

Multiply it by $e^{\int P\,dx}$ $\quad\therefore\quad e^{\int P\,dx}(Py - Q)\, dx + e^{\int P\,dx} \cdot dy = 0$

Now, $M = e^{\int P\,dx}(Py - Q)$ and $N = e^{\int P\,dx}$.

$$\therefore\quad \frac{\partial M}{\partial y} = e^{\int P\,dx} P \text{ and } \frac{\partial N}{\partial x} = e^{\int P\,dx} \cdot P$$

$$\left[\frac{d}{dx} e^{\int P\,dx} = e^{\int P\,dx} \cdot \frac{d}{dx}\int P\,dx \text{ and } \frac{d}{dx}\int P\,dx = P\right]$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation (1) is exact and hence $e^{\int P\,dx}$ is the I.F.of $\dfrac{dy}{dx} + Py = Q$

(c) To solve the equation $\dfrac{dy}{dx} + Py = Q$

As before multiplying the given equation by the I.F. $e^{\int P\,dx}$, we get,

$$e^{\int P\,dx}(Py - Q)\,dx + e^{\int P\,dx}\,dy = 0$$

Since it is exact, we have,

$$\int M\,dx = \int e^{\int P\,dx}(Py - Q)\,dx = y\int e^{\int P\,dx}\cdot P\,dx - \int e^{\int P\,dx}\cdot Q\,dx$$

$$= y\cdot e^{\int P\,dx} - \int e^{\int P\,dx}\cdot Q\,dx \qquad \left[\text{Put } \int P\,dx = t \quad \therefore \quad P\,dx = dt\right]$$

Since, there is no term in N free from x, the solution is

$$y\cdot e^{\int P\,dx} - \int e^{\int P\,dx}\cdot Q\,dx = c \qquad \text{i.e.} \qquad y\cdot e^{\int P\,dx} = \int e^{\int P\,dx}\cdot Q\,dx + c.$$

Remark ....

To solve a linear differential equation first write the equation with the coefficient of $\dfrac{dy}{dx}$ unity

i.e. in the form $\dfrac{dy}{dx} + Py = Q$.

Then find $\int P\,dx$ and further find $e^{\int P\,dx}$. The solution then is

$$\boxed{y\cdot e^{\int P\,dx} = \int e^{\int P\,dx}\cdot Q\,dx + c}$$

(a) Linear Equations $\dfrac{dy}{dx} + Py = Q$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $\dfrac{dy}{dx} + \left(\dfrac{1-2x}{x^2}\right)y = 1.$ **(M.U. 1990)**

Sol. : This is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$.

Now, $\displaystyle\int P\,dx = \int\left(\frac{1-2x}{x^2}\right)dx = \int\frac{dx}{x^2} - 2\int\frac{dx}{x} = -\frac{1}{x} - 2\log x$

$$\therefore \quad e^{\int P\,dx} = e^{-(1/x) - 2\log x} = e^{-1/x}\cdot e^{-2\log x} = e^{-1/x}\cdot\frac{1}{x^2}$$

$\therefore$ The solution is

$$y e^{-1/x}\cdot\frac{1}{x^2} = \int e^{-1/x}\cdot\frac{1}{x^2}Q\,dx + c = \int e^{-1/x}\cdot\frac{1}{x^2}dx + c = e^{-1/x} + c$$

(For integration put $e^{-1/x} = t$)

$\therefore$ The solution is $y e^{-1/x}\cdot\dfrac{1}{x^2} = e^{-1/x} + c \qquad \therefore \quad y = x^2 + c e^{1/x}\cdot x^2.$

Example 2 (b) : Solve $(1-x^2)\frac{dy}{dx}+2xy = x\sqrt{1-x^2}$. **(M.U. 1996)**

Sol. : The equation can be written as $\frac{dy}{dx}+\frac{2x}{1-x^2}\cdot y = \frac{x\sqrt{1-x^2}}{1-x^2} = \frac{x}{\sqrt{1-x^2}}$.

This is a linear equation of the form $\frac{dy}{dx}+Py = Q$.

$$e^{\int P\,dx} = e^{\int 2x/(1-x^2)\,dx} = e^{-\log(1-x^2)} = \frac{1}{1-x^2}$$

$\therefore$ The solution is,

$$y\cdot\frac{1}{1-x^2} = \int\frac{x}{\sqrt{1-x^2}}\cdot\frac{1}{1-x^2}\,dx + c = \int(1-x^2)^{-3/2}\cdot x\,dx + c$$

$$= -\int t^{-3/2}\frac{dt}{2}+c \qquad \left[\text{Put } 1-x^2 = t,\ x\,dx = -\frac{dt}{2}\right]$$

$$\therefore\ y\cdot\frac{1}{1-x^2} = (1-x^2)^{-1/2}+c \qquad \therefore\ y = \sqrt{(1-x^2)}+c\,(1-x^2).$$

Example 3 (b) : Solve $x(x-1)\frac{dy}{dx}-(x-2)\,y = x^3(2x-1)$. **(M.U. 2000, 04)**

Sol. : The equation can be written as $\frac{dy}{dx}-\frac{x-2}{x(x-1)}\cdot y = \frac{x^2(2x-1)}{x-1}$, which is linear.

$$\int P\,dx = \int\frac{x-2}{x(x-1)}\,dx = \int\left(\frac{2}{x}-\frac{1}{x-1}\right)dx$$

By inspection or partial fractions

$$\therefore\ \int P\,dx = 2\log x-\log(x-1) = \log\frac{x^2}{x-1}$$

$$\therefore\ \text{I.F.} = e^{-\int P\,dx} = e^{-\log x^2/(x-1)} = e^{\log(x-1)/x^2} = \frac{x-1}{x^2}$$

$\therefore$ The solution is

$$y\cdot\frac{x-1}{x^2} = \int\frac{x-1}{x^2}\cdot\frac{x^2(2x-1)}{x-1}\,dx + c$$

$$= \int(2x-1)\,dx + c = x^2-x+c = x(x-1)+c$$

$$\therefore\ y = x^3+\frac{cx^2}{x-1}.$$

Example 4 (b) : Solve $\frac{dy}{dx}+\frac{4x}{x^2+1}y = \frac{1}{(x^2+1)^3}$. **(M.U. 2002, 15)**

Sol. : This is a linear differential equation of the form $\frac{dy}{dx}+Py = Q$.

$$\text{Now, } \int P\,dx = \int\frac{4x}{x^2+1}\,dx = 2\int\frac{2x}{x^2+1}\,dx$$

$$= 2\log(x^2+1) = \log(x^2+1)^2$$

$$\therefore\ e^{\int P\,dx} = e^{\log(x^2+1)^2} = (x^2+1)^2$$

$\therefore$ The solution is $y(x^2+1)^2 = \int (x^2+1)^2 \cdot Q\,dx + c.$

$$= \int (x^2+1)^2 \cdot \frac{1}{(x^2+1)^3}\,dx + c$$

$$= \int \frac{1}{x^2+1}\,dx + c = \tan^{-1} x + c.$$

Hence, the solution is $y(x^2+1)^2 = \tan^{-1} x + c.$

Example 5 (b) : Solve $\sin 2x \cdot \dfrac{dy}{dx} = y + \tan x.$

Sol. : The equation can be written as $\dfrac{dy}{dx} - \dfrac{1}{\sin 2x} \cdot y = \dfrac{\tan x}{\sin 2x}$, which is linear.

Now, $\displaystyle\int P\,dx = -\int \frac{1}{2\sin x \cos x}\,dx = -\frac{1}{2}\int \frac{\sec^2 x}{\tan x}\,dx = -\frac{1}{2}\log \tan x$

$$\therefore\ e^{\int P\,dx} = e^{-\frac{1}{2}\log \tan x} = e^{\log(1/\sqrt{\tan x})} = \frac{1}{\sqrt{\tan x}}$$

$\therefore$ The solution is $y \cdot e^{\int P\,dx} = \int e^{\int P\,dx} \cdot Q\,dx + c.$

$$\frac{y}{\sqrt{\tan x}} = \int \frac{1}{\sqrt{\tan x}} \cdot \frac{\tan x}{\sin 2x}\,dx + c$$

$$\therefore\ \frac{y}{\sqrt{\tan x}} = \int \frac{1}{\sqrt{\tan x}} \cdot \frac{\sin x}{\cos x \cdot 2\sin x \cos x}\,dx + c$$

$$= \int \frac{1}{2\sqrt{\tan x}} \cdot \sec^2 x\,dx + c = \sqrt{\tan x} + c$$

$\therefore$ The solution is $y = c\sqrt{\tan x} + \tan x.$

Example 6 (b) : Solve $\dfrac{dy}{dx}\cos hx = 2\cos h^2 x \cdot \sin hx - y \sin hx.$ **(M.U. 2001, 02, 08)**

Sol. : We have $\dfrac{dy}{dx} + \dfrac{\sin hx}{\cos hx} \cdot y = 2\sin hx \cos hx$

This is linear of the form $\dfrac{dy}{dx} + Py = Q.$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int (\sin hx)/(\cos hx)} \cdot dx = e^{\log \cos hx} = \cos hx.$$

$\therefore$ The solution is $y \cos hx = \int \cos hx \cdot 2\sin hx \cos hx\,dx + c$

Put $\cos hx = t$ $\quad\therefore\ \sin hx\,dx = dt$

$$\therefore\ \int 2\cos h^2 x \cdot \sin hx\,dx = \int 2t^2\,dt = \frac{2t^3}{3}$$

$\therefore$ The solution is $y \cdot \cos hx = \dfrac{2}{3}\cos h^3 x + c.$

Example 7 (b) : Solve $x\cos x \dfrac{dy}{dx} + y(x\sin x + \cos x) = 1.$

Sol. : By Dividing by $x\cos x$ the equation can be written as

$$\frac{dy}{dx}+\left(\frac{x\sin x+\cos x}{x\cos x}\right)y=\frac{1}{x\cos x}$$

i.e., $$\frac{dy}{dx}+\left(\tan x+\frac{1}{x}\right)y=\frac{1}{x}\cdot\sec x.$$

Now, $$\int P\,dx=\int\left(\tan x+\frac{1}{x}\right)dx=\log\sec x+\log x=\log(x\sec x)$$

$$e^{\int P\,dx}=e^{\log x\sec x}=x\sec x$$

$\therefore$ The solution is $y\,e^{\int P\,dx}=\int e^{\int P\,dx}\cdot Q\,dx+c$

$$\therefore\ y\cdot x\sec x=\int x\sec x\cdot\frac{1}{x}\cdot\sec x\,dx+c=\int\sec^2 x\,dx+c$$

$$\therefore\ yx\sec x=\tan x+c.$$

EXERCISE - VII

Solve the following equations : Class (b) : 6 Marks

1. $\frac{dy}{dx}-\frac{1}{x(x-1)}\cdot y=x(x-1)$

2. $\frac{dy}{dx}+\frac{2x}{x^2+1}\cdot y=\frac{1}{(x^2+1)^2}$

3. $\frac{dy}{dx}+\frac{2x}{(x^2+1)}\cdot y=\frac{4x^2}{(x^2+1)}$

4. $x\log x\cdot\frac{dy}{dx}+y=2\log x$

5. $\cos x\cdot\frac{dy}{dx}+y\sin x=\sec^2 x$

6. $\cos^2 x\frac{dy}{dx}+y=\tan x$

7. $dr+(2r\cot\theta+\sin 2\theta)\,d\theta=0$

[**Ans.:** (1) $xy=\left(\frac{x^3}{3}+c\right)(x-1)$, (2) $y(x^2+1)=\tan^{-1}x+c$,

(3) $y(x^2+1)=\frac{4x^3}{3}+c$, (4) $y\log x=(\log x)^3+c$

(5) $y\sec x=\tan x+\frac{\tan^3 x}{3}+c$ (6) $y\cdot e^{\tan x}=e^{\tan x}(\tan x-1)+c$

(7) $r\sin^2\theta=-\frac{\sin^4\theta}{2}+c$]

(b) Equations Reducible To Linear Forms (Type I) (By Rearraning Terms)

A differential equation of the form $\boxed{\frac{dx}{dy}+P'x=Q'}$

where P' and Q' are functions of y only is also a linear differential equation with x and y having interchanged the positions.

Its solution is, $\boxed{x\cdot e^{\int P'dy}=\int e^{\int P'dy}\cdot Q'\,dy+c}$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $(1+x+xy^2)\,dy+(y+y^3)\,dx=0$. **(M.U. 1989, 93, 95)**

Sol. : We have,

$$1+x(1+y^2)+y(1+y^2)\frac{dx}{dy}=0 \qquad \therefore \frac{dx}{dy}+\frac{x}{y}=-\frac{1}{y(1+y^2)}$$

This is a linear differential equation of the form $\frac{dx}{dy}+P'x=Q'$.

Now, $\int P'\,dy=\int\frac{dy}{y}=\log y \qquad \therefore e^{\int P'dy}=e^{\log y}=y$

$\therefore$ The solution is $x\cdot e^{\int P'dy}=\int e^{\int P'dy}\cdot Q'\,dy+c$.

$$\therefore xy=\int y\left(-\frac{1}{y(1+y^2)}\right)dy+c=-\int\frac{dy}{1+y^2}=-\tan^{-1}y+c$$

$$\therefore xy+\tan^{-1}y=c.$$

Example 2 (b) : Solve $(1+y^2)\,dx=\left(e^{\tan^{-1}y}-x\right)dy$. **(M.U. 1992, 97, 98, 2002, 16)**

Sol. : The equation can be written as $\frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{e^{\tan^{-1}y}}{1+y^2}$

This is a linear differential equation of the form $\frac{dx}{dy}+P'x=Q'$.

Now, $\int P'\,dy=\int\frac{1}{1+y^2}\,dy=\tan^{-1}y \qquad \therefore e^{\int P'dy}=e^{\tan^{-1}y}$.

$\therefore$ The solution is $x\cdot e^{\int P'dy}=\int e^{\int P'dy}\cdot Q'\,dy+c$.

$$\therefore xe^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}}{1+y^2}\cdot dy+c.$$

(For integrating the r.h.s. put $\tan^{-1}y=t$)

$$\therefore xe^{\tan^{-1}y}=e^{\tan^{-1}y}+c.$$

Example 3 (b) : Solve $(y+1)\,dx+[\,x-(y+2)\,e^y]\,dy=0$. **(M.U. 1992)**

Sol. : The equation can be written as $\frac{dx}{dy}+\frac{1}{y+1}\cdot x=\frac{y+2}{y+1}\cdot e^y$.

It is of the form $\frac{dx}{dy}+P'x=Q'$.

Now, $\int P'\,dy=\int\frac{1}{y+1}\,dy=\log(y+1) \qquad \therefore e^{\int P'dy}=y+1$

$\therefore$ The solution is $x\,e^{\int P'dy}=\int e^{\int P'dy}\cdot Q'\,dy+c$.

$$x\cdot(y+1)=\int(y+1)\cdot\frac{(y+2)}{(y+1)}e^y\,dy+c$$

$\therefore \quad x \cdot (y+1) = \int (y+2) \cdot e^y = (y+2)\, e^y - \int e^y \cdot dy + c \qquad$ [By parts]

$$= (y+2)\, e^y - e^y + c = (y+1) \cdot e^y + c$$

$\therefore \quad (y+1)(x - e^y) = c.$

Example 4 (b) : Solve $(1 + \sin y) \dfrac{dx}{dy} = [2y \cos y - x(\sec y + \tan y)]$.

(M.U. 1995, 97, 99, 2010)

Sol. : The given equation can be written as

$$\frac{dx}{dy} + \frac{(\sec y + \tan y)}{1 + \sin y} \cdot x = \frac{2y \cos y}{1 + \sin y}$$

$$\therefore \quad \frac{dx}{dy} + \sec y \cdot \frac{(1 + \sin y)}{(1 + \sin y)} \cdot x = \frac{2y \cos y}{1 + \sin y} \qquad \text{[Note this]}$$

$$\therefore \quad \frac{dx}{dy} + \sec y \cdot x = \frac{2y \cos y}{1 + \sin y}$$

$$\therefore \quad e^{\int P' dy} = e^{\int \sec y} = e^{\log (\sec y + \tan y)} = \sec y + \tan y = \frac{1 + \sin y}{\cos y}$$

$\therefore$ The solution is

$$x \cdot \frac{(1 + \sin y)}{\cos y} = \int \frac{2y \cos y}{(1 + \sin y)} \cdot \frac{(1 + \sin y)}{\cos y}\, dy + c$$

$$= \int 2y\, dy + c = y^2 + c$$

$\therefore$ The solution is $x \cdot (1 + \sin y) = y^2 \cos y + c \cos y$.

Example 5 (b) : Solve $y \log y\, dx + (x - \log y)\, dy = 0$.

Sol. : The equation can be written as

$$y \log y \frac{dx}{dy} + x = \log y \qquad \therefore \quad \frac{dx}{dy} + \left(\frac{1}{y \log y}\right) x = \frac{1}{y}$$

It is of the form $\dfrac{dx}{dy} + P' x = Q'$.

Now, $\displaystyle\int P' dy = \int \frac{dy}{y \log y} = \log \log y \qquad$ [Put $\log y = t$] $\qquad \therefore \quad e^{\int P' dy} = e^{\log \log y} = \log y$

$\therefore$ The solution is $x \cdot e^{\int P' dy} = \int e^{\int P' dy} \cdot Q'\, dy + c$

$\therefore \quad x \log y = \displaystyle\int \log y \cdot \frac{1}{y}\, dy + c = \frac{1}{2} (\log y)^2 + c \qquad$ [Put $\log y = t$]

(M.U. 2013)

Example 6 (b) : Solve $(x + 2y^3)\, dy = y\, dx$.

Sol. : The equation can be written as

$$y \frac{dx}{dy} = x + 2y^3 \qquad \therefore \quad \frac{dx}{dy} - \frac{1}{y} x = 2y^2$$

This is of the form $\dfrac{dx}{dy} + P' x = Q'$.

Now, $\displaystyle\int P' dy = \int -\frac{dy}{y} = -\log y \qquad \therefore \quad e^{\int P' dy} = e^{-\log y} = e^{\log (1/y)} = \frac{1}{y}$

$\therefore$ The solution is $x \cdot e^{\int P' dy} = \int e^{\int P' dy} Q' \, dy + c$

$\therefore \quad x \cdot \frac{1}{y} = \int \frac{1}{y} \cdot 2y^2 \, dy = \int 2y \, dy = y^2 + c \qquad \therefore \quad x = y^3 + cy.$

EXERCISE - VIII

Solve the following equations : Class (b) : 6 Marks

1. $(1 + y^2) \, dx = (\tan^{-1} y - x) \, dy$ **(M.U. 1994)**
2. $dx + x \, dy = e^{-y} \sec^2 y \, dy$
3. $(x + y + 1) \frac{dy}{dx} = 1$
4. $\sin 2y \, dx = (\tan y - x) \, dy$

[**Ans. :** **(1)** $x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$ **(2)** $xe^y = \tan y + c$ **(3)** $x + y + 2 = ce^y$
(4) $x\sqrt{\tan y} = \frac{1}{3}(\tan y)^{3/2} + c$ **(5)** $x = y^3 + cy$]

(c) Equations Reducible To Linear Form (Type II) (By Substitution)

The equation of the type $\boxed{f'(y) \frac{dy}{dx} + P \cdot f(y) = Q}$

where P and Q are functions of x only can be reduced to linear form as follows.

Let us put $f(y) = v$ then $f'(y) \frac{dy}{dx} = \frac{dv}{dx}$.

$\therefore$ The equation reduces to $\frac{dv}{dx} + Pv = Q$, which is linear.

Its solution is $v \cdot e^{\int P dx} = \int e^{\int P dx} \cdot Q \, dx + c.$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y.$ **(M.U. 1988, 93, 99, 2002, 05, 12, 14)**

Sol. : Dividing by $\cos^2 y$

$$\sec^2 y \frac{dy}{dx} + \sec^2 y \cdot \sin 2y \cdot x = x^3 \qquad \therefore \sec^2 y \frac{dy}{dx} + 2 \tan y \cdot x = x^3 \qquad \text{.......... (1)}$$

Put $\tan y = v$ and differentiate w.r.t. x, $\qquad \therefore \sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$

Hence, from (1), we get $\frac{dv}{dx} + 2v \cdot x = x^3$

$\therefore \int P \, dx = \int 2x \, dx = x^2 \qquad \therefore$ I.F. $= e^{\int P dx} = e^{\int 2x dx} = e^{x^2}$

$\therefore$ The solution is $v e^{x^2} = \int e^{x^2} x^3 \, dx + c.$

To find the integral put $x^2 = t$, $x \, dx = \frac{dt}{2}$.

$$\therefore \quad I = \int e^t \cdot t \cdot \frac{dt}{2} = \frac{1}{2}\left[t \cdot e^t - \int e^t \cdot dt\right] \qquad \text{[By parts]}$$

$$= \frac{1}{2}\left[t\,e^t - e^t\right] = \frac{1}{2}e^t(t-1) = \frac{1}{2}e^{x^2}(x^2-1)$$

$\therefore$ The solution is $v\,e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c.$

$$\therefore \quad \tan y \cdot e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c \qquad \therefore \quad \tan y = \frac{1}{2}(x^2-1) + ce^{-x^2}.$$

Example 2 (b) : Solve $\tan y \frac{dy}{dx} + \tan x = \cos y \cdot \cos^3 x.$

Sol. : Dividing by cos y, we get $\quad \sec y \tan y \frac{dy}{dx} + \tan x \sec y = \cos^3 x$

Put sec $y = v$ and differentiate w.r.t. x, $\quad \therefore \quad \sec y \tan y \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore$ The equation reduce to $\quad \frac{dv}{dx} + \tan x \cdot v = \cos^3 x$

Now, $\int P\,dx = \int \tan x\,dx = \log \sec x$

$\therefore$ I.F. $= e^{\int P dx} = e^{\log \sec x} = \sec x$

$\therefore$ The solution is

$$v \sec x = \int \sec x \cos^3 x\,dx + c = \int \cos^2 x\,dx + c$$

$$= \frac{1}{2}\int (1 + \cos 2x)\,dx + c$$

$$\therefore \quad \sec y \cdot \sec x = \frac{x}{2} + \frac{\sin 2x}{4} + c.$$

Example 3 (b) : Solve $\frac{dy}{dx} = e^{x-y} \cdot (e^x - e^y).$ **(M.U. 1995, 2001, 02, 03, 11)**

Sol. : The equation can be written as

$$\frac{dy}{dx} = \frac{e^x}{e^y}(e^x - e^y) \quad i.e. \quad e^y \frac{dy}{dx} + e^y \cdot e^x = e^{2x}$$

Now, put $e^y = v$ and differentiate w.r.t. x,

$$e^y \frac{dy}{dx} = \frac{dv}{dx} \qquad \therefore \quad \frac{dv}{dx} + e^x \cdot v = e^{2x}$$

This is a linear equation of the form $\frac{dy}{dx} + Py = Q.$

Now, $\int P\,dx = \int e^x\,dx = e^x \qquad \therefore \quad$ I.F. $= e^{\int P dx} = e^{e^x}$

Its solution is $v\,e^{e^x} = \int e^{e^x} \cdot e^{2x} \cdot dx + c.$

To find the integral on r.h.s. put $e^x = t$

$$\therefore \quad \int e^{e^x} e^x \cdot e^x \cdot dx = \int e^t \cdot t\,dt = t \cdot e^t - \int e^t \cdot 1 \cdot dt$$

$$= t \cdot e^t - e^t = e^t(t-1)$$

$\therefore$ The solution is $ve^{e^x} = e^{e^x}(e^x - 1) + c$.

$\therefore \quad v = (e^x - 1) + ce^{-e^x} \qquad \therefore \quad e^y = e^x - 1 + ce^{-e^x}$.

Example 4 (b) : Solve $y(x^2 y + e^x)\,dx - e^x\,dy = 0$. **(M.U. 20...)**

Sol. : By dividing by dx, we write the equation as

$$y^2x^2 + ye^x = e^x \frac{dy}{dx}$$

Now, dividing by e^x, we get $\dfrac{dy}{dx} - y = \dfrac{x^2 y^2}{e^x}$.

Dividing by y^2, we get $\dfrac{1}{y^2}\cdot\dfrac{dy}{dx} - \dfrac{1}{y} = \dfrac{x^2}{e^x}$.

Now, put $-\dfrac{1}{y} = v$, $\quad \dfrac{1}{y^2}\cdot\dfrac{dy}{dx} = \dfrac{dv}{dx} \qquad \therefore \quad \dfrac{dv}{dx} + v = \dfrac{x^2}{e^x}$.

This is a linear equation of the form $\dfrac{dy}{dx} + Py = Q$.

Now, $\quad e^{\int P\,dx} = e^{\int 1\cdot dx} = e^x$.

Hence, the solution is $\quad v\cdot e^x = \int e^x \cdot \dfrac{x^2}{e^x}\,dx + c$.

$$\therefore \quad v\cdot e^x = \int x^2\,dx + c = \frac{x^3}{3} + c$$

Putting $v = -\dfrac{1}{y}$, we get the solution is $-\dfrac{1}{y}e^x = \dfrac{x^3}{3} + c$.

Example 5 (b) : Solve $\dfrac{dy}{dx} + x^3\sin^2 y + x\sin 2y = x^3$.

Sol. : We have **(M.U. 1987)**

$$\frac{dy}{dx} + x\sin 2y = x^3(1 - \sin^2 y) \qquad \therefore \quad \frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$$

Dividing by $\cos^2 y$, $\quad \sec^2 y\dfrac{dy}{dx} + 2\tan y\cdot x = x^3$

Put $\tan y = v$ and differentiate w.r.t. x,

$$\sec^2 y\frac{dy}{dx} = \frac{dv}{dx} \qquad \therefore \quad \frac{dv}{dx} + 2x\cdot v = x^3$$

This a linear equation

$$\therefore \quad \int P\,dx = \int 2x\,dx \qquad \therefore \quad e^{\int P\,dx} = e^{x^2}$$

$\therefore$ The solution is $ve^{x^2} = \int e^{x^2}\cdot x^3\,dx + c$.

To find the integral put $x^2 = t$, $x\,dx = \dfrac{dt}{2}$.

$$\therefore \quad \int e^t\cdot\frac{t}{2}\,dt = \frac{1}{2}[te^t - e^t] = \frac{e^t}{2}(t-1) \qquad \text{[By parts]}$$

$\therefore \quad v e^{x^2} = \frac{1}{2} e^{x^2}(x^2 - 1) + c$

Resubstituting $v = \tan y$, we get $\quad \tan y \cdot e^{x^2} = \frac{1}{2} e^{x^2}(x^2 - 1) + c.$

Example 6 (b) : Solve $\frac{dy}{dx} + (2x \tan^{-1} y - x^3)(1 + y^2) = 0.$ **(M.U. 1996, 2006)**

Sol. : The equation can be written as $\quad \frac{1}{1+y^2}\frac{dy}{dx} + 2x \cdot \tan^{-1} y = x^3$

Putting $\tan^{-1} y = v$, $\frac{1}{1+y^2}\frac{dy}{dx} = \frac{dv}{dx}$, we get $\frac{dv}{dx} + 2x \cdot v = x^3$, which is linear.

$\therefore \quad e^{\int P\,dx} = e^{\int 2x\,dx} = e^{x^2}$

$\therefore$ The solution is $v e^{x^2} = \int e^{x^2} \cdot x^3 dx + c.$

To find the integral put $x^2 = t$, $x\,dx = \frac{dt}{2}$. $\quad \therefore \quad I = \int e^t\, t \frac{dt}{2} = \frac{e^t}{2}(t - 1)$ [By parts]

$\therefore$ The solution is $v e^{x^2} = \frac{e^{x^2}}{2}(x^2 - 1) + c. \quad \therefore \quad \tan^{-1} y = \frac{(x^2 - 1)}{2} + ce^{-x^2}.$

Example 7 (b) : Solve $y\frac{dy}{dx} + \frac{4x}{3} - \frac{y^2}{3x} = 0.$ **(M.U. 1990, 92, 94, 2002)**

Sol. : The equation can be written as $\quad y\frac{dy}{dx} - \frac{y^2}{3x} = -\frac{4x}{3}.$

Putting $y^2 = v$, $2y\frac{dy}{dx} = \frac{dv}{dx}$, we get

$$\frac{1}{2}\frac{dv}{dx} - \frac{1}{3x} \cdot v = -\frac{4x}{3}. \quad \therefore \quad \frac{dv}{dx} - \frac{2}{3x} \cdot v = -\frac{8}{3}x$$

This is a linear equation.

$$e^{\int P\,dx} = e^{\int (-2/3x)\,dx} = e^{(-2/3)\log x} = e^{\log(x^{-2/3})} = x^{-2/3}$$

$\therefore$ The solution is,

$$v\,x^{-2/3} = \int x^{-2/3}\left(-\frac{8}{3}x\right)dx + c = -\frac{8}{3}\int x^{1/3}\,dx + c$$

$$= -\frac{8}{3} \cdot \frac{x^{4/3}}{(4/3)} + c = -2x^{4/3} + c$$

$\therefore \quad v\,x^{-2/3} + 2x^{4/3} = c \qquad \therefore \quad y^2 x^{-2/3} + 2x^{4/3} = c.$

Alternatively : The equation can also be solved by converting it into exact equation by multiplying it by the integrating factor as follows.

The equation can be written as $\quad (4x^2 - y^2)\,dx + 3xy\,dy = 0.$

Now, $\quad \frac{\partial M}{\partial y} = -2y, \; \frac{\partial N}{\partial x} = 3y \qquad \therefore \quad \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ N = \frac{-2y - 3y}{3xy} = -\frac{5}{3x}$

$\therefore$ I.F. $= e^{(-5/3)\int (1/x)\,dx} = e^{-(5/3)\log x} = \dfrac{1}{x^{5/3}}$

Multiplying by the I.F. the equation becomes

$$\left(\frac{4x^2 - y^2}{x^{5/3}}\right) dx + \left(\frac{3xy}{x^{5/3}}\right) dy = 0$$

i.e. $(4x^{1/3} - y^2 \cdot x^{-5/3})\,dx + (3x^{-2/3} \cdot y)\,dy = 0$, which is exact

$$\therefore \int M\,dx = \int (4x^{1/3} - y^2 \cdot x^{-5/3})\,dx = \frac{4x^{4/3}}{(4/3)} - \frac{y^2 \cdot x^{-2/3}}{(-2/3)}$$

$$= 3x^{4/3} + \frac{3}{2} y^2 \cdot x^{-2/3} \quad \text{and} \quad \int N\,dy = 0$$

$\therefore$ The solution is $3x^{4/3} + \dfrac{3}{2} y^2 \cdot x^{-2/3} = c'$. *i.e.* $2x^{4/3} + y^2 \cdot x^{-2/3} = c$.

Alternatively : Since the equation is homogeneous it can also be solved by putting $y = vx$.

Example 8 (b) : Solve $\dfrac{dy}{dx} = 1 - 2x(y - x) + x^3$.

Sol. : Putting $y - x = v$, $\dfrac{dy}{dx} - 1 = \dfrac{dv}{dx}$.

$\therefore$ The equation then reduces to $\dfrac{dv}{dx} + 2xv = x^3$, which is linear.

$\therefore$ I.F. $= e^{\int P\,dx} = e^{\int 2x\,dx} = e^{x^2}$

$\therefore$ The solution is $v e^{x^2} = \int e^{x^2} \cdot x^3\,dx + c$.

$\therefore$ $v e^{x^2} = \dfrac{1}{2}(x^2 - 1) e^{x^2} + c$ [Put $x^2 = t$ and integrate it by parts.]

$\therefore$ $y - x = \dfrac{1}{2}(x^2 - 1) + c e^{-x^2}$.

Example 9 (b) : Solve $\dfrac{dy}{dx} - \dfrac{\tan y}{1 + x} = (1 + x)\, e^x \sec y$.

Sol. : Dividing by sec y, we get

$$\frac{1}{\sec y}\frac{dy}{dx} - \frac{1}{\sec y} \cdot \frac{\sin y}{\cos y} \cdot \frac{1}{1 + x} = (1 + x)\, e^x$$

$$\therefore \cos y \frac{dy}{dx} - \sin y \cdot \frac{1}{1 + x} = (1 + x)\, e^x$$

Putting $\sin y = v$, $\cos y \dfrac{dy}{dx} = \dfrac{dv}{dx}$, we get $\dfrac{dv}{dx} - \dfrac{1}{1 + x} \cdot v = (1 + x)\, e^x$, which is linear.

$\therefore$ I.F. $= e^{\int -\frac{1}{1+x} dx} = e^{-\log(1+x)} = \dfrac{1}{1 + x}$

$\therefore$ The solution is

$$v \cdot \frac{1}{1 + x} = \int \frac{1}{1 + x} \cdot (1 + x) \cdot e^x dx + c. \quad \textit{i.e.} \quad \frac{\sin y}{1 + x} = e^x + c.$$

EXERCISE - IX

Solve the following equations : Class (b) : 6 Marks

1. $\sec y \dfrac{dy}{dx} + 2x \sin y = 2x \cos y$

2. $\tan y \dfrac{dy}{dx} + \tan x (1 - \cos y) = 0$

3. $\dfrac{dy}{dx} + \dfrac{1}{x} \tan y = \dfrac{1}{x^2} \tan y \sin y$

4. $\sec^2 y \dfrac{dy}{dx} + 2 \tan x \tan y = \sin x$

5. $x \dfrac{dy}{dx} - 1 = x e^{-y}$

6. $x \cos y \dfrac{dy}{dx} - \sin y = x \sin^2 y$

7. $3y^2 \dfrac{dy}{dx} + 2y^3 x = 4x^3 e^{x^2}$.

[Ans. : (1) $\tan y = 1 + c e^{-x^2}$ (2) $\sec y = 1 + c \cos x$

(3) $2x \operatorname{cosec} y = 1 + cx^2$ (4) $\sec^2 x \cdot \tan y = \sec x + c$

(5) $e^y = x (\log x + c)$ (6) $\operatorname{cosec} y + x (\log x + c) = 0$

(7) $2y^3 e^{x^2} = e^{2x^2}(2x^2 - 1) + c$.]

(d) Equation Reducible To Linear Form (Type III) (By Substitution)

The equation of the form $\boxed{f'(x) \dfrac{dx}{dy} + P f(x) = Q}$

where P and Q are functions of y only can also be reduced to linear form by putting $f(x) = v$.

If we put $f(x) = v$, then $f'(x) \dfrac{dx}{dy} = \dfrac{dv}{dy}$.

The given equation then becomes $\dfrac{dv}{dy} + Pv = Q$ which is linear. Its solution is

$$v \cdot e^{\int P\,dy} = \int e^{\int P\,dy} \cdot Q\,dy + c$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $e^x (x+1)\,dx + (ye^y - xe^x)\,dy = 0$.

Sol. : The equation can be written as $e^x (x+1) \dfrac{dx}{dy} - xe^x = -ye^y$

Putting $v = xe^x$ and differentiating w.r.t. y,

$$\frac{dv}{dy} = e^x (x+1) \frac{dx}{dy}, \quad \therefore \frac{dv}{dy} - v = -ye^y, \text{ which is linear.}$$

Now, $\int P\,dy = \int (-1)\,dy = -y \quad \therefore e^{\int P\,dy} = e^{-y}$

$\therefore$ The solution is,

$$v e^{-y} = \int e^{-y}(-ye^y)\,dy + c' = \int -y\,dy + c' = -\frac{y^2}{2} + c'$$

$\therefore 2v e^{-y} + y^2 = c \quad \therefore 2x e^{x-y} + y^2 = c$ is the required solution.

Example 2 (b) : Solve $(x^3 y^3 + xy)\, dy = dx$. **(M.U. 1998, 2003, 13)**

Sol. : The equation can be written as

$$\frac{dx}{dy} = x^3 y^3 + xy \qquad \therefore \quad \frac{dx}{dy} - xy = x^3 y^3$$

Dividing by x^3, we get $\frac{1}{x^3} \cdot \frac{dx}{dy} - \frac{1}{x^2} \cdot y = y^3$.

Now put $-\frac{1}{x^2} = v$.

$$\therefore \quad \frac{2}{x^3} \cdot \frac{dx}{dy} = \frac{dv}{dy} \qquad \therefore \quad \frac{1}{2} \cdot \frac{dv}{dy} + yv = y^3 \qquad \therefore \quad \frac{dv}{dy} + 2y \cdot v = 2y^3$$

This is linear of the form $\frac{dv}{dy} + Pv = Q$.

$$\therefore \quad \text{I.F.} = \int e^{P\,dy} = \int e^{2y}\, dy = e^{y^2}$$

$\therefore$ The solution is

$$v \cdot e^{\int P\,dy} = \int Q e^{\int P\,dy} + c$$

$$\therefore \quad v \cdot e^{y^2} = \int e^{y^2} 2y^3\, dy + c = \int e^{y^2} \cdot 2y \cdot y^2\, dy + c$$

Put $y^2 = t$, $2y\, dy = dt$.

$$\therefore \quad \int e^{y^2} \cdot 2y \cdot y^2\, dy = \int e^t \cdot t \cdot dt = t e^t - e^t = e^t (t-1)$$

$\therefore$ The solution is $v \cdot e^{y^2} = e^{y^2}(y^2 - 1) + c$

$$\therefore \quad -\frac{1}{x^2} \cdot e^{y^2} = e^{y^2}(y^2 - 1) + c \qquad \therefore \quad \frac{1}{x^2} = (1 - y^2) + c\, e^{-y^2}.$$

*** Example 3 (b) :** Solve $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$. **(M.U. 1994)**

Sol. : The equation can be written as $(1 + y^2) \frac{dx}{dy} = \tan^{-1} y - x$

Put $\tan^{-1} y = v$,

$$\therefore \quad \frac{1}{1+y^2} \frac{dy}{dx} = \frac{dv}{dx} \qquad \therefore \quad (1+y^2)\frac{dx}{dy} = \frac{dx}{dv} \qquad \therefore \quad \frac{dx}{dv} + x = v$$

This is a linear differential equation.

$$\therefore \quad \text{I.F.} = e^{\int P\,dv} = e^{\int 1 \cdot dv} = e^v$$

$\therefore$ The solution is $x e^v = \int e^v \cdot v\, dv + c = v e^v - e^v + c$ [By parts]

$$\therefore \quad x = v - 1 + c e^{-v} \qquad \therefore \quad x = \tan^{-1} y - 1 + c\, e^{-\tan^{-1} y}.$$

EXERCISE - X

Solve the following equations : Class (b) : 6 Marks

1. $e^x (x+1)\, dx + (y^2 e^{2y} - x e^x)\, dy = 0$
2. $y \frac{dx}{dy} = x + y x^2 \log y$

3. $\dfrac{dx}{dy} = e^{y-x}(e^y - e^x)$

4. $y \sin x \dfrac{dx}{dy} - \cos x = 2y^3 \cos^2 x$

[Ans. : (1) $xe^x + (y-1) = c\,e^{-y}$ (2) $\dfrac{y}{x} + \dfrac{y^2}{2}\log y - \dfrac{y^2}{4} = c$

(3) $e^x = e^y - 1 + c\,e^{-e^y}$ (4) $\sec x = y^3 + cy$]

9. Bernoulli's Equation

Equations Reducible To Linear Form (Type IV)

An equation of the form $$\frac{dy}{dx} + Py = Qy^n$$

where P and Q are functions of x alone, is called **Bernoulli's equation and is reducible to linear form by dividing by** y^n and by putting $\dfrac{1}{y^{n-1}} = v$. After simplification it can be solved by the previous method.

To solve $\dfrac{dy}{dx} + Py = Qy^n$

Dividing by y^n we get, $$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}} \cdot P = Q$$

Putting $\dfrac{1}{y^{n-1}} = v$, $(1-n)\dfrac{1}{y^n}\dfrac{dy}{dx} = \dfrac{dv}{dx}$, we get $\dfrac{1}{(1-n)}\dfrac{dv}{dx} + Pv = Q$

i.e. $$\frac{dv}{dx} + (1-n)Pv = (1-n)Q$$

which is linear in v. After solving it we resubstitute $v = \dfrac{1}{y^{n-1}}$ and get the solution of the given equation.

Jacob Bernoulli (1654 - 1705)

Jacob Bernoulli a member of the most distinguished family of mathematicians was born in Basel, Switzerland. Bernoulli received M.A. in Philosophy in 1671 and a degree in Theology in 1676. During this period he studied mathematics and astronomy which his father did not like. In 1687 he became professor of mathematics at the university of Basel remaining there until his death. He was the first to use the term integral in the sense we use in calculus today. Bernoulli's most famous work **'Ars Conjectandi'** (The Art of Conjecture) was published in 1713 after his death. It contains significant contribution

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $\frac{dy}{dx} = x^3y^3 - xy.$

(M.U. 2007, 11)

Sol. : Since $\frac{dy}{dx} + xy = x^3y^3$, this is a Bernoulli's equation.

Dividing by y^3, we get, $\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}x = x^3$

Putting $\frac{1}{y^2} = v$, $-\frac{2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$ the equation becomes

$$-\frac{1}{2}\frac{dv}{dx} + xv = x^3 \quad i.e. \quad \frac{dv}{dx} - 2xv = -2x^3$$

$$\therefore \int P\,dx = \int -2x\,dx = -x^2 \qquad \therefore \quad \text{I.F.} = e^{\int P dx} = e^{-x^2}$$

$\therefore$ The solution is $ve^{-x^2} = \int e^{-x^2}(-2x^3)\,dx + c.$

To find the integral put $x^2 = t \quad \therefore \; 2x\,dx = dt$

$$\therefore \int e^{-x^2}(x^2)(-2x)\,dx = -\int e^{-t}\cdot t\,dt$$

Integrating by parts $= te^{-t} - \int e^{-t}\,dt = te^{-t} + e^{-t} = x^2e^{-x^2} + e^{-x^2}$

$$\therefore \; ve^{-x^2} = x^2e^{-x^2} + e^{-x^2} + c$$

$$\therefore \; v = x^2 + 1 + ce^{x^2} \quad \therefore \; \frac{1}{y^2} = x^2 + 1 + ce^{x^2}.$$

Example 2 (b) : Solve $x\frac{dy}{dx} + y = x^3y^6.$

(M.U. 2004, 2013)

Sol. : Dividing by y^6, we get $\frac{x}{y^6}\frac{dy}{dx} + \frac{1}{y^5} = x^3$. Again dividing by x, $\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y^5} = x^2.$

Put $\frac{1}{y^5} = v \quad \therefore -5y^{-6}\frac{dy}{dx} = \frac{dv}{dx}$

$$\therefore \; -\frac{1}{5}\frac{dv}{dx} + \frac{1}{x}\cdot v = x^2 \quad \therefore \; \frac{dv}{dx} - \frac{5}{x}\cdot v = -5x^2.$$

This is a linear equation.

$$\int P\,dx = \int -\frac{5}{x}\,dx = -5\log x = \log x^{-5} \quad \therefore \; \text{I.F.} = e^{\int P dx} = e^{\log x^{-5}} = x^{-5}$$

$\therefore$ The solution is $\quad v\cdot e^{\int P dx} = \int Q\cdot e^{\int P dx} + c.$

$$\therefore \; vx^{-5} = \int -5x^{-5}\cdot x^2dx + c = -5\int x^{-3}\,dx + c = \frac{5}{2}x^{-2} + c.$$

Putting $v = 1/y^5$, the solution is

$$\frac{1}{y^5}\frac{1}{x^5} = \frac{5}{2}\frac{1}{x^2} + c \quad \therefore \; \frac{1}{y^5} = \frac{5}{2}x^3 + cx^5.$$

Example 3 (b) : Solve $y - \cos x \dfrac{dy}{dx} = y^2(1 - \sin x)\cos x$. Given that $y = 2$ when $x = 0$. **(M.U. 1989)**

Sol. : The given equation can be written as $\cos x \dfrac{dy}{dx} - y = -y^2(1 - \sin x)\cos x$

Dividing by $-y^2 \cos x$ $\quad \therefore \; -\dfrac{1}{y^2}\dfrac{dy}{dx} + \dfrac{\sec x}{y} = 1 - \sin x$

Putting $\dfrac{1}{y} = v$ and $-\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$, we get, $\dfrac{dv}{dx} + \sec x \cdot v = 1 - \sin x$

This is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$.

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \sec x\,dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

$\therefore$ The solution is

$$v \cdot (\sec x + \tan x) = \int (\sec x + \tan x)(1 - \sin x)\,dx + c$$

$$= \int \frac{(1 + \sin x)}{\cos x} \cdot (1 - \sin x)\,dx = \int \frac{(1 - \sin^2 x)}{\cos x}\,dx$$

$$= \int \frac{\cos^2 x}{\cos x}\,dx = \int \cos x\,dx = \sin x + c$$

$$\therefore \; \frac{\tan x + \sec x}{y} = \sin x + c$$

When $x = 0$, $y = 2$. $\quad \therefore \; \dfrac{1}{2} = c \quad \therefore \; \tan x + \sec x = y\left(\sin x + \dfrac{1}{2}\right)$.

Example 4 (b) : Solve $\dfrac{dy}{dx} = xy + y^2 e^{(-x^2/2)} \cdot \log x$. **(M.U. 1998)**

Sol. : The equation can be written as $\dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{1}{y} \cdot x = e^{-x^2/2} \cdot \log x$.

Putting $-\dfrac{1}{y} = v$, $\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$, we get $\dfrac{dv}{dx} + xv = e^{-x^2/2} \cdot \log x$.

This is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$.

$$\therefore \; \text{I.F.} = e^{\int x\,dx} = e^{x^2/2}$$

$\therefore$ The solution is

$$v\,e^{x^2/2} = \int e^{x^2/2} \cdot e^{-x^2/2} \cdot \log x\,dx + c'$$

$$= \int \log x\,dx + c' = \log x \cdot x - \int x \cdot \frac{1}{x}\,dx + c' \qquad \text{[By parts]}$$

$$= x \log x - x + c'$$

$\therefore$ The solution is $-\dfrac{1}{y} e^{x^2/2} = x(\log x - 1) + c'$. i.e. $\dfrac{e^{x^2/2}}{y} = x(1 - \log x) + c$.

Example 5 (b) : Solve $\dfrac{dz}{dx}+\dfrac{z}{x}\log z=\dfrac{z}{x^2}(\log z)^2$. **(M.U. 1990, 94, 2005)**

Sol. : The equation can be written as $\dfrac{1}{z}\cdot\dfrac{dz}{dx}+\dfrac{1}{x}\log z=\dfrac{1}{x^2}\cdot(\log z)^2$.

Putting $\log z=y$, $\dfrac{1}{z}\cdot\dfrac{dz}{dx}=\dfrac{dy}{dx}$ in the given equation, we get $\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{y^2}{x^2}$.

Dividing by y^2, we get, $\dfrac{1}{y^2}\dfrac{dy}{dx}+\dfrac{1}{x\cdot y}=\dfrac{1}{x^2}$

Now, we put $-\dfrac{1}{y}=v$, $\dfrac{1}{y^2}\dfrac{dy}{dx}=\dfrac{dv}{dx}$. $\quad\therefore\ \dfrac{dv}{dx}-\dfrac{1}{x}\cdot v=\dfrac{1}{x^2}$.

This is linear differential equation of the form $\dfrac{dy}{dx}+Py=Q$.

$$\therefore\ \text{I.F.}=e^{\int P\,dx}=e^{\int-(1/x)\,dx}=e^{-\log x}=\frac{1}{x}$$

$\therefore$ The solution is $v\cdot\dfrac{1}{x}=\displaystyle\int\frac{1}{x}\cdot\frac{1}{x^2}\,dx+c=-\frac{1}{2x^2}+c.$

Resubstituting $v=-\dfrac{1}{y}=-\dfrac{1}{\log z}$, we get,

$$-\frac{1}{x\log z}=-\frac{1}{2x^2}+c\quad i.e.\quad \frac{1}{x\log z}=\frac{1}{2x^2}+c'$$

This is the required solution.

Example 6 (b) : Solve $\dfrac{dr}{d\theta}=r\tan\theta-\dfrac{r^2}{\cos\theta}$. **(M.U. 1994, 2007, 12)**

or $\dfrac{dr}{d\theta}=\dfrac{r\sin\theta-r^2}{\cos\theta}$. **(M.U. 1997)**

Sol. : The equation can be written as $\dfrac{1}{r^2}\dfrac{dr}{d\theta}-\dfrac{1}{r}\tan\theta=-\sec\theta$

Putting $-\dfrac{1}{r}=v$, $\dfrac{1}{r^2}\dfrac{dr}{d\theta}=\dfrac{dv}{d\theta}$ $\quad\therefore\ \dfrac{dv}{d\theta}+\tan\theta\cdot v=-\sec\theta$

This is a linear equation.

$$e^{\int P\,d\theta}=e^{\int\tan\theta\,d\theta}=e^{\log\sec\theta}=\sec\theta$$

$\therefore$ The solution is

$$v\sec\theta=\int(-\sec\theta)\cdot\sec\theta\ d\theta+c$$
$$=\int-\sec^2\theta\ d\theta+c=-\tan\theta+c$$

$\therefore\ -\dfrac{1}{r}\sec\theta=-\tan\theta+c\qquad\therefore\ \dfrac{1}{r}=\sin\theta+c'\cos\theta.$

Example 7 (b) : Solve $\dfrac{dy}{dx}+\left(\dfrac{x}{1-x^2}\right)y = x\sqrt{y}$. **(M.U. 1997, 99)**

Sol. : Dividing by $\sqrt{y}$, we get $\dfrac{1}{\sqrt{y}}\dfrac{dy}{dx}+\left(\dfrac{x}{1-x^2}\right)\sqrt{y} = x$.

Now, put $\sqrt{y} = z$. $\therefore \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} = \dfrac{dz}{dx}$.

$$\therefore 2\frac{dz}{dx}+\left(\frac{x}{1-x^2}\right)z = x \qquad \therefore \frac{dz}{dx}+\frac{1}{2}\cdot\frac{x}{(1-x^2)}z = \frac{x}{2}$$

This is a linear differential equation of the form $\dfrac{dy}{dx}+Py = Q$.

$$\therefore \text{ I.F.} = e^{\int x/2(1-x^2)}dx = e^{-\frac{1}{4}\log(1-x^2)} = \frac{1}{(1-x^2)^{1/4}}.$$

$\therefore$ The solution is

$$z\cdot\frac{1}{(1-x^2)^{1/4}} = \int\frac{1}{(1-x^2)^{1/4}}\cdot\frac{x}{2}\,dx + c$$

$$= -\frac{1}{3}(1-x^2)^{3/4}+c \qquad [\text{ Put } x^2 = t\,]$$

$$\therefore z = -\frac{1}{3}(1-x^2)+c(1-x^2)^{1/4}$$

$$\sqrt{y}+\frac{1}{3}(1-x^2) = c(1-x^2)^{1/4}.$$

Example 8 (b) : Solve $\dfrac{dy}{dx}+\dfrac{2}{x}y = \dfrac{y^3}{x^3}$. **(M.U. 2000)**

Sol. : Dividing by y^3, we get $\dfrac{1}{y^3}\dfrac{dy}{dx}+\dfrac{2}{x}\cdot\dfrac{1}{y^2} = \dfrac{1}{x^3}$

Put $\dfrac{1}{y^2} = v \quad \therefore -\dfrac{2}{y^3}\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$$-\frac{1}{2}\frac{dv}{dx}+\frac{2}{x}\cdot v = \frac{1}{x^3} \qquad \therefore \frac{dv}{dx}-\frac{4}{x}\cdot v = -\frac{2}{x^3}$$

This is a linear differential equation of the form $\dfrac{dy}{dx}+Py = Q$.

$$\text{I.F.} = e^{\int-(4/x)\,dx} = e^{-4\log x} = e^{\log(1/x^4)} = \frac{1}{x^4}$$

$\therefore$ The solution is

$$v\cdot\frac{1}{x^4} = \int\frac{1}{x^4}\left(-\frac{2}{x^3}\right)dx + c = \int -2x^{-7}dx + c = \frac{-2}{-6}x^{-6}+c = \frac{1}{3x^6}+c$$

$$\therefore \frac{1}{y^2} = \frac{1}{3x^2}+cx^2.$$

(a) Another form of Bernoulli's Equation

A differential equation of the form $\frac{dx}{dy} + P'x = Q'x^n$

is also of Bernoulli's form with x any y having changed their positions. By dividing by x^n and putting $\frac{1}{x^{n-1}}$, we can reduce it to linear form as above.

Example 1 (b) : Solve $y\,dx + x(1-3x^2y^2)\,dy = 0$. **(M.U. 1997, 2009)**

Sol. : The equation can be written as

$$y\frac{dx}{dy} + x = 3x^3y^2 \quad \therefore \quad \frac{dx}{dy} + \frac{1}{y}x = 3x^3y$$

Dividing by x^3, we get $\quad \frac{1}{x^3}\frac{dx}{dy} + \frac{1}{y}\cdot\frac{1}{x^2} = 3y$

Putting $\frac{1}{x^2} = v$ and $-\frac{2}{x^3}\frac{dx}{dy} = \frac{dv}{dy}$, we get,

$$-\frac{1}{2}\frac{dv}{dy} + \frac{1}{y}\cdot v = 3y \quad \therefore \quad \frac{dv}{dy} - \frac{2}{y}\cdot v = -6y$$

This is a linear differential equation.

$$\therefore \quad \text{I.F.} = e^{\int(-2/y)dy} = e^{-2\log y} = e^{\log(1/y^2)} = \frac{1}{y^2}$$

$\therefore$ The solution is $v\cdot\frac{1}{y^2} = \int\frac{1}{y^2}\cdot(-6y)\,dy + c = -6\log y + c$

$$\therefore \quad \frac{1}{x^2y^2} = -6\log y + c \quad \therefore \quad \frac{1}{x^2y^2} + 6\log y = c.$$

Example 2 (b) : Solve $y\frac{dx}{dy} = x - yx^2\sin y$. **(M.U. 1991)**

Sol. : We have $\frac{dx}{dy} - \frac{x}{y} = -x^2\sin y$

Dividing by x^2, $\quad \frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}\cdot\frac{1}{y} = -\sin y$

Now, putting $-\frac{1}{x} = v$ and $\frac{1}{x^2}\frac{dx}{dy} = \frac{dv}{dy}$, we get $\frac{dv}{dy} + \frac{1}{y}v = -\sin y$.

This is a linear differential equation.

$$\therefore \quad e^{\int P\,dy} = e^{\int(1/y)dy} = e^{\log y} = y$$

$\therefore$ The solution is

$$vy = \int -\sin y\cdot y\,dy + c = [y\cos y - \int\cos y\cdot 1\cdot dy] + c \qquad \text{[By parts]}$$

$$\therefore \quad vy = y\cos y - \sin y + c$$

$\therefore$ The solution is $-\frac{y}{x} = y\cos y - \sin y + c$.

Example 3 (b) : Solve $xy(1+xy^2)\dfrac{dy}{dx}=1.$ **(M.U. 1995, 2002, 03, 14)**

Sol. : We have $\dfrac{dx}{dy}=xy+x^2y^3$ $\quad\therefore\quad \dfrac{1}{x^2}\dfrac{dx}{dy}-\dfrac{1}{x}y=y^3.$

Putting $-\dfrac{1}{x}=v$ and $\dfrac{1}{x^2}\dfrac{dx}{dy}=\dfrac{dv}{dy}$, we get $\dfrac{dv}{dy}+vy=y^3.$

This is a linear differential equation.

$$\therefore\quad e^{\int P'dy}=e^{\int y\,dy}=e^{y^2/2}$$

$\therefore$ The solution is $v\,e^{y^2/2}=\int e^{y^2/2}\cdot y^3\,dy+c.$

$$=e^{y^2/2}(y^2-2)\qquad\left(\text{Put }\frac{y^2}{2}=t\right)$$

$\therefore$ The solution is $v\,e^{y^2/2}=e^{y^2/2}(y^2-2)+c$

$$\therefore\quad -\frac{1}{x}e^{y^2/2}=e^{y^2/2}(y^2-2)+c$$

$$\therefore\quad -\frac{1}{x}=y^2-2+c\,e^{-y^2/2}\qquad\therefore\quad \frac{1}{x}=2-y^2+c'e^{-y^2/2}.$$

(b) Equations reducible to Bernoulli's Form

Example 1 (c) : Solve $\dfrac{dy}{dx}=1-x(y-x)-x^3(y-x)^2.$ ***(M.U. 2000, 02, 08)***

Sol. : Putting $y-x=v$, we get $\dfrac{dy}{dx}-1=\dfrac{dv}{dx}.$

Hence, the given equation reduces to

$$\frac{dv}{dx}+xv=-x^3v^2\qquad\therefore\quad -\frac{1}{v^2}\frac{dv}{dx}-\frac{x}{v}=x^3$$

Putting $\dfrac{1}{v}=t,\ -\dfrac{1}{v^2}\dfrac{dv}{dx}=\dfrac{dt}{dx}$, we get $\dfrac{dt}{dx}-xt=x^3$

This is a linear differential equation.

$$\therefore\quad \text{I.F.}=e^{\int -x\,dx}=e^{-x^2/2}$$

$\therefore$ The solution is $t\cdot e^{-x^2/2}=\int e^{-x^2/2}\cdot x^3dx+c.$

Put $-\dfrac{x^2}{2}=u$ $\quad\therefore\quad x^2=-2u,\ x\,dx=-du$

$$\therefore\quad t\,e^{-x^2/2}=\int e^u(-2u)(-du)+c=2\int e^u u\,du+c=2[u\,e^u-e^u]+c\qquad\text{[By parts]}$$

$$=2\,e^{-x^2/2}\left(-\frac{x^2}{2}-1\right)+c$$

$$\therefore\quad t=2\left(-\frac{x^2}{2}-1\right)+c\,e^{x^2/2}.$$

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Resubstituting for t and u, $\dfrac{1}{y-x} = -x^2 - 2 + c\,e^{x^2/2}$.

Example 2 (c) : Solve $\dfrac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1$. **(M.U. 2013)**

Sol. : Putting $x + y = v$, $1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$.

The equation then becomes

$$\frac{dv}{dx} - 1 + x \cdot v = x^3 v^3 - 1 \qquad \therefore \quad \frac{dv}{dx} + xv = x^3 v^3$$

This is a Bernoulli's equation. Dividing by v^3 throughout, we get

$$\frac{1}{v^3} \cdot \frac{dv}{dx} + \frac{1}{v^2} \cdot x = x^3$$

Putting $\dfrac{1}{v^2} = t$, $-\dfrac{2}{v^3} \cdot \dfrac{dv}{dx} = \dfrac{dt}{dx}$, we get

$$-\frac{1}{2} \cdot \frac{dt}{dx} + t\,x = x^3 \qquad \therefore \quad \frac{dt}{dx} - 2x\,t = -2x^3$$

This is a linear equation.

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{-\int 2x\,dx} = e^{-x^2}$$

$\therefore$ The solution is $t \cdot e^{-x^2} = \int e^{-x^2}(-2x^3)\,dx + c$.

For integration, we put $-x^2 = t$, then $-2x\,dx = dt$.

$$\therefore \quad \int e^{-x^2} \cdot 2x^3\,dx = \int e^t\,t\,dt = t\,e^t - e^t = -x^2 e^{-x^2} - e^{-x^2}$$

Hence, the solution is $\quad t \cdot e^{-x^2} = x^2 e^{-x^2} + e^{-x^2} + c$.

Putting $t = \dfrac{1}{v^2} = \dfrac{1}{(x+y)^2}$, we get the solution as $\dfrac{1}{(x+y)^2} = x^2 + 1 + c\,e^{x^2}$.

EXERCISE - XI

Solve the following equations : Class (b) : 6 Marks

1. $\dfrac{dy}{dx} = 2y(1 - 2xy)$

2. $2x\,dx - y^2(y^3 + x^2)\,dy = 0$

3. $3x(1 - x^2)\,y^2 \dfrac{dy}{dx} + (2x^2 - 1)\,y^3 = ax^3$

4. $x\dfrac{dy}{dx} + y = y^3 x^{n+1}$ **(M.U. 1992)**

5. $\dfrac{dz}{dx} + \dfrac{z}{x}\log z = \dfrac{z}{x^2}(\log z)^3$

6. $\dfrac{dy}{dx} - xy = y^2 e^{-(x^2/2)} \cdot \log x$

7. $x\dfrac{dy}{dx} + 2y = y^2 x^3$

8. $x\dfrac{dy}{dx} + y = y^2 \log x$

9. $y\dfrac{dx}{dy} = x - y\,x^2 \cos y$ **(M.U. 1991)**

10. $xy - \dfrac{dy}{dx} = y^3 e^{-x^2}$

11. $\dfrac{dy}{dx} + \dfrac{y}{x} = y^3 x^n$ **(M.U. 1989)**

[Ans. : (1) $\frac{1}{y} = 2x - 1 + ce^{-2x}$ (2) $x^2 + y^3 + 3 = ce^{y^3/3}$ (3) $y^3 = ax + cx\sqrt{1-x^2}$

(4) $\frac{(n-1)}{y^2} = cx^2 - 2x^{n+1}$ (5) $\frac{1}{(\log z)^2} = \frac{2}{3}\cdot\frac{1}{x} + cx^2$ (6) $\frac{1}{y}e^{x^2/2} + x\log x - x = c$

(7) $\frac{1}{y} = x^2(c - x)$ (8) $\frac{1}{y} = \log x + 1 + cx$ (9) $\frac{y}{x} = y\sin y + \cos y + c$

(10) $e^{x^2} = y^2(2x + c)$ (11) $\frac{(n-1)}{y^2} = cx^2 - 2x^{n+1}$.]

Procedure

The differential equation of the first order and first degree is of the form

$$\frac{dy}{dx} = f(x, y).$$

To solve the equation you *may* proceed as follows.

1. Write the equation in the form $M\,dx + N\,dy = 0$

   Test whether $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$. If so, the equation is exact.

   $\therefore$ The solution is

   $$\int M\,dx \text{ (treating } y \text{ constant)} + \int (\text{Terms in } N \text{ free from } x)\,dy = c$$

2. See whether the equation is of the form

   $$\frac{dy}{dx} + Py = Q \quad \text{or} \quad \frac{dx}{dy} + P'x = Q'$$

   If so, apply the formula of linear differential equation

   $$y\,e^{\int P\,dx} = \int e^{P\,dx}\cdot Q\,dx + c \quad \text{or} \quad x\,e^{\int P'\,dy} = \int e^{P'\,dy}\cdot Q'\,dy + c.$$

3. See whether the equation can be written as

   $$f'(y)\frac{dy}{dx} + P\,f(y) = Q$$

   then put $f(y) = v$, the equation reduces to

   $$\frac{dv}{dx} + Pv = Q, \text{ which is linear.}$$

4. See whether the equation can be written as

   $$f'(x)\frac{dx}{dy} + P'f(x) = Q'$$

   then put $f(x) = v$, the equation reduces to $\frac{dv}{dy} + P'v = Q'$, which is linear.

5. See whether the equation can be written as Bernoulli's equation

   $$\frac{dy}{dx} + Py = Qy^n \quad \text{or} \quad \frac{dx}{dy} + P'x = Q'x^n.$$

   If yes solve it by standard methods.

6. If all above methods fail try to find the integrating factor by applying the four rules.

Miscellaneous Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $\frac{dy}{dx} = -\frac{4x^3y^2 + y\cos(xy)}{2x^4y + x\cos(xy)}$.

Sol. : The equation can be written as

$$(4x^3y^2 + y\cos(xy))\,dx + (2x^4y + x\cos(xy))\,dy = 0$$

Here, $M = 4x^3y^2 + y\cos(xy)$; $N = 2x^4y + x\cos(xy)$

$$\therefore \quad \frac{\partial M}{\partial y} = 8x^3y + \cos(xy) - xy\sin(xy)\,; \quad \frac{\partial N}{\partial x} = 8x^3y + \cos(xy) - xy\sin(xy)$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$ the equation is exact.

$$\int M\,dx = \int [4x^3y^2 + y\cos(xy)]\,dx = x^4y^2 + \sin xy$$

$$\int (\text{Terms in } N \text{ free from } x)\,dy = 0$$

$\therefore$ The solution is $x^4y^2 + \sin xy = c$.

Example 2 (b) : Solve $[2x\sin h\left(\frac{y}{x}\right) + 3y\cos h\left(\frac{y}{x}\right)]\,dx - 3x \cdot \cos h\left(\frac{y}{x}\right) \cdot dy = 0$.

Sol. : Here, $M = 2x\sin h\left(\frac{y}{x}\right) + 3y\cos h\left(\frac{y}{x}\right)$ and $N = -3x\cos h\left(\frac{y}{x}\right)$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x \cdot \cos h\left(\frac{y}{x}\right) \cdot \frac{1}{x} + 3\cos h\left(\frac{y}{x}\right) + 3y\sin h\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$\therefore \quad \frac{\partial M}{\partial y} = 5\cos h\left(\frac{y}{x}\right) + \frac{3y}{x}\sin h\left(\frac{y}{x}\right)$$

$$\frac{\partial N}{\partial x} = -3\cos h\left(\frac{y}{x}\right) - 3x\sin h\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right) = -3\cos h\left(\frac{y}{x}\right) + \frac{3y}{x}\sin h\left(\frac{y}{x}\right)$$

$$\therefore \quad \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ N = \frac{8\cos h(y/x)}{-3x\cos h(y/x)} = -\frac{8}{3x} = f(x)$$

$$\therefore \quad \text{I.F.} = e^{\int f(x)\,dx} = e^{\int -(8/3x)\,dx} = e^{-(8/3)\log x} = x^{-(8/3)}$$

Multiplying by the I.F., we get,

$$\left[2x^{-5/3} \cdot \sin h\left(\frac{y}{x}\right) + 3x^{-8/3} \cdot y \cdot \cos h\left(\frac{y}{x}\right)\right] dx - 3x^{-5/3} \cdot \cos h\left(\frac{y}{x}\right) \cdot dy = 0$$

Since $\int M\,dx$ is rather a complex integral we use **the rule 2** given in note on page 1-7.

$$\therefore \quad \int N\,dy \text{ (treating } x \text{ constant)} = \int -3x^{-5/3} \cdot \cos h\left(\frac{y}{x}\right) dy$$

$$= -3x^{-5/3} \cdot \sin h\left(\frac{y}{x}\right) \cdot x = -3x^{-2/3} \cdot \sin h\left(\frac{y}{x}\right)$$

$$\int (\text{Terms in } M \text{ free from } y)\,dx = 0$$

$\therefore$ The solution is $x^{-2/3} \cdot \sin h\left(\frac{y}{x}\right) = -\frac{c'}{3} = c$.

Example 3 (b) : Solve $\frac{dy}{dx} = \frac{e^y}{x^2} - \frac{1}{x}$.

Sol. : Dividing by e^y, $e^{-y}\frac{dy}{dx} + e^{-y}\frac{1}{x} = \frac{1}{x^2}$.

Put $e^{-y} = v$ $\therefore -e^{-y}\frac{dy}{dx} = \frac{dv}{dx}$ $\therefore \frac{dv}{dx} - \frac{1}{x}\cdot v = -\frac{1}{x^2}$, which is linear.

$\therefore \int P\,dx = \int -\frac{dx}{x} = -\log x = \log\left(\frac{1}{x}\right)$ $\therefore e^{\int P\,dx} = e^{\log(1/x)} = \frac{1}{x}$.

$\therefore$ The solution is $v \cdot \frac{1}{x} = \int \frac{1}{x}\left(-\frac{1}{x^2}\right)dx + c = \frac{1}{2x^2} + c$

$\therefore 2vx = 1 + 2cx^2$ $\therefore 2x\,e^{-y} = 2cx^2 + 1$.

Example 4 (b) : Solve $4x^2 y\frac{dy}{dx} = 3x(3y^2+2) + (3y^2+2)^3$. **(M.U. 2003)**

Sol. : We put $3y^2 + 2 = v$ $\therefore 6y\frac{dy}{dx} = \frac{dv}{dx}$ $\therefore y\frac{dy}{dx} = \frac{1}{6}\frac{dv}{dx}$.

$\therefore$ The equation reduces to $4x^2 \cdot \frac{1}{6}\frac{dv}{dx} - 3xv = v^3$.

Dividing by $\frac{2x^2}{3}$, throughout, $\frac{dv}{dx} - \frac{9}{2x}v = \frac{3}{2x^2}v^3$

This is Bernoulli's equation $\therefore \frac{1}{v^3}\frac{dv}{dx} - \frac{9}{2x}\cdot\frac{1}{v^2} = \frac{3}{2x^2}$

Putting $-\frac{1}{v^2} = t$ $\therefore \frac{2}{v^3}\frac{dv}{dx} = dt$

$\therefore \frac{1}{2}\cdot\frac{dt}{dx} + \frac{9}{2x}\cdot t = \frac{3}{2x^2}$ $\therefore \frac{dt}{dx} + \frac{9}{x}\cdot t = \frac{3}{x^2}$

$\therefore$ I.F. $= e^{\int (9/x)dx} = e^{9\log x} = x^9$

$\therefore$ The solution is $t \cdot x^9 = \int x^9 \cdot \frac{3}{x^2}dx + c = \int 3x^7 dx + c = \frac{3}{8}x^8 + c$

$\therefore -\frac{x^9}{(3y^2+2)^2} = \frac{3}{8}x^8 + c$ $\therefore \frac{x^9}{(3y^2+2)^2} + \frac{3}{8}x^8 = c$.

Example 5 (b) : Solve $4xy\frac{dy}{dx} = (y^2+3) + x^3(y^2+3)^3$. **(M.U. 2008)**

Sol. : Put $y^2 + 3 = v$ $\therefore 2y\frac{dy}{dx} = \frac{dv}{dx}$.

$\therefore$ The equation becomes, $2x\frac{dv}{dx} = v + x^3 v^3$

Dividing by v^3, we get, $\frac{2}{v^3}\cdot x\frac{dv}{dx} - \frac{1}{v^2} = x^3$

Dividing by x, we get, $\dfrac{2}{v^3}\dfrac{dv}{dx}+\dfrac{1}{x}\cdot\left(-\dfrac{1}{v^2}\right)=x^2$

Putting $-\dfrac{1}{v^2}=z$, $\dfrac{2}{v^3}\dfrac{dv}{dx}=\dfrac{dz}{dx}$, we get $\dfrac{dz}{dx}+\dfrac{1}{x}\cdot z=x^2$, which is linear.

$\therefore\ e^{\int P\,dx}=e^{\int (1/x)\,dx}=e^{\log x}=x$

$\therefore$ The solution is $z\cdot x=\int x\cdot x^2\,dx+c=\int x^3\,dx+c=\dfrac{x^4}{4}+c$

$\therefore\ -\dfrac{1}{(y^2+3)^2}\cdot x=\dfrac{x^4}{4}+c$

$\therefore\ 4x+x^4\cdot(y^2+3)^2+c'(y^2+3)^2=0.$

Example 6 (b) : Solve $(xy^2-e^{1/x^3})\,dx-x^2y\,dy=0.$ **(M.U. 2004, 07, 14)**

Sol. : The equation can be written as $y\dfrac{dy}{dx}-\dfrac{1}{x}y^2=-\dfrac{1}{x^2}e^{1/x^3}$.

Putting $y^2=v$ and $2y\dfrac{dy}{dx}=\dfrac{dv}{dx}$, we get $\dfrac{dv}{dx}-\dfrac{2}{x}v=-\dfrac{2}{x^2}e^{1/x^3}$, which is linear.

$\therefore\ \int P\,dx=\int-\dfrac{2}{x}\,dx=-2\log x=\log\dfrac{1}{x^2}$

$\therefore$ I.F. $=e^{\int P\,dx}=e^{\log(1/x^2)}=\dfrac{1}{x^2}$

$\therefore$ The solution is

$$v\cdot\frac{1}{x^2}=\int-\frac{2}{x^2}e^{1/x^3}\cdot\frac{1}{x^2}\,dx+c=-2\int e^{1/x^3}\cdot\frac{1}{x^4}\,dx+c$$

To obtain the integral on the r.h.s. put $x^{-3}=t$.

$$\therefore\ \int e^{1/x^3}\cdot\frac{1}{x^4}\,dx=\int e^t\cdot\frac{dt}{-3}=-\frac{e^t}{3}=-\frac{1}{3}e^{1/x^3}$$

$\therefore$ The solution is

$$v\cdot\frac{1}{x^2}=\frac{2}{3}e^{1/x^3}+c\quad\therefore\ \frac{y^2}{x^2}=\frac{2}{3}e^{1/x^3}+c.$$

Note ....

See Ex. 5, page 1-18 for another method.

Example 7 (b) : Solve $y^4\,dx=(x^{-3/4}-y^3x)\,dy.$ **(M.U. 2015)**

Sol. : Dividing by dy, the equation becomes

$$y^4\frac{dx}{dy}=x^{-3/4}-y^3x\qquad\therefore\ y^4\frac{dx}{dy}+y^3x=x^{-3/4}$$

Dividing by y^4, we get

$$\frac{dx}{dy}+\frac{1}{y}\cdot x=\frac{1}{y^4}\cdot x^{-3/4}$$

Multiplying by $x^{3/4}$, we get

$$x^{3/4}\frac{dx}{dy}+\frac{1}{y}\cdot x^{7/4}=\frac{1}{y^4} \qquad \text{.............. (1)}$$

Putting $x^{7/4}=v$, we get

$$\frac{7}{4}x^{3/4}\frac{dx}{dy}=\frac{dv}{dy} \qquad \therefore \quad x^{3/4}\frac{dx}{dy}=\frac{4}{7}\cdot\frac{dv}{dy}$$

$\therefore$ The equation (1), becomes

$$\frac{4}{7}\cdot\frac{dv}{dy}+\frac{1}{y}\cdot v=\frac{1}{y^4} \qquad \therefore \quad \frac{dv}{dy}+\frac{7}{4}\cdot\frac{1}{y}v=\frac{7}{4}\cdot\frac{1}{y^4}$$

This is a linear differential equation of the form

$\frac{dv}{dy}+Pv=Q$ where P and Q are functions of y.

$$\therefore \quad \text{I.F.}=e^{\int P\,dy}=e^{\int\frac{7}{4}\cdot\frac{1}{y}dy}=e^{\frac{7}{4}\log y}=e^{\log(y^{7/4})}=y^{7/4}$$

$\therefore$ The solution is

$$v\cdot y^{7/4}=\int y^{7/4}\,\frac{7}{4}\cdot\frac{1}{y^4}\,dy+c$$

$$=\frac{7}{4}\int y^{(7/4)-4}\,dy+c=\frac{7}{4}\int y^{-9/4}\,dy+c$$

$$=\frac{7}{4}\cdot\frac{y^{-5/4}}{-5/4}+c=-\frac{7}{5}\cdot y^{-5/4}+c$$

Putting $v=x^{7/4}$, the solution is

$$x^{7/4}\cdot y^{7/4}=-\frac{7}{5}\cdot y^{-5/4}+c.$$

Example 8 (b) : Solve $\frac{dy}{dx}+y=y^2(\cos x-\sin x)$.

Sol. : Dividing by y^2, we get $\quad \frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}\cdot 1=\cos x-\sin x.$

Now, putting $\frac{1}{y}=v$ and $-\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$, we get $\frac{dv}{dx}-v\cdot 1=\sin x-\cos x$, which is linear.

$$\therefore \quad \text{I.F.}=e^{\int -1\cdot dx}=e^{-x}.$$

$\therefore$ The solution is

$$ve^{-x}=\int e^{-x}(\sin x-\cos x)\,dx$$

$$=\int e^{-x}(-\cos x)\,dx+\int e^{-x}\sin x\,dx \qquad \text{[By parts]}$$

$$=-e^{-x}\sin x-\int \sin x\,e^{-x}\,dx+\int e^{-x}\sin x\,dx$$

$$=-e^{-x}\sin x+c$$

$\therefore \quad \frac{1}{y}+\sin x=ce^{x}$ is the required solution.

Example 9 (b) : Solve $\dfrac{dy}{dx} = \dfrac{y^3}{e^{2x} + y^2}$. **(M.U. 2006)**

Sol. : The equation can be written as $e^{2x} + y^2 = y^3 \dfrac{dx}{dy}$ $\quad \therefore \dfrac{dx}{dy} - \dfrac{1}{y} = e^{2x} \cdot \dfrac{1}{y^3}$

Dividing by e^{-2x}, $e^{-2x} \dfrac{dx}{dy} - e^{-2x} \cdot \dfrac{1}{y} = \dfrac{1}{y^3}$.

Putting $e^{-2x} = v$, $-2e^{-2x} \dfrac{dx}{dy} = \dfrac{dv}{dy}$, we get $-\dfrac{1}{2} \cdot \dfrac{dv}{dy} - \dfrac{1}{y} \cdot v = \dfrac{1}{y^3}$ *i.e.* $\dfrac{dv}{dy} + \dfrac{2}{y} \cdot v = -\dfrac{2}{y^3}$

This is a linear differential equation.

Now, $\text{I.F.} = e^{\int P\,dy} = e^{\int (2/y)\,dy} = e^{2 \log y} = e^{\log y^2} = y^2$

$\therefore$ The solution is $v \cdot y^2 = \int y^2 \left(-\dfrac{2}{y^3} \right) dy + c$.

$\therefore \; vy^2 = \int -\dfrac{2}{y}\, dy + c = -2 \log y + c \qquad \therefore \; e^{-2x} y^2 + 2 \log y = c.$

Example 10 (b) : Solve $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}} \right) \dfrac{dx}{dy} = 1$. **(M.U. 2000, 05)**

Sol. : The equation can be written as $\dfrac{dy}{dx} + \dfrac{y}{\sqrt{x}} = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$, which is linear.

$\therefore \; \text{I.F.} = e^{\int P\,dx} = e^{\int (1/\sqrt{x})\,dx} = e^{2\sqrt{x}}$

$\therefore$ The solution is $y \cdot e^{2\sqrt{x}} = \int e^{2\sqrt{x}} \cdot \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}\, dx + c = \int \dfrac{dx}{\sqrt{x}} + c = 2\sqrt{x} + c.$

Example 11 (b) : Solve $\left(\dfrac{y}{x} \sec y - \tan y \right) dx - (x - \sec y \log x)\, dy = 0$. **(M.U. 2006, 08)**

Sol. : $\dfrac{\partial M}{\partial y} = \dfrac{1}{x} \sec y + \dfrac{y}{x} \sec y \tan y - \sec^2 y$

$= \dfrac{1}{x} \sec y + \dfrac{y}{x} \sec y \tan y - 1 - \tan^2 y$

$\dfrac{\partial N}{\partial x} = -1 + \dfrac{\sec y}{x}$

$\therefore \; \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} \right) \Big/ M = \dfrac{-\tan y\,[(y/x) \sec y - \tan y]}{[(y/x) \sec y - \tan y]} = -\tan y = f(y)$

$\therefore \; \text{I.F.} = e^{\int f(y)\,dy} = e^{-\int \tan y\,dy} = e^{\log \cos y} = \cos y$

Multiplying by $\cos y$,

$\left(\dfrac{y}{x} - \sin y \right) dx - (x \cos y - \log x)\, dy = 0$

$$\int M\,dx = \int\left(\frac{y}{x} - \sin y\right)dx = y\log x - x\sin y$$

$$\int N\,dy = 0$$

$\therefore$ The solution is $y\log x - x\sin y = c$.

Example 12 (b) : Solve

$$(x^2-1)\sin x\frac{dy}{dx} + [\,2x\sin x + (x^2-1)\cos x\,]\,y = (x^2-1)\cos x.$$ **(M.U. 2010)**

Sol. : Dividng throughout by $(x^2-1)\sin x$, we get

$$\frac{dy}{dx} + \left[\frac{2x}{x^2-1} + \frac{\cos x}{\sin x}\right]y = \frac{\cos x}{\sin x}$$

This is a linear differentiatial equation of the form $\frac{dy}{dx} + Py = Q$.

Now, $$\int P\,dx = \int\left[\left(\frac{2x}{x^2-1}\right) + \frac{\cos x}{\sin x}\right]dx$$

$$= \log(x^2-1) + \log\sin x = \log[(x^2-1)\sin x]$$

$$\therefore\ e^{\int P\,dx} = e^{\log[(x^2-1)\sin x]} = (x^2-1)\sin x$$

Hence, the solution is $y \cdot e^{\int P\,dx} = \int e^{\int P\,dx} \cdot Q\,dx + c$

$$\therefore\ y(x^2-1)\sin x = \int (x^2-1)\cos x\,dx + c$$

$$= (x^2-1)\sin x - \int 2x\sin x\,dx + c$$ [Integrating by parts]

$$= (x^2-1)\sin x + 2x\cos x - 2\int\cos x\,dx + c$$

$$= (x^2-1)\sin x + 2x\cos x - 2\sin x + c$$

15. $(1+y^2)\,dx = \left(\sqrt{1+y^2}\cdot\sin y - xy\right)dy$

16. $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{xy}$

17. $y\,dx - x\,dy + \log x\,dx = 0$

18. $(1-x^2)\dfrac{dy}{dx} - xy = (1-x^2)^{3/2}\cdot\sin x$

19. $[\cos x\tan y + \cos(x+y)]\,dx + [\sin x\sec^2 y + \cos(x+y)]\,dy = 0.$

20. $x\dfrac{dy}{dx} - y = x^3\cos x$, given $y = 0$ when $x = \pi$.

21. $(2x+3y-5)\,dy + (3x+2y-5)\,dx = 0$

22. $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2+y^2} = 0$

23. $x(x^2+1)\dfrac{dy}{dx} = y(1-x^2) + x^3\log x$

24. $dy + (2x\tan^{-1}y - x^3)(1+y^2)\,dx = 0$

25. $(x^3+3xy^2)\,dx + (y^3+3x^2y)\,dy = 0$

26. $xy - \dfrac{dy}{dx} = y^3e^{-x^2}$

27. $(y^2e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0$

28. $(2x^3y+3)\,dy + (3x^2y^2+2x)\,dx = 0$

29. $\dfrac{dy}{dx} = \dfrac{\tan y - 2xy - y}{x^2 - x\tan^2 y + \sec^2 y}$

(M.U. 2002, 04, 2009)

30. $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$

31. $\sin x\dfrac{dy}{dx} + 2y = \tan^3\left(\dfrac{x}{2}\right)$

32. $y\log y\,dx + (x-\log y)\,dy = 0$

33. $x^2\dfrac{dy}{dx} = 3x^2 - 2xy + 1$

34. $\cos x\dfrac{dy}{dx} + y\sin x = \sqrt{y\sec x}$ **(M.U. 1997)**

35. $(y+e^y-e^{-x})\,dx + x(1+e^y)\,dy = 0$

36. $\sin x\dfrac{dy}{dx} + y\cos x = 2\sin^2 x\cos x$

37. $[(e^y+1)2x + e^xy^2]\,dx + [x^2e^y + (e^x+1)2y]\,dy = 0$

38. $\dfrac{dy}{dx} - \dfrac{\tan y}{x+1} = (x+1)e^x\sec y$

39. $\dfrac{dy}{dx} - y = y^2(\sin x + \cos x)$

40. $(x^3+xy^4)\,dx + 2y^3\,dy = 0$

41. $(xy^3+y)\,dx + 2(x^2y^2+x+y^4)\,dy = 0$

42. $(1+xy^2)\,dx + (1+x^2y)\,dy = 0$

43. $(x-y^2)\,dx + 2xy\,dy = 0$

44. $x\dfrac{dy}{dx} + \sin y\cos y - x^3\cos^2 y = 0$

45. $\sin y\dfrac{dy}{dx} = (1-x\cos y)\cos y$

46. $\dfrac{dy}{dx} + \dfrac{1}{x}\tan y = \dfrac{1}{x^2}\tan y\cdot\sin y$

47. $(x+1)\dfrac{dy}{dx} - y = e^x(x+1)^2$ **(M.U. 2003)**

48. $\left[\dfrac{\log(\log y)}{x} + \dfrac{2}{3}xy^3\right]dx + \left[\dfrac{\log x}{y\log y} + x^2y^2\right]dy = 0$ **(M.U. 2010)**

49. $x(x+y)\,dy - y^2\,dx = 0$

[Ans. : (1) Exact $x^4 + 3x^2y^2 - 7x^2 + y^4 - 8y^2 = 0$

(2) Divide by $\cos^2 y$. Linear. $\sec y = (x+1) + ce^{x}$.

(3) Divide by y^3. Linear. $e^{x^2} = (c + x)\, y^2$.

(4) Divide by xy. Linear. $\log y = cx + 2x^2$.

(5) Divide by xy^2. Linear. $\dfrac{x^2}{y^3} + \dfrac{2}{3}x^3(1 + \log x) - \dfrac{2}{9}x^3 + c = 0$.

(6) Exact. $x^2 - xy - y^2 + 5y = c$.

(7) Homo. I.F. $= \dfrac{1}{xy^2}$ $\therefore \log x - \dfrac{x}{y} = c$.

(8) Put $x + y = v$. Bernoulli's. $(x+y)^2\,[\,x^2 + 1 + ce^{x^2}\,] = 1$.

(9) Divide by xy. Linear. $e^x \cdot \log y = e^x (x - 1) + c$.

(10) $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/ N = \dfrac{3}{x}$ $\therefore x^4 y (3 + y^2) + x^6 = c$.

(11) Divide by xy^2. Linear. $y\,(1 + \log x) + cxy = 1$.

(12) $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/ N = -\dfrac{2}{x}$ $\therefore e^x + m\left(\dfrac{y}{x}\right) = c$.

(13) $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/ N = 1$ $\therefore e^x (x^2 + y^2) = c$.

(14) $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/ N = -\dfrac{4}{x}$ $\therefore 3\left(\dfrac{y^2}{x^2}\right) - 2\,e^{1/x^3} = c$.

(15) Linear. $x\sqrt{1 + y^2} + \cos y = c$.

(16) Homo. I.F. $= \dfrac{1}{x^3}$ $\therefore cx^2 = e^{y^2/x^2}$.

(17) $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/ N = \dfrac{1}{x^2}$ $\therefore (cx + y) + \log x + 1 = 0$.

(18) Linear. $y\sqrt{1 - x^2} = c - (1 - x^2)\cos x - 2x \sin x - 2\cos x$.

(19) Exact. $\sin x \tan y + \sin(x + y) = c$.

(20) Linear. $\dfrac{y}{x} = x \sin x + \cos x - 1$.

(21) Exact. $3x^2 + 4xy + 3y^2 - 10x - 10y + 10 = c$

(22) Exact. $x^2 + y^2 + 2\tan^{-1}\left(\dfrac{y}{x}\right) = c$.

(23) Linear. $y\dfrac{(x^2 + 1)}{x} = \dfrac{1}{2}\left(\log x - \dfrac{1}{2}\right)x^2 + c$.

(24) Linear. $2\tan^{-1} y = x^2 - 1 + ce^{-x^2}$.

(25) Exact. $x^4 + 6x^2y^2 + y^4 = c$.

(26) Linear. $y^2(c+2x)=e^{x^2}$.

(27) Exact. $e^{xy^2}+x^4-y^3=c$.

(28) Exact. $x^3y^2+x^2+3y=c$.

(29) $x\tan y-x^2y-xy-\tan y=c$.

(30) $\left(\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}\right)\Big/M=-\dfrac{3}{y}$ $\quad\therefore\ xy+\dfrac{2x}{y^2}+y^2=c$.

(31) Linear. I.F. $=e^{\int 2\operatorname{cosec} x\,dx}=e^{2\log\tan(x/2)}$

$5y\cdot\tan^2(x/2)=\tan^5(x/2)+c$.

(32) I.F. $=e^{\int f(y)\,dy}=\dfrac{1}{y}$ $\quad\therefore\ 2x\log y=c+(\log y)^2$.

(33) Linear. $x^2y=x^3+x+c$.

(34) Bernoulli's. $\sqrt{y\sec x}=\dfrac{1}{2}\tan x+c$

(35) Exact. $x(y+e^y)+e^{-x}=c$.

(36) Linear. $y\sin x=\dfrac{2}{3}(\sin x)^3+c$.

(37) Exact. $(e^y+1)x^2+(e^x+1)y^2=c$.

(38) Linear. $\sin y=(x+1)(c+e^x)$.

(39) $\dfrac{1}{y}e^x+e^x\sin x=c$.

(40) $\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)\Big/N=2x$ $\quad\therefore\ e^{x^2}(x^2-1+y^4)=c$.

(41) $\left(\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}\right)\Big/M=y$ $\quad\therefore\ \dfrac{x^2y^4}{2}+xy^2+\dfrac{y^6}{3}=c$.

(42) Exact. $2(x+y)+x^2y^2=c$.

(43) $\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)\Big/N=-\dfrac{2}{x}$ $\quad\therefore\ y^2+x\log x=cx$.

(44) Linear. $4x\tan y=x^4+c$.

(45) Bernoulli's. $\sec y=x+1+e^x$.

(46) Linear. $\dfrac{1}{x\sin y}=\dfrac{1}{2x^2}+c$.

(47) Linear $y=(x+1)(e^x+c)$.

(48) Exact. $\log\log y\cdot\log x+\dfrac{x^2y^3}{3}=c$.

(49) Homo. $\dfrac{1}{x^2y}\cdot\dfrac{y}{x}+\log y=c$.

EXERCISE - XII

(A) Solve the following differential equations : Class (a) : 3 or 4 Marks

1. $(\sec x\tan x\cdot\tan y-e^x)\,dx+\sec x\sec^2 y\,dy=0$ **[Ans. :** $\sec x\tan y-e^x=c$ **]**
2. $(2x-y+1)\,dx+(2y-x-1)\,dy=0$ **[Ans. :** $x^2-xy+x+y^2-y=c$ **]**
3. $(2xy\cos x^2-2xy+1)\,dx+(\sin x^2-x^2)\,dy=0$ **(M.U. 1990, 97)** **[Ans. :** $y\sin x^2-x^2y+x=c$ **]**
4. $(e^x+2xy^2+y^3)\,dx+(2x^2y+3xy^2)\,dy=0$ **[Ans. :** $e^x+x^2y^2+xy^3=c$ **]**
5. $\dfrac{dy}{dx}+2y\tan x=\sin x$ **[Ans. :** $y\sec^2 x=\sec x+c$ **]**

6. $\frac{dy}{dx} - \frac{1}{x} \cdot y = 1$ [Ans. : $\frac{y}{x} = \log x + c$]

(B) Find the integrating factor : Class (a) : 3 or 4 Marks

1. $(12y + 4y^3 + 6x^2)\, dx + 3\,(x + xy^2)\, dy = 0$ [Ans. : x^3]
2. $(2x \log x - xy)\, dy + 2y\, dx = 0$ [Ans. : $1/x$]
3. $(xy^3 + y)\, dx + 2\,(x^2 y^2 + x + y^4)\, dy = 0$ [Ans. : y]
4. $(2x y^4 e^y + 2xy^3 + y)\, dx + (x^2 y^4 e^y - x^2 y^2 - 3x)\, dy = 0$ [Ans. : $1/y^4$]
5. $y\,(1 + xy)\, dx + x\,(1 + xy + x^2 y^2)\, dy = 0$ [Ans. : $-\frac{1}{x^3 y^3}$]
6. $y(1 + xy + x^2y^2 + x^3y^3)\, dx + x(1 - xy - x^2y^2 + x^3y^3)\, dy = 0$ [Ans. : $\frac{1}{2x^2y^2(1+xy)}$]
7. $(xy \sin xy + \cos xy)\, y\, dx + (xy \sin xy - \cos xy)\, x\, dy = 0$ [Ans. : $\frac{1}{2xy \cos xy}$]
8. $(3xy - 2ay^2)\, dx + (x^2 - 2axy)\, dy = 0$ [Ans. : $\frac{1}{4xy\,(x - ay)}$]
9. $(x^2 - xy + y^2)\, dx - xy\, dy = 0$ [Ans. : $\frac{1}{x^2(x-y)}$]
10. $x(x-1)\frac{dy}{dx} - (x-2)\, y = x^3(2x-1)$ [Ans. : $\frac{x-1}{x^2}$]
11. $(1 + x + xy^2)\, dy + (y + y^3)\, dx = 0.$ [Ans. : y]
12. $[\, 2y \cos y - x\,(\sec y + \tan y)\,]\, dy = (1 + \sin y)\, dx$ [Ans. : $\sec y + \tan y$]
13. $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$ [Ans. : e^{x^2}]
14. $\frac{dy}{dx} + (2x \tan^{-1} y - x^3)(1 + y^2) = 0$ [Ans. : e^{x^2}]
15. $e^x(x+1)\, dx + (y e^y - x e^x)\, dy = 0$ [Ans. : e^{-y}]
16. $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$ [Ans. : $e^{\tan^{-1} y}$]
17. $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$ [Ans. : $e^{2\sqrt{x}}$]

(C) Reduce the following equations to linear form : Class (a) : 3 or 4 Marks

1. $\frac{dy}{dx} = x^3 y^3 - xy$ [Ans. : $\frac{dv}{dx} - 2xv = -2x^3$]
2. $y - \cos x \frac{dy}{dx} = y^2 (1 - \sin x) \cos x$ [Ans. : $\frac{dv}{dx} + \sec x \cdot v = 1 - \sin x$]
3. $\frac{dr}{d\theta} = r \tan\theta - \frac{r^2}{\cos\theta}$ [Ans. : $\frac{dv}{d\theta} + \tan\theta \cdot v = -\sec\theta$]
4. $x^3 \frac{dy}{dx} + 2x^2 y = y^3$ [Ans. : $\frac{dv}{dx} - \frac{4}{x} \cdot v = -\frac{2}{x^3}$]

5. $xy(1+xy^2)\dfrac{dy}{dx}=1$ [Ans. : $\dfrac{dv}{dy}+v\cdot y=y^3$

6. $\dfrac{dy}{dx}=1-x(y-x)-x^3(y-x)^2$ [Ans. : $\dfrac{dt}{dx}-t\cdot x=x^3$

7. $\dfrac{dy}{dx}=\dfrac{e^y}{x^2}-\dfrac{1}{x}$ [Ans. : $\dfrac{dv}{dx}-\dfrac{1}{x}\cdot v=-\dfrac{1}{x^2}$

8. $4x^2y\dfrac{dy}{dx}=3x(3y^2+2)+(3y^2+2)^3$ [Ans. : $\dfrac{dt}{dx}-\dfrac{9}{x}\cdot t=\dfrac{3}{x^2}$

Summary

1. $M\,dx+N\,dy$ is exact if and only if $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$.

2. If $\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)\Big/N$ is a function of x only, say $f(x)$, then $e^{\int f(x)\,dx}$ is an integrating factor.

3. If $\left(\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}\right)\Big/M$ is a function of y only, say $f(y)$, then $e^{\int f(y)\,dy}$ is an integrating factor.

4. If the equation is of the form $f_1(xy)\cdot y\,dx+f_2(xy)\cdot x\,dy=0$ and if $Mx-Ny\neq 0$ then $\dfrac{1}{Mx-Ny}$ is an integrating factor.

5. If $M\,dx+N\,dy=0$ is homogeneous and $Mx+Ny\neq 0$ then $\dfrac{1}{Mx+Ny}$ is an integrating factor.

6. The solution of $\dfrac{dy}{dx}+Py=Q$ is $y\cdot e^{\int P\,dx}=\int e^{\int P\,dx}\cdot Q\,dx+c$.

7. The solution of $\dfrac{dx}{dy}+P'x=Q'$ is $x\cdot e^{\int P'dy}=\int e^{\int P'dy}\cdot Q'\,dy+c$.

8. In $f'(y)\dfrac{dy}{dx}+P\,f(y)=Q$ put $f(y)=v$.

9. In $f'(x)\dfrac{dx}{dy}+P\,f(x)=Q$, put $f(x)=v$.

10. **Bernoulli's Equation**

 (a) $\dfrac{dy}{dx}+Py=Qy^n$. Divide by y^n and put $\dfrac{1}{y^{n-1}}=v$.

 (b) $\dfrac{dx}{dy}+P'x=Q'x^n$ Divide by x^n and put $\dfrac{1}{x^{n-1}}=v$.

CHAPTER

2

Applications of Differential Equations

1. Introduction

In the previous chapters we have learnt some techniques of solving differential equations of certain types. We have also remarked that the differential equations arise in the study of some engineering, science and social problems. However, formulating a differential equation in a particular situation needs detailed study of the subject to which the problem belongs. We shall remain satisfied here with solving differential equations related to electrical and mechanical problems when the equations are given.

2. Applications of Differential Equations of First Order and First Degree (Mechanical Engineering)

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : An equation in the theory of stability of an aeroplane is $\dfrac{dv}{dt} = g\cos\alpha - kv$, v being velocity and g, k being constants. It is observed that at time $t = 0$, the velocity $v = 0$. Solve the equation completely.

Sol. : The given differential equation is of variable separable type.

$$\int \frac{dv}{g\cos\alpha - kv} = \int dt + c \quad \therefore -\frac{1}{k}\log(g\cos\alpha - kv) = t + c$$

By data when $t = 0$, $v = 0$ $\quad \therefore -\dfrac{1}{k}\log g\cos\alpha = c$

$$\therefore \quad t = \frac{1}{k}\log g\cos\alpha - \frac{1}{k}\log(g\cos\alpha - kv) = \frac{1}{k}\log\left(\frac{g\cos\alpha}{g\cos\alpha - kv}\right)$$

$$\therefore \frac{g\cos\alpha}{g\cos\alpha - kv} = e^{kt} \quad \therefore \frac{g\cos\alpha - kv}{g\cos\alpha} = e^{-kt}$$

$$\therefore \quad v = \frac{g\cos\alpha}{k}(1 - e^{-kt}).$$

Example 2 (b) : The equation of motion of a body falling under gravity is given by $\dfrac{dv}{dt} = g - kv^2$. Find the velocity and distance travelled as a function of time. Given $v = 0$ at $t = 0$.

Sol. : We have $\dfrac{dv}{dt} = g - kv^2$. Now taking $k = \dfrac{g}{\lambda^2}$ for convenience

$$\frac{dv}{dt} = g - \frac{g}{\lambda^2}v^2 = \frac{g}{\lambda^2}(\lambda^2 - v^2)$$

This is a differential equation of variable separable type.

$$\therefore \quad \frac{dv}{\lambda^2 - v^2} = \frac{g}{\lambda^2} dt$$

By integration, $\frac{1}{2\lambda}\log\left(\frac{\lambda+v}{\lambda-v}\right) = \frac{g}{\lambda^2}t + c$ $\quad \therefore \quad \frac{1}{\lambda}\cdot\frac{1}{2}\log\left(\frac{1+v/\lambda}{1-v/\lambda}\right) = \frac{g}{\lambda^2}t + c$

Since $\tan h^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$, $\quad \frac{1}{\lambda}\tan h^{-1}\frac{v}{\lambda} = \frac{g}{\lambda^2}t + c$

But by data, when $t = 0$, $v = 0$ $\quad \therefore \; c = 0$

$$\therefore \quad \frac{1}{\lambda}\tan h^{-1}\frac{v}{\lambda} = \frac{g}{\lambda^2}t \qquad \therefore \; \tan h^{-1}\frac{v}{\lambda} = \frac{gt}{\lambda}$$

$$\therefore \quad v = \lambda \tan h\left(\frac{gt}{\lambda}\right) \qquad \text{.............. (1)}$$

But $v = \frac{dx}{dt}$, $\quad \therefore \; \frac{dx}{dt} = \lambda \tan h\left(\frac{gt}{\lambda}\right) = \lambda \cdot \frac{\sin h(gt/\lambda)}{\cos h(gt/\lambda)}$

By integration, $x = \lambda \cdot \frac{\lambda}{g}\log\cos h\left(\frac{gt}{\lambda}\right) + c.$

But by data, when $t = 0$, $x = 0$ $\quad \therefore \; c = 0$

$$\therefore \quad x = \frac{\lambda^2}{g}\log\cos h\left(\frac{gt}{\lambda}\right) \qquad \text{.............. (2)}$$

Thus, the velocity of the body is given by (1) and the distance travelled is given by (2).

Example 3 (b) : In the above example show further that the velocity of the body approaches a limiting value as $t \to \infty$.

Sol. : As proved above

$$v = \lambda \tan h\left(\frac{gt}{\lambda}\right) = \lambda\left(\frac{e^{gt/\lambda} - e^{-gt/\lambda}}{e^{gt/\lambda} + e^{-gt/\lambda}}\right)$$

$$\therefore \quad v = \lambda\left(\frac{1 - e^{-2gt/\lambda}}{1 + e^{-2gt/\lambda}}\right) \qquad \therefore \quad \lim_{t\to\infty} v = \lim_{t\to\infty} \lambda\left(\frac{1 - e^{-2gt/\lambda}}{1 + e^{-2gt/\lambda}}\right)$$

As $t \to \infty$, $e^{-2gt/\lambda} \to 0$ $\quad \therefore \; \lim_{t\to\infty} v = \lambda$. If as $t \to \infty$, $v \to v_0$, then we have

$$v_0 = \lambda = \sqrt{\frac{g}{k}}.$$

Example 4 (b) : The distance x descended by a parashuter satisfied the differential equation $\frac{dv}{dt} = g\left(1 - \frac{v^2}{k^2}\right)$ where v is the velocity, k, g are constants. If $v = 0$ and $x = 0$ at $t = 0$, show that $x = \frac{k^2}{g}\log\cos h\left(\frac{gt}{k}\right)$.

Sol. : We have $\frac{dv}{dt} = g\left(1 - \frac{v^2}{k^2}\right) = \frac{g}{k^2}(k^2 - v^2)$

This is a differential equation of variable separable type.

$$\therefore \quad \frac{dv}{k^2 - v^2} = \frac{g}{k^2} dt$$

Now proceeding as in Example 2, we get

$$x = \frac{k^2}{g} \log \cosh\left(\frac{gt}{k}\right).$$

Example 5 (b) : The differential equation of a body of mass m falling from rest subjected to the force of gravity and an air resistance proportional to the square of the velocity is given by $mv\frac{dv}{dx} = ka^2 - kv^2$. If it falls through a distance x and possesses a velocity v at that instant, prove that $\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$ where $mg = ka^2$.

Sol. : We have $mv\frac{dv}{dx} = ka^2 - kv^2 = k(a^2 - v^2)$

This is a differential equation of variable separable type.

$$\therefore \frac{v}{a^2 - v^2}\,dv = \frac{k}{m}\,dx \qquad \text{.....(1)}$$

By integration, $-\frac{1}{2}\log(a^2 - v^2) = \frac{k}{m}x + \log c$

But by data when $t = 0$, $x = 0$, $v = 0$.

$\therefore \log c = -\frac{1}{2}\log a^2$ $\qquad \therefore -\frac{1}{2}\log(a^2 - v^2) = \frac{kx}{m} - \frac{1}{2}\log a^2$

$\therefore \frac{2kx}{m} = \log a^2 - \log(a^2 - v^2)$ $\qquad \therefore \frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right).$

Note ....

Compare this example with the solved Ex. No. 2 above.

Example 6 (b) : The differential equation of a moving body opposed by a force per unit mass of value cx and resistance per unit mass of value bv^2 where x and v are the displacement and velocity of the particle at that time is given by $v\frac{dv}{dx} = -cx - bv^2$. Find the velocity of the particle in terms of x, if it starts from rest.

Sol. : We have $v\frac{dv}{dx} = -cx - bv^2$. Putting $v^2 = y$, $v\frac{dv}{dx} = \frac{1}{2}\frac{dy}{dx}$

$\therefore \frac{1}{2}\cdot\frac{dy}{dx} + by = -cx \qquad \therefore \frac{dy}{dx} + 2by = -2cx$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$.

$$\therefore \text{I.F.} = e^{\int P\,dx} = e^{\int 2b\,dx} = e^{2bx}$$

$\therefore$ The solution is $y\,e^{2bx} = \int e^{2bx}(-2cx)\,dx + c'$

$$\therefore y\,e^{2bx} = -2c\left[x\cdot\frac{e^{2bx}}{2b} - \int\frac{e^{2bx}}{2b}\cdot(1)\cdot dx\right] + c' \qquad \text{[Integrate by parts]}$$

$$= -2c\left[x\cdot\frac{e^{2bx}}{2b} - \frac{e^{2bx}}{4b^2}\right] + c'.$$

Resubstituting $y = v^2$,

$$v^2 e^{2bx} = -\frac{cx}{b} \cdot e^{2bx} + \frac{c}{2b^2} e^{2bx} + c' \qquad \text{.....(1)}$$

By data, when $x = 0$, $v = 0$ $\quad \therefore c' = -\frac{c}{2b^2}$

$$\therefore \; v^2 e^{2bx} = -\frac{cx}{b} e^{2bx} + \frac{c}{2b^2} e^{2bx} - \frac{c}{2b^2}$$

$$\therefore \; v^2 = -\frac{cx}{b} + \frac{c}{2b^2} - \frac{c}{2b^2} e^{-2bx} = \frac{c}{2b^2}(1 - e^{-2bx}) - \frac{cx}{b}.$$

Example 7 (b) : The distance x descended by a parashuter satisfies the differential equation

$$\left(\frac{dx}{dt}\right)^2 = k^2 [1 - e^{-2gx/k^2}]$$

where k and g are constants. If $x = 0$ when $t = 0$, show that $x = \frac{k^2}{g} \log \cos h\left(\frac{gt}{k}\right)$.

Sol. : We have $\frac{dx}{dt} = k\sqrt{1 - e^{-2gx/k^2}}$ $\qquad \therefore \frac{dx}{\sqrt{1 - e^{-2gx/k^2}}} = k\,dt$ .....(1)

Let $\sqrt{1 - e^{-2gx/k^2}} = u$ $\quad \therefore 1 - e^{-2gx/k^2} = u^2$

$$\therefore \; e^{-2gx/k^2} \cdot \frac{g}{k^2} \cdot dx = u\,du \quad \therefore \; (1 - u^2)\frac{g}{k^2} dx = u\,du$$

$$\therefore \; dx = \frac{k^2}{g} \cdot \frac{u}{1 - u^2} du$$

Hence, from (1), we get,

$$\frac{k^2}{g} \cdot \frac{u}{1 - u^2} \cdot \frac{1}{u} du = k\,dt \quad \therefore \; \frac{k}{g} \cdot \frac{du}{1 - u^2} = dt + c$$

By integration, $\quad \frac{k}{g} \cdot \frac{1}{2} \log\left(\frac{1 + u}{1 - u}\right) = t + c$

But $\frac{1}{2} \log\left(\frac{1 + u}{1 - u}\right) = \tan h^{-1} u$ $\quad \therefore \; \frac{k}{g} \tan h^{-1} u = t + c$

But by data when $t = 0$, $x = 0$ and hence $u = 0$ $\therefore c = 0$

$$\therefore \; \frac{k}{g} \tan h^{-1} u = t \quad \therefore \; \tan h^{-1} u = \frac{gt}{k}$$

$$\therefore \; u = \tan h \frac{gt}{k} \quad \therefore \; u^2 = \tan h^2\left(\frac{gt}{k}\right)$$

$$\therefore \; 1 - e^{-2gx/k^2} = \tan h^2\left(\frac{gt}{k}\right)$$

$$\therefore \; e^{-2gx/k^2} = 1 - \tan h^2\left(\frac{gt}{k}\right) = \sec h^2\left(\frac{gt}{k}\right)$$

$$\therefore \; e^{2gx/k^2} = \cos h^2\left(\frac{gt}{k}\right) \qquad \therefore \; \frac{2gx}{k^2} = 2 \log \cos h\left(\frac{gt}{k}\right)$$

$$\therefore \; x = \frac{k^2}{g} \log \cos h\left(\frac{gt}{k}\right).$$

Example 8 (b) : The differential equation of a particle moving in a straight line with acceleration $k\left(x+\frac{a^4}{x^3}\right)$ directed towards origin is $v\frac{dv}{dx}=-k\left(x+\frac{a^4}{x^3}\right)$. If it starts from rest at a distance a from the origin, prove that it will arrive at the origin at the end of time $\frac{\pi}{4\sqrt{k}}$.

Sol. : We have $v\frac{dv}{dx}=-k\left(x+\frac{a^4}{x^3}\right)$

(Negative sign because the acceleration is directed towards the origin.)

This is a differential equation of variable separable type.

$$\therefore \int v\,dv=-k\int\left(x+\frac{a^4}{x^3}\right)dx$$

$$\therefore \frac{v^2}{2}=-k\left(\frac{x^2}{2}-\frac{a^4}{2x^2}\right)+c=-k\left(\frac{x^4-a^4}{2x^2}\right)+c$$

By data when $x=0$, $v=0$ $\therefore c=0$

$$\therefore v^2=k\left(\frac{a^4-x^4}{x^2}\right) \qquad \therefore v=\pm\sqrt{k}\cdot\frac{\sqrt{a^4-x^4}}{x} \qquad \therefore v=-\sqrt{k}\cdot\frac{\sqrt{a^4-x^4}}{x}.$$

We take the negative sign, since acceleration is directed towards origin.

$$v=\frac{dx}{dt}=-\sqrt{k}\cdot\frac{\sqrt{a^4-x^4}}{x} \qquad \therefore \int\frac{x\,dx}{\sqrt{a^4-x^4}}=-\sqrt{k}\int dt+c$$

Putting $x^2=u$, $x\,dx=\frac{1}{2}du$,

$$\frac{1}{2}\cdot\int\frac{du}{\sqrt{(a^2)^2-u^2}}du=-\sqrt{k}\int dt+c$$

$$\therefore \frac{1}{2}\sin^{-1}\left(\frac{u}{a^2}\right)=-\sqrt{k}\cdot t+c \qquad \therefore \frac{1}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right)=-\sqrt{k}\cdot t+c$$

By data when $t=0$, $x=a$ $\therefore c=\frac{\pi}{4}$

$$\therefore \frac{1}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right)=-\sqrt{k}\cdot t+\frac{\pi}{4}$$

The particle will arrive at the origin where $x=0$.

$$\therefore 0=-\sqrt{k}\cdot t+\frac{\pi}{4} \qquad \therefore t=\frac{\pi}{4\sqrt{k}}.$$

Example 9 (b) : The differential equation of a body fired vertically from the earth, if it is acted upon by gravitational force only is given by $v\frac{dv}{dx}=-\frac{gr^2}{x^2}$. Find the initial velocity of a body supposed to escape. (r is the radius of the earth and x is the distance of the body from the earth.)

Sol. : We have $v\frac{dv}{dx}=-\frac{gr^2}{x^2}$.

This is a differential equation of variable separable type.

$$\therefore \quad v\,dv = -gr^2 \cdot \frac{dx}{x^2}$$

By integration, $\dfrac{v^2}{2} = \dfrac{gr^2}{x} + c$

If u is the required velocity on the surface of the earth where $x = r$ then

$$\frac{u^2}{2} = gr + c \quad \therefore \quad c = \frac{u^2}{2} - gr$$

$$\therefore \quad \frac{v^2}{2} = \frac{gr^2}{x} + \frac{u^2}{2} - gr \qquad \therefore \quad v^2 = \frac{2gr^2}{x} + u^2 - 2gr$$

This is the equation of motion of a body projected from the surface of the earth with initial velocity u.

If the body is not to return to the earth its velocity v must be always positive. (If the velocity v becomes zero the body will come to rest and then will start to descend.) As x increases, $2gr^2/x$ decreases. Hence, v will be positive if

$$u^2 - 2gr \geq 0 \quad \therefore \quad u^2 \geq 2gr \quad i.e. \quad u \geq \sqrt{2gr}.$$

$\therefore$ The **least** velocity of projection $= \sqrt{2gr}$

A particle projected with this velocity will never return to the earth. This is called the **escape velocity** from the earth.

Note ....

Taking the radius of the earth $r = 3960$ miles and $g = 32.17$ ft/sec^2, the escape velocity comes out to be **7 miles per sec.**

Example 10 (b) : The differential equation of a body of mass m projected vertically upwards with velocity V with air resistance k times the velocity is given by $\dfrac{dv}{dt} = -g - \dfrac{kv}{m}$. Show that the particle will reach maximum height in time $\dfrac{m}{k}\log\left(1 + \dfrac{kV}{mg}\right)$.

Sol. : We have $\dfrac{dv}{dt} = -g - \dfrac{kv}{m}$

This is a differential equation of variable separable type.

$$\therefore \quad \frac{dv}{g + (k/m)v} = -dt \qquad \text{.................... (1)}$$

By integration, $\dfrac{m}{k}\log\left(g + \dfrac{k}{m}v\right) = -t + c$

When $t = 0$, $v = V$, $\dfrac{m}{k}\log\left(g + \dfrac{k}{m}V\right) = c$

$$\therefore \quad t = \frac{m}{k}\log\left(g + \frac{k}{m}V\right) - \frac{m}{k}\log\left(g + \frac{k}{m}v\right) = \frac{m}{k}\log\left(\frac{g + (k/m)V}{g + (k/m)v}\right)$$

When the body attains maximum height, $v = 0$.

$$\therefore \quad t = \frac{m}{k}\log\left(\frac{g + (k/m)V}{g}\right) = \frac{m}{k}\log\left(1 + \frac{k}{mg}V\right).$$

Example 11 (b) : The differential equation of a body projected vertically upwards in air, considering air resistance, is given by $\frac{dv}{dt} = -g - kv$. Show that the distance travelled by the particle at any time t is given by $x = \left(\frac{g}{k^2} + \frac{u}{k}\right)(1 - e^{-kt}) - \frac{g}{k} \cdot t$ where u is the initial velocity.

Sol. : We have $\frac{dv}{dt} = -g - kv$. $\therefore \frac{dv}{g + kv} = -dt$

By integration, we get $\frac{1}{k}\log(g + kv) = -t + c$

Initially when $t = 0$, $v = u$. $\therefore \frac{1}{k}\log(g + ku) = c$

$$\therefore \frac{1}{k}\log(g + kv) = -t + \frac{1}{k}\log(g + ku) \qquad \therefore t = \frac{1}{k}\log\left(\frac{g + ku}{g + kv}\right)$$

$$\therefore \frac{g + ku}{g + kv} = e^{kt} \qquad \therefore (g + ku)e^{-kt} = g + kv$$

$$\therefore v = -\frac{g}{k} + \left(\frac{g + ku}{k}\right)e^{-kt} \qquad \therefore \frac{dx}{dt} = -\frac{g}{k} + \left(\frac{g + ku}{k}\right)e^{-kt}$$

By integration, we get $x = -\frac{g}{k}t - \left(\frac{g + ku}{k^2}\right)e^{-kt} + c$

Initially when $t = 0$, $x = 0$ $\quad \therefore c = \frac{g + ku}{k^2}$

$$\therefore x = -\frac{g}{k}t - \left(\frac{g + ku}{k^2}\right)e^{-kt} + \left(\frac{g + ku}{k^2}\right)$$

$$= \left(\frac{g + ku}{k^2}\right)(1 - e^{-kt}) - \frac{g}{k}t$$

$$\therefore x = \left(\frac{g}{k^2} + \frac{u}{k}\right)(1 - e^{-kt}) - \frac{g}{k}t.$$

EXERCISE - I

Solve the following examples : Class (b) : 6 Marks

1. The differential equation of a body in motion is given by $\frac{dv}{dt} = k\left(1 - \frac{t}{T}\right)$ where k and T are constants. Find the maximum speed and the distance travelled when the maximum speed is attained. (At $t = 0$, $s = 0$, $v = 0$).

[**Ans. :** $t = T$, $v = \frac{kT}{2}$, $s = \frac{kT^2}{2}$]

2. The velocity of a bullet fired in a sand tank is given by $\frac{dv}{dt} = -k\sqrt{v}$. If at $t = 0$, $v = V$, find how long it will take to come to rest.

[**Ans. :** $2\sqrt{V}/k$]

3. A chain coiled up near the edge of a smooth table starts to fall over the edge. The velocity v when a length x has fallen is given by $xv\frac{dv}{dx} + v^2 = gx$. Show that $v = 8\sqrt{x/3}$.

4. The equation of motion of a particle moving in a straight line is given by $v\frac{dv}{dx} = -\frac{k}{x^3}$. If initially the particle was at rest at a distance a from the origin, show that it will be at a distance $\frac{a}{2}$ from the origin at $t = \frac{a^2}{2}\sqrt{\frac{3}{k}}$.

5. The equation of motion of a particle moving in a straight line in a resisting medium is given by $\frac{dv}{ds} = -kv^2$. If u is the initial velocity, prove that $v = \frac{u}{1+ksu}$ and $t = \frac{s}{u} + \frac{ks^2}{2}$.

6. The differential equation of a body falling from rest subjected to the force of gravity and air resistance is given by $v\frac{dv}{dx} + \frac{n^2}{g}v^2 = g$. Prove that the velocity is given by $v^2 = \frac{g^2}{n^2}(1 - e^{-2n^2x/g})$. ($v = 0$ at $x = 0$)

3. Applications of Differential Equations of First Order and First Degree (Electrical Engineering)

We shall consider simple electrical circuits containing resistance (R), inductance (L), capacitance (C) and voltage (V) or electromotive force (E).

We denote the charge by q and the rate of flow of electricity *i.e.* current by i. With these notations, we know that

1. $i = \frac{dq}{dt}$ or $q = \int i\, dt$
2. Voltage drop across resistance $R = R_i$
3. Voltage drop across inductance $L = L\frac{di}{dt}$
4. Voltage drop across capacitance $C = \frac{q}{C}$.

The differential equation of an electrical circuit can be formed by using the following Kirchhoff's Laws.

1. The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit.
2. The algebraic sum of the currents flowing into or from any terminal of the circuit is zero.

We shall consider the following cases.

(1) R-L-E circuit

Consider a circuit containing resistance R and inductance L in series with a voltage source (battery) E. Let i be the current in the circuit at any time t. Then by Kirchhoff's law.

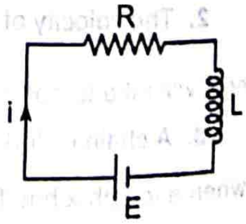

Fig. 2.1

Sum of voltage drops across R and $L = E$.

$$\therefore\ Ri + L\frac{di}{dt} = E. \qquad \therefore\ \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}.$$

This is the required differential equation of the circuit.

(2) R-L-C-E circuit

Consider a circuit containing resistance R, inductance L and capacitance C in series with a voltage source (battery) E. Let i be the current at any time t. Then by Kirchhoff's Law.

Sum of voltage drops across R, L, $C = E$.

$$\therefore \quad Ri + L\frac{di}{dt} + \frac{q}{C} = E$$

$$\therefore \quad \frac{di}{dt} + \frac{R}{L}i + \frac{q}{CL} = \frac{E}{L}$$

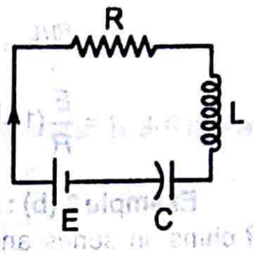

Fig. 2.2

Putting $i = \dfrac{dq}{dt}$, we get

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{CL} = \frac{E}{L}.$$

We need the following two integrals often in solving the differential equations of electrical circuits

1. $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\left(\frac{a}{\sqrt{a^2+b^2}}\sin bx - \frac{b}{\sqrt{a^2+b^2}}\cos bx\right) \quad \text{.................. (1)}$$

Putting $\dfrac{a}{\sqrt{a^2+b^2}} = \cos\Phi$, $\dfrac{b}{\sqrt{a^2+b^2}} = \sin\Phi$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin(bx-\Phi) \quad \text{................ (1A)}$$

2. $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\left(\frac{a}{\sqrt{a^2+b^2}}\cos bx + \frac{b}{\sqrt{a^2+b^2}}\sin bx\right) \quad \text{.................. (2)}$$

Putting $\dfrac{a}{\sqrt{a^2+b^2}} = \cos\Phi$, $\dfrac{b}{\sqrt{a^2+b^2}} = \sin\Phi$

$$\int e^{ax}\cos bx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos(bx-\Phi). \quad \text{................ (2A)}$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : In a circuit containing inductance L, resistance R, and voltage E, the current i is given by $L\dfrac{di}{dt} + Ri = E$. Find the current i at time t if at $t = 0$, $i = 0$ and L, R, E are constants. **(M.U. 1995, 2009, 12, 16)**

Sol. : The given equation $\dfrac{di}{dt} + \dfrac{Ri}{L} = \dfrac{E}{L}$ is linear of the type $\dfrac{dy}{dx} + Py = Q$.

$\therefore$ Its solution is $\quad i \cdot e^{\int (R/L)dt} = \int e^{\int (R/L)dt} \cdot \dfrac{E}{L} \cdot dt + c$

$$i \cdot e^{Rt/L} = \frac{E}{L}\int e^{Rt/L}\,dt + c = \frac{E}{L} \cdot e^{Rt/L} \cdot \frac{L}{R} + c = \frac{E}{R}e^{Rt/L} + c$$

When $t = 0$, $i = 0$ $\therefore c = -\frac{E}{R}$

$$\therefore i \cdot e^{Rt/L} = \frac{E}{R}e^{Rt/L} - \frac{E}{R} = \frac{E}{R}(e^{Rt/L} - 1)$$

$$\therefore i = \frac{E}{R}(1 - e^{-Rt/L}) \qquad \text{.................... (3)}$$

Example 2 (b) : A constant e.m.f. E volts is applied to a circuit containing a constant resistance R ohms. in series and a constant inductance L henries. The current i at any time t is given by $L\frac{di}{dt} + Ri = E$. If the initial current is zero, show that the current builds up to half its theoretical maximum value in $\frac{L}{R} \cdot \log 2$ seconds. **(M.U. 2000)**

Sol. : As in example 1 the current is given by $i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$

As $t \to \infty$, $e^{-Rt/L} \to 0$ and the current reaches its theoretical maximum value say I.

$$\therefore I = \frac{E}{R}$$

When $i = \frac{I}{2}$, we get $\frac{I}{2} = I\left(1 - e^{-Rt/L}\right)$ $\therefore \frac{1}{2} = 1 - e^{-Rt/L}$

$$\therefore e^{-Rt/L} = \frac{1}{2} \quad \therefore e^{Rt/L} = 2$$

$$\therefore \frac{Rt}{L} = \log 2 \quad \therefore t = \frac{L}{R}\log 2. \qquad \text{.................... (4)}$$

Example 3 (b) : A resistance of 100 ohms and inductance 0.5 henries are connected in series with a battery of 20 volts. Find the current at any instant if the relation between L, R and E is

$$L\frac{di}{dt} + Ri = E.$$

(M.U. 2015)

Sol. : The given equation can be written as

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \qquad \text{.................... (i)}$$

It is linear of the type $\frac{dy}{dx} + Py = Q$.

Putting the given values of $R = 100$, $L = 0.5$ and $E = 20$ in (i), we get

$$\frac{di}{dt} + \frac{100}{0 \cdot 5}i = \frac{20}{0 \cdot 5} \quad \text{i.e.,} \quad \frac{di}{dt} + 200\,i = 40$$

Now, $\int P\,dx = \int 200\,dt = 200\,t$

Hence, the solution is

$$y \cdot e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx}\,dx + c$$

$$\therefore i \cdot e^{200\,t} = \int 40 \cdot e^{200\,t}\,dt + c = 40 \cdot \frac{e^{200\,t}}{200} + c = \frac{1}{50} \cdot e^{200\,t} + c$$

When $t = 0$, $i = 0$ then $\frac{1}{50}e^0 + c = 0$ $\therefore$ $c = -\frac{1}{50}$.

$$\therefore \quad i \cdot e^{200t} = \frac{1}{50} \cdot e^{200t} - \frac{1}{50}$$

$$\therefore \quad i = \frac{1}{50} - \frac{1}{50}e^{-200t} = \frac{1}{50}(1 - e^{-200t}) = 0{\cdot}02(1 - e^{-200t})$$

Example 4 (b) : The differential equation of a circuit with inductance L and resistance R is given by $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}e^{-at}$. Show that the current at any time t is given by $i = \frac{E}{R - aL}(e^{-at} - e^{-Rt/L})$. (Given $i = 0$ at $t = 0$) **(M.U. 1998, 2010)**

Sol. : We have $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}e^{-at}$.

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$. Its solution is

$$i \cdot e^{\int (R/L)dt} = \int e^{\int (R/L)dt} \cdot \frac{E}{L}e^{-at}dt + c$$

$$\therefore \quad i \cdot e^{Rt/L} = \frac{E}{L}\int e^{(Rt/L)} \cdot e^{-at}\,dt + c = \frac{E}{L}\int e^{[(R/L)-a]t}dt + c$$

$$i \cdot e^{Rt/L} = \frac{E}{L} \cdot \frac{e^{[(R/L)-a]t}}{(R/L) - a} + c = \frac{E}{R - La} \cdot e^{[(R/L)-a]t} + c$$

By data, when $t = 0$, $i = 0$, $\therefore c = -\frac{E}{R - La}$

$$\therefore \quad i \cdot e^{Rt/L} = \frac{E}{R - La} \cdot e^{[(R/L)-a]t} - \frac{E}{R - La}$$

$$\therefore \quad i = \frac{E}{R - La} \cdot e^{-at} - \frac{E}{R - La} \cdot e^{-Rt/L} = \frac{E}{R - La}(e^{-at} - e^{-Rt/L}).$$

Example 5 (b) : The current in a circuit containing an inductance L, resistance R, and voltage $E \sin \omega t$ is given by $L\frac{di}{dt} + Ri = E \sin \omega t$. If $i = 0$ at $t = 0$, show that

$$i = \frac{E}{\sqrt{R^2 + L^2\omega^2}}\left[\sin(\omega t - \Phi)\, a + e^{-Rt/L}\sin \Phi\right] \text{ where } \Phi = \tan^{-1}\left(\frac{L\omega}{R}\right).$$ **(M.U. 1995, 2013)**

Sol. : The given equation $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\sin \omega t$ is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$.

Hence, the solution is

$$i \cdot e^{\int (R/L)dt} = \int e^{\int (R/L)dt} \cdot \frac{E}{L}\sin \omega t\, dt + c$$

$$\therefore \quad i \cdot e^{Rt/L} = \int e^{Rt/L} \cdot \frac{E}{L}\sin \omega t\, dt + c$$

Integrating by parts, *i.e.* by (1), page 2-9 we get,

$$i \cdot e^{Rt/L} = \frac{E}{L} \cdot \frac{1}{(R^2/L^2) + \omega^2} \cdot e^{Rt/L}\left(\frac{R}{L}\sin \omega t - \omega \cos \omega t\right) + c$$

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$$\therefore \quad i \cdot e^{Rt/L} = E \cdot \frac{1}{R^2 + \omega^2 L^2} \cdot e^{Rt/L}(R \sin \omega t - \omega L \cos \omega t) + c$$

When $t = 0, i = 0 \quad \therefore \quad c = E \cdot \dfrac{\omega L}{R^2 + \omega^2 L^2}$

$$\therefore \quad i = E \cdot \frac{1}{R^2 + \omega^2 L^2} \cdot (R \sin \omega t - \omega L \cos \omega t) + e^{-Rt/L} \cdot E \cdot \frac{\omega L}{R^2 + \omega^2 L^2}$$

$$= E \cdot \frac{1}{\sqrt{R^2 + \omega^2 L^2}} \cdot \left(\frac{R}{\sqrt{R^2 + \omega^2 L^2}} \sin \omega t - \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} \cos \omega t \right)$$

$$+ e^{-Rt/L} \cdot \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \cdot \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}.$$

Putting $\dfrac{R}{\sqrt{R^2 + \omega^2 L^2}} = \cos \Phi, \quad \dfrac{L\omega}{\sqrt{R^2 + \omega^2 L^2}} = \sin \Phi$

$$\therefore \quad i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} [\sin (\omega t - \Phi) + e^{-Rt/L} \sin \Phi].$$

Example 6 (b) : The equation of the electromotive force in terms of current i for an electrical circuit having resistance R and a condenser of capacity C in series is $E = Ri + \int \frac{i}{c} dt$. Find the current i at any time t, when $E = E_0 \sin \omega t$. **(M.U. 1991, 94)**

Sol. : Putting $E = E_0 \sin \omega t$ in the given equation we have $Ri + \int \frac{i}{c} dt = E_0 \sin \omega t$.

Differentiating it w.r.t. t, we get

$$R \frac{di}{dt} + \frac{i}{c} = E_0 \omega \cos \omega t \quad \therefore \quad \frac{di}{dt} + \frac{i}{Rc} = \frac{E_0 \omega}{R} \cdot \cos \omega t$$

This is a linear differential equation of the type $\dfrac{dy}{dx} + Py = Q$.

Its solution is

$$i \cdot e^{\int (1/Rc) dt} = \int e^{\int (1/Rc) dt} \cdot \frac{E_0 \omega}{R} \cdot \cos \omega t \, dt + k$$

$$\therefore \quad i \cdot e^{t/Rc} = \frac{E_0 \omega}{R} \int e^{t/Rc} \cos \omega t \, dt + k$$

$$\therefore \quad i \cdot e^{t/Rc} = \frac{E_0 \omega}{R} \cdot \frac{e^{t/Rc}}{\sqrt{(1/R^2 c^2) + \omega^2}} \cos (\omega t - \Phi) + k \qquad \text{[By (2A), page 2-9]}$$

where $\tan \Phi = \dfrac{\omega}{1/Rc} = Rc\omega$.

$$\therefore \quad i = \frac{E_0 \omega}{R} \cdot \frac{Rc}{\sqrt{1 + R^2 c^2 \omega^2}} \cos (\omega t - \Phi) + k e^{-t/Rc}$$

$$= \frac{E_0 \omega c}{\sqrt{1 + R^2 c^2 \omega^2}} \cdot \cos (\omega t - \Phi) + k e^{-t/Rc}$$

Example 7 (b) : The charge q on the plate of a condenser of capacity C charged through a resistance R by a steady voltage V satisfies the differential equation $R\dfrac{dq}{dt}+\dfrac{q}{C}=V$. If $q=0$ at $t=0$, show that $q=CV(1-e^{-t/RC})$. Find also the current flowing into the plate. **(M.U. 1995, 2015)**

Sol. : We are given that $\dfrac{dq}{dt}+\dfrac{1}{RC}\cdot q=\dfrac{V}{R}$.

This is a linear differential equation of the type $\dfrac{dy}{dx}+Py=Q$.

Its solution is $q\cdot e^{\int (1/RC)dt}=\int e^{\int (1/RC)dt}\cdot\dfrac{V}{R}dt+k$

$$\therefore\quad q\cdot e^{t/RC}=\int e^{t/RC}\cdot\frac{V}{R}dt+k=\frac{V}{R}\cdot\frac{e^{t/RC}}{(1/RC)}+k$$

$$=CV\cdot e^{t/RC}+k$$

By data when $t=0,\ q=0\quad\therefore k=-CV$

$$\therefore\quad q\cdot e^{t/RC}=CV\cdot e^{t/RC}-CV$$

$$\therefore\quad q=CV-CV\,e^{-t/RC}=CV(1-e^{-t/RC})$$

Further $i=\dfrac{dq}{dt}=CV\cdot e^{-t/RC}\cdot\dfrac{1}{RC}=\dfrac{V}{R}\cdot e^{-t/RC}$.

Example 8 (b) : In a circuit of resistance R, self inductance L, the current, i is given by

$$L\frac{di}{dt}+Ri=E\cos pt$$

when E and p are constants, find the current i at time t. **(M.U. 2013)**

Sol. : We have $\dfrac{di}{dt}+\dfrac{R}{L}i=\dfrac{E}{L}\cos pt$

This is a linear differential equation of the form $\dfrac{dy}{dx}+Py=Q$.

$$\therefore\quad \text{I.F.}=e^{\int (R/L)dt}=e^{(R/L)t}$$

$\therefore$ Its solution is

$$i\cdot e^{(R/L)t}=\int e^{(R/L)t}\cdot\frac{E}{L}\cos pt\ dt+c$$

$$=\frac{E}{L}\int e^{(R/L)t}\cdot\cos pt\ dt+c$$

$$=\frac{E}{L}\left[\frac{e^{(R/L)t}}{(R/L)^2+p^2}\left(\frac{R}{L}\cos pt+p\sin pt\right)\right]+c$$

[See (23), page F-6 in the list of formulae]

$$=\frac{EL}{R^2+L^2p^2}\cdot e^{(R/L)t}\left[\frac{R}{L}\cos pt+p\sin pt\right]+c$$

$$\therefore\quad i=\frac{EL}{R^2+L^2p^2}\left[\frac{R}{L}\cos pt+p\sin pt\right]+c\,e^{-(R/L)t}$$

$$=\frac{E}{R^2+L^2p^2}[R\cos pt+Lp\sin pt]+c\,e^{-(R/L)t}.$$

EXERCISE - II

Solve the following examples : Class (b) : 6 Marks

1. The current i in a circuit containing a resistance R and a condenser of capacity C farads and connected to a constant e.m.f. E is given by $Ri + \frac{q}{c} = \frac{E}{R}$. Find q, given that $q = 0$ when $t = 0$.

[**Ans. :** $q = EC(1 - e^{-t/CR})$]

2. When the inner of two concentric spheres of radius r_1 and r_2 $(r_1 < r_2)$ carries an electric charge, the differential equation for the potential v at a distance r from the common centre is $\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$. Find v in terms or r. [**Ans. :** $vr = c_1 r - c_2$. Put $\frac{dv}{dr} = z$]

3. When a switch is closed, the current built up in an electric circuit is given by $E = Ri + L\frac{di}{dt}$. If $L = 640$, $R = 250$, $E = 500$ and $i = 0$ when $t = 0$ show that the current will approach 2 amp. when $t \to \infty$.

4. A resistance of 100 ohms and inductance of 0.5 henries are connected in series with a battery of 20 volts. Find the current at any instant if the relation between L, R, E is $L\frac{di}{dt} + Ri = E$.

(M.U. 2015) [**Ans. :** $i = 0.2(1 - e^{-200t})$]

* * *

CHAPTER 3

Linear Differential Equations with Constant Coefficients

1. Introduction

In this chapter we shall study linear differential equations with constant coefficients. In this type of differential equation the derivative of any order may occur but its power is unity and its coefficient is constant. For example, the following differential equations are of this type.

$$3\frac{d^3y}{dx^3}+2\frac{d^2y}{dx}+\frac{dy}{dx}+y=x^2$$

$$5\frac{d^4y}{dx^4}+2\frac{d^3y}{dx^3}+\frac{dy}{dx}+y=\sin x+e^x$$

Definition

An equation of the form $\dfrac{d^ny}{dx^n}+P_1\dfrac{d^{n-1}y}{dx^{n-1}}+P_2\dfrac{d^{n-2}y}{dx^{n-2}}+\ldots..+P_ny=X$

where $P_1, P_2, \ldots.. P_n$ are constants and X is a function of x only is called **a linear differential equation with constant coefficients.**

2. The Operator D

Let D be the symbol which denotes differentiation with respect to x, say, of the function which immediately follows it *i.e.* D stands for d/dx.

Thus, if y is a differentiable function of x then

$$D(y)=\frac{d}{dx}(y) \quad \text{or} \quad Dy=\frac{dy}{dx},$$

$$D(Dy)=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$$

Let us further denote the operation of D repeated twice, thrice, ... n times by $D^2, D^3, \ldots.., D^n$. With this notation,

$$D^2y=\frac{d^2y}{dx^2},\quad D^3y=\frac{d^3y}{dx^3},\quad \ldots.\quad D^ny=\frac{d^ny}{dx^n}$$

From this point of view the symbol D is called an ***operator*** and the function y on which it operates is called ***operand***. With this notation the differential equation

$$\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots..+P_nY=X \quad \text{can be written as} \qquad \textbf{(1)}$$

$$D^ny+P_1D^{n-1}+P_2D^{n-2}y+\ldots..+P_ny=X$$

i.e. $(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n) y = X$

or briefly $f(D)\, y = X$ ....(1A)

In particular if the right hand side of the above equation X is zero the equation becomes

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_n y = 0 \quad \text{..................(2)}$$

i.e. $D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + \ldots + P_n y = 0$

i.e. $(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n) y = 0$

or briefly $f(D)\, y = 0.$ ....(2A)

3. Solution of $f(D)\, y = X$

Without going into the theoretical discussion of solving the general linear differentiation equation $f(D)\, y = X$ we shall learn the method of solving it through an illustration.

The equation (2) or (2A) can be called an ***associated equation*** of the more general equation (1) or (1A).

Now, consider the equation

$$\frac{d^3 y}{dx^3} + 2\frac{d^2 y}{dx^2} - \frac{dy}{dx} - 2y = x^2 \quad \text{..................(3)}$$

Its associated equation is

$$\frac{d^3 y}{dx^3} + 2\frac{d^2 y}{dx^2} - \frac{dy}{dx} - 2y = 0 \quad \text{..................(4)}$$

Without going into the question, at this stage, as to how we got it, we can verify that $y = e^x$ is a solution of (4). (Verify it !). Further, it can also be verified that $y = c_1 e^x$ where c_1 is an arbitrary constant is also a solution of (4). Similarly we can verify that $y = e^{-x}$ and $y = e^{-2x}$ and more generally $y = c_2 e^{-x}$, $y = c_3 e^{-2x}$ satisfy the equation (4). For this reason $Y = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$ satisfies (4) and is a solution of (4).

Further we can also verify that $u = -\frac{1}{2}\left(x^2 - x + \frac{5}{2}\right)$ satisfies (3) and hence, is a solution of (3).

Now, $y = Y + u$ is also solution of (3) because by putting $y = Y + u$ in the l.h.s. of (3) we get x^2 which is the r.h.s. of (3) (Verify it !)

Thus, $y = Y + u$ is the ***general or complete solution*** of (3). The part $Y = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$ which also is the complete solution of (4) is called the ***complementary function (C.F.)*** and the part

$$u = -\frac{1}{2}\left(x^2 - x + \frac{5}{2}\right)$$ is called the ***particular integral (I.F.)*** of (3).

Thus, in general, if $f(D)y = X$ is a given differential equation then $f(D)\, y = 0$ is called its **associated equation**. If $y = Y$ is a solution of $f(D)\, y = 0$ and $y = u$ is a solution $f(D)\, y = X$ then $y = Y + u$ is the *general or complete solution* of $f(D)\, y = X$. The part $y = Y$ is called the **complementary function** and the part $y = u$ is called the **particular integral**.

The method of solving the differential equation $f(D)\,y = X$ depends upon the nature of the right hand side. The following cases arise.

1. **When the r.h.s. $X = 0$.**
2. **When the r.h.s. $X = e^{ax}$.**
3. **When the r.h.s. $X = \sin ax, \cos ax$.**
4. **When the r.h.s. $X = x^m$.**
5. **When the r.h.s. $X = e^{ax} V$ where V is a function of x.**
6. **When the r.h.s. $X = xV$ where V is a function of x.**
7. **When the r.h.s X does not belong partially or completely to any one of the above forms.**

4. To Solve the Differential Equation $f(D)\,y = 0$

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n)\,y = 0 \quad \ldots\ldots (1)$$

Suppose by solving the equation

$$D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n = 0 \quad \ldots\ldots (2)$$

we get the roots of the equation $m_1, m_2, \ldots, m_n$.

Now, suppose we put $y = e^{m_1 x}$ in (1) since

$$D e^{m_1 x} = \frac{d}{dx} e^{m_1 x} = m_1 e^{m_1 x}, \quad D^2 e^{m_1 x} = \frac{d^2}{dx^2} e^{m_1 x} = m_1^2 e^{m_1 x},$$

$$D^n e^{m_1 x} = \frac{d^n}{dx^n} e^{m_1 x} = m_1^n e^{m_1 x}.$$

$$\text{l.h.s.} = m_1^n + P_1 m_1^{n-1} + P_2 m_1^{n-2} + \ldots + P_n$$

and this is equal to zero because m_1 is a root of (2). In other words $y = e^{m_1 x}$ is a solution of (1). Similarly putting $y = c_1 e^{m_1 x}$ in (1) we can prove that it satisfies (1) because of (2) and hence is a solution of (1). Further, we can prove that $y = c_2 e^{m_2 x}$, $y = c_3 e^{m_3 x}$, ..., $y = c_n e^{m_n x}$ are also solutions of (1) because of (2). Hence, the complete solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x} \quad \ldots\ldots (3)$$

where $m_1, m_2, \ldots, m_n$ are the roots of the equation.

$$D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n = 0 \quad \ldots\ldots (4)$$

The equation (4) is called the **auxiliary equation**.

Solved Examples : Class (a) : 3 Marks

Example (a) : Solve $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0.$ **(M.U. 2013, 15)**

Sol. : The auxiliary equation is $D^3 - 6D^2 + 11D - 6 = 0$.

$\therefore\ D^3 - D^2 - 5D^2 + 5D + 6D - 6 = 0 \quad \therefore\ (D-1)(D^2 - 5D - 6) = 0$

$\therefore\ (D-1)(D-2)(D-3) = 0 \quad \therefore\ D = 1, 2, 3.$

$\therefore$ The solution is $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$.

EXERCISE - I

Solve the following equations : Class (a) : 3 Marks

1. $2\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}-12y=0$
2. $9\dfrac{d^2y}{dx^2}+18\dfrac{dy}{dx}-16y=0$
3. $6\dfrac{d^3x}{dt^3}+23\dfrac{d^2x}{dt^2}+29\dfrac{dx}{dt}+12x=0$
4. $\dfrac{d^3y}{dx^3}+2\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}-6y=0$
5. $\dfrac{d^3y}{dx^3}+4\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-6y=0$
6. $\dfrac{d^4y}{dx^4}-5\dfrac{d^3y}{dx^3}+5\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}-6y=0.$

[**Ans. :** (1) $y=c_1 e^{(3/2)\cdot x}+c_2 e^{-4x}$ (2) $y=c_1 e^{(2/3)\cdot x}+c_2 e^{(-8/3)\cdot x}$

(3) $x=c_1 e^{-t}+c_2 e^{(-3/2)\cdot t}+c_3 e^{(-4/3)\cdot t}$ (4) $y=c_1 e^{-x}+c_2 e^{2x}+c_3 e^{-3x}$

(5) $y=c_1 e^{x}+c_2 e^{-2x}+c_3 e^{-3x}$ (6) $y=c_1 e^{-x}+c_2 e^{2x}+c_3 e^{3x}+c_4 e^{x}$.]

5. Auxiliary Equation with Repeated Roots

If the auxiliary equation (2) of § 4 has repeated roots say m_1 repeating twice then the general solution 3 of § 4 becomes

$$y=(c_1+c_2)e^{m_1 x}+c_3 e^{m_3 x}+.....+c_n e^{m_n x}$$

But this solution has $n-1$ arbitrary constants and is not the general solution because the general solution of the differential equation of n-th order should have n arbitrary constants.

To find the general solution when roots are repeated

Suppose the auxiliary equations $f(D)=0$ has the root m_1 repeated twice. Hence, suppose.

$$f(D)=\Phi(D)(D-m_1)^2$$

The solution corresponding to the repeated roots is the solution of $(D-m_1)^2\, y=0$ *i.e.* of $(D-m_1)(D-m_1)\,y=0$ .... (5)

Let $(D-m_1)\,y=v$ .... (6)

Then (5) becomes $(D-m_1)\,v=0$

$\therefore\ \dfrac{dv}{dx}-m_1 v=0.$ Separating the variables $\dfrac{dv}{v}=m_1\,dx.$

$\therefore\ \log v=m_1 x+\log c \qquad \therefore\ \log\dfrac{v}{c}=m_1 x \qquad \therefore\ v=c_1 e^{m_1 x}.$

Putting this value of v in (6), we get $\dfrac{dy}{dx}-m_1 y=c_1 e^{m_1 x}.$

But this is a linear differential equation of the type $\dfrac{dy}{dx}+Py=Q$ and hence its solution is

$$y e^{-m_1 x}=\int e^{-m_1 x}c_1 e^{m_1 x}\,dx+c_2=\int c_1\,dx+c_2=c_1 x+c_2$$

$\therefore\ y=(c_1 x+c_2)\,e^{m_1 x}$ which is the solution of the equation (5).

Similarly, if the root m_1 is repeated thrice, the solution corresponding to these roots is the solution of $(D-m_1)^3\,y=0.$

Proceeding as above, we get, $y = (c_1 x^2 + c_2 x + c_3) e^{m_1 x}$.

If this root m_1 is repeated r times the corresponding part of the solution is

$$y = (c_1 x^{r-1} + c_2 x^{r-2} + \ldots.. + c_r) e^{m_1 x}.$$

or $$y = (c_1 + c_2 x + c_3 x^2 + \ldots.. + c_r x^{r-1}) e^{m_1 x}.$$

Solved Examples : Class (a) : 3 Marks

Example (a) : Solve $\dfrac{d^3y}{dx^3} - 5\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} - 4y = 0.$ **(M.U. 2002, 16)**

Sol. : The auxiliary equation is $D^3 - 5D^2 + 8D - 4 = 0$.

$\therefore\ D^3 - 2D^2 - 3D^2 + 6D + 2D - 4 = 0$ $\quad\therefore\ (D-2)(D^2 - 3D + 2) = 0$

$\therefore\ (D-2)(D-2)(D-1) = 0$ $\quad\therefore\ D = 1, 2, 2.$

$\therefore$ The solution is $y = (c_1 + c_2 x) e^{2x} + c_3 e^x$.

EXERCISE - II

Solve the following equations : Class (a) : 3 Marks

1. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$
2. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = 0$
3. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0$
4. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4y = 0$
5. $\dfrac{d^4y}{dx^4} - 18\dfrac{d^2y}{dx^2} + 81y = 0$
6. $\dfrac{d^4y}{dx^4} - 2\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = 0.$

[**Ans. :** (1) $y = (c_1 + c_2 x) e^{-2x}$ (2) $y = (c_1 + c_2 x) + c_3 e^{-x}$

(3) $y = (c_1 + c_2 x + c_3 x^2) e^x$ (4) $y = (c_1 + c_2 x) e^{2x} + c_3 e^{-x}$

(5) $y = (c_1 + c_2 x) e^{3x} + (c_3 + c_4 x) e^{-3x}$ (6) $y = (c_1 + c_2 x) + (c_3 + c_4 x) e^x$]

6. Auxiliary Equation with Imaginary Roots

Further, if a pair of imaginary roots is repeated twice the corresponding part of the solution can be written as

$$y = (c_1 + c_2 x)\, e^{(\alpha + i\beta)x} + (c_3 + c_4 x)\, e^{(\alpha - i\beta)x} \text{ which can be written as}$$

$$\boxed{y = e^{\alpha x}\left[(A_1 + A_2 x)\cos\beta x + (A_3 + A_4 x)\sin\beta x\right]}$$

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Solve $\dfrac{d^3y}{dx^3} + y = 0.$

Sol. : The auxiliary equation is $D^3 + 1 = 0$

$$\therefore \quad (D+1)(D^2 - D + 1) = 0 \quad \therefore \quad D = -1, \ \frac{1 \pm \sqrt{3} \cdot i}{2}$$

$$\therefore \text{ The solution is } y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\frac{\sqrt{3}}{2} x + c_3 \sin\frac{\sqrt{3}}{2} x\right).$$

Example 2 (a) : Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0.$ **(M.U. 2014)**

Sol. : The auxiliary equation is $D^4 + 2D^2 + 1 = 0.$

$$\therefore \quad (D^2 + 1)^2 = 0 \qquad \therefore \quad D^2 = -1, -1$$

$$\therefore \quad D = +i, -i, +i, -i.$$

The roots are replealed complex roots.

$\therefore$ The solution is $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x.$

Example 3 (a) : Solve $\dfrac{d^4y}{dx^4} + 6\dfrac{d^2y}{dx^2} + 9y = 0.$

Sol. : The auxiliary equation is $D^4 + 6D^2 + 9 = 0.$

$$\therefore \quad (D^2 + 3)^2 = 0 \quad \therefore \quad D^2 = -3 \quad \therefore \quad D = \pm\sqrt{3}i, \ \pm\sqrt{3}i$$

The roots are repeated complex roots.

$\therefore$ The solution is $y = (c_1 + c_2 x)\cos\sqrt{3} \cdot x + (c_3 + c_4 x)\sin\sqrt{3} \cdot x.$

Example 4 (a) : Solve $\dfrac{d^4y}{dx^4} + k^4 y = 0.$ **(M.U. 2003)**

Sol. : The auxiliary equation is $D^4 + k^4 = 0.$

$$\therefore \quad (D^4 + 2D^2k^2 + k^4) - (2D^2k^2) = 0 \qquad \therefore \quad (D^2 + k^2)^2 - (\sqrt{2} \cdot Dk)^2 = 0$$

$$\therefore \quad (D^2 - \sqrt{2} \cdot Dk + k^2)(D^2 + \sqrt{2} \cdot Dk + k^2) = 0$$

Now, $D^2 - \sqrt{2} \cdot Dk + k^2 = 0$ gives $D = \dfrac{k \pm ik}{\sqrt{2}}$

$D^2 + \sqrt{2} \cdot Dk + k^2 = 0$ gives $D = \dfrac{-k \pm ik}{\sqrt{2}}$

Since, we have two pairs of complex roots, the solution is

$$y = e^{(k/\sqrt{2})x}\left[c_1 \cos(k/\sqrt{2})x + c_2 \sin(k/\sqrt{2})x\right]$$
$$+ e^{(-k/\sqrt{2})x}\left[c_3 \cos(k/\sqrt{2})x + c_4 \sin(k/\sqrt{2})x\right]$$

Example 4 (A) (a) : Solve $\dfrac{d^4y}{dx^4} + 4y = 0.$ **(M.U. 2013)**

Sol. : Put $k = \sqrt{2}$, $k^2 = 2$ in the above example and solve.

Example 5 (a) : Solve $\dfrac{d^4y}{dx^4} - 2\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0.$

Sol. : The auxiliary equation is $D^4 - 2D^3 + 3D^2 - 2D + 1 = 0$

$\therefore \quad D^4 + D^2 + 1 - 2D^3 + 2D^2 - 2D = 0 \qquad \therefore \quad (D^2 - D + 1)^2 = 0$

$\therefore \quad D = \dfrac{1 \pm \sqrt{3}\cdot i}{2}, \ \dfrac{1 \pm \sqrt{3}\cdot i}{2}$

Since, the roots are complex and repeated

$$y = e^{x/2}\left\{(c_1 + c_2x)\cos\frac{\sqrt{3}}{2}\cdot x + (c_3 + c_4x)\sin\frac{\sqrt{3}}{2}x\right\}.$$

Example 6 (a) : Solve $(D^4 - 4D^3 + 8D^2 - 8D + 4)\,y = 0.$ **(M.U. 2014)**

Sol. : The auxiliary equation is

$D^4 - 4D^3 + 8D^2 - 8D + 4 = 0$

$\therefore \quad (D^2 - 2D + 2)^2 = 0 \qquad \therefore \quad D^2 - 2D + 2 = 0$

$\therefore \quad D = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$ and $1 \pm i.$

The roots are complex and repeated.

$\therefore \quad y = e^x\,[(c_1 + c_2\,x)\cos x + (c_3 + c_4\,x)\sin x\,].$

7. Auxiliary Equation with Real, Irrational Roots

If an equation has one root real and irrational of the form $\alpha + \sqrt{\beta}$ then we know that the other root is $\alpha - \sqrt{\beta}$. If the auxiliary equation (2) of § 4 has a pair of irrational roots $\alpha + \sqrt{\beta}$ and $\alpha - \sqrt{\beta}$ then the corresponding part of the solution (1) of § 4 is

$$y = c_1\,e^{(\alpha+\sqrt{\beta})x} + c_2\,e^{(\alpha-\sqrt{\beta})x} = c_1\,e^{\alpha x}\cdot e^{\sqrt{\beta}\cdot x} + c_2\,e^{\alpha x}\cdot e^{-\sqrt{\beta}\cdot x} \quad \text{.................... (A)}$$

$\therefore \quad y = e^{\alpha x}\,(c_1\,e^{\sqrt{\beta}\cdot x} + c_2\,e^{-\sqrt{\beta}\cdot x})$

But $\dfrac{e^{\sqrt{\beta}\cdot x} + e^{-\sqrt{\beta}\cdot x}}{2} = \cos h\sqrt{\beta}\cdot x$ and $\dfrac{e^{\sqrt{\beta}\cdot x} - e^{-\sqrt{\beta}\cdot x}}{2} = \sin h\sqrt{\beta}\cdot x$

$\therefore$ By addition $\quad e^{\sqrt{\beta}\cdot x} = \cos h\sqrt{\beta}\cdot x + \sin h\sqrt{\beta}\cdot x$

and by subtraction $\quad e^{-\sqrt{\beta}\cdot x} = \cos h\sqrt{\beta}\cdot x - \sin h\sqrt{\beta}\cdot x$

$\therefore \quad y = e^{\alpha x}\,[\,c_1\,(\cos h\sqrt{\beta}\cdot x + \sin h\sqrt{\beta}\cdot x) + c_2\,(\cos h\sqrt{\beta}\cdot x - \sin h\sqrt{\beta}\cdot x)]$

$$= e^{\alpha x}[(c_1 + c_2)\cos h\sqrt{\beta} \cdot x + (c_1 - c_2)\sin h\sqrt{\beta} \cdot x]$$

$$\therefore \quad y = e^{\alpha x}(A\cos h\sqrt{\beta} \cdot x + B\sin h\sqrt{\beta} \cdot x)$$

Replacing A and B which are arbitrary constants by c_1 and c_2.

$$y = e^{\alpha x}(c_1 \cos h\sqrt{\beta} \cdot x + c_2 \sin h\sqrt{\beta} \cdot x)$$

Further if **a pair of real irrational roots are repeated twice** the corresponding part of the solution can be written as

$$y = (c_1 + c_2 x)\,e^{(\alpha + \sqrt{\beta})x} + (c_3 + c_4 x)\,e^{(\alpha - \sqrt{\beta})x}$$

$$\therefore \quad y = e^{\alpha x}[(A_1 + A_2 x)\cos h\sqrt{\beta} \cdot x + (A_3 + A_4 x)\sin h\sqrt{\beta} \cdot x]$$

Replacing A's by c's, we get

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos h\sqrt{\beta} \cdot x + (c_3 + c_4 x)\sin h\sqrt{\beta} \cdot x]$$

Solved Example : Class (a) : 3 Marks

Example (a) : Solve $(D^2 - 2D - 4)\,y = 0$.

Sol. : The A.E. is $D^2 - 2D - 4 = 0$

$$\therefore \quad D = \frac{2 \pm \sqrt{4 + 16}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$$

Thus roots are real and irrational and hence the complementary function *i.e.* the complete solution is

$$y = e^x(c_1 \cos h\sqrt{5} \cdot x + c_2 \sin h\sqrt{5} \cdot x).$$

EXERCISE - III

Solve the following equations : Class (a) : 3 Marks

1. $\dfrac{d^3y}{dx^3} + 8y = 0$ 2. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0$ **(M.U. 2014)** 3. $\dfrac{d^4y}{dx^4} - m^4 y = 0$

4. $\dfrac{d^4y}{dx^4} + 13\dfrac{d^2y}{dx^2} + 36y = 0$ 5. $\dfrac{d^4y}{dx^4} + y = 0$

6. $\dfrac{d^4y}{dx^4} + 4\dfrac{d^3y}{dx^3} + 8\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 4y = 0$ **(M.U. 2014)**

7. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$

8. $\dfrac{d^6y}{dx^6} - 64y = 0$ 9. $(D^3 - D^2 + D - 1)^2 y = 0$

10. $\left\{(D-1)^4(D^2 + 2D + 2)^2\right\} y = 0$ 11. $\left\{(D^2 + 1)^3(D^2 + D + 1)^2\right\} y = 0$ **(M.U. 2002)**

12. $(D^4 + 8D^2 + 16)\,y = 0$ **(M.U. 2003)**

[**Ans. : (1)** $y = c_1 e^{-2x} + e^x(c_2 \cos\sqrt{3} \cdot x + c_3 \sin\sqrt{3} \cdot x)$

(2) $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

(3) $y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$

(4) $y = c_1 \cos 3x + c_2 \sin 3x + c_3 \cos 2x + c_4 \sin 2x$

(5) $y = e^{x/\sqrt{2}}\left[c_1 \cos (x/\sqrt{2}) + c_2 \sin (x/\sqrt{2})\right]$
$$+ e^{-x/\sqrt{2}}\left[c_3 \cos (x/\sqrt{2}) + c_4 \sin (x/\sqrt{2})\right]$$

(6) $y = e^{-x}[(c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x]$

(7) $y = e^{-x/2}\left[(c_1 + c_2 x)\cos \frac{\sqrt{3}}{2} \cdot x + (c_3 + c_4 x)\sin \frac{\sqrt{3}}{2} \cdot x\right]$

(8) $y = c_1 e^{2x} + c_2 e^{-2x} + e^{-x}(c_3 \cos \sqrt{3} \cdot x + c_4 \sin \sqrt{3} \cdot x)$
$$+ e^{x}(c_5 \cos \sqrt{3} \cdot x + c_6 \sin \sqrt{3} \cdot x)$$

(9) $y = (c_1 + c_2 x)e^{x} + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x$

(10) $y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^{x} + e^{-x}\{(c_5 + c_6 x)\cos x + (c_7 + c_8 x)\sin x\}$

(11) $y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x)\sin x$
$$+ e^{-x/2}[(c_7 + c_8 x)\cos(\sqrt{3} \cdot x/2) + (c_9 + c_{10} x)\sin(\sqrt{3} \cdot x/2)]$$

(12) $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x.$]

8. Solution of $f(D)\, y = X$ where X is a Function of x

Consider the equation

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots\ldots + P_n)\, y = X \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

i.e. $\quad f(D)\, y = X \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$

where $P_1, P_2, \ldots\ldots P_n$ are constants and X is a function of x only. The general solution of (1) is the sum of the complementary function and the particular integral. The complementary function is obtained by solving the auxiliary equation.

$$D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots\ldots + P_n = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Since $f(D)\, y = X$, the **particular integral** can be written as $y = \frac{1}{f(D)} X$.

To find $\frac{1}{f(D)} X$ we **define** $\frac{1}{f(D)}$ as the inverse operator of $f(D)$.

(a) Definition : $\frac{1}{f(D)} X$ is **defined** as that function of x which when operated upon by $f(D)$ gives X.

(b) Meaning of $\frac{1}{D} X$: By the above definition $\frac{1}{D} X$ is that function of x which when operated on by D gives X. But D stands for $\frac{d}{dx}$ i.e. differentiation w.r.t. x and since integration is inverse of differentiation $\frac{1}{D}$ must stand for integration.

Thus, $$\boxed{\frac{1}{D} X = \int X\, dx}$$

This means the particular integral of the equation $D(y) = X$ is given by

$$\boxed{\text{P.I.} = \frac{1}{D} X = \int X\, dx}$$

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Find the particular integral of $D(y) = x^2$.

Sol. : As discussed above

$$\text{Particular Integral, P.I.} = \frac{1}{D} x^2 = \int x^2\, dx = \frac{x^3}{3}.$$

Example 2 (a) : Find the particular integral of $D(y) = e^x$.

Sol. : $$\text{Particular Integral, P.I.} = \frac{1}{D} e^x = \int e^x\, dx = e^x$$

Example 3 (a) : Find the particular integral of $D(y) = \sec^2 x$.

Sol. : $$\text{Particular Integral, P.I.} = \frac{1}{D} \sec^2 x = \int \sec^2 x\, dx = \tan x.$$

(c) Meaning of $\frac{1}{D-a} X$: Consider the equation $\frac{dy}{dx} - ay = X$

i.e. $(D - a)\, y = X$ .... (1)

Its auxiliary equation is $D - a = 0$ $\therefore$ $D = a$.

Its complementary function is $y = ce^{ax}$. Its particular integral is $\frac{1}{D-a} X$.

Hence, its general solution is $y = ce^{ax} + \frac{1}{D-a} X$ .... (2)

But the equation (1) can be looked upon as linear differential equation of the form $\frac{dy}{dx} + Py = Q$ and its solution is

$$y\, e^{-ax} = \int e^{-ax} \cdot X \cdot dx + c \qquad \therefore \quad y = ce^{ax} + e^{ax} \int e^{-ax} \cdot X \cdot dx \qquad \text{................... (3)}$$

Comparing (2) and (3) we find that

$$\boxed{\frac{1}{D-a} \cdot X = e^{ax} \int e^{-ax} \cdot X \cdot dx} \qquad \text{................... (4)}$$

Similarly we can prove that

$$\boxed{\frac{1}{D+a} \cdot X = e^{-ax} \int e^{ax} \cdot X \cdot dx} \qquad \text{................... (5)}$$

This means the particular integral (P.I.) of the equation $\frac{dy}{dx} - ay = X$ *i.e.* $(D - a)\, y = x$ is given by

$$\boxed{\text{P.I.} = \frac{1}{D-a} X = e^{ax} \int e^{-ax}\, X\, dx} \qquad \text{................... (6)}$$

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Find the particular integral of $(D-a)\,y = k$.

Sol. : As discussed above,

$$\text{P.I.} = \frac{1}{D-a} \cdot k = e^{ax} \int e^{-ax} \cdot k\,dx = e^{ax} \cdot k \cdot \int e^{-ax}\,dx$$

$$= e^{ax} \cdot k \cdot \left[\frac{e^{-ax}}{-a}\right] = -\frac{k}{a}.$$

Example 2 (a) : Find the particular integral of $(D-3)\,y = x$.

Sol. :

$$\text{P.I.} = \frac{1}{D-3}\,x = e^{3x} \int e^{-3x} \cdot x\,dx$$

$$= e^{3x}\left[x\left(\frac{e^{-3x}}{-3}\right) - \int \frac{e^{-3x}}{-3} \cdot 1 \cdot dx\right] \qquad \text{[By parts]}$$

$$= e^{3x}\left[-\frac{1}{3}\,x\,e^{-3x} + \left(-\frac{e^{-3x}}{9}\right)\right] = -\frac{x}{3} - \frac{1}{9}.$$

Example 3 (a) : Find the particular integral of $(D+3)\,y = \sin 2x$.

Sol. :

$$\text{P.I.} = \frac{1}{D+3}\sin 2x = e^{-3x}\int e^{3x}\sin 2x\,dx$$

$$= e^{-3x}\left[e^{3x} \cdot \frac{1}{3^2+2^2}(3\sin 2x - 2\cos 2x)\right]$$

$$= \frac{1}{13}(3\sin 2x - 2\cos 2x)$$

$$\left[\because \int e^{ax}\sin bx\,dx = e^{ax} \cdot \frac{1}{a^2+b^2}(a\sin bx - b\cos bx)\right.$$

$$\left.\text{and } \int e^{ax}\cos bx\,dx = e^{ax} \cdot \frac{1}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

Example 4 (a) : Find the particular integral of $(D^2 - 4D + 3)\,y = e^{2x}$.

Sol. :

$$\text{P.I.} = \frac{1}{(D^2-4D+3)}\,e^{2x} = \frac{1}{(D-1)(D-3)} \cdot e^{2x} = \frac{1}{D-1} \cdot \frac{1}{D-3} \cdot e^{2x}$$

$$= \frac{1}{D-1} \cdot e^{3x}\int e^{-3x} \cdot e^{2x}\,dx = \frac{1}{D-1} \cdot e^{3x}\int e^{-x}\,dx = \frac{1}{D-1} \cdot e^{3x}(-e^{-x})$$

$$= -\frac{1}{D-1} \cdot e^{2x} = -e^{x}\int e^{-x} \cdot e^{2x}\,dx = -e^{x}\int e^{x}\,dx = -e^{x} \cdot e^{x} = -e^{2x}.$$

Alternatively we can find the particular integral by using partial fractions as shown below.

$$\text{P.I.} = \frac{1}{D^2-4D+3}\,e^{2x} = \frac{1}{2}\left[\frac{1}{D-3} - \frac{1}{D-1}\right]e^{2x}$$

$$= \frac{1}{2}\left\{\frac{1}{D-3} \cdot e^{2x} - \frac{1}{D-1} \cdot e^{2x}\right\}$$

$$\therefore \quad \text{P.E.} = \frac{1}{2}\left\{e^{3x}\int e^{-3x} \cdot e^{2x}\,dx - e^{x}\int e^{-x} \cdot e^{2x}\,dx\right\}$$

$$= \frac{1}{2}\left\{e^{3x}\int e^{-x}\,dx - e^{x}\int e^{x}\,dx\right\} = \frac{1}{2}\left\{e^{3x}(-e^{-x}) - e^{x} \cdot e^{x}\right\}$$

$$= \frac{1}{2}\left[-e^{2x} - e^{2x}\right] = -e^{2x}.$$

Remark ....

The particular integral $\frac{1}{f(D)}X$ can be evaluated as illustrated in the above Ex. 4 by application of each factor successively or by application of partial fractions.

Thus, in general we have

$$\boxed{\begin{aligned} \frac{1}{f(D)}X &= \frac{1}{D-m_1} \cdot \frac{1}{D-m_2} \cdots\cdots \frac{1}{D-m_n}X \\ \text{or} \quad \frac{1}{f(D)}X &= \left\{\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \cdots\cdots + \frac{A_n}{D-m_n}\right\}X. \end{aligned}}$$

However, we have short methods for finding particular integrals in some certain standard cases. They are discussed in the following articles.

EXERCISE - IV

Find the particular integrals of the following differential equations : Class (a) : 3 Marks

1. $D(y) = \sin x$
2. $D^2(y) = e^{2x}$
3. $D(y) = x^3$
4. $D^2(y) = x^3$
5. $(D-1)\,y = \sin x$
6. $(D+2)\,y = x^2$
7. $(D-2)\,y = \sin x$
8. $(D+2)\,y = \cos x$
9. $(D^3 - D^2)\,y = x$
10. $(D^4 - 4D^2)\,y = 5e^{2x}$

[**Ans. :** **(1)** $-\cos x$ **(2)** $\frac{e^{2x}}{4}$ **(3)** $\frac{x^4}{4}$ **(4)** $\frac{x^5}{20}$ **(5)** $-\frac{\sin x + \cos x}{2}$ **(6)** $\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}$

(7) $-\frac{2\sin x + \cos x}{5}$ **(8)** $\frac{2\cos x + \sin x}{5}$ **(9)** $-\frac{x^3}{6} - \frac{x^2}{2}$ **(10)** $\frac{e^{3x}}{9}$.]

EXERCISE - V

Solve the following differential equations : Class (b) : 6 Marks

1. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = e^{2x}$
2. $\frac{d^2y}{dx^2} - y = 2 + 5x$
3. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{2x}$
4. $\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = 3 + x$
5. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 2x$
6. $(D^2 - 2D + 1)\,y = x^2 - 1$

[**Ans. :** **(1)** $y = c_1 e^{3x} + c_2 e^{4x} + \frac{1}{2}e^{2x}$ **(2)** $y = c_1 e^{x} + c_2 e^{-x} - 2 - 5x$

(3) $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{6} e^{2x}$ (4) $y = c_1 e^{x} + c_2 e^{2x} + \frac{x}{2} + \frac{9}{4}$

(5) $y = c_1 e^{-2x} + c_2 e^{x} + \left(x + \frac{1}{2}\right)$ (6) $y = (c_1 + c_2 x) e^{x} + (x^2 + 4x + 1))]$

9. Particular Integral when $X = e^{ax}$

When $X = e^{ax}$ the particular integral is given by

$$\boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}} \quad \text{if } f(a) \neq 0$$

Proof : Since $D e^{ax} = a e^{ax}$, $D^2 e^{ax} = a^2 e^{ax}$, ... $D^n e^{ax} = a^n e^{ax}$,

we get
$$f(D) e^{ax} = (D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n) e^{ax}$$
$$= (a^n + P_1 a^{n-1} + P_2 a^{n-2} + \ldots + P_n) e^{ax}$$
$$= f(a) e^{ax}.$$

Operating on both sides by $\frac{1}{f(D)}$

$$\frac{1}{f(D)} \cdot f(D) \cdot e^{ax} = \frac{1}{f(D)} \cdot f(a) \cdot e^{ax}$$

Since $\frac{1}{f(D)}$ and $f(D)$ are inverse operators they cancel each other in effect.

$$\therefore \; e^{ax} = \frac{1}{f(D)} \cdot f(a) \cdot e^{ax} \qquad \therefore \; \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

e.g. $\frac{1}{D^2 + 2D + 1} e^{2x} = \frac{1}{4 + 4 + 1} \cdot e^{2x} = \frac{1}{9} e^{2x}.$

(a) Particular case

The method, however, fails if $f(a) = 0$ i.e. $D - a$ is a factor of $f(D)$ because in this case

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} = \frac{1}{0} e^{ax}.$$

Now, since $D - a$ is a factor of $f(D)$, suppose

$$f(D) = (D - a)\, \Phi(D) \text{ and } \Phi(a) \neq 0.$$

Then
$$\text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{1}{D - a} \cdot \frac{1}{\Phi(D)} e^{ax}$$
$$= \frac{1}{D - a} \cdot \frac{1}{\Phi(D)} e^{ax} = \frac{1}{\Phi(a)} \cdot \frac{1}{D - a} e^{ax}$$
$$= \frac{1}{\Phi(a)} \cdot e^{ax} \int e^{-ax} \cdot e^{ax}\, dx = \frac{1}{\Phi(a)} \cdot e^{ax} \int dx$$

$$\therefore \; \boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{\Phi(a)} \cdot e^{ax} \cdot x} \qquad \text{.................... (1)}$$

If $D-a$ is a factor repeated twice *i.e.* if $f(D)=(D-a)^2\,\Psi(D)$ where $\Psi(a)\neq 0$ then proceeding as above, we can prove that,

$$\text{P.I.}=\frac{1}{f(D)}e^{ax}=\frac{x^2}{2\,!}\cdot\frac{1}{\Psi(a)}e^{ax}$$

In general, if $D-a$ is a factor repeated r times then

$$\boxed{\text{P.I.}=\frac{1}{f(D)}e^{ax}=\frac{x^r}{r\,!}\cdot\frac{1}{\Psi(a)}e^{ax}}\quad\text{where } \Psi(a)\neq 0 \qquad \text{.................... (2)}$$

Corollary 1 : $$\boxed{\frac{1}{f(D)}\cdot k=\frac{k}{f(0)}} \qquad \text{.................... (2A)}$$

We can write $\dfrac{1}{f(D)}\cdot k=\dfrac{1}{f(D)}e^{0x}\cdot k=k\dfrac{1}{f(D)}\cdot e^{0\,x}=k\cdot\dfrac{1}{f(0)}$.

Another Form : The above results (1) and (2) can be put in another form as follows:

Since, $f(D)=(D-a)\,\Phi(D)$, differentiating it w.r.t. D, we get,

$$f'(D)=(D-a)\,\Phi'(D)+\Phi(D)$$

$$\therefore\quad f'(a)=\Phi(a)\quad i.e.\quad \Phi(a)=f'(a)$$

Hence, (1) becomes $$\boxed{\frac{1}{f(D)}e^{ax}=x\cdot\frac{1}{f'(a)}\cdot e^{ax}}\quad\text{if } f'(a)\neq 0 \qquad \text{.................... (3)}$$

Proceeding in this way, if $f'(a)=0$

$$\frac{1}{f(D)}e^{ax}=x^2\cdot\frac{1}{f''(a)}\cdot e^{ax}\quad\text{if } f''(a)\neq 0$$

If $f''(a)=0$ then $\dfrac{1}{f(D)}e^{ax}=x^3\cdot\dfrac{1}{f'''(a)}\cdot e^{ax}$ etc.

Corrollary 2 : $$\boxed{\frac{1}{f(D)}\cdot a^x=\frac{1}{f(\log a)}\cdot a^x} \qquad \text{.................... (2B)}$$

Proof : Since $a^x=e^{x\log a}$, in the above formula, we have to replace a by $\log a$.

$$\therefore\quad \frac{1}{f(D)}\cdot a^x=\frac{1}{f(\log a)}\cdot e^{x\log a}\qquad\therefore\quad \boxed{\frac{1}{f(D)}\cdot a^x=\frac{1}{f(\log a)}\cdot a^x}$$

(See Ex. 6, page 3-16.)

Note ....

Linear differential equation of the first order and first degree can also be solved by the method discussed above. See Ex. 1 of the exercise.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^2}+4y=e^{3x}$.

Sol. : The auxiliary equation is $D^3-3D^2+4=0$

$\therefore\ D^3 - 2D^2 - D^2 + 2D - 2D + 4 = 0 \quad \therefore\ (D-2)(D^2 - D - 2) = 0$

$\therefore\ (D-2)(D-2)(D+1) = 0 \quad \therefore\ D = -1, 2, 2.$

$\therefore$ Complementary Function, C.F. is $y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x}$.

Particular Integral, $\text{P.I.} = \dfrac{1}{D^3 - 3D^2 + 4} \cdot e^{3x} = \dfrac{1}{3^3 - 3 \cdot 3^2 + 4} = \dfrac{e^{3x}}{4}$.

The complete solution is $y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x} + \dfrac{e^{3x}}{4}$.

Example 2 (b) : Solve $(D^3 - 2D^2 - 5D + 6)\, y = e^{3x} + 8$. **(M.U. 1991)**

Sol. : The auxiliary equation is $D^3 - 2D^2 - 5D + 6 = 0$.

$\therefore\ (D-1)(D^2 - D - 6) = 0 \quad \therefore\ (D-1)(D-3)(D+2) = 0$

$\therefore\ D = 1, -2, 3.$

$\therefore$ Complementary Function, C.F. is $y = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x}$.

$$\text{Particular Integral, P.I.} = \frac{1}{(D-1)(D+2)(D-3)} \cdot e^{3x} + \frac{8}{(D-1)(D+2)(D-3)} \cdot e^{0x}$$

$$= \frac{1}{(2)(5)} \cdot \frac{1}{D-3} \cdot e^{3x} + \frac{8}{(-1)(2)(-3)} \cdot e^{0x}$$

$$= \frac{1}{10} \cdot x \cdot e^{3x} + \frac{4}{3}.$$

$\therefore$ The complete solution is $y = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x} + \dfrac{x}{10} e^{3x} + \dfrac{4}{3}$.

Example 3 (b) : Solve $(D^3 - 2D^2 - 5D + 6)\, y = (e^{2x} + 3)^2$. **(M.U. 1993)**

Sol. : The auxiliary equation is $D^3 - 2D^2 - 5D + 6 = 0$.

As in the above example

$\therefore\ (D-1)(D-3)(D+2) = 0 \quad \therefore\ D = 1, -2, 3.$

$\therefore$ Complementary Function, C.F. is $y = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x}$.

$$\text{Particular Integral, P.I.} = \frac{1}{D^3 - 2D^2 - 5D + 6} (e^{4x} + 6e^{2x} + 9)$$

$$= \frac{1}{D^3 - 2D^2 - 5D + 6} e^{4x} + 6 \frac{1}{D^3 - 2D^2 - 5D + 6} e^{2x} + 9 \frac{1}{D^3 - 2D^2 - 5D + 6} e^{0x}$$

$$= \frac{e^{4x}}{18} - \frac{3}{2} e^{2x} + \frac{3}{2}.$$

$\therefore$ The complete solution is

$$y = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x} + \frac{e^{4x}}{18} - \frac{3}{2} e^{2x} + \frac{3}{2}.$$

Example 4 (b) : Solve $(D^3 + 3D^2 + 3D + 1)\, y = e^{-x}$.

Sol. : The auxiliary equation is $D^3 + 3D^2 + 3D + 1 = 0$

$\therefore\ (D+1)^3 = 0 \quad \therefore\ D = -1, -1, -1.$

$\therefore$ Complementary Function, C.F. is $y = (c_1 + c_2 x + c_3 x^2) e^{-x}$.

Particular Integral, P.I.$= \dfrac{1}{(D+1)^3} e^{-x} = \dfrac{x^3}{3!} e^{-x}$.

$\therefore$ The complete solution is $y = (c_1 + c_2 x + c_3 x^2) e^{-x} + \dfrac{x^3}{3!} e^{-x}$.

Example 5 (b) : Solve $\dfrac{d^3y}{dx^3} - 4\dfrac{dy}{dx} = 2\cos h^2 2x$. **(M.U. 1993, 94, 2002, 09)**

Sol. : The auxiliary equation is $D^3 - 4D = 0$

$\therefore \; D(D^2-4) = 0 \quad \therefore \; D = 0, 2, -2.$

$\therefore$ C.F. is $y = c_1 + c_2 e^{2x} + c_3 e^{-2x}$.

$$\text{P.I.} = \frac{1}{D^3-4D} 2\cos h^2 (2x) = \frac{1}{D^3-4D} 2\left(\frac{e^{2x}+e^{-2x}}{2}\right)^2$$

$$= \frac{1}{2} \cdot \frac{1}{D^3-4D}(e^{4x} + 2 + e^{-4x})$$

$$= \frac{1}{2}\left[\frac{1}{D^3-4D} e^{4x} + 2\frac{1}{D(D^2-4)} e^{0x} + \frac{1}{D^3-4D} e^{-4x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{48} e^{4x} - \frac{x}{2} - \frac{1}{48} e^{-4x}\right]$$

$$\therefore \; \text{P.I.} = -\frac{x}{4} + \frac{1}{48}\left(\frac{e^{4x}-e^{-4x}}{2}\right) = -\frac{x}{4} + \frac{1}{48}\sin h\, 4x.$$

$\therefore$ The complete solution is $y = c_1 + c_2 e^{2x} + c_3 e^{-2x} - \dfrac{x}{4} + \dfrac{1}{48}\sin h\, 4x$.

Example 6 (b) : Solve $6\dfrac{d^2y}{dx^2} + 17\dfrac{dy}{dx} + 12y = e^{-3x/2} + 2^x$. **(M.U. 1999)**

Sol. : The auxiliary equation is $6D^2 + 17D + 12 = 0$.

$\therefore \; (3D+4)(2D+3) = 0 \quad \therefore \; D = -4/3, \; D = -3/2.$

$\therefore$ The C.F. is $y = c_1 e^{-4x/3} + c_2 e^{-3x/2}$.

$$\therefore \; \text{P.I.} = \frac{1}{(3D+4)(2D+3)}(e^{-3x/2} + 2^x)$$

$$= \frac{1}{(3D+4)(2D+3)} e^{-3x/2} + \frac{1}{(3D+4)(2D+3)} e^{x\log 2} \qquad [\because 2^x = e^{x\log 2}]$$

$$= \frac{1}{[-(9/2)+4]} \cdot x \cdot e^{-3x/2} + \frac{e^{x\log 2}}{6(\log 2)^2 + 17\log 2 + 12} \qquad [\text{By (2B), page 3-14}]$$

$$= -2x e^{-3x/2} + \frac{2^x}{6(\log 2)^2 + 17\log 2 + 12}$$

$\therefore$ The complete solution is

$$y = c_1 e^{-4x/3} + c_2 e^{-3x/2} - 2x e^{-3x/2} + \frac{2^x}{6(\log 2)^2 + 17\log 2 + 12}.$$

EXERCISE - VI

Solve the following differential equations : Class (b) : 6 Marks

1. $(D-1)\,y = e^{3x}$
2. $(D^3 - 1)\,y = (e^x + 1)^2$
3. $(D^2 + 4D + 4)\,y = \cos h\, 2x$ **(M.U. 1988, 93, 97)**
4. $(D^2 - D - 6)\,y = e^x \cos h\, 2x.$
5. $\dfrac{d^2y}{dx^2} + 2a\dfrac{dy}{dx} + (a^2 + b^2)\,y = e^{mx}$
6. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{-x}$ **(M.U. 1994)**
7. $(2D+1)^2 y = 4e^{-x/2}$
8. $(D^4 + 1)\,y = \cos h\, 4x \sin h\, 3x$ **(M.U. 2003)**
9. $(D^3 - 4D^2 + 5D - 2)\,y = e^x$
10. $(D^2 - 9)\,y = e^{-3x} + 1 + e^{3x}$
11. $(D^2 - 2D + 1)\,y = e^x + 1$ **(M.U. 1989)**
12. $(D^3 - 4D)\,y = 2\cos h\, 2x$ **(M.U. 1989, 90)**
13. $(D^3 - a^2 D) = 2\cos h\, ax$
14. $(D^4 - 4D^3 + 8D^2 - 8D + 4)\,y = e^x + 1$ **(M.U. 2011)**

[**Ans.:** (1) $y = c_1 e^x + \dfrac{1}{2} e^{3x}.$

(2) $y = c_1 e^x + e^{-x/2}\left\{c_2 \cos(\sqrt{3}/2)\,x + c_3 \sin(\sqrt{3}/2)\,x\right\} + \left(\dfrac{1}{7}\right)e^{2x} + \left(\dfrac{2}{3}\right)x - 1$

(3) Hint : $\cos h\, 2x = \dfrac{e^{2x} + e^{-2x}}{2}$ $\quad\therefore\; y = (c_1 + c_2 x)\,e^{-2x} + \dfrac{1}{32} e^{2x} + \dfrac{x^2}{4} \cdot e^{-2x}$

(4) $y = c_1 e^{3x} + c_2 e^{-2x} + \dfrac{1}{10} x e^{3x} - \dfrac{1}{8} e^{-x}$

(5) $y = e^{-ax}(c_1 \cos bx + c_2 \sin bx) + \dfrac{e^{mx}}{(a+m)^2 + b^2}$

(6) $y = c_1 e^{-x} + c_2 e^{-2x} + x e^{-x}$ (7) $y = (c_1 + c_2 x)\,e^{-x/2} + \dfrac{x^2}{2} e^{-x/2}$

(8) Hint : $D^4 + 1 = (D^2+1)^2 - (\sqrt{2} \cdot D)^2$

$$y = e^{x/\sqrt{2}}\left\{c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2})\right\} + e^{-x/\sqrt{2}}\left\{c_3 \cos(x/\sqrt{2}) + c_4 \sin(x/\sqrt{2})\right\}$$
$$+ \frac{1}{9608}(e^{7x} - e^{-7x}) - \frac{1}{8}(e^x - e^{-x})$$

(9) $y = (c_1 + c_2 x)\,e^x + c_3 e^{2x} - \dfrac{x^2}{2} e^x$ (10) $y = c_1 e^{3x} + c_2 e^{-3x} + \dfrac{x}{6} e^{3x} - \dfrac{1}{9} - \dfrac{x}{6} e^{-3x}$

(11) $y = (c_1 + c_2 x)\, e^x + \dfrac{x^2}{2} e^x + 1$ (12) $y = c_1 + c_2\, e^{2x} + c_3\, e^{-2x} + \dfrac{x}{8}(e^{2x} + e^{-2x})$

(13) $y = c_1 + c_2\, e^{ax} + c_3\, e^{-ax} + \dfrac{x}{2a^2}(e^{ax} + e^{-ax})$]

(14) $y = e^{2x}[(C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x] + e^x + \dfrac{1}{4}$.

10. Particular Integral when $X = \sin ax$

When $X = \sin ax$, suppose $f(D) = \Phi(D^2)$, then the particular integral is given by

$$\boxed{\frac{1}{\Phi(D^2)} \sin ax = \frac{1}{\Phi(-a^2)} \sin ax}$$

Since $D(\sin ax) = a\cos ax$; $D^2(\sin ax) = -a^2 \sin ax$

$D^3(\sin ax) = -a^3 \cos ax$; $D^4(\sin ax) = (-a^2)^2 \sin ax$

....

If $\Phi D^2 = (D^2)^n + P_1(D^2)^{n-1} + P_2(D^2)^{n-2} + \ldots + P_n$, then

$$\Phi(D^2)\sin ax = \left\{(D^2)^n + P_1(D^2)^{n-1} + \ldots + P_n\right\}\sin ax$$

$$\Phi(D^2)\sin ax = \left\{(-a^2)^n + P_1(-a^2)^{n-1} + \ldots + P_n\right\}\sin ax = \Phi(-a^2)\sin ax$$

Operating on both sides by $\dfrac{1}{\Phi(D^2)}$, we get $\dfrac{1}{\Phi(D^2)} \cdot \Phi(D^2)\sin ax = \dfrac{1}{\Phi(D^2)} \cdot \Phi(-a^2)\sin ax$.

Since, $\Phi(D^2)$ and $1/\Phi(D^2)$ are inverse operators they cancel each other in effect. Hence if $\Phi(-a^2) \neq 0$.

$$\therefore \quad \sin ax = \frac{1}{\Phi(D^2)} \cdot \Phi(-a^2)\sin ax \qquad \therefore \quad \frac{1}{\Phi(D^2)}\sin ax = \frac{1}{\Phi(-a^2)}\sin ax$$

e.g. $$\frac{1}{D^4 + 3D^2 + 2}\sin 2x = \frac{1}{(-2^2)^2 + 3(-2^2) + 2}\sin 2x = \frac{1}{16 - 12 + 2}\sin 2x = \frac{1}{6}\sin 2x$$

Similarly, we can prove that

$$\boxed{\frac{1}{\Phi(D^2)}\cos ax = \frac{1}{\Phi(-a^2)}\cos ax}$$

e.g. $$\frac{1}{2D^4 - 5D^2 + 6}\cos 2x = \frac{1}{2(-2^2)^2 - 5(-2^2) + 6}\cos 2x$$

$$= \frac{1}{32 - 20 + 6}\cos 2x = \frac{1}{18}\cos 2x$$

(a) Particular Case : The above method fails if $\Phi(-a^2) = 0$ *i.e.* if $D^2 + a^2$ is a factor of $\Phi(D^2)$ because in this case

In this case, $$\frac{1}{\Phi(D^2)}\sin ax = \frac{1}{\Phi(-a^2)}\sin ax = \frac{1}{0}\sin ax \text{ is indeterminate.}$$

$$\boxed{\begin{aligned}&\frac{1}{\Phi(D^2)}\sin ax = x\cdot\frac{1}{\Phi'(D^2)}\sin ax \\ \text{and}\quad &\frac{1}{\Phi(D^2)}\cos ax = x\cdot\frac{1}{\Phi'(D^2)}\cos ax\end{aligned}}\quad \text{if } \Phi'(-a^2)\neq 0$$

.... (A)

.... (B)

Proof : Since $\cos ax + i\sin ax = e^{iax}$, we consider

$$\frac{1}{\Phi(D^2)}[\cos ax + i\sin ax] = \frac{1}{\Phi(D^2)}e^{iax}$$

$$= x\cdot\frac{1}{\Phi'(D^2)}e^{iax}$$ [By § 9 (a) (3), page 3-14]

$$= x\cdot\frac{1}{\Phi'(D^2)}[\cos ax + i\sin ax]$$

Equating imaginary and real parts, we get,

$$\frac{1}{\Phi(D^2)}\sin ax = x\cdot\frac{1}{\Phi'(D^2)}\sin ax$$

$$\frac{1}{\Phi(D^2)}\cos ax = x\cdot\frac{1}{\Phi'(D^2)}\cos ax$$

If $\Phi'(-a^2) = 0$, we can prove that

$$\frac{1}{\Phi(D^2)}\sin ax = x^2\cdot\frac{1}{\Phi''(D^2)}\sin ax$$

and $$\frac{1}{\Phi(D^2)}\cos ax = x^2\cdot\frac{1}{\Phi''(D^2)}\sin ax \quad \text{if} \quad \Phi''(-a^2)\neq 0$$

For Example,

$$\frac{1}{D^2+a^2}\cos ax = x\cdot\frac{1}{2D}\cos ax = \frac{x}{2}\int \cos ax\,dx$$

$$\therefore \quad \boxed{\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax}$$ .... (1)

and $$\frac{1}{D^2+a^2}\sin ax = x\cdot\frac{1}{2D}\sin ax = \frac{x}{2}\int \sin ax\,dx$$

$$\therefore \quad \boxed{\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax}$$ **(M.U. 1991)** .... (2)

Similarly, we can show that

$$\boxed{\begin{aligned}&\frac{1}{(D^2+a^2)^2}\cos ax = -\frac{x^2}{2(2a)^2}\cos ax \\ &\frac{1}{(D^2+a^2)^2}\sin ax = -\frac{x^2}{2(2a)^2}\sin ax\end{aligned}}$$

.... (3)

.... (4)

Note ....

In general we have

$$\frac{1}{(D^2+a^2)^r}\sin(ax+b) = \left(-\frac{x}{2a}\right)^r \cdot \frac{1}{r!}\sin\left(ax+b+\frac{r\pi}{2}\right)$$

$$\frac{1}{(D^2+a^2)^r}\cos(ax+b) = \left(-\frac{x}{2a}\right)^r \cdot \frac{1}{r!}\cos\left(ax+b+\frac{r\pi}{2}\right)$$

(b) When $X = \sin h\, ax$ or $\cos h\, ax$

We can write

$$\sin h\, ax = \frac{e^{ax}-e^{-ax}}{2} \quad \text{and} \quad \cos h\, ax = \frac{e^{ax}+e^{-ax}}{2} \quad \text{and apply the formula of § 9.}$$

$$\frac{1}{D^2+a^2}(\sin h\, ax) = \frac{1}{D^2+a^2}\left(\frac{e^{ax}-e^{-ax}}{2}\right)$$

$$= \frac{1}{2a^2}\left(\frac{e^{ax}-e^{-ax}}{2}\right) = \frac{1}{2a^2}\sin hx$$

$$\frac{1}{D^2+a^2}(\cos h\, ax) = \frac{1}{D^2+a^2}\left(\frac{e^{ax}+e^{-ax}}{2}\right)$$

$$= \frac{1}{2a^2}\left(\frac{e^{ax}+e^{-ax}}{2}\right) = \frac{1}{2a^2}\cos hx.$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Find the particular integral of $(D^2-4D+4)\,y = e^x + \cos 2x$. **(M.U. 2015)**

Sol. :

$$\text{P.I.} = \frac{1}{D^2-4D+4}(e^x+\cos 2x)$$

$$= \frac{1}{D^2-4D+4}e^x + \frac{1}{D^2-4D+4}\cos 2x$$

$$= \frac{e^x}{1-4+4} + \frac{1}{-4-4D+4}\cos 2x$$

$$= e^x - \frac{1}{4D}\cos 2x = e^x - \frac{1}{4}\cdot\frac{1}{D}\cos 2x$$

$$= e^x - \frac{1}{4}\int \cos 2x\, dx = e^x - \frac{1}{4}\cdot\frac{\sin 2x}{2} = e^x - \frac{1}{8}\sin 2x.$$

Example 2 (b) : Solve $(D^2+4)\,y = \cos 2x$. **(M.U. 2003, 15)**

Sol. : The auxiliary equation is $D^2+4=0 \quad \therefore \; D = 2i, -2i$.

The C.F. is $y = c_1\cos 2x + c_2 \sin 2x$.

$$\therefore \; \text{P.I.} = \frac{1}{D^2+4}\cos 2x = \frac{x}{2(2)}\sin 2x \qquad \text{[By (1), page 3-19]}$$

$\therefore$ The complete solution is

$$y = c_1\cos 2x + c_2\sin 2x + \frac{x}{4}\sin 2x.$$

Example 3 (b) : Solve $(D^2 - 5D + 6)\,y = \sin 3x$.

Sol. : The auxiliary equation is $D^2 - 5D + 6 = 0$

$\therefore\ (D-3)(D-2) = 0 \quad \therefore\ D = 2, 3.$

$\therefore$ The C.F. is $y = c_1 e^{2x} + c_2 e^{3x}$.

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6}\sin 3x = \frac{1}{-9 - 5D + 6}\sin 3x = -\frac{1}{5D+3}\sin 3x$$

$$= -\frac{1}{5D+3}\cdot\frac{5D-3}{5D-3}\sin 3x \qquad \textbf{[Note this]}$$

$$= -\frac{5D-3}{25D^2 - 9}\sin 3x = -\frac{5D-3}{25(-9)-9}\sin 3x$$

$$= \frac{1}{234}(5D-3)\sin 3x = \frac{1}{234}(15\cos 3x - 3\sin 3x)$$

$$= \frac{1}{78}(5\cos 3x - \sin 3x)$$

$\therefore$ The complete solution is $\quad y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{78}(5\cos 3x - \sin 3x)$.

Example 4 (b) : Solve $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 9\dfrac{dy}{dx} - 27y = \cos 3x$. **(M.U. 2005)**

Sol. : The auxiliary equation is $D^3 - 3D^2 + 9D - 27 = 0$

$\therefore\ D^2(D-3) + 9(D-3) = 0$

$\therefore\ (D-3)(D^2+9) = 0 \qquad \therefore \qquad D = 3, 3i, -3i.$

$\therefore$ The C.F. is $y = c_1 e^{3x} + (c_2\cos 3x + c_3 \sin 3x)$.

Now, $\text{P.I} = \dfrac{1}{(D-3)(D^2+9)}\cos 3x$

Since, $D^2 + 9$ is a factor of $\Phi(D^2)$, the general method fails.

$$\therefore\ \text{P.I.} = \frac{1}{D^2+9}\cdot\frac{D+3}{D^2-9}\cdot\cos 3x = \frac{1}{D^2+9}\cdot\frac{1}{-9-9}\cdot(D+3)\cos 3x$$

$$= \frac{1}{D^2+9}\cdot\frac{(-3\sin 3x + 3\cos 3x)}{-18} = \frac{1}{6}\cdot\frac{1}{D^2+9}\cdot(\sin 3x - \cos 3x)$$

Now, by using the formulae (A) and (B) of § 10 (a), page 3-19,

$$\frac{1}{D^2+9}\sin 3x = x\cdot\frac{1}{2\cdot D}\sin 3x = \frac{x}{2}\int \sin 3x\,dx = -\frac{x}{6}\cos 3x$$

and $\quad \dfrac{1}{D^2+9}\cos 3x = x\cdot\dfrac{1}{2\cdot D}\cos 3x = \dfrac{x}{2}\int \cos 3x\,dx\,\dfrac{x}{6}\sin 3x$

(Or you can use formulae (1) and (2) of page 3-19 directly).

$\therefore$ The complete solution is

$$y = c_1 e^{3x} + (c_2\cos 3x + c_3\sin 3x) - \frac{x}{36}\cos 3x - \frac{x}{36}\sin 3x.$$

Aliter : Consider $\quad (D^3 - 3D^2 + 9D - 27)\,y = (\cos 3x + i\sin 3x) = e^{3ix}$

$$\therefore \text{ P.I.} = \frac{1}{D^3 - 3D^2 + 9D - 27} \cdot e^{3ix} = \frac{1}{(D-3)(D^2+9)} \cdot e^{3ix}$$

$$= \frac{1}{(D-3)(D+3i)(D-3i)} \cdot e^{3ix} = \frac{1}{(3i-3)(3i+3i)} \cdot x \cdot e^{3ix}$$

$$= \frac{1}{3(i-1)\cdot 6i} \cdot x \cdot e^{3ix} = \frac{(i+1)\,i}{3(i^2-1)\,6i^2} \cdot x \cdot e^{3ix}$$

$$\therefore \text{ P.I.} = \frac{(-1+i)}{36} x(\cos 3x + i \sin 3x)$$

$$= \frac{-x(\cos 3x + \sin 3x) + ix(\cos 3x - \sin 3x)}{36}$$

Equating real parts, we get,

$$\text{P.I.} = -\frac{x}{36}(\cos 3x + \sin 3x).$$

Example 5 (b) : Solve $(D^3 - 3D^2 + 9D - 27)\, y = \sin 3x$.

Sol. : The C.F. is as above and P.I. is obtained by equating the imaginary parts.

$$\therefore \; y = c_1 e^{3x} + (c_2 \cos 3x + c_3 \sin 3x) + \frac{x}{36}(\cos 3x - \sin 3x)$$

Example 6 (b) : Solve $(D^3 + D)\, y = \cos x$.

Sol. : The auxiliary equation is $D^3 + D = 0$

$\therefore \; D(D^2+1) = 0 \qquad \therefore \; D = 0, +i, -i.$

$\therefore$ The C.F. is $y = c_1 + c_2 \cos x + c_3 \sin x$.

$$\text{P.I.} = \frac{1}{D^3 + D} \cos x = \frac{1}{D^2+1} \cdot \frac{1}{D} \cos x = \frac{1}{D^2+1} \cdot \frac{D}{D^2} \cos x$$

$$= \frac{1}{D^2+1} \cdot \frac{-\sin x}{(-1)} = \frac{1}{D^2+1} \cdot \sin x$$

$$= x \cdot \frac{1}{2D} \sin x \qquad \text{[By (A), § 10 (a), page 3-19]}$$

$$\therefore \text{ P.I.} = -\frac{x}{2} \cos x$$

(Or you can use formulae (2) of page 3-19 directly).

$\therefore$ The complete solution is $\; y = c_1 + c_2 \cos x + c_3 \sin x - \frac{x}{2} \cos x.$

Example 7 (b) : Solve $\dfrac{d^2y}{dx^2} + 9y = e^x - \cos 2x.$ **(M.U. 1992)**

Sol. : The auxiliary equation is $D^2 + 9 = 0 \qquad \therefore \; D = 3i, -3i.$

$\therefore$ The C.F. is $y = c_1 \cos 3x + c_2 \sin 3x$.

$$\text{P.I.} = \frac{1}{D^2+9}(e^x - \cos 2x) = \frac{1}{D^2+9} e^x - \frac{1}{D^2+9} \cos 2x$$

$$= \frac{1}{10} e^x - \frac{1}{5} \cos 2x$$

$\therefore$ The complete solution is $\; y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{10} e^x - \frac{1}{5} \cos 2x.$

Example 8 (b) : Solve $(D^4 - 1)y = e^x + \cos x \cos 3x$. **(M.U. 1993)**

Sol. : The auxiliary equation is $D^4 - 1 = 0$

$\therefore\ (D^2 - 1)(D^2 + 1) = 0 \qquad \therefore\ D = 1, -1, +i, -i$.

$\therefore$ The C.F. is $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_3 \sin x$.

$$\text{P.I.} = \frac{1}{D^4 - 1}(e^x + \cos x \cos 3x)$$

$$= \frac{1}{D^4 - 1}\left[e^x + \frac{1}{2}(\cos 4x + \cos 2x)\right] \qquad \textbf{[Note this.]}$$

$$= \frac{1}{(D-1)(D+1)(D^2+1)} e^x + \frac{1}{2} \cdot \frac{1}{(D^4 - 1)} \cos 4x + \frac{1}{2(D^4 - 1)} \cos 2x$$

$$= \frac{x}{4} e^x + \frac{1}{510} \cos 4x + \frac{1}{30} \cos 2x.$$

Example 9 (b) : Solve $(D^3 + D)y = \sin x$.

Sol. : The auxiliary equation is $D^3 + D = 0$

$\therefore\ D(D^2 + 1) = 0 \qquad \therefore\ D = 0, i, -i$.

$\therefore$ The C.F. is $y = c_1 + c_2 \cos x + c_3 \sin x$.

$$\text{P.I.} = \frac{1}{D^3 + D} \sin x = \frac{1}{D(D^2 + 1)} \sin x$$

$$= \frac{1}{D} \cdot \frac{x}{2D} \cdot \sin x \qquad \text{[By (A), § 10 (a), page 3-19]}$$

$$\text{P.I.} = \frac{x}{2D^2} \sin x = \frac{x}{2(-1)} \sin x$$

(Or you can use formulae (2) of page 3-19 directly).

$\therefore$ The complete solution is $\quad y = c_1 + c_2 \cos x + c_3 \sin x - \dfrac{x}{2} \sin x$.

Example 10 (b) : Solve $(D-1)^2 (D^2 + 1)y = e^x + \sin^2 (x/2)$. **(M.U. 2008, 12)**

Sol. : The auxiliary equation is $(D-1)^2 (D^2 + 1) = 0 \qquad \therefore\ D = 1, 1, +i, -i$.

$\therefore$ The C.F. is $y = (c_1 + c_2 x) e^x + (c_3 \cos x + c_4 \sin x)$.

$$\text{P.I.} = \frac{1}{(D-1)^2 (D^2+1)} \left[e^x + \sin^2 \frac{x}{2}\right]$$

Now, $\quad \dfrac{1}{(D-1)^2 (D^2+1)} e^x = \dfrac{x^2}{2!} \cdot \dfrac{1}{2} e^x \qquad$ [By (2), page 3-14]

and $\quad \dfrac{1}{(D-1)^2 (D^2+1)} \sin^2 \dfrac{x}{2} = \dfrac{1}{(D-1)^2 (D^2+1)} \left[\dfrac{1 - \cos x}{2}\right]$

$$= \frac{1}{(D-1)^2 (D^2+1)} \left(\frac{1}{2} e^{0x}\right) - \frac{1}{(D-1)^2 (D^2+1)} \left(\frac{1}{2} \cos x\right)$$

$$= \frac{1}{(-1)^2 (1)} \cdot \frac{1}{2} - \frac{1}{(D^2 - 2D + 1)(D^2 + 1)} \left(\frac{1}{2} \cos x\right)$$

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$$= \frac{1}{2} - \frac{1}{-2D} \cdot \frac{1}{(D^2+1)}\left(\frac{\cos x}{2}\right) = \frac{1}{2} - \frac{1}{(D^2+1)} \cdot \frac{D}{(-2D^2)}\left(\frac{\cos x}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{(D^2+1)} \frac{1}{(-1)}(-\sin x) = \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{(D^2+1)}(\sin x)$$

$$= \frac{1}{2} + \frac{1}{4} \cdot \frac{x}{2D} \sin x \qquad \text{[By (A), § 10 (a), page 3-19]}$$

$$= \frac{1}{2} + \frac{x}{8} \int \sin x \, dx = \frac{1}{2} - \frac{x}{8} \cos x$$

(Or you can use formulae (1) of page 3-19 directly).

Hence, the complete solution is

$$y = (c_1 + c_2 x)\, e^x + c_3 \cos x + c_4 \sin x + \frac{1}{2} \cdot \frac{x^2}{2\,!} e^x + \frac{1}{2} - \frac{x}{8} \cos x.$$

Example 11 (b) : Solve $(D^4 - a^4)\, y = \cos ax$.

Sol. : The auxiliary equation is $D^4 - a^4 = 0$.

$\therefore\ (D^2 - a^2)(D^2 + a^2) = 0 \qquad \therefore\ D = -a, +a, +ai, -ai.$

$\therefore$ The C.F. is $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$.

$$\text{P.I.} = \frac{1}{(D^2+a^2)(D^2-a^2)} \cos ax = \frac{1}{(D^2+a^2)} \cdot \frac{1}{(-2a^2)} \cos ax$$

$$= x \cdot \frac{1}{2D} \cdot \frac{1}{(-2a^2)} \cos ax \qquad \text{[By (B), § 10 (a), page 3-19]}$$

$$= -\frac{x}{4a^2} \int \cos ax \, dx = -\frac{x}{4a^3} \sin ax$$

(Or you can use formulae (2) of page 3-19 directly).

$\therefore$ The complete solution is

$$y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax - \frac{x}{4a^3} \sin ax.$$

Example 12 (b) : Solve $(D^4 + 8D^2 + 16)\, y = \sin^2 x$. **(M.U. 2002, 03)**

Sol. : The auxiliary equation is $D^4 + 8D^2 + 16 = 0$

$\therefore\ (D^2 + 4)^2 = 0 \qquad \therefore\ D = 2i, -2i, 2i, -2i.$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)(c_3 \cos 2x + c_4 \sin 2x)$.

$$\text{P.I.} = \frac{1}{(D^2+4)^2} \sin^2 x = \frac{1}{(D^2+4)^2}\left(\frac{1 - \cos 2x}{2}\right)$$

Now, $\displaystyle \frac{1}{(D^2+4)^2} \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{(0+4)^2} = \frac{1}{32}$ [By (2A) page 3-14]

And $\displaystyle \frac{1}{(D^2+4)^2}\left(-\frac{1}{2}\cos 2x\right) = -\frac{1}{2} \cdot \frac{x}{2(D^2+4) \cdot 2D} \cos 2x$

$$= -\frac{x}{8(D^2+4)} \int \cos 2x \, dx = -\frac{x}{8(D^2+4)} \cdot \frac{\sin 2x}{2}$$

$$= -\frac{x}{16} \cdot \frac{x}{2D} \sin 2x = -\frac{x^2}{32} \int \sin 2x \, dx$$

$$= -\frac{x^2}{32} \cdot \left(-\frac{\cos 2x}{2}\right) = \frac{x^2}{64} \cos 2x$$

(Or you can use formulae (3) of page 3-19 directly).

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x)(c_3 \cos 2x + c_4 \sin 2x) + \frac{1}{32} + \frac{x^2}{64} \cos 2x.$$

Example 13 (b) : Solve $(D^2 + D + 1)\, y = (1 + \sin x)^2$. **(M.U. 2006)**

Sol. : The auxiliary equation is $D^2 + D + 1 = 0 \quad \therefore \; D = \frac{-1 \pm \sqrt{3} \cdot i}{2}$

$\therefore$ C.F. is $y = e^{-x/2}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right)$

$$\therefore \text{ P.I.} = \frac{1}{D^2 + D + 1}(1 + \sin x)^2 = \frac{1}{D^2 + D + 1}(1 + 2 \sin x + \sin^2 x)$$

$$= \frac{1}{D^2 + D + 1}\left(1 + 2 \sin x + \frac{1 - \cos 2x}{2}\right)$$

$$= \frac{1}{D^2 + D + 1}\left(\frac{3}{2} + 2 \sin x - \frac{1}{2} \cos 2x\right)$$

Now, $\frac{1}{D^2 + D + 1} \cdot \left(\frac{3}{2}\right) = \frac{3}{2} \cdot \frac{1}{D^2 + D + 1} e^0 x = \frac{3}{2} \cdot \frac{1}{0 + 0 + 1} \cdot e^0 x = \frac{3}{2}$

$$\frac{1}{D^2 + D + 1} \sin x = \frac{1}{-1 + D + 1} \sin x = \frac{1}{D} \sin x$$

$$= \int \sin x \, dx = -\cos x$$

$$\frac{1}{D^2 + D + 1} \cos 2x = \frac{1}{-4 + D + 1} \cos 2x = \frac{D + 3}{D^2 - 9} \cos 2x$$

$$= \frac{-2 \sin 2x + 3 \cos 2x}{-13}$$

$\therefore$ The complete solution is

$$y = e^{-x/2}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right) + \frac{1}{3} - 2 \cos x - \frac{1}{26}(2 \sin 2x - 3 \cos 2x)$$

EXERCISE - VII

Solve the following differential equations : Class (b) : 6 Marks

1. $(D^2 - D + 1)\, y = \cos 2x$
2. $\frac{d^3 y}{dx^3} + y = \cos ax$
3. $(D^3 + D)\, y = \cos x$
4. $\operatorname{cosec} x \frac{d^4 y}{dx^4} + \operatorname{cosec} x \cdot y = \sin 2x$
5. $\frac{d^4 y}{dx^4} - a^4 y = \sin ax$ **(M.U. 1988, 2008)**

6. $(D^4 + 8D^2 + 16)\, y = \cos^2 x$ 7. $(D-1)^2 (D^2+1)^2 y = \sin^2 \dfrac{x}{2} + e^x$ **(M.U. 2001, 10)**

8. $(D^4 + 10\,D^2 + 9)\, y = 96 \sin 2x \cos x$ 9. $D^2 (D^2 + 1)\, y = \sin x + e^{-x}$

10. $D^2 (D^2 + 9)\, y = \cos 3x + 5$ 11. $(D^4 + 10\,D^2 + 9)\, y = \cos (2x + 3)$ **(M.U. 1988, 2004)**

12. $(D^2 - 4)\, y = \sin^2 x.$ **(M.U. 1988)**

[**Ans. :** (1) $y = e^{x/2} \left[c_1 \cos(\sqrt{3}/2)\, x + c_2 \sin(\sqrt{3}/2)\, x\right] - \dfrac{1}{13}(2 \sin 2x + 3 \cos 2x)$

(2) $y = c_1 e^{-x} + e^{x/2} \left[c_2 \cos(\sqrt{3}/2)\, x + c_3 \sin(\sqrt{3}/2)\, x\right] + \left[\dfrac{1}{1+a^6}\right]\left[\cos ax - a^3 \sin ax\right]$

(3) $y = c_1 + c_2 \cos x + c_3 \sin x - \dfrac{x}{2} \cos x$

(4) (**Hint :** Multiply by $\sin x$; Note $D^4 + 1 = (D^2+1)^2 - 2D^2$)

$$y = e^{x/\sqrt{2}} \left[c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2})\right]$$

$$+ e^{-x/\sqrt{2}} \left[c_3 \cos(x/\sqrt{2}) + c_4 \sin(x/\sqrt{2})\right] + \frac{1}{4} \cos x - \frac{1}{164} \cos 3x.$$

(5) $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax + \dfrac{1}{4a^3} \cdot x \cos ax.$

(6) $y = (c_1 + c_2 x)(c_3 \cos 2x + c_4 \sin 2x) + \dfrac{1}{32} - \dfrac{1}{64} \cdot x^2 \cdot \cos 2x$

(7) $y = (c_1 + c_2 x) e^x + (c_3 + c_4 x)(c_5 \cos x + c_6 \sin x) + \dfrac{1}{2} - \dfrac{1}{32}\, x^2 \sin x + \dfrac{1}{8} x^2 e^x$

(8) $y = c_1 \cos x + c_2 \sin x + c_3 \cos 3x + c_4 \cos 3x + x(\cos 3x - 3 \cos x)$

(9) $y = (c_1 + c_2 x) + (c_3 \cos x + c_4 \sin x) + \dfrac{1}{2} e^{-x} + \dfrac{x}{2} \cos x$

(10) $y = c_1 + c_2 x + c_3 \cos 3x + c_4 \sin 3x - \dfrac{x}{54} \sin 3x + \dfrac{5x^2}{18}$

(11) $y = c_1 \cos x + c_2 \sin x + c_3 \cos 3x + c_4 \sin 3x - \dfrac{1}{15} \cos (2x + 3)$

(12) $y = c_1 e^x + c_2 e^{2x} - \dfrac{x}{8} \sin 2x - \dfrac{1}{8}.$]

11. Particular Integral when $X = x^m$ where m is a Positive Integer

When $X = x^m$, we write $f(D)$ in ascending powers of D by putting it in the form $1 + \Phi(D)$. Then,

$$\text{P.I.} = \frac{1}{f(D)} x^m = \frac{1}{1+\Phi(D)} x^m \qquad \therefore \quad \boxed{\text{P.I.} = [1 + \Phi(D)]^{-1} x^m}$$

By expanding the bracket by the formula,

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots..$$

or $$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots..$$

and by operating each term of the expansion on x^m we get the required particular integral. It is obvious that in the expansion, terms beyond the m-th power of D need not be written since the derivatives of x^m of powers higher than m are zero.

Solved Examples : Class (b) : 6 Marks

Example 1 : Solve $(D^3 - 3D + 2)\, y = x$.

Sol. : The auxiliary equation is $D^3 - 3D + 2 = 0$

$\therefore\ D^3 - D^2 + D^2 - D - 2D + 2 = 0 \qquad \therefore\ (D-1)(D^2 + D - 2) = 0$

$\therefore\ (D-1)(D-1)(D+2) = 0 \qquad \therefore\ D = 1, 1, -2.$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)\, e^x + c_3 e^{-2x}$.

$$\text{P.I.} = \frac{1}{D^3 - 3D + 2}\, x = \frac{1}{2\left\{1 - \left[(3D - D^3)/2\right]\right\}}\, x = \frac{1}{2}\left[1 - \frac{(3D - D^3)}{2}\right]^{-1} x$$

$$= \frac{1}{2}\left[1 + \frac{3D - D^3}{2} + \ldots\ldots\right] x = \frac{1}{2}\left[x + \frac{3}{2}\right]$$

$\therefore$ The complete solution is $y = (c_1 + c_2 x)\, e^x + c_3 e^{-2x} + \frac{1}{2}\left[x + \frac{3}{2}\right]$.

Example 2 (b) : Solve $\frac{d^3 y}{dx^3} - 2\frac{dy}{dx} + 4y = 3x^2 - 5x + 2$. **(M.U. 1996, 99)**

Sol. : The auxiliary equation is $D^3 - 2D + 4 = 0$.

$\therefore\ D^3 + 2D^2 - 2D^2 - 4D + 2D + 4 = 0$

$\therefore\ (D+2)(D^2 - 2D + 2) = 0 \qquad \therefore\ D = -2,\ 1 \pm i$.

$\therefore$ The C.F. is $y = c_1 e^{-2x} + e^x (c_2 \cos x + c_3 \sin x)$.

$$\text{P.I.} = \frac{1}{D^3 - 2D + 4}(3x^2 - 5x + 2) = \frac{1}{4\left[1 - \frac{2D - D^3}{4}\right]}(3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 - \frac{2D - D^3}{4}\right]^{-1}(3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 + \frac{2D - D^3}{4} + \frac{4D^2}{16} + \ldots\right](3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[3x^2 - 5x + 2 + \frac{1}{2}(6x - 5) + \frac{1}{4}(6)\right] = \frac{1}{4}\left[3x^2 - 2x + 1\right]$$

$\therefore$ The complete solution is $y = c_1 e^{-2x} + e^x (c_2 \cos x + c_3 \sin x) + \frac{1}{4}\left[3x^2 - 2x + 1\right]$.

Example 3 (b) : Solve $\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x + \cos 2x$. **(M.U. 1995, 2005, 10, 11)**

Sol. : The auxiliary equation is $D^2 - 4D + 4 = 0 \quad \therefore\ (D-2)^2 = 0 \quad \therefore\ D = 2, 2.$

$\therefore$ The C.F. is $y = (c_1 + c_2 x) e^{2x}$.

$$\text{P.I.} = \frac{1}{(D-2)^2}(x^2 + e^x + \cos 2x)$$

Now, $$\frac{1}{D^2 - 4D + 4} x^2 = \frac{1}{4\left[1 - \dfrac{4D - D^2}{4}\right]} x^2 = \frac{1}{4}\left[1 - \left(\frac{4D - D^2}{4}\right)\right]^{-1} x^2$$

$$= \frac{1}{4}\left[1 + \left(\frac{4D - D^2}{4}\right) + D^2\right] x^2 = \frac{1}{4}\left[x^2 + \frac{1}{4}(8x - 2) + 2\right]$$

$$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right]$$

$$\frac{1}{D^2 - 4D + 4} e^x = \frac{1}{1 - 4 + 4} = e^x$$

$$\frac{1}{D^2 - 4D + 4} \cos 2x = -\frac{1}{4D} \cos 2x = -\frac{1}{4} \int \cos 2x \, dx = -\frac{1}{8} \sin 2x$$

$\therefore$ The complete solution is $y = (c_1 + c_2 x) e^{2x} + \dfrac{1}{4}\left[x^2 + 2x + \dfrac{3}{2}\right] + e^x - \dfrac{1}{8} \sin 2x.$

Example 4 (b) : Solve $(D^3 - 2D^2 + D) y = x^2 + x$. **(M.U. 1992)**

Sol. : The auxiliary equation is $D(D^2 - 2D + 1) = 0$

$\therefore \ D(D-1)^2 = 0 \qquad \therefore \ D = 0, 1, 1.$

$\therefore$ The C.F. is $y = c_1 + (c_2 + c_3 x) e^x$.

$$\text{P.I.} = \frac{1}{D - 2D^2 - D^3}(x^2 + x) = \frac{1}{D(1 - 2D + D^2)}(x^2 + x)$$

$$= \frac{1}{D} \cdot \frac{1}{[1 - (2D - D^2)]}(x^2 + x)$$

$$= \frac{1}{D}\left[1 + (2D - D^2) + 4D^2 \ldots\right](x^2 + x) = \frac{1}{D}\left[1 + 2D + 3D^2 \ldots\right](x^2 + x)$$

$$= \frac{1}{D}\left[(x^2 + x) + 2(2x + 1) + 3(2)\right] = \frac{1}{D}\left[x^2 + 5x + 8\right]$$

$$= \int (x^2 + 5x + 8)\, dx = \frac{x^3}{3} + \frac{5x^2}{2} + 8x.$$

$\therefore$ The complete solution is $y = c_1 + (c_2 + c_3 x) e^x + \dfrac{x^3}{3} + \dfrac{5x^2}{2} + 8x.$

Example 5 (b) : Solve $\dfrac{d^3 y}{dt^3} + \dfrac{dy}{dt} = \cos t + t^2 + 3.$ **(M.U. 1992)**

Sol. : The auxiliary equation is $D(D^2 + 1) = 0 \quad \therefore \ D = 0, i, -i.$

$\therefore$ The C.F. is $y = c_1 + c_2 \cos t + c_3 \sin t$.

$$\text{P.I.} = \frac{1}{D + D^3}(\cos t + t^2 + 3)$$

$$\frac{1}{D+D^3}\cos t = \frac{1}{D(1+D^2)}\cos t = \frac{1}{D}\cdot\frac{t}{2}\sin t \qquad \text{[By (1), page 3-19]}$$

$$= \frac{1}{2}\int t\sin t\, dt = \frac{1}{2}[-t\cos t + \sin t] \qquad \text{[By parts]}$$

$$\frac{1}{D+D^3}t^2 = \frac{1}{D(1+D^2)}t^2 = \frac{1}{D}(1-D^2+.....)t^2$$

$$= \frac{1}{D}[t^2-2] = \int (t^2-2)\,dt = \frac{t^3}{3} - 2t$$

$$\frac{1}{D+D^3}\cdot 3 = 3\cdot\frac{1}{D(1+D^2)}e^{0t} = 3\cdot\frac{1}{D}\cdot 1 = 3\int dt = 3t \qquad \text{[Note this]}$$

$\therefore$ The complete solution is

$$y = c_1 + c_2\cos t + c_3\sin t + \frac{1}{2}[-t\cos t + \sin t] + \frac{t^3}{3} + t.$$

Example 6 (b) : Solve $(D^3 - D^2 - 6D)\,y = x^2 + 1.$ **(M.U. 2009)**

Sol. : The auxiliary equation is $D^3 - D^2 - 6D = 0$

$\therefore\ D(D^2 - D - 6) = 0 \qquad \therefore\ D(D+2)(D-3) = 0 \qquad \therefore\ D = 0, -2, 3.$

$\therefore$ The C.F. is $y = c_1 + c_2 e^{-2x} + c_3 e^{3x}$.

$$\text{P.I.} = \frac{1}{D^3 - D^2 - 6D}(x^2+1) = -\frac{1}{6D}\cdot\frac{1}{\left\{1+\left[(D-D^2)/6\right]\right\}}(x^2+1)$$

$$= -\frac{1}{6D}\cdot\left[1+\frac{D-D^2}{6}\right]^{-1}(x^2+1)$$

$$= -\frac{1}{6D}\left[1 - \frac{(D-D^2)}{6} + \left\{\frac{D-D^2}{6}\right\}^2 - .....\right](x^2+1)$$

$$= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36} + ...\right](x^2+1)$$

$$= -\frac{1}{6D}\left[x^2 + 1 - \frac{x}{3} + \frac{1}{3} + \frac{1}{18}\right] = -\frac{1}{6D}\left[x^2 - \frac{x}{3} + \frac{25}{18}\right]$$

$$= -\frac{1}{6}\int\left(x^2 - \frac{x}{3} + \frac{25}{18}\right)dx = -\frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18}\right]$$

$\therefore$ The complete solution is $y = c_1 + c_2 e^{-2x} + c_3 e^{3x} - \dfrac{1}{6}\left[\dfrac{x^3}{3} - \dfrac{x^2}{6} + \dfrac{25x}{18}\right].$

EXERCISE - VIII

Solve the following differential equations : Class (b) : 6 Marks

1. $(D^3 + 2D^2 + D)\,y = x^2 + x$
2. $(D^3 + 3D^2 + 2D)\,y = x^2$

3. $(D^4 - 2D^3 + D^2)\,y = x^3$ **(M.U. 1996)**

4. $(D^2 + 2D + 3)\,y = x - x^2$

5. $(D^3 - D^2 + 6D)\,y = x^2 + \sin x$

6. $(D^2 + D - 2)\,y = x + e^{-2x}$

7. $(D^2 - 4D + 4)\,y = 8\,(x^2 + \sin 2x + e^{2x})$ **(M.U. 1997)**

8. $(D^3 - D)\,y = 2e^x + 2x + 1 - 4\cos x$ **(M.U. 2006)**

9. $(D^2 + 4D + 4)\,y = x^2$

10. $(D^2 + 4)\,y = x^2 + \sin 2x$ **(M.U. 1998)**

11. $(D^2 + 2D + 2)\,y = x^2 + 1$ **(M.U. 2004)**

[**Ans. :** **(1)** $y = c_1 + (c_2 + c_3 x)\,e^{-x} + \dfrac{x^3}{3} - \dfrac{3x^2}{2} + 4x$

(2) $y = c_1 + c_2\,e^{-x} + c_3\,e^{-2x} + \dfrac{x}{12}(2x^2 - 9x + 21)$

(3) $y = c_1 + c_2\,x + (c_3 + c_4 x)\,e^x + \dfrac{x^5}{20} + \dfrac{x^4}{2} + 3x^3 - 12x^2$

(4) $y = e^{-x}\left(c_1 \cos\sqrt{2}\cdot x + c_2 \sin\sqrt{2}\cdot x\right) + \dfrac{1}{3}\left(-x^2 + \dfrac{7}{3}x - \dfrac{8}{9}\right)$

(5) $y = c_1 + c_2\,e^{-2x} + c_3\,e^{-3x} - \dfrac{1}{6}\left(\dfrac{x^3}{3} - \dfrac{x^2}{6} + \dfrac{7x}{18}\right) + \dfrac{1}{50}(\sin x + 7\cos x)$

(6) $y = c_1\,e^x + c_2\,e^{-2x} - \dfrac{1}{2}\left(x + \dfrac{1}{2}\right) - \dfrac{x}{3}\,e^{-2x}$

(7) $y = (c_1 + c_2 x)\,e^{2x} + 2x^2 + 4x + 3 + \cos 2x + 4x^2\,e^{2x}$

(8) $y = c_1 + c_2\,e^x + c_3\,e^{-x} - x^2 - x + 2\sin x + xe^x$

(9) $y = (c_1 + c_2 x)\,e^{-2x} + \dfrac{1}{4}\left[x^2 - 2x + \dfrac{3}{2}\right]$

(10) $y = c_1 \cos 2x + c_2 \sin 2x - \dfrac{x}{4}\cos 2x + \dfrac{1}{4}\left(x^2 - \dfrac{1}{2}\right).$

(11) $y = (c_1 \cos x + c_2 \sin x)\,e^{-x} + \dfrac{1}{2}(x^2 - 2x + 2)$]

12. Particular Integral when $X = e^{ax}\,V$ where V is a function of x

Let V' be a function of x, then

$$D(e^{ax}V') = e^{ax}DV' + a\,e^{ax}V' = e^{ax}(D + a)\,V'$$

$$D^2(e^{ax}V') = D[\,e^{ax}(D + a)\,V'\,]$$

$$= e^{ax}D\cdot(D + a)\,V' + ae^{ax}(D + a)\,V'$$

$$= e^{ax}(D + a)^2\,V'$$

In general $\quad D^n(e^{ax}V') = e^{ax}(D + a)^n\,V'$

Hence, $f(D)\,e^{ax}\,V' = e^{ax}\,f(D+a)\,V'$ .... (1)

Now, let $f(D+a)\,V' = V$. $\therefore\ V' = \dfrac{1}{f(D+a)}V$ .... (2)

Hence, putting V' from (2), we get, from (1)

$$\therefore\ f(D)\cdot e^{ax}\cdot\frac{1}{f(D+a)}V = e^{ax}\ V$$

Operating by $\dfrac{1}{f(D)}$ on both sides

$$\frac{1}{f(D)}\cdot f(D)\,e^{ax}\cdot\frac{1}{f(D+a)}\cdot V = \frac{1}{f(D)}\,e^{ax}\,V$$

Hence, $\boxed{\dfrac{1}{f(D)}\,e^{ax}\,V = e^{ax}\cdot\dfrac{1}{f(D+a)}\cdot V}$ .... (3)

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $(D^2-3D+2)\,y = x^2e^{2x}$. **(M.U. 1994)**

Sol. : The auxiliary equation is $D^2-3D+2=0$.

$\therefore\ (D-1)(D-2)=0 \quad \therefore\ D=1,2.$

$\therefore$ The C.F. is $y = c_1e^x + c_2e^{2x}$.

$$\text{P.I.} = \frac{1}{D^2-3D+2}\,x^2\ e^{2x} = e^{2x}\cdot\frac{1}{(D+2)^2-3(D+2)+2}\cdot x^2$$

$$= e^{2x}\,\frac{1}{D^2+D}\,x^2 = e^{2x}\cdot\frac{1}{D\cdot(1+D)}\,x^2 = e^{2x}\,\frac{1}{D}\,(1+D)^{-1}x^2$$

$$= e^{2x}\,\frac{1}{D}\left[1-D+D^2-D^3+.....\right]x^2$$

$$= e^{2x}\,\frac{1}{D}\left[x^2-2x+2\right] = e^{2x}\int(x^2-2x+2)\,dx$$

$$= e^{2x}\left(\frac{x^3}{3}-x^2+2x\right)$$

$\therefore$ The complete solution is $y = c_1e^x + c_2e^{2x} + e^{2x}\left(\dfrac{x^3}{3}-x^2+2x\right)$.

Example 2 (b) : Solve $(D^3-3D^2+3D-1)\,y = xe^x + e^x$.

Sol. : The auxiliary equation is $(D-1)^3=0 \quad \therefore\ D=1,1,1.$

$\therefore$ The C.F. is $y = (c_1+c_2x+c_3x^2)\,e^x$.

$$\text{P.I.} = \frac{1}{(D-1)^3}\,x\,e^x + \frac{1}{(D-1)^3}\,e^x$$

Now, $\dfrac{1}{(D-1)^3}\ xe^x = \dfrac{1}{(D-1)^3}\ e^x\ x = e^x\cdot\dfrac{1}{(D+1-1)^3}\,x = e^x\cdot\dfrac{1}{D^3}\,x$

$$= e^x \cdot \frac{1}{D^2} \cdot \frac{1}{D}\, x = e^x \cdot \frac{1}{D^2} \int x\, dx = e^x \cdot \frac{1}{D^2} \cdot \frac{x^2}{2} = e^x \cdot \frac{1}{D} \cdot \frac{1}{D} \cdot \frac{x^2}{2}$$

$$= e^x \cdot \frac{1}{D} \int \frac{x^2}{2}\, dx = e^x \cdot \frac{1}{D} \frac{x^3}{6} = e^x \cdot \int \frac{x^3}{6}\, dx = e^x \cdot \frac{x^4}{24}$$

And $\dfrac{1}{(D-1)^3} \cdot e^x = \dfrac{x^3}{3!} e^x$ [By § 9 (a), page 3-13]

$\therefore$ The complete solution is $y = (c_1 + c_2 x + c_3 x^2)\, e^x + e^x \cdot \dfrac{x^4}{24} + e^x \cdot \dfrac{x^3}{3!}$.

Example 3 (b) : Solve $(D^2 - 2aD + a^2)\, y = \dfrac{e^{ax}}{x^r}$.

Sol. : The auxiliary equation is $D^2 - 2aD + a^2 = 0$

$\therefore$ $(D-a)^2 = 0$ $\therefore$ $D = a, a$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)\, e^{ax}$

$$\text{P.I.} = \frac{1}{(D-a)^2} \cdot \frac{e^{ax}}{x^r} = e^{ax} \cdot \frac{1}{(D+a-a)^2} \cdot \frac{1}{x^r} = e^{ax} \cdot \frac{1}{D^2} \cdot \frac{1}{x^r}$$

$$= e^{ax} \cdot \frac{1}{D} \int x^{-r}\, dx = e^{ax} \cdot \frac{1}{D} \left(\frac{x^{-r+1}}{-r+1} \right) = e^{ax} \int \frac{x^{-r+1}}{-r+1}\, dx$$

$$= e^{ax} \cdot \frac{x^{-r+2}}{(-r+1)(-r+2)}$$

$\therefore$ The complete solution is $y = (c_1 + c_2 x) e^{ax} + e^{ax} \cdot \dfrac{x^{-r+2}}{(-r+1)(-r+2)}$.

Example 4 (b) : Solve $(D^2 - 4D + 3)\, y = 2x\, e^{3x} + 3e^x \cos 2x$. **(M.U. 2015)**

Sol. : The auxiliary equation is

$D^2 - 4D + 3 = 0$ $\therefore$ $(D-1)(D-3) = 0$ $\therefore$ $D = 1, 3$.

$\therefore$ The C.F. is $y = c_1\, e^x + c_2\, e^{3x}$.

$$\text{P.I.} = \frac{1}{D^2 - 4D + 3} \cdot (2x\, e^{3x} + 3e^x \cos 2x)$$

$$= 2e^{3x} \cdot \frac{1}{(D+3)^2 - 4(D+3) + 3} \cdot x + 3\, e^x \cdot \frac{1}{(D+1)^2 - 4(D+1) + 3} \cdot \cos 2x$$

$$= 2e^{3x} \cdot \frac{1}{D^2 + 2D} \cdot x + 3e^x \cdot \frac{1}{D^2 - 2D} \cdot \cos 2x$$

$$= 2e^{3x} \cdot \frac{1}{2D[1 + (D/2)]} \cdot x + 3\, e^x \cdot \frac{1}{-4 - 2D} \cdot \cos 2x$$

$$= \frac{1}{D} e^{3x} \left[1 + \frac{D}{2} \right]^{-1} \cdot x - \frac{3}{2} e^x \cdot \frac{1}{D+2} \cdot \cos 2x$$

$$= \frac{1}{D} e^{3x} \left[1 - \frac{D}{2} + \frac{D^2}{4} - \ldots \right] \cdot x - \frac{3}{2} \cdot e^x\, \frac{D-2}{D^2 - 4} \cdot \cos 2x$$

$$= \frac{1}{D} e^{3x} \left[x - \frac{1}{2} \right] - \frac{3}{2} e^x \cdot \frac{D-2}{-8} \cos 2x$$

$$= e^{3x} \int \left(x - \frac{1}{2} \right) dx + \frac{3}{16} e^x (D-2) \cos 2x$$

$$= e^{3x} \left(\frac{x^2}{2} - \frac{x}{2} \right) + \frac{3}{16} e^x (-2 \sin 2x - 2 \cos 2x)$$

$$\therefore \text{ P.I.} = \frac{e^{3x}}{2} (x^2 - x) - \frac{3}{8} e^x (\sin 2x + \cos 2x)$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{3x} + \frac{e^{3x}}{2} (x^2 - x) - \frac{3}{8} e^x (\sin 2x + \cos 2x).$$

Example 5 (b) : Solve $\frac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x - \cos 2x$. **(M.U. 1993, 2003, 06, 12, 13, 14)**

Sol. : The auxiliary equation is $(D^2 + 2) = 0$ $\therefore$ $D = \sqrt{2} \cdot i, -\sqrt{2} \cdot i$

$\therefore$ The C.F. is $y = c_1 \cos \sqrt{2} \cdot x + c_2 \sin \sqrt{2} \cdot x$.

$$\text{P.I.} = \frac{1}{D^2 + 2} (x^2 e^{3x} + e^x - \cos 2x)$$

Now, $$\frac{1}{D^2+2} e^{3x} x^2 = e^{3x} \cdot \frac{1}{(D+3)^2 + 2} \cdot x^2 = e^{3x} \cdot \frac{1}{D^2 + 6D + 11} x^2$$

$$= \frac{e^{3x}}{11} \left[1 + \frac{6D + D^2}{11} \right]^{-1} x^2 = \frac{e^{3x}}{11} \left[1 - \frac{(6D + D^2)}{11} + \frac{36D^2}{121} + \ldots\ldots \right] x^2$$

$$= \frac{e^{3x}}{11} \left[x^2 - \frac{12x}{11} - \frac{2}{11} + \frac{72}{121} \right] = \frac{e^{3x}}{11} \left[x^2 - \frac{12x}{11} + \frac{50}{121} \right]$$

$$\frac{1}{D^2 + 2} e^x = \frac{1}{3} e^x$$

$$\frac{1}{D^2 + 2} \cos 2x = -\frac{1}{2} \cos 2x$$

$\therefore$ The complete solution is

$$y = c_1 \cos \sqrt{2} \cdot x + c_2 \sin \sqrt{2} \cdot x + \frac{e^{3x}}{11} \left[x^2 - \frac{12x}{11} + \frac{50}{121} \right] + \frac{1}{3} e^x + \frac{1}{2} \cos 2x.$$

Example 6 (b) : Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^x \cos \frac{x}{2}$. **(M.U. 1995, 2005, 10)**

Sol. : The auxiliary equation is $D^2 - 3D + 2 = 0$

$\therefore$ $(D-1)(D-2) = 0$ $\therefore$ $D = 1, 2$.

$\therefore$ The C.F. is $y = c_1 e^x + c_2 e^{2x}$.

$$\text{P.I.} = 2 \cdot \frac{1}{D^2 - 3D + 2} \cdot e^x \cos \left(\frac{x}{2} \right)$$

$$= 2 \cdot e^x \frac{1}{(D+1)^2 - 3(D+1) + 2} \cdot \cos \left(\frac{x}{2} \right)$$

$$\therefore \text{ P.I.} = 2 \cdot e^x \frac{1}{D^2 - D} \cos\left(\frac{x}{2}\right) = 2 \cdot e^x \frac{1}{-(1/4) - D} \cos\left(\frac{x}{2}\right)$$

$$= -8\, e^x \cdot \frac{1}{4D+1} \cos\left(\frac{x}{2}\right) = -8\, e^x \cdot \frac{4D-1}{16D^2 - 1} \cdot \cos\left(\frac{x}{2}\right)$$

$$= \frac{8}{5} e^x \left[-2 \sin\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right)\right]$$

$\therefore$ The complete solution is $y = c_1 e^x + c_2 e^{2x} - \frac{8}{5} e^x \left[2 \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right]$.

Example 7 (b) : Solve $(D^2 - 1) = \cos hx \cos x$. **(M.U. 2002)**

Sol. : The auxiliary equation is $D^2 - 1 = 0$

$\therefore$ $(D-1)(D+1) = 0$ $\therefore$ $D = 1, -1$.

$\therefore$ The C.F. is $y = c_1 e^x + c_2 e^{-x}$.

$$\text{P.I.} = \frac{1}{D^2 - 1} \cos hx \cos x = \frac{1}{D^2 - 1}\left(\frac{e^x + e^{-x}}{2}\right) \cos x \cdot$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 1} e^x \cos x + \frac{1}{D^2 - 1} e^{-x} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \cdot \frac{1}{(D+1)^2 - 1} \cos x + e^{-x} \cdot \frac{1}{(D-1)^2 - 1} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \cdot \frac{1}{D^2 + 2D} \cos x + e^{-x} \cdot \frac{1}{D^2 - 2D} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \cdot \frac{1}{2D - 1} \cos x - e^{-x} \cdot \frac{1}{2D + 1} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \cdot \frac{2D+1}{4D^2 - 1} \cos x - e^{-x} \cdot \frac{2D-1}{4D^2 - 1} \cos x\right]$$

$$= \frac{1}{2}\left[-\frac{e^x}{5}(-2 \sin x + \cos x) + \frac{e^{-x}}{5}(-2 \sin x - \cos x)\right]$$

$$= \frac{1}{5}\left[2 \sin x \left(\frac{e^x - e^{-x}}{2}\right) - \cos x \left(\frac{e^x + e^{-x}}{2}\right)\right]$$

$$\therefore \text{ P.I.} = \frac{1}{5}[2 \sin x \sin hx - \cos x \cos hx]$$

$\therefore$ The complete solution is $y = c_1 e^x + c_2 e^{-x} + \frac{1}{5}[2 \sin x \sin hx - \cos x \cos hx]$.

Example 8 (b) : Solve $(D^2 + 2)\, y = e^x \cos x + x^2 e^{3x}$. **(M.U. 2001, 08, 12, 14)**

Sol. : The auxiliary equation is $D^2 + 2 = 0$ $\therefore$ $D = +\sqrt{2} \cdot i, -\sqrt{2} \cdot i$.

$\therefore$ The C.F. is $y = c_1 \cos \sqrt{2} \cdot x + c_2 \sin \sqrt{2} \cdot x$.

$$\text{P.I.} = \frac{1}{D^2 + 2} e^x \cos x = e^x \cdot \frac{1}{(D+1)^2 + 2} \cos x$$

$$\text{P.I.} = e^x \cdot \frac{1}{D^2+2D+3} \cdot \cos x = e^x \cdot \frac{1}{2D+2} \cos x$$

$$= e^x \cdot \frac{1}{2} \cdot \frac{D-1}{D^2-1} \cdot \cos x = e^x \cdot \frac{1}{2} \cdot \frac{1}{-2} \cdot (-\sin x - \cos x)$$

$$= e^x \cdot \frac{1}{4}(\sin x + \cos x)$$

As in Example 4 above P.I. corresponding to the second part

$$= \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$$

$\therefore$ The complete solution is

$$y = c_1 \cos\sqrt{2} \cdot x + c_2 \sin\sqrt{2} \cdot x + e^x \cdot \frac{1}{4}(\sin x + \cos x) + \frac{e^{3x}}{11}\left(x^2 - \frac{12}{x} + \frac{50}{121}\right).$$

Example 9 (b) : Solve $(D^3 - 7D - 6)\, y = \cos hx \cos x$. **(M.U. 2002)**

Sol. : The auxiliary equation is $D^3 - 7D - 6 = 0$.

$\therefore\ D^3 + D^2 - D^2 - D - 6D - 6 = 0 \qquad \therefore\ (D+1)(D^2 - D - 6) = 0$

$\therefore\ (D+1)(D+2)(D-3) = 0 \qquad \therefore\ D = -1, -2, 3.$

$\therefore$ The C.F. is $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$.

$$\therefore\ \text{P.I.} = \frac{1}{D^3 - 7D - 6} \cos hx \cos x = \frac{1}{D^3 - 7D - 6}\left(\frac{e^x + e^{-x}}{2}\right) \cdot \cos x$$

$$\text{Now,}\quad \frac{1}{D^3 - 7D - 6} \cdot e^x \cos x = e^x \cdot \frac{1}{(D+1)^3 - 7(D+1) - 6} \cos x$$

$$= e^x \cdot \frac{1}{D^3 + 3D^2 - 4D - 12} \cos x$$

$$= e^x \cdot \frac{1}{-D - 3 - 4D - 12} \cos x \qquad [\text{ Putting } D^2 = -1\ ]$$

$$= -\frac{1}{5}\, e^x \cdot \frac{1}{D+3} \cos x = -\frac{1}{5} e^x \cdot \frac{(D-3)}{(D^2 - 9)} \cos x$$

$$= -\frac{1}{5}\, e^x \cdot \frac{1}{(-1-9)} \cdot (D-3) \cos x$$

$$= \frac{e^x}{50}(-\sin x - 3\cos x)$$

Similarly, we find that

$$\frac{1}{D^3 - 7D - 6} \cdot e^{-x} \cos x = \frac{e^{-x}}{34}(3\cos x - 5\sin x)$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{100} \cdot e^x (\sin x + 3\cos x) + \frac{1}{68} \cdot e^{-x}(3\cos x - 5\sin x).$$

EXERCISE - IX

Solve the following differential equations : Class (b) : 6 Marks

1. $(D^2-4D+4)\,y = e^{2x}x^2$
2. $(D^2+3D+2)\,y = e^{2x}\sin x$
3. $(D^2-4D+1)\,y = e^{2x}\sin 2x$
4. $(D^3-7D-6)\,y = e^{2x}(x+1)$ **(M.U. 1992, 96)**
5. $(D^3-7D-6)\,y = (1+x^2)\,e^{2x}$ **(M.U. 1999, 2007)**
6. $(D^2+4D+4)\,y = \dfrac{e^{-2x}}{x^5}$ **(M.U. 2004)**
7. $(D^2+D-6)\,y = e^{2x}\sin 3x$ **(M.U. 1996)**
8. $(D^2-4)\,y = x^2e^{3x}$ **(M.U. 1997)**
9. $(D^2-1)\,y = x\sin hx$ **(M.U. 2003)**
10. $(D^2-2D+1)\,y = \dfrac{3e^x}{x^2}$
11. $(D^4+D^2+1)\,y = ax^2+be^{-x}\sin 2x$
12. $(D^2-1)\,y = e^x\sin 3x$
13. $(D^2-2D+4)\,y = e^x\cos^2 x$
14. $(D^2-4D+3)\,y = e^x\cos 2x+\cos 3x$ **(M.U. 1999, 99)**
15. $(D^2+1)\,y = \sin x\sin hx$
16. $(D^2-3D+2)\,y = 2e^x\sin\left(\dfrac{x}{2}\right)$ **(M.U. 2004, 07)**
17. $(D^4-1)\,y = \cos x\cos hx.$ **(M.U. 2002)**

[**Ans. :** (1) $y = (c_1+c_2x)\,e^{2x}+e^{2x}\cdot\dfrac{x^4}{12}$

(2) $y = c_1e^{-x}+c_2e^{-2x}+e^{2x}\dfrac{1}{170}(11\sin x-7\cos x)$

(3) $y = c_1e^{(2+\sqrt{3})x}+c_2e^{(2-\sqrt{3})x}-e^{2x}\cdot\dfrac{1}{7}\sin 2x$

(4) $y = c_1e^{-x}+c_2e^{-2x}+c_3e^{3x}-e^{2x}\cdot\dfrac{1}{12}\left(x+\dfrac{17}{12}\right)$

(5) $y = c_1e^{-x}+c_2e^{-2x}+c_3e^{3x}-e^{2x}\cdot\dfrac{1}{12}\cdot\left(x^2+\dfrac{5}{6}x+\dfrac{169}{72}\right)$

(6) $y = (c_1+c_2x)\,e^{-2x}+\dfrac{e^{-2x}}{12x^3}$

(7) $y = c_1e^{2x}+c_2e^{-3x}-\dfrac{e^{2x}}{102}(5\cos 3x+3\sin 3x)$

(8) $y = c_1e^{2x}+c_2e^{-2x}+\dfrac{e^{3x}}{5}\left(x^2-\dfrac{12x}{5}+\dfrac{62}{25}\right)$

(9) $y = c_1e^x+c_2e^{-x}+\dfrac{x^2}{4}\cos hx-\dfrac{x}{4}\sin hx$

(10) $y = (c_1 + c_2 x) e^x - 3e^x \log x$

(11) $y = e^{x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$

$$+ e^{-x/2}\left(c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right) + ax^2 - 2a - b\frac{e^{-x}}{481}(20\cos 2x + 9\sin 2x)$$

(12) $y = c_1 e^x + c_2 e^{-x} - \frac{e^x}{117}(6\cos 3x + 9\sin 3x)$

(13) $y = e^x\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right) + \frac{1}{8}\cdot e^x - \frac{1}{2}\cdot e^x \cos 2x$

(14) $y = c_1 e^{3x} + c_2 e^x - \frac{1}{30}(2\sin 3x + \cos 3x) - e^x \frac{1}{8}\cdot(\sin 2x + \cos 3x)$

(15) $y = c_1 \cos x + c_2 \sin x + \frac{1}{5}[-2\cos x \cos hx + \sin x \sin hx]$.

(16) $y = c_1 e^x + c_2 e^{2x} - \frac{8}{5}e^x\left(\sin\frac{x}{2} - 2\cos\frac{x}{2}\right)$

(17) $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}\cos x \cos hx$]

13. Particular Integral when $X = xV$ where V is a function of x

Let V' be a function of x, then $D(xV') = x D V' + V'$

$$D^2 (xV') = D(x DV' + V') = x D^2 V' + 2 DV'$$

Similarly, $D^3 (x V') = x D^3 V' + 3 D^2 V'$

In general, $D^n(xV') = x D^n V' + n D^{n-1} V' = x D^n V' + \left\{\frac{d}{dD} D^n\right\} V'$

Thus, $f(D)\, x V' = x f(D) V' + f'(D) V'$ .... (1)

Now let $f(D)\ V' = V$

$\therefore\ V' = \frac{1}{f(D)}\cdot V.$

Operating by $\frac{1}{f(D)}$ on both sides, from (1).

$$\frac{1}{f(D)}\cdot f(D)\, x V' = \frac{1}{f(D)}\cdot x f(D) V' + \frac{1}{f(D)}\cdot f'(D)\cdot V'$$

$$\therefore\ xV' = \frac{1}{f(D)}\cdot x f(D) V' + \frac{1}{f(D)}\cdot f'(D)\cdot V'$$

$$\therefore\ x\frac{1}{f(D)}V = \frac{1}{f(D)}\cdot xV + \frac{1}{f(D)}\cdot f'(D)\cdot\frac{1}{f(D)}\cdot V$$

$$\therefore\ \frac{1}{f(D)}xV = x\frac{1}{f(D)}\cdot V - \frac{1}{f(D)}\cdot\left[f'(D)\cdot\frac{1}{f(D)}\cdot V\right]$$

$$\therefore \quad \boxed{\frac{1}{f(D)} xV = \left\{x - \frac{1}{f(D)} \cdot f'(D)\right\} \cdot \frac{1}{f(D)} \cdot V}$$

Caution

Do not in any way simplify the above formula as it may lead you to wrong result.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $(D^2 + 4) = x \sin x$. **(M.U. 2005)**

Sol. : The auxiliary equation is $D^2 + 4 = 0 \quad \therefore D = 2i, -2i.$

$\therefore$ The C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$.

$$\text{P.I.} = \frac{1}{D^2+4} \cdot (x \sin x) = \left\{x - \frac{1}{D^2+4} \cdot 2D\right\} \cdot \frac{1}{D^2+4} \sin x$$

$$= \left\{x - \frac{1}{D^2+4} \cdot 2D\right\} \cdot \frac{1}{3} \sin x = \frac{x}{3} \cdot \sin x - \frac{1}{D^2+4} \cdot \frac{2}{3} \cos x$$

$$= \frac{x}{3} \cdot \sin x - \frac{2}{3} \cdot \frac{1}{3} \cos x$$

$\therefore$ The complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{3} \sin x - \frac{2}{9} \cdot \cos x.$$

Remark ....

The above method is tedius to apply. However, we can use the method of § 12 when r.h.s. is $x^m \sin ax$ or $x^m \cos ax$ by writing the r.h.s. as $x^m e^{iax}$ and then considering the real part or imaginary part. (See Ex. 5, below)

Example 2 (b) : Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x.$ **(M.U. 2011)**

Sol. : The A.E. is $D^2 - 2D + 1 = 0 \quad \therefore (D-1)^2 = 0 \quad \therefore D = 1, 1.$

$\therefore$ The C.F. is $y = (c_1 + c_2 x) e^x$.

$$\therefore \quad \text{P.I.} = \frac{1}{(D-1)^2} \cdot e^x \cdot x \sin x = e^x \cdot \frac{1}{[(D+1)-1]^2} x \sin x \quad \text{[By § 12]}$$

$$= e^x \cdot \frac{1}{D^2} \cdot x \sin x = e^x \left[x - \frac{1}{D^2} \cdot 2D\right] \cdot \frac{1}{D^2} \sin x \quad \text{[By § 13]}$$

$$= e^x \left[x - \frac{1}{D^2} \cdot 2D\right]\left(\frac{1}{-1}\right) \sin x = -e^x \left[x - \frac{1}{D^2} \cdot 2D\right] \sin x$$

$$= -e^x \left[x \sin x - \frac{1}{D^2} \cdot 2 \cos x\right] = -e^x \left[x \sin x - \frac{2}{(-1)} \cos x\right]$$

$$\therefore \quad \text{P.I.} = -e^x [x \sin x + 2 \cos x]$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x) e^x - e^{-x}(x \sin x + 2 \cos x).$$

Example 3 (b) : Solve $(D^2 - 4)\, y = x \sin hx$. **(M.U. 1991)**

Sol. : The auxiliary equation is $D^2 - 4 = 0$ $\quad \therefore \; D = 2, -2$.

$\therefore$ The C.F. is $y = c_1 e^{2x} + c_2 e^{-2x}$.

$$\text{P.I.} = \frac{1}{D^2-4} x \sin hx = \frac{1}{D^2-4} x \cdot \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4} \cdot xe^x - \frac{1}{D^2-4} x \cdot e^{-x}\right]$$

$$= \frac{1}{2}\left[x - \frac{1}{D^2-4} \cdot 2D\right]\frac{1}{D^2-4} e^x - \frac{1}{2}\left[x - \frac{1}{D^2-4} \cdot 2D\right]\frac{1}{D^2-4} e^{-x}$$

$$= \frac{1}{2}\left[x - \frac{1}{D^2-4} \cdot 2D\right]\left(-\frac{1}{3} e^x\right) - \frac{1}{2}\left[x - \frac{1}{D^2-4} \cdot 2D\right]\left(-\frac{1}{3} e^{-x}\right)$$

$$= -\frac{1}{6}\left[x \cdot e^x - \frac{1}{D^2-4} \cdot 2e^x\right] + \frac{1}{6}\left[x \cdot e^{-x} - \frac{1}{D^2-4} \cdot 2(-e^{-x})\right]$$

$$= -\frac{1}{6}\left[x \cdot e^x + \frac{2}{3} e^x\right] + \frac{1}{6}\left[x \cdot e^{-x} - \frac{2}{3} \cdot e^{-x}\right]$$

$$= -\frac{x}{6}\left(e^x - e^{-x}\right) - \frac{1}{6} \cdot \frac{2}{3}\left(e^x + e^{-x}\right)$$

$$= -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right) = -\frac{x}{3} \sin hx - \frac{2}{9} \cos hx$$

$\therefore$ The complete solution is $\quad y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{x}{3} \sin hx - \dfrac{2}{9} \cos hx$.

Alternatively we may use the method of § 12 to find the particular integral. But, you may find it a little more tedius than the above method. Try it.

Example 4 (b) : Solve $(D^2 - 1)\, y = x \sin 3x + \cos x$. **(M.U. 1987)**

Sol. : The auxiliary equation is $\quad D^2 - 1 = 0 \quad \therefore \; D = +1, -1$.

$\therefore$ The C.F. is $y = c_1 e^x + c_2 e^{-x}$.

$$\text{P.I.} = \frac{1}{D^2-1}\left[x \sin 3x + \cos x\right]$$

$$= \left[x - \frac{1}{D^2-1} \cdot 2D\right]\frac{1}{D^2-1} \sin 3x + \frac{1}{D^2-1} \cos x$$

$$= \left[x - \frac{1}{D^2-1} \cdot 2D\right]\left(-\frac{1}{10}\right) \sin 3x - \frac{1}{2} \cos x$$

$$= -\frac{1}{10}\left[x \sin 3x - \frac{1}{D^2-1} 6 \cos 3x\right] - \frac{1}{2} \cos x$$

$$= -\frac{1}{10}\left[x \sin 3x + \frac{6}{10} \cos 3x\right] - \frac{1}{2} \cos x$$

$$\therefore \quad \text{P.I.} = -\frac{1}{10}\left[x \sin 3x + \frac{3}{5} \cos 3x\right] - \frac{1}{2} \cos x$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{10}\left[x \sin 3x + \frac{3}{5}\cos x\right] - \frac{1}{2}\cos x.$$

Example 5 (b) : Solve $(D^2 - 1)\, y = x^2 \sin 3x$. **(M.U. 2002)**

Sol. : The auxiliary equation is $D^2 - 1 = 0 \quad \therefore D = 1, -1$

$\therefore$ The C.F. is $y = c_1 e^x + c_2 e^{-x}$.

$$\text{P.I.} = \text{Imaginary Part of } \frac{1}{D^2 - 1} \cdot x^2 e^{3ix}$$

$$= \text{I.P. of } e^{3ix} \cdot \left\{\frac{1}{(D + 3i)^2 - 1}\right\} x^2 \qquad [\text{ By § 12 }]$$

$$= \text{I.P. of } e^{3ix} \cdot \left\{\frac{1}{D^2 + 6Di - 10}\right\} x^2$$

$$= \text{I.P. of } e^{3ix} \cdot \frac{1}{(-10)}\left\{1 - \frac{6Di + D^2}{10}\right\}^{-1} x^2$$

$$= \text{I.P. of } e^{3ix} \cdot \frac{1}{(-10)}\left\{1 + \left(\frac{6Di + D^2}{10}\right) + \frac{36D^2 i^2}{100} \ldots\right\} x^2$$

$$= \text{I.P. of } e^{3ix} \cdot \frac{1}{(-10)}\left\{1 + \frac{6Di}{10} - \frac{26}{100} D^2\right\} x^2$$

$$= \text{I.P. of } e^{3ix} \cdot \frac{1}{-10}\left[x^2 + \frac{6}{5} x i - \frac{13}{25}\right]$$

$$= \text{I.P. of } (\cos 3x + i \sin 3x) \cdot \frac{1}{(-10)} \cdot \left[x^2 + \frac{6}{5} x i - \frac{13}{25}\right]$$

$$\therefore \quad \text{P.I.} = \frac{1}{-10}\left\{x^2 \sin 3x + \frac{6}{5} x \cos 3x - \frac{13}{25} \sin 3x\right\}$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{10}\left\{x^2 \sin 3x + \frac{6}{5} x \cos 3x - \frac{13}{25} \sin 3x\right\}.$$

Alternatively we may use the method of § 13, but it may be a little more tedius.

Example 6 : Solve $(D^4 + 2D^2 + 1)\, y = x^2 \cos x$. **(M.U. 2012)**

Sol. : The auxiliary equation is $D^4 + 2D^2 + 1 = 0 \qquad \therefore (D^2 + 1)^2 = 0$

$\therefore \; D = i, i, -i, -i.$

$\therefore$ The C.F. is $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$.

$$\therefore \quad \text{P.I.} = \frac{1}{D^4 + 2D^2 + 1} \cdot x^2 \cos x = \text{Real Part of } \frac{1}{(D^2 + 1)^2} \cdot x^2 e^{ix}$$

$$= \text{R. P. of } e^{ix} \frac{1}{\left[(D + i)^2 + 1\right]^2} x^2 \qquad [\text{ By § 12 }]$$

$$\therefore \text{P.I.} = \text{R. P. of } e^{ix}\frac{1}{(D^2+2Di)^2}x^2 = \text{R. P. of } e^{ix}\frac{1}{D^2(D+2i)^2}x^2$$

$$= \text{R. P. of } e^{ix}\frac{1}{D^2}\cdot\frac{1}{-4+4iD+D^2}x^2$$

$$= \text{R. P. of } e^{ix}\cdot\frac{1}{D^2}\left(-\frac{1}{4}\right)\left[1-\frac{4iD+D^2}{4}\right]^{-1}x^2$$

$$= \text{R. P. of } e^{ix}\frac{1}{D^2}\cdot\left(-\frac{1}{4}\right)\left[1+\frac{4iD+D^2}{4}-D^2+\ldots..\right]x^2$$

$$= \text{R. P. of } e^{ix}\cdot\left(-\frac{1}{4D^2}\right)\left[1+iD-\frac{3}{4}D^2+\ldots..\right]x^2$$

$$= \text{R. P. of } e^{ix}\cdot\left(-\frac{1}{4D^2}\right)\left(x^2+2ix-\frac{3}{2}\right)$$

$$= \text{R. P. of } e^{ix}\cdot\left(-\frac{1}{4D}\right)\int\left(x^2+2ix-\frac{3}{2}\right)dx$$

$$= \text{R. P. of } e^{ix}\cdot\left(-\frac{1}{4D}\right)\left(\frac{x^3}{3}+ix^2-\frac{3}{2}x\right)$$

$$= \text{R. P. of } e^{ix}\cdot\left(-\frac{1}{4}\right)\int\left(\frac{x^3}{3}+ix^2-\frac{3}{2}x\right)dx$$

$$= \text{R. P. of } \left(-\frac{e^{ix}}{4}\right)\left(\frac{x^4}{12}+\frac{ix^3}{3}-\frac{3}{4}x^2\right)$$

$$= \text{R. P. of } \left(-\frac{1}{4}\right)(\cos x+i\sin x)\left(\frac{x^4}{12}+\frac{ix^3}{3}-\frac{3}{4}x^2\right)$$

$$= -\frac{1}{4}\left(\frac{x^4}{12}\cos x-\frac{3}{4}x^2\cos x-\frac{x^3}{3}\sin x\right)$$

$$\therefore \text{P.I.} = -\frac{1}{48}(x^4-9x^2)\cos x+\frac{x^3}{12}\sin x$$

$\therefore$ The complete solution is

$$y=(c_1+c_2x)\cos x+(c_3+c_4x)\sin x-\frac{1}{48}(x^4-9x^2)\cos x+\frac{x^3}{12}\sin x.$$

(M.U. 2003, 08)

Example 7 (b) : Solve $(D^2+4)\,y = x\sin^2 x$.

Sol. : The auxiliary equation is $D^2+4=0 \quad \therefore D = 2i, -2i$.

$\therefore$ The C.F. is $y = c_1\cos 2x + c_2\sin 2x$.

$$\text{P.I.} = \frac{1}{D^2+4}(x\sin^2 x) = \frac{1}{D^2+4}x\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{D^2+4}x - \frac{1}{2}\cdot\frac{1}{D^2+4}x\cos 2x.$$

Now, $\dfrac{1}{2}\cdot\dfrac{1}{D^2+4}\,x=\dfrac{1}{2}\cdot\dfrac{1}{4}\left(1-\dfrac{D^2}{4}\cdots\right)x=\dfrac{1}{8}\,x$

And $\dfrac{1}{2}\cdot\dfrac{1}{D^2+4}\,x\cos 2x=\dfrac{1}{2}\text{R. P. of}\dfrac{1}{D^2+4}\,x\cdot e^{2ix}$

$$=\frac{1}{2}\text{R. P. of } e^{2ix}\cdot\frac{1}{(D+2i)^2+4}\,x$$

$$=\frac{1}{2}\text{R. P. of } e^{2ix}\cdot\frac{1}{D^2+4iD}\,x$$

$$=\frac{1}{2}\text{ R. P. of } e^{2ix}\cdot\frac{1}{4iD}\cdot\frac{1}{[1+(D/4i)]}\,x$$

$$=\frac{1}{2}\text{R. P. of } e^{2ix}\cdot\frac{1}{4iD}\cdot\left[1-\frac{D}{4i}\cdots\right]x$$

$$=\frac{1}{2}\text{R. P. of } e^{2ix}\cdot\frac{1}{4iD}\cdot\left[x-\frac{1}{4i}\right]$$

$$=\frac{1}{2}\text{R. P. of } e^{2ix}\cdot\frac{1}{4i}\left[\frac{x^2}{2}-\frac{x}{4i}\right]\qquad\left[\because\ \frac{1}{D}=\int dx\right]$$

$$=\frac{1}{2}\text{R. P. of } e^{2ix}\left(\frac{x^2}{8i}+\frac{x}{16}\right)$$

$$=\frac{1}{2}\text{R. P. of } (\cos 2x+i\sin 2x)\left(\frac{x^2}{8i}+\frac{x}{16}\right)$$

$$=\frac{x}{32}\cos 2x+\frac{x^2}{16}\sin 2x$$

$\therefore$ The complete solution is $y=c_1\cos 2x+c_2\sin 2x+\dfrac{x}{8}-\dfrac{x}{32}\cos 2x-\dfrac{x^2}{16}\sin 2x.$

EXERCISE - X

Solve the following differential equations : Class (b) : 6 Marks

1. $(D^2-2D+1)\,y=x\sin x$
2. $(D^2+2D+1)\,y=x\cos x$
3. $(D^2+1)\,y=x^2\sin 2x$
4. $(D^2-1)\,y=x^2\cos 2x$
5. $(D^2-1)\,y=x\sin x+(1+x^2)\,e^x$
6. $(D^4+2D^2+1)\,y=x^2\sin x.$

[Ans. : **(1)** $y=(c_1+c_2x)\,e^x+\dfrac{1}{2}(x\cos x+\cos x-\sin x)$

(2) $y=(c_1+c_2x)\,e^{-x}+\dfrac{1}{2}(x\sin x-\sin x+\cos x)$

(3) $y=c_1\cos x+c_2\sin x-\dfrac{1}{27}\left[24x\cos 2x+(9x^2-26)\sin 2x\right]$

(4) $y=c_1e^x+c_2e^{-x}+x\sin x+\dfrac{1}{2}(1-x^2)\cos x$

(5) $y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}(x \sin x + \cos x) + \frac{1}{12} x e^x (2x^2 - 3x + 9)$

(6) $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x - \frac{1}{4}\left(\frac{x^4}{12} \sin x + \frac{x^3}{3} \cos x - \frac{3}{4} x^2 \sin x\right)$.

14. When r.h.s. X does not belong partially or completely to any one of the above forms

When any of the above method fails to give the particular integral we apply the definition of $\frac{1}{f(D)} X$ discussed in the article 7. We once again note that

(i) $\boxed{\frac{1}{D} X = \int x \, dx}$ and (ii) $\boxed{\frac{1}{D-a} X = e^{ax} \int e^{-ax} X \, dx}$

The method will be clear from the following illustrative examples.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $(D^2 + a^2) y = \sec ax$. **(M.U. 1991)**

Sol. : The auxiliary equation is $D^2 + a^2 = 0 \quad \therefore \quad D = + ai, - ai$.

$\therefore$ The C.F. is $y = c_1 \cos ax + c_2 \sin ax$.

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax = \frac{1}{(D + ai)(D - ai)} \sec ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D - ai} - \frac{1}{D + ai}\right] \sec ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D - ai} \sec ax - \frac{1}{D + ai} \sec ax\right]$$

$$= \frac{1}{2ai}\left[e^{aix} \int e^{-aix} \sec ax \, dx - e^{-aix} \int e^{aix} \sec ax \, dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix} \int (\cos ax - i \sin ax) \sec ax \, dx - e^{-aix} \int (\cos ax + i \sin ax) \sec ax \, dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix} \int (1 - i \tan ax) \, dx - e^{-aix} \int (1 + i \tan ax) \, dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\left\{x - \frac{i}{a} \log \sec ax\right\} - e^{-aix}\left\{x + \frac{i}{a} \log \sec ax\right\}\right]$$

$$= \frac{1}{2ai}\left[(\cos ax + i \sin ax)\left\{x - \frac{i}{a} \log \sec ax\right\} - (\cos ax - i \sin ax)\left\{x + \frac{i}{a} \log \sec ax\right\}\right]$$

$$= \frac{1}{2ai}\left\{2 i x \sin ax - \frac{2i}{a} \cos x \log \sec ax\right\}$$

$$= \frac{x}{a} \sin ax - \frac{1}{a^2} \cos ax \log \sec ax \qquad \left[\because \sec ax = \frac{1}{\cos ax}\right]$$

$$\therefore \text{P.I.} = \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \log \cos ax$$

$\therefore$ The complete solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \log \cos ax.$$

(For another method see Ex. 1, page 3-52.)

Example 2 (b) : Solve $(D^2 + a^2) = 2a \tan ax.$ **(M.U. 2003)**

Sol. : The auxiliary equation is $D^2 + a^2 = 0 \quad \therefore \quad D = ai, -ai.$

$\therefore$ The C.F. is $y = c_1 \cos ax + c_2 \sin ax.$

$$\text{P.I.} = \frac{2a}{D^2 + a^2} \tan ax = \frac{1}{i}\left[\frac{1}{D - ai} - \frac{1}{D + ai}\right] \tan ax$$

Now, $\boxed{\frac{1}{D - ai} \tan ax = e^{aix} \int e^{-aix} \tan ax \, dx}$

$$= e^{aix} \int (\cos ax - i \sin ax) \tan ax \, dx$$

$$= e^{aix} \int \left(\sin ax - i \frac{\sin^2 ax}{\cos ax}\right) dx$$

$$= e^{aix} \int \left(\sin\ ax - i \frac{(1 - \cos^2 ax)}{\cos\ ax}\right) dx$$

$$= e^{aix} \int (\sin ax - i \sec ax + i \cos ax) \, dx$$

$$= e^{aix} \left[-\frac{\cos ax}{a} - \frac{i}{a} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + \frac{i}{a} \sin ax\right]$$

$$= -e^{aix} \left[\frac{1}{a}(\cos ax - i \sin ax) + \frac{i}{a} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right]$$

$$= -\frac{e^{aix}}{a}\left[e^{-aix} + i \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right]$$

$$= -\frac{1}{a}\left[1 + i e^{aix} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right]$$

Changing i to $-i$,

$$\frac{1}{D + ai} \tan ax = -\frac{1}{a}\left[1 - i e^{-aix} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right]$$

$$\text{P.I.} = \frac{1}{i}\left[-\frac{i}{a}\left(e^{aix} + e^{-aix}\right) \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right]$$

$$= -\frac{2}{a}\left(\frac{e^{aix} + e^{-aix}}{2}\right) \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

$$= -\frac{2}{a} \cos ax \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

$\therefore$ The complete solution is

$$y = c_1 \cos ax + c_2 \sin ax - \frac{2}{a} \cos ax \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right).$$

Example 3 (b) : Solve $(D^2+3D+2)\,y=\sin e^x$. **(M.U. 1997, 2000, 05)**

Sol. : The auxiliary equation is

$$D^2+3D+2=0 \qquad \therefore\ (D+2)(D+1)=0$$

$\therefore$ The C.F. is $y=c_1e^{-x}+c_2e^{-2x}$.

$$\text{P.I.}=\frac{1}{(D+2)(D+1)}\sin e^x=\left(\frac{1}{D+1}-\frac{1}{D+2}\right)\sin e^x$$

$$=\frac{1}{(D+1)}\sin e^x-\frac{1}{D+2}\cdot\sin e^x$$

$$=e^{-x}\int e^x\sin e^x\cdot dx-e^{-2x}\int e^{2x}\sin e^x\cdot dx$$

To evaluate the integrals put $e^x=t$, $e^x dx=dt$.

$$\therefore\ \text{P.I.}=e^{-x}\int\sin t\,dt-e^{-2x}\int t\sin t\,dt$$

$$=e^{-x}(-\cos t)-e^{-2x}[(t)(-\cos t)-(1)\ (-\sin t)]$$

$$=-e^{-x}\cos e^x-e^{-2x}\left[-e^x\cos e^x+\sin e^x\right]$$

$$=-e^{-2x}\sin e^x$$

$\therefore$ The complete solution is $y=c_1e^{-x}+c_2e^{-2x}-e^{-2x}\sin e^x$.

Example 4 (b) : Solve $(D^2+5D+6)\,y=e^{-2x}\sec^2 x(1+2\tan x)$. **(M.U. 1996, 97, 2002, 05)**

Sol. : The auxiliary equation is $D^2+5D+6=0$

$\therefore\ (D+3)(D+2)=0 \qquad \therefore\ D=-2,-3.$

$\therefore$ The C.F. is $y=c_1e^{-2x}+c_2e^{-3x}$.

$$\therefore\ \text{P.I.}=\frac{1}{(D+3)}\cdot\frac{1}{(D+2)}\,e^{-2x}\sec^2 x(1+2\tan x)$$

$$=\frac{1}{(D+3)}\cdot e^{-2x}\int e^{2x}\cdot e^{-2x}\sec^2 x(1+2\tan x)\,dx \qquad [\text{ Put } t=\tan x\,]$$

$$=\frac{1}{D+3}\cdot e^{-2x}\int\sec^2 x(1+2\tan x)\,dx=\frac{1}{D+3}\cdot e^{-2x}(\tan x+\tan^2 x)$$

$$=e^{-3x}\int e^{3x}\cdot e^{-2x}(\tan x+\tan^2 x)\,dx$$

$$=e^{-3x}\int e^x\left\{(\tan x+\sec^2 x)-1\right\}dx$$

$$=e^{-3x}\left[\int e^x(\tan x+\sec^2 x)\,dx-\int e^x\,dx\right]$$

$$=e^{-3x}\left[e^x\tan x-e^x\right]=e^{-2x}\left[\tan x-1\right] \qquad \left[\because\ \int e^x\left[f(x)+f'(x)\right]dx=e^x f(x)\right]$$

$\therefore$ The complete solution is $y=c_1e^{-2x}+c_2e^{-3x}+e^{-2x}\left[\tan x-1\right]$.

(For another method see Ex. 12, page 3-58.)

The two answers differ slightly. But both are correct. They can be shown to be equal. They differ because of arbitrary constant c_1.

Example 5 (b) : Solve $(D^2 - 6D + 9)y = e^{3x}(1+x)$. **(M.U. 1990)**

Sol. : The auxiliary equation is $D^2 - 6D + 9 = 0$ $\therefore$ $(D-3)^2 = 0$ $\therefore$ $D = 3, 3$.

$\therefore$ The C.F. is $y = (c_1 + c_2 x)e^{3x}$.

$$\text{P.I.} = \frac{1}{D^2 - 6D + 9} e^{3x}(1+x) = \frac{1}{(D-3)^2} e^{3x} + \frac{1}{(D-3)^2} e^{3x} \cdot x$$

$$= \frac{x^2}{2!} e^{3x} + e^{3x} \cdot \frac{1}{(D+3-3)^2} x \qquad \text{[By (3), page 3-31]}$$

$$= \frac{x^2}{2} e^{3x} + e^{3x} \cdot \frac{1}{D^2} x$$

But $\frac{1}{D^2} x = \frac{1}{D} \int x\, dx = \frac{1}{D} \frac{x^2}{2} = \int \frac{x^2}{2}\, dx = \frac{x^3}{6}$

$\therefore$ The complete solution is $y = (c_1 + c_2 x)e^{3x} + \frac{x^2}{2} e^{3x} + \frac{x^3}{6} \cdot e^{3x}$.

Example 6 (b) : Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$. **(M.U. 1987, 95, 2008)**

Sol. : The auxiliary equation is $D^2 - 2D + 1 = 0$. $\therefore$ $(D-1)^2 = 0$ $\therefore$ $D = 1, 1$.

$\therefore$ The C.F. is $y = (c_1 + c_2 x)e^x$.

$$\text{P.I.} = \frac{1}{(D-1)^2} e^x (x \sin x) = e^x \frac{1}{(D+1-1)^2} x \sin x \qquad \text{[By (3), page 3-31]}$$

$$= e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D} \int x \sin x\, dx$$

$$= e^x \frac{1}{D}\left[x(-\cos x) - \int (-\cos x) \cdot 1 \cdot dx\right] \qquad \text{[By parts]}$$

$$= e^x \frac{1}{D}\left[-x \cos x + \sin x\right] = e^x \int \left[-x\cos x + \sin x\right] dx$$

$$= e^x \left[(-x)\sin x - \int \sin x(-1)\, dx - \cos x\right]$$

$$= e^x\left[-x \sin x - \cos x - \cos x\right]$$

$\therefore$ The complete solution is $y = (c_1 + c_2 x)e^x - e^x(x \sin x + 2\cos x)$.

Alternatively To find the P.I. we may use the formulae derived in § 12 and § 13 successively.

$$\therefore \text{P.I.} = \frac{1}{(D-1)^2} e^x \cdot (x \sin x) = e^x \frac{1}{(D+1-1)^2} x \sin x = e^x \cdot \frac{1}{D^2} x \sin x$$

$$= e^x \left[x - \frac{1}{D^2} \cdot 2D\right] \frac{1}{D^2} \sin x = e^x \left[x - \frac{2}{D}\right](-\sin x)$$

$$= e^x\left[-x \sin x - 2\cos x\right].$$

Example 7 (b) : Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = xe^{-x} \cos x$. **(M.U. 1997, 2009)**

Sol. : The auxiliary equation is $D^2 + 2D + 1 = 0$

$\therefore\ (D+1)^2 = 0 \qquad \therefore\ D = -1, -1.$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)\, e^{x}$.

$$\text{P.I.} = \frac{1}{(D+1)^2} e^{-x}\, x\cos x = e^{-x} \cdot \frac{1}{(D-1+1)^2} x\cos x \quad \text{[By (3), page 3-31]}$$

$$= e^{-x} \cdot \frac{1}{D^2} x\cos x = e^{-x} \frac{1}{D}\int x\cos x\, dx e^{-x} \cdot \frac{1}{D}[x\sin x + \cos x \cdot 1]$$

(By generalised rule of integration by parts)

$$= e^{-x}\int [x\sin x + \cos x]\,dx = e^{-x}[x(-\cos x) - (-\sin x)\cdot 1 + \sin x]$$

$$= e^{-x}[-x\cos x + 2\sin x]$$

$\therefore$ The complete solution is $y = (c_1 + c_2 x)\, e^{-x} + e^{-x}(-x\cos x + 2\sin x)$

$$= e^{-x}(c_1 + c_2 x - x\cos x + 2\sin x).$$

Alternatively the P.I. can also obtained by using the results of § 12 and § 13 successively as follows.

$$\text{P.I.} = \frac{1}{(D+1)^2} e^{-x} \cdot x\cos x = e^{-x} \cdot \frac{1}{(D-1+1)^2} x\cos x$$

$$= e^{-x} \cdot \frac{1}{D^2} x\cos x = e^{-x}\left[x - \frac{1}{D^2}\cdot 2D\right]\frac{1}{D^2}\cos x$$

$$= e^{-x}\left[x - \frac{1}{D^2}\cdot 2D\right](-1)\cos x = e^{-x}\left[-x\cos x - \frac{1}{D^2} 2\sin x\right]$$

$$= e^{-x}[-x\cos x + 2\sin x].$$

Example 8 (b) : Solve $(D^2 + 4D + 4)\, y = e^{-2x} x\cos x$. **(M.U. 1990, 93)**

Sol. : The auxiliary equation is $D^2 + 4D + 4 = 0$.

$\therefore\ (D+2)^2 = 0 \qquad \therefore\ D = -2, 2.$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)\, e^{-2x}$.

$$\text{P.I.} = \frac{1}{(D+2)^2} e^{-2x} x\cos x = e^{-2x} \frac{1}{(D-2+2)^2} x\cos x \quad \text{[By (3), page 3-31]}$$

$$= e^{-2x} \frac{1}{D^2} \cdot x\cos x = e^{-2x} \cdot \frac{1}{D}\int x\cos x\, dx$$

$$= e^{-2x} \cdot \frac{1}{D}[x(\sin x) + \cos x] = e^{-2x}\int (x\sin x + \cos x)\, dx$$

$$= e^{-2x}[x(-\cos x) - (-\sin x) + \sin x]$$

$$\therefore\ \text{P.I.} = e^{-2x}[-x\cos x + 2\sin x]$$

Alternatively : $\text{P.I.} = e^{-2x} \cdot \dfrac{1}{D^2} x\cos x = e^{-2x}\left[x - \dfrac{1}{D^2}\cdot 2D\right]\dfrac{1}{D^2}\cos x$

$$= e^{-2x}[-x\cos x + 2\sin x] \quad \text{(as in the previous example)}$$

$\therefore$ The complete solution is $y = (c_1 + c_2 x)\, e^{-2x} + e^{-2x}(-x\cos x + 2\sin x)$.

Example 9 (b) : Solve $(D^2-4D+4)y = 8 \cdot x^2 e^{2x} \sin 2x$.

Sol. : The auxiliary equation is $D^2-4D+4=0$ $\quad \therefore \; (D-2)^2=0 \quad \therefore \; D=2, 2.$

$\therefore$ The C.F. is $y=(c_1+c_2 x)\, e^{2x}$.

$$\text{P.I.} = \frac{1}{(D-2)^2}\, 8 \cdot x^2 e^{2x} \sin 2x = 8 \cdot \frac{1}{(D-2)^2} \cdot e^{2x} \cdot x^2 \sin 2x$$

$$= 8 \cdot e^{2x} \frac{1}{(D+2-2)^2}\, x^2 \sin 2x = 8 \cdot e^{2x} \cdot \frac{1}{D^2}\, x^2 \sin 2x$$

$$= 8 \cdot e^{2x} \cdot \frac{1}{D} \int x^2 \sin 2x \, dx$$

$$= 8 \cdot e^{2x} \cdot \frac{1}{D}\left[x^2\left(\frac{-\cos 2x}{2}\right) - (2x)\left(\frac{-\sin 2x}{4}\right) + 2 \cdot \frac{\cos 2x}{8}\right]$$

(By applying generalised rule of integration by parts.)

$$\text{P.I.} = 8 \cdot e^{2x} \cdot \int \left[-\frac{x^2}{2}\cos 2x + \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x\right] dx$$

Applying the rule again

$$= 8 \cdot e^{2x}\left[-\frac{1}{2}\left(x^2\frac{\sin 2x}{2} - 2x\left(\frac{-\cos 2x}{4}\right) + 2\left(\frac{-\sin 2x}{8}\right)\right)\right.$$

$$\left. + \frac{1}{2} \cdot x\left(\frac{-\cos 2x}{2}\right) - \frac{1}{2} \cdot \left(\frac{-\sin 2x}{4}\right) + \frac{1}{4} \cdot \frac{\sin 2x}{2}\right]$$

$$= 8 \cdot e^{2x}\left[-\frac{x^2}{4}\sin 2x - \frac{1}{2}x \sin 2x + \frac{3}{8}\sin 2x\right]$$

$\therefore$ The complete solution is $\; y=(c_1+c_2x)\,e^{2x} - e^{2x}(2x^2 \sin 2x + 4x \sin 2x - 3 \sin 2x)$.

Example 10 (b) : Solve $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{e^x}$. **(M.U. 1996, 99, 2002)**

Sol. : The auxiliary equation is $D^2+3D+2=0$ $\therefore$ $(D+1)(D+2)=0$ $\therefore$ $D=-1, -2$.

$\therefore$ The C.F. is $y=c_1 e^{-x} + c_2 e^{-2x}$.

$$\text{P.I.} = \frac{1}{(D+2)(D+1)}\, e^{e^x} = \frac{1}{D+2} \cdot e^{-x} \int e^{e^x} e^x \, dx$$

To find the integral, put $e^x = t$ $\quad \therefore \; e^x\, dx = dt$.

$$\therefore \; \int e^{e^x} e^x \, dx = \int e^t \, dt = e^t = e^{e^x}$$

$$\therefore \; \frac{1}{D+2} e^{-x} \int e^{e^x} e^x \, dx = \frac{1}{D+2} e^{-x} \cdot e^{e^x} = e^{-2x} \int e^{e^x} \cdot e^{2x} \cdot e^{-x}\, dx$$

$$= e^{-2x} \int e^{e^x} e^x \, dx = e^{-2x} \cdot e^{e^x} \text{ as above.}$$

$\therefore$ The complete solution is $\; y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \cdot e^{e^x}$.

Note ....

For another method see Ex. 4, page 3-54.

Example 11 (b) : Solve $(D^2 + D)\,y = \dfrac{1}{1+e^x}$. **(M.U. 2009)**

Sol. : The auxiliary equation is $D(D+1) = 0 \quad \therefore \; D = 0, -1.$

$\therefore$ The C.F. is $y = c_1 + c_2 e^{-x}$.

$$\text{P.I.} = \frac{1}{D+1}\cdot\frac{1}{D}\cdot\frac{1}{1+e^x} = \frac{1}{D+1}\int\frac{dx}{1+e^x}$$

$$= \frac{1}{D+1}\cdot\int\frac{e^{-x}}{e^{-x}+1}\,dx \qquad \left[\text{Put } e^{-x}+1 = t\right]$$

$$= \frac{1}{D+1}\left[-\log(e^{-x}+1)\right] = -e^{-x}\int e^x\cdot\log(e^{-x}+1)\,dx$$

$$\text{P.I.} = -e^{-x}\left[\log(e^{-x}+1)\cdot e^x - \int e^x\cdot\frac{(-e^{-x})}{e^{-x}+1}\,dx\right]$$

(By integrating by parts)

$$= -e^{-x}\left[e^x\log(e^{-x}+1) + \int\frac{dx}{e^{-x}+1}\right]$$

$$= -e^{-x}\left[e^x\log(e^{-x}+1) + \int\frac{e^x}{e^x+1}\,dx\right] \qquad [\text{Put } e^x + 1 = t]$$

$$= -e^{-x}\left[e^x\log(e^{-x}+1) + \log(1+e^x)\right]$$

$\therefore$ The complete solution is

$$y = c_1 + c_2 e^{-x} - e^{-x}\left[e^x\log(e^{-x}+1) + \log(1+e^x)\right].$$

Example 12 (b) : Solve $(D^2 - D - 2)\,y = 2\log x + \dfrac{1}{x} + \dfrac{1}{x^2}$. **(M.U. 2000, 08, 10, 11, 16)**

Sol. : The auxiliary equation is $(D^2 - D - 2) = 0$

$\therefore \; (D-2)(D+1) = 0 \qquad \therefore \; D = -1, 2.$

$\therefore$ The C.F. is $y = c_1 e^{-x} + c_2 e^{2x}$.

$$\text{P.I.} = \frac{1}{(D-2)(D+1)}\cdot\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$= \frac{1}{D-2}\cdot e^{-x}\int e^x\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= \frac{1}{D-2}\cdot e^{-x}\left[\int e^x\left(2\log x + \frac{2}{x}\right)dx + \int e^x\left(-\frac{1}{x} + \frac{1}{x^2}\right)dx\right] \qquad \textbf{[Note this]}$$

$$= \frac{1}{D-2}\cdot e^{-x}\cdot\left[e^x\,2\log x - e^x\cdot\frac{1}{x}\right] \qquad \left[\because \int e^x[f(x) + f'(x)]\,dx = e^x f(x)\right]$$

$$= \frac{1}{D-2}\cdot\left[2\log x - \frac{1}{x}\right] = e^{2x}\int e^{-2x}\left(2\log x - \frac{1}{x}\right)dx$$

$$= e^{2x}\left[2\log x\left(-\frac{e^{-2x}}{2}\right)-\int\left(-\frac{e^{-2x}}{2}\cdot\frac{2}{x}\right)dx-\int e^{-2x}\frac{1}{x}\,dx\right]$$

[Or you may use $\int e^{ax}\left[a\,f(x)+f'(x)\right]dx = e^{ax}f(x)$]

$$= -e^{2x}\cdot e^{-2x}\cdot\log x = -\log x.$$

$\therefore$ The complete solution is $y = c_1 e^{-x} + c_2 e^{2x} - \log x$.

Example 13 (b) : Solve $(D^2-1)\,y = (1+e^{-x})^2$.

Sol. : The auxiliary equation is $(D-1)(D+1)=0 \quad \therefore D = 1, -1$.

$\therefore$ The C.F. is $y = c_1 e^{x} + c_2 e^{-x}$.

$$\text{P.I.} = \frac{1}{(D^2-1)}\cdot\frac{1}{(1+e^{-x})^2} = \frac{1}{(D+1)}\cdot\frac{1}{(D-1)}\cdot\frac{e^{2x}}{(e^x+1)^2}$$

$$= \frac{1}{D+1}\cdot e^x\int\frac{e^{-x}\cdot e^{2x}}{(e^x+1)^2}dx = \frac{1}{D+1}\cdot e^x\int\frac{e^x}{(e^x+1)^2}dx$$

$$= \frac{1}{D+1}\cdot e^x(-1)\cdot\frac{1}{(e^x+1)} = -e^{-x}\int e^x\cdot\frac{e^x}{e^x+1}dx$$

$$= -e^{-x}\int\frac{e^{2x}}{(e^x+1)}dx$$

To find the integral put $e^x+1=t$

$$\therefore\ I = \int(t-1)\frac{dt}{t} = \int\left(1-\frac{1}{t}\right)dt = t-\log t$$

$$\therefore\ \text{P.I.} = -e^{-x}\left[e^x+1-\log(e^x+1)\right] = -1-e^{-x}+e^{-x}\log(e^x+1)$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-x} - 1 - e^{-x} + e^{-x}\log(e^x+1).$$

EXERCISE - XI

Solve the following differential equations : Class (b) : 6.Marks

1. $(D^2+a^2)\,y = \operatorname{cosec} ax$ **(M.U. 1997)**

2. $(D^2+2D+1)\,y = 4e^{-x}\log x$ **(M.U. 1997, 99)**

3. $(D^2-3D+2)\,y = \dfrac{1}{e^{e^{-x}}} + \cos\left(\dfrac{1}{e^x}\right)$

4. $(D^2+6D+9)\,y = \dfrac{e^{-3x}}{1+x^2}$

5. $(D^2-1)\,y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$ **(M.U. 2002, 06)**

6. $(D^2-1)\,y = \dfrac{2}{1+e^x}$ **(M.U. 2001)**

[**Ans. : (1) Hint :** Compare with solved Ex. 1 above

$$y = c_1\cos ax + c_2\sin ax + \frac{1}{a^2}\log(\sin ax) - \frac{x}{a}\cos ax$$

(2) $y = (c_1 + c_2 x) e^{-x} + e^{-x} x^2 (2 \log x - 3)$

(3) $y = c_1 e^x + c_2 e^{2x} - x \cdot \frac{1}{e^{e^{-x}}} - e^{2x} \cos e^{-x}$

(4) $y = (c_1 + c_2 x) e^{-3x} + e^{-3x} \left[x \tan^{-1} x - \frac{1}{2} \log (1 + x^2) \right]$

(5) $y = c_1 e^{-x} + c_2 e^x - e^x \sin e^{-x}$

(**Hint :** $\text{P.I.} = \frac{1}{D-1} \cdot \frac{1}{D+1} \left[\cos (e^{-x}) + e^{-x} \sin (e^{-x}) \right]$

$\text{P.I.} = \frac{1}{D-1} \cdot e^{-x} \int e^x \left[\cos (e^{-x}) + e^{-x} \sin (e^{-x}) \right] dx$

$= \frac{1}{D-1} e^{-x} \cdot e^x \cos (e^{-x}) \quad \left[\because \int e^x [f(x) + f'(x)] dx = e^x f(x) \right]$

$= \frac{1}{D-1} \cos (e^{-x}) = e^x \int e^{-x} \cos (e^{-x}) dx = - e^x \sin (e^{-x})$

(6) $y = c_1 e^{-x} + c_2 e^x - e^{-x} \log (1 + e^x) - 1 + e^x \log (e^{-x} + 1)$

(**Hint :** The part e^{-x} coming from P.I. can be absorbed in c_2 of C.F.)

15. Method of Variation of Parameters

The method of variation of parameters which is due to Lagrange is used when other methods fail to give us the particular integral.

Consider the linear differential equation of the second order with constant coefficients.

$$\frac{d^2y}{dx^2} + l \frac{dy}{dx} + my = X \quad \text{.................... (1)}$$

Let the complementary function be .... (2)

$$y = c_1 y_1 + c_2 y_2$$

$$\therefore \quad y_1 \text{ and } y_2 \text{ satisfy } \frac{d^2y}{dx^2} + l \frac{dy}{dx} + m y = 0 \quad \text{.................... (2A)}$$

.... (3)

We **assume** that the particular integral is $y = uy_1 + vy_2$ where u and v are functions of x to be determined. Since in place of the arbitrary constants c_1 and c_2 we use two functions of x viz. u and v which vary the method is called the **variation of parameters.**

Differentiating (3) .... (4)

$$y' = uy'_1 + vy'_2 + u'y_1 + v'y_2$$

Since to determine two unknown functions u, v we need two equations. We assume that .... (5)

$$u'y_1 + v'y_2 = 0$$

This is the second condition, the first being given by (3). Hence from (4) we get, ... (6)

$$y' = uy'_1 + vy'_2$$

Differentiating (6) again, ... (7)

$$y'' = uy_1'' + u'y'_1 + vy_2'' + v'y'_2$$

Putting y, y', y'' from (3), (6), (7) in (1), we get

$$(uy_1''+u'y'_1+vy_2''+v'y'_2)+l(uy'_1+vy'_2)+m(uy_1+vy_2)=X$$

$$u(y_1''+l\,y_1'+my_1)+v(y_2''+l\,y_2'+my_2)+u'y'_1+v'y'_2=X.$$

Since y_1, y_2 satisfy 1(A), we get $\quad u'y'_1+v'y'_2-X=0$ .... (8)

Solving (8) and (5) by Cramer's rule, we get

$$\frac{u'}{y_2X}=-\frac{v'}{y_1X}=\frac{1}{y'_1y_2-y_1y'_2}=-\frac{1}{y_1y'_2-y'_1y_2}$$

If we write $W=\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$

then the above equalities give us

$$u'=-\frac{y_2X}{W} \quad \text{and} \quad v'=\frac{y_1X}{W}$$

$$\therefore \quad \boxed{u=-\int\frac{y_2X}{W}dx, \quad v=\int\frac{y_1X}{W}dx}$$

Note ....

You are advised to use the method of variation of parameters only when asked to do so explicitly.

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2}+a^2y=\sec ax.$$

(M.U. 1995, 99, 2003, 15)

Sol. : The auxiliary equation is $D^2+a^2=0 \quad \therefore \quad D=ai, -ai.$

$\therefore$ The C.F. is $y=c_1\cos ax+c_2\sin ax$.

Here, $y_1=\cos ax$, $y_2=\sin ax$, $X=\sec ax$.

Let P.I. be $y=uy_1+vy_2$

$$\text{Now,} \quad W=\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}=\begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}=a$$

$$\therefore \quad u=-\int\frac{y_2X}{W}dx=-\frac{1}{a}\int\sin ax\cdot\sec ax\,dx$$

$$=-\frac{1}{a}\int\tan ax\,dx=\frac{1}{a^2}\log\cos ax$$

$$\text{and} \quad v=\int\frac{y_1X}{W}dx=\frac{1}{a}\int\cos\ x\cdot\sec ax\,dx=\frac{1}{a}\int dx=\frac{x}{a}$$

$$\therefore \quad \text{P.I.}=u\,y_1+v\,w_2=\frac{1}{a^2}\log\cos ax\cdot\cos ax+\frac{x}{a}\cdot\sin ax$$

$\therefore$ The complete solution is

$$y=c_1\cos ax+c_2\sin ax+\frac{1}{a^2}\log\cos ax\cdot\cos ax+\frac{x}{a}\cdot\sin ax.$$

(For another method see Ex. 1, page 3-43.)

Example 2 (c) : Apply the method of variation of parameters to solve

$$(D^2 - 2D + 2)\, y = e^x \tan x.$$ **(M.U. 2002, 09, 11, 12)**

Sol. : The auxiliary equation is $D^2 - 2D + 2 = 0 \quad \therefore \quad D = 1 \pm i.$

$\therefore$ The C.F. is $y = e^x (c_1 \cos x + c_2 \sin x)$.

Here $y_1 = e^x \cos x,\ y_2 = e^x \sin x,\ X = e^x \tan x.$

Let P.I. be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\sin x + \cos x) \end{vmatrix}$

$$= e^x \cos x \cdot e^x (\sin x + \cos x) - e^x \sin x \cdot e^x (\cos x - \sin x)$$

$\therefore W = e^{2x} (\sin^2 x + \cos^2 x) = e^{2x}$

$$\therefore \quad u = -\int \frac{y_2 X}{W}\, dx = -\int \frac{e^x \sin x \cdot e^x \tan x}{e^{2x}}\, dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{(1 - \cos^2 x)}{\cos x}\, dx = -\int \sec x\, dx + \int \cos x\, dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

and $$v = \int \frac{y_1 X}{W}\, dx = \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}}\, dx = \int \sin x\, dx = -\cos x$$

$\therefore$ P.I. $= uy_1 + vy_2 = -\log(\sec x + \tan x) \cdot e^x \cos x + e^x \sin x \cos x - e^x \cos x \sin x$

$\therefore$ The complete solution is

$$y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \cdot \log(\sec x + \tan x).$$

Example 3 (c) : Solve by the method of variation of parameters **(M.U. 2015)**

$$\frac{d^2 y}{dx^2} + y = \sec x \tan x$$

Sol. : The auxiliary equation is

$$D^2 + 1 = 0 \qquad \therefore \quad D = +i, -i.$$

$\therefore$ The C.F. is $y = c_1 \cos x + c_2 \sin x$.

$\therefore$ $y_1 = \cos x,\ y_2 = \sin x$ and $X = \sec x \tan x$.

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$$\therefore \quad u = -\int \frac{y_2 X}{W}\, dx = -\int \sin x (\sec x \tan x)\, dx$$

$$= -\int \sin x \left(\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \right) dx = -\int \frac{\sin^2 x}{\cos^2 x}\, dx$$

$$= -\int \tan^2 x\, dx = -\int (\sec^2 x - 1)\, dx$$

$$= -\int \sec^2 x\, dx + \int dx = -\tan x + x$$

$$v = \int \frac{y_1 X}{W} dx = \int \cos x \bullet \sec x \tan x \, dx$$

$$= \int \tan x \, dx = \log \sec x$$

$\therefore$ P.I. $= uy_1 + vy_2 = (-\tan x + x)\cos x + \log \sec x \cdot \sin x$

$\therefore$ The complete solution is

$$y = c_1 \cos x + c_2 \sin x + (-\tan x + x)\cos x + \log \sec x \sin x.$$

Example 4 (c) : Use the method of variation of parameters to solve

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}.$$ **(M.U. 1995, 96, 99, 2002, 05, 09, 13)**

Sol. : The auxiliary equation is $D^2 + 3D + 2 = 0$

$\therefore \ (D+1)(D+2) = 0 \quad \therefore \ D = -1, 2.$

$\therefore$ The C.F. is $y = c_1 e^{-x} + c_2 e^{-2x}$.

Here $y_1 = e^{-x}$, $y_2 = e^{-2x}$, $X = e^{e^x}$

Let P.I. be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$

$\therefore \ u = -\int \frac{y_2 X}{W} dx = -\int \frac{e^{-2x} \bullet e^{e^x}}{-e^{-3x}} dx = \int e^{e^x} e^x \, dx = e^{e^x}$ [Put $e^x = t$]

and $v = \int \frac{y_1 X}{W} dx = \int \frac{e^{-x} \bullet e^{e^x}}{e^{-3x}} dx = \int e^{2x} e^{e^x} dx$

Putting $e^x = t$, $v = \int e^t \bullet t \, dt = t e^t - e^t$ $\therefore \ v = e^x e^{e^x} - e^{e^x}$

$\therefore$ P.I. $= uy_1 + vy_2 = e^{e^x} \bullet e^{-x} - (e^x e^{e^x} - e^{e^x}) \bullet e^{-2x} = e^{-2x} \bullet e^{e^x}$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-2x} + e^{-2x} \bullet e^{e^x}.$$

(For another method see Ex. 10, page 3-48.)

Example 5 (c) : Solve the following by the method of variation of parameters·

$$\frac{d^2y}{dx^2} - y = e^{-x}\sin(e^{-x}) + \cos(e^{-x}).$$ **(M.U. 2003)**

Sol. : The auxiliary equation is $D^2 - 1 = 0 \quad \therefore \ D = -1, 1.$

$\therefore$ The C.F. is $y = c_1 e^{-x} + c_2 e^x$.

Here $y_1 = e^{-x}$, $y_2 = e^x$, $X = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

Let P.I. be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = e^0 + e^0 = 2$

$\therefore\ u = -\int \frac{y_2 X}{W} dx = -\frac{1}{2}\int e^x\left[\cos(e^{-x}) + e^{-x}\sin(e^{-x})\right]dx$

$= -\frac{1}{2}e^x \cos(e^{-x}) \qquad \left[\because \int e^x\left[f(x) + f'(x)\right]dx = e^x f(x)\right]$

and $\quad v = \int \frac{y_1 X}{W} dx = \frac{1}{2}\int e^{-x}\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]dx$

For integration, put $e^{-x} = t \quad \therefore -e^{-x}dx = dt$

$v = -\frac{1}{2}\int (t\sin t + \cos t)\ dt = -\frac{1}{2}\left[t(-\cos t) - (1)(-\sin t) + \sin t\right]$

$= \frac{1}{2}t\cos t - \sin t = \frac{1}{2}e^{-x}\cos(e^{-x}) - \sin(e^{-x})$

$\therefore$ P.I. $= uv_1 + vy_2$

$= -\frac{1}{2}e^x\cos(e^{-x})\cdot e^{-x} + \left[\frac{1}{2}\cdot e^{-x}\cos(e^{-x}) - \sin(e^{-x})\right]e^x$

$= -e^x \cdot \sin(e^{-x})$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-x} - e^x\cdot\sin(e^{-x}).$$

Example 6 (c) : Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}. \qquad \textbf{(M.U. 2005, 07)}$$

Sol. : The auxiliary equation is $D^2 + 1 = 0 \qquad \therefore D = i, -i.$

$\therefore$ The C.F. is $y = c_1\cos x + c_2\sin x$.

Here $\quad y_1 = \cos x,\ \ y_2 = \sin x,\ \ X = \frac{1}{1+\sin x}.$

Let P.I. be $\ y = uy_1 + vy_2$

Now, $\quad W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$\therefore\ u = -\int \frac{y_2 X}{W} dx = -\int \frac{\sin x}{1}\cdot\frac{1}{1+\sin x}dx$

$= -\int \frac{\sin x}{1+\sin x}\cdot\frac{(1-\sin x)}{(1-\sin x)}dx = -\int\frac{\sin x(1-\sin x)}{\cos^2 x}dx$ **[Note this]**

$= -\int(\sec x\tan x - \tan^2 x)\,dx = -\int(\sec x\tan x - \sec^2 x + 1)\,dx$

$= -\left[\sec x - \tan x + x\right]$

and $\quad v = \int\frac{y_1 X}{W}dx = \int\frac{\cos x}{1}\cdot\frac{1}{(1+\sin x)}dx = \log(1+\sin x)$

$\therefore$ P.I. $= uy_1 + vy_2 = -\left[\sec x - \tan x + x\right]\cos x + \log(1+\sin x)\cdot\sin x$

$\therefore$ The complete solution is

$$y = c_1\cos x + c_2\sin x - \left[1 - \sin x + x\cos x\right] + \sin x\cdot\log(1+\sin x).$$

Example 7 (c) : Solve by the method of variation of parameters $(D^2+1)\,y = \cot x$.

Sol. : The auxiliary equation is $D^2+1=0$ $\quad\therefore\; D = i, -i.$

$\therefore$ The C.F. is $y = c_1\cos x + c_2 \sin x$.

Here $y_1 = \cos x,\; y_2 = \sin x,\; X = \cot x$.

Let P.I. be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$$\therefore\quad u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{\sin x}{1}\bullet \cot x\,dx = -\int \cos x\,dx = -\sin x$$

and $$v = \int \frac{y_1 X}{W}\,dx = \int \frac{\cos x}{1}\bullet \cot x\,dx = \int \frac{\cos^2 x}{\sin x}\,dx = \int \frac{(1-\sin^2 x)}{\sin x}\,dx$$

$$= \int (\operatorname{cosec} x - \sin x)\,dx = \log(\operatorname{cosec} x - \cot x) + \cos x$$

$\therefore$ P.I. $= uy_1 + vy_2 = -\sin x\cos x + \log(\operatorname{cosec} x - \cot x)\cdot \sin x + \sin x \cos x$.

$\therefore$ The complete solution is $y = c_1 \cos x + c_2 \sin x + \sin x \cdot \log(\operatorname{cosec} x - \cot x)$.

Example 8 (c) : Solve by the method of variation of parameters

$$(D^2 - 6D + 9)\,y = \frac{e^{3x}}{x^2}.$$

(M.U. 2000, 08, 13)

Sol. : The auxiliary equation is $(D-3)^2 = 0$ $\quad\therefore\; D = 3, 3.$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)\,e^{3x} = c_1 e^{3x} + c_2 xe^{3x}$.

Here $y_1 = e^{3x},\; y_2 = xe^{3x},\; X = e^{3x}/x^2$.

Let P.I. be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}$

$$\therefore\quad u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{xe^{3x}\bullet (e^{3x}/x^2)}{e^{6x}}\,dx = -\int \frac{dx}{x} = -\log x$$

and $$v = \int \frac{y_1 X}{W}\,dx = \int \frac{e^{3x}\bullet (e^{3x}/x^2)}{e^{6x}} = \int \frac{dx}{x^2} = -\frac{1}{x}$$

$$\therefore\quad \text{P.I.} = uy_1 + vy_2 = -e^{3x}\bullet \log x - xe^{3x}\bullet \frac{1}{x} = -e^{3x}(\log x + 1)$$

$\therefore$ The complete solution is $y = c_1 e^{3x} + c_2 xe^{3x} - e^{3x}(\log x + 1)$.

Example 9 (c) : Solve by the method of variation of parameters

$$(D^2 - 4D + 4)\,y = e^{2x}\sec^2 x.$$

(M.U. 2008, 10, 14)

Sol. : The auxiliary equation is $(D-2)^2 = 0$ $\quad\therefore\; D = 2, 2.$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)\,e^{2x} = c_1 e^{2x} + c_2 xe^{2x}$.

Here $y_1 = e^{2x},\; y_2 = xe^{2x},\; X = e^{2x}\sec^2 x$.

Let P.I. be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix} = e^{4x}$

$\therefore \quad u = -\int \frac{y_2 X}{W} dx = -\int \frac{xe^{2x} \cdot e^{2x} \sec^2 x}{e^{4x}} dx$

$= -\int x \sec^2 x \, dx = -\left[x \tan x - \int \tan x \cdot 1 \cdot dx\right]$

$= -[x \tan x - \log \sec x]$

and $\quad v = \int \frac{y_1 X}{W} dx = \int \frac{e^{2x} \cdot e^{2x} \sec^2 x}{e^{4x}} dx = \int \sec^2 x \, dx = \tan x$

$\therefore \quad$ P.I. $= uy_1 + vy_2 = e^{2x} \cdot \log \sec x - x e^{2x} \tan x - x e^{2x} \tan x = e^{2x} \cdot \log \sec x$

$\therefore$ The complete solution is $\quad y = c_1 e^{2x} + c_2 xe^{2x} + e^{2x} \cdot \log \sec x$.

Example 10 (c) : Solve $(D^2 - 1) y = \frac{2}{\sqrt{1-e^{-2x}}}$. **(M.U 2007)**

Sol. : The auxiliary equation is $D^2 - 1 = 0 \quad \therefore \; D = +1, -1$.

$\therefore$ The C.F. $y = c_1 e^x + c_2 e^{-x}$.

Here $y_1 = e^x, \; y_2 = e^{-x}, \quad X = \frac{2}{\sqrt{1-e^{-2x}}}$. Let P.I. $= uy_1 + vy_2$.

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$

$\therefore \quad u = -\int \frac{y_2 X}{W} dx = -\int e^{-x} \cdot \frac{2}{\sqrt{1-e^{-2x}}} \cdot \frac{1}{-2} dx$

$= \int \frac{e^{-x}}{\sqrt{1-e^{-2x}}} dx = \int \frac{-dt}{\sqrt{1-t^2}}$. $\qquad$ Put $e^{-x} = t$.

$= -\sin^{-1}(t) = -\sin^{-1}(e^{-x})$.

$\therefore \quad uy_1 = -e^x \sin^{-1}(e^{-x})$

$v = \int \frac{y_1 X}{W} dx = \int e^x \cdot \frac{2}{\sqrt{1-e^{-2x}}} \cdot \frac{1}{-2} dx$

$= \int \frac{e^x}{\sqrt{1-e^{-2x}}} dx = \int \frac{e^x \cdot e^x}{\sqrt{e^{2x}+1}} dx$

(Multiply by e^x in the numerator and denominator)

Put $e^x = t, \;\; \therefore \quad I = \int \frac{t\,dt}{\sqrt{t^2+1}} = \sqrt{t^2+1} = \sqrt{e^{2x}+1}$

$v \cdot y_2 = e^{-x} \sqrt{e^{2x}+1} = \sqrt{1+e^{-2x}}$.

$\therefore \quad$ P.I. $= uy_1 + vy_2 = -e^x \cdot \sin^{-1} e^{-x} + \sqrt{1+e^{-2x}}$

$\therefore$ The complete solution is

$y = c_1 e^x + c_2 e^{-x} - e^x \sin(e^{-x}) + \sqrt{1+e^{-2x}}$.

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Example 11 (c) : Solve $(D^2-1)\,y=\dfrac{2}{1+e^x}$. **(M.U. 1997, 2002, 03,**

Sol. : The auxiliary equation is

$$D^2-1=0 \quad \therefore\ (D-1)(D+1)=0 \quad \therefore\ D=1,-1.$$

$\therefore$ The C.F. is $y=c_1e^x+c_2e^{-x}$.

Hence, $y_1=e^x$, $y_2=e^{-x}$, and $X=\dfrac{2}{1+e^x}$.

Now, $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}=-2e^0=-2.$

$$\therefore\quad u=-\int\frac{y_2X}{W}dx=-\int\frac{e^{-x}}{-2}\cdot\frac{2}{1+e^x}dx=\int\frac{e^{-x}}{1+e^x}dx$$

Put $e^{-x}=t \quad \therefore\ -e^{-x}dx=dt$

$$\therefore\quad u=-\int\frac{dt}{1+(1/t)}=-\int\frac{t}{1+t}dt$$

$$=-\int\frac{(t+1)-1}{t+1}dt=-\int 1\,dt+\int\frac{dt}{1+t}$$

$$=-t+\log(1+t)=-e^{-x}+\log(1+e^{-x})$$

$$v=\int\frac{y_1X}{W}dx=\int\frac{e^x}{-2}\cdot\frac{2}{1+e^x}dx=-\int\frac{e^x}{1+e^x}dx=-\log(1+e^x)$$

$$\therefore\quad \text{P.I.}=uy_1+vy_2=[-e^{-x}+\log(1+e^{-x})]e^x+[-\log(1+e^x)]e^{-x}$$

$\therefore$ The complete solution is

$$y=c_1e^x+c_2e^{-x}-1+e^x\cdot\log(1+e^{-x})-e^{-x}\cdot\log(1+e^x).$$

Example 12 (c) : Use the method of variation of parameters to solve the equation

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=e^{-2x}\cdot\sec^2x\,(1+2\tan x).$$

(M.U. 2010, 14)

Sol. : The auxiliary equation is $D^2+5D+6=0$

$\therefore\ (D+2)(D+3)=0 \quad \therefore\ D=-2,-3.$

$\therefore$ C.F. is $y=c_1e^{-2x}+c_2e^{-3x}$.

Here $y_1=e^{-2x}$, $y_2=e^{-3x}$, $X=e^{-2x}\sec^2x\,(1+2\tan x)$. Let P.I. $=uy_1+vy_2$

Now, $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix}=-e^{-5x}$

$$\therefore\quad u=-\int\frac{y_2X}{W}dx=-\int\frac{e^{-3x}\cdot e^{-2x}}{-e^{-5x}}\sec^2x\cdot(1+2\tan x)\,dx$$

$$=\int(1+2\tan x)\sec^2x\,dx=\frac{1}{4}(1+2\tan x)^2 \qquad [\text{Put } 1+2\tan x=t]$$

$$v=\int\frac{y_1X}{W}dx=\int\frac{e^{-2x}\cdot e^{-2x}\cdot\sec^2x\,(1+2\tan x)}{-e^{-5x}}$$

$$=-\int e^x\cdot[1+2\tan x]\cdot\sec^2x\,dx$$

$$=-\int e^x(\sec^2x+2\sec^2x\tan x)\,dx$$

Let $f(x) = \sec^2 x$ $\quad \therefore \quad f'(x) = 2\sec^2 x \tan x.$

Now, $\int e^x [f(x) + f'(x)]\,dx = e^x f(x)$ $\quad \therefore \quad v = -e^x \sec^2 x$

$$\therefore \text{ P.I. } = uy_1 + vy_2$$

$$= \frac{1}{4}(1+2\tan x)^2 \cdot e^{-2x} - e^x \sec^2 x \cdot e^{-3x}$$

$$= \frac{e^{-2x}}{4}(1+4\tan x+4\tan^2 x) - e^{-2x}(1+\tan^2 x)$$

$$\therefore \text{ P.I. } = \frac{e^{-2x}}{4}(1+4\tan x+4\tan^2 x-4-4\tan^2 x) = \frac{e^{-2x}}{4}(4\tan x-3)$$

$\therefore$ The complete solution is $y = c_1 e^{-2x} + c_2 e^{-3x} + \dfrac{e^{-2x}}{4}(4\tan x - 3)$.

Example 13 (c) : Apply the method of variation of parameters to solve $(D^3 + D)\,y = \operatorname{cosec} x$.

(M.U. 1997, 2005, 08)

Sol. : The auxiliary equation is $D(D^2+1) = 0$ $\quad \therefore \quad D = 0, i, -i.$

$\therefore$ The C.F. is $y = c_1 + c_2 \cos x + c_3 \sin x$.

Here $y_1 = 1,\ y_2 = \cos x,\ y_3 = \sin x,\ X = \operatorname{cosec} x.$

Let P.I. be $y = uy_1 + vy_2 + wy_3$.

$$\text{Now, } W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= \sin^2 x + \cos^2 x = 1$$

$$\therefore \quad u = \int \frac{(y_2 y'_3 - y_3 y'_2)X}{W}\,dx = \int (\cos^2 x + \sin^2 x)\operatorname{cosec} x\,dx$$

$$= \int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x)$$

$$v = \int \frac{(y_3 y'_1 - y_1 y'_3)X}{W}\,dx = \int (\sin x \cdot 0 - 1 \cdot \cos x) \cdot \operatorname{cosec} x\,dx$$

$$= -\int \cot x\,dx = -\log \sin x$$

$$\text{and} \quad w = \int \frac{(y_1 y'_2 - y_2 y'_1)X}{W}\,dx = \int [1 \cdot (-\sin x) - 0 \cdot \cos x] \operatorname{cosec} x\,dx$$

$$= \int -dx = -x$$

$$\therefore \text{ P.I. } = uy_1 + vy_2$$

$$= \log(\operatorname{cosec} x - \cot x)\,1 - \log \sin x \cdot \cos x - x\sin x.$$

$\therefore$ The complete solution is

$$y = c_1 + c_2 \cos x + c_3 \sin x + \log(\operatorname{cosec} x - \cot x) - \log \sin x \cdot \cos x - x \sin x.$$

Note ....

Because the above equation is of third order the formulae used have got to be modified as above.

Example 14 (c) : Apply the method of variation of parameters to solve $(D^3 + 4D)\, y = 4 \cot 2x$

Sol. : The A.E. is $D^3 + 4D = 0 \quad \therefore \; D(D^2+4) = 0 \quad \therefore \; D = 0, 2i, -2i.$

$\therefore$ C.F. is $y = c_1 + c_2 \cos 2x + c_3 \sin 2x.$

Here, $y_1 = 1,\; y_2 = \cos 2x,\; y_3 = \sin 2x,\; X = 4 \cot 2x.$

Let P.I. be $y = uy_1 + vy_2 + wy_3$.

$$\text{Now, } w = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= 8(\sin^2 2x + \cos^2 2x) = 8$$

$$\therefore \; u = \int \frac{(y_2 y_3' - y_3 y_2')}{W} \cdot X\, dx = \int \frac{2(\cos^2 2x + \sin^2 2x)}{8} \cdot 4 \cot 2x\, dx$$

$$\therefore \; u = \int \cot 2x\, dx = \frac{1}{2} \log \sin 2x$$

$$\text{and} \quad v = \int \frac{(y_3 y_1' - y_1 y_3')}{W} \cdot X\, dx = \int \frac{(\sin 2x \cdot 0 - 1 \cdot 2\cos 2x)}{8} \cdot 4 \cot 2x\, dx$$

$$= -\int \frac{\cos^2 2x}{\sin 2x}\, dx = -\int \frac{(1-\sin^2 2x)}{\sin 2x}\, dx = -\int (\operatorname{cosec} 2x - \sin 2x)\, dx$$

$$\therefore \; v = -\left(\frac{1}{2} \log \tan x + \frac{1}{2} \cos 2x\right)$$

$$\text{and} \quad w = \int \frac{(y_1 y_2' - y_2 y_1')}{W} \cdot X\, dx = \int \frac{(-2\sin 2x - 0)}{8} \cdot 4 \cot 2x\, dx$$

$$= -\int \cos 2x\, dx = -\frac{1}{2} \sin 2x$$

$$\therefore \; \text{P.I.} = uy_1 + vy_2 + wy_3$$

$$= \frac{1}{2} \log \sin 2x \cdot 1 - \left(\frac{1}{2} \log \tan x + \frac{1}{2} \cos 2x\right) \cdot \cos 2x - \frac{1}{2} \sin 2x (\sin 2x)$$

$\therefore$ The complete solution is

$$y = c_1 + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{2} \cdot 1 \cdot \log \sin 2x$$

$$- \frac{1}{2} \log \tan x \cdot \cos 2x - \frac{1}{2} \cos^2 2x - \frac{1}{2} \sin^2 2x$$

$$= c_1 + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{2} \log \sin 2x - \frac{1}{2} \log \tan x \cdot \cos 2x - \frac{1}{2}$$

$$= c_1 + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{2} \log \sin 2x - \frac{1}{2} \log \tan x \cdot \cos 2x \qquad \left[\because \; c_1 - \frac{1}{2} = c_1\right]$$

Example 15 (c) : Apply the method of variation of parameters to solve

$$(D^3 - 6D^2 + 12D - 8)\, y = \frac{e^{2x}}{x}.$$

Sol. : The auxiliary equation is

$$D^3 - 6D^2 + 12D - 8 = 0 \quad \therefore \; (D-2)^3 = 0 \quad \therefore \; D = 2, 2, 2.$$

$\therefore$ The C.F. is $y = (c_1 + c_2 x + c_3 x^2)\, e^{2x}$

$$= c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}.$$

Here, $y_1 = e^{2x}$, $y_2 = x\,e^{2x}$, $y_3 = x^2\,e^{2x}$ and $X = \dfrac{e^{2x}}{x}$.

Now, $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{2x} & x\,e^{2x} & x^2\,e^{2x} \\ 2\,e^{2x} & (2x+1)\,e^{2x} & (2x^2+2x)\,e^{2x} \\ 4\,e^{2x} & 4\,(x+1)\,e^{2x} & (4x^2+8x+2)\,e^{2x} \end{vmatrix}$

$$= e^{6x} \begin{vmatrix} 1 & x & x^2 \\ 2 & 2x+1 & 2x^2+2x \\ 4 & 4\,(x+1) & 4x^2+8x+2 \end{vmatrix} = 2\,e^{6x} \begin{vmatrix} 1 & x & x^2 \\ 2 & 2x+1 & 2x^2+2x \\ 2 & 2x+2 & 2x^2+4x+1 \end{vmatrix}$$

By $R_2 - 2R_1$ and $R_3 - R_2$

$$W = 2\,e^{6x} \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 1 & 2x+1 \end{vmatrix} = 2\,e^{6x}\,(2x+1-2x) = 2\,e^{6x}$$

Now, P.I. $= uy_1 + vy_2 + wy_3$

$$\therefore \quad u = \int \frac{(y_2\,y_3' - y_3\,y_2')\,X}{W}\,dx\,.$$

$$= \int \frac{[x\,e^{2x}\,(2x^2+2x)\,e^{2x} - x^2\,e^{2x}\,(2x+1)\,e^{2x}]}{2\,e^{6x}} \cdot \frac{e^{2x}}{x}\,dx$$

$$= \int \frac{x}{2}\,dx = \frac{x^2}{4}$$

$$v = \int \frac{(y_3\,y_1' - y_1\,y_3')\,X}{W}\,dx$$

$$= \int \frac{[x^2\,e^{2x} \cdot 2\,e^{2x} - e^{2x}\,(2x^2+2x)\,e^{2x}]}{2\,e^{6x}} \cdot \frac{e^{2x}}{x}\,dx$$

$$= \int -\,dx = -\,x$$

and $\quad w = \displaystyle\int \frac{(y_1\,y_2' - y_2\,y_1')\,X}{W}\,dx$

$$= \int \frac{e^{2x}\,(2x+1)\,e^{2x} - x\,e^{2x} \cdot 2\,e^{2x}}{2\,e^{6x}} \cdot \frac{e^{2x}}{x}\,dx$$

$$= \int \frac{1}{2x}\,dx = \frac{1}{2}\log x$$

$\therefore$ P.I. $= uy_1 + vy_2 + wy_3$

$$= \frac{x^2}{4} \cdot e^{2x} - x \cdot x\,e^{2x} + \frac{1}{2}\log x \cdot x^2\,e^{2x}$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2\,x + c_3\,x^2)\,e^{2x} - \frac{3x^2}{4} \cdot e^{2x} + \frac{x^2}{2}\log x \cdot e^{2x}$$

EXERCISE - XII

Solve the following differential equations by the method of variation of parameters : Class (: 8 Marks

1. $\dfrac{d^2y}{dx^2}+k^2y=\tan kx$ **(M.U. 1998, 2004)**
2. $\dfrac{d^2y}{dx^2}+4y=\tan 2x$ **(M.U. 2013)**
3. $\dfrac{d^2y}{dx^2}+y=x\sin x$
4. $(D^2+1)\,y=\operatorname{cosec} x$
5. $(D^2+2D+5)\,y=4e^{-x}\tan 2x+5\,e^{-x}$ **(M.U. 1997, 2002, 03, 12)**
6. $(D^2-1)\,y=\operatorname{sec}h\,x$
7. $(D^2+1)\,y=3x-8\cot x$
8. $(D^2+D)\,y=\dfrac{1}{1+e^x}$ **(M.U. 1997, 2003)**
9. $(D^2-1)\,y=e^x\sin 2x$
10. $(D^2-3D+2)\,y=\dfrac{e^x}{1+e^x}$
11. $(D^2-4D+4)\,y=\dfrac{e^{2x}}{x}$
12. $(D^2+2D+1)\,y=e^{-x}\log x$
13. $(D^2+6D+5)\,y=16\,e^{3x}$
14. $(D^2+1)\,y=x-\cot x$ **(M.U. 1997, 2003)**
15. $(D^2+3D+2)\,y=\sin e^x$
16. $(D^2+a^2)\,y=a^2\sec^2 ax$
17. $(D^2-4D+4)\,y=e^{2x}\sec^2 x$
18. $(D^2+3D+2)\,y=\dfrac{1}{1+e^x}$ **(M.U. 2011)**

[**Ans. :** (1) $y=c_1\cos kx+c_2\sin kx-\dfrac{1}{k^2}\cos kx\cdot\log(\sec kx+\tan kx)$

(2) $y=c_1\cos 2x+c_2\sin 2x-\dfrac{1}{4}\cos 2x\cdot\log(\sec 2x+\tan 2x)$

(3) $y=c_1\cos x+c_2\sin x+\dfrac{x}{4}\sin x-\dfrac{x^2}{4}\cos x+\dfrac{1}{8}\cos x$

(4) $y=c_1\cos x+c_2\sin x-x\cos x+\sin x\cdot\log\sin x$

(5) $y=e^{-x}(c_1\cos 2x+c_2\sin 2x)-e^{-x}\cos 2x\cdot\log(\sec 2x+\tan 2x)+\dfrac{5}{4}e^{-x}$

(6) (**Hint :** Take C.F. as $y=c_1\cos hx+c_2\sin hx$)

$y=c_1\cos hx+c_2\sin hx-\cos hx\log\cos hx+x\sin hx$

(7) $y=c_1\cos x+c_2\sin x+3x-8\sin x\cdot\log(\operatorname{cosec} x-\cot x)$

(8) $y=c_1+c_2e^{-x}-\log(1+e^{-x})-e^{-x}\log(1+e^x)$

or $y=c_1+c_2e^{-x}-(1+e^{-x})\log(1+e^x)+x$

(9) $y=c_1e^x+c_2e^{-x}-\dfrac{1}{8}e^x(\sin 2x+\cos 2x)$

(10) $y=c_1e^x+c_2e^{2x}-e^x+(e^x+e^{2x})\log(e^{-x}+1)$

(11) $y=c_1e^{2x}+c_2xe^{2x}-xe^{2x}+x\log x\cdot e^{2x}$

(12) $y = c_1 e^{-x} + c_2 xe^{-x} - e^{-x}\left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) + xe^{-x}(x \log x - x)$

(13) $y = c_1 e^{-x} + c_2 e^{-5x} + \frac{1}{2}e^{3x}$

(14) $y = c_1 \cos x + c_2 \sin x + x - \sin x \cdot \log(\operatorname{cosec} x - \cot x)$

(15) $y = c_1 e^{-x} + c_2 e^{2x} - e^{-2x} \cdot \sin e^{x}$

(16) $y = c_1 \cos ax + c_2 \sin ax - 1 + \sin ax \cdot \log(\sec ax + \tan ax)$

(17) $y = c_1 e^{2x} + c_2 xe^{2x} + e^{2x} \log \sec x.$

(18) $y = c_1 e^{-x} + c_2 e^{-x} + (e^{-x} - e^{-2x}) \log(1 + e^{x}) + e^{-2x}(1 + e^{x}).$]

Miscellaneous Examples : Class (b) : 6 Marks

Example 1 (b) : Solve $(D^3 + 1) = e^{x/2} \sin\left(\frac{\sqrt{3}}{2}x\right)$. **(M.U. 2007)**

Sol. : The auxiliary equation is $D^3 + 1 = 0$.

$\therefore\ (D+1)(D^2 - D + 1) = 0 \quad \therefore\ D = -1, \frac{1 \pm \sqrt{3} \cdot i}{2}$

$\therefore$ The C.F. is $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$.

$$\text{P.I.} = \frac{1}{D^3+1} e^{x/2} \cdot \sin\frac{\sqrt{3}x}{2} = e^{x/2} \frac{1}{[D + (1/2)]^3 + 1} \sin\frac{\sqrt{3}}{2}x$$

$$\therefore\ \text{P.I.} = e^{x/2} \frac{1}{D^3 + (3/2)D^2 + (3/4)D + (9/8)} \sin\frac{\sqrt{3}}{2}x$$

If we put $D^2 = -\frac{3}{4}$ the denominator vanishes. **[Note this]**

Hence by § 9 (a), since $\Phi'(D)^2 = 3D^2 + 3D + \frac{3}{4}$, putting $D^2 = -\frac{3}{4}$, we get

$$\text{P.I.} = e^{x/2} \frac{x}{3(-3/4) + 3D + (3/4)} \sin\frac{\sqrt{3}}{2}x$$

$$= e^{x/2} \frac{x}{3D - (3/2)} \sin\frac{\sqrt{3}}{2}x = e^{x/2} \cdot x \frac{3D + (3/2)}{9D^2 - (9/4)} \sin\frac{\sqrt{3}}{2}x$$

$$= \frac{e^{x/2} \cdot x \cdot \left[3 \cdot (\sqrt{3}/2)\cos(\sqrt{3}/2)x + (3/2)\sin(\sqrt{3}/2)x\right]}{-9}$$

$$= -\frac{x\, e^{x/2}}{6}\left[\sqrt{3}\cos(\sqrt{3}/2)x + \sin(\sqrt{3}/2)x\right]$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right) - \frac{x e^{x/2}}{6}\left[\sqrt{3} \cos\left(\sqrt{3}/2\right) x + \sin\left(\sqrt{3}/2\right) x\right].$$

Example 2 (b) : Solve $(D^3 + D^2 + D + 1) = \sin^2 x.$ **(M.U. 1990, 2008)**

Sol. : The auxiliary equation is $D^3 + D^2 + D + 1 = 0$

$\therefore\ (D^2 + 1)(D + 1) = 0 \qquad \therefore\ D = \pm i, -1.$

$\therefore$ The C.F. is $y = c_1 \cos x + c_2 \sin x + c_3 e^{-x}.$

$$\text{P.I.} = \frac{1}{D^3 + D^2 + D + 1} \sin^2 x = \frac{1}{D^3 + D^2 + D + 1} \cdot \frac{(1 - \cos 2x)}{2}$$

$$= \frac{1}{D^3 + D^2 + D + 1} \cdot \frac{1}{2} e^0 x + \frac{1}{D^3 + D^2 + D + 1}\left(-\frac{1}{2}\right) \cos 2x$$

$$= \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{-4D - 4 + D + 1} \cos 2x$$

$$= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3D + 3} \cos 2x = \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{D + 1} \cdot \frac{D - 1}{D - 1} \cos 2x$$

$$= \frac{1}{2} + \frac{1}{6} \cdot \frac{D - 1}{D^2 - 1} \cos 2x = \frac{1}{2} + \frac{1}{6} \cdot \frac{-2 \sin 2x - \cos 2x}{-4 - 1}$$

$$= \frac{1}{2} + \frac{1}{30} (2 \sin 2x + \cos 2x)$$

$\therefore$ The complete solution is

$$y = c_1 \cos x + c_2 \sin x + c_3 e^{-x} + \frac{1}{2} + \frac{1}{30} (2 \sin 2x + \cos 2x).$$

Example 3 (c) : Solve $(D^2 + a^2)\, y = x \sin ax.$

Sol. : The auxiliary equation is $D^2 + a^2 = 0 \qquad \therefore\ D = +ai, -ai.$

$\therefore$ The C.F. is $y = c_1 \cos ax + c_2 \sin ax.$

$$\text{P.I.} = \frac{1}{D^2 + a^2} x \sin ax = \text{I.P. of } \frac{1}{D^2 + a^2} e^{iax} \cdot x$$

$$= \text{I.P. of } e^{iax} \frac{1}{(D + ia)^2 + a^2} \cdot x = \text{I.P. of } e^{iax} \frac{1}{D^2 + 2aiD} \cdot x$$

$$= \text{I.P. of } e^{iax} \frac{1}{2aiD}\left[1 + \frac{D}{2ai}\right]^{-1} x = \text{I.P. of } e^{iax} \frac{1}{2aiD}\left[1 - \frac{D}{2ai}\right] x$$

$$= \text{I.P. of } e^{iax} \frac{1}{2ai} \int\left(x - \frac{1}{2ai}\right) dx = \text{I.P. of } e^{iax} \frac{1}{2ai}\left[\frac{x^2}{2} - \frac{x}{2ai}\right]$$

$$= \text{I.P. of } (\cos ax + i \sin ax)\left(-\frac{x^2}{4a} i + \frac{x}{4a^2}\right)$$

$$= -\frac{x^2}{4a} \cos ax + \frac{x}{4a^2} \sin ax.$$

$\therefore$ The complete solution is

$$y = c_1 \cos ax + c_2 \sin ax - \frac{x^2}{4a} \cos ax + \frac{x}{4a^2} \sin ax.$$

Example 4 (b) : Solve $(D^2 - 1)\, y = xe^x \sin x$.

Sol. : The auxiliary equation is $D^2 - 1 = 0$ $\qquad \therefore \; D = 1, -1$.

$\therefore$ The C.F. is $y = c_1 e^x + c_2 e^{-x}$.

$$\text{P.I.} = \frac{1}{D^2 - 1} e^x x \sin x = e^x \cdot \frac{1}{(D+1)^2 - 1} x \sin x$$

$$= e^x \cdot \frac{1}{D^2 + 2D} x \sin x = \text{I.P. of } e^x \cdot \frac{1}{D^2 + 2D} e^{ix}\, x$$

$$= \text{I.P. of } e^x \cdot e^{ix} \frac{1}{(D+i)^2 + 2(D+i)} x$$

$$= \text{I.P. of } e^x \cdot e^{ix} \frac{1}{D^2 + 2iD - 1 + 2D + 2i} x$$

$$= \text{I.P. of } e^x \cdot e^{ix} \frac{1}{(-1+2i) + 2D(1+i) + D^2} x$$

$$= \text{I.P. of } e^x \cdot e^{ix} \frac{1}{(-1+2i)} \left[1 + \frac{2D(1+i)}{(-1+2i)} + \ldots\right]^{-1} x$$

$$= \text{I.P. of } e^x \cdot e^{ix} \frac{1}{(-1+2i)} \left[x - \frac{2(1+i)}{(-1+2i)}\right]$$

$$= \text{I.P. of } e^x \cdot e^{ix} \frac{(1+2i)}{(-1+2i)(1+2i)} \left[x - \frac{2(1+i)(1+2i)}{(-1+2i)(1+2i)}\right]$$

$$= \text{I.P. of } e^x \cdot e^{ix} \frac{(1+2i)}{-5} \left[x - \frac{2(-1+3i)}{-5}\right]$$

$$= \text{I.P. of } e^x \cdot e^{ix} \frac{(1+2i)}{-5} \left[x + \frac{2(-1+3i)}{5}\right]$$

$$= \text{I.P. of } \frac{e^x}{-5} (\cos x + i \sin x)(1+2i) \cdot \left[x + \frac{2(-1+3i)}{5}\right]$$

$$= \text{I.P. of } \frac{e^x}{-5} (\cos x + i \sin x) \left[x + 2ix + \frac{2(-7+i)}{5}\right]$$

$$\therefore \quad \text{P.I.} = \frac{-e^x}{5} \left[x(\sin x + 2\cos x) + \frac{2}{5}(\cos x - 7\sin x)\right]$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-x} - \frac{e^x}{5} \left[x(\sin x + 2\cos x) + \frac{2}{5}(\cos x - 7\sin x)\right].$$

(M.U. 1992, 2002)

Example 5 (b) : Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = \left(x^2 e^x\right)^2$.

Sol. : The auxiliary equation is $D^2 - 4D + 3 = 0$

$\therefore\ (D-1)(D-3)=0 \qquad \therefore\ D=1,3.$

$\therefore$ The C.F. is $y=c_1e^x+c_2e^{3x}$.

$$\text{P.I.} = \frac{1}{D^2-4D+3}\,e^{2x}\cdot x^4 = e^{2x}\cdot\frac{1}{(D+2)^2-4(D+2)+3}\cdot x^4$$

$$= e^{2x}\cdot\frac{1}{D^2-1}\,x^4 = -e^{2x}(1-D^2)^{-1}\cdot x^4$$

$$= -e^{2x}\cdot\left[1+D^2+D^4+\ldots\right]\cdot x^4$$

$$= -e^{2x}\cdot(x^4+12x^2+24)$$

$\therefore$ The complete solution is $y=c_1e^x+c_2e^{3x}-e^{2x}\cdot(x^4+12x^2+24)$.

Example 6 (b) : Solve $\dfrac{d^2y}{dx^2}+y=\sin x\sin 2x+2^x$. **(M.U. 1992)**

Sol. : The auxiliary equation is $D^2+1=0 \qquad \therefore\ D=i,-i$.

$\therefore$ The C.F. is $y=c_1\cos x+c_2\sin x$.

$$\text{P.I.}=\frac{1}{D^2+1}\left[\sin x\sin 2x+2^x\right]$$

Now, $$\frac{1}{D^2+1}(\sin x\sin 2x)=\frac{1}{D^2+1}\left[-\frac{1}{2}(\cos 3x-\cos x)\right]$$

$$=-\frac{1}{2}\cdot\frac{1}{D^2+1}\cos 3x+\frac{1}{2}\cdot\frac{1}{D^2+1}\cos x$$

$$=-\frac{1}{2}\cdot\frac{1}{(-8)}\cos 3x+\frac{1}{2}\cdot\frac{x}{2}\sin x$$

And $$\frac{1}{D^2+1}\cdot 2^x=\frac{1}{D^2+1}e^{x\log 2}=\frac{1}{(\log 2)^2+1}e^{x\log 2}=\frac{1}{(\log 2)^2+1}\cdot 2^x$$

$\therefore$ The complete solution is

$$y=c_1\cos x+c_2\sin x+\frac{1}{16}\cos 3x+\frac{x}{4}\sin x+\frac{1}{(\log 2)^2+1}\cdot 2^x.$$

Example 7 (b) : Solve $(D^3+2D^2+D)\,y=x^2e^{3x}+\sin^2 x+2^x$. **(M.U. 2013)**

Sol. : The auxiliary equation is

$D^3+2D^2+D=0 \qquad \therefore\ D(D^2+2D+1)=0$

$\therefore\ D(D+1)^2=0 \qquad \therefore\ D=0,-1,-1.$

$\therefore$ C.F. is $y=c_1+(c_2+c_3x)\,e^{-x}$

$$\therefore\ \text{P.I.}=\frac{1}{D^3+2D^2+D}(x^2e^{3x}+\sin^2 x+2^x)$$

Now, $$\frac{1}{D^3+2D^2+D}e^{3x}\cdot x^2=e^{3x}\cdot\frac{1}{(D+3)^3+2(D+3)+(D+3)}\cdot x^2$$

$$=e^{3x}\cdot\frac{1}{D^3+9D^2+27D+27+2D+6+D+3}\cdot x^2$$

$$=e^{3x}\cdot\frac{1}{D^3+9D^2+30D+36}\cdot x^2$$

$$\frac{1}{D^3+2D^2+D}e^{3x}\cdot x^2 = e^{3x}\cdot\frac{1}{D^3+9D^2+30D+36}\cdot x^2$$

$$= e^{3x}\cdot\frac{1}{36\left(1+\frac{5}{6}D+\frac{1}{4}D^2+\frac{D^3}{36}\right)}\cdot x^2$$

$$= \frac{e^{3x}}{36}\left[1+\frac{5}{6}D+\frac{1}{4}D^2+\frac{1}{36}D^3\right]^{-1}\cdot x^2$$

$$= \frac{e^{3x}}{36}\left[1-\frac{5}{6}D-\frac{1}{4D^2}-\frac{1}{36}D^3+\frac{25}{36}D^2+\ldots\right]x^2+$$

$$= \frac{e^{3x}}{36}\left[1-\frac{5}{6}D+\frac{4}{9}D^2-\ldots\right]x^2$$

$$= \frac{e^{3x}}{36}\left[x^2-\frac{5}{3}x+\frac{8}{9}\right]$$

And $\frac{1}{D^3+2D^2+D}\sin^2 x = \frac{1}{D^3+2D^2+D}\left(\frac{1-\cos 2x}{2}\right)$

Now, $\frac{1}{D^3+2D^2+D}\cdot\frac{1}{2}e^{0x} = \frac{1}{2}\cdot\frac{1}{D(D+1)(D+1)}\cdot 1$

[The method of § 9, page 3-13 fails here. Hence, we use (5), page 3-10.]

$$= \frac{1}{2}\cdot\frac{1}{D(D+1)}\cdot e^{-x}\int e^x\cdot 1\,dx$$

$$= \frac{1}{2}\cdot\frac{1}{D(D+1)}\cdot e^{-x}\cdot e^x = \frac{1}{2}\cdot\frac{1}{D(D+1)}\cdot 1$$

$$= \frac{1}{2}\cdot\frac{1}{D}\cdot e^{-x}\int e^x\,dx = \frac{1}{2}\cdot\frac{1}{D}e^{-x}\cdot e^x$$

$$= \frac{1}{2}\cdot\frac{1}{D}\cdot 1 = \frac{1}{2}\int 1\cdot dx = \frac{x}{2}$$

And $\frac{1}{D^3+2D^2+D}\cdot\cos 2x = \frac{1}{D(-4)+2(-4)+D}\cdot\cos 2x$

$$= \frac{1}{-3D-8}\cdot\cos 2x = -\frac{3D-8}{9D^2-64}\cdot\cos 2x$$

$$= -\frac{3D-8}{9(-4)-64}\cdot\cos 2x = -\frac{-6\sin 2x-8\cos 2x}{-100}$$

$$= -\frac{1}{100}(6\sin 2x+8\cos 2x)$$

And $\frac{1}{D^3+2D^2+D}\cdot 2^x = \frac{1}{(\log 2)^3+2(\log 2)^2+\log 2}\cdot 2^x$ [By 2 (B), page 3-14]

$\therefore$ The complete solution is

$$y = c_1+(c_2+c_3x)e^{-x}+\frac{e^{3x}}{36}\left[x^2-\frac{5}{3}x+\frac{8}{9}\right]+\frac{1}{2}x-\frac{1}{100}(3\sin 2x+4\cos 2x)$$
$$+\frac{1}{(\log 2)^3+2(\log 2)^2+\log 2}\cdot 2^x.$$

MISCELLANEOUS EXERCISE

Solve the following differential equations : Class (b) : 6 Marks

1. $(D^4+4)\,y=0$
2. $(D^4-2D^3+3D^2-2D+1)\,y=0$
3. $(D^4+2D^3+3D^2+2D+1)\,y=0$
4. $(D^2-4D+4)\,y=4(e^{2x}-\cos 2x)$
5. $(D^2+1)\,y=\sin x\sin 2x$ **(M.U. 2009)**
6. $(D^2-(a+b)\,D+ab)\,y=e^{ax}+e^{bx}$ **(M.U. 1998, 2001,**
7. $(D^4-2D^3+D^2)\,y=x^3$ **(M.U. 1994)**
8. $(D^2-5D+6)\,y=x(x+e^x)$ **(M.U. 19**
9. $(D^3-3D^2+4D-2)\,y=e^x+\cos x$
10. $(D^2+1)^2y=\sin x+\cos x$
11. $(D^2-2D+1)\,y=e^x+\sin(\sqrt{3}\,)\,x$ **(M.U. 1992)**
12. $(D^2-2D+1)\,y=e^x$
13. $(D^2+2D+1)\,y=\cos^2 x$
14. $(D^2+D+1)\,y=(1+\sin x)^2$
15. $(D^2-2D+4)\,y=e^x\cos^2 x$
16. $(D^3+3D)\,y=\cos h2x\ \sin h3x$
17. $(D^4-1)\,y=e^x\cos x$
18. $(D^2+D-6)\,y=e^{2x}\sin 3x$ **(M.U. 1997**
19. $(D^2+3D+2)\,y=e^{2x}\sin x$
20. $(D^3-3D-2)\,y=540\,x^3e^{-x}$
21. $(D^2+2D+2)\,y=2e^{-x}\sin x$
22. $(D^2+a^2)\,y=x\cos ax$
23. $(D^3-2D^2+5D+26)\,y=e^x\cos^2 2x$
24. $(D^2-1)\,y=\cos hx\ \cos x+a^x$
25. $\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+5y=e^x\cos 3x$ **(M.U. 1996)**
26. $\dfrac{d^3y}{dx^3}-y=(1+e^x)^2$ **(M.U. 1995)**
27. $\dfrac{d^2y}{dx^2}+2y=x^2e^{3x}+e^x\cos 3x$ **(M.U. 1993)**
28. $(D^2+3D+2)\,y=xe^{-x}\sin x$
29. $(D^2-8D+16)\,y=\dfrac{e^{4x}}{x^2}$ **(M.U. 1994)**
30. $(D^2-6D+9)\,y=\dfrac{e^{3x}}{x^2}$ **(M.U. 2002)**
31. $(D^2-6D+13)\,y=8e^{3x}\sin 4x+2^x$
32. $(D^2+6D+9)\,y=\dfrac{1}{x^3}e^{-3x}+2^x$ **(M.U. 2006)**
33. $(D^2+5D+6)\,y=e^{-2x}\sin 2x+4x^2e^x$
34. $(D^2+9)\,y=4[\cos(\pi/3)+x]$. Given that $y=0$ at $x=0$ and $y=2$ at $x=\pi/6$.
35. $(D^2+4D+4)\,y=xe^{-x}\sin hx$
36. $(D^4-1)\,y=\cos x\cos hx$
37. $(D^2-2D+2)\,y=e^x(x+\sin x)$ **(M.U. 1991)**
38. Find $\dfrac{1}{D^2-2D+2}e^x(x+\sin x)$ **(M.U. 1991)**
39. Find $\dfrac{1}{D^2+a^2}(\sin ax+\cos ax)$ **(M.U. 1991)**
40. $(D^2-4D+4)\,y=\dfrac{e^{2x}}{1+x^2}$. **(M.U. 2004)**

41. $(D^2+6D+9)\,y = \sin h\,3x$ (M.U. 2004)

42. $(D-2)^2 y = 8\,[e^{2x} + \sin 2x + x^2]$ (M.U. 2002)

43. $(D^3-7D-6)\,y = (1+x^2)\,e^{2x}$ (M.U. 2007)

44. $(D^2-4D+4)\,y = e^{2x} + x^3 + \cos 2x.$ (M.U. 2003)

45. $(D^2+4)\,y = \cos x \cos 2x \cos 3x$

46. $(D^2-3D+2)\,y = \sin e^{-x}$

[Ans. : (1) $y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^{x}(c_3 \cos x + c_4 \sin x)$

(2) $y = e^{-x/2}\left[(c_1 + c_2 x) \cos \frac{\sqrt{3}}{2} \cdot x + (c_3 + c_4 x) \sin \frac{\sqrt{3}}{2} \cdot x\right]$

(3) $y = e^{x/2}\left[(c_1 + c_2 x) \cos \frac{\sqrt{3}}{2} \cdot x + (c_3 + c_4 x) \sin \frac{\sqrt{3}}{2} \cdot x\right]$

(4) $y = (c_1 + c_2 x)\, e^{2x} + 2x^2 e^{2x} + \frac{1}{2} \sin 2x$

(5) $y = c_1 \cos x + c_2 \sin x + \frac{1}{4} x \sin x + \frac{1}{16} \cos 3x$

(6) $y = c_1 e^{ax} + c_2 e^{-ax} + \frac{x}{a-b}\left[e^{ax} - e^{bx}\right]$

(7) $y = (c_1 + c_2 x) + (c_3 + c_4 x)\, e^{x} + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$

(8) $y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{6}\left(x^2 + \frac{5}{3} x + \frac{19}{18}\right) + \frac{xe^x}{2} + \frac{3}{4} e^x$

(9) $y = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x) + xe^x + \frac{3 \sin x + \cos x}{10}$

(10) $y = (c_1 + c_2 x)(c_3 \cos x + c_4 \sin x) - \frac{x^2}{8} (\sin x + \cos x)$

(11) $y = (c_1 + c_2 x)\, e^x + \frac{1}{8}\left(\sqrt{3} \cdot \cos \sqrt{3} \cdot x - \sin \sqrt{3} \cdot x\right)$

(12) $y = (c_1 + c_2 x)\, e^x + \frac{x^2}{2} e^{2x}$

(13) $y = (c_1 + c_2 x)\, e^{-x} + \frac{1}{50} (4 \sin 2x - 3 \cos 2x) + \frac{1}{2}$

(14) $y = e^{-x/2}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right) + \frac{3}{2} - 2 \cos x + \frac{1}{26} (3 \cos 2x - 2 \sin 2x)$

(15) $y = e^x (c_1 \cos \sqrt{3}\, x + c_2 \sin \sqrt{3}\, x) + \frac{1}{6} e^x - \frac{1}{2} e^x \cos 2x$

(16) $y = c_1 + c_2 \cos \sqrt{3}\, x + c_3 \sin \sqrt{3}\, x + \frac{1}{4}\left[\frac{1}{140}\left(e^{5x} + e^{-5x}\right) + \frac{1}{4}\left(e^x + e^{-x}\right)\right]$

(17) $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{e^x}{5} \cos x$

(18) $y = c_1 e^{2x} + c_2 e^{-3x} - \frac{1}{306} e^{2x}(15\cos 3x + 9\sin 3x)$

(19) $y = c_1 e^{-x} + c_2 e^{-2x} - \frac{e^{2x}}{170}(7\cos x - 11\sin x)$

(20) $y = c_1 e^{2x} + (c_2 + c_3 x) e^{-x} - (20x^2 + 20x^3 + 15x^4 + 9x^5) e^{-x}$

(21) $y = e^{-x}(c_1 \cos x + c_2 \sin x - x\cos x)$

(22) $y = c_1 \cos ax + c_2 \sin ax + \frac{1}{4a^2}\left[ax^2 \sin ax + x\cos ax\right]$

(23) $y = c_1 e^{-2x} + e^{2x}(c_2 \cos 3x + c_3 \sin 3x) - \frac{1}{2500} e^{x}(7\cos 4x - 24\sin 4x) + \frac{1}{60} e^{x}$

(24) $y = c_1 e^{x} + c_2 e^{-x} + \frac{2}{5}\sin hx \sin x - \frac{1}{5}\cos hx \cos x + \frac{a^x}{(\log\ a)^2 - 1}$

(25) $y = c_1 e^{-x} + e^{x}(c_2 \cos 2x + c_3 \sin 2x) - \frac{e^x}{65}(3\sin 3x + 2\cos 3x)$

(26) $y = c_1 e^{x} + e^{-x/2}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\right)x + c_3 \sin\left(\frac{\sqrt{3}}{2}\right)x\right] - 1 + \frac{2}{3} xe^{x} + \frac{1}{7} e^{2x}$

(27) $y = c_1 \cos\sqrt{2}\, x + c_2 \sin\sqrt{2}\, x + \frac{e^{3x}}{11}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right] + \frac{e^x}{6}(\sin 3x - \cos 3x)$

(28) $y = c_1 e^{-x} + c_2 e^{-2x} - \frac{e^{-x}}{2}\left[x(\sin x + \cos x) + (\cos x - 2\sin x)\right]$

(29) $y = c_1 \cos 4x + c_2 \sin 4x - e^{4x}\log x$ (30) $y = (c_1 + c_2 x) e^{3x} - e^{3x}\log x$

(31) $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - \frac{2}{3} e^{3x}\sin 4x + \frac{1}{(\log\ 2)^2 - 6\log 2 + 13} \cdot 2^x$

(32) $y = (c_1 + c_2 x) e^{-3x} + \frac{1}{2x} \cdot e^{-3x} + \frac{1}{(3 + \log\ 2)^2} \cdot 2^x$

(33) $y = c_1 e^{-2x} + c_2 e^{-3x} - \frac{e^{-2x}}{20}(2\cos 2x + \sin 2x) + \frac{1}{3} e^{-x}\left(x^2 - \frac{7}{6}x + \frac{25}{144}\right)$

(34) $y = -\frac{1}{4}\cos 3x + \frac{8 + \sqrt{3}}{4}\sin 3x + \frac{1}{4}\cos x - \frac{\sqrt{3}}{4}\sin x$

(35) $y = (c_1 + c_2 x) e^{-2x} + \frac{x}{8} - \frac{1}{8} - \frac{1}{12} e^{-2x} \cdot x^3$

(36) $y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}\cos x \cos hx$

(37) $y = e^{x}(c_1 \cos x + c_2 \sin x) + xe^{x} - \frac{x}{2} e^{x}\cos x$

(38) $y = xe^{x}\left(1 - \frac{1}{2}\sin x\right)$ (39) $y = \frac{x}{2a}(\sin ax - \cos ax)$

(40) $y = (c_1 + c_2 x) e^{2x} + e^{2x}\left[x\tan^{-1} x - \frac{1}{2}\log(1 + x^2)\right]$

(41) $y = (c_1 + c_2 x)\, e^{-3x} + \frac{1}{2}\left[\frac{e^{3x}}{36} + \frac{x^2}{2} e^{-3x}\right]$

(42) $y = (c_1 + c_2 x)\, e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$

(43) $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{12} \cdot e^{2x}\left(\frac{169}{72} + \frac{5x}{6} + x^2\right)$

(44) $y = (c_1 + c_2 x)\, e^{2x} + \frac{x^2}{2} e^{2x} + \frac{1}{4}\left[x^3 + 3x^2 + \frac{9x}{2} - 3\right] - \frac{\sin 2x}{8}$.

(45) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{16} - \frac{1}{128} \cos 6x - \frac{1}{48} \cos 4x + \frac{x}{16} \sin 2x$

(46) $y = c_1 e^x + c_2 e^{2x} + e^{2x}\,(e^{-x} \cos e^{-x} - \sin e^{-x}) - e^x \cos e^{-x}$.

EXERCISE - XIII

(A) Solve the following differential Equations : Class (a) : 3 marks

1. $(D^3 - 3D^2 + 3D - 1)\, y = 0$
2. $(D^4 - 2D^3 + 3D^2 - 2D + 1)\, y = 0$
3. $[(D-1)^4 (D^2 + 1)^2]\, y = 0$
4. $(D^4 - 2D^3 + D^2)\, y = 0$
5. $(D^6 + 2D^4 + D^2)\, y = 0$

[Ans.: (1) $y = (c_1 + c_2 x + c_3 x^2)\, e^x$

(2) $y = e^{x/2}\left[(c_1 + c_2 x) \cos \frac{\sqrt{3}}{2} \cdot x + (c_3 + c_4 x) \sin \frac{\sqrt{3}}{2} \cdot x\right]$

(3) $y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)\, e^x + (c_5 + c_6 x) \cos x + (c_7 + c_8 x) \sin x$

(4) $y = c_1 + c_2 x + (c_3 + c_4 x)\, e^x$

(5) $y = c_1 + c_2 x + (c_3 + c_4 x) \cos x + (c_5 + c_6 x) \sin x$]

(B) Find the Particular Integrals of : Class (a) : 3 Marks

1. $(D-4)\, y = x$
2. $(D^2 - 5D + 6)\, y = c^{2x}$
3. $(D+2)\, y = \sin x$
4. $(D+1)\, y = \log x + \frac{1}{x}$
5. $(D+1)\, y = \frac{1-x}{x^2}$
6. $(D^3 - 3D^2)\, y = e^{4x}$
7. $(D^3 - D^2)\, y = x$
8. $(D^3 - 3D^2 + 4)\, y = e^{2x}$
9. $(D+1)^3\, y = e^{-x}$
10. $\left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y\right) = 2e^{3x}$
11. $(D^3 - 1)\, y = 2 \cos h\, x$
12. $(D^4 - 81)\, y = \sin 3x$
13. $(D^2 - D + 1)\, y = \cos x$
14. $(D^4 - 16)\, y = \sin 2x$
15. $(D^2 - D + 1) \sin 2x$
16. $(D-1)^2 (D^2 + 1)^2\, y = \sin^2 \frac{x}{2}$
17. $(D^2 + 1)^2\, y = \sin x$
18. $(D^3 - D)\, y = 2x + 1$
19. $(D^4 - 2D^3 + D^2)\, y = x^2$
20. $(D-1)^3\, y = x e^x$
21. $(D^2 + D - 6)\, y = e^{2x} \sin 3x$
22. $(D^2 - 2D + 1)\, y = \frac{e^x}{x^2}$.
23. $(D+1)\, y = e^{e^x}$

[Ans. : (1) $-\dfrac{x}{4}-\dfrac{1}{16}$ (2) $e^{2x}-e^{2x}\cdot x$] (3) $e^{-2x}\left[\dfrac{1}{4+1}e^{2x}(2\sin x-\cos x)\right]$

(4) $\log x$ (5) $-\dfrac{1}{x}$ (6) $\dfrac{e^{4x}}{16}$ (7) $-\left(\dfrac{x^2}{6}+\dfrac{x^2}{2}\right)$ (8) $\dfrac{1}{6}\cdot e^{2x}$

(9) $\dfrac{x^3}{3!}\cdot e^{-x}$ (10) $\dfrac{1}{8}e^{3x}$ (11) $x\cos hx$ (12) $\dfrac{x}{162}\cos 3x$

(13) $-\sin x$ (14) $\dfrac{x}{32}\cos 2x$ (15) $\dfrac{1}{13}(2\cos 2x-3\sin 2x)$

(16) $\dfrac{1}{2}-\dfrac{x^2}{16}\sin x$ (17) $-\dfrac{x^2}{8}\sin x$ (18) $-(x^2+x)$ (19) $x^4+\dfrac{2x^3}{3}+3x^2$

(20) $\dfrac{e^x\cdot x^4}{24}$ (21) $-\dfrac{1}{102}\cdot e^{2x}(5\cos 3x+3\sin 3x)$

(22) $-e^x\log x$ (23) $e^{-x}\cdot e^{e^x}$]

Summary

1. If the roots of the **auxiliary equation** $D^n+P_1D^{n-1}+\ldots\ldots+P_n=0$ are $m_1, m_2, \ldots\ldots, m_n$ then the **complementary function** is

$$y=c_1e^{m_1x}+c_2e^{m_2x}+\ldots\ldots+c_ne^{m_nx}$$

(a) If the roots of the auxiliary equation are repeated then

$$y=(c_1+c_2x)e^{m_1x}$$

$$y=(c_1+c_2x+c_3x^2)e^{m_1x} \quad \text{etc.}$$

(b) If the roots of the auxiliary equation are imaginary then

$$y=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$$

(c) If imaginary roots are repeated then

$$y_1=e^{\alpha x}[(c_1+c_2x)\cos\beta x+(c_3+c_4x)\sin\beta x]$$

(d) If the roots of the auxiliary equation are irrational then

$$y_1=e^{\alpha x}[c_1\cos h\sqrt{\beta}\cdot x+c_2\sin h\sqrt{\beta}\cdot x]$$

(e) If the irrational roots are repeated then

$$y=e^{\alpha x}[(c_1+c_2x)\cos h\sqrt{\beta}\cdot x+(c_3+c_4x)\sin h\sqrt{\beta}\cdot x]$$

2. **Particular Integral,**

$$\text{P.I.}=\frac{1}{f(D)}\cdot X$$

(a) $\text{P.I.}=\dfrac{1}{D}\cdot X=\int X\,dx$ **(b)** $\text{P.I.}=\dfrac{1}{D-a}\cdot X=e^{ax}\int e^{-ax}X\,dx$

and $\text{P.I.}=\dfrac{1}{D+a}\cdot X=e^{-ax}\int e^{ax}X\,dx$

3. If $X = e^{ax}$ then $\frac{1}{f(D)} \cdot e^{ax} = \frac{1}{f(a)} \cdot e^{ax}$.

(a) If $(D-a)$ is a factor of $f(D)$, then $\frac{1}{f(D)} \cdot e^{ax} = x \cdot \frac{1}{\Phi(a)} \cdot e^{ax}$

(b) If $(D-a)$ is a factor of $f(D)$ repeated r times, then $\frac{1}{f(D)} \cdot e^{ax} = \frac{x^r}{r!} \cdot \frac{1}{\Psi(a)} \cdot e^{ax}$

(c) $\frac{1}{f(D)} \cdot k = \frac{k}{f(0)}$, k is a constant.

4. If $X = \sin ax$ (or $\cos ax$), then

$$\frac{1}{\Phi(D^2)} \sin ax = \frac{1}{\Phi(-a^2)} \cdot \sin ax$$

$$\frac{1}{\Phi(D^2)} \cdot \cos ax = \frac{1}{\Phi(-a^2)} \cdot \cos ax$$

(a) If $(D^2 + a^2)$ is a factor of $\Phi(D^2)$ then

$$\frac{1}{\Phi(D^2)} \cdot \sin ax = x \cdot \frac{1}{\Phi'(D^2)} \cdot \sin ax$$

$$\frac{1}{\Phi(D^2)} \cdot \cos ax = x \cdot \frac{1}{\Phi'(D^2)} \cdot \cos ax$$

(b) In particular

$$\frac{1}{D^2 + a^2} \cdot \sin ax = -\frac{x}{2a} \cdot \cos ax$$

$$\frac{1}{D^2 + a^2} \cdot \cos ax = \frac{x}{2a} \cdot \sin ax$$

5. If $X = x^m$, then

$$\frac{1}{f(D)} \cdot x^m = \frac{1}{1 + \Phi(D)} \cdot x^m = [1 + \Phi(D)]^{-1} \cdot x^m$$

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + ...$$

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + ...$$

6. If $X = e^{ax} V$, then $\frac{1}{f(D)} \cdot e^{ax} V = e^{ax} \cdot \frac{1}{f(D+a)} V$

7. If $X = xV$, then $\frac{1}{f(D)} xV = \left\{x - \frac{1}{f(D)} \cdot f'(D)\right\} \cdot \frac{1}{f(D)} V$

8. **Variation of Parameters**

$$y = u\, y_1 + v\, y_2$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad u = -\int \frac{y_2 X}{W} dx, \quad v = \int \frac{y_1 X}{W} dx$$

CHAPTER

4

Cauchy's Homogeneous Linear Differential Equations

1. Introduction

In this chapter we shall study linear differential equations whose coefficients are not constants but powers of x of a particular type. For example, we shall study differential equations of the type

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+3y=X \qquad \text{where } X \text{ is a function of } x \text{ only.}$$

Definition

An equation of the form

$$x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\ldots\ldots+a_{n-1}x\frac{dy}{dx}+a_ny=X$$

where $a_1, a_2, \ldots\ldots a_n$ are constants and X is a function of x is called **Cauchy's homogeneous linear differential equation** of order n. The equation is also known as Cauchy's equation.

Augustin Louis (Baron de) Cauchy (1789 - 1857)

A French mathematician of great repute who contributed to various branches of mathematics. He wanted to be an engineer but because of poor health he was advised to pursue mathematics. His mathematical work began in 1811 when he gave brilliant solutions to some difficult problems of that time. In the next 35 years he published 700 papers in various branches of mathematics. He is supposed to have initiated the era of modern analysis.

2. Method of Solution

The equation can be transformed into an equation with constant coefficients by the substitution

$z=\log x$ or $x=e^z$.

Now, $\because z=\log x$, $\dfrac{dz}{dx}=\dfrac{1}{x}$ and $\dfrac{dy}{dx}=\dfrac{dy}{dz}\cdot\dfrac{dz}{dx}=\dfrac{1}{x}\cdot\dfrac{dy}{dz}$

$$\frac{d^2y}{dx^2}=-\frac{1}{x^2}\cdot\frac{dy}{dz}+\frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}=-\frac{1}{x^2}\cdot\frac{dy}{dz}+\frac{1}{x^2}\cdot\frac{d^2y}{dz^2}$$

$$=\frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$$

Similarly, it can be shown that

$$\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) \quad \text{.................... (iii)}$$

and so on .

If we put $D = \frac{d}{dz}$ then we get, from (i), (ii), (iii),

$$x\frac{dy}{dx} = \frac{dy}{dz} = Dy$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D^2y - Dy = D(D-1)\,y$$

$$x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz} = D^3y - 3D^2y + 2Dy$$

$$= D(D^2 - 3D + 2)\,y = D(D-1)(D-2)\,y$$

and so on.

Further the r.h.s. X by the substitution of $x = e^z$ changes to a function of z only say Z. Thus, the given equation by the substitution $x = e^z$ changes to a linear differential equation with constant coefficients of the form $f(D)\,y = Z$ and can be solved by the methods studied in the previous chapter.

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Find the complementary function of $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - k^2y = 0.$

Sol. : Putting $z = \log x$ and $x = e^z$, we get, as seen above

$[D(D-1) + D - k^2]\,y = 0 \quad \therefore \ (D^2 - k^2)\,y = 0$

$\therefore$ A.E. is $D^2 - k^2 = 0 \quad \therefore \ D = k, -k.$

$\therefore$ The C.F. is $y = c_1e^{kz} + c_2e^{-kz}.$

Resubstituting in terms of x, the C.F. and hence the complete solution is

$$y = c_1x^k + c_2x^{-k}.$$

Example 2 (a) : Find the complementary function of $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0.$

Sol. : Putting $z = \log x$ and $x = e^z$, we get (as seen above)

$[D(D-1) + 4D + 2]\,y = 0 \quad \therefore \ (D^2 + 3D + 2)\,y = 0$

$\therefore$ A.E. is $D^2 + 3D + 2 = 0$

$\therefore$ $(D+2)(D+1) = 0 \quad \therefore \ D = -1, -2.$

$\therefore$ The C.F. is $y = c_1e^{-z} + c_2e^{-2z}.$

Resubstituting in terms of x, we get $y = \frac{c_1}{x} + \frac{c_2}{x^2}.$

Example 3 (a) : Find the complementary function of $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2.$

Sol. : Putting $z = \log x$ and $x = e^z$, we get (as seen above)

Example 2 (c) : Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x.$

Sol. : Putting $z = \log x$ and $x = e^z$, we get

$[D(D-1) - D + 1]\, y = 2z \qquad \therefore\ (D^2 - 2D + 1)\, y = 2z$

$\therefore$ The A.E. is $D^2 - 2D + 1 = 0 \qquad \therefore\ (D-1)^2 = 0 \quad \therefore\ D = 1, 1.$

$\therefore$ The C.F. is $y = (c_1 + c_2 z)\, e^z$.

$$\text{P.I.} = 2 \cdot \frac{1}{1 - 2D + D^2} z = 2[1 - (2D - D^2)]^{-1} z$$

$$= 2[1 + 2D - D^2 + \ldots\ldots]\, z = 2[z + 2]$$

$\therefore$ The complete solution is $y = (c_1 + c_2 z)\, e^z + 2(z+2)$.

Resubstituting in terms of x, we get,

$$y = (c_1 + c_2 \log x) \cdot x + 2(\log x + 2).$$

Example 3 (c) : Solve $x^3 \frac{d^3y}{dx^3} - x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = x^3 + 3x.$ **(M.U. 1991)**

Sol. : Putting $z = \log x$ and $x = e^z$, we get

$$[D(D-1)(D-2) - D(D-1) + 2D - 2]\, y = e^{3z} + 3e^z$$

$\therefore\ (D^3 - 3D^2 + 2D - D^2 + D + 2D - 2)\, y = e^{3z} + 3e^z$

$\therefore\ (D^3 - 4D^2 + 5D - 2)\, y = e^{3z} + 3e^z$

$\therefore$ The A.E. is $D^3 - 4D^2 + 5D - 2 = 0.$

$\therefore\ D^3 - D^2 - 3D^2 + 3D + 2D - 2 = 0 \qquad \therefore\ (D-1)(D^2 - 3D + 2) = 0$

$\therefore\ (D-1)(D-1)(D-2) = 0 \qquad \therefore\ D = 1, 1, 2$

$\therefore$ The C.F. is $y = (c_1 + c_2 z)\, e^z + c_3 e^{2z}$.

$$\therefore\ \text{P.I.} = \frac{1}{(D-1)^2 (D-2)} e^{3z} + \frac{1}{(D-1)^2 (D-2)} 3e^z$$

$$= \frac{1}{(3-1)^2 (3-2)} e^{3z} + \frac{z^2}{2} \cdot \frac{1}{(1-2)} 3e^z$$

$$= \frac{e^{3z}}{4} - \frac{z^2}{2} 3e^z$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 z)\, e^z + c_3 e^{2z} + \frac{e^{3z}}{4} - \frac{3z^2}{2} e^z$$

$$\therefore\ y = (c_1 + c_2 \log x)\, x + c_3 x^2 + \frac{x^3}{4} - \frac{3x}{2} (\log x)^2.$$

Example 4 (c) : Solve $x^2 \frac{d^2y}{dx} - x\frac{dy}{dx} + 2y = x \log x.$

Sol. : Putting $z = \log x$ and $x = e^z$, we get

$[D(D-1) - D + 2]\, y = ze^z \quad \therefore\ (D^2 - 2D + 2)\, y = ze^z$

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$\therefore$ The A.E. is $D^2 - 2D + 2 = 0$. $\therefore$ $D = 1 \pm i$

$\therefore$ The C.F. is $y = e^z (c_1 \cos z + c_2 \sin z)$.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2} e^z \cdot z$$

[By (3), page 3-31]

$$= e^z \cdot \frac{1}{(D+1)^2 - 2(D+1) + 2} \cdot z$$

$$= e^z \cdot \frac{1}{D^2 + 1} z = e^z \cdot [1 + D^2]^{-1} z$$

$$= e^z [1 - D^2 + ...] z = e^z \cdot z$$

$\therefore$ The complete solution is

$$y = e^z (c_1 \cos z + c_2 \sin z) + e^z \cdot z$$

$$\therefore \quad y = x (c_1 \cos \log x + c_2 \sin \log x) + x \log x.$$

Example 5 (c) : Solve $x^2 \dfrac{d^3y}{dx^3} + 3x \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 \log x$. **(M.U. 2002, 16)**

Sol. : We are required to multiply the equation throughout by x in order to make it **homogeneous** linear.

$$\therefore \quad x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = x^3 \log x$$

Putting $z = \log x$, $x = e^z$, we get

$$[D(D-1)(D-2) + 3D(D-1) + D]\, y = e^{3z} \cdot z$$

$$\therefore \quad [D(D^2 - 3D + 2) + 3D(D-1) + D]\, y = e^{3z} \cdot z$$

$$\therefore \quad [D^3 - 3D^2 + 2D + 3D^2 - 3D + D]\, y = e^{3z} \cdot z$$

$$\therefore \quad D^3 y = e^{3z} \cdot z$$

$\therefore$ The auxiliary equation is $D^3 = 0$ $\quad\therefore$ $D = 0, 0, 0$.

The complementary function is $y = c_1 + c_2 z + c_3 z^2$.

$$\therefore \quad \text{P.I.} = \frac{1}{D^3} \cdot e^{3z} \cdot z = e^{3z} \cdot \frac{1}{(D+3)^3} \cdot z$$

[By (3), page 3-31]

$$= e^{3z} \cdot \frac{1}{D^3 + 9D^2 + 27D + 27} \cdot z = e^{3z} \cdot \frac{1}{27\left(1 + D + \frac{1}{3}D^2 + \frac{D^3}{27}\right)} \cdot z$$

$$= e^{3z} \cdot \frac{1}{27}\left[1 + D + \frac{1}{3}D^2 + \frac{D^3}{27}\right]^{-1} z = e^{3z} \cdot \frac{1}{27}\left(1 - D - \frac{1}{3}D^2 .....\right) z$$

$$= e^{3z} \cdot \frac{1}{27}(z - 1)$$

$\therefore$ The complete solution is

$$y = c_1 + c_2 z + c_3 z^2 + \frac{1}{27} \cdot e^{3z}(z - 1)$$

$$\therefore \quad y = c_1 + c_2 \log x + c_3 (\log x)^2 + \frac{1}{27} x^3 (\log x - 1).$$

Example 6 (c) : Solve $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 12y = x^3 \log x$.

Sol. : Putting $z = \log x$, $x = e^z$, we get

$$[\,D(D-1) + 2D - 12\,]\,y = e^{3z} \cdot z \qquad \therefore\ (D^2 + D - 12)\,y = e^{3z} \cdot z$$

$\therefore$ The A.E. is $D^2 + D - 12 = 0$

$\therefore\ (D+4)(D-3) = 0$ $\qquad \therefore\ D = 3, -4.$

$\therefore$ C.F. is $y = c_1 e^{3z} + c_2 e^{-4z}$.

$$\text{P.I.} = \frac{1}{(D^2 + D - 12)} e^{3z} \cdot z = e^{3z} \cdot \frac{1}{(D+3)^2 + (D+3) - 12} \cdot z \qquad [\text{By (3), page 3-31}]$$

$$= e^{3z} \cdot \frac{1}{D^2 + 7D} \cdot z = e^{3z} \cdot \frac{1}{7D[1 + (D/7)]} \cdot z$$

$$= e^{3z} \cdot \frac{1}{7D}\left[1 + \frac{D}{7}\right]^{-1} z = e^{3z} \cdot \frac{1}{7D}\left[1 - \frac{D}{7} + \ldots\right] z$$

$$= e^{3z} \cdot \frac{1}{7D}\left[z - \frac{1}{7}\right] = e^{3z} \cdot \frac{1}{7}\left[\int z\,dz - \frac{1}{7}\int dz\right]$$

$$= e^{3z} \cdot \frac{1}{7}\left[\frac{z^2}{2} - \frac{z}{7}\right] = e^{3z} \cdot \frac{z}{98}(7z - 2)$$

$\therefore$ The complete solution is

$$y = c_1 e^{3z} + c_2 e^{-4z} + e^{3z} \cdot \frac{z}{98}(7z - 2)$$

$$\therefore\ y = c_1 x^3 + c_2 x^{-4} + \frac{x^3 \log x}{98}(7 \log x - 2).$$

Example 7 (c) : Solve $(x^2 D^2 + 4x D + 2)\, y = \sin x$.

Sol. : Putting $z = \log x$, $x = e^x$, we get

$$[\,D(D-1) + 4D + 2\,]\,y = \sin e^z \qquad \therefore\ (D^2 + 3D + 2)\,y = \sin e^z$$

$\therefore$ The A.E. is $D^2 + 3D + 2 = 0$

$\therefore\ (D+1)(D+2) = 0$ $\qquad \therefore\ D = -1, -2.$

$\therefore$ C.F. is $y = c_1 e^{-z} + c_2 e^{-2z}$.

$$\therefore\ \text{P.I.} = \frac{1}{D^2 + 3D + 2} \cdot \sin e^z = \frac{1}{(D+1)(D+2)} \cdot \sin e^z$$

$$= \frac{1}{D+2} \cdot \frac{1}{D+1} \cdot \sin e^z$$

$$= \frac{1}{D+2} \cdot e^{-z} \int e^z \sin e^z\, dz \qquad [\text{Put } e^z = t \text{ and } e^z\,dz = dt\,]$$

$$= \frac{1}{D+2} \cdot e^{-z}(-\cos e^z) = -e^{-2z} \int e^{2z} \cdot e^{-z} \cos e^z\, dz$$

$$= -e^{-2z} \int e^z \cos e^z dz = -e^{-2z} \sin e^z \qquad [\text{Put } e^z = t\,]$$

$\therefore$ The complete solution is

$$y = c_1 e^{-z} + c_2 e^{-2z} - e^{-2z} \sin e^z \qquad \therefore\ y = \frac{c_1}{x} + \frac{c_2}{x^2} - \frac{\sin x}{x^2}.$$

Example 8 (c) : Solve $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = x^2 \sin\log x.$ **(M.U. 1988, 91, 94)**

Sol. : As seen in the example 1 above

$$\text{C.F.} = e^{2z}(c_1 \cos z + c_2 \sin z)$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 5} e^{2z} \cdot \sin z = e^{2z} \cdot \frac{1}{(D+2)^2 - 4(D+2) + 5} \sin z$$

$$= e^{2z} \cdot \frac{1}{D^2+1} \sin z = e^{2z} \cdot \left(\frac{-z}{2}\right) \cos z \qquad \text{[By (2), page 3-19]}$$

$\therefore$ The complete solution is

$$y = e^{2z}(c_1 \cos z + c_2 \sin z) - \frac{1}{2} e^{2z} \cdot z \cos z$$

$$\therefore \quad y = x^2 (c_1 \cos\log x + c_2 \sin\log x) - \frac{1}{2} x^2 \log x \cos\log x.$$

Example 9 (c) : Solve $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 4y = \cos\log x + x \sin\log x.$ **(M.U. 1998, 2010)**

Sol. : Putting $z = \log x$ and $x = e^z$, we get

$$[D(D-1) - D + 4]\, y = \cos z + e^z \sin z \qquad \therefore \quad (D^2 - 2D + 4)\, y = \cos z + e^z \sin z$$

$\therefore$ The A.E. is $D^2 - 2D + 4 = 0.$ $\qquad \therefore \quad D = 1 \pm \sqrt{3} \cdot i$

$\therefore$ The C.F. is $y = e^z (c_1 \cos\sqrt{3} \cdot z + c_2 \sin\sqrt{3} \cdot z)$.

$$\text{P.I. for } \cos z = \frac{1}{D^2 - 2D + 4} \cos z = \frac{1}{3 - 2D} \cos z$$

$$= \frac{1}{9 - 4D^2}(3 + 2D)\cos z = \frac{1}{13}(3 + 2D)\cos z$$

$$= \frac{1}{13}(3\cos z - 2\sin z)$$

$$\text{P.I. for } e^z \sin z = \frac{1}{D^2 - 2D + 4} e^z \sin z$$

$$= e^z \frac{1}{(D+1)^2 - 2(D+1) + 4} \cdot \sin z$$

$$= e^z \frac{1}{D^2 + 3} \sin z = e^z \cdot \frac{1}{2} \sin z.$$

$\therefore$ The complete solution is

$$y = e^z (c_1 \cos\sqrt{3} \cdot z + c_2 \sin\sqrt{3} \cdot z) + \frac{1}{13}(3\cos z - 2\sin z) + e^z \cdot \frac{1}{2} \sin z$$

i.e. $\quad y = x[c_1 \cos(\sqrt{3}\log x) + c_2 \sin(\sqrt{3}\log x)] + \dfrac{1}{13}(3\cos\log x - 2\sin\log x) + \dfrac{x}{2}\sin(\log x).$

Example 10 (c) : Solve $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{(\sin\log x) + 1}{x}.$ **(M.U. 2002, 09)**

Sol. : Putting $z = \log x$ and $x = e^z$, we get

$$[D(D-1)-3D+1]\,y=(\sin z+1)\cdot e^{-z}$$

$\therefore$ $(D^2-4D+1)\,y=e^{-z}\sin z+e^{-z}$

$\therefore$ The A.E. is $D^2-4D+1=0$ $\quad\therefore\ D=2\pm\sqrt{3}$.

$\therefore$ The C.F. is $y=Ae^{(2+\sqrt{3})z}+Be^{(2-\sqrt{3})z}$.

$\therefore$ $y=e^{2z}(Ae^{\sqrt{3}\cdot z}+Be^{-\sqrt{3}\cdot z})$ which can be expressed as

$y=e^{2z}(c_1\cosh\sqrt{3}\cdot z+c_2\sinh\sqrt{3}\cdot z)$

P.I. for $e^{-z}=\dfrac{1}{D^2-4D+1}e^{-z}=\dfrac{1}{6}e^{-z}$ [See § 7, page 3-7]

P.I. for $e^{-z}\sin z=e^{-z}\cdot\dfrac{1}{(D-1)^2-4(D-1)+1}\sin z$

$$=e^{-z}\cdot\frac{1}{D^2-6D+6}\sin z=e^{-z}\cdot\frac{1}{5-6D}\sin z$$

$$=e^{-z}\cdot\frac{5+6D}{25-36D^2}\sin z=e^{-z}\frac{(5\sin z+6\cos z)}{61}$$

$\therefore$ The complete solution is

$$y=e^{2z}(c_1\cosh\sqrt{3}\cdot z+c_2\sinh\sqrt{3}\cdot z)+\frac{1}{6}e^{-z}+\frac{e^{-z}}{61}(5\sin z+6\cos z)$$

$$\therefore\ y=x^2[c_1\cosh(\sqrt{3}\log x)+c_2\sinh(\sqrt{3}\log x)]+\frac{1}{6x}+\frac{1}{61x}[5\sin(\log x)+6\cos(\log x)].$$

Example 11 (c) : Solve $(x^2D^2+5xD+3)\,y=\left(1+\dfrac{1}{x}\right)^2\log x.$ **(M.U. 2003, 06)**

Sol. : Putting $z=\log x$ and $x=e^z$, we get

$[D(D-1)+5D+3]\,y=(1+e^{-z})^2\cdot z$ $\quad\therefore\ [D^2+4D+3]\,y=(1+e^{-z})^2 z$

$\therefore$ The A.E. is $D^2+4D+3=0$. $\quad\therefore\ (D+1)(D+3)=0$ $\quad\therefore\ D=-1,-3$.

$\therefore$ The C.F. is $y=c_1e^{-z}+c_2e^{-3z}$.

$$\text{P.I.}=\frac{1}{D^2+4D+3}(z+2e^{-z}z+e^{-2z}z)$$

Now, $\dfrac{1}{D^2+4D+3}z=\dfrac{1}{3}\left[1+\dfrac{4D+D^2}{3}\right]^{-1}z$

$$=\frac{1}{3}\left[1-\frac{4D}{3}.....\right]z=\frac{1}{3}\left[z-\frac{4}{3}.....\right]$$

$$\frac{1}{D^2+4D+3}2e^{-z}z=2e^{-z}\cdot\frac{1}{(D-1)^2+4(D-1)+3}z$$

$$=2\cdot\frac{e^{-z}}{D^2+2D}z=2\cdot\frac{e^{-z}}{2D}\left[1+\frac{D}{2}.....\right]^{-1}z$$

$$=\frac{e^{-z}}{D}\left[z-\frac{1}{2}\right]=e^{-z}\int\left[z-\frac{1}{2}\right]dz=e^{-z}\left(\frac{z^2}{2}-\frac{z}{2}\right)$$

$$\frac{1}{D^2+4D+3}e^{-2z}\cdot z = e^{-2z}\cdot\frac{1}{(D-2)^2+4(D-2)+3}z$$

$$=\frac{e^{-2z}}{D^2-1}z = e^{-2z}\cdot(-1)[1-D^2]^{-1}z$$

$$=-e^{-2z}[1+D^2+.....]z = -e^{-2z}z.$$

$$\therefore \text{ P.I.} = \frac{z}{3}-\frac{4}{9}+e^{-z}\left(\frac{z^2}{2}-\frac{z}{2}\right)-e^{-2z}\cdot z$$

$\therefore$ The complete solution is

$$y = c_1e^{-z}+c_2e^{-3z}+\frac{z}{3}-\frac{4}{9}+e^{-z}\left(\frac{z^2}{2}-\frac{z}{2}\right)-e^{-2z}\cdot z$$

$$y = \frac{c_1}{x}+\frac{c_2}{x^3}+\frac{\log x}{3}-\frac{4}{9}-\frac{1}{x}\left[\frac{(\log x)^2}{2}-\frac{(\log x)}{2}\right]-\frac{1}{x^2}\cdot\log x.$$

Example 12 (c) : Solve $x^2\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+6y=-x^4\sin x.$

Sol. : Putting $z=\log x$ and $x=e^z$, we get

$[D(D-1)-4D+6]y=-e^{4z}\cdot\sin e^z.$ $\therefore$ $(D^2-5D+6)y=-e^{4z}\cdot\sin e^z.$

$\therefore$ The A.E. is $(D^2-5D+6)=0.$ $\therefore$ $(D-2)(D-3)=0$ $\therefore$ $D=2,3.$

$\therefore$ The C.F. is $y=c_1e^{2z}+c_2e^{3z}.$

$$\text{P.I.} = \frac{1}{D^2-5D+6}(-e^{4z}\sin e^z)$$

$$=-e^{4z}\cdot\frac{1}{(D+4)^2-5(D+4)+6}\sin e^z$$

$$\therefore \text{ P.I.} = -e^{4z}\cdot\frac{1}{D^2+3D+2}\sin e^z = -e^{4z}\cdot\frac{1}{(D+2)(D+1)}\sin e^z$$

$$=-e^{4z}\cdot\frac{1}{D+2}\cdot e^{-z}\cdot\int e^z\sin e^z\,dz \quad [\text{Put } e^z=t]$$

$$=-e^{4z}\cdot\frac{1}{D+2}\cdot e^{-z}(-\cos e^z)$$

$$=+e^{4z}\cdot e^{-2z}\cdot\int e^{2z}e^{-z}\cos e^z\,dz$$

$$=e^{2z}\int e^z\cos e^z\,dz = e^{2z}\cdot\sin e^z \quad [\text{Put } e^z=t]$$

$\therefore$ The complete solution is

$$y=c_1e^{2z}+c_2e^{3z}+e^{2z}\sin e^z = c_1x^2+c_2x^3+x^2\sin x.$$

Example 13 (c) : Solve $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=\log x\cdot\sin(\log x).$ **(M.U. 1987)**

Sol. : Putting $z=\log x$ and $x=e^z$, we get

$$[D(D-1)+D+1]y = z \sin z \qquad \therefore\ [D^2+1]y = z\sin z$$

$\therefore$ The A.E. is $D^2+1=0 \quad \therefore\ D = i, -i$

$\therefore$ The C.F. is $y = c_1 \cos z + c_2 \sin z$.

$$\text{P.I.} = \frac{1}{D^2+1} z \sin z = \text{I.P. of } \frac{1}{D^2+1} e^{iz} \cdot z$$

$$= \text{I.P. of } e^{iz} \frac{1}{(D+i)^2+1} z = \text{I.P. of } e^{iz} \frac{1}{D^2+2iD} \cdot z$$

$$= \text{I.P. of } e^{iz} \frac{1}{D^2+2iD} \cdot z = \text{I.P. of } e^{iz} \cdot \frac{1}{2iD}\left[1+\frac{D}{2i}\right]^{-1} \cdot z$$

$$= \text{I.P. of } e^{iz} \cdot \frac{1}{2iD}\left[1-\frac{D}{2i}+\ldots\ldots\right] z = \text{I.P. of } e^{iz} \cdot \frac{1}{2iD}\left[z-\frac{1}{2i}\right]$$

$$= \text{I.P. of } e^{iz} \cdot \frac{1}{2i}\int\left(z-\frac{1}{2i}\right)dz = \text{I.P. of } e^{iz} \cdot \frac{1}{2i}\left[\frac{z^2}{2}-\frac{z}{2i}\right]$$

$$= \text{I.P. of } (\cos z + i \sin z)\ \frac{1}{2i}\left(\frac{z^2}{2}-\frac{z}{2i}\right)$$

$$= \text{I.P. of } (\cos z + i \sin z)\left(-\frac{i}{2}\right)\left(\frac{z^2}{2}+\frac{zi}{2}\right)$$

$$= \text{I.P. of } (\cos z + i \sin z) \cdot \left(-\frac{z^2 i}{4}+\frac{z}{4}\right) = -\frac{z^2}{4}\cos z + \frac{z}{4}\sin z$$

$\therefore$ The complete solution is

$$y = c_1 \cos z + c_2 \sin z - \frac{z^2}{4}\cos z + \frac{z}{4}\sin z$$

$$\therefore\ y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \frac{(\log x)}{4}\sin(\log x).$$

Example 14 (c) : Solve $\left(\frac{d}{dx}+\frac{1}{x}\right)^2 y = \frac{1}{x^4}$. **(M.U. 2003, 2007)**

Sol. : We have $\left(\frac{d}{dx}+\frac{1}{x}\right)\left(\frac{dy}{dx}+\frac{y}{x}\right) = \frac{1}{x^4}$

$$\therefore\ \frac{d}{dx}\left(\frac{dy}{dx}+\frac{y}{x}\right)+\frac{1}{x}\left(\frac{dy}{dx}+\frac{y}{x}\right) = \frac{1}{x^4}$$

$$\therefore\ \frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}+\frac{1}{x}\frac{dy}{dx}+\frac{y}{x^2} = \frac{1}{x^4}$$

$$\therefore\ \frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx} = \frac{1}{x^4}$$

Multiplying by x^2, we get $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx} = \frac{1}{x^2}$.

Putting $z = \log x$ and $x = e^z$, we get $[D(D-1) + 2D]\, y = e^{-2z}$.

$\therefore$ The A.E. is $D^2 + D = 0 \quad \therefore D(D+1) = 0 \quad \therefore D = 0, -1$

$\therefore$ The C.F. is $y = c_1 + c_2 e^{-z}$.

$$\text{P.I.} = \frac{1}{D(D+1)}\, e^{-2z} = \frac{1}{-2(-2+1)}\, e^{-2z} = \frac{1}{2}\, e^{-2z}$$

$\therefore$ The complete solution is

$$y = c_1 + c_2 e^{-z} + \frac{1}{2} e^{-2z} = c_1 + \frac{c_2}{x} + \frac{1}{2x^2}.$$

Example 15 (c) : Solve $\left(\dfrac{d^2}{dx^2} - \dfrac{2}{x^2}\right)^2 y = x^2$.

Sol. : Let $\left(\dfrac{d^2}{dx^2} - \dfrac{2}{x^2}\right) y = v$ .... (1)

$\therefore$ The equation becomes $\left(\dfrac{d^2}{dx^2} - \dfrac{2}{x^2}\right) v = x^2$

$$\therefore \frac{d^2v}{dx^2} - \frac{2v}{x^2} = x^2 \quad \therefore x^2 \frac{d^2v}{dx^2} - 2v = x^4$$

Putting $z = \log x$ and $x = e^z$, we get

$$[D(D-1) - 2]\, v = e^{4z} \quad \therefore (D^2 - D - 2)\, v = e^{4z}$$

$\therefore$ The A.E. is $D^2 - D - 2 = 0 \quad \therefore (D-2)(D+1) = 0 \quad \therefore D = -1, 2.$

$\therefore$ The C.F. is $v = c_1 e^{-z} + c_2 e^{2z}$.

$$\text{P.I.} = \frac{1}{D^2 - D - 2} \cdot e^{4z} = \frac{1}{16 - 4 - 2} \cdot e^{4z} = \frac{1}{10}\, e^{4z}$$

$\therefore$ The complete solution is

$$v = c_1 e^{-z} + c_2 e^{2z} + \frac{1}{10} e^{4z} = \frac{c_1}{x} + c_2 x^2 + \frac{x^4}{10}.$$

Putting the value of v in (1), we get,

$$\frac{d^2y}{dx^2} - \frac{2y}{x^2} = \frac{c_1}{x} + c_2 x^2 + \frac{x^4}{10} \qquad \therefore x^2 \frac{d^2y}{dx^2} - 2y = c_1 x + c_2 x^4 + \frac{x^6}{10}$$

Putting again $z = \log x$ and $x = e^z$, we get

$$[D(D-1) - 2]\, y = c_1 e^z + c_2 e^{4z} + \frac{1}{10} e^{6z}$$

$\therefore$ The A.E. is $D^2 - D - 2 = 0. \quad \therefore (D+1)(D-2) = 0 \quad \therefore D = -1, 2.$

$\therefore$ The C.F. is $y = c_3 e^{-z} + c_4 e^{2z}$.

$$\text{P.I.} = \frac{1}{D^2 - D - 2}\left(c_1 e^z + c_2 e^{4z} + \frac{1}{10} e^{6z}\right)$$

$$\text{P.I.} = \frac{c_1}{1-1-2}e^z + \frac{c_2}{16-4-2}e^{4z} + \frac{1}{10}\cdot\frac{1}{36-6-2}\cdot e^{6z}$$

$$= -\frac{c_1}{2}e^z + \frac{c_2}{10}e^{4z} + \frac{1}{280}e^{6z}$$

$\therefore$ The complete solution of the given equation is

$$y = c_3 e^{-z} + c_4 e^{2z} - \frac{c_1}{2}e^z + \frac{c_2}{10}e^{4z} + \frac{1}{280}e^{6z}$$

$$\therefore\ y = \frac{c_3}{x} + c_4 x^2 - \frac{c_1}{2}x + \frac{c_2}{10}x^4 + \frac{1}{280}x^6$$

Since the constants are arbitrary we can write the solution as

$$y = \frac{a}{x} + bx + cx^2 + dx^4 + \frac{1}{280}x^6.$$

Example 16 (c) : Solve $u = r\dfrac{d}{dr}\left(r\dfrac{du}{dr}\right) + ar^3$.

Sol. : The equation can be written as

$$u = r\left[r\frac{d^2u}{dr^2} + 1\cdot\frac{du}{dr}\right] + ar^3 \qquad \therefore\ r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = -ar^3$$

Putting $z = \log r$ and $r = e^z$, we get

$$[D(D-1) + D - 1]\,u = -ae^{3z} \qquad \therefore\ (D^2-1)\,u = -ae^{3z}$$

$\therefore$ The A.E. is $D^2 - 1 = 0$

$\therefore$ $(D-1)(D+1) = 0$ $\qquad \therefore$ $D = 1, -1.$

$\therefore$ The C.F. is $u = c_1 e^z + c_2 e^{-z}$.

$$\text{P.I.} = \frac{1}{D^2-1}(-ae^{3z}) = \frac{a}{10}e^{3z}$$

$\therefore$ The complete solution is

$$u = c_1 e^z + c_2 e^{-z} + \frac{a}{10}e^{3z} = c_1 r + \frac{c_2}{r} + \frac{a}{10}r^3.$$

Example 17 (c) : Solve $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$.

Sol. : Putting $z = \log x$ and $x = e^z$, we get

$$[D(D-1) + 3D + 1]\,y = \frac{1}{(1-e^z)^2} \qquad \therefore\ [D^2 + 2D + 1]\,y = \frac{1}{(1-e^z)^2}$$

$\therefore$ The A.E. is $(D+1)^2 = 0$ $\qquad \therefore$ $D = -1, -1.$

$\therefore$ The C.F. is $y = (c_1 + c_2 z)\,e^{-z}$.

$$\text{P.I.} = \frac{1}{D+1}\cdot\frac{1}{D+1}\cdot\frac{1}{(1-e^z)^2} = \frac{1}{D+1}e^{-z}\int\frac{e^z}{(1-e^z)^2}dz$$

$$= \frac{1}{D+1}e^{-z}\cdot\frac{1}{(1-e^z)} = e^{-z}\int\frac{e^z\cdot e^{-z}}{1-e^z}dz = e^{-z}\cdot\int\frac{dz}{1-e^z}$$

$$\therefore \text{ P.I.} = e^{-z}\int \frac{e^{-z}}{e^{-z}-1}\,dx = -e^{-z}\cdot \log(e^{-z}-1) \qquad \text{[Note this]}$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 z)e^{-z} - e^{-z}\cdot \log(e^{-z}-1)$$

$$\therefore \; y = (c_1 + c_2 \log x)\cdot \frac{1}{x} - \frac{1}{x}\log\left(\frac{1}{x}-1\right)$$

$$\therefore \; y = (c_1 + c_2 \log x)\cdot \frac{1}{x} - \frac{1}{x}\log\left(\frac{1-x}{x}\right).$$

Example 18 (c) : Solve $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + 3y = \dfrac{\log x \cdot \cos \log x}{x}$.

Sol. : Putting $z = \log x$ and $x = e^z$, we get

$$[D(D-1) + 3D + 3]\,y = e^{-z}\cdot z \cdot \cos z \quad \therefore \; (D^2 + 2D + 3)\,y = e^{-z}\cdot z\cdot \cos z$$

$\therefore$ The A.E. is $D^2 + 2D + 3 = 0$.

$$\therefore \; D = \frac{-2 \pm 2\sqrt{2}\cdot i}{2} = -1 \pm \sqrt{2}\cdot i$$

$\therefore$ The C.F. is $y = e^{-z}(c_1 \cos\sqrt{2}\,z + c_2 \sin\sqrt{2}\,z)$.

$$\text{P.I.} = \frac{1}{D^2+2D+3}e^{-z}\cdot z\cdot \cos z$$

$$= e^{-z}\cdot \frac{1}{(D-1)^2 + 2(D-1)+3}\cdot z\cos z = e^{-z}\cdot \frac{1}{D^2+2}\cdot z\cos z$$

$$= e^{-z}\cdot\left[z - \frac{1}{D^2+2}\cdot 2D\right]\cdot\frac{1}{D^2+2}\cos z \qquad \text{[By § 13, page 3-37]}$$

$$= e^{-z}\left[z - \frac{1}{D^2+2}\cdot 2D\right]\cos z = e^{-z}\left[z\cos z + \frac{2}{D^2+2}\cdot \sin z\right]$$

$$= e^{-z}[z\cos z + 2\sin z]$$

$\therefore$ The complete solution is

$$y = e^{-z}(c_1\cos\sqrt{2}\,z + c_2 \sin\sqrt{2}\,z) + e^{-z}(z\cos z + 2\sin z)$$

$$\therefore \; y = \frac{1}{x}[c_1\cos(\sqrt{2}\log x) + c_2 \sin(\sqrt{2}\log x)] + \frac{1}{x}[\log x \cdot \cos(\log x) + 2\sin(\log x)]$$

Example 19 (c) : Solve $(x^2D^2 - xD + 4)\,y = \cos \log x$. **(M.U. 2013)**

Sol. : Putting $z = \log x$ and $x = e^z$, we get

$$[D(D-1) - D + 4]\,y = \cos z$$

$\therefore$ The A.E. is $D^2 - 2D + 4 = 0$. $\quad \therefore \; D = \dfrac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{3}\cdot i$

$\therefore$ The C.F. is $y = e^z(c_1\cos\sqrt{3}\cdot z + c_2 \sin\sqrt{3}\cdot z)$

$$\therefore \; \text{P.I.} = \frac{1}{D^2 - 2D + 4}\cdot \cos z = \frac{1}{3-2D}\cdot \cos z$$

$\therefore$ P.I. $= \dfrac{3+2D}{9-4D^2} \cdot \cos z = \dfrac{1}{13}(3\cos z - 2\sin z)$

$\therefore$ The complete solution is

$$y = e^z(c_1 \cos\sqrt{3} \cdot z + c_2 \sin\sqrt{3} \cdot z) + \frac{1}{13}(3\cos z - 2\sin z)$$

$$\therefore\ y = x(c_1 \cos\sqrt{3}\log x + c_2 \sin\sqrt{3}\log x) + \frac{1}{13}(3\cos\log x - 2\sin\log x)$$

Example 20 (c) : The radial displacement u in a rotating disc at a distance r from the axis is given by

$$\frac{d^2u}{dr^2} + \frac{1}{r} \cdot \frac{du}{dr} - \frac{u}{r^2} + kr = 0$$

Find the displacement, if $u = 0$ when $r = 0$ and $r = a$. **(M.U. 1996, 2005, 08, 14)**

Sol. : The given equation can be written as

$$r^2 \frac{d^2u}{dr^2} + r \cdot \frac{du}{dr} - u = -kr^3$$

Putting $z = \log r$ and $r = e^z$, we get

$$[D(D-1) + D - 1]\, u = -k\, e^{3z} \qquad \therefore\ (D^2 - 1)\, u = -k\, e^{3z}.$$

$\therefore$ The auxiliary equation is $D^2 - 1 = 0$ $\quad \therefore\ D = 1, -1.$

$\therefore$ The complementary function is $u = c_1 e^z + c_2 e^{-z}$.

$$\therefore\ \text{P.I.} = -\frac{k}{D^2 - 1} \cdot e^{3z} = -\frac{k}{8} \cdot e^{3z}$$

$\therefore$ The complete solution is

$$u = c_1 e^z + c_2 e^{-z} - \frac{k}{8} e^{3z}$$

$$\therefore\ u = c_1 r + \frac{c_2}{r} - \frac{k}{8} \cdot r^3$$

$$\therefore\ ur = c_1 r^2 + c_2 - \frac{k}{8} r^4 \qquad \text{.............. (1)}$$

Now, by data when $r = 0$, $u = 0$ and also when $r = a$, $u = 0$.

$$\therefore\ 0 = c_2 \quad \text{and} \quad 0 = c_1 a^2 - \frac{k}{8} a^4$$

$$\therefore\ 0 = c_1 - \frac{k}{8} a^2 \qquad \therefore\ c_1 = \frac{k}{8} a^2.$$

Hence, from (1), we get

$$u = \frac{ka^2}{8} r - \frac{k}{8} r^3 = \frac{k}{8} r(a^2 - r^2)$$

Example 21 (c) : Solve by the method of variation of parameters.

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^3 e^x.$$

Sol. : Putting $z = \log x$, $x = e^z$, we get

$$[D(D-1)+D-1]\,y = e^{3z}\cdot e^{e^z} \qquad \therefore\ (D^2-1)\,y = e^{3z}\cdot e^{e^z}$$

$\therefore$ The A.E. is $D^2-1=0$ $\qquad \therefore\ (D-1)(D+1)=0$

$\therefore$ The C.F. is $y = c_1 e^z + c_2 e^{-z} = c_1 x + \dfrac{c_2}{x}$.

Here, $y_1 = x,\ y_2 = \dfrac{1}{x},\ X = x^3 e^x$. Let P.I. be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\dfrac{2}{x}$

$$\therefore\quad u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{(1/x)(x^3 e^x)}{-2/x}\,dx = \frac{1}{2}\int x^3 e^x\,dx$$

$$= \frac{1}{2}\left[x^3\cdot e^x - 3x^2\cdot e^x + 6x\cdot e^x - 6\cdot e^x\right] \quad \text{[Using chain rule of integration by parts]}$$

and $\quad v = \displaystyle\int \frac{y_1 X}{W}\,dx = \int \frac{x\cdot x^3 e^x}{-2/x}\,dx = -\frac{1}{2}\int x^5 e^x\,dx$

$$= -\frac{1}{2}\left[x^5\cdot e^x - 5x^4\cdot e^x + 20x^3\cdot e^x - 60x^2\cdot e^x + 120x\cdot e^x - 120\cdot e^x\right] \quad \text{[By parts]}$$

$\therefore$ P.I. $= uy_1 + vy_2$

$$= \frac{1}{2}\left[x^3\cdot e^x - 3x^2\cdot e^x + 6x\cdot e^x - 6\cdot e^x\right]x$$

$$-\frac{1}{2}\left[x^5\cdot e^x - 5x^4\cdot e^x + 20x^3\cdot e^x - 60x^2\cdot e^x + 120x\cdot e^x - 120\cdot e^x\right]\times\frac{1}{x}$$

$$= \frac{e^x}{2}\left[x^4 - 3x^3 + 6x^2 - 6x - x^4 + 5x^3 - 20x^2 + 60x - 120 + \frac{120}{x}\right]$$

$$= \frac{e^x}{2}\left[2x^3 - 14x^2 + 54x - 120 + \frac{120}{x}\right]$$

$\therefore$ The complete solution is

$$y = c_1 x + \frac{c_2}{x} + \frac{e^x}{2}\left[2x^3 - 14x^2 + 54x - 120 + \frac{120}{x}\right].$$

Example 22 (c) : Solve $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = \dfrac{x^3}{x^2+1}$ by the method of variation of parameters.

Sol. : Putting $z = \log x$ and $x = e^z$, we get $(D^2-1)\,y = \dfrac{e^{3z}}{e^{2z}+1}$.

$\therefore$ The A.E. is $D^2-1=0$ $\quad\therefore\ D = 1, -1$.

$\therefore$ The C.F. is $y = c_1 e^z + c_2 e^{-z}$.

Here $y_1 = e^z,\ y_2 = e^{-z},\ X = \dfrac{e^{3z}}{e^{2z}+1}$

Let P.I. be $y = uy_1 + vy_2$.

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^z & e^{-z} \\ e^z & -e^{-z} \end{vmatrix} = -2$

$$u = -\int \frac{y_2 X}{W}\,dz = -\int \frac{e^{-z}}{-2} \cdot \frac{e^{3z}}{e^{2z}+1}\,dz$$

$$= \frac{1}{2}\int \frac{e^{2z}}{e^{2z}+1} = \frac{1}{4}\log(e^{2z}+1) \qquad [\text{Put } e^{2z} = t]$$

and $v = \int \frac{y_1 X}{W}\,dz = \int \frac{e^z}{-2} \cdot \frac{e^{3z}}{(e^{2z}+1)}\,dz = -\frac{1}{2}\int \frac{e^{4z}}{e^{2z}+1}\,dz$

$$= -\frac{1}{2}\int \frac{e^{4z}}{e^{2z}+1}\,dz = -\frac{e^{2z}}{4} + \frac{1}{4}\log(e^{2z}+1) \qquad [\text{Put } e^{2z} = t \text{ and integrate}]$$

$\therefore$ The complete solution is

$$y = c_1 e^z + c_2 e^{-z} + \frac{e^z}{4}\log(e^{2z}+1) - \frac{e^z}{4} + \frac{e^{-z}}{4}\log(e^{2z}+1)$$

$$\therefore\ y = c_1 x + \frac{c_2}{x} + \frac{x}{4}\log(x^2+1) + \frac{1}{4x}\log(x^2+1).$$

Example 23 (c) : Solve $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x^3\sec^2 x$ by the method of variation of parameters.

Sol. : Putting $z = \log x$ and $x = e^z$, we get

$$[D(D-1) - 2D + 2]\,y = e^{3z}\sec^2(e^z) \qquad \therefore\ [D^2 - 3D + 2]\,y = e^{3z}\sec^2(e^z)$$

$\therefore$ The A.E. is $D^2 - 3D + 2 = 0$.

$\therefore\ (D-1)(D-2) = 0 \qquad \therefore\ D = 1, 2.$

$\therefore$ The C.F. is $y = c_1 e^z + c_2 e^{2z}$.

Here, $y_1 = e^z$, $y_2 = e^{2z}$, $X = e^{3z} \cdot \sec^2(e^z)$

Let P.I. be $y = uy_1 + vy_2$.

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^z & e^{2z} \\ e^z & 2e^z \end{vmatrix} = e^{3z}$

$$\therefore\ u = -\int \frac{y_2 X}{W}\,dz = -\int \frac{e^{2z} \cdot e^{3z}\sec^2(e^z)}{e^{3z}}\,dz$$

$$= -\int e^{2z}\sec^2(e^z)\,dz \qquad [\text{Put } e^z = t]$$

$$\therefore\ u = -\int t\sec^2 t\,dt = -\left[t \cdot \tan t - \int \tan t\,dt\right]$$

$$= -[t \cdot \tan t - \log\sec t] = -e^z\tan(e^z) + \log\sec(e^z)$$

$$v = \int \frac{y_1 X}{W}\,dz = \int \frac{e^z \cdot e^{3z}\sec^2(e^z)}{e^{3z}}\,dz$$

$$= \int e^z\sec^2(e^z)\,dz = \tan(e^z) \qquad [\text{Put } e^z = t]$$

$\therefore$ P.I. $= [-e^z \tan(e^z) + \log\sec(e^z)]\, e^z + \tan(e^z) \cdot e^{2z}$

$= e^z \log\sec(e^z)$

$\therefore$ The complete solution is $y = c_1 e^z + c_2 e^{2z} + e^z \log(\sec e^z)$

$\therefore$ $y = c_1 x + c_2 x^2 + x \log(\sec x)$.

EXERCISE - I

Solve the following differential equations : Class (c) : 8 Marks

1. $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} + 2y = x^{-1}$. **(M.U. 1994)**

2. $x^3 \dfrac{d^3y}{dx^3} + 6x^2 \dfrac{d^2y}{dx^2} + 8x \dfrac{dy}{dx} + 2y = x^2 + 3x - 4$

3. $x^4 \dfrac{d^4y}{dx^4} + 2x^3 \dfrac{d^3y}{dx^3} + x^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + y = x + \log x$

4. $x^3 \dfrac{d^3y}{dx^3} + x^2 \dfrac{d^2y}{dx^2} = x^2 + x + 1$

5. $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$ **(M.U. 1989, 95)**

6. $[x^2D^2 - (2m-1)xD + (m^2 + n^2)]y = n^2 x^m \log x$

7. $(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)\, y = (1 + \log x)^2$

8. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$

9. $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = (\log x)^2 + x \sin(\log x)$

10. $x^3 \dfrac{d^2y}{dx^2} + 3x^2 \cdot \dfrac{dy}{dx} + xy = \sin\log x$ **(M.U. 1989)**

11. $x^2 \dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} + 2y = e^x$

12. $x^3 \dfrac{d^3y}{dx^3} + 3x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = \sin\log x$

13. $(x^3D^3 + 2x^2D^2 - xD + 1)\, y = x^3 + x$

14. $x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 6y = x^3 \log x$

15. $x^2 \dfrac{d^2y}{dx^2} + 3x \dfrac{dy}{dx} + 5y = 5 + \dfrac{4}{x}$

16. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = 8 + 4\log x$

17. $x^2 \dfrac{d^2y}{dx^2} + 3x \dfrac{dy}{dx} + y = \dfrac{1}{(1+x)^2}$

18. $x^2 \dfrac{d^3y}{dx^3} + 3x \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + \dfrac{y}{x} = 4\log x$ **(M.U. 2007)**

19. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = A + B\log x$

20. $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + y = \dfrac{\log x \sin\log x}{x} + \dfrac{1}{x}$

21. $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = 2\sin\log x$

22. $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} - 4y = x^2 + 2\log x$

23. $(x^3D^3 + x^2D^2 - 2)\, y = x + \dfrac{1}{x^3}$

24. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = 2 + 5\log x$

25. $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = x \log x$

26. $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2 \log x$

27. $x^2 \dfrac{d^3y}{dx^3} + 3x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 1 + \dfrac{1}{x^2}$

28. $(x^3D^3 + x^2D^2 - xD + 1)\, y = x \log x$

29. Find the equation of the curve which satisfies the differential equation

$$4x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0$$

and crosses the x-axis at an angle of 60° at $x = 1$. (M.U. 2004)

[**Ans. :** (1) $y = c_1x + c_2x^2 + \dfrac{1}{6x}$

(2) $y = \dfrac{c_1}{x} + \dfrac{1}{x}(c_2 \cos \log x + c_3 \sin \log x) + \dfrac{1}{30}x^2 + \dfrac{3}{10}x - 2$

(3) $y = x[c_1 + c_2 \log x + c_3 (\log x)^2 + c_4 (\log x)^3] + \dfrac{(\log x)^4}{4!} \cdot x + \log x + 4$

(4) $y = c_1 + x(c_2 + c_3 \log x) + x + \dfrac{(\log x)^2}{2} \cdot x + \dfrac{x^2}{2}$

(5) $y = \dfrac{c_1}{x} + x(c_2 \cos \log x + c_3 \sin \log x) + 5x + \dfrac{2}{x}\log x$

(6) $y = x^m[c_1 \cos n \log x + c_2 \sin n \log x] + x^m \log x$

(7) $y = (c_1 + c_2 \log x) \cos \log x + (c_3 + c_4 \log x) \sin \log x + (\log x)^2 + 2 \log x - 3$

(8) $y = c_1 + c_2 \log x + 2(\log x)^3$

(9) $y = c_1 \cos \log x + c_2 \sin \log x + (\log x)^2 - \dfrac{1}{5}x[2 \cos \log x - \sin \log x] - 2$

(10) $y = \dfrac{1}{x}[c_1 + c_2 \log x - \sin \log x]$

(11) $y = \dfrac{c_1}{x} + \dfrac{c_2}{x^2} + \dfrac{e^x}{x^2}$

(12) $y = \dfrac{c_1}{x} + \sqrt{x}\, c_2 \cos\left(\dfrac{\sqrt{3}}{2} \cdot \log x\right) + c_3 \sin\left(\dfrac{\sqrt{3}}{2} \cdot \log x\right) + \dfrac{1}{2}(\sin \log x + \cos \log x)$

(13) $y = [c_1 \log x + c_2]\, x + \dfrac{c_3}{x} + \dfrac{x^3}{16} + \dfrac{1}{4}x(\log x)^2$

(14) $y = c_1 x^2 + c_2 x^{-3} + \dfrac{x^3}{6}\left[\log x - \dfrac{7}{6}\right]$

(15) $y = \dfrac{1}{x}[c_1 \cos(2 \log x) + c_2 \sin(2 \log x)] + 1 + \dfrac{1}{x}$

(16) $y = c_1 + c_2 \log x + x^2(\log x + 1)$

(17) $y = (c_1 + c_2 \log x) \cdot \dfrac{1}{x} + \dfrac{1}{x} \log\left(\dfrac{1+x}{x}\right)$

(18) $y = \frac{c_1}{x} + \sqrt{x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + x(2\log x - 3)$

(19) $y = c_1 + c_2 \log x + \frac{x^2}{4}[A + B(\log x - 1)]$

(20) $y = x^2(c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}) + \frac{1}{6x} + \frac{\log x}{61x}(5 \sin \log x + 6 \cos \log x)$

$$+ \frac{2}{3721x}(27 \sin \log x + 191 \cos \log x)$$

(21) $y = c_1 \cos \log x + c_2 \sin \log x - \log x \cdot \cos \log x$

(22) $y = \frac{c_1}{x} + c_2 x^4 - \frac{1}{6}x^2 - \frac{1}{2}\log x + \frac{3}{8}$.

(23) $y = c_1 x^2 + c_2 \cos \log x + c_3 \sin \log x - \frac{x}{2} - \frac{1}{50x^3}$.

(24) $y = c_1 + c_2 \log x + 2x + 5x(\log x - 2)$

(25) $y = (c_1 + c_2 \log x)\, x + \frac{x}{6} \cdot (\log x)^3$

(26) $y = \frac{c_1}{x} + c_2 x^3 - \frac{x^2}{3}\left(\log x - \frac{2}{3}\right)$

(27) $y = c_1 + c_2 \log x + c_3 (\log x)^2 + x - \frac{1}{x}$

(28) $y = c_1 x + \sqrt{x}\left[c_2 \cos h\left(\frac{\sqrt{5}}{2}\right)\log x + c_3 \sin h\left(\frac{\sqrt{5}}{2}\right)\log x\right] - x\left[\frac{(\log x)^2}{2} + \log x\right]$

(29) $y = x^{(2+\sqrt{3})/2} - x^{(2-\sqrt{3})/2}$]

3. Legendre's Linear Equation

An equation of the type

$$p_0 (a + bx)^n \frac{d^n y}{dx^n} + p_1 (a + bx)^{n-1} \frac{d^{n-1}}{dx^{n-1}} + ... + p_n y = X \quad \text{.....................} (1)$$

where $p_1, p_2, ..., p_n$ are constants and X is a function of x only can by the substitution $a + bx = v$ be reduced to homogeneous linear differential equation of the form

$$a_0 v^n \frac{d^n y}{dv^n} + a_1 v^{n-1} \frac{d^{n-1} y}{dv^{n-1}} + ... + a_n y = V$$

where $a_0, a_1, ..., a_n$ are constants and V is a function of v only.

This equation in tern by the substitution $z = \log v$ *i.e.* $v = e^z$ can be transformed to linear differential equation with constant coefficients in the form

$$P_0 \frac{d^n y}{dz^n} + P_1 \frac{d^{n-1} y}{dz^{n-1}} + ... + P_n y = Z$$

where Z is a function of z only as explained before.

The equation is known **Legendre's Linear Differential Equation.**

Adrien-Marie Legendre (1752 - 1833)

Legendre was one of the great French Mathematicians. He got his Ph.D. in 1770. He was a member of French academy of sciences. He was made an officer Légion d'Honner. Most of his work was brought to perfection by others like Galois, Abel, Gauss. He made significant contributions in the fields of roots of polynomials, elliptic functions, elliptic integrals, curve fitting, number theory, etc. He is known for Legendre's differential equation, Legendre's polynomials and Legendre's transformations. He is best known as the author of "Éléments de géométrie" which was a leading text for over hundred years.

His name is one of the 72 names inscribed on the Eiffel Tower.

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Solve $(x+2)^2\dfrac{d^2y}{dx^2}-(x+2)\dfrac{dy}{dx}+y=3x+4.$ **(M.U. 1988, 90, 2002, 03)**

Sol. : Put $x+2=v$ $\quad\therefore\quad \dfrac{dv}{dx}=1$

$$\therefore\quad \frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}=\frac{dy}{dv}.$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{dy}{dv}\right)=\frac{d}{dv}\left(\frac{dy}{dv}\right)\frac{dv}{dx}=\frac{d^2y}{dv^2}\cdot 1$$

$\therefore$ The given equation changes to

$$v^2\frac{d^2y}{dv^2}-v\frac{dy}{dv}+y=3(v-2)+4=3v-2 \qquad \text{.......... (1)}$$

Now, put $z=\log v$ and $v=e^z$.

Then as in § 2, $\quad v\dfrac{dy}{dv}=Dy,\quad v^2\dfrac{d^2y}{dv^2}=D(D-1)\,y$

The equation (1) then changes to

$$[\,D(D-1)-D+1\,]\;y=3e^z-2 \qquad i.e.,\quad (D^2-2D+1)\;y=3e^z-2$$

$\therefore$ A.E. is $(D-1)^2=0 \quad \therefore\quad D=1,1.$

$\therefore$ The C.F. is $y=(c_1+c_2\,z)\;e^z.$

$$\text{P.I.}=\frac{1}{(D-1)^2}\,(3e^z-2)=3\cdot\frac{1}{(D-1)^2}\,e^z-2\,\frac{1}{(D-1)^2}\,e^{0z}$$

$$\therefore\quad \text{P.I.}=3\cdot\frac{z^2}{2}\cdot e^z-2$$

$\therefore$ The complete solution is $\quad y=(c_1+c_2z)\;e^z+\dfrac{3}{2}\,z^2\,e^z-2.$

Putting $z=\log v=\log(x+2)$ and $e^z=v=x+2$, we get the solution as

$$y=[c_1+c_2\log(x+2)](x+2)+\frac{3}{2}[\log(x+2)]^2\,(x+2)-2.$$

Example 2 (c) : Solve $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\log(1+x)$.

(M.U. 1997, 2007, 08, 11)

Sol. : Put $x + 1 = v$

$\therefore \quad \dfrac{dv}{dx} = 1 \qquad \therefore \quad \dfrac{dy}{dx} = \dfrac{dy}{dv}, \; \dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dv^2}$ as in the above example.

$\therefore$ The given equation changes to

$$v^2\frac{d^2y}{dv^2} + v\frac{dy}{dv} + y = 4\cos\log v$$

Now put $\log v = z$, $v = e^z$

$\therefore \quad [D(D-1) + D + 1]\,y = 4\cos z \qquad \therefore \quad (D^2+1)\,y = 4\cos z.$

$\therefore$ The A.E. is $D^2 + 1 = 0 \qquad \therefore \quad D = i, -i.$

$\therefore$ The C.F. is $y = c_1\cos z + c_2\sin z$.

$\therefore$ P.I. $= \dfrac{4}{D^2+1}\cos z = 4\,\dfrac{z}{2}\cdot\sin z = 2z\sin z$ [By (1), page 3-19]

$\therefore$ The complete solution is $y = c_1\cos z + c_2\sin z + 2z\sin z$.

Putting $z = \log v = \log(1+x)$, we get,

$$y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] + 2\log(1+x)\sin[\log(1+x)].$$

Example 3 (c) : Solve $(5+2x)^2\dfrac{d^2y}{dx^2} - 6(5+2x)\dfrac{dy}{dx} + 8y = 6x$. **(M.U. 2004, 08)**

Sol. : Put $5 + 2x = v \qquad \therefore \quad \dfrac{dv}{dx} = 2$

$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx} = 2\frac{dy}{dv}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2\frac{dy}{dv}\right) = \frac{d}{dv}\left(2\frac{dy}{dv}\right)\cdot\frac{dv}{dx}$$

$$= 2\cdot\frac{d^2y}{dv^2}\cdot 2 = 4\frac{d^2y}{dv^2}$$

The given equation then changes to

$$v^2\cdot 4\frac{d^2y}{dv^2} - 6\cdot v\cdot 2\frac{dy}{dv} + 8y = 6\cdot\left(\frac{v-5}{2}\right)$$

$$4v^2\frac{d^2y}{dv^2} - 12v\frac{dy}{dv} + 8y = 3(v-5)$$

Putting $z = \log v$ and $v = e^z$,

$$[4D(D-1) - 12D + 8]\,y = 3(e^z - 5)$$

$\therefore \quad (4D^2 - 16D + 8)\,y = 3(e^z - 5) \qquad \therefore \quad (D^2 - 4D + 2)\,y = \dfrac{3}{4}(e^z - 5)$

$\therefore$ The A.E. is $D^2 - 4D + 2 = 0 \qquad \therefore \quad D = 2 \pm \sqrt{2}$.

$\therefore$ The C.F. is $y = e^{2z}\left(c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z}\right)$.

$\therefore \quad y = e^{2\log v}\left(c_1 e^{\sqrt{2}\log v} + c_2 e^{-\sqrt{2}\log v}\right)$

$\therefore \quad y = v^2\left(c_1 v^{\sqrt{2}} + c_2 v^{-\sqrt{2}}\right)$

$= (5+2x)^2\,[c_1(5+2x)^{\sqrt{2}} + c_2(5+2x)^{-\sqrt{2}}]$

$\therefore$ P.I. $= \dfrac{1}{D^2 - 4D + 2}\cdot\dfrac{3}{4}(e^z - 5) = \dfrac{3}{4}\left[\dfrac{1}{D^2 - 4D \ldots}\right.$

$= \dfrac{3}{4}\left[\dfrac{e^z}{-1} - \dfrac{5e^{0z}}{2}\right] = \dfrac{3}{4}\left[-e^z - \dfrac{5}{2}\right] =$

$= -\dfrac{3}{4}(5+2x) - \dfrac{15}{8} = -\dfrac{3}{2}x - \dfrac{45}{8}$.

$\therefore$ The complete solution is

$$y = (5+2x)^2\,[c_1(5+2x)^{\sqrt{2}} + c_2(5+2\ldots$$

Example 4 : Solve $(2x+1)^2\dfrac{d^2y}{dx^2} - 2(2x+1)\ldots$

Sol. : Put $2x + 1 = v$.

$\therefore \quad \dfrac{dv}{dx} = 2 \qquad \therefore \quad \dfrac{dy}{dx} = \dfrac{dy}{dv}\cdot\dfrac{dv}{dx} = 2\dfrac{dy}{dv}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2\frac{dy}{dv}\right) = \frac{d}{dv}\left(2\ldots\right.$$

$$= 2\cdot\frac{d^2y}{dv^2}\cdot 2 = 4\frac{d^2y}{dv^2}$$

The given equation then changes to

$$v^2\cdot 4\frac{d^2y}{dv^2} - 2\cdot v\cdot 2\frac{dy}{dv} - 12y = 6\ldots$$

$$\therefore \quad 4v^2\frac{d^2y}{dv^2} - 4v\frac{dy}{dv} - 12y = 3(v - \ldots$$

Putting $z = \log v$, $v = e^z$.

$[4D(D-1) - 4D - 12]\,y = 3(\ldots$

$\therefore$ The A.E. is $4D^2 - 8D - 12 = 0$

$\therefore \quad (D-3)(D+1) = 0$

$\therefore$ The C.F. is $y = c_1 e^{3z} + c_2 e^{-z}$.

$\therefore$ P.I. $= \dfrac{1}{4(D^2 - 2D - 3)}\cdot 3(e^z \ldots$

Hence, the complete solution

$$y = c_1 e^{3z} + c_2 e^{-z} + \frac{3}{4}\left(-\ldots\right.$$

$\therefore$ The C.F. is $y = e^{2z}\,(c_1 e^{\sqrt{2}\,z} + c_2 e^{-\sqrt{2}\,z})$.

$\therefore\ y = e^{2\log v}(c_1 e^{\sqrt{2}\log v} + c_2 e^{-\sqrt{2}\log v})$ [By (A), page 3-19]

$\therefore\ y = v^2\,(c_1 v^{\sqrt{2}} + c_2 v^{-\sqrt{2}})$

$= (5+2x)^2\,[c_1(5+2x)^{\sqrt{2}} + c_2(5+2x)^{-\sqrt{2}}]$

$$\therefore\ \text{P.I.} = \frac{1}{D^2-4D+2}\cdot\frac{3}{4}(e^z-5) = \frac{3}{4}\left[\frac{1}{D^2-4D+2}(e^z-5e^{0z})\right]$$

$$= \frac{3}{4}\left[\frac{e^z}{-1} - \frac{5e^{0z}}{2}\right] = \frac{3}{4}\left[-e^z-\frac{5}{2}\right] = \frac{3}{4}\left(-v-\frac{5}{2}\right)$$

$$= -\frac{3}{4}(5+2x) - \frac{15}{8} = -\frac{3}{2}x - \frac{45}{8}.$$

$\therefore$ The complete solution is

$$y = (5+2x)^2\,[c_1(5+2x)^{\sqrt{2}} + c_2(5+2x)^{-\sqrt{2}}] - \frac{3}{2}x - \frac{45}{8}.$$

Example 4 : Solve $(2x+1)^2\dfrac{d^2y}{dx^2} - 2(2x+1)\dfrac{dy}{dx} - 12y = 6x.$ **(M.U. 2001, 02, 09, 12)**

Sol. : Put $2x+1 = v$.

$$\therefore\ \frac{dv}{dx} = 2 \qquad \therefore\ \frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx} = 2\frac{dy}{dv}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2\frac{dy}{dv}\right) = \frac{d}{dv}\left(2\frac{dy}{dv}\right)\cdot\frac{dv}{dx}$$

$$= 2\cdot\frac{d^2y}{dv^2}\cdot 2 = 4\frac{d^2y}{dv^2}$$

The given equation then changes to

$$v^2\cdot 4\frac{d^2y}{dv^2} - 2\cdot v\cdot 2\frac{dy}{dv} - 12y = 6\left(\frac{v-1}{2}\right)$$

$$\therefore\ 4v^2\frac{d^2y}{dv^2} - 4v\frac{dy}{dv} - 12y = 3(v-1)$$

Putting $z = \log v,\ v = e^z$.

$[\,4D(D-1) - 4D - 12\,]\,y = 3\,(e^z - 1)$

$\therefore$ The A.E. is $4D^2 - 8D - 12 = 0$ $\qquad\therefore\ D^2 - 2D - 3 = 0$

$\therefore\ (D-3)(D+1) = 0$ $\qquad\therefore\ D = 3, -1.$

$\therefore$ The C.F. is $y = c_1 e^{3z} + c_2 e^{-z}$.

$$\therefore\ \text{P.I.} = \frac{1}{4(D^2-2D-3)}\cdot 3\,(e^z - e^{0z}) = \frac{3}{4}\left(-\frac{e^z}{4} + \frac{1}{3}e^{0z}\right)$$

Hence, the complete solution

$$y = c_1 e^{3z} + c_2 e^{-z} + \frac{3}{4}\left(-\frac{e^z}{4} + \frac{1}{3}\right)$$

$$\therefore \quad y = c_1(2x+1)^3 + c_2(2x+1)^{-1} + \frac{3}{4}\left[-\frac{2+1}{4} + \frac{1}{3}\right].$$

Note ....

In general if $y = a + bx$, $\quad \dfrac{dy}{dx} = b\dfrac{dy}{dv}, \quad \dfrac{d^2y}{dx^2} = b^2\dfrac{d^2y}{dv^2}$ ..... etc.

EXERCISE - II

Solve the following differential equations : Class (c) : 8 Marks

1. $(5+2x)^2 \dfrac{d^2y}{dx^2} - 6(5+2x)\dfrac{dy}{dx} + 8y = 0$
2. $(2x-1)^3 \dfrac{d^3y}{dx^3} + (2x-1)\dfrac{dy}{dx} - 2y = 0$
3. $(1+2x)^2 \dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2$ **(M.U. 2011, 12)**
4. $(x+2)^2 \dfrac{d^2y}{dx^2} - 4(x+2)\dfrac{dy}{dx} + 6y = x$
5. $(x+a)^2 \dfrac{d^2y}{dx^2} - 4(x+a)\dfrac{dy}{dx} + 6y = x$ **(M.U. 1997)**
6. $(3x+2)^2 \dfrac{d^2y}{dx^2} + 5(3x+2)\dfrac{dy}{dx} - 3y = x^2 + x + 1$ **(M.U. 1998, 2006)**
7. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\cos\log(1+x)$ **(M.U. 2002)**
8. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin\log(1+x)$ **(M.U. 1992)**
9. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = \sin[2\log(1+x)]$
10. $(3x+1)^2 \dfrac{d^2y}{dx^2} - 3(3x+1)\dfrac{dy}{dx} - 12y = 9x$ **(M.U. 1999)**
11. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = [\log(1+x)]^4 + \cos\log(1+x)$
12. $(x+1)^2 \dfrac{d^2y}{dx^2} - 3(x+1)\dfrac{dy}{dx} + 4y = x^2$ **(M.U. 1993)**
13. $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$ **(M.U. 1988, 2002, 10)**
14. $(2x+1)^2 \dfrac{d^2y}{dx^2} - 2(2x+1)\dfrac{dy}{dx} - 12y = x^2$ **(M.U. 2004)**

[**Ans. : (1)** $y = (5+2x)^2\,[c_1(5+2x)^{\sqrt{2}} + c_2(5+2x)^{-\sqrt{2}}]$

(2) $y = c_1(2x-1) + (2x-1)\left[c_2(2x-1)^{(\sqrt{3}/2)} + c_3(2x-1)^{-(\sqrt{3}/2)}\right]$

(3) $y = (1+2x)^2\,[c_1 + \{\log(1+2x)\}\,(c_2 + \log(1+2x))]$

(4) $y = c_1(x+2)^2 + c_2(x+2)^3 + \frac{1}{2}(x+2) - \frac{1}{3}$

(5) $y = c_1(x+a)^2 + c_2(x+a)^3 + \frac{1}{2}(x+a) - \frac{a}{6}$

(6) $y = c_1(3x+2)^{1/3} + c_2(3x+2)^{-1} + \frac{1}{27}\left[\frac{1}{15}(3x+2)^2 + \frac{1}{4}(3x+2) - 7\right]$

(7) $y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) + 2\log(1+x) \cdot \sin\log(1+x)$

(8) $y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - \log(1+x)\cos\log(1+x)$

(9) $y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - \frac{1}{3}\sin\{2\log(1+x)\}$

(10) $y = (3x+1)\left[c_1(3x+1)^{\sqrt{7/3}} + c_2(3x+1)^{-\sqrt{7/3}}\right] - 3\left[\frac{3x+1}{7} - \frac{1}{4}\right]$

(12) $y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) + [\log(1+x)]^4$

$$- 12\,[\log(1+x)]^2 + 24 + \frac{1}{2}\log(1+x) \cdot \sin\log(1+x)$$

(13) $y = [c_1 + c_2 \log(x+1)]\,(x+1) + \left[\frac{\log(x+1)}{2}\right]^2 (x+1) + 2(x+1) + \frac{1}{4}$

(14) $y = c_1(3x+2)^2 + \frac{c_2}{(3x+2)^2} + \frac{1}{108}(3x+2)^2 \log(3x+2) + \frac{1}{108}.$

(15) $y = c_1(2x+1)^3 + c_2(2x+1)^{-1} + \frac{1}{16}\left[-\frac{1}{3}(2x+1)^2 + \frac{1}{2}(2x+1) - \frac{1}{3}\right]$

EXERCISE - III

(A) Solve the following differential equations : Class (a) : 3 Marks

1. $(x^2D^2 - xD)\,y = 0$
2. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$
3. $(x^2D^2 - xD + 1)\,y = 0$
4. $(x^2D^2 - xD + 2)\,y = 0$
5. $(x^2D^2 - 4x\,D + 6)\,y = 0$
6. $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 0.$
7. $(1+x)^2\frac{d^2y}{dx^2} - 4(1+x)\frac{dy}{dx} + 6 = 0.$
8. $(x+3)^2\frac{d^2y}{dx^2} - 2(x+3)\frac{dy}{dx} - 4y = 0.$

[**Ans.:** (1) $y = c_1 + c_2x^2$ (2) $y = c_1 + c_2\log x$ (3) $y = (c_2 + c_2\log x)\,x$

(4) $y = x(c_1\cos\log x + c_2\sin\log x)$ (5) $y = c_1x^2 + c_2x^3$

(6) $y = c_1\cos\log(1+x) + c_2\sin\log(1+x)$ (7) $y = c_1(1+x)^2 + c_2(1+x)^3$

(8) $y = c_1(x+3)^4 + c_2(x+3)^{-1}$]

(B) Find the complementary function of the solution of the following differential equations : Class (a) : 3 Marks

1. $(x^2 D^2 - 3x D + 4)\, y = 2x^2.$ [**Ans. :** $y = (c_1 + c_2 \log x)\, x^2$]
2. $(x^2 D^2 - 3x D + 3)\, y = \sin \log x.$ [**Ans. :** $y = (c_1 \cos \log x + c_2 \sin \log x)\, x^2$]

(C) Find the particular integrals of the following differential equations : Class (a) : 3 Marks

1. $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = (\log x)^2.$
2. $(x^2 D^2 - 3x D + 4)\, y = 2x^2.$
3. $(x^2 D^2 - xD + 1)\, y = 2x.$
4. $(x^2 D^2 - xD + 1)\, y = \log x.$
5. $(x^2 D^2 + 2x D)\, y = x^{-2}.$
6. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = \sin\log x.$

[**Ans. :** **(1)** $(\log x)^2 - 2$ **(2)** $x^2 (\log x)^2$ **(3)** $(\log x)^2 \cdot x$ **(4)** $\log x + 2$
(5) $\dfrac{1}{2x^2}$ **(6)** $\dfrac{1}{8}(\sin\log x + \cos\log x)$]

Summary

1. To solve Cauchy's Homogeneous Linear Differential Equation,
   Put $z = \log x$ or $x = e^z$.
   Then $x\dfrac{dy}{dx} = Dy$, $x^2 \dfrac{d^2y}{dx^2} = D(D-1)\, y$, $x^3 \dfrac{d^3y}{dx^3} = D(D-1)(D-2)\, y$
2. To solve Legendre's Linear Equation,
   Put $a + bx = v$.
   Then put $z = \log v$ or $v = e^z$.

❋ ❋ ❋

CHAPTER 5

Numerical Solutions of Ordinary Differential Equations

1. Introduction

The analytical methods of solutions of differential equations studied so far are applicable to a particular type in which the function is integrable. In many physical and engineering problems these methods cannot be used *e.g.*, $\int \sqrt{\sin x}\, dx$, $\int e^{x^2} dx$ and we have to use numerical methods.

We shall consider in this chapter the methods of solving the differential equation of the first order and first degree *i.e.* of the type

$$\frac{dy}{dx} = f(x, y) \quad \text{with the initial condition when } x = x_0,\ y = y_0.$$

These methods can be divided into two classes.

(i) We get y in terms of a series of powers of x in the form

$$y = a_0 + a_1 x + a_2 x^2 + \ldots..$$

By directly putting the value of x we get the required value of y.

(ii) We get the value y_1 of y for a value x_1 of x near x_0, then treating these values x_1 and y_1 as initial values we obtain the value y_2 of y near x_1 and so on we get the value of y for a given value of x. These methods are called **step-by-step methods or marching methods**.

Of the methods discussed in this chapter the first one *i.e.*, Taylor's method belongs to the first type and the remaining two *viz.* (i) Euler's method and (ii) Runge-Kutta method belong to the second type.

2. Euler's Method

Let the differential equation be $\frac{dy}{dx} = f(x, y)$ where $y = y_0$ when $x = x_0$. Let $y = F(x)$ be the solution and let its graph be as shown in the figure. Let P be (x_0, y_0) and Q_1 be $(x_0 + h, y_1)$. Let the tangent at P make an angle θ with the x-axis. Then, from the figure

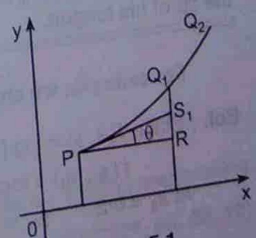


Fig. 5.1

$$\tan\theta = \frac{S_1 R}{PR} \qquad \therefore\ S_1 R = \tan\theta \cdot PR$$

But $\frac{dy}{dx} = \tan\theta$ $\qquad \therefore\ S_1 R = \left(\frac{dy}{dx}\right) PR$

But $S_1 R \approx Q_1 R$ (Approximately)

$$\therefore\ Q_1 R = \left(\frac{dy}{dx}\right) \cdot PR$$

But $Q_1 R = y_1 - y_0$ and $PR = h$

$$\therefore \quad y_1 - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} \cdot h$$

$$\therefore \quad y_1 = y_0 + h f(x_0, y_0) \qquad \left[\because \ f(x_0, y_0) = \left(\frac{dy}{dx}\right)_{(x_0, y_0)}\right]$$

Similarly, if Q_2 is $(x_0 + 2h, y_2)$, by the same reasoning we can get $y_2 = y_1 + hf(x_1, y_1)$ and so on.

In general, $\boxed{y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})}$

Note ....

In the above method we find the y - coordinate of S_1 and assume it to be the y - coordinate of Q_1 approximately. In other words we approximate the curve PQ_1 by a segment PS_1 of a straight line. Now from y - coordinate of S_1 we find by the same argument y - coordinate of S_2 and assume it to be approximately equal to the y - coordinate of Q_2 and so on. **If the slope of the curve changes slowly this method gives fairly good results. But if the slope of the curve changes rapidly this approximation does not give a good result.** The difference in the actual value and the approximated value may become appreciable. In the figure $PQ_1 \ Q_2 \ Q_3$ ..... Q is the true solution curve. See that the y - coordinate of the point Q differs widely from the y - coordinate of the point 'S'.

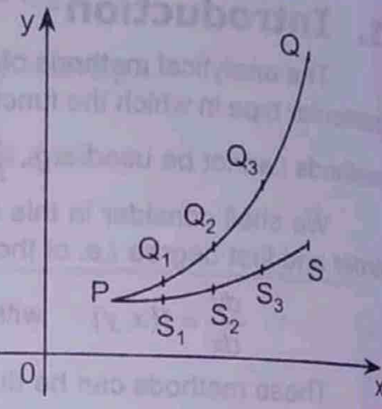


Fig. 5.2

Leonhard Euler (1707 - 1783)

One of the great mathematicians of Switzerland. His father wanted him to become a pastor (a priest). But Bernoulli persuaded his father to allow his son to pursue mathematics. He studied under his fellow countryman, mathematician Joham Bernoulli and had published his first paper when he was 18. The word function first suggested by Leibnitz was generalised further by Bernoulli and Euler. Euler is supposed to be the most prolific mathematical writer in history. He has written a number of text books which are known for his clarity, detail and completeness. Although he had lost his eye-sight for the last 17 years of his life, he did not allow his work to be hampered because all the formulae from trigonometry and analysis (and many poems including the entire Latin epic-Aeneid) were on the tip of his tongue.

For example, we shall find y_1, where $\frac{dy}{dx} = \log(x + y)$, $y(0) = 2$ at $x = 0{\cdot}2$ taking $h = 0{\cdot}2$.

Sol. : Since $f(x, y) = \log(x + y)$,

$$f(x_0, y_0) = \log(0 + 2) = \log 2$$

At $x_1 = 0{\cdot}2$, $\quad y_1 = y_0 + h f(x_0, y_0) = 2 + 0{\cdot}2 \log 2$

$$= 2 + 0{\cdot}2\,(0{\cdot}0310) = 2{\cdot}0602.$$

Again for example, we shall find y_1 where $\frac{dy}{dx} = \frac{(x^2+1)\,y^2}{2}$, $y(0{\cdot}2) = 1{\cdot}1114$, taking $h = 0{\cdot}1$.

Sol. : Since $f(x, y) = \frac{1}{2}(x^2+1)\,y^2$, $x_0 = 0{\cdot}2$, $y_0 = 1{\cdot}1114$, and we want y at $x = 0{\cdot}3$, $h = 0{\cdot}1$.

At $x_1 = 0{\cdot}3$,
$$\begin{aligned} y_1 &= y_0 + h\,f(x_0, y_0) \\ &= 1{\cdot}1114 + 0{\cdot}1 \times \frac{1}{2}(0{\cdot}2^2+1)\times(1{\cdot}1114)^2 \\ &= 1{\cdot}1114 + \frac{0{\cdot}1}{2}[\,1{\cdot}2846184\,] \\ &= 1{\cdot}1756. \end{aligned}$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Using Euler's Method find the approximate value of y at $x = 0{\cdot}1$ in 5 steps, given that $\frac{dy}{dx} = \frac{y-x}{y+x}$ and $y(0) = 1$.

Sol. : We divide the interval (a, b) *i.e.* $(0, 0{\cdot}1)$ in 5 steps and take $h = \frac{0{\cdot}1 - 0}{5} = 0{\cdot}02$.

Since $f(x, y) = \frac{y-x}{y+x}$, $f(x_0, y_0) = \frac{1-0}{1+0} = 1$

$\therefore$ At $x_1 = 0{\cdot}02$,
$$\begin{aligned} y_1 &= y_0 + h\,f(x_0, y_0) \\ &= 1 + (0{\cdot}02)\,(1) = 1{\cdot}02 \end{aligned}$$

Now, $f(x_1, y_1) = \frac{y_1 - x_1}{y_1 + x_1} = \frac{1{\cdot}02 - 0{\cdot}02}{1{\cdot}02 + 0{\cdot}02} = \frac{1}{1{\cdot}04} = 0{\cdot}9615$

$\therefore$ At $x_2 = 0{\cdot}04$,
$$\begin{aligned} y_2 &= y_1 + h\,f(x_1, y_1) \\ &= 1{\cdot}02 + 0{\cdot}02\,(0{\cdot}9615) = 1{\cdot}0392 \end{aligned}$$

Now, $f(x_2, y_2) = \frac{y_2 - x_2}{y_2 + x_2} = \frac{1{\cdot}0392 - 0{\cdot}04}{1{\cdot}0392 + 0{\cdot}04} = 0{\cdot}9259$

$\therefore$ At $x_3 = 0{\cdot}06$,
$$\begin{aligned} y_3 &= y_2 + h\,f(x_2, y_2) \\ &= 1{\cdot}0392 + 0{\cdot}02\,(0{\cdot}9259) = 1{\cdot}0577 \end{aligned}$$

Now, $f(x_3, y_3) = \frac{y_3 - x_3}{y_3 + x_3} = \frac{1{\cdot}0577 - 0{\cdot}06}{1{\cdot}0577 + 0{\cdot}06} = 0{\cdot}8926$

$\therefore$ At $x_4 = 0{\cdot}08$,
$$\begin{aligned} y_4 &= y_4 + h\,f(x_3, y_3) \\ &= 1{\cdot}0577 + 0{\cdot}02\,(0{\cdot}8926) = 1{\cdot}0755 \end{aligned}$$

Now, $f(x_4, y_4) = \frac{y_4 - x_4}{y_4 + x_4} = \frac{1{\cdot}0755 - 0{\cdot}08}{1{\cdot}0755 + 0{\cdot}08} = 0{\cdot}8615$

$\therefore$ At $x_5 = 0{\cdot}10$,
$$\begin{aligned} y_5 &= y_4 + hf(x_4, y_4) \\ &= 1{\cdot}0755 + 0{\cdot}02\,(0{\cdot}8615) = 1{\cdot}0927. \end{aligned}$$

Example 2 (b) : Using Euler's method find approximate value of y at $x = 1$ in five steps taking $h = 0{\cdot}2$. Given $\frac{dy}{dx} = x + y$ and $y(0) = 1$. **(M.U. 2004, 05, 08, 12)**

Sol. : Since $f(x, y) = x + y$, $f(x_0, y_0) = 0 + 1 = 1$

$$\text{At } x_1 = 0{\cdot}2, \quad y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0{\cdot}2\,(1) = 1{\cdot}2$$

$$\text{Now,} \quad f(x_1, y_1) = x_1 + y_1 = 0{\cdot}2 + 1{\cdot}2 = 1{\cdot}4$$

$$\therefore \text{ At } x_2 = 0{\cdot}4, \quad y_2 = y_1 + h\, f(x_1, y_1) = 1{\cdot}2 + 0{\cdot}2\,(1{\cdot}4) = 1{\cdot}48$$

$$f(x_2, y_2) = x_2 + y_2 = 0{\cdot}4 + 1{\cdot}48 = 1{\cdot}88$$

$$\text{At } x_3 = 0{\cdot}6, \quad y_3 = y_2 + h\, f(x_2, y_2) = 1{\cdot}48 + 0{\cdot}2\,(1{\cdot}88) = 1{\cdot}856$$

$$f(x_3, y_3) = 0{\cdot}6 + 1{\cdot}856 = 2{\cdot}456$$

$$\text{At } x_4 = 0{\cdot}8, \quad y_4 = y_3 + h\, f(x_3, y_3) = 1{\cdot}856 + 0{\cdot}2\,(2{\cdot}456) = 2{\cdot}3472$$

$$\therefore \quad f(x_4, y_4) = x_4 + y_4 = 0{\cdot}8 + 2{\cdot}3472 = 3{\cdot}1472$$

$$\therefore \text{ At } x_5 = 1, \quad y_5 = y_4 + h\, f(x_4, y_4) = 2{\cdot}3472 + 0{\cdot}2\,(3{\cdot}1472) = 2{\cdot}97664.$$

Example 3 (b) : Using Euler's Method find the approximate value of y when $x = 1{\cdot}5$ in five steps taking $h = 0{\cdot}1$ given $\dfrac{dy}{dx} = \dfrac{y - x}{\sqrt{xy}}$ and $y(1) = 2$. **(M.U. 2010)**

Sol. : We divide the interval (1, 1·5) in five steps.

$$\therefore \quad h = \frac{1.5 - 1.0}{5} = 0.1. \text{ Also } x_0 = 1,\ y_0 = 2$$

$$\text{Since } f(x, y) = \frac{y - x}{\sqrt{xy}}, \quad f(x_0, y_0) = \frac{y_0 - x_0}{\sqrt{x_0 y_0}} = \frac{2 - 1}{\sqrt{(1)(2)}} = 0.7071$$

$$\therefore \text{ At } x_1 = 1{\cdot}1, \quad y_1 = y_0 + h\, f(x_0, y_0) = 2 + (0{\cdot}1)\,(0{\cdot}7071) = 2{\cdot}07071$$

$$\text{Now} \quad f(x_1, y_1) = \frac{y_1 - x_1}{\sqrt{x_1 y_1}} = \frac{2.07071 - 1.1}{\sqrt{(1.1)\,(2.07071)}} = 0.6432$$

$$\therefore \text{ At } x_2 = 1{\cdot}2, \quad y_2 = y_1 + h\, f(x_1, y_1) = 2{\cdot}07071 + (0{\cdot}1)\,(0{\cdot}6432) = 2{\cdot}13503$$

$$\text{Now,} \quad f(x_2, y_2) = \frac{y_2 - x_2}{\sqrt{x_2 y_2}} = \frac{2.13503 - 1.2}{\sqrt{(1.2)\,(2.13503)}} = 0.5842$$

$$\therefore \text{ At } x_3 = 1{\cdot}3, \quad y_3 = y_2 + h\, f(x_2, y_2) = 2{\cdot}13503 + (0{\cdot}1)\,(0{\cdot}5842) = 2{\cdot}19345$$

$$\text{Now,} \quad f(x_3, y_3) = \frac{y_3 - x_3}{\sqrt{x_3 y_3}} = \frac{2.19345 - 1.3}{\sqrt{(1.3)(2.19345)}} = 0.5291$$

$$\therefore \text{ At } x_4 = 1{\cdot}4, \quad y_4 = y_3 + h\, f(x_3, y_3) = 2{\cdot}19345 + 0{\cdot}1\,(0{\cdot}5291) = 2{\cdot}24636$$

$$\text{Now,} \quad f(x_4, y_4) = \frac{y_4 - x_4}{\sqrt{x_4 y_4}} = \frac{2.24636 - 1.4}{\sqrt{(1.4)(2.24636)}} = 0.4773$$

$\therefore$ At $x_5 = 1{\cdot}5$, $\quad y_5 = y_4 + hf(x_4, y_4)$

$= 2{\cdot}24636 + (0{\cdot}1)(0{\cdot}4773) = 2{\cdot}29409$

We tabulate the above calculations as follows.

x_n	y_n	$f(x_n, y_n) = \dfrac{(y_n - x_n)}{\sqrt{x_n y_n}}$	$y_{n+1} = y_n + h f(x_n, y_n)$
1	2	$\dfrac{2-1}{\sqrt{1\times 2}} = 0{\cdot}7071$	$2 + 0{\cdot}1(0{\cdot}7071) = 2{\cdot}0707$
1·1	2·0707	$\dfrac{2{\cdot}0707 - 1{\cdot}1}{\sqrt{(1{\cdot}1)(2{\cdot}0707)}} = 0{\cdot}6432$	$2{\cdot}0707 + 0{\cdot}1(0{\cdot}6432) = 2{\cdot}13503$
1·2	2·13503	$\dfrac{2{\cdot}13503 - 1{\cdot}2}{\sqrt{(1{\cdot}2)(2{\cdot}13503)}} = 0{\cdot}5842$	$2{\cdot}13503 + 0{\cdot}1(0{\cdot}5842) = 2{\cdot}19345$
1·3	2·19345	$\dfrac{2{\cdot}19345 - 1{\cdot}3}{\sqrt{(1{\cdot}3)(2{\cdot}19345)}} = 0{\cdot}5291$	$2{\cdot}19345 + 0{\cdot}1(0{\cdot}5291) = 2{\cdot}24636$
1·4	2·24636	$\dfrac{2{\cdot}24636 - 1{\cdot}4}{\sqrt{(1{\cdot}4)(2{\cdot}24636)}} = 0{\cdot}4773$	$2{\cdot}24636 + 0{\cdot}1(0{\cdot}4773) = 2{\cdot}29409$
1·5	2·29409.		

Example 4 (b) : Use Euler's Method to find an approximate value of y correct to 4 decimal places for $x = 0{\cdot}1$ given $\dfrac{dy}{dx} = x - y^2$ at $x = 0$, $y = 1$. Take $h = 0{\cdot}02$. **(M.U. 2013)**

Sol. : We divide the interval (0, 0·1) in 5 steps

$\therefore\ h = \dfrac{0{\cdot}1 - 0}{5} = 0{\cdot}02$. Also $x_0 = 0$, $y_0 = 1$.

Since, $f(x, y) = x - y^2$, $\quad f(x_0, y_0) = x_0 - y_0^2 = 0 - (1)^2 = -1$

$\therefore$ At $x_1 = 0{\cdot}02$, $\quad y_1 = y_0 + h f(x_0, y_0)$

$= 1 + (0{\cdot}02)(-1) = 0{\cdot}98$

Now, $f(x_1, y_1) = x_1 - y_1^2 = 0{\cdot}02 - (0{\cdot}98)^2 = 0{\cdot}9404$

$\therefore$ At $x_2 = 0{\cdot}04$, $\quad y_2 = y_1 + h f(x_1, y_1)$

$= 0{\cdot}98 + (0{\cdot}02)(-0{\cdot}9404) = 0{\cdot}9612$

Now, $f(x_2, y_2) = x_2 - y_2^2 = 0{\cdot}04 - (0{\cdot}9612)^2 = -0{\cdot}8839$

$\therefore$ At $x_3 = 0{\cdot}06$, $\quad y_3 = y_2 + h f(x_2, y_2)$

$= 0{\cdot}9612 + (0{\cdot}02)(-0{\cdot}8839) = 0{\cdot}9435$

Now, $f(x_3, y_3) = x_3 - y_3^2 = 0{\cdot}06 - (0{\cdot}9435)^2 = -0{\cdot}8302$

$\therefore$ At $x_4 = 0{\cdot}08$, $\quad y_4 = y_3 + h f(x_3, y_3)$

$= 0{\cdot}9435 + (0{\cdot}02)(-0{\cdot}8302) = 0{\cdot}9269$

Now, $f(x_4, y_4) = x_4 - y_4^2 = 0{\cdot}08 - (0{\cdot}9269)^2 = -0{\cdot}7791$

$\therefore$ At $x_5 = 0{\cdot}10$, $\quad y_5 = y_4 + hf(x_4, y_4)$

$= 0{\cdot}9269 + (0{\cdot}02)(-0{\cdot}7791) = 0{\cdot}9113$

We tabulate the above calculations as follows.

x_n	y_n	$f(x_n, y_n) = x_n - y_n^2$	$y_{n+1} = y_n + h\,f(x_n, y_n)$
0	1	$0 - 1^2 = -1$	$1 + (0.2)(-1) = 0.98$
0·02	0·98	$0.02 - 0.98^2 = -0.9404$	$0.98 + (0.02)(-0.9404) = 0.9612$
0·04	0·9612	$0.04 - 0.9612^2 = -0.8839$	$0.9612 + (0.02)(-0.8839) = 0.9435$
0·06	0·9435	$0.06 - 0.9435^2 = -0.8302$	$0.9435 + (0.02)(-0.8302) = 0.9369$
0·08	0·9269	$0.08 - 0.9269^2 = -0.7791$	$0.9269 + (0.02)(-0.7791) = 0.9113$
0·10	0·9113		

Example 5 (b) : Solve the differential equation given in Ex. 2 page 4-5 by taking $h = 0.1$.

Sol. : Calculations involved can be expressed in a tabulate form as follows.

x_n	y_n	$f(x_n, y_n) = x_n + y_n$	$y_{n+1} = y_n + h\,f(x_n, y_n)$
0·0	1	$0 + 1 = 1$	$1 + (0.1)(1) = 1.1$
0·1	1·1	$0.1 + 1.1 = 1.2$	$1.1 + 0.1\,(1.2) = 1.22$
0·2	1·22	$0.2 + 1.22 = 1.42$	$1.22 + 0.1\,(1.42) = 1.36$
0·3	1·36	$0.3 + 1.36 = 1.66$	$1.36 + 0.1\,(1.66) = 1.53$
0·4	1·53	$0.4 + 1.53 = 1.93$	$1.53 + 0.1\,(1.93) = 1.72$
0·5	1·72	$0.5 + 1.72 = 2.22$	$1.72 + 0.1\,(2.22) = 1.94$
0·6	1·94	$0.6 + 1.94 = 2.54$	$1.94 + 0.1\,(2.54) = 2.19$
0·7	2·19	$0.7 + 2.19 = 2.89$	$2.19 + 0.1\,(2.89) = 2.48$
0·8	2·48	$0.8 + 2.48 = 3.29$	$2.48 + 0.1\,(3.29) = 2.81$
0·9	2·81	$0.9 + 2.81 = 3.71$	$2.81 + 0.1\,(3.71) = 3.18$
1·0	3·18		

Note ....

Comparing the values obtained in Ex. 2, page 5-3 and in Ex. 5, we find that the Euler's method gives better results if h is small. Exact solution is $y = -x - 1 + 2e^x$ and exact value of y when $x = 1$ is 3·4366.

Example 6 (b) : Using Euler's method, find the approximate value of y when $\frac{dy}{dx} = x^2 + y^2$ and $y = 1$ when $x = 0$ at $y = 2$ in five steps *i.e.* $h = 0.2$. **(M.U. 2006)**

Sol. : We tabulate the calculations as follows.

x_n	y_n	$f(x_n, y_n) = x_n^2 + y_n^2$	$y_{n+1} = y_n + h\,f(x_n, y_n)$
0·0	1	$0^2 + 1^2 = 1$	$1 + (0.2)(1) = 1.2$
0·2	1·2	$0.2^2 + 1.2^2 = 1.48$	$1.2 + 0.2\,(1.48) = 1.496$
0·4	1·496	$0.4^2 + 1.496^2 = 2.3980$	$1.496 + 0.2\,(2.3980) = 1.9756$
0·6	1·9756	$0.6^2 + 1.9756^2 = 4.2630$	$1.9756 + 0.2\,(4.2630) = 2.8282$
0·8	2·8282	$0.8^2 + 2.8282^2 = 8.6387$	$2.8282 + 0.2\,(8.6387) = 4.5559$
1·0	4·5559	$1.0^2 + 4.5559^2 = 21.7562$	

Example 7 (b) : Using Euler's method find the approximate value of y where $\frac{dy}{dx} = xy$, $y(0) = 2$ taking $h = 0.2$ at $x = 1$.

Sol. : We tabulate the calculations as follows.

x_n	y_n	$f(x_n, y_n) = x_n \times y_n$	$y_{n+1} = y_n + h\,f(x_n, y_n)$
0·0	2	$0 \times 2 = 0$	$2 + 0{\cdot}2 \times 0 = 2$
0·2	2	$0{\cdot}2 \times 2 = 0{\cdot}4$	$2 + 0{\cdot}2\,(0{\cdot}4) = 2{\cdot}08$
0·4	2·08	$0{\cdot}4 \times 2{\cdot}08 = 0{\cdot}832$	$2{\cdot}08 + 0{\cdot}2\,(0{\cdot}832) = 2{\cdot}2464$
0·6	2·2464	$0{\cdot}6 \times 2{\cdot}2464 = 1{\cdot}3478$	$2{\cdot}2464 + 0{\cdot}2\,(1{\cdot}3478) = 2{\cdot}5160$
0·8	2·5160	$0{\cdot}8 \times 2{\cdot}5160 = 2{\cdot}0128$	$2{\cdot}5160 + 0{\cdot}2\,(2{\cdot}0128) = 2{\cdot}9186$
1·0	2·9186		

Example 8 (b) : Using Euler's method find the approximate value of y where $\frac{dy}{dx} = x - y$ and $y(0) = 1$ at $x = 1$, taking $h = 0.2$.

Sol. : We tabulate the calculations as follows.

x_n	y_n	$f(x_n, y_n) = x_n - y_n$	$y_{n+1} = y_n + h\,f(x_n, y_n)$
0	1	$0 - 1 = -1$	$1 + 0{\cdot}2\,(-1) = 0{\cdot}8$
0·2	0·8	$0{\cdot}2 - 0{\cdot}8 = -0{\cdot}6$	$0{\cdot}8 + 0{\cdot}2\,(-0{\cdot}6) = 0{\cdot}68$
0·4	0·68	$0{\cdot}4 - 0{\cdot}68 = -0{\cdot}28$	$0{\cdot}68 + 0{\cdot}2\,(-0{\cdot}28) = 0{\cdot}624$
0·6	0·624	$0{\cdot}6 - 0{\cdot}624 = -0{\cdot}024$	$0{\cdot}624 + 0{\cdot}2\,(-0{\cdot}024) = 0{\cdot}6192$
0·8	0·6192	$0{\cdot}8 - 0{\cdot}6192 = -0{\cdot}1808$	$0{\cdot}6192 + 0{\cdot}2\,(-0{\cdot}1808) = 0{\cdot}6554$
1·0	0·6554		

Example 9 (b) : Using Euler's method find the approximate value of y where $\frac{dy}{dx} = x + 2y$ and $y(1) = 1$ at $x = 2$ taking $h = 0{\cdot}2$.

Sol. : We tabulate the calculations as follows.

x_n	y_n	$f(x_n, y_n) = x_n + 2y_n$	$y_{n+1} = y_n + h\,f(x_n, y_n)$
1	1	$1 + 2(1) = 3$	$1 + 0{\cdot}2\,(3) = 1{\cdot}6$
1·2	1·6	$1{\cdot}2 + 2\,(1{\cdot}6) = 4{\cdot}4$	$1{\cdot}6 + 0{\cdot}2\,(4{\cdot}4) = 2{\cdot}48$
1·4	2·48	$1{\cdot}4 + 2\,(2{\cdot}48) = 6{\cdot}36$	$2{\cdot}48 + 0{\cdot}2\,(6{\cdot}36) = 3{\cdot}752$
1·6	3·752	$1{\cdot}6 + 2\,(3{\cdot}752) = 9{\cdot}104$	$3{\cdot}752 + 0{\cdot}2\,(9{\cdot}104) = 5{\cdot}5728$
1·8	5·5728	$1{\cdot}8 + 2\,(5{\cdot}5728) = 12{\cdot}9456$	$5{\cdot}5728 + 0{\cdot}2\,(12{\cdot}9456) = 8{\cdot}1619$
1·0	8·1619		

Example 10 (b) : Using Euler's method find the approximate value of y where $\frac{dy}{dx} = \frac{y - x}{x}$ and $y(1) = 2$ at $x = 2$ taking $h = 0{\cdot}2$ and compare it with exact value.

Sol. : We tabulate the calculations as follows.

x_n	y_n	$f(x_n, y_n) = \dfrac{(y_n - x_n)}{x_n}$	$y_{n+1} = y_n + h\,f(x_n, y_n)$
1	2	$\dfrac{2-1}{1} = 1$	$2 + 0{\cdot}2\,(1) = 2{\cdot}2$
1·2	2·2	$\dfrac{2{\cdot}2 - 1{\cdot}2}{1{\cdot}2} = 0{\cdot}8333$	$2{\cdot}2 + 0{\cdot}2\,(0{\cdot}8333) = 2{\cdot}3667$
1·4	2·3667	$\dfrac{2{\cdot}3667 - 1{\cdot}4}{1{\cdot}4} = 0{\cdot}6905$	$2{\cdot}3667 + 0{\cdot}2\,(0{\cdot}6905) = 2{\cdot}5048$
1·6	2·5048	$\dfrac{2{\cdot}5048 - 1{\cdot}6}{1{\cdot}6} = 0{\cdot}5655$	$2{\cdot}5048 + 0{\cdot}2\,(0{\cdot}5655) = 2{\cdot}6179$
1·8	2·6179	$\dfrac{2{\cdot}6179 - 1{\cdot}8}{1{\cdot}8} = 0{\cdot}4544$	$2{\cdot}6179 + 0{\cdot}2\,(0{\cdot}4544) = 2{\cdot}7088$
2·0	2·7088.		

Now, consider $\dfrac{dy}{dx} = \dfrac{y}{x} - 1 \quad \therefore \quad \dfrac{dy}{dx} - \dfrac{y}{x} = -1.$ It is of the form $\dfrac{dy}{dx} + Py = Q$

$\therefore$ Its solution is $y\,e^{\int(-1/x)\,dx} = \int e^{\int(-1/x)\,dx}(-1)\,dx + c.$

$\therefore \; ye^{-\log x} = \int e^{-\log x}(-1)\,dx + c \qquad \therefore \; \dfrac{y}{x} = -\int \dfrac{1}{x}\,dx + c = -\log x + c$

But when $x = 1$, $y = 2 \;\therefore\; c = 2$.

$\therefore \; \dfrac{y}{x} = 2 - \log x \quad \therefore \; y = x(2 - \log x)$

Putting $x = 2$, we get the exact value of $y = 2{\cdot}6137$.

EXERCISE - I

Using Euler's method find the approximate value of y from the following : Class (b) : 6 Marks

1. $\dfrac{dy}{dx} = y^2 - \dfrac{y}{x}$ and $y(1) = 1$, taking $h = 0{\cdot}1$, at $x = 1{\cdot}3, 1{\cdot}5$. [**Ans. :** 1·0268, 1·0889]
2. $\dfrac{dy}{dx} = x + y,\; y(0) = 1$, taking $h = 0{\cdot}2$, at $x = 1$. **(M.U. 20004)** [**Ans. :** 2·97664]
3. $\dfrac{dy}{dx} = x + y^2,\; y(0) = 1$, taking $h = 0{\cdot}01$, at $x = 0{\cdot}05$. [**Ans. :** 1·0530]
4. $\dfrac{dy}{dx} = 1 - 2xy,\; y(0) = 0$, taking $h = 0{\cdot}2$, at $x = 0{\cdot}6$ [**Ans. :** 0·52256]
5. $\dfrac{dy}{dx} = 2 + \sqrt{xy},\; y(1) = 1$, taking $h = 0{\cdot}2$, at $x = 2$. [**Ans. :** 4·8031]
6. $\dfrac{dy}{dx} = x + \sqrt{y},\; y(2) = 4$, taking $h = 0{\cdot}2$, at $x = 3$. [**Ans. :** 8·7839]
7. $\dfrac{dy}{dx} = x^2 + y^2,\; y(0) = 0$, taking $h = 0{\cdot}1$, at $x = 0{\cdot}5$. [**Ans. :** 0·030]

8. $\dfrac{dy}{dx} = xy,\; y(0) = 1$, taking $h = 0{\cdot}1$, at x
9. $\dfrac{dy}{dx} = 1 - y^2,\; y(0) = 0$, taking $h = 0{\cdot}2$
10. $\dfrac{dy}{dx} = x + y + xy,\; y(0) = 1$, taking $h =$
11. $\dfrac{dy}{dx} = -y$ with $y(0) = 1$, taking $h = 0$
12. $\dfrac{dy}{dx} = x + y$ with $y(0) = 0$, taking h
13. $\dfrac{dy}{dx} = 1 + y^2$ with $y(0) = 0$, taking

3. Euler's Modified Meth... Or Runge-Kutta Meth...

To understand the Euler's modified ...

order consider again the differential equ... $y = f(x)$ and let P be (x_0, y_0). Let Q_1 be t... dy.

As seen earlier in Euler's met... coordinate of S_1 where S_1 is the point ... tangent PT_1 at P and the ordinate at Q_1. ... taken as the y-coordinate of Q_1. Let us...

i.e. $y_1 = y_0 + h\,f(x_0, y_0)$

Let $S_1 T_2$ be the line through S... slope $f(x_1, y_1)$. Let $S_1\,T$ be the line th... having slope equal to the average of th... (i.e. of PT_1) and $S_1\,T_2$ *i.e.* having the...

$$= \frac{f(x_0, y_0) + f(x_1, y_1)}{2}$$

We draw a line PR through $P(x_0$... value of y - coordinate of Q_1. Let t... coordinate of B is taken as the appr...

Now, the equation of PBR is

$$y - y_0 = (x - x_0)\Big\{ f(x_0 \ldots$$

Since we assume $BM = y_1$ ar...

$\therefore \; \boxed{y_1^{(1)} = y_0 + \dfrac{h}{2}[f(x_0 \ldots}$

8. $\frac{dy}{dx} = xy$, $y(0) = 1$, taking $h = 0{\cdot}1$, at $x = 0{\cdot}5$. [**Ans. :** 1·1035]

9. $\frac{dy}{dx} = 1 - y^2$, $y(0) = 0$, taking $h = 0{\cdot}2$, at $x = 1$. [**Ans. :** 0·8007]

10. $\frac{dy}{dx} = x + y + xy$, $y(0) = 1$, taking $h = 0{\cdot}025$, at $x = 0{\cdot}1$. [**Ans. :** 1·1468]

11. $\frac{dy}{dx} = -y$ with $y(0) = 1$, taking $h = 0{\cdot}01$, at $x = 0{\cdot}04$. Compare it with the exact value. [**Ans. :** 0·9606, 0·9608]

12. $\frac{dy}{dx} = x + y$ with $y(0) = 0$, taking $h = 0{\cdot}2$, at $x = 0{\cdot}6$. [**Ans. :** 0·128]

13. $\frac{dy}{dx} = 1 + y^2$ with $y(0) = 0$, taking $h = 0{\cdot}2$, at $x = 1$. [**Ans. :** 1·2941]

3. Euler's Modified Method Or Runge-Kutta Method of Second Order

To understand the Euler's modified method also known as Runge - Kutta method of Second Order consider again the differential equation $\frac{dy}{dx} = f(x, y)$. Let P be a point on the solution curve $y = f(x)$ and let P be (x_0, y_0). Let Q_1 be the point $(x_0 + h, y)$. We want to find the approximate value of y.

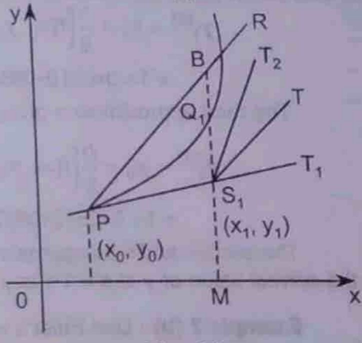


Fig. 5.3

As seen earlier in Euler's method we find the y coordinate of S_1 where S_1 is the point of intersection of the tangent PT_1 at P and the ordinate at Q_1. This is approximately taken as the y-coordinate of Q_1. Let us denote it by y_1.

i.e. $y_1 = y_0 + h\,f(x_0, y_0)$

Let $S_1\,T_2$ be the line through $S_1\,(x_1, y_1)$ and having slope $f(x_1, y_1)$. Let $S_1\,T$ be the line through $S_1\,(x_1, y_1)$ and having slope equal to the average of the slopes of $S_1\,(x_1, y_1)$ (*i.e.* of PT_1) and $S_1\,T_2$ *i.e.* having the slope

$$= \frac{f(x_0, y_0) + f(x_1, y_1)}{2}$$

We draw a line PR through $P\,(x_0, y_0)$ and parallel to $S_1\,T$. This line is used to find the approximate value of y - coordinate of Q_1. Let the ordinate of Q_1 intersect the above line PR in B. The y - coordinate of B is taken as the approximate value of y - coordinate of Q_1.

Now, the equation of PBR is

$$y - y_0 = (x - x_0)\left\{\frac{f(x_0, y_0) + f(x_1, y_1)}{2}\right\}$$

Since we assume $BM = y_1$ and $x_1 - x_0 = h$

$$\therefore \quad \boxed{y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0 + y_0) + f(x_1, y_1)\right]} \quad \text{where } y_1 = y_0 + h\,f(x_0, y_0)$$

This is the first approximation of y. Using this approximate value $y_1^{(1)}$ in the place of y_1 we get the second approximation

$$\boxed{y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0 + y_0) + f(x_1, y_1^{(1)})\right]}$$

We continue this process till we do not find any difference between two successive approximations.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Use Euler's modified method to find $y\,(0{\cdot}1)$ from $\dfrac{dy}{dx} = y - \dfrac{2x}{y}$ given, $y\,(0) = 1$ taking $h = 0{\cdot}1$.

Sol. : We have $\dfrac{dy}{dx} = y - \dfrac{2x}{y}$ $\quad \therefore \quad f(x, y) = y - \dfrac{2x}{y}$

and $\quad x_0 = 0,\ y_0 = 1,\ h = 0{\cdot}1$ and $x_1 = 0{\cdot}1$.

$\therefore\ y_1 = y_0 + h\,f(x_0, y_0) = 1 + 0{\cdot}1\,(1 - 0) = 1{\cdot}1$

$\therefore$ The first approximation gives

$$y_1^{(1)} = y_0 + \frac{h}{2}[\,f(x_0, y_0) + f(x_1, y_1)\,] = 1 + \frac{0{\cdot}1}{2}\left[(1 - 0) + \left(1{\cdot}1 - \frac{2\,(0{\cdot}1)}{1{\cdot}1}\right)\right]$$

$$= 1 + 0{\cdot}05\,[\,2{\cdot}1 - 0{\cdot}1818\,] = 1{\cdot}0959$$

Now, second approximation gives

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 1 + \frac{0{\cdot}1}{2}\left[(1 - 0) + 1{\cdot}0959 - \frac{2(0{\cdot}1)}{1{\cdot}0959}\right]$$

$$= 1 + 0{\cdot}05\,[\,2{\cdot}0959 - 0{\cdot}18250\,] = 1{\cdot}09567 = 1{\cdot}0957$$

The third approximation gives,

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] = 1 + \frac{0{\cdot}1}{2}\left[(1 - 0) + 1{\cdot}0957 - \frac{2(0{\cdot}1)}{1{\cdot}0957}\right]$$

$$= 1 + 0{\cdot}05\,[\,2{\cdot}0957 - 0{\cdot}1825\,] = 1{\cdot}09566 = 1{\cdot}0957.$$

The second and third approximation gives the same result upto four places of decimals. Hence, the correct value of y at $x = 1{\cdot}1$ is $y = 1{\cdot}0957$.

Example 2 (b) : Use Euler's modified method *i.e.* Runge - Kutta Method of Second Order to find the value of y satisfying the equation $\dfrac{dy}{dx} = \log\,(x + y),\ y\,(1) = 2$ for (i) $x = 1{\cdot}2$ and (ii) $x = 1{\cdot}4$ correct to four decimal places by taking $h = 0{\cdot}2$. **(M.U. 2009, 11)**

Sol. : (i) We have $\dfrac{dy}{dx} = \log\,(x + y)$ $\quad \therefore\ f(x, y) = \log\,(x + y)$ and $x_0 = 1,\ y_0 = 2$ and $h = 0{\cdot}2,\ x_1 = 1{\cdot}2.$

$\therefore\ y_1 = y_0 + h\,f(x_0, y_0)$

Now, $\ f(x_0, y_0) = \log\,(x_0 + y_0) = \log\,(1 + 2) = 1{\cdot}0986$

$\therefore\ y_1 = 2 + (0{\cdot}2)(1{\cdot}0986) = 2{\cdot}2197$

Now, first approximation gives

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0 + y_0) + f(x_1, y_1)\right]$$

But $f(x_0, y_0) + f(x_1, y_1) = \log(1+2) + \log(1{\cdot}2 + 2{\cdot}2197)$
$= 1{\cdot}0986 + 1{\cdot}2295 = 2{\cdot}3281$

$\therefore\ y_1^{(1)} = 2 + \frac{0{\cdot}2}{2}(2{\cdot}3281) = 2{\cdot}2328$

Now, second approximation gives

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

But $f(x_0, y_0) + f(x_1, y_1^{(1)}) = \log(1+2) + \log(1{\cdot}2 + 2{\cdot}2328)$
$= 1{\cdot}0986 + 1{\cdot}2334 = 2{\cdot}332$

$\therefore\ y_1^{(2)} = 2 + \frac{0{\cdot}2}{2}(2{\cdot}332) = 2{\cdot}2332$

Now, the third approximation gives,

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

But $f(x_0, y_0) + f(x_1, y_1^{(2)}) = \log(1+2) + \log(1{\cdot}2 + 2{\cdot}2332)$
$= 1{\cdot}0986 + 1{\cdot}2335 = 2{\cdot}3321$

$\therefore\ y_1^{(3)} = 2 + \frac{0{\cdot}2}{2}(2{\cdot}3321) = 2{\cdot}2332$

Thus, there is no change in the second and third approximation. Hence, the correct value of y at $x = 1{\cdot}2$ is $y(1{\cdot}2) = 2{\cdot}2332$.

(ii) To find the value of y at $x = 1{\cdot}4$ we take $x_0 = 1{\cdot}2$ and $y_0 = 2{\cdot}2332$ as obtained above and use the above procedure.

Now, $y_1 = y_0 + h\, f(x_0, y_0)$

But $f(x_0, y_0) = \log(x_0, y_0) = \log(1{\cdot}2 + 2{\cdot}2332) = 1{\cdot}2335$

$\therefore\ y_1 = 2{\cdot}2332 + (0{\cdot}2)(1{\cdot}2335) = 2{\cdot}4799$

Now, first approximation gives

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$
$$= 2{\cdot}2332 + \frac{0{\cdot}2}{2}\left[\log(1{\cdot}2 + 2{\cdot}2332) + \log(1{\cdot}4 + 2{\cdot}4799)\right]$$
$$= 2{\cdot}2332 + (0{\cdot}1)(2{\cdot}5893) = 2{\cdot}4921$$

Further, second approximation gives

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 2{\cdot}2332 + \frac{0{\cdot}2}{2}\left[\log(1{\cdot}2 + 2{\cdot}2332) + \log(1{\cdot}4 + 2{\cdot}4921)\right]$$
$$= 2{\cdot}2332 + (0{\cdot}1)(2{\cdot}5924) = 2{\cdot}4924$$

Further, the third approximation gives

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$
$$= 2{\cdot}2332 + \frac{0{\cdot}2}{2}\left[\log(1{\cdot}2 + 2{\cdot}2332) + \log(1{\cdot}4 + 2{\cdot}4924)\right]$$
$$= 2{\cdot}2332 + (0{\cdot}1)(2{\cdot}5925) = 2{\cdot}4924$$

Since the second and third approximations give the same value.

The correct value of y at $x = 1 \cdot 4$ is $y(1 \cdot 4) = 2 \cdot 4924$.

Example 3 (b) : Solve $\frac{dy}{dx} = x + 3y$ with $x_0 = 0$, $y_0 = 1$ by Euler's modified formula *i.e.* by Runge-Kutta Method of Second Order for (i) $x = 0 \cdot 05$, (ii) $x = 0 \cdot 1$ correct to four places of decimals. Also find the exact value and then find absolute and relative errors.

Sol. : We have $\frac{dy}{dx} = x + 3y$

(i) $f(x, y) = x + 3y$ and $x_0 = 0$, $y_0 = 1$, $h = 0 \cdot 05$, $x_1 = 0 \cdot 05$.

$$y_1 = y_0 + h\, f(x_0 + y_0) = 1 + 0 \cdot 05\,(0 + 3\,(1)) = 1 \cdot 15$$

Now, first approximation gives

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 1 + \frac{0.05}{2}[(0 + 3) + (0.05 + 3(1.15))]$$

$$= 1 + 0.1625 = 1.1625$$

Second approximation gives

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + \frac{0.05}{2}[(0 + 3) + (0.05 + 3(1.1625))]$$

$$= 1 + 0.1634 = 1.1634$$

Third approximation gives

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1 + \frac{0.05}{2}[(0 + 3) + (0.05 + 3(1.1634))]$$

$$= 1 + 0.1635 = 1.1635$$

Fourth approximation gives

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 1 + \frac{0.05}{2}[(0 + 3) + (0.05 + 3(1.1635))]$$

$$= 1 + 0.1635 = 1.1635$$

Since upto 4 places of decimal the third and the fourth approximations give the same value. The correct value of y at $x = 0 \cdot 05$ is $y(0 \cdot 05) = 1 \cdot 1635$.

(ii) To find the value of y at $x = 0 \cdot 1$, we take $x_0 = 0 \cdot 05$ and $y_0 = 1 \cdot 1635$, $h = 0 \cdot 05$, $x_1 = 0 \cdot 1$

$$\therefore\ y_1 = y_0 + h\, f(x_0, y_0)$$

$$= 1 \cdot 1635 + 0 \cdot 05\,(0 \cdot 05 + 3\,(1{,}1635))$$

$$= 1 \cdot 1635 + 1 \cdot 1770 = 1 \cdot 3405$$

Now, first approximation gives

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 1.1635 + \frac{0.05}{2}[(0.05) + 3(1.1635) + (0.05) + 3(1.3405)]$$

$$= 1.1635 + 0.1903 = 1.3538$$

Second approximation gives

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1.1635 + \frac{0.05}{2}[(0.05) + 3(1.1635) + (0.05) + 3(1.3538)]$$

$$= 1.1635 + 0.1913 = 1.3548$$

Third approximation gives

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1.1635 + \frac{0.05}{2}[(0.05) + 3(1.1635) + (0.05) + 3(1.3548)]$$

$$= 1.1635 + 0.1913 = 1.3548$$

Since the second and third approximation give the same value of y.

The correct value of y at $x = 0.1$ is $y(0.1) = 1.3548$.

To obtain the exact value we solve the given differential equation which is of the form $\frac{dy}{dx} + Py = Q$ whose solution is $y e^{\int P\,dx} = \int e^{\int P\,dx} \cdot Q\,dx + c$.

Now, $\frac{dy}{dx} - 3y = x \qquad \therefore \quad y e^{\int -3\,dx} = \int e^{\int -3\,dx} \cdot x\,dx + c$

$$\therefore \quad y e^{-3x} = \int e^{-3\,dx} \cdot x\,dx + c$$

$$= x \cdot \left(\frac{-e^{-3x}}{3}\right) - \int \left(\frac{-e^{-3x}}{3}\right) \cdot 1 \cdot dx + c \qquad \text{[By parts]}$$

$$= -x\frac{-e^{-3x}}{3} - \frac{e^{-3x}}{9} + c$$

By data when $x = 0$, $y = 1$. $\quad \therefore \quad 1 = -\frac{1}{9} + c \quad \therefore \quad c = \frac{10}{9}$

$\therefore$ The exact solution is $y = -\frac{x}{3} - \frac{1}{9} + \frac{10}{9}e^{x}$.

When $x = 0.1$, $y = -\frac{0.1}{3} - \frac{1}{9} + \frac{10}{9}e^{0.1} = 1.0835$

$$\text{Absolute Error} = \text{Exact Value} - \text{Estimated Value}$$

$$= 1.0835 - 1.3548 = -0.2713.$$

$$\text{Relative Error} = \frac{\text{Absolute Error}}{\text{Exact Value}} = -\frac{0.2713}{1.0835} = -0.2504$$

Example 4 (b) : Solve $\dfrac{dy}{dx} = 2 + \sqrt{xy}$ with $x_0 = 1{\cdot}2$, $y_0 = 1{\cdot}6403$ by Euler's modified formula *i.e.* by Runge-Kutta Method of Second Order for (i) $x = 1{\cdot}4$, (ii) $x = 1{\cdot}6$ correct to four places of decimals by taking $h = 0{\cdot}2$. **(M.U. 2011, 15)**

Sol. : We have $\dfrac{dy}{dx} = 2 + \sqrt{xy}$

(i) $f(x, y) = 2 + \sqrt{xy}$, $x_0 = 1{\cdot}2$, $y_0 = 1{\cdot}6403$, $h = 0{\cdot}2$ and $x_1 = 1{\cdot}2 + 0{\cdot}2 = 1{\cdot}4$.

$$\therefore \quad y_1 = y_0 + h\,f(x_0, y_0) = 1{\cdot}6403 + 0{\cdot}2\left(2 + \sqrt{1{\cdot}2 \times 1{\cdot}6403}\right)$$

$$= 1{\cdot}6403 + 0{\cdot}6806 = 2{\cdot}3209$$

Now, first approximation gives

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$\therefore \quad y_1^{(1)} = 1{\cdot}6403 + \frac{0{\cdot}2}{2}\left[\left(2 + \sqrt{1{\cdot}2 \times 1{\cdot}6403}\right) + \left(2 + \sqrt{1{\cdot}4 \times 2{\cdot}3209}\right)\right]$$

$$= 1{\cdot}6403 + 0{\cdot}7206 = 2{\cdot}3609$$

Second approximation gives

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1{\cdot}6403 + \frac{0{\cdot}2}{2}\left[\left(2 + \sqrt{1{\cdot}2 \times 1{\cdot}6403}\right) + \left(2 + \sqrt{1{\cdot}4 \times 2{\cdot}3609}\right)\right]$$

$$= 1{\cdot}6403 + 0{\cdot}7221 = 2{\cdot}3624$$

Third approximation gives

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1{\cdot}6403 + \frac{0{\cdot}2}{2}\left[\left(2 + \sqrt{1{\cdot}2 \times 1{\cdot}6403}\right) + \left(2 + \sqrt{1{\cdot}4 \times 2{\cdot}3624}\right)\right]$$

$$= 1{\cdot}6403 + 0{\cdot}7222 = 2{\cdot}3625$$

Fourth approximation gives

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 1{\cdot}6403 + \frac{0{\cdot}2}{2}\left[\left(2 + \sqrt{1{\cdot}2 \times 1{\cdot}6403}\right) + \left(2 + \sqrt{1{\cdot}4 \times 2{\cdot}3625}\right)\right]$$

$$= 1{\cdot}6403 + 0{\cdot}7222 = 2{\cdot}3625$$

Since upto 4 decimal places the third and the fourth approximations give the same value, the correct value of y at $x = 1{\cdot}4$ is

$$y(1{\cdot}4) = 2{\cdot}3625.$$

(ii) To find the value of y at $x = 1{\cdot}6$, we take $x_0 = 1{\cdot}4$, $y_0 = 2{\cdot}3625$, $h = 0{\cdot}2$, $x_1 = 1{\cdot}6$.

$$\therefore \quad y_1 = y_0 + h\,f(x_0, y_0)$$

$$= 2{\cdot}3625 + 0{\cdot}2\left(2 + \sqrt{1{\cdot}4 \times 2{\cdot}3625}\right)$$

$$= 2{\cdot}3625 + 0{\cdot}7637 = 3{\cdot}1262$$

Now, first approximation gives

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$
$$= 2.3625 + \frac{0.2}{2}\left[(2 + \sqrt{1.4 \times 2.3625}) + (2 + \sqrt{1.6 \times 3.1262})\right]$$
$$= 2.3625 + 0.8055 = 3.1680$$

Second approximation gives

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 2.3625 + \frac{0.2}{2}\left[(2 + \sqrt{1.4 \times 2.3625}) + (2 + \sqrt{1.6 \times 3.1680})\right]$$
$$= 2.3625 + 0.8070 = 3.1695$$

Third approximation gives

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$
$$= 2.3625 + \frac{0.2}{2}\left[(2 + \sqrt{1.4 \times 2.3625}) + (2 \cdot \sqrt{1.6 \times 3.1695})\right]$$
$$= 2.3625 + 0.8070 = 3.1695$$

Since upto 4 places of decimal the second and third approximations give the same value.

The correct value of y at $x = 1.6$ is $y(1.6) = 3.1695$.

Example 5 (b) : Solve $\frac{dy}{dx} = x - y^2$ with $x_0 = 0$, $y_0 = 1$ by Euler's modified formula *i.e.* by Runge-Kutta Method of Second Order for (i) $x = 0.05$, (ii) $x = 0.1$ correct to four places of decimals.

Sol. : We have $\frac{dy}{dx} = x - y^2$

(i) $f(x, y) = x - y^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.05$ and $x_1 = 0 + 0.05 = 0.05$

$$\therefore \quad y_1 = y_0 + h\,f(x_0, y_0) = 1 + \frac{0.05}{2}(0 - 1^2) = 0.975$$

Now, first approximation gives

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] = 1 + \frac{0.05}{2}\left[(0 - 1^2) + (0.05 - 0.975^2)\right]$$
$$= 1 - 0.0475 = 0.9525$$

Second approximation gives

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 1 + \frac{0.05}{2}\left[(0 - 1^2) + (0.05 - 0.9525^2)\right]$$
$$= 1 - 0.0464 = 0.9536$$

Third approximation gives

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] = 1 + \frac{0.05}{2}\left[(0 - 1^2) + (0.05 - 0.9536^2)\right]$$
$$= 1 - 0.0465 = 0.9535$$

Fourth approximation gives

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] = 1 + \frac{0.05}{2}\left[(0 - 1^2) + (0.05 - 0.9535^2)\right]$$

$$= 1 - 0.0465 = 0.9535$$

Since the third and fourth approximations give the same value upto 4 places of decimal, the correct value of y at $x = 0{\cdot}05$ is

$$y\,(0{\cdot}05) = 0{\cdot}9535.$$

(ii) To find the value of y at $x = 0.1$, we take $x_0 = 0.05$, $y_0 = 0.9535$, $h = 0.05$, $x_1 = 0.1$.

$$\therefore\ y_1 = y_0 + h\,f(x_0, y_0)$$

$$= 0{\cdot}9535 + 0{\cdot}05\,(0{\cdot}05 - 0{\cdot}9535^2)$$

$$= 0{\cdot}9535 - 0{\cdot}0430 = 0{\cdot}9105$$

Now, first approximation gives

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$= 0{\cdot}9535 + \frac{0{\cdot}05}{2}\left[(0{\cdot}05 - 0{\cdot}9535^2) + (0{\cdot}1 - 0{\cdot}9135^2)\right]$$

$$= 0{\cdot}9535 - 0{\cdot}0398 = 0{\cdot}9137$$

Second approximation gives

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 0{\cdot}9535 + \frac{0{\cdot}05}{2}\left[(0{\cdot}05 - 0{\cdot}9535^2) + (0{\cdot}1 - 0{\cdot}9137^2)\right]$$

$$= 0{\cdot}9535 - 0{\cdot}0398 = 0{\cdot}9137$$

Since the two approximations give the same value, the correct value of y at $x = 0{\cdot}1$ is

$$y\,(0{\cdot}1) = 0{\cdot}9137.$$

Example 6 (b) : Solve $\frac{dy}{dx} = x + y$ with $x_0 = 0$, $y_0 = 1$ by Euler's modified formula *i.e.* by Runge-Kutta Method of Second Order for (i) $x = 0{\cdot}05$, (ii) $x = 0{\cdot}1$ correct to four places of decimals.

Sol. : We have $\frac{dy}{dx} = x + y$

(i) $f(x, y) = x + y$ and $x_0 = 0$, $y_0 = 1$ and $h = 0{\cdot}05$, $x_1 = 0{\cdot}05$.

$$\therefore\ y_1 = y_0 + h\,f(x_0, y_0) = 1 + 0{\cdot}05\,(0 + 1) = 1{\cdot}05$$

Now, first approximation gives,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right] = 1 + \frac{0{\cdot}05}{2}\left[(0 + 1) + (0{\cdot}05 + 1{\cdot}05)\right]$$

$$= 1 + 0{\cdot}0525 = 1{\cdot}0525$$

Second approximation gives,

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 1 + \frac{0{\cdot}05}{2}\left[(0 + 1) + (0{\cdot}05 + 1{\cdot}0525)\right]$$

$$= 1 + 0{\cdot}0526 = 1{\cdot}0526$$

The third approximation gives,

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] = 1 + \frac{0\cdot05}{2}[(0+1)+(0\cdot05+1\cdot0526)]$$

$$= 1 + 0\cdot0526 = 1\cdot0526$$

Since upto 4 decimal places the second and the third approximations give the same value, the correct value of y at $x = 0\cdot05$ is

$$y(0\cdot05) = 1\cdot0526.$$

(ii) To find the value of y at $x = 0\cdot01$, we take $x_0 = 0\cdot05$, $y_0 = 1\cdot0526$ and $h = 0\cdot05$, $x_1 = 0\cdot1$.

$$\therefore\ y_1 = y_0 + h\,f(x_0, y_0) = 1\cdot0526 + 0\cdot05\,(0\cdot05 + 1\cdot0526) = 1\cdot1077$$

Now, first approximation gives,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$= 1\cdot0526 + \frac{0\cdot05}{2}[(0\cdot05 + 1\cdot0526) + (0\cdot1 + 1\cdot1077)]$$

$$= 1\cdot0526 + 0\cdot05775 = 1\cdot1103$$

Further, second approximation gives,

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1\cdot0526 + \frac{0\cdot05}{2}[(0\cdot05 + 1\cdot0526) + (0\cdot1 + 1\cdot1103)]$$

$$= 1\cdot0526 + 0\cdot0578 = 1\cdot1104$$

Now, third approximation gives

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1\cdot0526 + \frac{0\cdot05}{2}[(0\cdot05 + 1\cdot0526) + (0\cdot1 + 1\cdot1104)]$$

$$= 1\cdot0526 + 0\cdot0578 = 1\cdot1104$$

Since the second and third approximations give the same value of y, the correct value of y at $x = 0\cdot1$ is $y(0\cdot1) = 1\cdot1104$.

Example 7 (b) : Determine the value of y for (i) $x = 0\cdot05$, (ii) $x = 0\cdot1$ given that $y(0) = 1$ and $\frac{dy}{dx} = x^2 + y$ by Euler's modified method *i.e.* by Runge-Kutta Method of Second Order correct to four places of decimlas. **(M.U. 2009)**

Sol. : We have $\frac{dy}{dx} = x^2 + y$

$$\therefore\ f(x, y) = x^2 + y \text{ and } x_0 = 0,\ y_0 = 1 \text{ and } h = 0\cdot05,\ x_1 = 0\cdot05.$$

(i) $y_1 = y_0 + h\,f(x_0, y_0) = 1 + 0\cdot05\,[(0)^2 + 1] = 1\cdot05$

Now, first approximation gives,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right] = 1 + \frac{0\cdot05}{2}\left[(0^2+1) + \{(0\cdot05)^2 + 1\cdot05\}\right]$$

$$= 1 + 0\cdot0507 = 1\cdot0507$$

Second approximation gives,

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 1 + \frac{0{\cdot}05}{2}\left[(0^2+1) + \{(0{\cdot}05)^2 + 1{\cdot}0507\}\right]$$

$$= 1 + 0{\cdot}0513 = 1{\cdot}0513$$

Third approximation gives

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] = 1 + \frac{0{\cdot}05}{2}\left[(0^2+1) + \{(0{\cdot}05)^2 + 1{\cdot}0513\}\right]$$

$$= 1 + 0{\cdot}0513 = 1{\cdot}0513$$

Since the second and the third approximations give the same value, we take the correct value of y at $x = 0.05$ as $y(0{\cdot}05) = 1{\cdot}0513$.

(ii) To find the value of y at $x = 0{\cdot}1$, we take $x_0 = 0{\cdot}05$, $y_0 = 1{\cdot}0513$, $h = 0{\cdot}05$ and $x_1 = 0{\cdot}1$.

$$\therefore\ y_1 = y_0 + h\, f(x_0, y_0) = 1{\cdot}0513 + 0{\cdot}05\ [\ (0{\cdot}05)^2 + 1{\cdot}0513\ ]$$

$$= 1{\cdot}0513 + 0{\cdot}05269 = 1{\cdot}10399$$

Now, the first approximation gives,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$= 1{\cdot}0513 + \frac{0{\cdot}05}{2}\left[\{(0{\cdot}05)^2 + 1{\cdot}0513\} + \{(0{\cdot}1)^2 + 1{\cdot}10399\}\right]$$

$$= 1{\cdot}0513 + 0{\cdot}054195 = 1{\cdot}10545$$

Second approximation gives

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1{\cdot}0513 + \frac{0{\cdot}05}{2}\left[\{(0{\cdot}05)^2 + 1{\cdot}0513\} + \{(0{\cdot}1)^2 + 1{\cdot}10545\}\right]$$

$$= 1{\cdot}0513 + 0{\cdot}05423 = 1{\cdot}10553$$

Third approximation gives

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1{\cdot}0513 + \frac{0{\cdot}05}{2}\left[\{(0{\cdot}05)^2 + 1{\cdot}0513\} + \{(0{\cdot}1)^2 + 1{\cdot}10553\}\right]$$

$$= 1{\cdot}0513 + 0{\cdot}05423 = 1{\cdot}10553$$

Since, the second and third approximations give the same value, the correct value of y is $y(0{\cdot}1) = 1{\cdot}10553$.

EXERCISE - II

Use Euler's modified method *i.e.* Runge-Kutta Method of Second Order to find the value of y upto 4 places of decimals satisfying the following equations : Class (b) : 6 Marks

1. $\frac{dy}{dx} = x - y^2$, $y(0) = 1$, for $x = 0{\cdot}4$ by taking $h = 0{\cdot}2$. **(M.U. 2005)** [**Ans. :** 0·7586]
2. $\frac{dy}{dx} = x^2 + y$, $y(0) = 0{\cdot}94$ for $x = 0{\cdot}1$. [**Ans. :** 1·0394]

3. $\frac{dy}{dx} = x + y,\ y(0 \cdot 1) = 1 \cdot 1104$ for $x = 0 \cdot 15$ and $x = 0 \cdot 2$. [**Ans. :** 1·3107 ; 1·3869]

4. $\frac{dy}{dx} = 1 - y,\ y(0) = 0$ for (i) $x = 0 \cdot 1$, (ii) $x = 0 \cdot 2$. [**Ans. :** 0·1812]

5. $\frac{dy}{dx} = x + y,\ y(0) = 1$ for $x = 0 \cdot 5$. [**Ans. :** 1·999]

6. $\frac{dy}{dx} = -xy^2,\ y(0) = 2$ for $x = 0 \cdot 2$ by taking $h = 0 \cdot 1$. [**Ans. :** 1·9234]

7. $\frac{dy}{dx} = 1 + \frac{y}{x},\ y(1) = 2$ for $x = 1 \cdot 2$ and compare it with its exact value. [**Ans. :** 2·6182, Exact value = 2·6188]

8. $\frac{dy}{dx} = 2 + \sqrt{xy},\ y(1) = 1$ for $x = 1 \cdot 2$. [**Ans. :** 1·64029]

9. $\frac{dy}{dx} = x + \sqrt{y},\ y(0) = 1$, for $x = 0 \cdot 2$. **(M.U. 2003)** [**Ans. :** 1·2309]

10. $\frac{dy}{dx} = y - x,\ y(0) = 2$ for $x = 0 \cdot 2$. Also compare it with exact value. [**Ans. :** 2·4222, Exact value = 2·4214]

11. $\frac{dy}{dx} = 2x - y,\ y(0) = -1$ for $x = 0 \cdot 2$. Also compare with exact value. [**Ans. :** – 0·7818, Exact value = – 0·7813]

12. $\frac{dy}{dx} = x^2 + y,\ y(0) = 1$ for $x = 0 \cdot 02, 0 \cdot 04, 0 \cdot 06$. [**Ans. :** 1·020, 1·0406, 1·0616]

13. $\frac{dy}{dx} = y^2 - \frac{y}{x},\ y(1) = 1$ for (i) $x = 0 \cdot 05$, (ii) $x = 1 \cdot 1$. [**Ans. :** 1·0073]

14. $\frac{dy}{dx} = x + y,\ x_0 = 0,\ y_0 = 1$ at $x = 1$. **(M.U. 2004)** [**Ans. :** 3·55]

15. $\frac{dy}{dx} = 1 + xy$ with $y = 1$ at $x = 0$ for $x = 0 \cdot 1$ and $x = 0 \cdot 2$ **(M.U. 2003)** [**Ans. : (i)** 1·1053, **(ii)** 1·2229]

4. Taylor's Series Method

Consider again the differential equation $\frac{dy}{dx} = f(x, y)$ with the initial conditions $y = y_0$ when $x = x_0$ where $f(x, y)$ can be differentiated as required.

Let the solution of the given differential equation be $y = F(x)$. Suppose we want the solution $y = F(x)$ at (x_1, y_1) where $x_1 = x_0 + h$.

Now, by Taylor's series we have

$$y_1 = F(x_0 + h) = F(x_0) + hF'(x_0) + \frac{h^2}{2!} F''(x_0) + \frac{h^3}{3!} F'''(x_0) + \ldots$$

Since $y = F(x)$, we can write the above series as

$$y_1 = y_0 + h\, y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

The coefficients y_0, y_0', y_0'', y_0''', ... can be obtained by successive differentiation of $y' = f(x, y)$.

Treating the increment h as x and the value of y when $x = x_0 + h$ as y we can write the above series as

$$\boxed{y = y_0 + x\, y_0' + \frac{x^2}{2!}\, y_0'' + \frac{x^3}{3!}\, y_0''' + \ldots..}$$

Putting the given value of x in the above series we get the required value of y.

Remark ....

The method of Taylor's series can also be used to solve second order differential equations as illustrated in Ex. 8 and Ex. 9.

Brook Taylor (1685 - 1731)

He was an English mathematician best known for Taylor's Theorem and Taylor's Series. Initially he had interest in law and got doctorate in law in 1714. But he had also keen interest in mathematics. His publication **'Methodus Incrementorum Directa et Inversa'** is considered as the beginning of new branch of mathematics called **"Calculus of finite differences"**. The famous Taylor's theorem remained unrecognised untill 1712 when Lagrange realised its powers. He was elected to Royal Society in the same year. From 1715 he took interest in the studies of religion and philosopy.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Employ Taylor's series method to obtain correct to four places of decimals, solution of the differential equation with $\frac{dy}{dx} = x^2 + y^2$ with $y = 0$ when $x = 0$ for $x = 0{\cdot}4$.

Sol. : The Taylor's series is given by

$$y = y_0 + x\, y_0' + \frac{x^2}{2!}\, y_0'' + \frac{x^3}{3!}\, y_0''' + \ldots.. \quad \text{where } x_0 = 0,\ y_0 = 0. \qquad \ldots\ldots\ldots\ldots (1)$$

Now, $\quad y' = x^2 + y^2$

$\therefore \quad y_0' = x_0^2 + y_0^2 = 0 + 0 = 0$

$y'' = 2x + 2yy'$

$\therefore \quad y_0'' = 2x_0 + 2y_0\, y_0' = 0 + 0 = 0$

$y''' = 2 + 2\,[\,yy'' + y'^2\,]$

$\therefore \quad y_0''' = 2 + 2\,[\,y_0\, y_0'' + y_0'^2\,] = 2 + 0 = 2$

$y^{iv} = 2\,[y'\,y'' + yy''' + 2\,y'\,y''\,] = 2\,[3\,y'\,y'' + yy'''\,] = 0$

$y^{v} = 2\,[3\,(y'\,y''' + y''^2) + yy^{iv} + y'\,y'''\,]$

$\quad = 8y'\,y''' + 6y''^2 + 2yy^{iv} = 0$

$y^{vi} = 8\,[\,y'\,y^{iv} + y''\,y''' + 12\,y''\,y''' + 2yy^{v} + 2y'\,y^{iv}\,] = 0$

$$y^{viii} = 8\,[\,y'\,y^{v} + y''\,y^{iv} + y''\,y^{iv} + y'''^{\,2} + 12\,y''\,y^{iv} + 12y'''^{\,2} + 2yy^{vi} + 2y'\,y^{v} + 2y'\,y^{v} + 2y''\,y^{iv}\,]$$

$$= 8\,(52) = 416.$$

Putting these values in (1), we get for any x.

$$\therefore\quad y = \frac{x^3}{3!}\,y_0''' + \frac{x^7}{7!}\,y_0^{vii} + \ldots\ldots = \frac{x^3}{3!}\cdot 2 + \frac{x^7}{7!}\cdot 418 + \ldots\ldots$$

Putting $x = 0{\cdot}4$, we get the value of y as

$$y = \frac{(0{\cdot}4)^3\cdot 2}{3!} + \frac{(0{\cdot}4)^7\cdot 418}{7!} + \ldots\ldots = 0{\cdot}021469 \approx 0{\cdot}02147.$$

Example 2 (b) : Solve using Taylor's series method, the differential equation $\dfrac{dy}{dx} = x + y$ numerically. Start from $x = 1$, $y = 0$ and carry to $x = 1{\cdot}2$ with $h = 0{\cdot}1$. Compare the final result with the value of the exact solution. **(M.U. 2009)**

Sol. : The Taylor's series is given by

$$y = y_0 + xy_0' + \frac{x^2}{2!}\,y_0'' + \frac{x^3}{3!}\,y_0''' + \ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

where $x_0 = 1$, $y_0 = 0$, and $x = 0{\cdot}1$.

(a) Now, $y' = x + y \quad \therefore \quad y_0' = x_0 + y_0 = 1 + 0 = 1$

$y'' = 1 + y' \quad \therefore \quad y_0'' = 1 + y_0' = 1 + 1 = 2$

$y''' = y'' \quad \therefore \quad y_0''' = y_0'' = 2$

$y^{iv} = y''' \quad \therefore \quad y_0^{iv} = y_0''' = 2$

Putting these values in (1), we get,

$$\therefore\quad y_1 = 0 + (0{\cdot}1)1 + \frac{(0{\cdot}1)^2}{2!}\cdot 2 + \frac{(0{\cdot}1)^3}{!}\cdot 2 + \frac{(0{\cdot}1)^4}{4!}\cdot 2 + \frac{(0{\cdot}1)^5}{5!}\cdot 2 + \ldots\ldots$$

$$= 0{\cdot}11034$$

(b) To find the value of y at $x = 1{\cdot}2$, we put $x_0 = 1{\cdot}1$, $y_0 = 0{\cdot}11034$, $x = 0{\cdot}1$.

As above $y_0' = x_0 + y_0 = 1{\cdot}21034$

$y_0'' = 1 + y_0' = 2{\cdot}21034, \quad y_0''' = 2{\cdot}21034, \quad y_0^{iv} = 2{\cdot}21034$

$$\therefore\quad y_1 = 0{\cdot}11034 + (0{\cdot}1)(1{\cdot}21034) + \frac{(0{\cdot}1)^2}{2!}\cdot(2{\cdot}21034) + \frac{(0{\cdot}1)^3}{3!}(2{\cdot}21034) + \frac{(0{\cdot}1)^4}{4!}\cdot(2{\cdot}21034)$$

$$= 0{\cdot}2428$$

Further $\dfrac{dy}{dx} - y = x$, which is linear. Its solution is $ye^{-x} = \int xe^{-x}dx + c$

$\therefore\ ye^{-x} = -(x+1)\,e^{-x} + c \qquad \therefore\ y = -(x+1) + c\,e^{x}.$

But when $x = 1$, $y = 0 \qquad \therefore\ c = 2/e$

$\therefore\ y = -(x+1) + 2e^{x-1}$

When $x = 1{\cdot}1$, $y = 0{\cdot}11034$. When $x = 1{\cdot}2$, $y = 0{\cdot}2428$.

Thus, Taylor's series method gives exact values in this example.

Example 3 (b) : If y satisfies the equation $\frac{dy}{dx} = x^2y - 1$ and $y = 1$ when $x = 0$, using Taylor's expansion for y about $x = 0$, find y when $x = 0{\cdot}1$, $0{\cdot}2$ to five decimal places. **(M.U. 2014)**

Sol. : The Taylor's series is given by

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \ldots.. \qquad \ldots\ldots\ldots\ldots (1)$$

where $x_0 = 0$, $y_0 = 1$, and $x = 0{\cdot}1$.

(a) Now, $y' = x^2 y - 1 \qquad \therefore \quad y_0' = 0 - 1 = -1$

$y'' = x^2 y' + 2xy \qquad \therefore \quad y_0'' = 0 + 0 = 0$

$y''' = x^2 y'' + 2xy' + 2y + 2xy' = x^2 y'' + 4xy' + 2y \qquad \therefore \quad y_0''' = +2$

$y^{iv} = x^2 y''' + 2xy'' + 4xy'' + 4y' + 2y' = x^2 y''' + 6xy'' + 6y' \qquad \therefore \quad y_0^{iv} = -6$

Now, $y = y_0 + x\,y_0' + x^2\,y_0'' + \frac{x_3}{3}y_0''' + \frac{x^4}{4!}y_0^{iv} + \ldots..$

$$\therefore \quad y = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \ldots.$$

$$\therefore \quad y = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.$$

Putting $x = 0{\cdot}1$, $\quad y = 1 - 0{\cdot}1 + \frac{(0{\cdot}1)^3}{3} - \frac{(0{\cdot}1)^4}{4} + \ldots. = 0{\cdot}90031$

(b) To find y at $x = 0{\cdot}2$, we start with $x_0 = 0{\cdot}1$, $y_0 = 0{\cdot}90031$ and $x = 0{\cdot}1$.

$$\begin{aligned}
\therefore \quad y_0' &= x_0^2\,y_0 - 1 = (0{\cdot}1)^2 \times (0{\cdot}9001) - 1 \\
&= -\,0{\cdot}99099 \\
y_0'' &= x_0^2\,y_0' + 2\,x_0\,y_0 \\
&= (0{\cdot}1)^2\,(-\,0{\cdot}99099) + 2\,(0{\cdot}1)\,(0{\cdot}90031) - 1 \\
&= -\,0.82982 \\
y_0''' &= x_0^2\,y_0'' + 4\,x_0\,y_0' + 2y_0 \\
&= (0{\cdot}1)^2 \cdot (-\,0{\cdot}82985) + 4\,(0{\cdot}1)\,(-\,0{\cdot}99099) + 2\,(0{\cdot}90031) \\
&= 1{\cdot}42836 \\
y_0^{iv} &= x_0^2\,y_0''' + 6\,x_0\,y_0'' + 6\,y_0' \\
&= (0{\cdot}1)^2\,(1{\cdot}42836) + 6\,(0{\cdot}1)\,(-\,0{\cdot}82985) + 6\,(-\,0{\cdot}99099) \\
&= -\,6{\cdot}42957
\end{aligned}$$

Now, $y = y_0 + x\,y_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{iv} + \ldots.$

Putting $x = 0{\cdot}1$ and the above values, we get

$$y = 0{\cdot}90031 + (0{\cdot}1)\,(-\,0{\cdot}99099) + \frac{(0{\cdot}1)^2}{2!}(-\,0{\cdot}82985) + \frac{(0{\cdot}1)^3}{3!}(1{\cdot}42836) + \frac{(0{\cdot}1)^4}{4!}(-\,6{\cdot}42957)$$

$$\therefore \quad y = 0{\cdot}79732$$

Example 4 (b) : Using Taylor's series method solve $\frac{dy}{dx} = y - xy$ with $x_0 = 0$, $y_0 = 2$. And compare the solution with the exact expression.

Sol. : The Taylor's series is given by,

$$y = y_0 + xy_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \ldots \quad \text{where } x_0 = 0, y_0 = 2$$

Now, $y' = y - xy \qquad \therefore \quad y_0' = 2 - 0 \times 2 = 2$

$y'' = y' - xy' - y \qquad \therefore \quad y_0'' = 2 - 0 \times 2 - 2 = 0$

$y''' = y'' - xy'' - 2y' \qquad \therefore \quad y_0''' = 0 - 0 - 4 = -4$

$y^{iv} = y''' - xy''' - 3y'' \qquad \therefore \quad y_0^{iv} = -4 - 0 - 0 = -4$

$$\therefore \quad y = 2 + 2x - 0 \cdot \frac{x^2}{2!} - 4 \cdot \frac{x^3}{3!} - 4 \cdot \frac{x^4}{4!} + \ldots$$

$$= 2 + 2x - \frac{2}{3} x^3 - \frac{1}{6} x^4 + \ldots \qquad \ldots\ldots\ldots\ldots (1)$$

Now, $\frac{dy}{dx} = y - xy = y(1 - x) \qquad \therefore \quad \frac{dy}{y} = (1 - x)\, dx$

$$\therefore \quad \log y = x - \frac{x^2}{2} + \log c \qquad \therefore \quad y = c\, e^{[x - (x^2/2)]}$$

But when $x = 0$, $y = 2$ $\therefore$ $c = 2$.

$\therefore$ Exact solution is $y = 2 e^{x - (x^2/2)}$

$$= 2\left[1 + \left(x - \frac{x^2}{2}\right) + \frac{1}{2!}\left(x - \frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(x - \frac{x^2}{2}\right)^3 + \frac{1}{4!}\left(x - \frac{x^2}{2}\right)^4 + \ldots\right]$$

$$= 2\left[1 + x - \frac{x^3}{3} - \frac{x^4}{12} - \ldots\right] = 2 + 2x - \frac{2x^3}{3} - \frac{x^4}{6} - \ldots \qquad \ldots\ldots\ldots\ldots (2)$$

From (1) and (2) we see that Taylor's series method gives fairly accurate solution.

Example 5 (b) : Using Taylor's series method solve $\frac{dy}{dx} = -xy^2$ with $x = 0$, $y = 2$ and compare it with the exact solution.

Sol. : The Taylor's series is given by

$$y \quad y_0 + xy_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \ldots \quad \text{where } x_0 = 0, y_0 = 2$$

$y' = -xy^2 \qquad \therefore \quad y_0' = 0$

$y'' = -y^2 - 2xy\,y' \qquad \therefore \quad y_0'' = -4$

$y''' = -2yy' - 2yy' - 2xy'^2 - 2xy\,y''$

$\quad = -4yy' - 2xy'^2 - 2xy\,y'' \qquad \therefore \quad y_0''' = -0$

$y^{iv} = -4y'^2 - 4y\,y'' - 2y'^2 - 4xy'\,y'' - 2y\,y'' - 2xy''^2 - 2xy\,y'''$

$\quad = 32 + 16 = 48.$

$$\therefore \quad y = 2 + x(0) + \frac{x^2}{2!}(-4) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(48) + \ldots$$

$$= 2 - 2x^2 + 2x^4 + \ldots = 2(1 - x^2 + x^4 + \ldots) \qquad \ldots\ldots\ldots\ldots (1)$$

Now, $\dfrac{dy}{y^2} = -x\,dx \quad \therefore -\dfrac{1}{y} = -\dfrac{x^2}{2} + c \quad \therefore \dfrac{1}{y} = \dfrac{x^2}{2} + c'$.

When $x = 0$, $y = \dfrac{1}{2}$ $\therefore c' = \dfrac{1}{2}$. $\quad \therefore \dfrac{1}{y} = \dfrac{x^2}{2} + \dfrac{1}{2} = \dfrac{x^2+1}{2} \quad \therefore y = \dfrac{2}{x^2+1}$

$\therefore y = 2(1+x^2)^{-1} = 2(1 - x^2 + x^4 - x^6 + \ldots.)$ .... (2)

From (1) and (2) we see that Taylor's series method gives fairly accurate solution.

Example 6 (b) : Solve $\dfrac{dy}{dx} = 2x - y$ with initial conditions $x_0 = 0$, $y_0 = 0$ by Taylor's method.

(M.U. 2013)

Obtain y as a series in powers of x. Find the approximate value of y for $x = 0{\cdot}2, 0{\cdot}4$. Compare your results with the exact values.

Sol. : The Taylor's series is given by

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \ldots. \qquad \text{where } x_0 = 0,\ y_0 = 0$$

Now, $y' = 2x - y \quad \therefore y_0' = 0;\qquad y'' = 2 - y' \quad \therefore y_0'' = 2$

$y''' = -y'' \quad \therefore y_0''' = -2;\qquad y^{iv} = -y''' \quad \therefore y_0^{iv} = 2$

$$\therefore y = 0 + x(0) + \frac{x^2}{2!}\cdot 2 + \frac{x^3}{3!}\cdot(-2) + \frac{x^4}{4!}\cdot(2) + \ldots.$$

$$y = x^2 - \frac{x^3}{3} + \frac{x^4}{12} + \ldots. \qquad \ldots\ldots (1)$$

Now, $\dfrac{dy}{dx} + y = 2x$. This is a linear equation. Its solution is

$$y \cdot e^x = \int e^x \cdot 2x\,dx + c = [xe^x - e^x] + c \qquad \text{[By parts]}$$

But $y = 0$ when $x = 0$ $\therefore c = 2$.

$$\therefore ye^x = 2xe^x - 2e^x + 2 \quad \therefore y = 2x - 2 + 2e^{-x} \qquad \ldots\ldots (2)$$

$$\therefore y = 2x - 2 + 2\left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots.\right] = x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \ldots.$$

Thus, the two series are the same.

Now, from (1) when $x = 0{\cdot}2$, $y(0{\cdot}2) = 0{\cdot}03747$.

Again from (2), when $x = 0{\cdot}2$, $y(0{\cdot}2) = 0{\cdot}03747$.

Example 7 (b) : Solve $\dfrac{dy}{dx} = 1 + y^2$ with initial conditions $x_0 = 0$, $y_0 = 0$ by Taylor's method.

Obtain y as a series in powers of x. Find the approximate values of y for $x = 0{\cdot}2, 0{\cdot}4$.

Compare your results with the exact values. **(M.U. 2010, 15)**

Sol. : The value of y at any point x is given by

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0'' + \ldots. \text{ with } x_0 = 0,\ y_0 = 0$$

Now, $y' = 1 + y^2$ $\therefore$ $y_0' = 1$

$y'' = 2yy'$. $\therefore$ $y_0'' = 0$

$y''' = 2yy'' + 2y'^2$ $\therefore$ $y_0''' = 2$

$y^{iv} = 2y'y'' + 2yy''' + 4y'y'' = 2yy''' + 6y'y''$ $\therefore$ $y^{iv} = 0$

$y^{v} = 2yy^{iv} + 8y'y''' + 6y''^2$ $\therefore$ $y_0^{v} = 16$.

Hence, $y = x + \dfrac{x^3}{3!}\cdot 2 + \dfrac{x^5}{5!}\cdot(16) + \ldots = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \ldots$

Now, when $x_0 = 0$, $y_0 = 0$ and $x = 0{\cdot}2$, we get

$$y = (0\cdot 2) + \frac{(0\cdot 2)^3}{3} + \frac{2(0\cdot 2)^5}{15} = 0\cdot 2027$$

To find y, when $x = 0{\cdot}4$, we take $x_0 = 0{\cdot}2$ $y_0 = 0{\cdot}2027$ and h (*i.e.* x) $= 0{\cdot}2$.

From the above results

$y_0' = 1 + (0{\cdot}2027)^2 = 1{\cdot}0411$

$y_0'' = 2\,(0{\cdot}2027)\,(1{\cdot}0411) = 0{\cdot}4221$

$y_0'' = 2\,(0{\cdot}2027)\,(0{\cdot}4221) + 2\,(1{\cdot}0411)^2 = 2{\cdot}3389$

$$\therefore\ y = y_0 + xy' + \frac{x^2}{2}\cdot y_0'' + \frac{x^3}{3!}y_0''' + \ldots = 0{\cdot}4225$$

Now, the solution of the differential equation $\dfrac{dy}{dx} = 1 + y^2$ *i.e.* of $\dfrac{dy}{1+y^2} = dx$ is $\tan^{-1} y = x + c$.

But when $x = 0$, $y = 0$. Hence, $c = 0$.

$$\therefore\ \tan^{-1} y = x \qquad \therefore\ y = \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

Now, when $x = 0{\cdot}2$, $y = \tan 0{\cdot}2^c = 0{\cdot}2027$

when $x = 0{\cdot}4$, $y = \tan 0{\cdot}4^c = 0{\cdot}4228$

Thus, the approximate values differ from the exact values by $0{\cdot}0000$ and $0{\cdot}0003$ respectively.

Example 8 (b) : Using Taylor's method solve $\dfrac{dy}{dx} = x^2 - y$, with $x_0 = 0$, $y_0 = 1$. Find y when $x = 0{\cdot}1$. **(M.U. 2012)**

Sol. : The Taylor's series is given by

$$y = y_0 + x\,y_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \ldots \qquad \text{where } x_0 = 0,\ y_0 = 1,\ x = 0{\cdot}1.$$

Now, $y' = x^2 - y$ $\therefore$ $y_0' = 0 - 1 = -1$

$y'' = 2x - y'$ $\therefore$ $y_0'' = 2\,(0) - (-1) = 1$

$y''' = 2 - y''$ $\therefore$ $y_0''' = 2 - 1 = 1$

$y^{iv} = -y'''$ $\therefore$ $y_0^{iv} = -1$

$$\therefore\ y = 1 - x + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(-1) + \ldots$$

$$= 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}$$

Putting $x = 0{\cdot}1$, $\quad y = 1 - 0{\cdot}1 + \dfrac{(0{\cdot}1)^2}{2} + \dfrac{(0{\cdot}1)^3}{6} - \dfrac{(0{\cdot}1)^4}{24} = 0{\cdot}905163$

Example 9 (b) : Using Taylor's method solve $\frac{dy}{dx} = x^3 + y$ with $x_0 = 1$, $y_0 = 1$ at $x = 0{\cdot}1$.

Sol. : The Taylor's series is given by

$$y = y_0 + x\,y_0' + \frac{x^2}{2!}\,y_0'' + \frac{x^3}{3!}\,y_0''' + \ldots \qquad \text{where } x_0 = 1,\ y_0 = 1.$$

Now, $y' = x^3 + y \qquad \therefore\ y_0' = 1 + 1 = 2$

$y'' = 3x^2 + y' \qquad \therefore\ y_0'' = 3\,(1) + 2 = 5$

$y''' = 6x + y'' \qquad \therefore\ y_0''' = 6\,(1) + 5 = 11$

$$\therefore\ y = 1 + x\,(2) + \frac{x^2}{2!}\cdot 5 + \frac{x^3}{3!}\cdot 11 + \ldots$$

$$= 1 + 2x + \frac{5}{2}x^2 + \frac{11}{6}x^3 + \ldots$$

Putting $x = 1$,

$$y = 1 + 2(0{\cdot}1) + \frac{5}{2}(0{\cdot}1)^2 + \frac{11}{6}(0{\cdot}1)^3 = 1{\cdot}22683.$$

Example 10 (b) : Using Taylor's series method solve $\frac{dy}{dx} = 1 + xy$, $x_0 = 0$, $y_0 = 0{\cdot}2$ at $x = 0{\cdot}4$. **(M.U. 2012)**

Sol. : The Taylor's series is given by

$$y = y_0 + x\,y_0' + \frac{x^2}{2!}\,y_0'' + \frac{x^3}{3!}\,y_0''' + \ldots \qquad \ldots\ldots\ldots\ldots (1)$$

with $x_0 = 0$, $y_0 = 0{\cdot}2$, $x = 0{\cdot}4$.

Now, $y' = 1 + xy \qquad \therefore\ y_0' = 1$

$y'' = y + xy' \qquad \therefore\ y_0'' = y_0 = 0{\cdot}2$

$y''' = y' + y' + xy'' = 2y' + xy'' \qquad \therefore\ y_0''' = 2y_0' = 2$

$y^{iv} = 2y'' + y'' + xy''' \qquad \therefore\ y_0^{iv} = 3y_0'' + xy_0''' = 3 \times 0{\cdot}2 = 0{\cdot}6$

Putting these values in (1), we get

$$y = 0{\cdot}2 + (0{\cdot}4)\,(1) + \frac{(0{\cdot}4)^2}{2} \times (0{\cdot}2) + \frac{(0{\cdot}4)^3}{6} \times (2) + \frac{(0{\cdot}4)^4}{24} \times (0{\cdot}6) + \ldots$$

$$= 0{\cdot}2 + 0{\cdot}4 + 0{\cdot}016 + 0{\cdot}02133 + 0{\cdot}00064$$

$$\therefore\ y = 0{\cdot}63797.$$

Example 11 (b) : Solve $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$ using Taylor's series method. Find approximate values of y for $x = 1$ and $x = 1{\cdot}1$. (Take six terms.) **(M.U. 2013)**

Sol. : The Taylor's series is given by

$$y = y_0 + x\,y_0' + \frac{x^2}{2!}\,y_0'' + \frac{x^3}{3!}\,y_0''' + \ldots \qquad \ldots\ldots\ldots\ldots (1)$$

where $x_0 = 0$, $y_0 = 0$.

(a) Now, $y' = 2y + 3e^x \qquad \therefore\ y_0' = 3$

$y'' = 2y' + 3e^x \qquad \therefore\ y_0'' = 9$

$$y''' = 2y'' + 3e^x \qquad \therefore\ y_0''' = 21$$
$$y^{iv} = 2y''' + 3e^x \qquad \therefore\ y_0^{iv} = 45$$
$$y^{v} = 2y^{iv} + 3e^x \qquad \therefore\ y_0^{v} = 93$$
$$y^{vi} = 2y^{v} + 3e^x \qquad \therefore\ y_0^{vi} = 189$$

Putting these values in (1), we get

$$y = 0 + x \cdot 3 + \frac{x^2}{2} \cdot 9 + \frac{x^3}{6} \cdot 21 + \frac{x^4}{24} \cdot 45 + \frac{x^5}{120} \cdot 93 + \frac{x^6}{720} \cdot 189 + \ldots$$

When $x = 1$, $\quad y = 3 + 4\cdot5 + 3\cdot5 + 1\cdot875 + 0\cdot775 + 0\cdot2625 = 13\cdot9125$

(b) Now, to find y at x 1·1, we start with $x_0 = 1$, $y_0 = 13\cdot9125$ and $x = 0\cdot1$.

Now, $y' = 2y_0 + 3e^{x_0} = 2(13\cdot9125) + 3e = 35\cdot9798$

$$y'' = 2y_0' + 3e^{x_0} = 2(35\cdot9798) + 3e = 80\cdot1145$$
$$y''' = 2y_0'' + 3e^{x_0} = 2(80\cdot1145) + 3e = 168\cdot3838$$
$$y^{iv} = 2y_0''' + 3e^{x_0} = 2(168\cdot3838) + 3e = 344\cdot9224$$
$$y^{v} = 2y_0^{iv} + 3e^{x_0} = 2(344\cdot9224) + 3e = 687\cdot9996$$
$$y^{vi} = 2y_0^{v} + 3e^{x_0} = 2(687\cdot9996) + 3e = 1384\cdot1540$$

Putting these values and $x = 0\cdot1$ in (1), we get

$$y = 13\cdot9125 + 0\cdot1(35\cdot9798) + \frac{(0\cdot1)^2}{2}(80\cdot1145) + \frac{(0\cdot1)^3}{6}(168\cdot3838) + \frac{(0\cdot1)^4}{24}(344\cdot9224) + \frac{(0\cdot1)^5}{120}(687\cdot9996) + \frac{(0\cdot1)^6}{720}(1384\cdot1540)$$

$$= 17\cdot9737$$

(The exact values from the solution obtained on the next page are 14·01 and 18·06.)

Example 12 (b) : Using Taylor's series method solve $\frac{dy}{dx} = 2y + 3e^x$ with $y_0 = 0$ when $x_0 = 0$ for $x = 0\cdot1, 0\cdot2$. Compare your values with exact value. **(M.U. 2010, 16)**

Sol. : The Taylor's series is given by

$$y = y_0 + x\,y_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \ldots \qquad \ldots\ldots\ldots\ldots (1)$$

(a) We have $x_0 = 0$, $y_0 = 0$, $x = 0\cdot1$.

Now, $y' = 2y + 3e^x \qquad \therefore\ y_0' = 2(0) + 3e^0 = 3$

$$y'' = 2y' + 3e^x \qquad \therefore\ y_0'' = 2(3) + 3 = 9$$
$$y''' = 2y'' + 3e^x \qquad \therefore\ y_0''' = 2(9) + 3 = 21$$
$$y^{iv} = 2y''' + 3e^x \qquad \therefore\ y_0^{iv} = 2(21) + 3 = 45$$

Putting these values in (1), we get,

$$y = 0 + x(3) + \frac{x^2}{2!}(9) + \frac{x^3}{3!}(21) + \frac{x^4}{4!}(45) + \ldots$$
$$\therefore\ y = 0 + 3x + 4\cdot5\,x^2 + 3\cdot5\,x^3 + 1\cdot875\,x^4 + \ldots$$

When $x = 0\cdot1$,

$$y = 0 + 3\,(0.1) + 4.5\,(0.1)^2 + 3.5\,(0.1)^3 + 1.875\,(0.1)^4 = 0.34869$$

(b) To find y at $x = 0{\cdot}2$, we start with $x_0 = 0{\cdot}1$, $y_0 = 0{\cdot}34869$ and $x = 0{\cdot}1$.

$\therefore\ y' = 2y + 3e^x \qquad \therefore\ y_0' = 2\,(0.34869) + 3\,e^{0{\cdot}1} = 4{\cdot}01289$

$y'' = 2y' + 3e^x \qquad \therefore\ y_0'' = 2\,(4{\cdot}01289) + 3\,e^{0{\cdot}1} = 11{\cdot}3412$

$y''' = 2y'' + 3e^x \qquad \therefore\ y_0''' = 2\,(11{\cdot}3412) + 3\,e^{0{\cdot}1} = 25{\cdot}9979$

When $x = 0{\cdot}2$,

$$y = 0{\cdot}34869 + 0{\cdot}1\,(4{\cdot}01289) + \frac{(0{\cdot}1)^2}{2!}\,(11{\cdot}3412) + \frac{(0{\cdot}1)^3}{3!}\,(25{\cdot}9979) + \ldots$$

$$= 0{\cdot}81102$$

Now, we solve the equation $\dfrac{dy}{dx} - 2y = 3\,e^x$. This is a linear equation.

$\therefore$ I.F. $= e^{\int -2\,dx} = e^{-2x}$

$\therefore$ The solution is

$$y \cdot e^{-2x} = 3\int e^{-2x} \cdot e^x\,dx + c = 3\int e^{-x}dx + c = -3\,e^{-x} + c$$

$\therefore\ y \cdot e^{-2x} = -3\,e^{-x} + c.$

When $x = 0$, $y = 0 \qquad \therefore\ 0 = -3 + c \qquad \therefore\ c = 3.$

$\therefore$ The particular solution is

$$y \cdot e^{-2x} = -3e^{-x} + 3 \qquad \therefore\ y = -3\,e^x + 3\,e^{2x}$$

When $x = 0{\cdot}1$, $\quad y = -\,3e^{0{\cdot}1} + 3e^{0{\cdot}2} = 0{\cdot}34869$

When $x = 0{\cdot}2$, $\quad y = -\,3e^{0{\cdot}2} + 3e^{0{\cdot}4} = 0{\cdot}8112$

Thus, the values given by Taylor's series are equal to exact values.

Example 13 (b) : Using Taylor's series method solve $\dfrac{dy}{dx} = 1 - 2xy$, given that $y(0) = 0$ and hence find $y\,(0{\cdot}2)$ and $y\,(0{\cdot}4)$. **(M.U. 2014)**

Sol. : The Taylor's series is given by

$$y = y_0 + x\,y_0' + \frac{x^2}{2!}\,y_0'' + \frac{x^3}{3!}\,y_0''' + \ldots \qquad \ldots\ldots\ldots\ldots (1)$$

We have $x_0 = 0$, $y_0 = 0$ and $x = 0{\cdot}2$.

(a) Now, $y' = 1 - 2xy \qquad \therefore\ y_0' = 1 - 0 = 1$

$y'' = 0 - 2\,[\,y + xy'\,] \qquad \therefore\ y_0'' = -\,2\,[\,0 + 0\,] = 0$

$y''' = -\,2\,[\,y' + y' + xy''\,] \qquad \therefore\ y_0''' = -\,2\,[\,2 + 0\,] = -\,4$

$y^{iv} = -\,2\,[\,2y'' + y'' + xy'''\,] \qquad \therefore\ y_0^{iv} = -\,2\,[\,0 - 0\,] = 0$

$y^{v} = -\,2\,[\,3y''' + y''' + xy^{iv}\,] \qquad \therefore\ y_0^{v} = 32$

Putting these values in (1), we get, when $x = 0{\cdot}2$

$$y = 0 + (0{\cdot}2)\,1 + 0 + \frac{(0{\cdot}2)^3}{3!} \cdot (-\,4) + 0 + \frac{(0{\cdot}2)^5}{5!} \cdot (32) + \ldots.$$

$$= 0{\cdot}19475$$

(b) To find y at $x = 0{\cdot}4$, we start with $x_0 = 0{\cdot}2$ and $y_0 = 0{\cdot}19475$ and $x = 0{\cdot}2$.

As above

$y' = 1 - 2xy \qquad \therefore\ y_0' = 1 - 2\,(0{\cdot}2)\,(0{\cdot}19475)\,5 = 0{\cdot}92210$

$y'' = -2\,[\,y + xy'\,]$ $\quad\therefore\quad y_0'' = -2\,[\,0{\cdot}19475 + 0.2\,(0{\cdot}9221)\,] = -0{\cdot}75834$

$y''' = -2\,[\,2y' + xy''\,]$ $\quad\therefore\quad y_0''' = -2\,[\,2\,(0{\cdot}9221) + 0{\cdot}2\,(-0{\cdot}75834)\,] = -3{\cdot}38506$

$y^{iv} = -2\,[\,3y'' + xy'''\,]$ $\quad\therefore\quad y_0^{iv} = -2\,[\,3\,(-0{\cdot}75834) - 0{\cdot}2\,(-3{\cdot}38506)\,] = 5{\cdot}90406$

Putting these values in (1), we get, when $x = 0{\cdot}4$.

$$y = 0{\cdot}19475 + 0{\cdot}2(0{\cdot}9221) + \frac{(0{\cdot}2)^2}{2}(-0{\cdot}75834) + \frac{(0{\cdot}2)^3}{6}(-3{\cdot}38506) + \frac{(0{\cdot}2)^4}{24}(5{\cdot}90406) + \ldots.$$

$$= 0{\cdot}3598$$

EXERCISE - III

Using Taylor's series method obtain the solutions of the following differential equations (correct to four places of decimals) : Class (b) : 6 Marks

1. $\dfrac{dy}{dx} = 3x + y^2$ with $x_0 = 0$, $y_0 = 1$ at $x = 0{\cdot}1$. **(M.U. 2007)** [**Ans. :** $1{\cdot}127225$]

2. $\dfrac{dy}{dx} = -xy$ with $x_0 = 0$, $y_0 = 1$. **(M.U. 2003, 05, 08)** [**Ans. :** $y = 1 - \dfrac{x^2}{2} - \dfrac{x^4}{8} - \dfrac{x^6}{48} + \ldots..$]

3. $\dfrac{dy}{dx} = y - xy$ with $x_0 = 0$, $y_0 = 1$. [**Ans. :** $y = 1 + x - \dfrac{x^3}{3} - \dfrac{x^4}{12} + \dfrac{x^5}{20} + \ldots..$]

4. $\dfrac{dy}{dx} = x - y^2$ with $x_0 = 0$, $y_0 = 1$ at $0{\cdot}1$. [**Ans. :** $0{\cdot}9138$]

5. $\dfrac{dy}{dx} = (0{\cdot}1)(x^3 + y^2)$ with $x_0 = 0$, $y_0 = 1$.
   [**Ans. :** $y = 1 + 0{\cdot}1x + 0{\cdot}01x^2 + 0{\cdot}001x^3 + 0{\cdot}0251x^4 + \ldots.$]

6. $\dfrac{dy}{dx} = y\sin x + \cos x$ with $x_0 = 0$, $y_0 = 0$. [**Ans. :** $y = x + \dfrac{1}{6}x^3 + \dfrac{1}{120}x^5 + \ldots..$]

7. $\dfrac{dy}{dx} = xy - 1$ with $x_0 = 1$, $y_0 = 2$ at $x = 1{\cdot}02$. [**Ans. :** $2{\cdot}02061$]

8. $\dfrac{dy}{dx} = x^2 + y^2$ with $x_0 = 1$, $y_0 = 0$ at $x = 1{\cdot}3$. [**Ans. :** $0{\cdot}41205$]

9. $\dfrac{dy}{dx} = 4 + y^2$, $x_0 = 0$, $y_0 = 0$ at $x = 0{\cdot}2$ and $0{\cdot}3$. [**Ans. :** $0{\cdot}8454$, $1{\cdot}3683$]

10. $\dfrac{dy}{dx} = xy - 1$, $x_0 = 1$, $y_0 = 2$ at $x = 1{\cdot}02$. [**Ans. :** $2{\cdot}0206$]

11. $\dfrac{dy}{dx} = 1 + xy$, $x_0 = 0$, $y_0 = 1$ at $x = 0{\cdot}1$.
    [**Ans. :** $1{\cdot}1053$, $y = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{8} + \dfrac{x^5}{15} + \ldots..$]

12. $\dfrac{dy}{dx} = x + y^2$, $x_0 = 0$, $y_0 = 0$ at $x = 0{\cdot}2$. [**Ans. :** $0{\cdot}02002$, $y = \dfrac{1}{2}x^2 + \dfrac{1}{20}x^5 + \ldots..$]

13. $\dfrac{dy}{dx} = \dfrac{1}{x^2 + y^2}$, $x_0 = 4$, $y_0 = 4$ at $x = 4{\cdot}1$, $4{\cdot}2$. [**Ans. :** $4{\cdot}0031$, $4{\cdot}00609$]

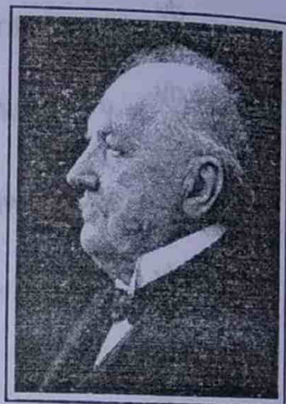
5. Runge-Kutta Method Or Runge-Kutta Method of Fourth Order

Runge-Kutta method is a general method. Euler's method, Euler's modified method and Runge's method are particular cases of Runge-Kutta method. They are called Runge-Kutta methods of first, second and third order. Runge-Kutta Method of Fourth Order is known as Runge-Kutta Method also.

Carl David Tolmé Runge (1856 - 1927)

He spent his early life in Havana, Cuba where his father worked. He got his Ph.D. in mathematics at Berlin where he worked under another giant mathematician Karl Weierstrass. In 1886 he became a professor in Hannover, Germany. Although he had interest in spectroscopy, geodesy and astrophysics, his main interest was pure mathematics. He is remembered for Runge-Kutta method for solving differential equations. He was a man who loved science, mathematics and life.

Martin Wilhelm Kutta (1867 - 1944)



Martin Wilhelm Kutta was born in Poland. He studied in Breslan University and Berlin University. He was professor in Aachen University and also in Stuttgart University, Germany. He co-developed Runge-Kutta method of solving ordinary differential equations numerically. He is also remembered for Kutta-Zhukovsky theorem, Zhukovsky-Kutta aerofoil and the Kutta condition in aerodynamics. He died in Germany in 1944.

Procedure

Let the given differential equation be $\frac{dy}{dx} = f(x, y)$ with the initial conditions $x = x_0$, $y = y_0$.

To find the value of $y = y_0 + k$ say at $x = x_0 + h$, we calculate,

$$k_1 = h\,f(x_0, y_0)\,; \qquad k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right); \qquad k_4 = h\,f(x_0 + h,\ y_0 + k_3)$$

Lastly, we calculate $k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

The required value of $y = y_0 + k$.

Solved Examples — Class (c) : 8 Marks

Example 1 (c) : Find an approximate value of y when $x = 0{\cdot}2$ given that $\frac{dy}{dx} = x + y$ when $y = 1$ at $x = 0$.

Sol. : We have $\frac{dy}{dx} = x + y$

$f(x, y) = x + y$ and $x_0 = 0$, $y_0 = 1$ and $h = 0{\cdot}2$.

$\therefore\ k_1 = h f(x_0, y_0) = 0{\cdot}2(x_0 + y_0) = 0{\cdot}2(0 + 1) = 0{\cdot}2$

$$k_2 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0{\cdot}2\left[(0 + 0{\cdot}1) + (1 + 0{\cdot}1)\right] = 0{\cdot}24$$

$$k_3 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0{\cdot}2\left[(0 + 0{\cdot}1) + (1 + 0{\cdot}12)\right] = 0{\cdot}244$$

$$k_4 = h f(x_0 + h,\ y_0 + k_3) = 0{\cdot}2\left[(0 + 0{\cdot}2) + (1 + 0{\cdot}244)\right] = 0{\cdot}2888$$

$$\therefore\ k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= \frac{1}{6}\left[0{\cdot}2 + 0{\cdot}2(0{\cdot}24) + 2(0{\cdot}244) + 0{\cdot}2888\right] = 0{\cdot}2428$$

$\therefore$ The approximate value of $y = y_0 + k = 1 + 0{\cdot}2428 = 1{\cdot}2428$.

Example 2 (c) : Solve $\dfrac{dy}{dx} = x^3 + y$, $x = 0$, $y = 2$ by Runge-Kutta method of fourth order for $x = 0.2$.

Sol. : We have $\dfrac{dy}{dx} = x^3 + y$.

$\therefore\ f(x, y) = x^3 + y,\ x_0 = 0,\ y_0 = 2,\ h = 0{\cdot}2.$

$$k_1 = h f(x_0, y_0) = 0{\cdot}2\ [\ 0^3 + 2\ ] = 0{\cdot}4$$

$$k_2 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0.2\ f(0{\cdot}1,\ 2{\cdot}2)$$

$$= 0{\cdot}2\ [\ 0{\cdot}1^3 + 2{\cdot}2\ ] = 0{\cdot}2\ [\ 2{\cdot}2201\ ] = 0{\cdot}4402.$$

$$k_3 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0{\cdot}2\, f\left(0{\cdot}1,\ 2 + \frac{0{\cdot}4402}{2}\right)$$

$$= 0{\cdot}2\ [\ 0{\cdot}1^3 + 2{\cdot}2201\ ] = 0{\cdot}2\ [\ 2{\cdot}2211\ ] = 0{\cdot}44422$$

$$k_4 = h f(x_0 + h,\ y_0 + k_3) = 0{\cdot}2\ f(0{\cdot}2,\ 2 + 0{\cdot}44422)$$

$$= 0{\cdot}2\ [\ 0{\cdot}2^3 + 2{\cdot}44422\ ] = 0{\cdot}2\ [\ 2{\cdot}45222\ ] = 0{\cdot}490444$$

$$\therefore\ k = \frac{1}{6}[\ k_1 + 2\,k_2 + 2\,k_3 + k_4\ ]$$

$$= \frac{1}{6}[\ 0{\cdot}4 + 2\,(0{\cdot}4402) + 2\,(0{\cdot}44422) + 0{\cdot}49444\ ]$$

$$= \frac{1}{6}[\ 2{\cdot}65928\ ] = 0{\cdot}44321.$$

$\therefore$ The approximate value of $y = y_0 + k = 2 + 0{\cdot}44321 = 2{\cdot}44321$.

Example 3 (c) : Given $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ with $y\ (0) = 1$. Using Runge-Kutta method of fourth order find y when $x = 0{\cdot}2,\ 0{\cdot}4$. **(M.U. 2015)**

Sol. : We have $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ $\quad\therefore\quad f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2},\ x_0 = 0,\ y_0 = 1,\ h = 0{\cdot}2.$

(a) $k_1 = h\,f(x_0,\ y_0) = 0{\cdot}2\left[\dfrac{1-0}{1+0}\right] = 0{\cdot}2$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0{\cdot}2\,f(0{\cdot}1,\ 1{\cdot}1)$$

$$= 0{\cdot}2\left[\frac{(1{\cdot}1)^2 - (0{\cdot}1)^2}{(1{\cdot}1)^2 + (0{\cdot}1)^2}\right] = 0{\cdot}2\,[\,0{\cdot}9836\,] = 0{\cdot}19672.$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0{\cdot}2\,f\left(0{\cdot}1,\ 1 + \frac{0{\cdot}19672}{2}\right)$$

$$= 0{\cdot}2\,f(0{\cdot}1,\ 1{\cdot}09836) = 0{\cdot}2\left[\frac{(1{\cdot}09836)^2 - (0{\cdot}1)^2}{(1{\cdot}09836)^2 + (0{\cdot}1)^2}\right]$$

$$= 0{\cdot}2\,[\,0{\cdot}9835\,] = 0{\cdot}1967$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = 0{\cdot}2\,f(0{\cdot}2,\ 1{\cdot}1967)$$

$$= 0{\cdot}2\left[\frac{(1{\cdot}1967)^2 - (0{\cdot}2)^2}{(1{\cdot}1967)^2 + (0{\cdot}2)^2}\right]$$

$$= 0{\cdot}2\,[\,0{\cdot}9455\,] = 0{\cdot}1891$$

$$\therefore\quad k = \frac{1}{6}[\,k_1 + 2\,k_2 + 2\,k_3 + k_4\,]$$

$$= \frac{1}{6}[\,0{\cdot}2 + 2(0{\cdot}19672) + 2(0{\cdot}1967) + 0{\cdot}1891\,]$$

$$= 0{\cdot}19598$$

$\therefore$ The approximate value of y

$$y = y_0 + k = 1 + 0{\cdot}19598 = 1{\cdot}19598.$$

(b) To find y at $x = 0{\cdot}4$, we have $x_0 = 0{\cdot}2$, $y_0 = 1{\cdot}19598$ and $h = 0{\cdot}2$.

$$k_1 = h\,f(x_0,\ y_0) = 0{\cdot}2\left[\frac{(1{\cdot}19598)^2 - (0{\cdot}2)^2}{(1{\cdot}19598)^2 + (0{\cdot}2)^2}\right] = 0{\cdot}1891$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0{\cdot}2\left[\frac{(1{\cdot}29055)^2 - (0{\cdot}3)^2}{(1{\cdot}29055)^2 + (0{\cdot}3)^2}\right] = 0{\cdot}17949$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0{\cdot}2\left[\frac{(1{\cdot}28572)^2 - (0{\cdot}3)^2}{(1{\cdot}28572)^2 + (0{\cdot}3)^2}\right] = 0{\cdot}1793$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = 0{\cdot}2\left[\frac{(1{\cdot}37528)^2 - (0{\cdot}4)^2}{(1{\cdot}37528)^2 + (0{\cdot}4)^2}\right] = 0{\cdot}1687$$

$$\therefore\quad k = \frac{1}{6}[\,k_1 + 2\,k_2 + 2\,k_3 + k_4\,]$$

$$= \frac{1}{6}[\,0{\cdot}1891 + 2(0{\cdot}17949) + 2(0{\cdot}1793) + 0{\cdot}1687\,]$$

$$= 0{\cdot}1792$$

Example 4 (c) : Solve $\frac{dy}{dx} = xy$ with initial conditions $y(1) = 2$ and find y at $x = 1{\cdot}2$, $1{\cdot}4$ by Runge-Kutta Method of Fourth Order. **(M.U. 2003, 04, 05)**

Sol. : We have $\frac{dy}{dx} = xy$ $\therefore f(x, y) = xy$ and $x_0 = 1$, $y_0 = 2$, $h = 0{\cdot}2$.

(a)
$$k_1 = h\,f(x_0, y_0) = 0{\cdot}2\,(1 \times 2) = 0{\cdot}4$$
$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0{\cdot}2\,[(1 + 0{\cdot}1)(2 + 0{\cdot}2)] = 0{\cdot}484.$$
$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0{\cdot}2\,[(1 + 0{\cdot}1)(2 + 0{\cdot}242)] = 0{\cdot}49324.$$
$$k_4 = h\,f(x_0 + h, y_0 + k_3) = 0{\cdot}2\,[(1 + 0{\cdot}2)(2 + 0{\cdot}49324)] = 0{\cdot}598378.$$
$$\therefore\ k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= \frac{1}{6}[0{\cdot}4 + 2(0{\cdot}484) + 2(0{\cdot}49324) + 0{\cdot}598378] = 0{\cdot}492143.$$

Hence, the approximate value of $y = y_0 + k = 2 + 0{\cdot}492143 = 2{\cdot}492143$.

(b) Again to find y at $x = 1{\cdot}4$, we have $x_0 = 1{\cdot}2$, $y_0 = 2{\cdot}492143$, $h = 0{\cdot}2$
$$k_1 = h\,f(x_0, y_0) = 0{\cdot}2\,(1{\cdot}2 \times 2{\cdot}492143) = 0{\cdot}598114$$
$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0{\cdot}2\left[(1{\cdot}2 + 0{\cdot}1)\left(2{\cdot}492143 + \frac{0{\cdot}598114}{2}\right)\right]$$
$$= 0{\cdot}725712$$
$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0{\cdot}2\left[(1{\cdot}2 + 0{\cdot}1)\left(2{\cdot}492143 + \frac{0{\cdot}725712}{2}\right)\right]$$
$$= 0{\cdot}7423$$
$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = 0{\cdot}2\,[(1{\cdot}2 + 0{\cdot}2)(2{\cdot}492143 + 0{\cdot}7423)]$$
$$= 0{\cdot}905644.$$
$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= \frac{1}{6}[0{\cdot}598114 + 2(0{\cdot}725712) + 2(0{\cdot}7423) + 0{\cdot}905644]$$
$$= 0{\cdot}739964.$$

Hence, the approximate value of $y = y_0 + k = 2{\cdot}492143 + 0{\cdot}739964 = 3{\cdot}23107$.

Example 5 (c) : Solve $\frac{dy}{dx} = xy$ with $x_0 = 1$, $y_0 = 1$ at $x = 1{\cdot}2$ taking $h = 0{\cdot}1$ by Runge-Kutta method. **(M.U. 2016)**

Sol. : We have $\frac{dy}{dx} = xy$, $f(x, y) = xy$ and $x_0 = 1$, $y_0 = 1$, $h = 0{\cdot}1$.

(a)
$$k_1 = h\,f(x_0, y_0) = 0{\cdot}1\,(1 \times 1) = 0{\cdot}1$$
$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0{\cdot}1\,(1{\cdot}05 \times 1{\cdot}05) = 0{\cdot}11025$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0{\cdot}1\ (1{\cdot}05 \times 1{\cdot}055125) = 0{\cdot}11079$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = 0{\cdot}1\ (1{\cdot}1 + 1{\cdot}11079) = 0{\cdot}2211$$

$$k = \frac{1}{6}[\,k_1 + 2k_2 + 2k_3 + k_4\,]$$

$$= \frac{1}{6}\ [\,0{\cdot}1 + 2\,(0{\cdot}11025) + 2\,(0{\cdot}11079) + 0{\cdot}2211\,]$$

$$\therefore\ k = \frac{1}{6}\ [\,0{\cdot}76318\,] = 0{\cdot}1272$$

$$\therefore\ y = y_0 + k = 1{\cdot}1272$$

(b) Again to find y at $x = 1{\cdot}2$, we have $x_0 = 1{\cdot}1$, $y_0 = 1{\cdot}1272$ and $h = 0{\cdot}1$.

$$k_1 = h\,f(x_0,\ y_0) = 0{\cdot}1\ (1{\cdot}1 \times 1{\cdot}1272) = 0{\cdot}1240$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0{\cdot}1\ (1{\cdot}15 \times 1{\cdot}1892) = 0{\cdot}1368$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0{\cdot}1\ (1{\cdot}15 \times 1{\cdot}1956) = 0{\cdot}1375$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = 0{\cdot}1\ (1{\cdot}2 \times 1{\cdot}2647) = 0{\cdot}1518$$

$$k = \frac{1}{6}[\,k_1 + 2k_2 + 2k_3 + k_4\,]$$

$$= \frac{1}{6}\ [\,0.1240 + 2\,(0{\cdot}1368) + 2\,(0{\cdot}1375) + 0{\cdot}1518\,]$$

$$\therefore\ k = \frac{1}{6}\ [\,0{\cdot}8244\,] = 0{\cdot}1374$$

$$\therefore\ y = y_0 + k = 1{\cdot}1 + 0{\cdot}1374 = 1{\cdot}2374.$$

Example 6 (c) : Solve numerically (using Runge-Kutta Fourth Order Formula) the differential equation $\frac{dy}{dx} = x^2 + y^2$ with the given condition $x = 1$, $y = 1{\cdot}5$ in the interval $(1,\ 1{\cdot}2)$ with $h = 0{\cdot}1$.

(M.U. 2007, 14)

Sol. : We have $\frac{dy}{dx} = x^2 + y^2$ $\quad\therefore\ f(x, y) = x^2 + y^2$ and $x_0 = 1$, $y_0 = 1{\cdot}5$, $h = 0{\cdot}1$.

(a)

$$k_1 = hf(x_0,\ y_0) = 0{\cdot}1\ [\,(1)^2 + (1{\cdot}5)^2\,] = 0{\cdot}325$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0{\cdot}1\left[(1 + 0{\cdot}05)^2 + (1{\cdot}5 + 0{\cdot}1625)^2\right] = 0{\cdot}38664$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0{\cdot}1\left[(1 + 0{\cdot}05)^2 + (1{\cdot}5 + 0{\cdot}19332)^2\right] = 0{\cdot}39698$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = 0{\cdot}1\ [\,(1 + 0{\cdot}1)^2 + (1{\cdot}5 + 0{\cdot}39698)^2\,] = 0{\cdot}48085$$

$$\therefore\ k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= \frac{1}{6}\left[0{\cdot}325 + 2\,(0{\cdot}38664) + 2\,(0{\cdot}39698) + 0{\cdot}48085\right]$$

$$= 0{\cdot}3955$$

Hence, the approximate value of $y = y_0 + k = 1{\cdot}5 + 0{\cdot}3955 = 1{\cdot}8955.$

(b) Again to find y at $x = 1{\cdot}2$, we have $x_0 = 1{\cdot}1$, $y_0 = 1{\cdot}8955$, $h = 0{\cdot}1$, $f(x, y) = x^2 + y^2$.

$$\therefore \quad k_1 = h\,f(x_0, y_0) = 0{\cdot}1[(1{\cdot}1)^2 + (1{\cdot}8955)^2] = 0{\cdot}48029$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0{\cdot}1\left[(1{\cdot}1 + 0{\cdot}05)^2 + (1{\cdot}8955 + 0{\cdot}24014)^2\right] = 0{\cdot}58835$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0{\cdot}1\left[(1{\cdot}1 + 0{\cdot}05)^2 + (1{\cdot}8955 + 0{\cdot}29417)^2\right] = 0{\cdot}61171$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = 0{\cdot}1\left[(1{\cdot}1 + 0{\cdot}1)^2 + (1{\cdot}8955 + 0{\cdot}61171)^2\right] = 0{\cdot}77261$$

$$\therefore \quad k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] = 0{\cdot}6088$$

$\therefore$ The approximate value of $y = y_0 + k = 1{\cdot}8955 + 0{\cdot}6088 = 2{\cdot}5043$.

Example 7 (c) : Apply Runge-Kutta Method of Fourth Order to find an approximate value of y at $x = 0{\cdot}2$ if $\dfrac{dy}{dx} = x + y^2$, given that $y = 1$ when $x = 0$ in steps of $h = 0{\cdot}1$. **(M.U. 2005, 11, 12, 13, 14)**

Sol. : We have $\dfrac{dy}{dx} = x + y^2 \quad \therefore \quad f(x, y) = x + y^2$ and $x_0 = 0$, $y_0 = 1$, $h = 0{\cdot}1$.

(a)
$$k_1 = h\,f(x_0, y_0) = 0{\cdot}1[0^2 + 1^2] = 0{\cdot}1$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0{\cdot}1\left[(0 + 0{\cdot}05) + (1 + 0{\cdot}05)^2\right] = 0{\cdot}11525$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0{\cdot}1\left[(0{\cdot}05) + (1 + 0{\cdot}05762)^2\right] = 0{\cdot}11686$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = 0{\cdot}1\left[(0 + 0{\cdot}1) + (1 + 0{\cdot}11686)^2\right] = 0{\cdot}13474$$

$$\therefore \quad k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$\therefore \quad k = \frac{1}{6}\left[0{\cdot}1 + 2(0{\cdot}11525) + 2(0{\cdot}11686) + 0{\cdot}13474\right] = 0{\cdot}1165$$

$\therefore$ The approximate value of $y = y_0 + k = 1 + 0{\cdot}1165 = 1{\cdot}1165$.

(b) Again to find y at $x = 0{\cdot}2$, we have $x_0 = 0{\cdot}1$, $y_0 = 1{\cdot}1165$, $h = 0{\cdot}1$ and $f(x, y) = x + y^2$.

$$\therefore \quad k_1 = h\,f(x_0, y_0) = 0{\cdot}1[0{\cdot}1^2 + (1{\cdot}1165)^2] = 0{\cdot}13466$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0{\cdot}1\left[(0{\cdot}1 + 0{\cdot}05) + (1{\cdot}1165 + 0{\cdot}06733)^2\right] = 0{\cdot}15514$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0{\cdot}1\left[(0{\cdot}1 + 0{\cdot}05) + (1{\cdot}1165 + 0{\cdot}07757)^2\right] = 0{\cdot}15758$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = 0{\cdot}1\left[(0{\cdot}1 + 0{\cdot}1) + (1{\cdot}1165 + 0{\cdot}15758)^2\right] = 0{\cdot}18233$$

$$\therefore \quad k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$\therefore \quad k = \frac{1}{6}\left[0{\cdot}13466 + 2(0{\cdot}15514) + 2(0{\cdot}15758) + 0{\cdot}18233\right] = 0{\cdot}1571$$

$\therefore$ The approximate value of $y = y_0 + k = 1{\cdot}1165 + 0{\cdot}1571 = 1{\cdot}2736$.

Example 8 (c) : Solve the differential equation $\frac{dy}{dx} = \frac{1}{(x+y)}$, $x_0 = 0$, $y_0 = 1$ for the interval (0, 1) choosing $h = 0 \cdot 5$ by Runge-Kutta Method of Fourth Order. **(M.U. 2009, 10, 11, 12, 15)**

Sol. : We have $\frac{dy}{dx} = \frac{1}{(x+y)}$

$\therefore \quad f(x, y) = \frac{1}{(x+y)}$, $x_0 = 0$, $y_0 = 1$ and $h = 0 \cdot 5$.

(a)
$$k_1 = h f(x_0, y_0) = 0 \cdot 5 \left[\frac{1}{0+1}\right] = 0 \cdot 5$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0 \cdot 5 \left[\frac{1}{(0+0 \cdot 25)+(1+0 \cdot 25)}\right] = 0 \cdot 33333$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0 \cdot 5 \left[\frac{1}{(0+0 \cdot 25)+(1+0 \cdot 16666)}\right] = 0 \cdot 35294$$

$$k_4 = h f(x_0 + h, y_0 + k_3) = 0 \cdot 5 \left[\frac{1}{(0+0 \cdot 5)+(1+0 \cdot 35294)}\right] = 0 \cdot 26984$$

$$\therefore \quad k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$\therefore \quad k = \frac{1}{6}[0 \cdot 5 + 2(0 \cdot 33333) + 2(0 \cdot 35294) + 0 \cdot 26984]$$

$$= 0 \cdot 3571$$

$\therefore$ The approximate value of $y = y_0 + k = 1 + 0 \cdot 3571 = 1 \cdot 3571$.

(b) To find y at $x = 1$, we have $x_0 = 0 \cdot 5$, $y_0 = 1 \cdot 3571$ and $h = 0 \cdot 5$.

$$\therefore \quad k_1 = h f(x_0, y_0) = 0 \cdot 5 \left[\frac{1}{(0 \cdot 5 + 1 \cdot 3571)}\right] = 0 \cdot 26924$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0 \cdot 5 \left[\frac{1}{(0 \cdot 5 + 0 \cdot 25)+(1 \cdot 3571 + 0 \cdot 13462)}\right] = 0 \cdot 22304$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0 \cdot 5 \left[\frac{1}{(0 \cdot 5 + 0 \cdot 25)+(1 \cdot 3571 + 0 \cdot 11152)}\right]$$

$$= 0 \cdot 22536$$

$$k_4 = h f(x_0 + h, y_0 + k_3) = 0 \cdot 5 \left[\frac{1}{(0 \cdot 5 + 0 \cdot 5)+(1 \cdot 3571 + 0 \cdot 22536)}\right]$$

$$= 0 \cdot 19361$$

$$\therefore \quad k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0 \cdot 26924 + 2(0 \cdot 22304) + 2(0 \cdot 22536) + 0 \cdot 19361]$$

$$= 0 \cdot 22661$$

$\therefore$ The approximate value of $y = y_0 + k = 1 \cdot 3571 + 0 \cdot 22661 = 1 \cdot 5837$.

Example 9 (c) : Compute $y(0\cdot2)$ given $\dfrac{dy}{dx} + y + xy^2 = 0$, $y(0) = 1$ by taking $h = 0\cdot1$ using Runge-Kutta method of fourth order; correct to 4 decimal places. **(M.U. 2013)**

Sol. : We have $\dfrac{dy}{dx} = -(xy+1)\,y$ and $x_0 = 0$, $y_0 = 1$ and $h = 0\cdot1$.

(a)

$$k_1 = h\,f(x_0, y_0) = -\,0\cdot1\,(0+1) = -\,0\cdot1$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = -\,0\cdot1\,[\,(0\cdot05)(0\cdot95) + 1\,]\,(0\cdot95) = -\,0\cdot0995$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = -\,0\cdot1\,[\,-(0\cdot05)(0\cdot95025) + 1\,]\,(0\cdot95025) = -\,0\cdot0995$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = -\,0\cdot1\,[\,(0\cdot1)(0\cdot9005) + 1\,]\,(0\cdot9005) = -\,0\cdot0982$$

$$k = \frac{1}{6}[\,k_1 + 2k_2 + 2k_3 + k_4\,]$$

$$= \frac{1}{6}\,[\,-(0\cdot1) + 2\,(-\,0\cdot0995) + 2\,(-\,0\cdot0995) - 0\cdot0982\,]$$

$$\therefore\ k = -\,0\cdot0994$$

$$\therefore\ y_1 = y_0 + k = 1 + (-\,0\cdot0994) = 0\cdot9006$$

(b) Again to find y at $x = 0\cdot2$, we have $x_0 = 0\cdot1$, $y_0 = 0\cdot9006$ and $h = 0\cdot1$.

$$k_1 = h\,f(x_0, y_0) = -\,0\cdot1\,[\,(0\cdot1)(0\cdot9006) + 1\,]\,(0\cdot9006) = -\,0\cdot0982$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = -\,0\cdot1\,[\,(0\cdot15)(0\cdot8515) + 1\,]\,(0\cdot8515) = -\,0\cdot0960$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = -\,0\cdot1\,[\,(0\cdot15)(0\cdot08526) + 1\,]\,(0\cdot8526) = -\,0\cdot0962$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = -\,0\cdot1\,[\,(0\cdot2)(0\cdot8044) + 1\,]\,(0\cdot8044) = -\,0\cdot0934$$

$$k = \frac{1}{6}[\,k_1 + 2k_2 + 2k_3 + k_4\,]$$

$$= \frac{1}{6}\,[\,-\,0.0982 + 2\,(-\,0\cdot0960) + 2\,(-\,0\cdot0962) + (-\,0\cdot0934)\,]$$

$$\therefore\ k = -\,0\cdot096$$

$$\therefore\ y_2 = y_0 + k = 0\cdot9006 + (-\,0\cdot096) = 0\cdot8046.$$

EXERCISE - IV

Apply Runge-Kutta Method of Fourth Order to find an approximate value of y given that : Class (c) : 8 Marks

1. $\dfrac{dy}{dx} = 3x + y^2$, $x_0 = 1$, $y_0 = 1\cdot2$ at $x = 1\cdot1$. **(M.U. 2008) [Ans. : 1·7271]**
2. $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$ at $x = 0\cdot2$. **[Ans. : 1·1678]**
3. $\dfrac{dy}{dx} = 1 + y^2$, $x_0 = 0$, $y_0 = 0$ at $x = 0\cdot2, 0\cdot4, 0\cdot6$. **(M.U. 2004)**

   [Ans. : 0·2027, 0·4228, 0·6891]

4. $\frac{dy}{dx} = xy^2$, $x_0 = 2$, $y_0 = -1$ for $x = 2.2$, taking $h = 0.2$ and compare with exact value.
[**Ans. :** -0.7044, Exact value $= -0.7042$]

5. $\frac{dy}{dx} = x - y^2$, $x_0 = 0$, $y_0 = 1$ at $x = 0.2$, taking $h = 0.1$. [**Ans. :** 0.8512]

6. $\frac{dy}{dx} = \sqrt{x+y}$, $x_0 = 0.4$, $y_0 = 0.41$ at $x = 0.8$ in two steps. **(M.U. 2009)** [**Ans. :** 0.8311]

7. $\frac{dy}{dx} = \frac{y}{x}$, $x_0 = 1$, $y_0 = 1$ at $x = 1.2$, taking $h = 0.1$. **(M.U. 2008)** [**Ans. :** 1.2]

8. $\frac{dy}{dx} = \frac{1}{x+y}$, $x_0 = 0$, $y_0 = 2$ at $x = 0.4$, taking $h = 0.2$. [**Ans. :** 2.1775]

9. $\frac{dy}{dx} = 0.3 + 0.25y + 0.3x$, $x_0 = 0$, $y_0 = 0.72$ at $x = 0.2$. [**Ans. :** 0.8266]

10. $\frac{dy}{dx} = \frac{x+y}{xy}$, $x_0 = 1$, $y_0 = 1$ at $x = 1.1$. [**Ans. :** 1.19967]

11. $\frac{dy}{dx} = y - x$, $x_0 = 0$, $y_0 = 2$ at $x = 0.1$ and 0.2. [**Ans. :** 2.2052, 1.4222]

12. $\frac{dy}{dx} = \frac{1}{x+y}$ with $y(1) = 1$. Find $y(2)$ in two steps. [**Ans. :** 1.3779]

13. $(x+y)\frac{dy}{dx} = 1$, $y(0) = 1$ for $x = 1$. [**Ans. :** 1.5837]

14. $\frac{dy}{dx} = x^2 + y^2$, $x_0 = 0$, $y_0 = 1$ at $x = 0.2$ taking $h = 0.1$. [**Ans. :** 0.11146]

15. $\frac{dy}{dx} = 1 + xy$ with $y(0) = 2$ for $x = 0.1$, $x = 0.2$. **(M.U. 2004)**
[**Ans. :** (i) -1.3714, (ii) -1.2844]

EXERCISE - V

Solve the following examples : Class (a) : 3 or 4 Marks

1. If $\frac{dy}{dx} = y - xy$ with $x_0 = 0$, $y_0 = 2$ then find the first three terms of the Taylor's series for y.
[**Ans. :** $y = 2 + 2x - \frac{2}{3}x^3$]

2. If $\frac{dy}{dx} = -xy^2$ with $x_0 = 0$, $y_0 = 2$, find the first two terms of the Taylor's series for y.
[**Ans. :** $y = 2 - 2x^2$]

3. If $\frac{dy}{dx} = x^2y - 1$, with $x_0 = 0$, $y_0 = 1$, find the first three terms of Taylor's series for y.
[**Ans. :** $y = 1 - x + \frac{x^3}{3}$]

4. If $\frac{dy}{dx} = 3x + y^2$ with $x_0 = 1$, $y_0 = 1$, find the first four terms of Taylor's series for y.
[**Ans. :** $y = 1 + x + \frac{5}{2}x^2 + 2x^3$]

5. If $\frac{dy}{dx} = x - y^2$ with $x_0 = 0$, $y_0 = 1$, find the first four terms of Taylor's series for y.
[**Ans. :** $y = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3$]

6. Using Euler's method find the approximate value of y at $x = 0{\cdot}04$ in two steps, given that $\frac{dy}{dx} = \frac{y - x}{y + x}$ and $y(0) = 1$.
[**Ans. :** 1·0392]

7. Using Euler's method find the approximate value of y at 1·2 in two steps, given that $\frac{dy}{dx} = y^2 - \frac{y}{x}$ and $y(1) = 1$.
[**Ans. :** 1·00909]

8. Using Euler's method find the approximate value of y at $x = 0{\cdot}02$ in two steps given that $\frac{dy}{dx} = x + y^2$ and $y(0) = 1$.
[**Ans. :** 1·020301]

9. Using Euler's method find the approximate value of y at $x = 0{\cdot}6$, taking $h = 0{\cdot}2$, given $\frac{dy}{dx} = 1 - 2xy$ and $y(0) = 0$.
[**Ans. :** 0·52256]

10. Solve $\frac{dy}{dx} = \log(x + y)$, $y(1) = 2$ for $x = 1{\cdot}2$ taking only one approximation by Euler's modified method.
[**Ans. :** 2·2328]

11. Solve $\frac{dy}{dx} = x^2 + y$, $y(0) = 0{\cdot}94$ for $x = 0{\cdot}1$ by Euler's modified method taking only one approximation.
[**Ans. :** 1·0392]

12. Solve $\frac{dy}{dx} = x + y$, $y(0) = 1$ at $x = 0{\cdot}5$ by Euler's modified method taking only one approximation.
[**Ans. :** 1·75]

13. Solve $\frac{dy}{dx} = \frac{xy}{1 + x^2}$ with initial conditions $x_0 = 0$, $y_0 = 1$ for $x = 0{\cdot}1$ using Runge-Kutta method of fourth order, taking $h = 0{\cdot}1$.
[**Ans. :** 1·004992]

14. Using Runge-Kutta method of fourth order solve $\frac{dy}{dx} = x + y$ given $x = 0$, $y = 1$ for $x = 0{\cdot}1$.
[**Ans. :** 1·1305]

15. Solve $\frac{dy}{dx} = -y$, with initial conditions $x_0 = 0$, $y_0 = 1$ for $x = 0{\cdot}1$ using Runge-Kutta method of fourth order.
[**Ans. :** 0·90484]

Summary

1. **Euler's Method :** $y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})$

2. **Euler's Modified Method**

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0 + y_0) + f(x_1, y_1)\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0 + y_0) + f(x_1, y_1^{(1)})\right]$$

3. Taylor's Series Method

$$y = y_0 + x\, y_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \ldots\ldots \qquad \text{Then put } x = x_1.$$

4. Runge-Kutta Method

$$k_1 = h\, f(x_0,\ y_0)\ ; \qquad k_2 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right);$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right); \qquad k_4 = h\, f(x_0 + h,\ y_0 + k_3)\ ;$$

$$k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]; \qquad y = y_0 + k.$$

* * *

CHAPTER

6

Beta, Gamma Functions

1. Introduction

In this chapter we are going to study two important special functions viz. Gamma and Beta functions, which are related to each other. These functions are defined in terms of certain integrals which cannot be evaluated by usual methods. These functions occur frequently in Statistics, Physics and Engineering problems.

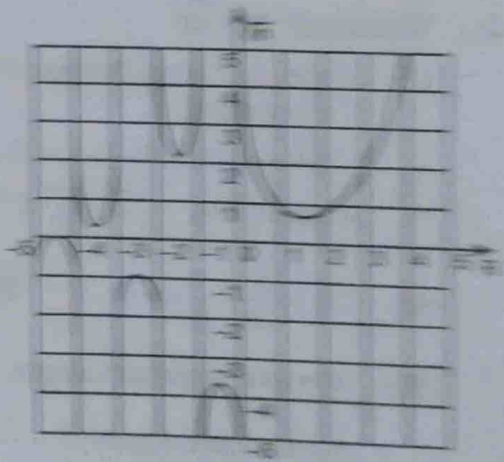
2. Definition

The function of n $(n > 0)$ defined by the integral $= \int_0^\infty e^{-x} x^{n-1}\,dx$ is called Gamma Function and is denoted by $\overline{|n}$. It can be shown that the above integral is convergent for $n > 0$. Thus,

$$\boxed{\overline{|n} = \int_0^\infty e^{-x}\, x^{n-1}\, dx} \qquad \text{——— (1)}$$

The values of $\overline{|n}$ and its graph are as follows :

n	$\overline{\vert n}$
1·00	1·0000
1·20	0·9182
1·40	0·8873
1·60	0·8935
1·80	0·9314
2·00	0·9618



Gamma function may also be remembered as

$$\boxed{\int_0^\infty e^{-x}\, x^{n}\, dx = \overline{|n+1}} \qquad \text{——— (1A)}$$

3. A Property of Gamma Function

$$\boxed{\overline{|n+1} = n\overline{|n}} \qquad \text{——— (2)}$$

Or $$\boxed{\overline{|n} = (n-1)\overline{|n-1}} \qquad \text{——— (2A)}$$

Proof : By changing n to $n+1$ in the definition (1)

$$\overline{|n+1} = \int_0^{\infty} e^{-x} x^n \, dx$$

Integrating by parts,

$$\overline{|n+1} = \left[-x^n e^{-x}\right]_0^{\infty} - \int_0^{\infty} -e^{-x} \cdot n x^{n-1} \, dx$$

But $\lim_{x \to \infty} \dfrac{x^n}{e^x} = \lim_{x \to \infty} \dfrac{n x^{n-1}}{e^x} = \ldots\ldots = \lim_{x \to \infty} \dfrac{n!}{e^x} = 0$ ...... [By Hospital's Rule]

$$\therefore \overline{|n+1} = n \int_0^{\infty} e^{-x} x^{n-1} \, dx = n \overline{|n}$$

Cor. If n is a positive integer, by successive application of (2), we get,

$$\overline{|n+1} = n \overline{|n} = n \cdot (n-1) \overline{|n-1} = n(n-1)(n-2) \overline{|n-2}$$

$$\therefore \boxed{\overline{|n+1} = n!} \qquad \text{.......... (3)}$$

Alternatively, the result means $\boxed{\overline{|n} = (n-1)!}$ .... (3A)

The result (3) is true for $n = 1, 2, 3, \ldots$ and for this reason the **Gamma Function** $\overline{|n}$ is often referred to as **Generalised Factorial**.

Combining (1A) and (3), we have, if n is a positive integer

$$\boxed{\int_0^{\infty} e^{-x} x^n \, dx = n!} \qquad \text{.......... (3B)}$$

4. Values of $\overline{|n}$

(i) **If n is a positive integer**, it is clear that we can use (3) and easily find the value of $\overline{|n}$. For example,

$\overline{|5} = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$; $\quad \overline{|3} = 2! = 2 \cdot 1 = 2$; $\quad \overline{|1} = 0! = 1$

Alternatively, $\overline{|1} = \int_0^{\infty} e^{-x} x^0 \, dx = \left[-e^{-x}\right]_0^{\infty} = 1$.

(ii) $\overline{|0} = \dfrac{\overline{|1}}{0} = \dfrac{1}{0} = \infty \quad \left[\because \text{By (2)}, \overline{|n} = \dfrac{\overline{|n+1}}{n}\right]$

(iii) **If n is a negative integer, $\overline{|n}$ is infinite.**

By repeated use of $\overline{|n} = \dfrac{\overline{|n+1}}{n}$, we get $\overline{|n} = \infty$ if $n = -1, -2, -3, \ldots$ For example,

$$\overline{|-4} = \frac{\overline{|-3}}{(-4)} = \frac{\overline{|-2}}{(-4)(-3)} = \frac{\overline{|-1}}{(-4)(-3)(-2)} = \frac{\overline{|0}}{(-4)(-3)(-2)(-1)} = \infty$$

(because $\overline{|0} = \infty$)

(iv) **If n is a positive fraction**, we can use (3) repeatedly and find $\overline{|n}$ in terms of $\overline{|x}$ where $0 < x < 1$. For example,

$$\overline{\left|\frac{5}{2}\right.} = \left(\frac{3}{2}\right) \overline{\left|\frac{3}{2}\right.} = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \overline{\left|\frac{1}{2}\right.}$$

(v) **If n is a negative fraction**, we can use $\overline{|n} = \dfrac{\overline{|n+1}}{n}$ repeatedly and find $\overline{|n}$ in terms of $\overline{|x}$ where $0 < x < 1$. For example,

$$\overline{|-5/3} = \frac{\overline{|-2/5}}{(-5/3)} = \frac{\overline{|3/5}}{(-5/3)(-2/5)}$$

(vi) **If $0 < n < 1$**, we find $\overline{|n}$ by numerical integration.

Thus, $\overline{|n}$ is defined for all n, except $n = 0, -1, -2, -3, \ldots$

5. Evaluation of Integrals

With the help of the definition of Gamma function, we can now evaluate the integrals of the following type very easily.

(i) $\displaystyle\int_0^\infty \frac{e^{-x}}{x}\,dx = \int_0^\infty e^{-x}x^{-1}\,dx = \overline{|-1+1} = \overline{|0} = \infty$

(ii) $\displaystyle\int_0^\infty \frac{e^{-x}}{x^2}\,dx = \int_0^\infty e^{-x}x^{-2}\,dx = \overline{|-2+1} = \overline{|-1} = \infty$

(iii) $\displaystyle\int_0^\infty e^{-x}x^{-1/2}\,dx = \overline{\left|\left(-\frac{1}{2}\right)+1\right.} = \overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$ [See (9), page 6-18]

(iv) $\displaystyle\int_0^\infty e^{-x^2}x^{3/2}\,dx = \overline{\left|\left(\frac{3}{2}\right)+1\right.} = \overline{\left|\frac{5}{2}\right.} = \frac{3}{2}\overline{\left|\frac{3}{2}\right.}$

$$= \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\overline{\left|\frac{1}{2}\right.} = \frac{3}{4}\sqrt{\pi}.$$

(v) $\displaystyle\int_0^\infty e^{-x}x^2\,dx = \overline{|2+1} = \overline{|3} = 2! = 2$

(vi) $\displaystyle\int_0^\infty e^{-x}\cdot x^3\,dx = \overline{|3+1} = \overline{|4} = 3! = 6$

6. Second form of Gamma Function

$$\boxed{\overline{|n} = 2\int_0^\infty e^{-x^2}\cdot x^{2n-1}\,dx} \qquad \ldots\ldots (4)$$

Proof : In (1) if we put $x = z^2$, $dx = 2z\,dz$

$$\overline{|n} = \int_0^\infty e^{-z^2}\cdot(z^2)^{n-1}\,2z\,dz$$

$$= 2\int_0^\infty e^{-z^2}\cdot z^{2n-1}\,dz$$

$$= 2\int_0^\infty e^{-x^2}\cdot x^{2n-1}\,dx$$

Thus, $$\boxed{\int_0^\infty e^{-x^2}\cdot x^{2n-1}\,dx = \frac{1}{2}\overline{|n}} \qquad \ldots\ldots (4A)$$

Proof : Put $ax^n = t$ $\quad \therefore \ x^n = t/a$

$$\therefore \ x = \left(\frac{t}{a}\right)^{1/n} \text{ and } dx = \frac{1}{an}\cdot\left(\frac{t}{a}\right)^{(1/n)-1}\cdot dt$$

$$\therefore \ \int_0^\infty e^{-ax^n}\, dx = \int_0^\infty e^{-t}\cdot\frac{1}{an}\cdot\left(\frac{t}{a}\right)^{(1/n)-1}\cdot dt$$

$$= \frac{1}{n\cdot\sqrt[n]{a}}\int_0^\infty e^{-t}\, t^{(1/n)-1}\, dt = \frac{1}{n\cdot\sqrt[n]{a}}\left|\overline{\frac{1}{n}}\right..$$

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Evaluate $\int_0^\infty e^{-h^2x^2}\, dx$.

Sol. : In the above result (5), put $h^2 = = a$ and $n = 2$ or independently put $h^2x^2 = t$.

$$\therefore \ x = \frac{\sqrt{t}}{h} \text{ and } dx = \frac{1}{h}\cdot\frac{1}{2\sqrt{t}}\, dt$$

$$\therefore \ \int_0^\infty e^{-h^2x^2}\, dx = \int_0^\infty \frac{1}{2h}\, e^{-t}\, t^{-1/2}\, dt = \frac{1}{2h}\left|\overline{\frac{1}{2}}\right. = \frac{\sqrt{\pi}}{2h}.$$

Cor. : $\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}$ and $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

Note ....

For the time being we shall assume the result that $\left|\overline{1/2}\right. = \sqrt{\pi}$ but shall prove it later at a proper place. See (9) page 6-18.

Example 2 (a) : Evaluate $\int_0^\infty e^{-x^4}\, dx$.

Sol. : In the above result (5), put $a = 1$, $n = 4$ or independently put $x^4 = t$.

$$\therefore \ x = t^{1/4} \quad \therefore \ dx = \frac{1}{4}\cdot t^{-3/4}\, dt$$

$$\therefore \ \int_0^\infty e^{-x^4}\, dx = \int_0^\infty \frac{1}{4}\cdot e^{-t}\, t^{-3/4}\, dt = \frac{1}{4}\left|\overline{\frac{1}{4}}\right..$$

Note ....

Putting $h = 1$ in Ex. 1, we get

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \quad \text{i.e. } \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

This is an important result and is often required in certain integrals.

Example 3 (a) : Evaluate $\int_0^\infty e^{-x^2}\,dx$

Solution : In the above result (5), put $a = 1$, $n = 2$ or independently put $x^2 = t$

$$\therefore\ x = t^{1/2} \quad \therefore\ dx = \frac{1}{2}t^{-1/2}\,dt$$

$$\therefore\ \int_0^\infty e^{-x^2}\,dx = \int_0^\infty \frac{1}{2}\cdot e^{-t}\cdot t^{-1/2}\,dt$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Remark ....

Type I is a particular case of type II (given below) when $m = 0$.

EXERCISE - I

Evaluate the following integrals : Class (a) : 3 Marks

1. $\int_0^\infty e^{-x^3}\,dx$ 2. $\int_0^\infty e^{-x^2/4}\,dx$ 3. $\int_0^\infty e^{-x^5}\,dx$

[**Ans : (1)** $\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$ (2) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (3) $\frac{1}{5}\cdot\Gamma\left(\frac{4}{5}\right)$]

Type II

Prove that $$\boxed{\int_0^\infty x^m \cdot e^{-ax^n}\,dx = \frac{1}{n}\cdot\frac{1}{a^{(m+1)/n}}\Gamma\left(\frac{m+1}{n}\right)} \quad \text{.............. (6)}$$

Proof : Put $ax^n = t$

$$\therefore\ x = \left(\frac{t}{a}\right)^{1/n} \quad \text{and} \quad dx = \frac{1}{an}\cdot\left(\frac{t}{a}\right)^{(1/n)-1}dt$$

$$\therefore\ \int_0^\infty x^m e^{-ax^n}\,dx = \int_0^\infty \left(\frac{t}{a}\right)^{m/n}\cdot e^{-t}\cdot\frac{1}{an}\cdot\left(\frac{t}{a}\right)^{(1/n)-1}dt$$

$$= \frac{1}{n}\cdot\frac{1}{a^{(m+1)/n}}\int_0^\infty e^{-t}\cdot t^{\left(\frac{m+1}{n}\right)-1}\,dt$$

$$= \frac{1}{n}\cdot\frac{1}{a^{(m+1)/n}}\Gamma\left(\frac{m+1}{n}\right)$$

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Evaluate $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\cdot dx.$ **(M.U. 1991)**

Sol. : In the above result (6) put $m = 1/2$, $a = 1$, $n = 1/3$ or independently put $x^{1/3} = t.$

$$\therefore\ x = t^3 \quad \therefore\ dx = 3t^2\,dt$$

$$\therefore \int_0^{\infty} \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx = \int_0^{\infty} t^{3/2} \cdot e^{-t}\, 3 \cdot t^2\, dt$$

$$= 3\int_0^{\infty} e^{-t}\, t^{7/2}\, dt = 3\,\overline{|9/2}$$

$$= 3 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\,\overline{\left|\frac{1}{2}\right.} = \frac{315}{16}\sqrt{\pi}.$$

Example 2 (a) : Evaluate $\int_0^{\infty} x^{1/4}\, e^{-\sqrt{x}}\, dx$. (M.U. 1992)

Sol. : In the above result (6), put $m = 1/4$, $a = 1$, $n = 1/2$ or independently put $\sqrt{x} = t$.

$$\therefore x = t^2 \quad \therefore dx = 2t\, dt$$

$$\int_0^{\infty} x^{1/4}\, e^{-\sqrt{x}}\, dx = \int_0^{\infty} t^{1/2} \cdot e^{-t} \cdot 2t \cdot dt$$

$$= \int_0^{\infty} 2e^{-t} \cdot t^{3/2}\, dt = 2\,\overline{\left|\frac{5}{2}\right.} = 2 \cdot \frac{3}{2} \cdot \frac{1}{2}\,\overline{\left|\frac{1}{2}\right.}$$

$$= 2 \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} = \frac{3}{2}\sqrt{\pi}.$$

Example 3 (a) : Evaluate $\int_0^{\infty} x^2\, e^{-x^8}\, dx$.

Sol. : Putting $x^8 = t$ i.e. $x = t^{1/8}$

$$dx = \frac{1}{8}\, t^{-7/8}\, dt, \text{ we get}$$

$$I = \int_0^{\infty} t^{1/4} \cdot e^{-t} \cdot \frac{1}{8} \cdot t^{-7/8}\, dt$$

$$= \frac{1}{8}\int_0^{\infty} e^{-t} \cdot t^{-5/8}\, dt = \frac{1}{8}\,\overline{\left|\frac{3}{8}\right.}.$$

Example 4 (a) : Evaluate $\int_0^{\infty} x\, e^{-x^4}\, dx$.

Sol. : Putting $x^4 = t$ i.e. $x = t^{1/4}$

$$dx = \frac{1}{4}\, t^{-3/4}\, dt, \text{ we get}$$

$$I = \int_0^{\infty} t^{1/4} \cdot e^{-t} \cdot \frac{1}{4}\, t^{-3/4}\, dt$$

$$= \frac{1}{4}\int_0^{\infty} e^{-t} \cdot t^{-1/2}\, dt = \frac{1}{4}\,\overline{\left|\frac{1}{2}\right.} = \frac{1}{4}\sqrt{\pi}$$ [By (9), page 6-18]

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Evaluate $\int_0^{\infty} (x^4 + 4)\, e^{-2x^2}\, dx$.

Sol. : We have

$$I = \int_0^{\infty} x^4\, e^{-2x^2}\, dx + 4\int_0^{\infty} e^{-2x^2}\, dx$$

Putting $2x^2 = t$, $x = \frac{1}{\sqrt{2}}\sqrt{t}$, we get $dx = \frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{t}} = \frac{1}{2\sqrt{2}} t^{-1/2}$.

$$\therefore \quad I = \int_0^\infty \frac{1}{4} t^2 \cdot e^{-t} \cdot \frac{1}{2\sqrt{2}} t^{-1/2}\, dt + 4\int_0^\infty e^{-t} \cdot \frac{1}{2\sqrt{2}} \cdot t^{-1/2}\, dt$$

$$= \int_0^\infty \frac{1}{8\sqrt{2}} t^{3/2} \cdot e^{-t}\, dt + \frac{2}{\sqrt{2}} \int_0^\infty t^{-1/2} e^{-t}\, dt$$

$$= \frac{1}{8\sqrt{2}} \Gamma\left(\frac{5}{2}\right) + \sqrt{2}\, \Gamma\left(\frac{1}{2}\right) = \frac{1}{8\sqrt{2}} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) + \frac{2}{\sqrt{2}} \Gamma\left(\frac{1}{2}\right)$$

$$= \left(\frac{3}{32\sqrt{2}} + \frac{2}{\sqrt{2}}\right) \Gamma\left(\frac{1}{2}\right) = \frac{67}{32}\sqrt{\frac{\pi}{2}} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Example 2 (c) : Show that $\int_0^\infty x e^{-x^8}\, dx \cdot \int_0^\infty x^2 e^{-x^4}\, dx = \frac{\pi}{16\sqrt{2}}$. **(M.U. 2003, 07, 12)**

Sol. : Let $I_1 = \int_0^\infty x e^{-x^8}\, dx$ and $I_2 = \int_0^\infty x^2 e^{-x^4}\, dx$

Putting $x^8 = t$ i.e. $x = t^{1/8}$; $dx = \frac{1}{8} t^{-7/8}\, dt$

$$\therefore \quad I_1 = \int_0^\infty t^{1/8} e^{-t} \cdot \frac{1}{8} t^{-7/8}\, dt = \frac{1}{8}\int_0^\infty e^{-t} t^{-6/8}\, dt$$

$$= \frac{1}{8}\int_0^\infty e^{-t} t^{-3/4}\, dt = \frac{1}{8} \Gamma\left(\frac{1}{4}\right)$$

Putting $x^4 = t$ i.e. $x = t^{1/4}$; $dx = \frac{1}{4} t^{-3/4}\, dt$

$$\therefore \quad I_2 = \int_0^\infty t^{1/2} e^{-t} \frac{1}{4} t^{-3/4}\, dt = \frac{1}{4}\int_0^\infty e^{-t} t^{-1/4}\, dt = \frac{1}{4} \Gamma\left(\frac{3}{4}\right)$$

$$\therefore \quad I_1 \cdot I_2 = \frac{1}{8} \Gamma\left(\frac{1}{4}\right) \cdot \frac{1}{4} \Gamma\left(\frac{3}{4}\right) = \frac{1}{32} \sqrt{2} \cdot \pi = \frac{\pi}{16\sqrt{2}}.$$

[By (13), page 6-19]

Example 3 (c) : Prove that $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}\, dx \cdot \int_0^\infty y^4 e^{-y^6}\, dy = \frac{\pi}{9}$. **(M.U. 1998, 2000, 04, 07)**

Sol. : In I_1, put $x^3 = t$, $\therefore$ $x = t^{1/3}$ $\therefore$ $dx = \frac{1}{3} t^{-2/3} dt$

$$\therefore \quad I_1 = \int_0^\infty e^{-t} \cdot t^{-1/6} \cdot \frac{1}{3} \cdot t^{-2/3}\, dt$$

$$= \frac{1}{3}\int_0^\infty e^{-t} \cdot t^{-5/6}\, dt = \frac{1}{3} \Gamma\left(\frac{1}{6}\right)$$

In I_2, put $y^6 = t$. $\therefore$ $y = t^{1/6}$ $\therefore$ $dy = \frac{1}{6} t^{-5/6}\, dt$

$$\therefore \quad I_2 = \int_0^\infty t^{4/6} \cdot e^{-t} \cdot \frac{1}{6} t^{-5/6}\, dt = \frac{1}{6}\int_0^\infty e^{-t} \cdot t^{-1/6}\, dt = \frac{1}{6} \cdot \Gamma\left(\frac{5}{6}\right)$$

$$\therefore \quad I = I_1 \times I_2$$

$$I = \frac{1}{3}\left|\overline{\frac{1}{6}} \cdot \frac{1}{6}\right|\overline{\frac{5}{6}} = \frac{1}{18}\left|\overline{\frac{1}{6}} \cdot \right|\overline{\frac{5}{6}} = \frac{1}{18} \cdot 2\pi = \frac{\pi}{9}$$ [By (2), page 6-53]

Example 4 (c) : Prove that $\int_0^\infty \sqrt{y} \cdot e^{-y^2} dy \cdot \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy = \frac{\pi}{2\sqrt{2}}$. **(M.U. 1990, 96)**

Sol. : Put $t = y^2$, $y = t^{1/2}$, $dy = \frac{1}{2} t^{-1/2} dt$.

$$\therefore \quad I_1 = \int_0^\infty t^{1/4} e^{-t} \cdot \frac{1}{2} t^{-1/2} dt = \frac{1}{2}\int_0^\infty t^{-1/4} e^{-t} dt = \frac{1}{2}\left|\overline{\frac{3}{4}}\right.$$

$$\therefore \quad I_2 = \int_0^\infty t^{-1/4} e^{-t} \cdot \frac{1}{2} t^{-1/2} dt = \frac{1}{2}\int_0^\infty t^{-3/4} e^{-t} dt = \frac{1}{2}\left|\overline{\frac{1}{4}}\right.$$

$$\therefore \quad I = I_1 \times I_2 = \frac{1}{4}\left|\overline{\frac{3}{4}} \cdot \right|\overline{\frac{1}{4}} = \frac{1}{4}\sqrt{2} \cdot \pi = \frac{\pi}{2\sqrt{2}}$$ [By (13), page 6-19]

EXERCISE - II

Evaluate the following integrals : Class (c) : 8 Marks

1. $\int_0^\infty (x^2 + 4) e^{-2x^2} dx$ **(M.U. 1990)**
2. $\int_0^\infty x^2 e^{-x^4} dx \cdot \int_0^\infty e^{-x^4} dx$ **(M.U. 1997)**
3. $\int_0^\infty x e^{-x^2} dx \cdot \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$ **(M.U. 2008)**
4. $\int_0^\infty x e^{-x^8} dx \cdot \int_0^\infty x^2 e^{-x^8} dx$ (**Hint :** Put $x^8 = t$.) **(M.U. 1998, 2007)**

(**Hint :** Use $\left|\overline{1/2}\right. = \sqrt{\pi}$ and $\left|\overline{1/4}\right. \left|\overline{3/4}\right. = \sqrt{2} \cdot \pi$ proved on page 6-18 and 6-19.)

[**Ans. :** (1) $\frac{9\sqrt{\pi}}{4\sqrt{2}}$, (2) $\frac{\pi}{8\sqrt{2}}$, (3) $\frac{\pi}{2\sqrt{2}}$, (4) $\frac{\pi}{32\sqrt{2}}$]

Type III

Prove that $$\boxed{\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n}{(m+1)^{n+1}} \left|\overline{n+1}\right.}$$ .... (7)

Proof : Put $\log x = -t$ $\quad \therefore \; x = e^{-t}$

$\therefore \; dx = -e^{-t} dt \quad \therefore \; x^m = e^{-mt}$

$\therefore \; (\log x)^n = (-t)^n$. When $x = 0$, $t = \infty$; when $x = 1$, $t = 0$.

$$\therefore \; \int_0^1 x^m (\log x)^n dx = \int_\infty^0 e^{-mt} (-t)^n (-e^{-t}) dt$$

$$= \int_0^\infty (-1)^n e^{-mt-t} \cdot t^n dt$$

Put $(m+1) t = u \quad \therefore \; (m+1) dt = du$

$$\therefore \quad I = \int_0^\infty (-1)^n \cdot e^{-u} \cdot \frac{u^n}{(m+1)^n} \cdot \frac{du}{(m+1)}$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-u}\, u^n\, du = \frac{(-1)^n}{(m+1)^{n+1}} \overline{|n+1}$$

Remark ....

If an integrand involves $\log x$ or $\log(1/x)$, put $\log x = -t$ i.e. $x = e^{-t}$.

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Evaluate $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx$. **(M.U. 1992, 99)**

Sol. : We have $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \int_0^1 x^m (\log 1 - \log x)^n\, dx$

$$= (-1)^n \int_0^1 x^m (\log x)^n\, dx = (-1)^n (-1)^n \frac{\overline{|n+1}}{(m+1)^{n+1}}$$

$$= \frac{\overline{|n+1}}{(m+1)^{n+1}}. \qquad \text{[Using the above result 7]}$$

Or independently, we put $\log x = -t$, $x = e^{-t}$.

$$\therefore \quad dx = -e^{-t}\, dt \quad \text{and} \quad \log\left(\frac{1}{x}\right) = \log 1 - \log x = -\log x = t$$

$$\therefore \quad \int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \int_\infty^0 (e^{-t})^m (t)^n (-e^{-t})\, dt$$

$$= \int_0^\infty (e^{-t})^{m+1} \cdot t^n\, dt$$

Now, put $(m+1)\, t = u \quad \therefore \; (m+1)\, dt = du$.

$$\therefore \quad I = \int_0^\infty e^{-u} \cdot \left(\frac{u}{m+1}\right)^n \frac{du}{m+1}$$

$$= \frac{1}{(m+1)^{n+1}} \cdot \int_0^\infty e^{-u} \cdot u^n\, du$$

$$= \frac{1}{(m+1)^{n+1}} \cdot \overline{|n+1}$$

Example 2 (a) : Evaluate $\int_0^1 x^{q-1} \left(\log \frac{1}{x}\right)^{p-1} dx$. **(M.U. 1999)**

Sol. : We have $\int_0^1 x^{q-1} (\log 1 - \log x)^{p-1} dx = (-1)^{p-1} \int_0^1 x^{q-1} (\log x)^{p-1}\, dx$

$$= (-1)^{p-1} (-1)^{p-1} \cdot \frac{\overline{|p}}{(q-1+1)^{p-1+1}} = \frac{\overline{|p}}{q^p}$$

Putting $n = p-1$, $m = q-1$ in (9) above or put $\log\left(\frac{1}{x}\right) = t$

$\therefore \quad \log 1 - \log x = t \qquad \therefore \; \log x = -t$

$\therefore \; x = e^{-t} \qquad \therefore \; dx = -e^{-t}\,dt \;$ and $\; \log\dfrac{1}{x} = \log 1 - \log x = t$

$$\int_0^1 x^{q-1}\left(\log\frac{1}{x}\right)^{p-1} dx = \int_\infty^0 (e^{-t})^{q-1}\, t^{p-1}(-e^{-t})\,dt$$

$$= \int_0^\infty e^{-tq}\, t^{p-1}\,dt.$$

Put $tq = u, \; q\,dt = du$

$$\therefore \qquad I = \int_0^\infty e^{-u}\left(\frac{u}{q}\right)^{p-1} \cdot \frac{du}{q} = \frac{1}{q^p}\int_0^\infty e^{-u} \cdot u^{p-1}\,du = \frac{1}{q^p}\overline{|p}.$$

Example 3 (a) : Evaluate $\int_0^1 (\log x)^4\,dx.$ **(M.U. 2001)**

Sol. : Put $m = 0$ and $n = 4$ in the above result, or try independently as above by putting $\log x = -t$.

Example 4 (a) : Evaluate $\int_0^1 \left(\log\dfrac{1}{x}\right)^{p-1} dx.$

Sol. : In the above Example 2, put $q = 1$.

$$\therefore \quad \int_0^1 \log\left(\frac{1}{x}\right)^{p-1} dx = \overline{|p}$$

or proceed independently by putting $\log\left(\dfrac{1}{x}\right) = t$ *i.e.* $\log x = -t$.

Example 5 (a) : Evaluate $\int_0^1 (x\log x)^3\,dx.$ **(M.U. 2003)**

Sol. : Put $m = 3$, $n = 3$ in the above result (9) or proceed independently.

Put $\log x = -t \;\therefore\; x = e^{-t} \qquad \therefore\; dx = -e^{-t}\,dt$

$$\int_0^1 x^3(\log x)^3\,dx = \int_\infty^0 e^{-3t}(-t)^3(-e^{-t})\,dt = -\int_0^\infty e^{-4t} \cdot t^3\,dt$$

Put $4t = u \therefore \quad dt = \dfrac{1}{4}du$

$$\therefore \qquad I = -\int_0^\infty e^{-u}\left(\frac{1}{4}u\right)^3 \frac{1}{4}du = -\frac{1}{256}\int_0^\infty e^{-u}u^3\,du$$

$$= -\frac{1}{256}\overline{|4} = -\frac{1}{256}3! = -\frac{3}{128}.$$

Example 6 (a) : Evaluate $\int_0^1 (x\log x)^4\,dx.$ **(M.U. 2009, 11, 12, 13)**

Sol. : Put $m = 4$, $n = 4$ in the result (7) or proceed independently.

Put $\log x = -t \qquad \therefore\; x = e^{-t} \qquad \therefore\; dx = -e^{-t}\,dt$

When $x = 0$, $t = \infty$; when $x = 1$, $t = 0$

$$\therefore \quad \int_0^1 x^4(\log x)^4\,dx = \int_\infty^0 e^{-4t}(-t)^4(-e^{-t})\,dt = \int_0^\infty e^{-5t}\,t^4\,dt.$$

Now, put $5t = u \quad \therefore \; 5\,dt = du$

$$\therefore \quad I = \int_0^{\infty} e^{-u}\left(\frac{u}{5}\right)^4 \frac{du}{5} = \frac{1}{5^5}\overline{|5} = \frac{4!}{5^5}.$$

Example 7 : Evaluate $\int_0^1 \frac{dx}{\sqrt{-\log x}}$. (M.U. 2003, 16)

Sol. : Put $\log x = -t \quad \therefore \; x = e^{-t} \quad \therefore \; dx = -e^{-t}\,dt$

When $x = 0,\; t = \infty \quad \left(\because x = e^{-t} = \frac{1}{e^t}\right)$ and when $x = 1,\; t = 0.$

$$\therefore \quad I = \int_{\infty}^{0} \frac{-e^{-t}dt}{\sqrt{t}} = \int_0^{\infty} \frac{e^{-t}dt}{\sqrt{t}} = \int_0^{\infty} e^{-t} \cdot t^{-1/2}\,dt = \overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}.$$

EXERCISE - III

Evaluate the following integrals : Class (a) : 3 Marks

1. $\int_0^1 (\log x)^5\,dx$ 2. $\int_0^1 x^3\left[\log\left(\frac{1}{x}\right)\right]^4 dx$ 3. $\int_0^1 \frac{dx}{\sqrt{x \cdot \log(1/x)}}$ **(M.U. 2000, 05)**

4. $\int_0^1 \sqrt{x \cdot \log(1/x)} \cdot dx$ 5. $\int_0^1 \sqrt[3]{\log(1/x)}\;dx$ 6. $\int_0^1 \sqrt{\log(1/x)} \cdot dx$ **(M.U. 1995)**

[**Ans. :** (1) 120, (2) $\frac{3}{128}$, (3) $\sqrt{2\pi}$, (4) $\frac{\sqrt{\pi}}{3\sqrt{3/2}}$, (5) $\frac{1}{3}\overline{\left|\frac{1}{3}\right.}$, (6) $\frac{\sqrt{\pi}}{2}$]

Type IV

Prove that $\boxed{\int_0^{\infty} \frac{x^a}{a^x}\,dx = \frac{1}{(\log a)^{a+1}}\overline{|a+1}}$ .... (8)

Proof : Let $a^x = e^t \quad \therefore \; t = x\log a \quad \therefore \; dt = \log a \cdot dx$

When $x = 0,\; t = 0$; when $x = \infty,\; t = \infty.$

$$\therefore \quad \int_0^{\infty} \frac{x^a}{a^x}\,dx = \int_0^{\infty}\left(\frac{t}{\log a}\right)^a \cdot e^{-t} \cdot \frac{1}{\log a} \cdot dt$$

$$= \frac{1}{(\log a)^{a+1}}\int_0^{\infty} e^{-t} \cdot t^a\,dt = \frac{1}{(\log a)^{a+1}}\overline{|a+1}.$$

Remark ....

If an integrand involves a^x or a^{kx^2}, put it equal to e^t.

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Evaluate $\int_0^{\infty} \frac{x^7}{7^x}\,dx$. (M.U. 1997)

Sol. : Put $a = 7$ in the above result (8) or independently put $7^x = e^t$.

$\therefore \quad t = x \log 7 \qquad \therefore \quad dt = \log 7 \cdot dx$

When $x = 0$, $t = 0$; when $x = \infty$, $t = \infty$.

$$\therefore \int_0^\infty \frac{x^7}{7^x}\,dx = \int_0^\infty \left(\frac{t}{\log 7}\right)^7 e^{-t} \cdot \frac{1}{(\log 7)}\,dt$$

$$= \frac{1}{(\log 7)^8} \int_0^\infty t^7 e^{-t}\,dt = \frac{\overline{|8}}{(\log 7)^8} = \frac{7!}{(\log 7)^8}.$$

Example 1 (A) (a) : Evaluate $\int_0^\infty \frac{x^4}{4^x}\,dx$. (M.U. 2015)

Sol. : Solve as above. [**Ans. :** $\frac{4!}{(\log 4)^5}$]

Example 2 (a) : Evaluate $\int_0^\infty 7^{-4x^2}\,dx$. (M.U. 2013)

Sol. : Put $7^{-4x^2} = e^{-t} \qquad \therefore \quad -4x^2 \log 7 = -t$

$$\therefore \quad x = \frac{\sqrt{t}}{2\sqrt{\log 7}} \qquad \therefore \quad dx = \frac{1}{4\sqrt{t}\,\sqrt{\log 7}} \cdot dt$$

When $x = 0$, $t = 0$; when $x = \infty$, $t = \infty$.

$$\therefore \int_0^\infty 7^{-4x^2}\,dx = \int_0^\infty e^{-t} \cdot \frac{1}{4\sqrt{\log 7}} \cdot \frac{1}{\sqrt{t}} \cdot dt$$

$$= \frac{1}{4\sqrt{\log 7}} \cdot \int_0^\infty e^{-t}\,t^{-1/2}\,dt = \frac{1}{4\sqrt{\log 7}} \cdot \overline{\left|\frac{1}{2}\right.} = \frac{\sqrt{\pi}}{4\sqrt{\log 7}}.$$

Example 3 (a) : Evaluate $\int_0^\infty x^2 \cdot 7^{-4x^2}\,dx$. (M.U. 2013)

Sol. : Put $7^{-4x^2} = e^{-t} \qquad \therefore \quad -4x^2 \log 7 = -t$

$$\therefore \quad x^2 = \frac{t}{4\log 7} \qquad \therefore \quad x = \frac{\sqrt{t}}{2\sqrt{\log 7}} \qquad \therefore \quad dx = \frac{1}{4\sqrt{\log 7}} \cdot \frac{1}{\sqrt{t}}\,dt.$$

When $x = 0$, $t = 0$; when $x = \infty$, $t = \infty$.

$$\therefore \quad I = \int_0^\infty \frac{t}{4\log 7} \cdot e^{-t} \cdot \frac{1}{4\sqrt{\log 7}} \cdot \frac{1}{\sqrt{t}}\,dt$$

$$= \frac{1}{16(\log 7)^{3/2}} \int_0^\infty e^{-t} \cdot t^{1/2}\,dt = \frac{1}{16(\log 7)^{3/2}} \cdot \overline{\left|\frac{3}{2}\right.}$$

$$= \frac{1}{16(\log 7)^{3/2}} \cdot \frac{1}{2}\overline{\left|\frac{1}{2}\right.} = \frac{1}{32(\log 7)^{3/2}}\sqrt{\pi}$$

Important Note ....

When an integrand involves e^{-ax^n} (put $ax^n = t$) or $(\log x)^n$ (put $\log x = t$) or a^x (put $a^x = e^t$), the integral can be put in the form $\int_0^\infty e^{-x} x^n\,dx$ *i.e.* as a Gamma Function $\overline{|n+1}$.

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EXERCISE - IV

Evaluate the following integrals : Class (a) : 3 Marks

1. $\int_0^{\infty} \frac{x^{m-1}}{(m-1)^x}\, dx$ 2. $\int_0^{\infty} 5^{-4x^2}\, dx$ 3. $\int_0^{\infty} 3^{-4x^2}\, dx$

[**Ans. : (1)** $\frac{\overline{|m}}{[\log (m-1)]^m}$, **(2)** $\frac{\sqrt{\pi}}{4\sqrt{\log 5}}$, **(3)** $\frac{\sqrt{\pi}}{4\sqrt{\log 3}}$]

Miscellaneous Examples : Class (a) : 3 Marks

Example 1 (a) : Given $\overline{|1{\cdot}8} = 0{\cdot}9314$, find the value of $\overline{|-2{\cdot}2}$.

Sol. : We have $\overline{|n+1} = n\overline{|n}$ $\quad \therefore \overline{|n} = \frac{\overline{|n+1}}{n}$.

Putting $n = -2{\cdot}2$, $\overline{|-2{\cdot}2} = \frac{\overline{|-1{\cdot}2}}{-2{\cdot}2}$. Using the result repeatedly.

$$\therefore \overline{|-2{\cdot}2} = \frac{1}{-2{\cdot}2}\cdot\frac{\overline{|-0{\cdot}2}}{(-1{\cdot}2)} = \frac{1}{(-2{\cdot}2)}\cdot\frac{1}{(-1{\cdot}2)}\cdot\frac{\overline{|0{\cdot}8}}{(-0{\cdot}2)}$$

$$= \frac{1}{(-2{\cdot}2)}\cdot\frac{1}{(-1{\cdot}2)}\cdot\frac{1}{(-0{\cdot}2)}\cdot\frac{1}{(0{\cdot}8)}\cdot\overline{|1{\cdot}8}$$

$$= \frac{0{\cdot}9314}{-0{\cdot}4224} = -2{\cdot}21.$$

Example 2 (a) : Compute $\overline{|-2{\cdot}5}$.

Sol. : We use $\overline{|n} = \frac{\overline{|n+1}}{n}$. Putting $n = -2{\cdot}5$ and using the result repeatedly.

$$\overline{|-2{\cdot}5} = \frac{\overline{|-2{\cdot}5+1}}{-2{\cdot}5} = \frac{\overline{|-1{\cdot}5}}{-2{\cdot}5} = \frac{1}{(-2{\cdot}5)}\cdot\frac{\overline{|-0{\cdot}5}}{(-1{\cdot}5)}$$

$$= \frac{1}{(-2{\cdot}5)}\cdot\frac{1}{(-1{\cdot}5)}\cdot\frac{\overline{|0{\cdot}5}}{(-0{\cdot}5)} = -\frac{8}{15}\sqrt{\pi}.$$

Example 3 (a) : Given $\overline{\left|\frac{3}{5}\right.} = 1.4891$, find $\overline{\left|-\frac{7}{5}\right.}$.

Sol. : We use $\overline{|n} = \frac{\overline{|n+1}}{n}$ and put $n = -\frac{7}{5}$.

$$\therefore \overline{\left|-\frac{7}{5}\right.} = \frac{\overline{\left|-\frac{7}{5}+1\right.}}{-\frac{7}{5}} = -\frac{5}{7}\overline{\left|-\frac{2}{5}\right.} = -\frac{5}{7}\cdot\frac{\overline{\left|-\frac{2}{5}+1\right.}}{-\frac{2}{5}}$$

$$= \frac{5}{7}\cdot\frac{5}{2}\overline{\left|\frac{3}{5}\right.} = \frac{25}{14}\times(1.4891) = 2.6591.$$

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Prove that $\overline{\left| n+\frac{1}{2} \right.} = \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)}{2^n}\sqrt{\pi}$. (M.U. 1997)

Hence or otherwise prove that $\overline{\left| n+\frac{1}{2} \right.} = \frac{(2n!)\sqrt{\pi}}{n!\,4^n}$.

Sol. : Clearly n must be a positive integer

$$\therefore \quad \overline{\left| n+\frac{1}{2} \right.} = \left(n-\frac{1}{2}\right)\overline{\left| n-\frac{1}{2} \right.}$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\overline{\left| n-\frac{3}{2} \right.} \text{ and so on}$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\ldots \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\overline{\left| \frac{1}{2} \right.}$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\ldots \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$= \frac{(2n-1)(2n-3)\ldots\ldots 5\cdot 3\cdot 1\sqrt{\pi}}{2^n}$$

Further multiply the numerator and denominator by

$2n\,(2n-2)\,(2n-4)\ldots\ldots 6\cdot 4\cdot 2$

$$\therefore \quad \overline{\left| n+\frac{1}{2} \right.} = \frac{2n(2n-1)(2n-2)\ldots\ldots 5\cdot 4\cdot 3\cdot 2\cdot 1\sqrt{\pi}}{2^n\cdot 2n(2n-2)(2n-4)\ldots\ldots 6\cdot 4\cdot 2}$$

$$= \frac{2n(2n-1)(2n-2)\ldots\ldots 3\cdot 2\cdot 1\sqrt{\pi}}{2^n\cdot 2^n\cdot n(n-1)(n-2)\ldots\ldots 3\cdot 2\cdot 1} = \frac{(2n)!}{4^n\cdot n!}\sqrt{\pi}.$$

Example 2 (c) : If $I_n = \dfrac{\frac{\sqrt{\pi}}{2}\overline{\left| \frac{n+1}{2} \right.}}{\overline{\left| \frac{n}{2}+1 \right.}}$, show that $I_{n+2} = \frac{n+1}{n+2} I_n$ and hence, find I_5.

(M.U. 1999, 2000)

Sol. : Replacing n by $n+2$ in I_n,

$$\therefore \quad I_{n+2} = \frac{\frac{\sqrt{\pi}}{2}\overline{\left| \frac{n+3}{2} \right.}}{\overline{\left| \frac{n+2}{2}+1 \right.}} = \frac{\frac{\sqrt{\pi}}{2}\overline{\left| \frac{n+1}{2}+1 \right.}}{\overline{\left| \frac{n+2}{2}+1 \right.}} = \frac{\frac{\sqrt{\pi}}{2}\cdot\frac{n+1}{2}\overline{\left| \frac{n+1}{2} \right.}}{\frac{n+2}{2}\overline{\left| \frac{n+2}{2} \right.}}$$

$$\therefore \quad I_{n+2} = \left(\frac{n+1}{n+2}\right)\cdot\frac{\frac{\sqrt{\pi}}{2}\overline{\left| \frac{n+1}{2} \right.}}{\overline{\left| \frac{n}{2}+1 \right.}} = \frac{n+1}{n+2}\cdot I_n$$

Putting $n = 3$ in the above result, we get $I_5 = \frac{4}{5} I_3$, Now, put $n = 1$, $I_3 = \frac{2}{3} I_1$ and $I_1 = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\overline{|3/2}}$.

$$\therefore \quad I_5 = \frac{4}{5} I_3 = \frac{4}{5} \cdot \frac{2}{3} I_1 = \frac{8}{15} \frac{(\sqrt{\pi}/2) \cdot 1}{\overline{|3/2}}$$

$$= \frac{8}{15} \cdot \frac{(\sqrt{\pi}/2)}{(1/2)\overline{|1/2}} = \frac{8}{15} \cdot \frac{\sqrt{\pi}}{\sqrt{\pi}} = \frac{8}{15}.$$

Example 3 (c) : Show that $\int_0^\infty x^{m-1} \cos ax \, dx = \frac{\overline{|m}}{a^m} \cos\left(\frac{m\pi}{2}\right)$. (M.U. 2002, 06, 08, 09)

Sol. : Since $e^{-iax} = \cos ax - i \sin ax$, we consider the real part of

$$I = \int_0^\infty x^{m-1} e^{-iax} dx. \quad \text{Put } iax = t, \; dx = \frac{dt}{ia}$$

$$\therefore \quad I = \int_0^\infty \frac{t^{m-1}}{(ia)^{m-1}} \cdot e^{-t} \cdot \frac{dt}{ia} = \frac{1}{i^m a^m} \int_0^\infty e^{-t} t^{m-1} dt$$

$$= \frac{\overline{|m}}{a^m} \cdot \frac{1}{i^m}. \qquad \text{But } i = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2}$$

$$= \frac{\overline{|m}}{a^m} (\cos m\frac{\pi}{2} - i \sin m\frac{\pi}{2}) \qquad \text{[By DeMoiver's Theorem]}$$

$$\therefore \quad \int_0^\infty x^{m-1} \cos ax \, dx = \text{Real Part of } I = \frac{\overline{|m}}{a^m} \cos\left(\frac{m\pi}{2}\right)$$

(Equating Imaginary part, we get $\int_0^\infty x^{m-1} \sin ax \, dx = -\frac{\overline{|m}}{a^m} \sin\frac{m\pi}{2}$).

Example 4 (c) : Show that $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\overline{|n}}{a^n}$ where a, n are positive.

Deduce that (i) $\int_0^\infty e^{-ax} x^{n-1} \cos bx \, dx = \frac{\overline{|n}}{r^n} \cdot \cos n\theta$

(ii) $\int_0^\infty e^{-ax} x^{n-1} \sin bx \, dx = \frac{\overline{|n}}{r^n} \cdot \sin n\theta$ where $r^2 = a^2 + b^2$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Sol. : Putting $ax = z$, $dx = dz / a$, we get,

$$I = \int_0^\infty e^{-z} \left(\frac{z}{a}\right)^{n-1} \cdot \frac{dz}{a} = \frac{1}{a^n} \int_0^\infty e^{-z} z^{n-1} dz = \frac{\overline{|n}}{a^n} \quad \text{.............. (1)}$$

Now, replacing a by $a + ib$, we get $\int_0^\infty e^{-(a+ib)x} x^{n-1} dx = \frac{\overline{|n}}{(a+ib)^n}$

Putting $a = r\cos\theta$, $b = r\sin\theta$

$$(a + ib)^n = [r(\cos\theta + i \sin\theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

Further, $e^{-(a+ib)x} = e^{-ax} \cdot e^{-ibx} = e^{-ax}(\cos bx - i \sin bx)$

Hence, from (1) we get,

$$\int_0^{\infty} e^{-ax}(\cos bx - i \sin bx) \cdot x^{n-1}\, dx = \frac{\overline{|n}}{r^n}(\cos n\theta - i \sin n\theta)$$

Equating real and in imaginary parts we get the required results.

Example 5 (c) : Evaluate $\int_0^{\infty} \cos(ax^{1/n})\, dx$.

Sol. : Put $ax^{1/n} = t \quad \therefore\ x^{1/n} = \frac{t}{a}. \quad \therefore\ x = \frac{t^n}{a^n} \quad \therefore\ dx = \frac{1}{a^n} \cdot nt^{n-1}\, dt$

$$\therefore\ \int_0^{\infty} \cos t \cdot \frac{n}{a^n} t^{n-1}\, dt = \text{R.P.} \int_0^{\infty} e^{-it} \frac{n}{a^n} t^{n-1}\, dt \qquad [\because e^{-it} = \cos t - i \sin t]$$

$$= \text{R.P.} \frac{n}{a^n} \int_0^{\infty} e^{-it}\, t^{n-1}\, dt$$

Now, put $it = u \quad \therefore\ i\, dt = u$

$$\therefore\ I = \text{R.P.} \frac{n}{a^n} \int_0^{\infty} e^{-u} \frac{u^{n-1}}{i^{n-1}} \cdot \frac{du}{i} = \text{R.P.} \frac{n}{a^n} \cdot \frac{1}{i^n} \int_0^{\infty} e^{-u} u^{n-1}\, du$$

$$= \text{R.P.} \frac{n}{a^n} \cdot \frac{1}{i^n} \overline{|n} = \text{R.P.} \frac{\overline{|n+1}}{a^n \cdot i^n} \qquad \left[\because n\overline{|n} = \overline{|n+1}\right]$$

$$= \text{R.P.} \frac{\overline{|n+1}}{a^n}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)^{-n} \qquad \left[\because i = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right]$$

$$= \text{R.P.} \frac{\overline{|n+1}}{a^n}\left(\cos\frac{n\pi}{2} - i \sin\frac{n\pi}{2}\right) = \frac{\overline{|n+1}}{a^n} \cos\left(\frac{n\pi}{2}\right).$$

Example 6 (a) : Prove that $\int_0^{\infty} xe^{-ax} \sin bx\, dx = \frac{2ab}{(a^2 + b^2)^2}$. (M.U. 2004)

Sol. : Consider $\int_0^{\infty} x e^{-(a-ib)x} \cdot dx$.

Put $(a - ib)\, x = t \quad \therefore\ (a - ib)\, dx = dt$

$$\therefore\ I = \int_0^{\infty} e^{-t} \cdot \frac{t}{(a-ib)^2}\, dt = \frac{1}{(a-ib)^2} \overline{|2} = \frac{1}{(a-ib)^2}$$

Now, $$\frac{1}{(a-ib)^2} = \frac{1}{(a^2-b^2) - 2aib} \cdot \frac{(a^2-b^2) + 2a\,ib}{(a^2-b^2) + 2a\,ib} = \frac{(a^2-b^2) + 2a\,ib}{(a^2+b^2)^2}$$

$$\therefore\ \int_0^{\infty} x \cdot e^{-(a-ib)x}\, dx = \int_0^{\infty} x \cdot e^{-ax}(\cos bx + i \sin bx)\, dx = \frac{(a^2-b^2) + 2aib}{(a^2+b^2)^2}$$

Equating real and imaginary parts, we get

$$\int_0^{\infty} x e^{-ax} \cos bx\, dx = \frac{a^2 - b^2}{(a^2+b^2)^2}$$

$$\int_0^{\infty} x e^{-ax} \sin bx\, dx = \frac{2ab}{(a^2+b^2)^2}.$$

7. Beta Functions

Definition : The function of m and n defined by the integral $(m, n > 0)$

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

is called the **Beta function** and is denoted by $B(m, n)$. Thus,

$$\boxed{B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx} \qquad \text{——— (1)}$$

Beta function may also be remembered as

$$\boxed{\int_0^1 x^m (1-x)^n\,dx = \beta(m+1, n+1)} \qquad \text{——— (1A)}$$

Properties of Beta Functions

(I) $\boxed{B(m, n) = B(n, m)}$ ——— (2)

The result is self evident.

(II) The Relation between Beta and Gamma Functions.

$$\boxed{B(m, n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)} = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}} \quad \text{if } m, n \text{ are integers.} \qquad \text{——— (3)}$$

We shall accept this result for the time being and shall prove it at a proper place while studying double integration in § 2 (a) in chapter 9, page 9-10.

(III) $\boxed{B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\,d\theta}$

Proof : Put $x = \sin^2\theta$ in (1) $\therefore\ dx = 2\sin\theta\cos\theta\,d\theta$

$$\therefore\ B(m, n) = \int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot 2 \cdot \sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\,d\theta \qquad \text{——— (4)}$$

This can be considered as **second form of Beta Function.**

(IV) $\boxed{\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$ ——— (5)

Proof : In (4) put $2m - 1 = p$ and $2n - 1 = q$

i.e. $m = (p+1)/2$ and $n = (q+1)/2$ and the result follows.

(V) $\boxed{\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}\,\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}}$ ——— (6)

Proof : The result follows from (5) and (3).

(VI) $\boxed{\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(m, n)}$ ——— (7)

$$\therefore \quad \boxed{\int_0^\infty \frac{x^m}{(1+x)^n}\,dx = B(m+1,\ n-m-1)} \qquad \text{.................... (7A)}$$

Remember the results (1A) given on the previous page and (7A) given above. They are highly useful. We shall prove this as an example later (See page 6-42). It may be noted that the above result can also be used **as a definition of Beta function.**

(VII) $\boxed{\overline{|p} \cdot \overline{|1-p} = \dfrac{\pi}{\sin p\pi}}$, $\quad 0 < p < 1 \qquad$ .... (8)

This result also we shall prove as an example later. (See Ex. 4 page 6-52)

(VIII) $\boxed{\overline{\left|\tfrac{1}{2}\right.} = \sqrt{\pi}} \qquad$ .... (9)

Proof : In (6) put $p = 0$, $q = 0$

$$\therefore \int_0^{\pi/2} d\theta = \frac{1}{2}\left(\overline{\left|\tfrac{1}{2}\right.}\right)^2 \qquad \therefore \left[\theta\right]_0^{\pi/2} = \frac{1}{2}\left(\overline{\left|\tfrac{1}{2}\right.}\right)^2$$

$$\therefore \frac{\pi}{2} = \frac{1}{2}\left(\overline{\left|\tfrac{1}{2}\right.}\right)^2 \qquad \therefore \overline{\left|\tfrac{1}{2}\right.} = \sqrt{\pi}$$

(IX) Duplication Formula of Gamma Function (M.U. 2002, 11, 12)

$$\boxed{2^{2m-1} \cdot \overline{|m} \cdot \overline{\left|m+\tfrac{1}{2}\right.} = \sqrt{\pi} \cdot \overline{|2m}} \qquad \text{.................... (10)}$$

Proof : In (6) put $p = q$,

$$\frac{\frac{1}{2}\left(\overline{\left|\frac{p+1}{2}\right.}\right)^2}{\overline{|p+1}} = \int_0^{\pi/2} \sin^p\theta\cos^p\theta\,d\theta = \frac{1}{2^p}\int_0^{\pi/2} \sin^p 2\theta\ d\theta$$

Put $2\theta = t \quad \therefore d\theta = \frac{1}{2} \cdot dt$ and when $\theta = \frac{\pi}{2}$, $t = \pi$.

$$\therefore \frac{\frac{1}{2}\left(\overline{\left|\frac{p+1}{2}\right.}\right)^2}{\overline{|p+1}} = \frac{1}{2^p}\int_0^{\pi} \sin^p t \cdot \frac{1}{2}\,dt = \frac{1}{2^p} \cdot \frac{1}{2} \cdot 2\int_0^{\pi/2} \sin^p t\,dt$$

$$\left[\because \int_0^{2a} f(x)\,dx = \int_0^{a} 2f(x)\,dx \ \text{ if } \ f(2a-x) = f(x)\right]$$

$$= \frac{1}{2^p}\int_0^{\pi/2} \sin^p t\cos^o t\,dt$$

$$= \frac{1}{2^p} \cdot \frac{1}{2} \cdot \frac{\overline{\left|\frac{p+1}{2}\right.} \cdot \overline{\left|\frac{1}{2}\right.}}{\overline{\left|\frac{p+2}{2}\right.}} \qquad [\text{By (6)}] \qquad \text{.................... (11)}$$

Now, put $\frac{p+1}{2} = m$, *i.e.* $p = 2m-1$

$$\therefore \frac{1}{2}\frac{(\overline{|m})^2}{\overline{|2m}} = \frac{1}{2^{2m-1}}\cdot\frac{1}{2}\frac{\overline{|m}\ \overline{|1/2}}{\overline{|m+(1/2)}}$$

$$\therefore \boxed{2^{2m-1}\cdot\overline{|m}\ \overline{\left|m+\frac{1}{2}\right.} = \sqrt{\pi}\cdot\overline{|2m}} \qquad \text{.................... (12)}$$

Duplication formula is also stated as

$$\boxed{\overline{|m}\ \overline{\left|m+\frac{1}{2}\right.} = \frac{\sqrt{\pi}\cdot\overline{|2m}}{2^{2m-1}}} \qquad \text{.................... (12A)}$$

Remark ....

The formula is called duplication formula because it gives $\overline{|2m}$ in terms of $\overline{|m}$.

Particular Cases

(i) Putting $m = 1/4$ in (12), we get, $2^{-1/2}\overline{|1/4}\ \overline{|3/4} = \sqrt{\pi}\ \overline{|1/2}$

$$\therefore \boxed{\overline{\left|\frac{1}{4}\right.}\cdot\overline{\left|\frac{3}{4}\right.} = \sqrt{2}\cdot\pi} \qquad \text{.................... (13)}$$

(ii) Putting $m = 3/4$ in (12), we get, $2^{1/2}\overline{|3/4}\ \overline{|5/4} = \sqrt{\pi}\ \overline{|3/2}$

$$\therefore \overline{\left|\frac{3}{4}\right.}\ \overline{\left|\frac{5}{4}\right.} = \frac{\sqrt{\pi}}{\sqrt{2}}\cdot\frac{1}{2}\overline{\left|\frac{1}{2}\right.} = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

$$\therefore \boxed{\overline{\left|\frac{3}{4}\right.}\ \overline{\left|\frac{5}{4}\right.} = \frac{\pi}{2\sqrt{2}}} \qquad \text{.................... (13A)}$$

(iii) Putting $m = 5/4$ in (12), $2^{3/2}\overline{|5/4}\ \overline{|7/4} = \sqrt{\pi}\ \overline{|5/2}$

$$\therefore 2\sqrt{2}\,\overline{\left|\frac{5}{4}\right.}\ \overline{\left|\frac{7}{4}\right.} = \sqrt{\pi}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\overline{\left|\frac{1}{2}\right.}$$

$$\therefore \boxed{\overline{\left|\frac{5}{4}\right.}\ \overline{\left|\frac{7}{4}\right.} = \frac{3}{16}\sqrt{2}\cdot\pi} \qquad \text{.................... (13B)}$$

Corollary : Using duplication formula, prove that

$$\boxed{\overline{\left|m+\frac{1}{2}\right.} = \frac{\sqrt{\pi}\cdot(2m)!}{2^{2m}\cdot m!}} \qquad \text{.................... (13C)}$$

where m is a positive integer.

Proof : By duplication formula

$$2^{2m-1}\cdot\overline{|m}\ \overline{\left|m+\frac{1}{2}\right.} = \sqrt{\pi}\ \overline{|2m} \qquad \therefore \overline{\left|m+\frac{1}{2}\right.} = \frac{\sqrt{\pi}\cdot\overline{|2m}}{2^{2m-1}\cdot\overline{|m}}$$

Since m is a positive integer, $\overline{|m} = (m-1)!$ and $\overline{|2m} = (2m-1)!$

$$\therefore \overline{\left| m+\frac{1}{2} \right.} = \frac{\sqrt{\pi}\cdot(2m-1)!}{2^{2m-1}\cdot(m-1)!}$$

Multiplying the numerator and denominator by $2m$, we get

$$\overline{\left| m+\frac{1}{2} \right.} = \frac{\sqrt{\pi}\cdot 2m(2m-1)!}{2^{2m-1}\cdot 2m\cdot(m-1)!} \qquad \therefore \overline{\left| m+\frac{1}{2} \right.} = \frac{\sqrt{\pi}\cdot(2m)!}{2^{2m}\cdot m!}$$

Cor. : Putting m = 0, we get

$$\overline{\left| \frac{1}{2} \right.} = \sqrt{\pi} \qquad [\because\ 0! = 1]$$

Illustrative Examples

Type I

Prove that $\boxed{\int_0^a x^m (a-x)^n\, dx = a^{m+n+1} B(m+1,\ n+1)}$ .... (14)

Proof : Put $x = at$, $\therefore$ $dx = a\, dt$

$$\therefore\ I = \int_0^1 a^m t^m a^n (1-t)^n a\, dt = a^{m+n+1}\int_0^1 t^m (1-t)^n\, dt$$

$$= a^{m+n+1} B(m+1,\ n+1)$$

Remark ....

If an integral involves $(a-x)^n$ put $x = at$. Examples of this type can also be solved by putting $x = a\sin^2\theta$ as shown in Ex. 7 and 8 on page 6-38.

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Evaluate $\int_0^4 \sqrt{x}\,(4-x)^{3/2}\, dx$.

Sol. : As above, put $x = 4t$ $\therefore$ $dx = 4\,dt$

$$\therefore\ I = \int_0^1 2\sqrt{t}\cdot 2^3\cdot(1-t)^{3/2}\, 4\, dt = 64\int_0^1 t^{1/2}(1-t)^{3/2}\, dt$$

$$= 64\ B\left(\frac{3}{2},\frac{5}{2}\right) = 64\cdot\frac{\overline{|3/2}\ \overline{|5/2}}{\overline{|4}}$$

$$= 64\cdot\frac{(1/2)\overline{|1/2}\cdot(3/2)(1/2)\overline{|1/2}}{3!} = 4\pi.$$

(For another method, see Ex. 8, page 6-38.)

Example 2 (a) : Evaluate $\int_0^2 x^3\sqrt{2-x}\, dx$.

Sol. : In the above result, put $m = 3$, $n = 1/2$, $a = 2$

or independently put $x = 2t$, $\therefore$ $dx = 2\, dt$

$$I = \int_0^1 8t^3\sqrt{2}\,(1-t)^{1/2}\, 2\, dt = 16\sqrt{2}\int_0^1 t^3(1-t)^{1/2}\, dt$$

$$\therefore \quad I = 16\sqrt{2} \cdot B\left(4, \frac{5}{2}\right) = 16\sqrt{2} \cdot \frac{\Gamma 4 \, \Gamma(5/2)}{\Gamma(13/2)}$$

$$= \frac{16\sqrt{2} \cdot 3! \, \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{\frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}} = \frac{512}{3465}\sqrt{2}$$

Example 3 (a) : Evaluate $\int_0^2 x^2 (2-x)^3 \, dx$ (M.U. 2010)

Sol. : Put $x = 2t$ $\quad \therefore \; dx = 2\,dt$

When $x = 0$, $t = 0$; when $x = 2$, $t = 1$

$$\therefore \quad I = \int_0^1 4t^2 \cdot 2^3 (1-t)^3 \cdot 2\,dt = 64 \int_0^1 t^2 (1-t)^3 \, dt$$

$$= 64 \cdot B(3, 4) = 64 \cdot \frac{\Gamma 3 \, \Gamma 4}{\Gamma 7} = 64 \cdot \frac{2! \, 3!}{6!} = \frac{16}{15}$$

Example 4 (a) : Evaluate $\int_0^2 x^3 (2-x)^3 \, dx$ (M.U. 2019)

Sol. : As above put $x = 2t$, $dx = 2\,dt$

$$\therefore \quad I = \int_0^1 8t^3 \cdot 2^3 (1-t)^3 \cdot 2\,dt = 2^7 \int_0^1 t^3 (1-t)^3 \, dt$$

$$= 2^7 \, B(4, 4)$$

Example 5 (a) : Evaluate $\int_0^8 x^4 (8-x)^{-1/3} \, dx$

Sol. : Put $x = 8t$ $\quad \therefore \; dx = 8\,dt$

$$\therefore \quad I = \int_0^1 (8t)^4 \cdot 8^{-1/3} (1-t)^{-1/3} \cdot 8\,dt$$

$$= \int_0^1 8^4 t^4 \cdot \frac{1}{2} (1-t)^{-1/3} \cdot 8\,dt$$

$$= \frac{8^5}{2} \int_0^1 t^4 (1-t)^{-1/3} \, dt = \frac{8^5}{2} \cdot B\left(5, \frac{2}{3}\right)$$

Example 6 (a) : Evaluate $\int_0^a \frac{x^m}{\sqrt{a-x}} \, dx$

Sol. : Put $n = -1/2$ in the result (14) or independently.

Put $x = at$ $\quad \therefore \; dx = a\,dt$

$$\therefore \quad I = \int_0^1 \frac{(at)^m}{\sqrt{a}\sqrt{1-t}} \cdot a\,dt = \int_0^1 a^m \cdot a^{-1/2} \cdot a \cdot t^m \cdot (1-t)^{-1/2} \, dt$$

$$= a^{m+(1/2)} \int_0^1 t^m (1-t)^{-1/2} \, dt = a^{m+(1/2)} \, B\left(m+1, \frac{1}{2}\right)$$

Type 11

Prove that $$\boxed{\int_0^1 x^m (1-x^n)^p \, dx = \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}} \quad \text{.......... (15)}$$

Proof : Put $x^n = t$ $\therefore x = t^{1/n}$ $\therefore dx = \frac{1}{n} t^{(1/n)-1} \, dt$

$$\therefore I = \int_0^1 t^{m/n} (1-t)^p \cdot \frac{1}{n} \cdot t^{(1/n)-1} \, dt = \frac{1}{n} \int_0^1 t^{(m/n)+(1/n)-1} (1-t)^p \, dt$$

$$= \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right) = \frac{1}{n} \cdot \frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$$

Note ...

$$\int_0^a x^m (a^n - x^n)^p \, dx$$

is a generalisation of the above integral (15). See Ex. 2 below.

Also $\int_0^{1/a} x^m [1-(ax)^n]^p$ is another generalisation of the above integral (15). See solved Ex. 4 below.

Example 2 (a) : Evaluate $\int_0^1 \sqrt{(\sqrt{x}-x)}\cdot dx$. (M.U. 1998, 2002, 14)

Sol. : Putting $\sqrt{x}=t$, $\frac{1}{2\sqrt{x}}dx=dt$ $\therefore$ $dx=2\sqrt{x}\cdot dt=2t\,dt$

When $x=0$, $t=0$, when $x=1$, $t=1$.

$$\therefore I=\int_0^1\sqrt{t-t^2}\cdot 2t\,dt=\int_0^1\sqrt{t}\sqrt{1-t}\cdot 2t\,dt$$

$$=2\int_0^1 t^{3/2}(1-t)^{1/2}dt=2B\left(\frac{5}{2},\frac{3}{2}\right)$$

Example 3 (a) : Evaluate $\int_0^2 y^4(8-y^3)^{-1/3}dy$. (M.U. 2014)

Sol. : Put $y^3=8t$ $\therefore$ $y=2t^{1/3}$ $\therefore$ $dy=\frac{2}{3}t^{-2/3}dt$

$$\therefore I=\int_0^1(8t)^{4/3}\cdot 8^{-1/3}\cdot(1-t)^{-1/3}\cdot\frac{2}{3}\cdot t^{-2/3}dt$$

$$=\int_0^1 16t^{2/3}\cdot\frac{1}{2}(1-t)^{-1/3}\cdot\frac{2}{3}dt$$

$$=\frac{16}{3}\int_0^1 t^{2/3}(1-t)^{-1/3}dt=\frac{16}{3}\cdot B\left(\frac{5}{3},\frac{2}{3}\right)$$

Example 4 (a) : Evaluate $\int_0^{1/2}x^3\sqrt{1-4x^2}\cdot dx$.

Sol. : Put $4x^2=t$ $\therefore$ $x=\frac{t^{1/2}}{2}$ $\therefore$ $dx=\frac{1}{2}\cdot\frac{t^{-1/2}}{2}\cdot dt$.

When $x=0$, $t=0$, when $x=1/2$, $t=1$.

$$\therefore I=\int_0^1\frac{t^{3/2}}{8}\cdot(1-t)^{1/2}\cdot\frac{t^{-1/2}}{4}\cdot dt$$

$$=\frac{1}{32}\int_0^1 t^1(1-t)^{1/2}dt=\frac{1}{32}B\left(2,\frac{3}{2}\right)$$

Example 5 (a) : Evaluate $\int_0^{2a}x^m\sqrt{2ax-x^2}\,dx$.

Sol. : Put $x=2at$, $dx=2a\,dt$

$$\therefore I=\int_0^{2a}x^m\cdot x^{1/2}\sqrt{2a-x}\,dx$$

$$=\int_0^1(2a)^{m+(1/2)}t^{m+(1/2)}\cdot\sqrt{2a}\cdot(1-t)^{1/2}\cdot 2a\,dt$$

$$=(2a)^{m+2}\int_0^1 t^{m+(1/2)}(1-t)^{1/2}dt$$

$$=(2a)^{m+2}B\left(m+\frac{3}{2},\frac{3}{2}\right)$$

Example 6 (a) : Show that $\int_0^{2a}x^2\sqrt{2ax-x^2}\,dx=\frac{5}{8}a^4\pi$. (M.U. 2002, 04)

Sol. : In the Ex. 3 put $m=2$ or independently put $x=2at$, $dx=2a\,dt$.

$$I = \int_0^{2a} x^2 \cdot x^{1/2} \sqrt{2a - x} \cdot dx$$

$$= \int_0^1 (2a)^{2+(1/2)} \cdot t^{5/2} \cdot \sqrt{2a} \cdot (1-t)^{1/2} \cdot 2a\,dt$$

$$= (2a)^4 \int_0^1 t^{5/2} (1-t)^{1/2}\,dt = (2a)^4\, B\left(\frac{7}{2}, \frac{3}{2}\right)$$

$$= 16a^4 \frac{\Gamma(7/2)\,\Gamma(3/2)}{\Gamma 5} = \frac{16a^4}{4!}\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\frac{1}{2}} \cdot \left(\frac{1}{2}\right)\sqrt{\frac{1}{2}}$$

$$= \frac{16a^4}{4!} \cdot \frac{15}{16}\left(\Gamma\frac{1}{2}\right)^2 = \frac{5}{8}a^4 \pi.$$

Example 7 (a) : Evaluate $\int_0^1 x^3 (1-\sqrt{x})^5\,dx$.

Sol. : In the above result (16), put $m = 3$, $n = 1/2$, $p = 5$ or independently put $\sqrt{x} = t$.

$\therefore\ x = t^2 \qquad \therefore\ dx = 2t\,dt$

$$\therefore\ I = \int_0^1 (t^2)^3 \cdot (1-t)^5 \cdot 2t\,dt = 2\int_0^1 t^7 (1-t)^5\,dt$$

$$= 2\,B(8, 6) = 2 \cdot \frac{\Gamma 8\,\Gamma 6}{\Gamma 14} = \frac{2 \cdot 7! \cdot 5!}{13!} = \frac{1}{5148}$$

Example 8 (a) : Evaluate $\int_0^2 y^4 (8 - y^3)^{-1/3}\,dy$.

Sol. : We put $y^3 = 8t \quad \therefore\ y = 2t^{1/3} \quad \therefore\ dy = \frac{2}{3}t^{-2/3}\,dt$

$$\therefore\ I = \int_0^1 (2t^{1/3})^4 \cdot 8^{-1/3} \cdot (1-t)^{-1/3} \cdot \frac{2}{3} t^{-2/3}\,dt$$

$$= \int_0^1 16\,t^{2/3} \cdot \frac{1}{2}(1-t)^{-1/3} \cdot \frac{2}{3}\,dt$$

$$= \frac{16}{3}\int_0^1 t^{2/3}(1-t)^{-1/3}\,dt = \frac{16}{3} \cdot B\left(\frac{5}{3}, \frac{2}{3}\right).$$

Example 9 (a) : Evaluate $\int_0^2 x^7 (16 - x^4)^{10}\,dx$.

Sol. : Put $x^4 = 16t \quad \therefore\ x = 2t^{1/4} \quad \therefore\ dx = 2 \cdot \frac{1}{4} t^{-3/4}\,dt$

$$\therefore\ I = \int_0^1 (2t^{1/4})^7 \cdot (16)^{10} (1-t)^{10} \cdot \frac{1}{2} t^{-3/4}\,dt$$

$$= \int_0^1 2^7 \cdot t^{7/4} \cdot (2^4)^{10} (1-t)^{10} \cdot \frac{1}{2} t^{-3/4}\,dt$$

$$= 2^{46} \int_0^1 t(1-t)^{10}\,dt = 2^{46}\, B(2, 11).$$

Note ...

Note that these examples can also be solved by putting $x = 2a \sin^2\theta$ and then the integrals reduce to the type $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\,d\theta$. [See Ex. 3, page 9-36; Ex. 8, page 9-38.] Solve these examples using this substitution.

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Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Prove that $\int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}\,dx \cdot \int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}} = \frac{432}{35}\pi.$ **(M.U. 1997, 2002)**

Sol. : In I_1, put $x = 3t$, $dx = 3\,dt$.

$$\therefore\ I_1 = \int_0^1 (3t)^{3/2} \frac{3\,dt}{\sqrt{3}\sqrt{1-t}} = 9\int_0^1 t^{3/2}(1-t)^{-1/2}\,dt = 9B\left(\frac{5}{2}, \frac{1}{2}\right)$$

In I_2, put $x^{1/4} = t$ i.e. $x = t^4$ $\quad\therefore\ dx = 4t^3\,dt$

$$\therefore\ I_2 = \int_0^1 4t^3 \frac{dt}{\sqrt{1-t}} = 4\int_0^1 t^3(1-t)^{-1/2}\,dt = 4B\left(4, \frac{1}{2}\right)$$

$$\therefore\ I = I_1 \cdot I_2 = 9B\left(\frac{5}{2}, \frac{1}{2}\right) \cdot 4B\left(4, \frac{1}{2}\right)$$

$$= 36 \cdot \frac{\overline{|5/2}\ \overline{|1/2}}{\overline{|3}} \cdot \frac{\overline{|4}\ \overline{|1/2}}{\overline{|9/2}}$$

$$= 36 \cdot \frac{\overline{|5/2}\ \overline{|1/2}}{2!} \cdot \frac{3!\ \overline{|1/2}}{(7/2)(5/2)\overline{|5/2}}$$

$$\therefore\ I = 36 \cdot \frac{3 \cdot 2 \cdot 1}{2 \cdot 1} \cdot \left(\overline{\left|\tfrac{1}{2}\right.}\right)^2 \cdot \frac{2}{7} \cdot \frac{2}{5}\pi = \frac{432}{35}\pi.$$

Example 2 (c) : Prove that $\int_0^1 \sqrt{1-\sqrt{x}}\,dx \cdot \int_0^{1/2} \sqrt{2y-4y^2}\,dy = \frac{\pi}{30}.$ **(M.U. 1998, 2001, 12)**

Sol. : In I_1, put $\sqrt{x} = t$ i.e. $x = t^2$ $\quad\therefore\ dx = 2t\,dt$

$$\therefore\ I_1 = \int_0^1 \sqrt{1-t} \cdot 2t\,dt = 2\int_0^1 t(1-t)^{1/2}\,dt = 2B\left(2, \frac{3}{2}\right)$$

In I_2, put $2y = t$ $\quad\therefore\ 2\,dy = dt$.

$$\therefore\ I_2 = \int_0^{1/2} \sqrt{2y}\sqrt{1-2y}\,dy = \int_0^1 t^{1/2}(1-t)^{1/2} \cdot \frac{1}{2}\,dt$$

$$= \frac{1}{2}\int_0^1 t^{1/2}(1-t)^{1/2}\,dt = \frac{1}{2} \cdot B\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$\therefore\ I = I_1 \cdot I_2 = 2B\left(2, \frac{3}{2}\right) \cdot \frac{1}{2}B\left(\frac{3}{2}, \frac{3}{2}\right) = B\left(2, \frac{3}{2}\right) \cdot B\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= \frac{\overline{|2}\ \overline{|3/2}}{\overline{|7/2}} \cdot \frac{\overline{|3/2}\ \overline{|3/2}}{\overline{|3}} = \frac{1!\ \overline{|3/2}}{(5/2)(3/2)\overline{|3/2}} \cdot \frac{\left[(1/2)\ \overline{|1/2}\right]^2}{2!}$$

$$= \frac{4}{15} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \pi = \frac{\pi}{30}.$$

Example 3 (c) : Prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \cdot \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{4}.$ **(M.U. 2013)**

Sol. : Put $x^4 = t$ $\quad\therefore\ x = t^{1/4}$ $\quad\therefore\ dx = \frac{1}{4}t^{-3/4}\,dt.$

When $x = 0$, $t = 0$; when $x = 1$, $t = 1$.

$$\therefore \; I = \int_0^1 \frac{t^{1/2}}{\sqrt{1-t}} \cdot \frac{1}{4} \cdot t^{-3/4}\, dt \cdot \int_0^1 \frac{1}{\sqrt{1-t}} \cdot \frac{1}{4} \cdot t^{-3/4}\, dt$$

$$= \int_0^1 \frac{1}{4} \cdot t^{-1/4} (1-t)^{-1/2}\, dt \cdot \int_0^1 \frac{1}{4} \cdot t^{-3/4} (1-t)^{-1/2}\, dt$$

$$= \frac{1}{16} B\left(-\frac{1}{4}+1, -\frac{1}{2}+1\right) \cdot B\left(-\frac{3}{4}+1, -\frac{1}{2}+1\right)$$

$$= \frac{1}{16} B\left(\frac{3}{4}, \frac{1}{2}\right) \cdot B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{16} \cdot \frac{\Gamma(3/4)\, \Gamma(1/2)}{\Gamma(5/4)} \cdot \frac{\Gamma(1/4)\, \Gamma(1/2)}{\Gamma(3/4)}$$

$$= \frac{1}{16} \cdot \frac{\Gamma(1/2)}{(1/4)\, \Gamma(1/4)} \cdot \Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{4} \left(\Gamma\left(\frac{1}{2}\right) \right)^2 = \frac{\pi}{4}$$

EXERCISE - VI

Evaluate the following : Class (a) : 3 Marks

1. $\int_0^1 x^2 (1-x^2)^4\, dx$
2. $\int_0^2 x^7 (16-x^4)^{10}\, dx$
3. $\int_0^1 \frac{x^n}{\sqrt{1-x^2}}\, dx$
4. $\int_0^1 \frac{dx}{\sqrt[n]{1-x^n}}$
5. $\int_0^1 \frac{x\, dx}{\sqrt{1-x^5}}$
6. $\int_0^1 \sqrt{x}\, (1-x^2)^{1/3}\, dx$
7. $\int_0^{2a} \frac{x^{9/2}}{\sqrt{2a-x}}\, dx$
8. $\int_0^2 x \cdot \sqrt{2x-x^2}\, dx$
9. $\int_0^4 x^2 \sqrt{4x-x^2}\, dx$

[Ans. : (1) $\frac{1}{2} B\left(\frac{3}{2}, 5\right)$ (2) $2^6 \cdot 16^{10}\, B(2, 11)$ (3) $\frac{1}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right)$ (4) $\frac{1}{n} B\left(-\frac{1}{n}+1, \frac{1}{n}\right)$

(5) $\frac{1}{5} B\left(\frac{2}{5}, \frac{1}{2}\right)$ (6) $\frac{1}{2} B\left(\frac{3}{4}, \frac{4}{3}\right)$ (7) $\frac{\pi}{4}$ (7) $(2a)^5 \cdot B\left(\frac{11}{2}, \frac{1}{2}\right)$ (8) $\frac{\pi}{2}$ (9) 10π]

Prove the following results : Class (a) : 3 Marks

10. $\int_0^1 \frac{x}{\sqrt{1-x^4}}\, dx = \frac{\pi}{4}$
11. $\int_0^a x^4 (a^2-x^2)^{1/2}\, dx = \frac{a^6 \pi}{32}$
12. $\int_0^a x^2 (a^2-x^2)^{3/2}\, dx = \frac{a^6 \pi}{32}$
13. $\int_0^a x^6 (a^4-x^4)^{1/4}\, dx = \frac{\sqrt{2}}{128} a^8 \pi$
14. $\int_0^a x^{10} (a^6-x^6)^{1/6}\, dx = \frac{5}{6^3} a^{12} \pi$

Evaluate the following : Class (c) : 8 Marks

15. $\int_0^1 \frac{x^2\, dx}{\sqrt{1-x^4}} \int_0^1 \frac{dx}{\sqrt{1-x^4}}$ (M.U. 1994,95) [Ans. : $\frac{\pi}{4}$]

Type III

Show that if m, n are positive integers. ...1)

$$\boxed{\int_0^1 (1-\sqrt[n]{x})^m\,dx = \frac{m!\,n!}{(m+n)!}} \quad \text{.......... (16)}$$

Proof : Put $x^{1/n} = t \quad \therefore\ x = t^n \quad \therefore\ dx = n\,t^{n-1}\,dt$

$$\therefore\ I = \int_0^1 n \cdot t^{n-1} \cdot (1-t)^m\,dt = n\,B(n,\ m+1)$$

$$= n\frac{\lfloor n \ \lfloor m+1}{\lfloor n+m+1} = \frac{n(n-1)!\ m!}{(m+n)!} = \frac{n!\ m!}{(m+n)!}$$

Note ....

Note that the result (16) is a particular case of the result (15) with $m = 0$, $1/n = n$, $p = m$.

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Evaluate $\int_0^1 (1-\sqrt[3]{x})^{11/2}\,dx$.

Sol. : In the above result (16), put $n = 3$, $m = 11/2$ or independently put $x^{1/3} = t$.

$$\therefore\ x = t^3 \quad \therefore\ dx = 3t^2\,dt$$

$$\therefore\ I = \int_0^1 3\,t^2\,(1-t)^{11/2}\,dt = 3\,B\left(3,\ \frac{13}{2}\right)$$

Example 2 (a) : Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^m}}$.

Sol. : In the above result (16), put $1/n = m$ and $m = -1/2$ or independently put $x^m = t$.

$$\therefore\ x = t^{1/m} \quad \therefore\ dx = \frac{1}{m}\,t^{(1/m)-1}\,dt.$$

$$\therefore\ I = \int_0^1 \frac{1}{m} \cdot t^{(1/m)-1}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{m}\int_0^1 t^{(1/m)-1} \cdot (1-t)^{-1/2}\,dt = \frac{1}{m}\,B\left(\frac{1}{m},\ \frac{1}{2}\right)$$

Example 3 (a) : Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$.

Sol. : In the above result (16) put $n = 1/4$ and $m = -1/2$ or independently put $x^4 = t$.

$$\therefore\ x = t^{1/4} \text{ and } dx = \frac{1}{4}\,t^{-3/4}\,dt$$

$$\therefore\ I = \int_0^1 (1-t)^{-1/2} \cdot \frac{1}{4}\,t^{-3/4}\,dt = \frac{1}{4}\,B\left(\frac{1}{4},\ \frac{1}{2}\right).$$

(M.U. 2008)

Example 4 (a) : Evaluate $\int_0^1 \sqrt{1-x^4}\,dx$.

Sol. : In the above result (16) put $n = 1/4$ and $m = 1/2$ or independently put $x^4 = t$

$$\therefore \quad x = t^{1/4} \qquad \therefore \quad dx = \frac{1}{4} t^{-3/4}\, dt$$

$$\therefore \quad I = \int_0^1 \frac{1}{4} \cdot t^{-3/4} (1-t)^{1/2}\, dt = \frac{1}{4} B\left(\frac{1}{4}, \frac{3}{2}\right)$$

EXERCISE - VII

Evaluate the following : Class (a) : 3 Marks

1. $\int_0^1 \frac{dx}{\sqrt{1-x^8}}$ 2. $\int_0^1 \sqrt{1-x^m} \cdot dx$ 3. $\int_0^1 \sqrt{1-x^8}\, dx$ 4. $\int_0^1 \sqrt{1-x^6}\, dx$

[**Ans. :** (1) $\frac{1}{8} B\left(\frac{1}{8}, \frac{1}{2}\right)$ (2) $\frac{1}{m} B\left(\frac{1}{m}, \frac{3}{2}\right)$ (3) $\frac{1}{8} B\left(\frac{1}{8}, \frac{1}{2}\right)$ (4) $\frac{1}{6} B\left(\frac{1}{6}, \frac{3}{2}\right)$]

Type IV

$$\boxed{\int_0^{\pi/2} \sin^p \theta \cos^q \theta \; d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)} \quad \text{.....................} (17)$$

Using Gamma functions

$$\boxed{\int_0^{\pi/2} \sin^p \theta \cos^q \theta \; d\theta = \frac{1}{2} \frac{\left|\overline{(p+1)/2}\right. \left|\overline{(q+1)/2}\right.}{\left|\overline{(p+q+2)/2}\right.}} \quad \text{.....................} (18)$$

(a) Integrals of the form $\int_0^{\pi/2} \sin^n \theta \; d\theta$

Cor. 1 : Putting $p = n$ and $q = 0$ in (I), we get,

$$\int_0^{\pi/2} \sin^n \theta \; d\theta = \frac{1}{2} \frac{\left|\overline{(n+1)/2}\right. \left|\overline{1/2}\right.}{\left|\overline{(n+2)/2}\right.} \quad \text{.....................} (19)$$

Case 1 : If n is an even positive integer then

$$\int_0^{\pi/2} \sin^n \theta \; d\theta = \frac{1}{2} \cdot \frac{\frac{(n-1)}{2} \frac{(n-3)}{2} \cdots\cdots \frac{3}{2} \cdot \frac{1}{2} \left|\overline{\tfrac{1}{2}}\right. \left|\overline{\tfrac{1}{2}}\right.}{\frac{n}{2} \frac{(n-2)}{2} \cdots\cdots \frac{4}{2} \cdot \frac{2}{2}}$$

$$\therefore \quad \int_0^{\pi/2} \sin^n \theta \; d\theta = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots\cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{.....................} (20)$$

For example, (1) $\int_0^{\pi/2} \sin^2 \theta \, d\theta = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$, (2) $\int_0^{\pi/2} \sin^4 \theta \, d\theta = \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{3}{16} \pi$,

(3) $\int_0^{\pi/2} \sin^6 \theta \, d\theta = \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{5}{32} \pi$.

Case 2 : If n is an odd positive integer then

$$\int_0^{\pi/2} \sin^n \theta \; d\theta = \frac{1}{2} \cdot \frac{\frac{(n-1)}{2} \frac{(n-3)}{2} \cdots\cdots \frac{4}{2} \cdot \frac{2}{1} \cdot \left|\overline{\tfrac{1}{2}}\right.}{\frac{n}{2} \frac{(n-2)}{2} \cdots\cdots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \left|\overline{\tfrac{1}{2}}\right.}$$

$$\therefore \int_0^{\pi/2} \sin^n \theta \, d\theta = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \qquad \text{.......... (21)}$$

For example, **(1)** $\int_0^{\pi/2} \sin^3 \theta \, d\theta = \frac{2}{3 \cdot 1} = \frac{2}{3}$, **(2)** $\int_0^{\pi/2} \sin^5 \theta \, d\theta = \frac{4 \cdot 2}{5 \cdot 3 \cdot 1} = \frac{8}{15}$.

Cor. : To find $\int_0^1 \frac{x^n}{\sqrt{1-x^2}} dx$. Put $x = \sin \theta$.

$$\therefore \int_0^1 \frac{x^n}{\sqrt{1-x^2}} dx = \int_0^{\pi/2} \frac{\sin^n \theta}{\cos \theta} \cos \theta \, d\theta = \int_0^{\pi/2} \sin^n \theta \, d\theta$$

$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1 & \text{if } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

(See Examples 3, 4, 5, 8 below)

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Evaluate $\int_0^{\pi/4} \sin^7 2\theta \, d\theta$.

Sol. : Put $2\theta = x \therefore 2 \, d\theta = dx$.

$$\therefore I = \int_0^{\pi/2} \frac{1}{2} \sin^7 x \, dx = \frac{1}{2} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{8}{35} \qquad \text{[By formula (21)]}$$

Example 2 (a) : Evaluate $\int_0^{\pi} (1 - \cos \theta)^3 \, d\theta$.

Sol. : We have $\int_0^{\pi} (1 - \cos \theta)^3 \, d\theta = \int_0^{\pi} 2^3 \cdot \sin^6 (\theta / 2) \cdot d\theta$

Putting $\theta / 2 = x \quad \therefore d\theta = 2 \, dx$

$$\therefore I = 8 \int_0^{\pi/2} 2 \cdot \sin^6 x \, dx = 16 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5}{2} \cdot \pi \qquad \text{[By formula (20)]}$$

Example 3 (a) : Evaluate $\int_0^a \frac{x^4}{\sqrt{a^2 - x^2}} dx$.

Sol. : Put $x = a \sin \theta \quad \therefore dx = a \cos \theta \, d\theta$

$$\therefore I = \int_0^{\pi/2} \frac{a^4 \sin^4 \theta}{a \cos \theta} a \cos \theta \, d\theta = a^4 \int_0^{\pi/2} \sin^4 \theta \, d\theta$$

$$= a^4 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3}{16} \pi a^4 \qquad \text{[By formula (20)]}$$

Example 4 (a) : Evaluate $\int_0^1 \frac{x^7}{\sqrt{1 - x^4}} dx$.

Sol. : Put $x^2 = \sin \theta \quad \therefore 2x \, dx = \cos \theta \, d\theta$

$$\therefore \int_0^1 \frac{x^7}{\sqrt{1-x^4}} dx = \int_0^1 \frac{(x^2)^3}{\sqrt{1-(x^2)^2}} x \, dx = \int_0^{\pi/2} \frac{\sin^3 \theta}{\cos \theta} \cdot \frac{1}{2} \cdot \cos \theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \sin^3 \theta \, d\theta = \frac{1}{2} \cdot \frac{2}{3} \cdot 1 = \frac{1}{3} \qquad \text{[By formula (21)]}$$

Example 5 (a) : Evaluate $\int_0^3 \frac{x^{3/2}}{(3-x)^{1/2}}\,dx$.

Sol. : Put $x = 3\sin^2\theta$ $\quad\therefore\ dx = 6\sin\theta\cos\theta\,d\theta$

$$\therefore\ \int_0^3 \frac{x^{3/2}}{(3-x)^{1/2}}\,dx = \int_0^{\pi/2} \frac{3\sqrt{3}\sin^3\theta}{\sqrt{3}\cdot\cos\theta}\cdot 6\sin\theta\cos\theta\,d\theta = 18\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$= 18\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = \frac{27\pi}{8}$$ [By (20), page 6-28]

Alternatively we put $x = 3t$ $\quad\therefore\ dx = 3\,dt$

$$\therefore\ I = \int_0^1 (3t)^{3/2}\cdot 3^{-1/2}(1-t)^{-1/2}\cdot 3\,dt = 3^2\int_0^1 t^{3/2}(1-t)^{-1/2}\,dt$$

$$= 9\cdot B\left(\frac{5}{2},\frac{1}{2}\right) = 9\cdot\frac{\Gamma(5/2)\,\Gamma(1/2)}{\Gamma(3)}$$

$$= 9\cdot\frac{(3/2)(1/2)\,\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(3)} = \frac{27\pi}{8}.$$

Example 6 (a) : Evaluate $\int_0^1 x^5\sqrt{\left(\frac{1+x^2}{1-x^2}\right)}\,dx$

Sol. : We have $\int_0^1 x^5\sqrt{\left(\frac{1+x^2}{1-x^2}\right)}\,dx = \int_0^1 x^4\cdot\frac{(1+x^2)}{\sqrt{1-x^4}}\cdot x\cdot dx$

Put $x^2 = \sin\theta$ $\quad\therefore\ 2x\,dx = \cos\theta\,d\theta$

$$\therefore\ I = \frac{1}{2}\int_0^{\pi/2}\sin^2\theta\cdot\frac{(1+\sin\theta)}{\cos\theta}\cdot\cos\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}(\sin^2\theta+\sin^3\theta)\,d\theta = \frac{1}{2}\left[\frac{1}{2}\cdot\frac{\pi}{2}+\frac{2}{3}\cdot 1\right]$$

[By (20), page 6-28 and (21), page 6-29]

$$\therefore\ I = \frac{1}{2}\left[\frac{\pi}{4}+\frac{2}{3}\right] = \frac{3\pi+8}{24}.$$

Example 7 (a) : Evaluate $\int_0^1 x^5\sin^{-1}x\,dx$.

Sol. : Integrating by parts, we have,

$$\int_0^1 x^5\sin^{-1}x\,dx = \left[\sin^{-1}x\cdot\frac{x^6}{6}\right]_0^1 - \int_0^1\frac{x^6}{6}\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

$$\therefore\ \int_0^1 x^5\sin^{-1}x\,dx = \frac{\pi}{2}\cdot\frac{1}{6} - \frac{1}{6}\int_0^1\frac{x^6}{\sqrt{1-x^2}}\,dx$$

Put $x = \sin\theta$ $\quad\therefore\ dx = \cos\theta\,d\theta$

$$\therefore\ I = \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2}\frac{\sin^6\theta}{\cos\theta}\cos\theta\,d\theta = \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2}\sin^6\theta\,d\theta$$

$$= \frac{\pi}{12} - \frac{1}{6}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{12} - \frac{5\pi}{192} = \frac{11}{192}\cdot\pi.$$ [By (20), page 6-28]

Aliter : Put $\sin^{-1} x = t$ $\therefore$ $x = \sin t$ $\therefore$ $dx = \cos t\, dt$.

When $x = 0$, $t = 0$; when $x = 1$, $t = \pi/2$.

$$\therefore \; I = \int_0^{\pi/2} \sin^5 t \cdot t \cdot \cos t\, dt = \int_0^{\pi/2} t(\sin^5 t \cos t)\, dt$$

Integrating by parts, we get,

$$I = \left[t \cdot \left(\frac{\sin^6 t}{6} \right) \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin^6 t}{6} \cdot 1 \cdot dt$$

$$= \left(\frac{\pi}{2} \cdot \frac{1}{6} - 0 \right) - \frac{1}{6} \cdot \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} \qquad \text{[By (20), page 6-28]}$$

$$= \frac{\pi}{12} - \frac{5\pi}{192} = \frac{11\pi}{192}.$$

Example 8 (a) : Prove that $\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\, dx = \frac{(2n)!}{2^{2n} \cdot (n!)^2} \cdot \frac{\pi}{2}$.

Sol. : Put $x = \sin\theta$ $\therefore$ $dx = \cos\theta\, d\theta$

$$\therefore \; \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\, dx = \int_0^{\pi/2} \frac{\sin^{2n}\theta}{\cos\theta} \cos\theta\, d\theta = \int_0^{\pi/2} \sin^{2n}\theta\, d\theta$$

$$= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdots\cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{2n \cdot (2n-1) \cdot (2n-2) \cdot (2n-3) \cdots\cdots 3 \cdot 2 \cdot 1}{[2n \cdot (2n-2) \cdot (2n-4) \cdots\cdots 2]^2} \cdot \frac{\pi}{2}$$

$$= \frac{(2n)!}{2^{2n} \cdot (n!)^2} \cdot \frac{\pi}{2}.$$

(If you put $x = \cos\theta$, you will get the same result.)

(For another method, see Ex. 7, page 6-56.)

EXERCISE - VIII

Evaluate the following integrals. (1 to 10) : Class (a) : 3 Marks

1. $\int_0^1 \frac{x^7}{\sqrt{1-x^2}}\, dx$

2. $\int_0^1 \frac{x^6}{\sqrt{1-x^2}}\, dx$

3. $\int_0^1 \frac{x^2(4-x^4)}{\sqrt{1-x^2}}\, dx$

4. $\int_0^{\pi/6} \sin^6 3\theta \cdot d\theta$

5. $\int_0^{\pi} (1-\cos\theta)^5 \cdot d\theta$

6. $\int_0^1 \frac{x^9}{\sqrt{1-x^4}}\, dx$

7. $\int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2} \cdot d\theta$ **(M.U. 2005)**

8. $\int_0^1 x^4 \cos^{-1} x\, dx$ **(M.U. 2006)**

9. $\int_0^{\infty} \frac{t^6}{(1+t^2)^4} \cdot dt$

10. $\int_0^1 x^5 \cdot \sqrt{\left\{ \frac{1-x^2}{1+x^2} \right\}}\, dx$

11. Prove that $\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n \cdot n!} \cdot \frac{\pi}{2}$

12. Prove that $\int_0^{\pi/2} \sin^p x \, dx \int_0^{\pi/2} \sin^{p+1} x \, dx = \frac{1}{(p+1)} \cdot \frac{\pi}{2}$ (M.U. 1999)

[Ans. : (1) $\frac{16}{35}$ (2) $\frac{5}{32}\pi$ (3) $\frac{27}{32}\pi$ (4) $\frac{5}{96}\pi$, (5) $\frac{63}{8}\pi$,

(6) $\frac{3}{32}\pi$ (7) $\frac{3}{2}\pi$ (8) $\frac{8}{75}$ (9) $\frac{5}{32}\pi$ (10) $\frac{\pi}{8} - \frac{1}{3}$.]

(b) Integrals of the form $\int_0^{\pi/2} \cos^n x \, dx$

Cor. 2 : Putting $p = 0$ and $q = n$ in (17) page 6-28, we get,

$$\int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{1}{2} \frac{\overline{|(n+1)/2} \ \overline{|1/2}}{\overline{|(n+2)/2}} \quad \text{.......... (22)}$$

Reasoning as for sine function (page 6-28), we get,

Case 1 : If n is even positive integer then

$$\int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots\cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{.......... (23)}$$

For example, (1) $\int_0^{\pi/2} \cos^2 \theta \, d\theta = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$

(2) $\int_0^{\pi/2} \cos^6 \theta \, d\theta = \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{5}{32}\pi$

(3) $\int_0^{\pi/2} \cos^8 \theta \, d\theta = \frac{7 \cdot 5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{35}{256}\pi$

Case 2 : If n is an odd positive integer then,

$$\int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots\cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \quad \text{.......... (24)}$$

For example, (1) $\int_0^{\pi/2} \cos^5 \theta \, d\theta = \frac{4 \cdot 2}{5 \cdot 3 \cdot 1} = \frac{8}{15}$

(2) $\int_0^{\pi/2} \cos^7 \theta \, d\theta = \frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3 \cdot 1} = \frac{16}{35}$

Cor. 3 : To find $\int_0^{\infty} \frac{dx}{(1+x^2)^{n+(1/2)}}$. Put $x = x \tan \theta$.

$$\int_0^{\infty} \frac{dx}{(1+x^2)^{n+(1/2)}} = \int_0^{\pi/2} \frac{\sec^2 \theta \, d\theta}{\sec^{2n+1} \theta \, d\theta} = \int_0^{\pi/2} \cos^{2n-1} \theta \, d\theta$$

$$= \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots\cdots \frac{2}{3} \cdot 1.$$

(See Examples 2 and 3)

Cor. 4 : To find $\int_0^{\infty} \frac{dx}{(1+x^2)^n} dx$. Put $x = \tan \theta$.

$$\int_0^{\infty} \frac{dx}{(1+x^2)^n} dx = \int_0^{\pi/2} \frac{\sec^2 \theta \, d\theta}{\sec^{2n} \theta \, d\theta} = \int_0^{\pi/2} \cos^{2n-2} \theta \, d\theta$$

$$= \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdots\cdots \frac{1}{2} \cdot \frac{\pi}{2}.$$

(See Examples 4 and 5)

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Evaluate $\int_0^a (a^2 - x^2)^{5/2}\, dx$.

Sol. : Put $x = a \sin\theta \qquad \therefore\ dx = a\cos\theta\, d\theta$

$$\therefore\ I = \int_0^{\pi/2} a^5 \cos^5\theta \cdot a\cos\theta\, d\theta = a^6 \int_0^{\pi/2} \cos^6\theta\, d\theta$$

$$= a^6 \cdot \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5}{32}\cdot\pi a^6. \qquad \text{[By formula (23)]}$$

Example 2 (a) : Evaluate $\int_0^\infty \frac{dx}{(1+x^2)^{9/2}}$.

Sol. : Put $x = \tan\theta \qquad \therefore\ dx = \sec^2\theta\, d\theta$

$$\therefore\ I = \int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{\sec^9\theta} = \int_0^{\pi/2} \cos^7\theta\, d\theta$$

$$= \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{16}{35} \qquad \text{[By formula (24)]}$$

Example 3 (a) : Evaluate $\int_0^\infty \frac{x^2}{(1+x^6)^{7/2}}\, dx$.

Sol. : Put $x^3 = t \qquad \therefore\ 3x^2\, dx = dt$

$$\therefore\ I = \frac{1}{3}\int_0^\infty \frac{dt}{(1+t^2)^{7/2}}. \quad \text{Put } t = \tan\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{\sec^7\theta} = \frac{1}{3}\int_0^{\pi/2} \cos^5\theta\, d\theta \qquad \text{[By formula (24)]}$$

$$= \frac{1}{3}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{8}{45}.$$

Example 4 (a) : Evaluate $\int_0^\infty \frac{dx}{(1+x^2)^5}$.

Sol. : By Cor. 4, put $x = \tan\theta \qquad \therefore\ dx = \sec^2\theta\, d\theta$

$$\therefore\ I = \int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{\sec^{10}\theta} = \int_0^{\pi/2} \cos^8\theta\, d\theta \qquad \text{[By formula (23)]}$$

$$= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35}{256}\cdot\pi.$$

Example 5 (a) : Evaluate $\int_0^\infty \frac{x^3}{(1+x^8)^4}\, dx$.

Sol. : Put $x^4 = t \qquad \therefore\ 4x^3\, dx = dt$

$$I = \frac{1}{4}\int_0^\infty \frac{dt}{(1+t^2)^4}. \quad \text{Put } t = \tan\theta$$

$$\therefore\ I = \frac{1}{4}\int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{\sec^8\theta} = \frac{1}{4}\int_0^{\pi/2} \cos^6\theta\, d\theta \qquad \text{[By formula (23)]}$$

$$= \frac{1}{4}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5}{128}\pi.$$

Example 6 (a) : Evaluate $\int_0^{\infty} \frac{1}{(x^2+1)^{n+1}}\, dx$.

Sol. : By Cor. 4, put $x = \tan\theta$ $\quad \therefore \quad dx = \sec^2\theta \; d\theta$

$$\int_0^{\infty} \frac{1}{(x^2+1)^{n+1}}\, dx = \int_0^{\pi/2} \frac{\sec^2\theta}{(\sec^2\theta)^{2n+1}} \cdot d\theta = \int_0^{\pi/2} \frac{1}{\sec^{2n}\theta} \cdot d\theta$$

$$= \int_0^{\pi/2} \cos^{2n}\theta \cdot d\theta = \frac{(2n)!}{2^{2n}(n!)^2} \cdot \frac{\pi}{2}. \qquad \text{(As in Ex. 8 on page 6-31)}$$

EXERCISE - IX

Evaluate the following integrals : Class (a) : 3 Marks

1. $\int_0^{\infty} \frac{dx}{(1+x^2)^{7/2}}$ 2. $\int_0^{\infty} \frac{dx}{(a^2+x^2)^5}$ 3. $\int_0^{\infty} \frac{x^2}{(1+x^6)^3}\, dx$

4. $\int_0^{\infty} \frac{x^2}{(a^2+x^6)^{5/2}}\, dx$ 5. $\int_0^{\pi} (1+\cos\theta)^3\, d\theta$ 6. $\int_0^{\pi/4} \cos^7 2\theta\, d\theta$

7. $\int_0^{\pi/4} (1+\cos 4\theta)^5\, d\theta$ 8. $\int_0^{\infty} \frac{dx}{(x^2+1)^5}$ 9. $\int_0^{\infty} (1+x^2)^{-n-(1/2)}\, dx$

[**Ans. :** (1) $\frac{8}{15}$, (2) $\frac{35}{256} \cdot \frac{1}{a^9}$, (3) $\frac{\pi}{16}$, (4) $\frac{2}{9a^4}$, (5) $\frac{5}{2} \cdot \pi$,

(6) $\frac{8}{35}$, (7) $\frac{63}{32} \cdot \pi$, (8) $\frac{35}{256} \cdot \pi$, (9) $\frac{2^{2(n-1)} \cdot [(n-1)!]^2}{(2n-1)!}$.]

(c) Integrals of the form $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$

Cor. 5 : Putting $p = m$ and $q = n$ in (18), (page 6-28) we get,

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \; d\theta = \frac{1}{2} \cdot \frac{\left\lceil \frac{m+1}{2} \right. \left\lceil \frac{n+1}{2} \right.}{\left\lceil \frac{m+n+2}{2} \right.}$$

Case 1 : If m and n both are even positive integers,

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \; d\theta = \frac{1}{2} \cdot \frac{\frac{m-1}{2} \cdot \frac{m-3}{2} \cdots\cdots \frac{3}{2} \cdot \frac{1}{2} \left\lceil \frac{1}{2} \right. \cdot \frac{n-1}{2} \cdot \frac{n-3}{2} \cdots\cdots \frac{3}{2} \cdot \frac{1}{2} \cdot \left\lceil \frac{1}{2} \right.}{\frac{m+n}{2} \cdot \frac{m+n-2}{2} \cdots\cdots \frac{6}{2} \cdot \frac{4}{2} \cdot \frac{2}{2}}$$

$$= \frac{(m-1)(m-3)\cdots\cdots(n-1)(n-3)\cdots\cdots}{(m+n)(m+n-2)\cdots\cdots} \cdot \frac{\pi}{2} \qquad \cdots\cdots\cdots\cdots\cdots\cdots \text{(25)}$$

Case 2 : If m or n or both are odd positive integers, we get,

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \; d\theta = \frac{(m-1)(m-3)\cdots\cdots(n-1)(n-3)\cdots\cdots}{(m+n)(m+n-2)\cdots\cdots} \cdot 1 \qquad \cdots\cdots\cdots\cdots\cdots\cdots \text{(26)}$$

The results (25) to (26) can be combined in the following working rule :

$$\boxed{\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(m-1)(m-3)\ldots\ldots(n-1)(n-3)\ldots\ldots}{(m+n)(m+n-2)\ldots\ldots} \cdot P} \qquad \ldots\ldots (27)$$

where, $P = \pi/2$ when m, n are both even
$= 1$ otherwise.

For example,

$$\int_0^{\pi/2} \sin^6 x \cos^6 x\, dx = \frac{5\cdot3\cdot1\cdot5\cdot3\cdot1}{12\cdot10\cdot8\cdot6\cdot4\cdot2}\cdot\frac{\pi}{2} = \frac{5\pi}{2048}$$

$$\int_0^{\pi/2} \sin^4 x \cos^4 x\, dx = \frac{3\cdot1\cdot3\cdot1}{8\cdot6\cdot4\cdot2}\cdot\frac{\pi}{2} = \frac{3\pi}{256}$$

$$\int_0^{\pi/2} \sin^5 x \cos^4 x\, dx = \frac{4\cdot2\cdot3\cdot1}{9\cdot7\cdot5\cdot3\cdot1} = \frac{8}{315}$$

$$\int_0^{\pi/2} \sin^4 x \cos^5 x\, dx = \frac{3\cdot1\cdot4\cdot2}{9\cdot7\cdot5\cdot3\cdot1} = \frac{8}{315}$$

$$\int_0^{\pi/2} \sin^5 x \cos^3 x\, dx = \frac{4\cdot2\cdot2\cdot1}{8\cdot6\cdot4\cdot2} = \frac{1}{24}$$

Note ....

The formulae (19), (20), (21), (22), (23), (24), (25), (26) are known as reduction formulae and are highly useful in evaluating integrals of the type $\int_0^{\pi/2} \sin^m \theta\, d\theta$, $\int_0^{\pi/2} \cos^m \theta\, d\theta$, $\int_0^{\pi/2} \sin^n \theta \cos^n \theta\, d\theta$. Particularly, we shall require them while evaluating areas and volumes.

$$\therefore \; I = 32\int_0^{\pi} \sin^2(\theta/2)\cos^2(\theta/2)\cos^6(\theta/2)\cdot d\theta$$

$$= 32\int_0^{\pi} \sin^2(\theta/2)\cos^8(\theta/2)\cdot d\theta$$

Put $(\theta/2) = t \quad \therefore \; d\theta = 2\,dt.$

When $\theta = 0,\ t = 0$; when $\theta = \pi,\ t = \pi/2$.

$$\therefore \; I = 32\int_0^{\pi/2} \sin^2 t\cos^8 t\cdot 2\,dt = 64\int_0^{\pi/2}\sin^2 t\cdot\cos^8 t\,dt$$

$$= \frac{64\cdot(1)\cdot(7\cdot5\cdot3\cdot1)}{10\cdot8\cdot6\cdot4\cdot2}\cdot\frac{\pi}{2} = \frac{7\pi}{8}.$$

Example 3 (a) : Evaluate $\int_{-\pi}^{\pi}\sin^2 x\cos^4 x\,dx.$ **(M.U. 2008)**

Sol. : We have,

$$I = \int_{-\pi}^{\pi}\sin^2 x\cos^4 x\,dx$$

$$= 2\int_0^{\pi}\sin^2 x\cos^4 x\,dx \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(x) \text{ is even};\ = 0 \text{ if } f(x) \text{ is odd}\right]$$

$$\therefore \; I = 2\left[\int_0^{\pi/2}\sin^2 x\cos^4 x\,dx + \int_0^{\pi/2}\sin^2(\pi - x)\cos^4(\pi - x)\,dx\right]$$

$$\left[\because \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx\right]$$

$$\therefore \; I = 2\left[\int_0^{\pi/2}\sin^2 x\cos^4 x + \int_0^{\pi/2}\sin^2 x\cos^4 x\,dx\right]$$

$$= 4\int_0^{\pi/2}\sin^2 x\cos^4 x\,dx = 4\cdot\frac{1\cdot3\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2} = \frac{\pi}{8}.$$

Example 4 (a) : Prove that $\int_0^{\pi} x\sin^5 x\cos^4 x\,dx = \frac{8\pi}{315}.$ **(M.U. 2008)**

Sol. : We have $\quad I = \int_0^{\pi}(\pi - x)\sin^5(\pi - x)\cos^4(\pi - x)\,dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$

$$= \int_0^{\pi}(\pi - x)\sin^5 x\cos^4 x\,dx$$

$$= \pi\int_0^{\pi}\sin^5 x\cos^4 x\,dx - \int_0^{\pi} x\sin^5 x\cos^4 x\,dx$$

$$\therefore \; 2I = \pi\int_0^{\pi}\sin^5 x\cos^4 x\,dx$$

$$= \pi\int_0^{\pi/2}\sin^5 x\cos^4 x\,dx + \pi\int_0^{\pi/2}\sin^5(\pi - x)\cos^4(\pi - x)\,dx$$

$$\left[\because \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx\right]$$

$$\therefore \; I = \left(\frac{2}{2}\right)\pi\int_0^{\pi/2}\sin^5 x\cos^4 x\,dx = \pi\cdot\frac{4\cdot2\cdot3\cdot1}{9\cdot7\cdot5\cdot3\cdot1} = \frac{8\pi}{315}. \qquad \text{[By Formula (26)]}$$

by proper substitution. If the given integral involves $\sqrt{a^2 - x^2}$, we put either $x = a\sin\theta$ or $x = a\cos\theta$ and if it involves $x^2 + a^2$, we put $x = a\tan\theta$.

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Evaluate $\int_0^1 x^4\sqrt{1-x^2}\,dx$.

Sol. : We put $x = \sin\theta$ $\therefore$ $dx = \cos\theta\,d\theta$.

When $x = 0, \theta = 0$; when $x = 1, \theta = \pi/2$.

$$\therefore \ I = \int_0^{\pi/2} \sin^4\theta \cdot \cos\theta \cdot \cos\theta\,d\theta = \int_0^{\pi/2} \sin^4\theta\cos^2\theta\,d\theta$$

$$= \frac{3\cdot1\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2} = \frac{\pi}{32}.$$ [By formula (25)]

Example 2 (a) : Evaluate $\int_0^{2a} x\sqrt{2ax - x^2}\,dx$.

Sol. : We have $I = \int_0^{2a} x^{3/2}\sqrt{2a - x}\,dx$

Put $x = 2a\sin^2\theta$ $\therefore$ $dx = 4a\sin\theta\cos\theta\,d\theta$.

When $x = 0, \theta = 0$; when $x = 2a$, $\theta = \pi/2$.

$$\therefore \ I = \int_0^{2a} x\cdot\sqrt{x}\sqrt{2a - x}\,dx = \int_0^{2a} x^{3/2}\sqrt{2a - x}\,dx$$

$$= \int_0^{\pi/2} (2a)^{3/2}(\sin^2\theta)^{3/2}\sqrt{2a - 2a\sin^2\theta}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= (2a)^2\cdot 4a\int_0^{\pi/2}\sin^4\theta\cdot\cos^2\theta\,d\theta$$

$$= 16a^3\cdot\frac{3\cdot1\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2} = \frac{\pi a^3}{2}.$$ [See also Ex. 5, page 6-23, $m = 1$]

Example 3 (a) : Evaluate $\int_0^{1/2} x^3\sqrt{1-4x^2}\cdot dx$.

Sol. : Put $2x = \sin\theta$ $\therefore$ $2\,dx = \cos\theta\,d\theta$.

When $x = 0, \theta = 0$; when $x = 1/2$, $\theta = \pi/2$.

$$\therefore \ I = \int_0^{\pi/2} \frac{1}{8}\sin^3\theta\cdot\sqrt{1-\sin^2\theta}\cdot\frac{1}{2}\cos\theta\,d\theta$$

$$= \frac{1}{16}\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta = \frac{1}{16}\cdot\frac{2\cdot1}{5\cdot3\cdot1} = \frac{1}{120}.$$

Example 4 (a) : Evaluate $\int_0^{\infty}\frac{x^2}{(1+x^2)^{7/2}}\,dx$.

Sol. : Put $x = \tan\theta$ $\therefore$ $dx = \sec^2\theta\,d\theta$.

When $x = 0, \theta = 0$; when $x = \infty$, $\theta = \pi/2$.

$$\therefore \ I = \int_0^{\pi/2}\tan^2\theta\cdot\frac{1}{\sec^7\theta}\cdot\sec^2\theta\,d\theta$$

While evaluating $\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta$, we use formulae (25), (26), (27) when m and n are integers and formula (17), (18) when m, n are fractions.

Example 5 (a) : Evaluate $\int_0^a (a^2 - x^2)^{5/2} \, dx$.

Sol. : Put $x = a \sin \theta$ $\quad \therefore \quad dx = a \cos \theta \, d\theta$

$$\therefore \quad I = \int_0^{\pi/2} a^5 \cos^5 \theta \cdot a \cos \theta \, d\theta = a^6 \int_0^{\pi/2} \cos^6 \theta \, d\theta$$

$$= a^6 \cdot \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{5}{32} \pi a^6. \qquad \text{[By (23), page 6-32]}$$

Example 6 (a) : Evaluate $\int_0^a x^4 \sqrt{a^2 - x^2} \cdot dx$. **(S.U. 2006)**

Sol. : Put $x = a \sin \theta$, $dx = a \cos \theta \, d\theta$

$$\therefore \quad I = \int_0^{\pi/2} a^4 \sin^4 \theta \cdot a \cos \theta \cdot a \cos \theta \, d\theta$$

$$= \int_0^{\pi/2} a^6 \sin^4 \theta \cos^2 \theta \, d\theta$$

$$= a^6 \cdot \frac{3 \cdot 1}{6 \cdot 4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^6}{32}. \qquad \text{[By (25), page 6-34]}$$

Remark ....

Examples 5, 6 can also be solved by putting $x = at$ as shown on page 6-20 without using trigonometric substitution.

$$I = \int_0^{\pi/2} 2\sin\theta \cdot 4^{3/2} \cdot \cos^5\theta \cdot 8\sin\theta\cos\theta\, d\theta$$

$$= 128 \int_0^{\pi/2} \sin^2\theta \cos^6\theta\, d\theta$$

$$= 128 \cdot \frac{1 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = 4\pi \qquad \text{[By (25), page 6-34]}$$

(Solve Ex. 3, 4, 5 given on page 6-21 by using trigonometric substitutions as shown above.)

Example 9 (a) : Express $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$ as a Gamma Function.

Sol. : We have $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$

Putting $p = 1/2$, $q = -1/2$ in (5), page 6-17, we get

$$= \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2} \cdot \frac{\Gamma(3/4)\,\Gamma(1/4)}{\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

[But by the particular case of the duplication formula,

$$\Gamma(3/4) \cdot \Gamma(1/4) = \sqrt{2} \cdot \pi. \qquad \text{[By (13), page 6-19]}$$

$$\therefore \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{1}{2} \cdot \sqrt{2} \cdot \pi = \frac{\pi}{\sqrt{2}}. \text{]}$$

Similarly, prove that $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi^2}{2}$.

Example 10 (a) : Evaluate $\int_0^{\pi/4} \sqrt{\tan 2x} \cdot dx$.

Sol. : Putting $2x = t$, $x = \frac{1}{2}t$, $dx = \frac{1}{2}dt$, we get

$$I = \int_0^{\pi/2} \sqrt{\tan t} \cdot \frac{1}{2}dt = \frac{1}{2}\int_0^{\pi/2} \sqrt{\tan t}\, dt$$

$$= \frac{1}{2}\int_0^{\pi/2} (\sin t)^{1/2} (\cos t)^{-1/2}\, dt = \frac{1}{2} \cdot \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{4} \cdot \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) \Big/ \Gamma(1) = \frac{1}{4} \cdot \sqrt{2}\,\pi = \frac{1}{2\sqrt{2}} \cdot \pi \text{ as above.} \qquad \text{[By (13), page 6-19]}$$

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Express $\int_{-\pi/4}^{\pi/4} (\sin\theta + \cos\theta)^{1/3}\, d\theta$ as a Gamma Function.

Sol. : $$I = \int_{-\pi/4}^{\pi/4} 2^{1/6} \cdot \left(\sin\theta \cdot \frac{1}{\sqrt{2}} + \cos\theta \cdot \frac{1}{\sqrt{2}}\right)^{1/3} d\theta$$

$$= 2^{1/6} \int_{-\pi/4}^{\pi/4} \left[\sin\left(\frac{\pi}{4} + \theta\right)\right]^{1/3} d\theta$$

Now, put $\frac{\pi}{4} + \theta = t \quad \therefore d\theta = dt$

When $\theta = -\pi/4$, $t = 0$; when $\theta = \pi/4$, $t = \pi/2$.

$$\therefore \; I = 2^{1/6} \int_0^{\pi/2} \sin^{1/3} t \, dt = 2^{1/6} \int_0^{\pi/2} \sin^{1/3} t \cos^0 t \, dt$$

$$= 2^{1/6} \cdot \frac{1}{2} B\left(\frac{2}{3}, \frac{1}{2}\right) = 2^{(1/6)-1} \frac{\Gamma(2/3)\,\Gamma(1/2)}{\Gamma(7/6)}$$

$$= \frac{1}{2^{5/6}} \cdot \frac{\Gamma(2/3)}{\Gamma(7/6)} \cdot \sqrt{\pi}.$$

EXERCISE - X

(A) Evaluate the following integrals : Class (a) : 3 Marks

1. $\int_0^{\pi/4} \cos^3 2\theta \cdot \sin^2 4\theta \, d\theta$
2. $\int_0^{\pi/6} \cos^6 3\theta \cdot \sin^2 6\theta \, d\theta$
3. $\int_0^{\pi} \sin^2 \theta \cdot (1 + \cos\theta)^4 \, d\theta$
4. $\int_{-\pi}^{\pi} \sin^4 x \cos^2 x \, dx$
5. $\int_0^{\pi} x \sin^7 x \cos^4 x \, dx$
6. $\int_{-\pi/2}^{\pi/2} \cos^3 \theta \cdot (1 + \sin\theta)^2 \, d\theta$
7. $\int_0^{2\pi} \sin^2 \theta \cdot (1 + \cos\theta)^4 \, d\theta$ (M.U. 2004)
8. $\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^2 x \, dx$
9. $\int_0^1 x^6 \sqrt{1 - x^2} \, dx$
10. $\int_0^1 x^2 \cdot (1 - x^2)^{3/2} \, dx$
11. $\int_0^1 x^2 \cdot (1 - \sqrt{x})^5 \, dx$
12. $\int_0^a x^4 \cdot \sqrt{a^2 - x^2} \, dx$
13. $\int_0^1 x^4 \cdot (1 - x^2)^{3/2} \, dx$
14. $\int_0^a x^2 \cdot \sqrt{(ax - x^2)} \, dx$
15. $\int_0^1 x^{3/2} \cdot (1 - x)^{3/2} \, dx$
16. $\int_0^{\pi} x \sin^5 x \cos^6 x \, dx$
17. $\int_0^{\pi} x \sin^4 x \cos^6 x \, dx$

[**Ans. :** (1) $\frac{16}{105}$, (2) $\frac{7\pi}{384}$, (3) $\frac{21\pi}{16}$, (4) $\frac{\pi}{8}$, (5) $\frac{16\pi}{1155}$, (6) $\frac{8}{5}$, (7) $\frac{21\pi}{8}$, (8) $\frac{\pi}{16}$, (9) $\frac{5\pi}{286}$, (10) $\frac{\pi}{32}$, (11) $\frac{1}{1386}$, (12) $\frac{\pi a^6}{32}$, (13) $\frac{3\pi}{256}$, (14) $\frac{5\pi a^4}{128}$, (15) $\frac{3\pi}{128}$, (16) $\frac{8\pi}{693}$, (17) $\frac{3\pi^2}{512}$.]

(B) Evaluate the following (1 to 8) : Class (a) : 3 Marks

1. $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \cdot \int_0^{\pi/2} \sqrt{\sin\theta} \, d\theta$ (M.U. 2006, 12)
2. $\int_0^{\pi/2} (\sin 2x)^{2t-1} \, dx$
3. $\int_0^{\pi} \sin^2 \theta \cdot (1 + \cos\theta)^4 \, d\theta$
4. $\int_0^{\pi/4} \cos^3 2x \cdot \sin^4 4x \, dx$
5. $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta$
6. $\int_0^{\pi/8} \sin^4 8\theta \cos^6 4\theta \, d\theta$
7. $\int_0^{\infty} \left(\frac{t}{1 + t^2}\right)^3 dt$
8. $\int_0^{\infty} \left(\frac{x}{1 + x^2}\right)^6 dx$ (M.U. 1994)
9. Prove that $\int_{-\pi/6}^{\pi/3} (\sqrt{3} \sin\theta + \cos\theta)^{1/4} \, d\theta = 2^{-3/4} B\left(\frac{1}{2}, \frac{5}{8}\right)$

10. Prove that

11. Express th

is $2^{-9/10} B(3/5, 1/2$

[Ans. : (1) π, (2)

(7) $\frac{1}{4}$ p

Type V

Prove that

Proof : Pu

$I =$

$\therefore I$

Note ....

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Solved Exa

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10. Prove that $\int_{-\pi/3}^{\pi/6} (\sin\theta + \sqrt{3}\cos\theta)^{1/6} d\theta = 2^{-5/6} B\left(\frac{1}{2}, \frac{7}{12}\right)$

11. Express the integral $\int_{-\pi/4}^{\pi/4} (\sin\theta + \cos\theta)^{1/5} d\theta$ as a Gamma function and show that its value is $2^{-9/10} B(3/5, 1/2)$.

[**Ans. :** **(1)** π, **(2)** $2^{2t-2} \cdot \frac{(\overline{|t})^2}{\overline{|2t}}$, **(3)** $\frac{21\pi}{16}$, **(4)** $4 \cdot B\left(4, \frac{5}{2}\right)$, **(5)** $\frac{\pi}{\sqrt{2}}$, **(6)** $2B\left(\frac{5}{2}, \frac{11}{2}\right)$,

(7) $\frac{1}{4}$ put $t = \tan\theta$, **(8)** $\frac{3\pi}{512}$.]

Type V

Prove that $\boxed{\int_0^\infty \frac{x^m}{(1+x^2)^n} dx = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{2n-m-1}{2}\right)}$ .... (28)

Proof : Putting $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, we get

$$I = \int_0^{\pi/2} \frac{\tan^m\theta}{(\sec^2\theta)^n} \cdot \sec^2\theta\, d\theta = \int_0^{\pi/2} \frac{\sin^m\theta}{\cos^m\theta} \cdot (\cos\theta)^{2n-2}\, d\theta$$

$$= \int_0^{\pi/2} \sin^m\theta \cos^{2n-m-2} d\theta$$

$$\therefore \quad I = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{2n-m-1}{2}\right)$$ [By (5), page 6-17]

Note ....

When the limits of integration are 0 to ∞ and the integrand involves the term $1 + x^2$ or $1 + x$ as **in the above type and the next type**, we put either $x = \tan\theta$ or $x = \tan^2\theta$.

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Evaluate $\int_0^\infty \frac{dx}{(1+x^2)^{7/2}}$.

Sol. : Putting $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, we get

$$I = \int_0^{\pi/2} \frac{1}{(\sec^2\theta)^{7/2}} \cdot \sec^2\theta\, d\theta = \int_0^{\pi/2} \frac{1}{\sec^7\theta} \cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \cos^5\theta\, d\theta = \frac{4 \cdot 2}{5 \cdot 3 \cdot 1} = \frac{8}{15}.$$ [By (24), page 6-32]

Example 2 (a) : Evaluate $\int_0^\infty \left(\frac{x}{1+x^2}\right)^4 dx$.

Sol. : Putting $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, we get

$$I = \int_0^{\pi/2} \left(\frac{\tan\theta}{\sec^2\theta}\right)^4 \cdot \sec^2\theta\, d\theta = \int_0^{\pi/2} \frac{\sin^4\theta}{\cos^4\theta} \cdot \frac{\cos^8\theta}{1} \cdot \frac{1}{\cos^2\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta = \frac{3 \cdot 1 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi}{32}.$$ [By (25), page 6-34]

Example 3 (a) : Evaluate $\int_0^{\infty} \frac{dx}{1+x^4}$.

Sol. : We put $x^2 = \tan\theta$, $x = \sqrt{\tan\theta}$, $dx = \frac{1}{2\sqrt{\tan\theta}} \cdot \sec^2\theta \, d\theta$.

$$\therefore \quad I = \int_0^{\pi/2} \frac{1}{1+\tan^2\theta} \cdot \frac{1}{2\sqrt{\tan\theta}} \cdot \sec^2\theta \; d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2} d\theta = \frac{1}{2} \cdot \frac{1}{2} \cdot B\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{4} \cdot \frac{\overline{|1/4} \; \overline{|3/4}}{\overline{|1}} = \frac{1}{4} \cdot \sqrt{2} \cdot \pi = \frac{\pi}{2\sqrt{2}}. \qquad \text{[By (13), page 6-19]}$$

EXERCISE - XI

Evaluate the following integrals : Class (a) : 3 Marks

1. $\int_0^{\infty} \frac{dx}{(1+x^2)^{5/2}}$ 2. $\int_0^{\infty} \left(\frac{x}{1+x^2}\right)^3 dx$ 3. $\int_0^{\infty} \left(\frac{x}{1+x^2}\right)^6 dx$

4. $\int_0^{\infty} \frac{x^2}{(1+x^6)^{5/2}} dx$ 5. $\int_0^{\infty} \frac{x^3}{(1+x^8)^4} dx$

[**Ans. : (1)** $\frac{2}{3}$, **(2)** $\frac{1}{4}$, **(3)** $\frac{3\pi}{512}$, **(4)** Put $x^3 = \tan\theta$, $\frac{2}{9}$, **(5)** Put $x^4 = \tan\theta$, $\frac{5\pi}{128}$.]

Type VI

Prove that $\boxed{\int_0^{\infty} \frac{x^m}{(1+x)^n} = B(m+1,\ n-m-1)}$ .... (29)

Proof : Putting $x = \tan^2\theta$, $dx = 2\tan\theta \cdot \sec^2\theta \, d\theta$, we get

$$I = \int_0^{\pi/2} \frac{\tan^{2m}\theta}{\sec^{2n}\theta} \cdot 2\tan\theta \cdot \sec^2\theta \; d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2m+1}\theta}{\cos^{2m+1}\theta} \cdot \frac{\cos^{2n-2}\theta}{1} \; d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m+1}\theta \cdot \cos^{2n-2m-3}\theta \; d\theta$$

$$= 2 \cdot \frac{1}{2} \; B(m+1,\ n-m-1) \qquad \text{[By (5), page 6-17]}$$

$$= B(m+1,\ n-m-1)$$

Aliter : Putting $x = \frac{t}{1-t}$, we get $1+x = 1+\frac{t}{1-t} = \frac{1}{1-t}$. $\quad \therefore \; dx = \frac{1}{(1-t)^2} dt$

When $x = 0$, $t = 0$; when $x = \infty$, $t = 1$.

$$\therefore \quad I = \int_0^1 \left(\frac{t}{1-t}\right)^m \cdot (1-t)^n \cdot \frac{dt}{(1-t)^2}$$

$$= \int_0^1 t^m (1-t)^{n-m-2} \, dt = B(m+1,\ n-m-1)$$

Note ....

The above result can be considered as another **definition of Beta Function and can be used as a formula.**

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Evaluate $\int_0^\infty \frac{x^5(1+x^4)}{(1+x)^{16}}\,dx$.

Sol. : We have $I = \int_0^\infty \frac{x^5}{(1+x)^{16}}\,dx + \int_0^\infty \frac{x^9}{(1+x)^{16}}\,dx$

$\therefore\ I = B(5+1,\ 16-5-1) + B(9+1,\ 16-9-1)$ [By (29)]

$= B(6, 10) + B(10, 6) = 2B(6, 10)$

Example 2 (a) : Evaluate $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}$.

Sol. : We have $I = \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$

$\therefore\ I = B(8+1,\ 24-8-1) - B(14+1,\ 24-14-1)$ [By (29)]

$= B(9, 15) - B(15, 9) = 0$

Example 3 (a) : Evaluate $\int_0^\infty \frac{x^{10}-x^{18}}{(1+x)^{30}}\,dx$.

Sol. : We have $I = \int_0^\infty \frac{x^{10}}{(1+x)^{30}}\,dx - \int_0^\infty \frac{x^{18}}{(1+x)^{30}}\,dx$

$\therefore\ I = B(10+1,\ 30-10-1) - B(18+1,\ 30-18-1)$ [By (29)]

$= B(11, 19) - B(19, 11) = 0$

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Evaluate $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}$. **(M.U. 2015)**

Sol. : We write $a + bx = a\left(1 + \frac{b}{a}x\right)$ and comparing with the above result (29), we put $\frac{b}{a}x = \tan^2\theta$.

$\therefore\ x = \frac{a}{b}\tan^2\theta \qquad \therefore\ dx = \frac{a}{b}\cdot 2\tan\theta\cdot\sec^2\theta\ d\theta$

When $x = 0,\ \theta = 0$; when $x = \infty,\ \theta = \frac{\pi}{2}$.

$$\therefore\ I = \int_0^{\pi/2} \frac{[(a/b)\tan^2\theta]^{m-1}}{(a\sec^2\theta)^{m+n}}\cdot\frac{2a}{b}\cdot\tan\theta\cdot\sec^2\theta\ d\theta$$

$$= \frac{2}{a^n b^m}\int_0^{\pi/2} \frac{\sin^{2m-2}\theta}{\cos\theta^{2m-2}}\cdot\frac{(\cos^2\theta)^{m+n-1}}{1}\cdot\frac{\sin\theta}{\cos\theta}\ d\theta$$

$$\therefore \quad I = \frac{2}{a^n b^m} \int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$$

$$= \frac{2}{a^n b^m} \cdot \frac{1}{2} B(m, n) = \frac{1}{a^n b^m} B(m, n) \qquad \text{[By (5), page 6-17]}$$

Example 2 (c) : Evaluate $\int_0^\infty \frac{\sqrt{x}}{(1+2x+x^2)} dx$.

Sol. : We have $I = \int_0^\infty \frac{x^{1/2}}{(1+x)^2} dx$

Putting $x = \tan^2\theta$, $dx = 2\tan\theta \cdot \sec^2\theta \, d\theta$.

When $x = 0$, $\theta = 0$; when $x = \infty$, $\theta = \frac{\pi}{2}$, we get

$$I = \int_0^{\pi/2} \frac{\tan\theta}{\sec^4\theta} \cdot 2\tan\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^2\theta \, d\theta = 2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{2}. \qquad \text{[By (20), page 6-28]}$$

Example 3 (c) : Evaluate $\int_0^\infty \frac{\sqrt{x}}{(a^2+2ax+x^2)} dx$.

Sol. : We have $I = \int_0^\infty \frac{\sqrt{x}}{(a+x)^2} dx$

Putting $x = a\tan\theta$, and proceeding as above, we get

$$I = \frac{\pi}{2\sqrt{a}}.$$

Example 4 (c) : Prove that $\int_0^\infty \frac{x^5}{(2+3x)^{15}} dx = \frac{1}{2^9 \cdot 3^6} B(9, 6)$.

Sol. : As in Ex. 1 above, we first write $2 + 3x = 2\left(1 + \frac{3}{2}x\right)$ and then put $\frac{3}{2}x = \tan^2\theta$

i.e. $\quad x = \frac{2}{3}\tan^2\theta \quad \therefore \quad dx = \frac{4}{3}\tan\theta \sec^2\theta \, d\theta$

$$\therefore \quad I = \int_0^{\pi/2} \frac{1}{2^{15}(\sec\theta)^{30}} \cdot \left(\frac{2}{3}\right)^5 \cdot \tan^{10}\theta \cdot \frac{4}{3}\tan\theta \cdot \sec^2\theta \, d\theta$$

$$= \frac{2^7}{2^{15} \cdot 3^6} \int_0^{\pi/2} \frac{\tan^{11}\theta}{\sec^{28}\theta} \, d\theta$$

$$= \frac{1}{2^8 \cdot 3^6} \int_0^{\pi/2} \sin^{11}\theta \cos^{17}\theta \, d\theta$$

$$= \frac{1}{2^8 \cdot 3^6} \cdot \frac{1}{2} B(6, 9). \qquad \text{[By (5), page 6-17]}$$

EXERCISE - XII

Prove that : Class (a) : 3 Marks

1. $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(m,n)$ (**Note : Memorise this result**)

2. $\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = B(n,m)$ (**Hint :** Interchange n and m.)

3. $\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} = 2B(m,n)$

4. $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = 0$

5. $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx = \frac{1}{5005}$

6. $\int_0^\infty \frac{x^6(1-x^{10})}{(1+x)^{24}}\,dx = 0$

7. $\int_0^\infty \frac{x^3(1+x^8)}{(1+x)^{13}}\,dx = \frac{83}{990}$

Prove that : Class (c) : 8 Marks

8. $\int_0^\infty \frac{x^6-x^3}{(1+x^3)^5}\,x^2\,dx = 0$ (**Hint :** Put $x^3 = t$, and separate the integrals.)

Type VII

Prove that $$\boxed{\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{B(m,n)}{(a+b)^m\cdot a^n}} \quad \text{.................. (30)}$$

Proof : We put $\frac{x}{a+bx} = \frac{t}{a+b}$ [**Note this**]

When $x = 0$, $t = 0$; when $x = 1$, $t = 1$.

$\therefore\ (a+b)\,x = at + bx\,t \quad \therefore\ (a+b-b\,t)\,x = at$ .... (A)

$$\therefore\ x = \frac{at}{a+b-bt}$$

$$\therefore\ 1-x = 1-\frac{a\,t}{a+b-bt} = \frac{(a+b)-(a+b)\,t}{a+b-b\,t} = \frac{(a+b)(1-t)}{a+b-bt}$$

and $$a+bx = a+\frac{abt}{a+b-bt} = \frac{a(a+b)}{a+b-bt}$$

Differentiating this, w.r.t. x, we get

$$b\,dx = -\frac{a(a+b)}{(a+b-b\,t)^2}\cdot(-b)\cdot dt \qquad \therefore\ dx = \frac{a(a+b)}{(a+b-b\,t)^2}\cdot dt$$

$$\therefore\ I = \int_0^1 \left(\frac{a\,t}{a+b-bt}\right)^{m-1}\cdot\frac{(a+b)^{n-1}(1-t)^{n-1}}{(a+b-b\,t)^{n-1}}\cdot\frac{(a+b-b\,t)^{m+n}}{a^{m+n}\cdot(a+b)^{m+n}}\cdot\frac{a(a+b)}{(a+b-b\,t)^2}\cdot dt$$

$$= \int_0^1 \frac{1}{(a+b)^m\cdot a^n}\cdot t^{m-1}\cdot(1-t)^{n-1}\,dt$$

$$= \frac{1}{(a+b)^m\cdot a^n}\cdot B(m,n).$$

Note ....

When the limits of integration are 0 and 1 and the integrand contains the denominator of the form $a + bx$, we put $\dfrac{x}{a+bx} = \dfrac{t}{a+b}$ as in the above type or the next type.

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Prove that $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(1+x)^{m+n}} = \frac{B(m,n)}{2^m}$

and hence evaluate $\displaystyle\int_0^1 \frac{x^3 - 2x^4 + x^5}{(1+x)^7}\,dx.$

Sol. : Putting $a = 1$, $b = 1$ in the above result (31) or putting $\dfrac{x}{1+x} = \dfrac{t}{2}$ and proceeding as above, we get

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(1+x)^{m+n}} = \frac{B(m,n)}{2^m}.$$

Now, $\displaystyle\int_0^1 \frac{x^3 - 2x^4 + x^5}{(1+x)^7}\,dx = \int_0^1 \frac{x^3(1-2x+x^2)}{(1+x)^7}\,dx = \int_0^1 \frac{x^3(1-x)^2}{(1+x)^7}\,dx.$

Putting $m = 4$, $n = 3$,

$$I = \frac{B(4,3)}{2^4} = \frac{\overline{|4}\ \overline{|3}}{16\ \overline{|7}} = \frac{1}{16}\cdot\frac{3!\ 2!}{6!} = \frac{1}{960}.$$

Example 2 (c) : Evaluate $\displaystyle\int_0^1 \frac{x - 2x^2 + x^3}{(1+x)^5}\,dx.$

Sol. : We have

$$I = \int_0^1 \frac{x(1-2x+x^2)}{(1+x)^5}\,dx = \int_0^1 \frac{x(1-x)^2}{(1+x)^5}\,dx$$

$$= \int_0^1 \frac{x^{2-1}(1-x)^{3-1}}{(1+x)^{2+3}}\,dx = \frac{B(2,3)}{2^2} = \frac{1}{48} \text{ by the above result.}$$

Or put $\dfrac{x}{1+x} = \dfrac{t}{2}$ and proceed.

Example 3 (c) : Evaluate $\displaystyle\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx.$

Sol. : Put $x^4 = t$, $x = t^{1/4}$, $dx = \dfrac{1}{4}t^{-3/4}\,dt$, we get

$$I = \int_0^1 \frac{(1-t)^{3/4}}{(1+t)^2}\cdot t^{-3/4}\cdot\frac{dt}{4} = \frac{1}{4}\int_0^1 \frac{t^{-3/4}(1-t)^{3/4}}{(1+t)^2}\,dt$$

Comparing this with the above form, we put $\dfrac{t}{1+t} = \dfrac{u}{2}$.

When $t = 0$, $u = 0$; when $t = 1$, $u = 1$ $\quad\therefore\ 2t = u + ut.$

$\therefore\ t(2-u) = u \qquad \therefore\ t = \dfrac{u}{2-u}$

$$\therefore \quad 1-t=1-\frac{u}{2-u}=\frac{2(1-u)}{2-u}, \qquad 1+t=1+\frac{u}{2-u}=\frac{2}{2-u}$$

$$\therefore \quad dt=\frac{2}{(2-u)^2}\,du$$

$$\therefore \quad I=\frac{1}{4}\int_0^1 \frac{u^{-3/4}}{(2-u)^{-3/4}}\cdot\frac{2^{3/4}(1-u)^{3/4}}{(2-u)^{3/4}}\cdot\frac{(2-u)^2}{2^2}\cdot\frac{2\,du}{(2-u)^2}$$

$$=\frac{2^{3/4}}{8}\int_0^1 u^{-3/4}(1-u)^{3/4}\,du=\frac{2^{3/4}}{8}\,B\left(\frac{1}{4},\frac{7}{4}\right)$$

$$=\frac{2^{3/4}}{8}\cdot\frac{\Gamma(1/4)\,\Gamma(7/4)}{\Gamma(2)}=\frac{2^{3/4}}{8\cdot 1}\cdot\Gamma\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right)$$

$$=\frac{3}{32}\cdot 2^{5/4}\,\pi \qquad \left[\because \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\sqrt{2}\cdot\pi\right] \qquad \text{[By (13), page 6-19]}$$

Example 4 (c) : Prove that $\int_0^1 \frac{x^{-1/3}(1-x)^{-2/3}}{(1+2x)}\,dx=\frac{1}{\sqrt[3]{9}}\,B\left(\frac{2}{3},\frac{1}{3}\right)$. **(M.U. 1999)**

Sol. : Comparing with the Type VII (page 6-45), we see that $a=1$, $b=2$.

Hence from (A), page 6-45, we put $x=\frac{t}{3-2t}$.

$$\therefore \quad dx=\frac{(3-2t)-t(-2)}{(3-2t)^2}\,dt=\frac{3}{(3-2t)^2}\,dt$$

When $x=0$, $t=0$; when $x=1$, $t=1$.

Further, $1-x=1-\frac{t}{3-2t}=\frac{3(1-t)}{3-2t}$; $\qquad 1+2x=1+\frac{2t}{3-2t}=\frac{3}{3-2t}$

$$\therefore \quad I=\int_0^1 \frac{t^{-1/3}}{(3-2t)^{-1/3}}\cdot\frac{3^{-2/3}(1-t)^{-2/3}}{(3-2t)^{-2/3}}\cdot\frac{(3-2t)}{3}\cdot\frac{3\,dt}{(3-2t)^2}$$

$$=\int_0^1 \frac{1}{3^{2/3}}\cdot t^{-1/3}(1-t)^{-2/3}\,dt=\frac{1}{\sqrt[3]{9}}\cdot B\left(\frac{2}{3},\frac{1}{3}\right).$$

Type VIII

Prove that $\boxed{\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx=B(m,n)}$ .... (31)

Prof : Let $I_1=\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$, $I_2=\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

In I_1, put $x=\frac{t}{1-t}$ $\quad\therefore\ 1+x=\frac{1}{1-t}$, $dx=\frac{1}{(1-t)^2}\,dt$

When $x=0$, $t=0$; when $x=1$, $t=1/2$.

$$\therefore \quad I_1=\int_0^{1/2}\left(\frac{t}{1-t}\right)^{m-1}\cdot(1-t)^{m+n}\cdot\frac{dt}{(1-t)^2}=\int_0^{1/2} t^{m-1}\cdot(1-t)^{n-1}\,dt$$

Similarly, $I_2=\int_0^{1/2} t^{n-1}\cdot(1-t)^{m-1}\,dt$

Now, put $t = 1 - z$ in I_2

$$\therefore \quad I_2 = \int_1^{1/2} (1-z)^{n-1} z^{m-1} (-dz) = \int_{1/2}^{1} t^{m-1} (1-t)^{n-1}\, dt$$

$$\therefore \quad I = I_1 + I_2$$

$$= \int_0^{1/2} t^{m-1} \cdot (1-t)^{n-1}\, dt + \int_{1/2}^{1} t^{m-1} \cdot (1-t)^{n-1}\, dt$$

$$= \int_0^1 t^{m-1} \cdot (1-t)^{n-1}\, dt = B(m, n)$$

Aliter : We have by (7), page 6-17

$$B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

In I_2, put $x = \frac{1}{t}$, $dx = -\frac{1}{t^2}\, dt$

When $x = 1$, $t = 1$; when $x = \infty$, $t = 0$.

$$\therefore \quad I_2 = \int_1^0 \frac{(1/t)^{m-1}}{[1+(1/t)]^{m+n}} \cdot \left(-\frac{1}{t^2}\right) dt = \int_0^1 \frac{1}{t^{m-1}} \cdot \frac{t^{m+n}}{(1+t)^{m+n}} \cdot \frac{1}{t^2}\, dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\, dt$$

$$\therefore \quad I_2 = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx \qquad \text{[Changing } t \text{ to } x \text{]}$$

$$\therefore \quad B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} \cdot dx$$

Solved Example : Class (c) : 8 Marks

Example 1 : Prove that $\int_0^1 \frac{x^2 + x^3}{(1+x)^7}\, dx = B(3, 4)$.

Sol. : Putting $x = \frac{t}{1-t}$, proceed as above.

Solved Example : Class (a) : 3 Marks

Example : Evaluate $\int_7^{11} \sqrt[4]{(x-7)(11-x)}\, dx$.

Sol. : Comparing this with the above type, we see that $a = 7$, $b = 11$ and $m = n = 1/4$. Hence, we put

$$(x-7) = (11-7)\,t \qquad i.e., \qquad x - 7 = 4t \qquad \therefore\ dx = 4\, dt$$

$$\therefore\ 11 - x = 11 - (7 - 4t) = 4(1-t)$$

When $x = 7$, $t = 0$; when $x = 11$, $t = 1$.

$$\therefore\ I = \int_0^1 \sqrt[4]{4t \cdot 4(1-t)} \cdot 4\, dt = 4 \cdot \sqrt[4]{16} \int_0^1 t^{1/4}(1-t)^{1/4}\, dt$$

$$= 8 \int_0^1 t^{1/4}(1-t)^{1/4}\, dt = 8 \cdot B\left(\frac{5}{4}, \frac{5}{4}\right)$$

EXERCISE - XIII

Prove that : Class (a) : 3 Marks

1. $\int_{-1}^{1} (1+x)^m (1-x)^n\, dx = 2^{m+n+1} B(m+1, n+1)$ **(M.U. 1990, 97, 98, 2002, 05)**

   Hence, evaluate $\int_{-1}^{1} \sqrt{\left(\frac{1+x}{1-x}\right)}\, dx$. **[Ans. :** $2B\left(\frac{3}{2}, \frac{1}{2}\right)$ **]**

2. $\int_a^b \sqrt[n]{(x-a)(b-x)} = (b-a)^{(2/n)+1} \cdot B\left(\frac{1}{n}+1, \frac{1}{n}+1\right)$

3. $\int_3^7 \sqrt[4]{(x-3)(7-x)} \cdot dx = \frac{2\left(\Gamma(1/4)\right)^2}{3\sqrt{\pi}}$ **(M.U. 1997, 2000)**

4. $\int_5^9 \sqrt[4]{(9-x)(x-5)} \cdot dx = \frac{2\left(\Gamma(1/4)\right)^2}{3\sqrt{\pi}}$

5. $\int_5^6 (x-5)^5 (6-x)^6\, dx = B(6, 7)$

Type X

$$\boxed{B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}} \qquad \text{.....................} \quad (32)$$

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Prove that $y\, B(x+1, y) = x\, B(x, y+1)$.

Sol. : We have

$$yB(x+1, y) = y\, \frac{\Gamma(x+1) \cdot \Gamma(y)}{\Gamma(x+y+1)} = \frac{yx\,\Gamma(x)\,\Gamma(y)}{\Gamma(x+y+1)}$$

$$= \frac{x \cdot \Gamma(x)\,\Gamma(y+1)}{\Gamma(x+y+1)} = x\, B(x, y+1)$$

Example 2 (a) : Prove that $B(m, n) = B(m, n+1) + B(m+1, n)$.

Sol. : $B(m, n+1) + B(m+1, n)$

$$= \frac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)} + \frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)} = \frac{\Gamma(m)\cdot n\Gamma(n) + m\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)}$$

$$= \frac{(m+n)\,\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = B(m, n)$$

Example 3 (a) : Prove that $B(m, m)\cdot B\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\pi}{m}\cdot 2^{1-4m}$. **(M.U. 2006, 08, 11, 12)**

Sol. : $$B(m, m)\cdot B\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)}\cdot\frac{\Gamma(m+(1/2))\cdot\Gamma(m+(1/2))}{\Gamma(2m+1)}$$

$$= \left[\frac{\Gamma(m)\,\Gamma(m+(1/2))}{\Gamma(2m)}\right]^2\cdot\frac{1}{2m} \qquad \left(\because \Gamma(2m+1) = 2m\Gamma(2m)\right)$$

$$= \frac{\pi}{2^{4m-2}}\cdot\frac{1}{2m} = \frac{\pi}{2^{4m-1}}\cdot\frac{1}{m} = \frac{\pi}{m}\cdot 2^{1-4m}.$$

[By duplication formula (10), page 6-18]

Example 4 (a) : Prove that $B(m+1, n) = \frac{m}{m+n}B(m, n)$.

Sol. : $$B(m+1, n) = \frac{\Gamma(m+1)\cdot\Gamma(n)}{\Gamma(m+n+1)} = \frac{m\Gamma(m)\,\Gamma(n)}{(m+n)\Gamma(m+n)} = \frac{m}{(m+n)}\cdot B(m, n)$$

Similarly, $B(m, n+1) = \frac{n}{n+m}B(m, n)$

Example 5 (a) : Prove that $B(x, x) = \frac{1}{2^{2x-1}}B\left(x, \frac{1}{2}\right)$. **(M.U. 1996, 97, 2002)**

Sol. : $B(x, x) = \frac{\Gamma(x)\,\Gamma(x)}{\Gamma(2x)}$

But duplication formula gives $\Gamma(m)\,\Gamma(m+(1/2)) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$ [By (10), page 6-18]

$$\therefore\ \frac{\Gamma(m)}{\Gamma(2m)} = \frac{\sqrt{\pi}}{2^{2m-1}\Gamma(m+(1/2))}$$

$$\therefore\ B(x, x) = \frac{1}{2^{2x-1}}\cdot\frac{\sqrt{\pi}}{\Gamma(x+(1/2))}\Gamma(x) = \frac{1}{2^{2x-1}}\cdot\frac{\Gamma(x)\,\Gamma(1/2)}{\Gamma(x+(1/2))}$$

$$= \frac{1}{2^{2x-1}}B\left(x, \frac{1}{2}\right)$$

Example 6 (a) : Show that $B(n, n+1) = \frac{1}{2}\cdot\frac{(\Gamma(n))^2}{\Gamma(2n)}$. Hence, deduce that

$$\int_0^{\pi/2}\left(\frac{1}{\sin^3\theta} - \frac{1}{\sin^2\theta}\right)^{1/4}\cdot\cos\theta\, d\theta = \frac{(\Gamma(1/4))^2}{2\sqrt{\pi}}.$$

Sol. : We have $B(n, n+1) = \dfrac{\Gamma(n)\,\Gamma(n+1)}{\Gamma(2n+1)} = \dfrac{\Gamma(n) \cdot n\,\Gamma(n)}{2n\,\Gamma(2n)}$

$$\therefore \quad B(n, n+1) = \frac{1}{2} \cdot \frac{(\Gamma(n))^2}{\Gamma(2n)} \qquad \text{....... (2)}$$

Now, $\displaystyle\int_0^{\pi/2} \left(\frac{1}{\sin^3\theta} - \frac{1}{\sin^2\theta}\right)^{1/4} \cos\theta \, d\theta = \int_0^{\pi/2} \frac{(1-\sin\theta)^{1/4}}{\sin^{3/4}\theta} \cdot \cos\theta \, d\theta$

Put $\sin\theta = t \quad \therefore \cos\theta\, d\theta = dt$

When $\theta = 0,\ t = 0$; when $\theta = \pi/2,\ t = 1$.

$$\therefore \quad I = \int_0^1 (1-t)^{1/4} \cdot t^{-3/4}\, dt = B\left(\frac{5}{4}, \frac{1}{4}\right) = \frac{1}{2} \cdot \frac{(\Gamma(1/4))^2}{\sqrt{\pi}} \qquad \left[\text{Putting } n = \frac{1}{4} \text{ in (2)}\right]$$

Miscellaneous Examples

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Prove that $B\left(\dfrac{1}{3}, \dfrac{2}{3}\right) = \dfrac{2\pi}{\sqrt{3}}$.

Sol. : $B\left(\dfrac{1}{3}, \dfrac{2}{3}\right) = \dfrac{\Gamma(1/3)\,\Gamma(2/3)}{\Gamma(1)} = \dfrac{2\pi}{\sqrt{3}}$. [By (4) page 6-53]

Example 2 (a) : State true or false with proper justification.

(i) if $m < n$, $\Gamma(m) < \Gamma(n)$.

(ii) $\Gamma\left(\dfrac{1}{6}\right)\Gamma\left(\dfrac{2}{6}\right)\Gamma\left(\dfrac{3}{6}\right)\Gamma\left(\dfrac{4}{6}\right)\Gamma\left(\dfrac{5}{6}\right) = 4\pi^2\sqrt{\dfrac{\pi}{3}}$. (M.U. 1998)

Sol. : (i) Let $m = 1/2,\ n = 1$ then $m < n$.

But $\Gamma(m) = \Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$.

Since $\sqrt{\pi} \nless 1$, the statement is false.

(ii) $$\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{6}\right)\Gamma\left(\frac{4}{6}\right)\Gamma\left(\frac{3}{6}\right) = \left(\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)\right)\left(\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)\Gamma\left(\frac{1}{2}\right)$$

$$= 2\pi \cdot \frac{2\pi}{\sqrt{3}} \cdot \sqrt{\pi} = 4\pi^2 \cdot \sqrt{\frac{\pi}{3}} \qquad \text{[By (1), (2), (3), page 6-53]}$$

$\therefore$ The statement is true.

(M.U. 2002)

Example 3 (a) : If $B(n, 3) = \dfrac{1}{105}$ and n is a positive integer, find n.

Sol. : $B(n, 3) = \dfrac{\Gamma(n)\,\Gamma(3)}{\Gamma(n+3)} = \dfrac{\Gamma(n)\,\Gamma(3)}{(n+2)(n+1)\,n\,\Gamma(n)} \qquad \left[\because \Gamma(n+1) = n\,\Gamma(n)\right]$

$= \dfrac{2!}{(n+2)(n+1)\,n} \qquad \left[\because \Gamma(n) = (n-1)!\right]$

By data this is equal to 1 / 105.

$$\therefore \quad \frac{2}{(n+2)(n+1)n} = \frac{1}{105}$$

$$\therefore \quad (n+2)(n+1)n = 210 = 7 \cdot 6 \cdot 5 \qquad \therefore \quad n = 5.$$

(Or $\quad n^3 + 3n^2 + 2n - 210 = 0$

$\therefore \quad (n-5)(n^2 + 8n - 42) = 0 \qquad \therefore \quad n = 5$ since n is an integer.

Example 4 (a) : Given $\int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}$, prove that $\overline{|\,p}\ \overline{|\,1-p} = \frac{\pi}{\sin p\pi}$ $(0 < p < 1)$.

(M.U. 1999, 2002)

Sol. : Putting $x = \tan^2 \theta$, we get,

$$I = \int_0^{\pi/2} \frac{\tan^{2p-2}\theta \cdot 2\tan\theta \sec^2\theta \, d\theta}{1 + \tan^2\theta}$$

$$= 2\int_0^{\pi/2} \tan^{2p-1}\theta \, d\theta = 2\int_0^{\pi/2} \sin^{2p-1}\theta \cdot \cos^{1-2p}\theta \, d\theta$$

$$= 2 \cdot \frac{1}{2} \cdot \frac{\overline{\left|\frac{2p-1+1}{2}\right.}\ \overline{\left|\frac{1-2p+1}{2}\right.}}{\overline{\left|\frac{2p-1+1-2p+2}{2}\right.}} = \frac{\overline{|\,p}\ \overline{|\,1-p}}{\overline{|\,1}} = \overline{|\,p}\ \overline{|\,1-p} \quad \text{[By (18), page 6-28]}$$

But $\quad I = \frac{\pi}{\sin p\pi} \qquad \therefore \quad \overline{|\,p}\ \overline{|\,1-p} = \frac{\pi}{\sin p\pi}$

Example 5 (a) : Prove that $\overline{\left|\frac{3}{2} - x\right.}\ \overline{\left|\frac{3}{2} + x\right.} = \left(\frac{1}{4} - x^2\right)\pi \sec \pi x$.

Sol. : Since $\overline{|\,n} = (n-1)\,\overline{|\,n-1}$, we have

$$\text{l.h.s.} = \left(\frac{1}{2} - x\right)\overline{\left|\frac{1}{2} - x\right.} \cdot \left(\frac{1}{2} + x\right)\overline{\left|\frac{1}{2} + x\right.}$$

$$= \left(\frac{1}{4} - x^2\right)\overline{\left|\frac{1}{2} - x\right.}\ \overline{\left|\frac{1}{2} + x\right.} \qquad \text{.......... (A)}$$

But $\quad \overline{|\,p}\ \overline{|\,1-p} = \frac{\pi}{\sin p\pi}$. Putting $p = \frac{1}{2} + x$, we get

$$\overline{\left|\frac{1}{2} + x\right.}\ \overline{\left|\frac{1}{2} - x\right.} = \frac{\pi}{\sin[(1/2) + x]\pi}$$

But $\quad \sin\left(\frac{1}{2} + x\right)\pi = \sin\left(\frac{\pi}{2} + \pi x\right) = \cos \pi x$

Hence, from (1), we get,

$$\text{l.h.s.} = \left(\frac{1}{4} - x^2\right) \cdot \frac{\pi}{\cos \pi x} = \left(\frac{1}{4} - x^2\right) \cdot \pi \sec \pi x.$$

Example 6 (a) : Given $\int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}$, prove that $\int_0^{\infty} \frac{dy}{1+y^4} = \frac{\pi}{2\sqrt{2}}$.

Sol. : Put $y^4 = x, \quad \therefore \quad y = x^{1/4} \quad \therefore \quad dy = \frac{1}{4} x^{-3/4}\, dx$

$$\therefore \int_0^\infty \frac{dy}{1+y^4} = \int_0^\infty \frac{1}{4} \cdot \frac{x^{-3/4}}{1+x}\,dx = \frac{1}{4} \cdot \int_0^\infty \frac{x^{(1/4)-1}}{1+x}\,dx$$

$$\therefore \int_0^\infty \frac{dy}{1+y^4} = \frac{1}{4} \cdot \frac{\pi}{\sin(\pi/4)} = \frac{\sqrt{2}\,\pi}{4} = \frac{\pi}{2\sqrt{2}}. \qquad [\,p = \frac{1}{4}\,]$$

Note .....

The result $\Gamma p \; \Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$ is highly useful in certain problems.

Particularly note that

1. When $p = \frac{1}{2}$, $\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \dfrac{\pi}{\sin(\pi/2)} = \pi \quad \therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ .... (1)

2. When $p = \frac{1}{6}$, $\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right) = \dfrac{\pi}{\sin(\pi/6)} = \dfrac{\pi}{1/2} = 2\pi$ .... (2)

3. When $p = \frac{1}{4}$, $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \dfrac{\pi}{\sin(\pi/4)} = \dfrac{\pi}{1/\sqrt{2}} = \sqrt{2}\,\pi$ .... (3)

4. When $p = \frac{1}{3}$, $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \dfrac{\pi}{\sin(\pi/3)} = \dfrac{\pi}{\sqrt{3}/2} = \dfrac{2\pi}{\sqrt{3}}$ .... (4)

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Prove that $\displaystyle\int_0^1 \sqrt{1-x^4}\,dx = \frac{(\Gamma(1/4))^2}{6\sqrt{2\pi}}$. (M.U. 2001)

Sol. : Put $x^4 = t$, $x = t^{1/4}$ $\quad \therefore dx = \frac{1}{4} t^{-3/4}\,dt$

$$\therefore I = \int_0^1 (1-t)^{1/2} \cdot \frac{1}{4} \cdot t^{-3/4}\,dt = \frac{1}{4}\int_0^1 t^{-3/4}(1-t)^{1/2}\,dt$$

$$= \frac{1}{4} B\left(\frac{1}{4}, \frac{3}{2}\right) = \frac{1}{4} \cdot \frac{\Gamma(1/4)\,\Gamma(3/2)}{\Gamma(7/4)} = \frac{1}{4} \cdot \frac{\Gamma(1/4)\,(1/2)\,\Gamma(1/2)}{(3/4)\,\Gamma(3/4)}$$

But $\Gamma(1/4)\,\Gamma(3/4) = \sqrt{2} \cdot \pi \quad \therefore \Gamma(3/4) = \dfrac{\sqrt{2}\,\pi}{\Gamma(1/4)}$ [By (3) above]

$$\therefore I = \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{\Gamma(1/4)}{\sqrt{2}\,\pi} \cdot \Gamma(1/4) \cdot \sqrt{\pi} = \frac{1}{6} \cdot \frac{1}{\sqrt{2\pi}} \cdot (\Gamma(1/4))^2$$

Example 2 (c) : Prove that $\displaystyle\int_0^\infty \frac{x}{(1+x^4)^{5/4}}\,dx \cdot \int_0^\infty \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{2\sqrt{2}}$ (M.U. 1995, 2006)

Sol. : Put $x^4 = t \quad \therefore x = t^{1/4} \quad \therefore dx = \frac{1}{4} t^{-3/4}\,dt$.

When $x = 0, t = 0$; when $x = \infty, t = \infty$.

$$\therefore I_1 = \int_0^\infty \frac{1}{(1+t)^{5/4}} \cdot t^{1/4} \cdot \frac{1}{4} \cdot t^{-3/4}\,dt = \frac{1}{4}\int_0^\infty \frac{t^{-1/2}}{(1+t)^{5/4}}\,dt$$

$$= \frac{1}{4}\int_0^\infty \frac{t^{(1/2)-1}}{(1+t)^{(1/2)+(3/4)}}\,dt = \frac{1}{4} \cdot B\left(\frac{1}{2}, \frac{3}{4}\right) \qquad [\text{ By (29), } m = \frac{1}{2},\ n = \frac{3}{4} \text{ page 6-42 }]$$

$$I_2 = \int_0^\infty \frac{1}{(1+t)^{1/2}} \cdot \frac{1}{4} \cdot t^{-3/4}\, dt = \frac{1}{4}\int_0^\infty \frac{t^{(1/4)-1}}{(1+t)^{(1/4)+(1/4)}}\, dt$$

$$= \frac{1}{4} \cdot B\left(\frac{1}{4}, \frac{1}{4}\right) \qquad \left[\text{By (7), } m = \frac{1}{4},\ n = \frac{1}{4}, \text{ page 6-17}\right]$$

$$\therefore\ I = I_1 \times I_2 = \frac{1}{4} \cdot B\left(\frac{1}{2}, \frac{3}{4}\right) \times \frac{1}{4} \cdot B\left(\frac{1}{4}, \frac{1}{4}\right)$$

$$\therefore\ I = \frac{1}{16} \cdot \frac{\Gamma(1/2)\,\Gamma(3/4)}{\Gamma(5/4)} \cdot \frac{\Gamma(1/4)\,\Gamma(1/4)}{\Gamma(1/2)}$$

$$= \frac{1}{16} \cdot \frac{\Gamma(3/4)\,(\Gamma(1/4))^2}{(1/4)\,\Gamma(1/4)} = \frac{1}{4} \cdot \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{4} \cdot \sqrt{2} \cdot \pi = \frac{\pi}{2\sqrt{2}} \qquad [\text{ By (13), page 6-19 }]$$

Alternatively : We may put $x^2 = \tan\theta$, $2x\,dx = \sec^2\theta\ d\theta$

$$\therefore\ dx = \frac{\sec^2\theta}{2x}\, d\theta = \frac{1}{2} \cdot \frac{\sec^2\theta}{\sqrt{\tan\theta}}\, d\theta$$

When $x = 0, \theta = 0$; when $x = \infty, \theta = \pi/2$.

$$\therefore\ I_1 = \int_0^{\pi/2} \frac{1}{2} \cdot \frac{\sec^2\theta\ d\theta}{(\sec^2\theta)^{5/4}} = \frac{1}{2}\int_0^{\pi/2} \sec^{-1/2}\theta\ d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^0\theta \cos^{1/2}\theta\ d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot B\left(\frac{1}{2}, \frac{3}{4}\right) = \frac{1}{4} \cdot B\left(\frac{1}{2}, \frac{3}{4}\right)$$

$$I_2 = \int_0^{\pi/2} \frac{1}{2} \cdot \frac{1}{\sec\theta} \cdot \frac{\sec^2\theta}{\sqrt{\tan\theta}}\, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{\sec\theta}{\sqrt{\tan\theta}}\, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{-1/2}\ d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot B\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{1}{4} \cdot B\left(\frac{1}{4}, \frac{1}{4}\right).$$ Now, proceed as above.

Example 3 (c) : Show that $\Gamma(p)\,\Gamma\left(\dfrac{1-p}{2}\right) = \dfrac{\sqrt{\pi}\,\Gamma(p/2)}{2^{1-p}\cos(\pi p/2)}$.

Sol. : We know that $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$ [By Ex. 4, page 6-52]

Putting $n = \dfrac{p+1}{2}$, we get $\Gamma\left(\dfrac{p+1}{2}\right)\Gamma\left(\dfrac{1-p}{2}\right) = \dfrac{\pi}{\sin(p+1)\pi/2} = \dfrac{\pi}{\cos(\pi p/2)}$ .... (1)

By duplication formula (12A), page 6-19, $\Gamma(n)\,\Gamma\left(n+\dfrac{1}{2}\right) = \dfrac{\sqrt{\pi}}{2^{2n-1}} \cdot \Gamma(2n)$

Putting $n = p/2$, we get $\Gamma\left(\dfrac{p}{2}\right)\Gamma\left(\dfrac{p+1}{2}\right) = \dfrac{\sqrt{\pi}}{2^{p-1}} \cdot \Gamma(p)$ .... (2)

Dividing (1) by (2), we get

$$\frac{\Gamma((1-p)/2)}{\Gamma(p/2)} = \frac{\pi}{\cos(p\pi/2)} \cdot \frac{2^{p-1}}{\sqrt{\pi}\,\Gamma(p)} \qquad \therefore\ \Gamma(p)\,\Gamma\left(1-\frac{p}{2}\right) = \frac{\sqrt{\pi} \cdot \Gamma(p/2)}{2^{1-p}\cos(p\pi/2)}$$

Example 4 (c) : Prove that $B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$. Hence, evaluate (i) $\int_0^1 \frac{x^5 + x^8}{(1+x)^{15}}\, dx$

(ii) $\int_0^1 \frac{x^2 + x^3}{(1+x)^7}\, dx$. (M.U. 1991, 2006, 92, 93, 96)

Sol. : For the first part see (31), page 6-47.

Putting $m = 6$, $n = 9$ in (i) and $m = 3$ and $n = 4$ in (ii), we get

(i) $\int_0^1 \frac{x^5 + x^8}{(1+x)^{15}}\, dx = B(6, 9)$. (ii) $\int_0^1 \frac{x^2 + x^3}{(1+x)^7}\, dx = B(3, 4)$.

Example 5 (c) : Show that

$$\int_0^{\pi/2} \frac{\cos^{2m-1}\theta \cdot \sin^{2n-1}\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{m+n}}\, dx = \frac{B(m, n)}{2 \cdot a^{2n} \cdot b^{2m}}$$ (M.U. 1997, 99, 02, 04)

Sol. : Dividing the numerator and denominator by $\cos^{2m+2n}\theta$, we get

$$I = \int_0^{\pi/2} \frac{\cos^{2m-1}\theta}{\cos^{2m}\theta} \cdot \frac{\sin^{2n-1}\theta}{\cos^{2n}\theta} \cdot \frac{1}{(a^2 + b^2\tan^2\theta)^{m+n}}\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\cos\theta} \cdot \frac{\sin^{2n-1}\theta}{\cos^{2n-1}\theta} \cdot \frac{1}{\cos\theta} \cdot \frac{1}{(a^2 + b^2\tan^2\theta)^{m+n}}\, d\theta$$

$$= \int_0^{\pi/2} \frac{\tan^{2n-1}\theta}{(a^2 + b^2\tan^2\theta)^{m+n}} \cdot \sec^2\theta\, d\theta$$

$$\therefore\ I = \int_0^{\pi/2} \frac{(\tan\theta)^{2m-2} \cdot \tan\theta\sec^2\theta}{(a^2 + b^2\tan^2\theta)^{m+n}} \cdot d\theta$$

Now, put $b^2\tan^2\theta = a^2 y$ $\quad \therefore\ b^2 \cdot 2\tan\theta\sec^2\theta\, d\theta = a^2\, dy$

When $\theta = 0$, $y = 0$; when $\theta = \pi/2$, $y = \infty$.

$$\therefore\ I = \int_0^{\infty} \left(\frac{a^2 y}{b^2}\right)^{m-1} \cdot \frac{1}{(a^2 + a^2 y)^{m+n}} \cdot \frac{a^2}{2b^2}\, dy$$

$$= \frac{1}{2 \cdot a^{2n} \cdot b^{2m}} \cdot \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

[By (29), page 6-42]

$$= \frac{1}{2 \cdot a^{2n} \cdot b^{2m}} \cdot B(m, n)$$

Example 6 (c) : Prove that $\int_0^{\infty} \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{4} B\left(\frac{n}{2}, \frac{n}{2}\right)$ and hence evaluate $\int_0^{\infty} \sec h^8 x\, dx$.

(M.U. 2002, 03, 06, 07, 11, 12)

Sol. : We have $I = \int_0^{\infty} \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(e^x + e^{-x})^n}$

Put $e^x = \tan\theta$ $\quad \therefore\ e^x dx = \sec^2\theta\, d\theta$ $\quad \therefore\ dx = \frac{\sec^2\theta\, d\theta}{\tan\theta}$

When $x = \infty$, $e^x = \infty$, $\tan\theta = \infty$ $\quad \therefore\ \theta = \pi/2$

When $x = -\infty$, $e^x = 0$, $\tan\theta = 0$ $\quad \therefore\ \theta = 0$

$$\therefore \quad I = \frac{1}{2}\int_0^{\pi/2} \frac{1}{(\tan\theta + \cot\theta)^n} \cdot \frac{\sec^2\theta}{\tan\theta} \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\left(\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}\right)^n} \cdot \frac{1}{\cos^2\theta} \cdot \frac{\cos\theta}{\sin\theta} \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin^n\theta \cos^n\theta}{\sin\theta\, \cos\theta} \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{n-1}\theta \cos^{n-1}\theta \cdot d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot B\left(\frac{n-1+1}{2}, \frac{n-1+1}{2}\right) = \frac{1}{4} B\left(\frac{n}{2}, \frac{n}{2}\right)$$

Since, $\dfrac{e^x + e^{-x}}{2} = \cos h\, x, \quad e^x + e^{-x} = 2\cos\, hx$

Putting $n = 8$ in the integral,

$$\therefore \quad \int_0^{\infty} \frac{dx}{(e^x + e^{-x})^8} = \int_0^{\infty} \frac{dx}{2^8 \cos h^8 x} = \frac{1}{4} B(4, 4)$$

$$\therefore \quad \int_0^{\infty} \sec h^8\, x\, dx = \frac{2^8}{4} \cdot \frac{\overline{|4}\ \overline{|4}}{\overline{|8}} = 2^6 \cdot \frac{3! \cdot 3!}{7!} = \frac{16}{35}.$$

Example 7 (c) : Prove that $\displaystyle\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\, dx = \frac{(2n)!}{2^{2n}(n!)^2} \cdot \frac{\pi}{2}.$

Sol. : Put $x^2 = t$ *i.e.* $x = \sqrt{t} \quad \therefore\ dx = \dfrac{1}{2\sqrt{t}}\, dt$

$$\therefore \quad I = \int_0^1 \frac{t^n}{\sqrt{1-t}} \cdot \frac{1}{2\sqrt{t}}\, dt = \frac{1}{2}\int_0^1 t^{n-(1/2)} \cdot (1-t)^{-1/2}\, dt$$

$$= \frac{1}{2} B\left(n + \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} \cdot \frac{\overline{|n+(1/2)}\ \overline{|1/2}}{\overline{|n+1}} \qquad \left[\because B(m, n) = \frac{\overline{|m}\ \overline{|n}}{\overline{|m+n}}\right]$$

$$= \frac{1}{2} \cdot \frac{\overline{|n+(1/2)}\ \sqrt{\pi}}{n!} \qquad \left[\because \overline{|1/2} = \sqrt{\pi},\ \overline{|n+1} = n!\right]$$

By duplication formula

$$2^{2n-1} \cdot \overline{|n}\ \overline{|n+(1/2)} = \sqrt{\pi}\ \overline{|2n}$$

$$\therefore \quad \overline{|n+(1/2)} = \frac{\sqrt{\pi} \cdot \overline{|2n}}{2^{2n-1} \cdot \overline{|n}} = \frac{\sqrt{\pi} \cdot (2n-1)!}{2^{2n-1} \cdot (n-1)!}$$

$$= \frac{\sqrt{\pi} \cdot 2n(2n-1)!}{2^{2n-1} \cdot 2n(n-1)!} = \frac{\sqrt{\pi} \cdot (2n)!}{2^{2n} \cdot n!}$$

$$\therefore \quad I = \frac{1}{2} \cdot \frac{\sqrt{\pi}}{n!} \cdot \frac{\sqrt{\pi} \cdot (2n)!}{2^{2n} \cdot n!} = \frac{(2n)!}{2^{2n} \cdot (n!)^2} \cdot \frac{\pi}{2}.$$

(For another method, see Ex. 8, page 6-31.)

Example 8 (c) : Prove that

$$\int_1^\infty \frac{dx}{x^{p+1}(x-1)^q} = B(p+q,\ 1-q),\ -p<q<1. \qquad \text{(M.U. 1997, 99)}$$

Sol. : Let $x-1=t$ $\quad \therefore\ dx=dt$

When $x=1$, $t=0$; when $x=\infty$, $t=\infty$.

$$\therefore\ I=\int_0^\infty \frac{dt}{(1+t)^{p+1}\cdot t^q}=\int_0^\infty \frac{t^{-q}}{(1+t)^{p+1}}\,dt$$

Comparing this with $\displaystyle\int_0^\infty \frac{x^m}{(1+x)^n}\,dx = B(m+1,\ n-m-1)$ [See (29), page 6-42]

we get, $\displaystyle I=\int_0^\infty \frac{t^{-q}}{(1+t)^{p+1}}\,dt = B(-q+1,\ p+1+q-1)$

$$= B(p+q,\ 1-q).$$

Alternatively : Putting $x=\dfrac{1}{t}$, $dx=-\dfrac{1}{t^2}\,dt$, we get,

$$I=\int_1^0 \frac{1}{\dfrac{1}{t^{p+1}}\left(\dfrac{1}{t}-1\right)^q}\cdot\left(-\frac{1}{t^2}\right)dt=\int_0^1 \frac{t^{p+q-1}}{(1-t)^q}\,dt$$

$$=\int_0^1 t^{p+q-1}\cdot(1-t)^{-q}\,dt = B(p+q,\ 1-q).$$

Example 9 (c) : Using duplication formula, prove that

$$1\cdot 3\cdot 5\cdot 7\ldots.(2n-1)=\frac{2^n\,\overline{|n+(1/2)}}{\sqrt{\pi}}.$$

Sol. : By duplication formula given on page 6-18, (replacing m by n), we have

$$2^{2n-1}\cdot\overline{|n}\ \overline{|n+(1/2)}=\sqrt{\pi}\cdot\overline{|2n} \qquad \therefore\ \frac{2^{2n-1}\ \overline{|n+(1/2)}}{\sqrt{\pi}}=\frac{\overline{|2n}}{\overline{|n}}$$

Clearly, n is a positive integer (why?)

$$\therefore\ \frac{2^{2n-1}\,\overline{|n+(1/2)}}{\sqrt{\pi}}=\frac{(2n-1)!}{(n-1)!}$$

$$=\frac{(2n-1)(2n-2)(2n-3)(2n-4)\ldots.4\cdot 3\cdot 2\cdot 1}{(n-1)(n-2)(n-3)(n-4)\ldots.4\cdot 3\cdot 2\cdot 1}$$

$$=\frac{(2n-1)(2n-3)(2n-5)\ldots.5\cdot 3\cdot 1\cdot 2(n-1)\cdot 2(n-2)\ldots.2(2)\cdot 2(1)}{(n-1)(n-2)(n-3)\ldots.\ 3\cdot 2\cdot 1}$$

$$=\frac{(2n-1)(2n-3)(2n-5)\ldots.5\cdot 3\cdot 1\cdot 2^{n-1}(n-1)(n-2)(n-3)\ldots.3\cdot 2\cdot 1}{(n-1)(n-2)(n-3)\ldots.\ 3\cdot 2\cdot 1}$$

$$\therefore\ \frac{2^n\cdot\overline{|n+(1/2)}}{\sqrt{\pi}}=1\cdot 3\cdot 5\cdot 7\ \ldots.\ (2n-3)(2n-1).$$

[The term 2^{n-1} is cancelled by 2^{n-1} on the l.h.s. and the denominator is cancelled by the last terms in the numerator.]

Example 10 (c) : Prove that $B\left(n+\frac{1}{2},\ n+\frac{1}{2}\right)=\frac{1}{2^{2n}}\cdot\frac{\Gamma(n+(1/2))}{\Gamma(n+1)}\cdot\sqrt{\pi}.$ **(M.U. 2010)**

Hence, deduce that $2^n\,\Gamma(n+(1/2))=1\cdot3\cdot5\ldots\ldots(2n-1)\sqrt{\pi}.$ **(M.U. 2002, 07)**

Sol. : We have, by definition,

$$B\left(n+\frac{1}{2},\ n+\frac{1}{2}\right)=\int_0^1 x^{n-(1/2)}\cdot(1-x)^{n-(1/2)}\,dx$$

Putting $x=\sin^2\theta,\quad dx=2\sin\theta\cos\theta\,d\theta$

$$B\left(n+\frac{1}{2},\ n+\frac{1}{2}\right)=\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2n-1}\theta\cdot2\sin\theta\cos\theta\,d\theta$$

$$=2\int_0^{\pi/2}\sin^{2n}\theta\cos^{2n}\theta\,d\theta=2\int_0^{\pi/2}\left(\frac{\sin2\theta}{2}\right)^{2n}d\theta$$

$$=\frac{1}{2^{2n-1}}\int_0^{\pi/2}(\sin2\theta)^{2n}\,d\theta$$

$$=\frac{1}{2^{2n}}\int_0^{\pi}\sin^{2n}\phi\,d\phi\quad\text{where }\phi=2\theta$$

$$=\frac{2}{2^{2n}}\int_0^{\pi/2}\sin^{2n}\phi\,d\phi\qquad\left[\because\int_0^{2a}f(x)\,dx=2\int_0^{a}f(x)\,dx\ \text{ if } f(2a-x)=f(x)\right]$$

$$=\frac{1}{2^{2n}}\cdot2\cdot\frac{1}{2}\frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2n+2}{2}\right)}=\frac{1}{2^{2n}}\cdot\frac{\Gamma\left(n+\frac{1}{2}\right)\cdot\sqrt{\pi}}{\Gamma(n+1)}\qquad\text{[By (19), page 6-28]}$$

But $$B\left(n+\frac{1}{2},\ n+\frac{1}{2}\right)=\frac{\left[\Gamma(n+(1/2))\right]^2}{\Gamma(2n+1)}$$

Equating the two results, we get,

$$\frac{\left[\Gamma(n+(1/2))\right]^2}{\Gamma(2n+1)}=\frac{1}{2^{2n}}\cdot\frac{\Gamma(n+(1/2))\cdot\sqrt{\pi}}{\Gamma(n+1)}$$

$$\therefore\quad 2^{2n}\,\Gamma(n+(1/2))=\frac{\Gamma(2n+1)}{\Gamma(n+1)}\cdot\sqrt{\pi}=\frac{2n(2n-1)\ldots\ldots3\cdot2\cdot1\sqrt{\pi}}{n(n-1)(n-2)\ldots\ldots3\cdot2\cdot1}$$

$$=\frac{2n(2n-1)\cdot2(n-1)(2n-3)\ldots\ldots3\cdot2\cdot1\sqrt{\pi}}{n(n-1)(n-2)\ldots\ldots3\cdot2\cdot1}$$

$$=2^n\cdot1\cdot3\cdot5\ldots\ldots(2n-1)\sqrt{\pi}$$

$$=1\cdot3\cdot5\ldots\ldots(2n-1)\sqrt{\pi}.$$

Example 11 (c) : Show that $\int_0^a\frac{dx}{(a^n-x^n)^{1/n}}=\frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right).$ **(M.U. 2003, 08, 09)**

Sol. : Putting $x^n=a^n\sin^2\theta.$ *i.e.* $x=a\sin^{2/n}\theta,\ dx=\frac{2a}{n}\sin^{(2/n)-1}\theta\cdot\cos\theta\cdot d\theta$

[or put $x^n = a^n t$ i.e. $x = at^{1/n}$

$\therefore\ I = \int_0^1 \frac{1}{a(1-t)^{1/n}} \cdot \frac{a}{n} \cdot t^{(1/n)-1}\, dt$. Now put $t = \sin^2\theta$]

$$\therefore\ I = \int_0^{\pi/2} \frac{2a}{n} \cdot \frac{1}{a\cos^{2/n}\theta} \cdot \sin^{(2/n)-1}\theta \cdot \cos\theta \cdot d\theta$$

$$= \frac{2}{n}\int_0^{\pi/2} \sin^{(2/n)-1}\theta \cdot \cos^{1-(2/n)}\theta \cdot d\theta$$

$$= \frac{2}{n} \cdot \frac{1}{2} \cdot \frac{\Gamma\left(\frac{(2/n)-1+1}{2}\right)\Gamma\left(\frac{1-(2/n)+1}{2}\right)}{\Gamma\left(\frac{(2/n)-1+1-(2/n)+2}{2}\right)}$$ [By (6), page 6-17]

$$= \frac{1}{n} \cdot \frac{\Gamma(1/n)\,\Gamma(1-(1/n))}{\Gamma(1)} = \frac{1}{n} \cdot \frac{\pi}{\sin(\pi/n)}$$ [By Ex. 4, page 6-52]

$$= \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right).$$

Example 12 (c) : Show that $\int_0^{\pi/2} \tan^n x\, dx = \frac{\pi}{2}\sec\left(\frac{\pi n}{2}\right)$. **(M.U. 1996, 97)**

Deduce that $\int_0^{\pi/2} \cot^n x\, dx = \frac{\pi}{2}\sec\left(\frac{\pi n}{2}\right)$.

Sol. : $$I = \int_0^{\pi/2} \sin^n x \cos^{-n} x\, dx = \frac{1}{2} B\left(\frac{n+1}{2}, \frac{-n+1}{2}\right)$$ [By (3), page 6-17]

$$\therefore\ I = \frac{1}{2} \cdot \frac{\Gamma((n+1)/2)\,\Gamma((-n+1)/2)}{\Gamma(1)}.$$

$$= \frac{1}{2}\Gamma(p)\,\Gamma(1-p) = \frac{1}{2} \cdot \frac{\pi}{\sin p\pi} \quad \text{where } p = \frac{n+1}{2}$$ [By Ex. 4, page 6-52]

$$= \frac{1}{2} \cdot \frac{\pi}{\sin\left(\frac{n+1}{2} \cdot \pi\right)} = \frac{\pi}{2}\sec\left(\frac{n\pi}{2}\right).$$

Cor : $$\int_0^{\pi/2} \cot^n x\, dx = \int_0^{\pi/2} \cot^n\left(\frac{\pi}{2} - x\right) dx$$
$$= \int_0^{\pi/2} \tan^n x\, dx = \frac{\pi}{2}\sec\left(\frac{\pi n}{2}\right)$$

Example 13 (c) : Prove that $\int_0^{\pi} \frac{\sin^{n-1} x}{(a + b\cos x)^n}\, dx = \frac{2^{n-1}}{(a^2 - b^2)^{n/2}}\, B\left(\frac{n}{2}, \frac{n}{2}\right)$. **(M.U. 2007)**

Sol. : Put $t = \tan\frac{x}{2}$, $\sin x = \frac{2t}{(1+t^2)}$, $\cos x \frac{(1-t^2)}{(1+t^2)}$, $dx = \frac{2\,dt}{(1+t^2)}$.

$$\therefore\ I = \int_0^{\infty} \frac{[2t/(1+t^2)]^{n-1}}{\left[a + b \cdot \frac{(1-t^2)}{(1+t^2)}\right]^n} \cdot \frac{2\,dt}{(1+t^2)} = 2^n \int_0^{\infty} \frac{t^{n-1}}{[(a+b) + (a-b)t^2]^n}\, dt$$

Put $t = \sqrt{\dfrac{a+b}{a-b}}\cdot\tan\theta$ $\quad\therefore\ dt = \sqrt{\dfrac{a+b}{a-b}}\cdot\sec^2\theta\,d\theta$

$$\therefore\ I = 2^n\int_0^{\pi/2}\frac{[\sqrt{(a+b)/(a-b)}]^{n-1}\tan^{n-1}\theta}{(a+b)^n\cdot(\sec^2\theta)^n}\cdot\sqrt{\frac{a+b}{a-b}}\sec^2\theta\,d\theta$$

$$= \frac{2^n}{(a^2-b^2)^{n/2}}\int_0^{\pi/2}\sin^{n-1}\theta\cdot\cos^{n-1}\theta\,d\theta$$

$$= \frac{2^n}{(a^2-b^2)^{n/2}}\cdot\frac{1}{2}\cdot B\left(\frac{n}{2},\frac{n}{2}\right) = \frac{2^{n-1}}{(a^2-b^2)^{n/2}}\,B\left(\frac{n}{2},\frac{n}{2}\right)$$

Example 14 (c) : Prove that $\displaystyle\int_0^\pi \frac{\sqrt{\sin x}}{(5+3\cos x)^{3/2}}\,dx = \frac{\left(\Gamma(3/4)\right)^2}{2\sqrt{2\pi}}$. **(M.U. 1989, 2001, 0**

Sol. : In the above Ex. 13, put $n = 3/2$, $a = 5$, $b = 3$ or independently as above put

$$t = \tan\frac{x}{2},\ \sin x = \frac{2t}{(1+t^2)},\ \cos x = \frac{(1-t^2)}{(1+t^2)},\ dx = \frac{2\,dt}{(1+t^2)}.$$

$$\therefore\ I = \int_0^\infty \frac{\sqrt{2t/(1+t^2)}}{\left[5+3\cdot\left(\dfrac{1-t^2}{1+t^2}\right)\right]^{3/2}}\cdot\frac{2dt}{(1+t^2)}$$

$$= \int_0^\infty \frac{2\sqrt{2}\cdot\sqrt{t}\,dt}{(8+2t^2)^{3/2}} = \int_0^\infty \frac{\sqrt{t}}{(4+t^2)^{3/2}}\,dt$$

Putting $t^2 = 4y$, $t = 2\sqrt{y}$ $\quad\therefore\ dt = \dfrac{dy}{\sqrt{y}}$

$$\therefore\ I = \frac{1}{8}\int_0^\infty \frac{\sqrt{2}\cdot y^{1/4}}{(1+y)^{3/2}}\cdot\frac{dy}{\sqrt{y}} = \frac{1}{4\sqrt{2}}\int_0^\infty \frac{y^{-1/4}}{(1+y)^{3/2}}\,dy$$

$$= \frac{1}{4\sqrt{2}}\,B\left(\frac{3}{4},\frac{3}{4}\right) \qquad \left[\because \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(m,n) \text{ By (30), page 1-41}\right]$$

$$= \frac{1}{4\sqrt{2}}\cdot\frac{\Gamma(3/4)\cdot\Gamma(3/4)}{\Gamma(3/2)} = \frac{1}{4\sqrt{2}}\cdot\frac{\left(\Gamma(3/4)\right)^2}{(1/2)\,\Gamma(1/2)} = \frac{\left(\Gamma(3/4)\right)^2}{2\sqrt{2\pi}}.$$

Example 15 (c) : Given $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$, prove that

$$\int_0^1 \frac{x^{n-1}}{(1+cx)(1-x)^n}\,dx = \frac{1}{(1+c)^n}\cdot\frac{\pi}{\sin(n\pi)},\quad 0<p<1.$$ **(M.U. 1996)**

Sol. : We put $\dfrac{x}{1+cx} = \dfrac{y}{1+c}$ $\quad\therefore\ (1+c)\,x = y\,(1+cx)$

$\therefore\ x(1+c-cy) = y$ $\quad\therefore\ x = \dfrac{y}{1+c-cy}$

$$\therefore\ dx = \frac{(1+c-cy)1-y(-c)}{(1+c-cy)^2}\,dy = \frac{1+c}{(1+c-cy)^2}\,dy$$

Further, $1-x = 1-\dfrac{y}{1+c-cy} = \dfrac{(1+c)(1-y)}{1+c-cy}$ and $1+cx = 1+\dfrac{cy}{1+c-cy} = \dfrac{1+c}{1+c-cy}$

$$\therefore\ I = \int_0^1 \frac{y^{n-1}}{(1+c-cy)^{n-1}}\cdot\frac{(1+c-cy)}{(1+c)}\cdot\frac{(1+c-cy)^n}{(1+c)^n(1-y)^n}\cdot\frac{1+c}{(1+c-cy)^2}\,dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+c)^n(1-y)^n}\,dy$$

To get the limits 0 to ∞, we put $\dfrac{y}{1-y} = t \quad \therefore\ y = t - ty.$

$$\therefore\ y(1+t) = t \quad \therefore\ y = \frac{t}{1+t} \quad \therefore\ dy = \frac{(1+t)\cdot 1 - t\cdot 1}{(1+t)^2}\,dt = \frac{1}{(1+t)^2}\,dt$$

And $1-y = 1-\dfrac{t}{1+t} = \dfrac{1}{1+t}$

$$\therefore\ I = \int_0^\infty \frac{t^{n-1}}{(1+t)^{n-1}}\cdot\frac{1}{(1+c)^n}\cdot\frac{(1+t)^n}{1}\cdot\frac{dt}{(1+t)^2} = \int_0^\infty \frac{t^{n-1}}{(1+c)^n\cdot(1+t)}\,dt$$

$$= \frac{1}{(1+c)^n}\int_0^\infty \frac{t^{n-1}}{(1+t)}\,dt = \frac{1}{(1+c)^n}\cdot\frac{\pi}{\sin(n\pi)} \qquad \text{[By data]}$$

MISCELLANEOUS EXERCISE

(A) Class (a) : 3 Marks

1. Find the value of

   (i) $B\left(\dfrac{4}{3},\dfrac{5}{3}\right)$ (ii) $B\left(\dfrac{3}{2},\dfrac{1}{2}\right)$ (iii) $\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{2}{4}\right)\Gamma\left(\dfrac{3}{4}\right)\Gamma\left(\dfrac{4}{4}\right)$

   [Ans. : (i) $\dfrac{2\pi}{9\sqrt{3}}$, (ii) $\dfrac{\pi}{4}$, (iii) $\sqrt{2}\cdot\pi$ **]**

2. (i) Given $\Gamma(1\cdot6) = 0\cdot8935$, find the value of $\Gamma(-2\cdot4)$. **[Ans. :** $-1\cdot11$ **]**

   (ii) Given $\sqrt{\pi} = 1\cdot772$, find the value $\Gamma(-4\cdot5)$. **[Ans. :** $-0\cdot06$ **]**

3. Show that : (i) $\Gamma(x)\,\Gamma(-x) = -\dfrac{\pi}{x\sin x\pi}$ (ii) $\Gamma\left(\dfrac{1}{2}+x\right)\Gamma\left(\dfrac{1}{2}-x\right) = \dfrac{\pi}{\cos\pi x}$

4. Show that $\Gamma\left(\dfrac{3}{2}-n\right)\Gamma\left(\dfrac{3}{2}+n\right) = \left(\dfrac{1}{4}-n^2\right)\pi\sec n\pi,\ (-1<2n<1).$ **(M.U. 2007)**

   (**Hint :** l.h.s. $= [(1/2)-n]\,\Gamma((1/2)-n)\cdot[(1/2)+n]\,\Gamma((1/2)+n)$

   $= \left[(1/4)-n^2\right]\Gamma(n+(1/2))\,\Gamma(1-[n+(1/2)])$

   $= \left(\dfrac{1}{4}-n^2\right)\dfrac{\pi}{\sin[n+(1/2)\pi]} = \left(\dfrac{1}{4}-n^2\right)\cdot\dfrac{\pi}{\cos n\pi}$)

5. Using Beta function, prove that $\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$.

6. Show that $\int_0^1 \left(\frac{1}{x} - 1\right)^{1/4} dx = \frac{\pi}{2\sqrt{2}}$.

(**Hint :** $I = \int_0^1 (1-x)^{1/4} \cdot x^{-1/4}\, dx = \frac{\overline{|3/4}\ \overline{|5/4}}{|2}$

$= \overline{|3/4} \cdot (1/4) \cdot \overline{|1/4} = \frac{1}{4} \cdot \sqrt{2} \cdot \pi = \frac{\pi}{2\sqrt{2}}$)

7. Prove that $\int_0^1 \frac{x^2}{(1-x^4)^{1/2}}\, dx \cdot \int_0^{\infty} \frac{1}{(1+x^4)^{1/2}}\, dx = \frac{\pi}{4\sqrt{2}}$ **(M.U. 2005)**

(**Hint :** In I_1 put $x^2 = \sin\theta$, in I_2 put $x^2 = \tan\theta$.)

8. Prove that $\iint_D x^{l-1} y^{m-1}\, dx\, dy = a^{l+m} \frac{\overline{|l}\ \overline{|m}}{|l+m+1}$

where D is the domain $x \geq 0,\ y \geq 0,\ x + y \leq a$.

(**Hint :** Put $x = au,\ y = av$

$\therefore\ I = a^{l+m} \int_0^1 \int_0^{1-u} u^{l-1} v^{m-1}\, du\, dv$

$= \frac{a^{l+m}}{m} \int_0^1 u^{l-1} (1-u)^m\, du = \frac{a^{l+m}}{m} B(l,\ m+1)$)

9. If $B(n, 2) = \frac{1}{42}$ and n is a positive integer find n. **[Ans. :** $n = 6$ **]**

10. If $B(n, 3) = \frac{1}{60}$ and n is a positive integer, find n. **[Ans. :** 4 **]**

(B) Class (c) : 8 Marks

1. Prove that $\int_0^{\infty} \frac{e^{2mx} + e^{-2mx}}{(e^x + e^{-x})^{2n}}\, dx = \frac{1}{2} B(m+n,\ n-m)$. **(M.U. 1990, 2002)**

(**Hint :** Multiply the numerator and denominator by e^{2nx} and put $e^{2x} = t$. Then put $t = 1/y$.)

2. Prove that $\int_0^{\infty} \frac{x^2\, dx}{(1+x^4)^{3/2}} \cdot \int_0^{\infty} \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4}$. (**Hint :** Put $x^4 = t$.) **(M.U. 1989)**

3. Express $\int_{-1}^{1} (1+x)^m (1-x)^n\, dx$ as a Beta Function. **(M.U. 1997)**

(**Hint :** Put $1 + x = 2t$.) **[Ans. :** $2^{m+n+1} B(m+1,\ n+1)$ **]**

4. Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{2^{(2-n)/n} \cdot (\overline{|1/n})^2}{n \cdot |2/n}$. **(M.U. 1997)**

(**Hint :** Put $x^n = t$ and then use duplication formula.)

5. Prove that $\int_0^{\infty} \sec h^6 x\, dx = \frac{8}{15}$. (**Hint :** Put $e^x = \tan\theta$.) **(M.U. 1998)**

6. Prove that $\int_0^2 x \cdot \sqrt[3]{8-x^3}\, dx = \dfrac{16\,\pi}{9\sqrt{3}}$. (M.U. 2005)

7. Prove that $B(n, n) = 2\int_0^{1/2} (t - t^2)^{n-1} dt$. (M.U. 2002)

(**Hint :** $B(n, n) = \int_0^{1/2} t^n (1-t)^{n-1} dt + \int_{1/2}^{1} t^n (1-t)^{n-1} dt$. In I_2 put $1 - t = x$.)

8. Find $\int_0^3 \dfrac{x^{3/2}}{\sqrt{3-x}}\, dx \int_0^1 \dfrac{dx}{\sqrt{1-x}}$. [**Ans. :** $\dfrac{9}{4} B\left(\dfrac{5}{2}, \dfrac{1}{2}\right) B\left(\dfrac{1}{4}, \dfrac{1}{2}\right)$]

9. Prove that $\int_0^1 \dfrac{(1-x^4)^{3/4}}{(1+x^4)^2}\, dx = \dfrac{3\pi}{2^{15/4}}$ (M.U. 1999)

10. Prove that $\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} B(m+1, n+1)$

Hence deduce that (i) $\int_{-1}^{1} (1+x)^m (1-x)^n dx = 2^{n+n+1} B(m+1, n+1)$

(ii) $\int_0^n x^n (1-x)^p dx = n^{p+n+1} B(n+1, p+1)$ (M.U. 2005)

EXERCISE - XIV

Class (a) : 3 marks

1. Evaluate $\int_0^{\infty} x^2 e^{-x^2} dx$.
2. Evaluate $\int_0^{\infty} \sqrt{x} \cdot e^{-x^2} dx$.
3. Evaluate $\int_0^{\infty} x^n e^{-\sqrt{ax}} dx$.
4. Evaluate $\int_0^1 (-\log x)^5\, dx$.
5. Evaluate $\int_0^1 x^3 \left[\log\left(\dfrac{1}{x}\right)\right]^4 dx$.
6. Evaluate $\int_0^1 \sqrt[3]{\log(1/x)} \cdot dx$.
7. Evaluate $\int_0^{\infty} 3^{-4x^2} dx$.
8. Evaluate $\int_0^4 \sqrt{x}\,(4-x)^{3/2}\, dx$.
9. Evaluate $\int_0^2 x^3 \sqrt{2-x} \cdot dx$.
10. Evaluate $\int_0^2 x^4 (8-x^3)^{-1/3} dx$.
11. Evaluate $\int_0^{2a} x\sqrt{2ax - x^2} \cdot dx$.
12. Evaluate $\int_0^1 \dfrac{1}{\sqrt{1-x^4}}$.
13. Evaluate $\int_0^3 \dfrac{x^{3/2}}{\sqrt{3-x}}\, dx$.
14. Find $B\left(\dfrac{5}{2}, \dfrac{3}{2}\right)$.
15. If $B(n, 3) = \dfrac{1}{3}$, find n.
16. If $\overline{|1\cdot 8} = 0\cdot 9314$, find $\overline{|4\cdot 8}$.
17. If $\overline{\left|\dfrac{3}{5}\right.} = 1\cdot 4891$, find $\overline{\left|\dfrac{18}{5}\right.}$.
18. Evaluate $\int_0^{\infty} e^{-x^2} dx$.
19. Evaluate $B\left(\dfrac{3}{2}, \dfrac{3}{2}\right)$.
20. Evaluate $\int_0^{\pi/4} \sin^5 2\theta\, d\theta$.
21. Evaluate $\int_0^{\pi/6} \cos^6 3\theta\, d\theta$.

[**Ans. :** (1) $\dfrac{\sqrt{\pi}}{4}$, (2) $\dfrac{1}{2}\overline{\left|\dfrac{3}{4}\right.}$, (3) $\dfrac{2}{a^{n+1}} \cdot \overline{|2n+2}$, (4) $\overline{|16}$, (5) $\dfrac{1}{4^5}\overline{|5}$, (6) $\overline{|4/3}$, (7) $\dfrac{1}{4\sqrt{\log 3}}\overline{\left|\dfrac{1}{2}\right.}$,

EXERCISE - XV

Theory : Class (a) : 3 Marks

1. State and prove the relation between Beta and Gamma functions. **(M.U. 1995, 2002, 03)**

2. Prove that $\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$. **(M.U. 1992)**

3. State and prove duplication formula. **(M.U. 1996, 97, 2002, 11, 12)**

4. Prove that $\overline{|1/4} \cdot \overline{|3/4} = \pi\sqrt{2}$. **(M.U. 1998, 2009)**

5. Prove that $\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2} \frac{\overline{\left|\frac{p+1}{2}\right.} \; \overline{\left|\frac{q+1}{2}\right.}}{\overline{\left|\frac{p+1+2}{2}\right.}}$. **(M.U. 2002)**

6. Prove that $\overline{|1/2} = \sqrt{\pi}$. **(M.U. 2002)**

7. Show that $\overline{|n+1} = n\overline{|n}$. **(M.U. 2003)**

8. Show that $B(n, n) = \frac{\sqrt{\pi}}{2^{2n-1}} \cdot \frac{\overline{|n}}{\overline{|n+(1/2)}}$.

9. State Gamma function and hence evaluate $\int_0^\infty x^{1/4} e^{-\sqrt{x}} \, dx$. **[Ans. : $2\,\overline{|5/2}$]**

10. Prove that $B(m, n) = \frac{\overline{|m}\;\overline{|n}}{\overline{|m+n}}$.

11. State duplication formula and prove that

$$B(n, n) \cdot B\left(n + \frac{1}{2}, n + \frac{1}{2}\right) = \frac{\pi}{n} \cdot 2^{1-4n}.$$ **(M.U. 2005, 12)**

Summary

1. Gamma Function

(a) Definition 1 : $\int_0^\infty e^{-x} \cdot x^{n-1}\, dx = \overline{|n}$

Definition 2 : $\int_0^\infty e^{-x^2} \cdot x^{2n-1}\, dx = \frac{1}{2}\overline{|n}$

(b) Properties of $\overline{|n}$

1. $\overline{|n} = (n-1)\overline{|n-1}$
2. $\overline{|n} = n!$ if n is a positive integer.
3. $\overline{|n} = \infty$ if $n = 0, -1, -2, -3, \ldots$
4. $\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$

2. Beta Function

(a) Definition 1 : $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx = B(m, n)$

Definition 2 : $\int_0^{\pi/2} \sin^p \theta \cos^q \theta\, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Definition 3 : $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = B(m, n)$

Definition 4 : $\int_0^\infty \frac{x^m}{(1+x)^n}\, dx = B(m+1, n-m-1)$

(b) Properties of Beta Function

1. $B(m, n) = B(n, m)$
2. $B(m, n) = \frac{\overline{|m}\ \overline{|n}}{\overline{|m+n}}$
3. $\overline{|m}\ \overline{\left|m+\frac{1}{2}\right.} = \frac{\sqrt{\pi} \cdot \overline{|2m}}{2^{2m-1}}$
4. $\overline{\left|m+\frac{1}{2}\right.} = \frac{\sqrt{\pi} \cdot (2m)!}{2^{2m} \cdot m!}$ if m is a positive integer.

Putting $m = 0$, we have $\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}.$

Corollaries from Duplication Formula

1. $\overline{\left|\frac{1}{4}\right.}\ \overline{\left|\frac{3}{4}\right.} = \sqrt{2} \cdot \pi$
2. $\overline{\left|\frac{3}{4}\right.}\ \overline{\left|\frac{5}{4}\right.} = \frac{\pi}{2\sqrt{2}}$
3. $\overline{\left|\frac{5}{4}\right.} \cdot \overline{\left|\frac{7}{4}\right.} = \frac{3}{16}\sqrt{2} \cdot \pi$

(c) Trigonometric Integration Formulae

1. $\int_0^{\pi/2} \sin^m \theta \cos^n \theta\, d\theta = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots} \cdot \frac{\pi}{2}$

if m, n **both** are even positive integers.

$\int_0^{\pi/2} \sin^m \theta \cos^n \theta\, d\theta = \frac{(m-1)(m-3)\ldots.(n-1)(n-3)\ldots.}{(m+n)(m+n-2)\ldots.} \cdot 1$

in other cases.

2. $\int_0^{\pi/2} \sin^n \theta \, d\theta = \frac{(n-1)\cdot(n-3)}{n\cdot(n-2)} \cdots\cdots \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$

if n is **even** positive integer.

$\int_0^{\pi/2} \sin^n \theta \, d\theta = \frac{(n-1)\cdot(n-3)}{n\cdot(n-2)} \cdots\cdots \frac{4}{5}\cdot\frac{2}{3}\cdot 1$

if n is **odd** positive integer.

3. $\int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{(n-1)\cdot(n-3)}{n\cdot(n-2)} \cdots\cdots \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$

if n is **even** positive integer.

$\int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{(n-1)\cdot(n-3)}{n\cdot(n-2)} \cdots\cdots \frac{4}{5}\cdot\frac{2}{3}\cdot 1$

if n is **odd** positive integer.

(d) Assuming $\int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}$, we get $\overline{|\,p}\;\overline{|\,1-p} = \frac{\pi}{\sin p\pi}$.

Corollaries :

(1) $\overline{\left|\,\frac{1}{2}\right.}\cdot\overline{\left|\,\frac{1}{2}\right.} = \pi$ (2) $\overline{\left|\,\frac{1}{6}\right.}\cdot\overline{\left|\,\frac{5}{6}\right.} = 2\pi$

(3) $\overline{\left|\,\frac{1}{4}\right.}\cdot\overline{\left|\,\frac{3}{4}\right.} = \sqrt{2}\cdot\pi$ (4) $\overline{\left|\,\frac{1}{3}\right.}\cdot\overline{\left|\,\frac{2}{3}\right.} = \frac{2\pi}{\sqrt{3}}$.

3. Special Integrals

$$\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

CHAPTER

7

Differentiation Under Integral Sign

1. Introduction

There are some definite integrals which cannot be effected by methods studied so far. However, if the function under integral sign satisfies certain conditions, then we can differentiate the given function under the integral sign and from the resulting function we can obtain the required integral. This is known as **differentiation under integral sign** abbreviated as **D.U.I.S**.

2. Rule

If $f(x, \alpha)$ is a continuous function of x, and α is a parameter and if $\partial f / \partial \alpha$ is a continuous function of x and α together throughout the interval $[a, b]$ where a, b are constants and independent of α and if

$$I(\alpha) = \int_a^b f(x, \alpha)\, dx \quad \text{then} \quad \boxed{\frac{dI}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}\, dx}$$

Proof : We have $I(\alpha) = \int_a^b f(x, \alpha)\, dx$

$$\therefore \quad I(\alpha + \delta\alpha) = \int_a^b f(x, \alpha + \delta\alpha)\, dx$$

$$\therefore \quad I(\alpha + \delta\alpha) - I(\alpha) = \int_a^b [f(x, \alpha + \delta\alpha) - f(x, \alpha)]\, dx$$

$$\therefore \quad \frac{I(\alpha + \delta\alpha) - I(\alpha)}{\delta\alpha} = \int_a^b \frac{f(x, \alpha + \delta\alpha) - f(x, \alpha)}{\delta\alpha}\, dx$$

Taking limits of both sides as $\delta\alpha \to 0$, we have $\frac{dI}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}\, dx$.

(A) Solved Examples with one parameter : Class (c) : 8 Marks

Example 1 (c) : Prove that $\int_0^1 \frac{x^\alpha - 1}{\log x}\, dx = \log(1 + \alpha),\ \alpha \geq 0$.

Hence, evaluate $\int_0^1 \frac{x^7 - 1}{\log x}\, dx$. **(M.U. 1992, 95, 2002, 03)**

Sol. : Let $I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x}\, dx$

By the rule of differentiation under integral sign we have, differentiating w.r.t. α,

$$\frac{dI}{d\alpha} = \int_0^1 \frac{\partial f}{\partial \alpha}\, dx = \int_0^1 \frac{1}{\log x} \cdot x^\alpha \log x \cdot dx$$

$$\therefore \quad \frac{dI}{d\alpha} = \int_0^1 x^\alpha\, dx = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$$

Integrating both sides w.r.t. α, $\quad I = \log(\alpha+1) + c$

To find, c, we put $\alpha = 0$. $\quad \therefore \quad I(0) = \log(1) + c = c$

But $\quad I(0) = \int_0^1 \frac{1-1}{\log x}\, dx = \int_0^1 0\, dx = 0 \quad \therefore \quad c = 0 \quad \therefore \quad I = \log(\alpha+1).$

For deduction, putting $\alpha = 7$, we get $\quad \int_0^1 \frac{x^7 - 1}{\log x}\, dx = \log 8.$

Corollary : Prove that $\int_0^1 \frac{x^\alpha - x^\beta}{\log x}\, dx = \log\left(\frac{1+\alpha}{1+\beta}\right).$

Hence, evaluate $\int_0^1 \frac{x^7 - x^3}{\log x}\, dx.$ **(M.U. 2009)**

Sol. : By Ex. (1), $\int_0^1 \frac{x^\alpha - 1}{\log x}\, dx = \log(1+\alpha) \quad \therefore \quad \int_0^1 \frac{x^\beta - 1}{\log x}\, dx = \log(1+\beta)$

By subtraction,

$$\int_0^1 \frac{x^\alpha - x^\beta}{\log x}\, dx = \log(1+\alpha) - \log(1+\beta) = \log\left(\frac{1+\alpha}{1+\beta}\right)$$

Putting $\alpha = 7, \beta = 3$, we get

$$\int_0^1 \frac{x^7 - x^3}{\log x}\, dx = \log\left(\frac{1+7}{1+3}\right) = \log 2.$$

Example 2 (c) : Evaluate $\int_0^\infty \frac{e^{-x}}{x}(1 - e^{-ax})\, dx, \quad a > -1.$ **(M.U. 2003)**

Hence, evaluate $\int_0^\infty \frac{e^{-x}}{x}(1 - e^{-7x})\, dx.$

Sol. : Let $I(a) = \int_0^\infty \frac{e^{-x}}{x}(1 - e^{-ax})\, dx$

By the rule of differentiation under integral sign, differentiating w.r.t. a, we get,

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\, dx = \int_0^\infty \frac{e^{-x}}{x} \cdot x \cdot e^{-ax}\, dx$$

$$= \int_0^\infty e^{-x(1+a)}\, dx = \left[\frac{e^{-x(1+a)}}{-(1+a)}\right]_0^\infty = \frac{1}{1+a}$$

Integrating both sides w.r.t. a, $\quad I = \log(1+a) + c$

To find , c, we put $a = 0 \quad \therefore \quad I(0) = \log(1) + c = c$

But $\quad I(0) = \int_0^\infty \frac{e^{-x}}{x}(1-1)\, dx = \int_0^\infty 0\, dx = 0 \quad \therefore \quad c = 0$

$\therefore \quad I = \log(1+a).$

For deduction, putting $a = 7$, we get, $\quad \int_0^\infty \frac{e^{-x}}{x}(1 - e^{-7x})\, dx = \log 8.$

Corollary : Prove that $\int_0^\infty \frac{e^{-x}}{x}(e^{-ax} - e^{-bx})\,dx = \log\left(\frac{1+b}{1+a}\right)$.

Sol. : Changing a to b in Ex. (2)

$$\int_0^\infty \frac{e^{-x}}{x}(1 - e^{-bx})\,dx = \log(1+b)$$

By subtraction, we get the result.

Example 3 (c) : Prove that

$$\int_0^\infty e^{-ax} \cdot \frac{\sin mx}{x}\,dx = \tan^{-1}\frac{m}{a}. \text{ (} a \text{ is a parameter.)}$$ **(M.U. 1989, 99)**

Hence, evaluate $\int_0^\infty e^{-x} \cdot \frac{\sin x}{x}\,dx$.

Sol. : Let $I(a) = \int_0^\infty e^{-ax} \cdot \frac{\sin mx}{x}\,dx$ .... (1)

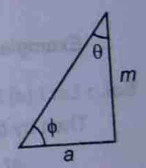
By the rule of differentiation under the integral sign

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\,dx = \int_0^\infty \frac{\sin mx}{x}(-e^{-ax} \cdot x)\,dx$$

$$= -\int_0^\infty e^{-ax} \sin mx \cdot dx = -\left[\frac{e^{-ax}}{a^2+m^2}(-a\sin mx - m\cos mx)\right]_0^\infty$$

$$\left[\because \int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$\therefore \frac{dI}{da} = -\frac{1}{a^2+m^2}[0 - (0 - m)] = -\frac{m}{a^2+m^2}$$



By integration, $I = -\tan^{-1}\frac{a}{m} + c$

To find c, we put $a = 0$. $\therefore I(0) = c$.

But by putting $a = 0$ in (1)

$$I(0) = \int_0^\infty \frac{\sin mx}{x}\,dx = \frac{\pi}{2} \quad \text{[See Ex. 6 page 7-18]} \quad \therefore c = \frac{\pi}{2}$$

$$\therefore \int_0^\infty e^{-ax}\frac{\sin mx}{x}\,dx = \frac{\pi}{2} - \tan^{-1}\frac{a}{m} = \tan^{-1}\frac{m}{a}$$

[From the above figure, it is clear that $\phi = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \tan^{-1}\frac{a}{m} = \tan^{-1}\frac{m}{a}$.]

For deduction, putting $a = 1$, $m = 1$, we get

$$\int_0^\infty e^{-x} \cdot \frac{\sin x}{x}\,dx = \tan^{-1}1 = \frac{\pi}{4}.$$

Corollary : Prove that **(M.U. 2006, 09)**

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin mx\,dx = \tan^{-1}\left(\frac{m}{a}\right) - \tan^{-1}\left(\frac{m}{b}\right)$$

Sol. : Changing a to b in Ex. (3) and then by subtraction, we get the required result.

Example 4 (c) : Show that $\int_0^\infty \frac{\log(1+ax^2)}{x^2}\,dx = \pi\sqrt{a}$, $(a > 0)$. **(M.U. 2015)**

Hence, evaluate $\int_0^\infty \frac{\log(1+x^2)}{x^2}\,dx$. **(M.U. 1997, 2007, 10, 11, 13)**

Sol. : Let $I(a)$ be the given integral. Then by the rule of differentiation under the integral sign

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\,dx = \int_a^\infty \frac{1}{x^2} \cdot \frac{1}{1+ax^2} \cdot x^2\,dx = \int_0^\infty \frac{dx}{1+ax^2}$$

$$= \frac{1}{a}\int_0^\infty \frac{dx}{(1/a)+x^2} = \frac{1}{a}\cdot(\sqrt{a})\left[\tan^{-1} x\sqrt{a}\right]_0^\infty = \frac{1}{\sqrt{a}}\cdot\frac{\pi}{2}$$

$$\therefore \quad \frac{dI}{da} = \frac{\pi}{2\sqrt{a}}$$

Integrating both sides, $\quad I(a) = \frac{\pi}{2}\int \frac{da}{\sqrt{a}} = \pi\sqrt{a} + c$

To find c, put $a = 0$, $\quad \therefore \quad I(0) = c$.

But $\quad I(0) = \int_0^\infty 0\,dx = 0 \quad \therefore\ c = 0 \quad \therefore\ I = \pi\sqrt{a}$

$$\therefore \quad \int_0^\infty \frac{\log(1+ax^2)}{x^2}\,dx = \pi\sqrt{a}.$$

For deduction, putting $a = 1$, we get,

$$\int_0^\infty \frac{\log(1+x^2)}{x^2}\,dx = \pi.$$

Example 5 (c) : Evaluate $\int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right)dx$. **(M.U. 2000, 06, 11)**

Sol. : Let $I(a)$ be the given integral.

Then by the rule of differentiation under integral sign

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\,dx = \int_0^\infty \frac{e^{-x}}{x}\left(1 - \frac{x}{x}e^{-ax}\right)dx = \int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx$$

(See that this is the same as Ex. 2)

Applying the rule again,

$$\frac{d^2I}{da^2} = \int_0^\infty \frac{e^{-x}}{x}\cdot xe^{-ax}\,dx = \int_0^\infty e^{-x(1+a)}\,dx = \left[\frac{e^{-x(1+a)}}{-(1+a)}\right]_0^\infty = \frac{1}{1+a}$$

Integrating both sides w.r.t. a, $\quad \frac{dI}{da} = \log(1+a) + c$

To find c, we put $a = 0 \quad \therefore \quad \frac{dI}{da}(0) = \log 1 + c = c$

But $\quad \frac{dI}{da}(0) = \int_0^\infty \frac{e^{-x}}{x}(1-1)\,dx = \int_0^\infty 0\,dx = 0 \quad \therefore\ c = 0$

$$\therefore \quad \frac{dI}{da} = \log(1+a)$$

Integrating again $I = \int \log(1+a)\, da$ .... (A)

Now, integrating by parts

$$I = \log(1+a) \cdot a - \int a \cdot \frac{1}{1+a}\, da \qquad \left[\because \frac{a}{1+a} = \frac{a+1-1}{1+a} = 1 - \frac{1}{1+a}\right]$$

$$= a\log(1+a) - \int da + \int \frac{da}{1+a}$$

$$= a\log(1+a) - a + \log(1+a) + c.$$

To find c we put $a = 0$ $\quad \therefore \; I(0) = c.$

But from (A), $\quad I(0) = \int \log(1)\, da = \int 0\, da = 0 \quad \therefore \; c = 0$

$$\therefore \; I = a\log(1+a) - a + \log(1+a)$$

$$= (1+a)\log(1+a) - a.$$

Example 6 (c) : Show that $\displaystyle\int_0^{\pi} \frac{\log(1 + a\cos x)}{\cos x}\, dx = \pi \sin^{-1} a, \quad 0 \le a \le 1.$ **(M.U. 1998, 2004)**

Hence, evaluate $\displaystyle\int_0^{\pi} \frac{\log(1+\cos x)}{\cos x}\, dx.$

Sol. : Let $I(a)$ be the given integral. By the rule of differentiation under integral sign.

$$\frac{dI}{da} = \int_0^{\pi} \frac{\partial f}{\partial a}\, dx = \int_0^{\pi} \frac{1}{\cos x} \cdot \frac{\cos x}{1 + a\cos x}\, dx = \int_0^{\pi} \frac{dx}{1 + a\cos x}.$$

Put $\quad t = \tan\dfrac{x}{2}, \quad dx = \dfrac{2\,dt}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}.$

When $x = 0,\ t = 0$; when $x = 0,\ t = \tan \pi/2 = \infty.$

$$\therefore \; \frac{dI}{da} = \int_0^{\infty} \frac{1}{1 + a \cdot \left(\dfrac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2} = \int_0^{\infty} \frac{2\,dt}{(1+t^2) + a(1-t^2)}$$

$$= \int_0^{\infty} \frac{2\,dt}{(1+a) + (1-a)t^2} = \frac{1}{1-a}\int_0^{\infty} \frac{2\,dt}{[(1+a)/(1-a)] + t^2}$$

$$= \frac{2}{1-a} \cdot \sqrt{\frac{1-a}{1+a}} \cdot \left[\tan^{-1}\sqrt{\frac{1-a}{1+a}} \cdot t\right]_0^{\infty} = \frac{2}{\sqrt{1-a^2}} \cdot \frac{\pi}{2}$$

$$\therefore \; \frac{dI}{da} = \frac{\pi}{\sqrt{1-a^2}} = \frac{\pi}{\sqrt{1-a^2}}$$

Integrating both sides w.r.t. a, we get $\quad I = \pi \sin^{-1} a + c.$

To find c, we put $a = 0$. $\quad \therefore \quad I(0) = \sin^{-1} 0 + c = 0 + c = c.$

But $\quad I(0) = \int_0^{\pi} 0\, dx = 0 \quad \therefore \; c = 0 \; \therefore \; I = \pi \sin^{-1} a$

$$\therefore \; \int_0^{\pi} \frac{\log(1 + a\cos x)}{\cos x}\, dx = \pi \sin^{-1} a.$$

For deduction, putting $a = 1$, we get

$$\int_0^{\pi} \frac{\log(1+\cos x)}{\cos x}\, dx = \pi \sin^{-1}(1) = \frac{\pi^2}{2}.$$

Example 7 (c) : Show that $\int_0^{\pi} \frac{\log(1+\sin\alpha\cos x)}{\cos x}\,dx = \pi\alpha$.

Sol. : Let $I(\alpha)$ be the given integral. By the rule of differentiation under integral sign.

$$\frac{dI}{d\alpha} = \int_0^{\pi} \frac{\cos\alpha\cos x}{(1+\sin\alpha\cos x)\cos x}\,dx = \int_0^{\pi} \frac{\cos\alpha}{1+\sin\alpha\cos x}\,dx$$

Put $t = \tan\frac{x}{2}$ $\quad\therefore\; dx = \frac{2\,dt}{\sec^2(x/2)} = \frac{2\,dt}{1+t^2}$

And $\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-t^2}{1+t^2}$

When $x = 0,\ t = 0$; when $x = \pi,\ t = \tan\pi/2 = \infty$.

$$\therefore\quad \frac{dI}{d\alpha} = \int_0^{\infty} \frac{\cos\alpha}{1+\sin\alpha\cdot\frac{(1-t^2)}{(1+t^2)}}\cdot\frac{2\,dt}{1+t^2}$$

$$= 2\cos\alpha\int_0^{\infty}\frac{dt}{(1+\sin\alpha)+(1-\sin\alpha)\cdot t^2}$$

$$= \frac{2\cos\alpha}{(1-\sin\alpha)}\int_0^{\infty}\frac{dt}{[(1+\sin\alpha)/(1-\sin\alpha)]+t^2}$$

$$= \frac{2\cos\alpha}{(1-\sin\alpha)}\cdot\frac{\sqrt{1-\sin\alpha}}{\sqrt{1+\sin\alpha}}\cdot\left[\tan^{-1}\left\{\frac{\sqrt{1-\sin\alpha}}{\sqrt{1+\sin\alpha}}\right\}\cdot t\right]_0^{\infty}$$

$$\therefore\quad \frac{dI}{d\alpha} = \frac{2\cos\alpha}{\sqrt{1-\sin^2\alpha}}\cdot\frac{\pi}{2} = \pi$$

Integrating both sides w.r.t. α, we get $\quad I = \pi\alpha + c$.

To find c, we put $\alpha = 0$. $\quad\therefore\; I(0) = \pi\cdot 0 + c = c$.

But $\quad I(0) = \int_0^{\pi}\log 1\cdot dx = \int_0^{\pi} 0\,dx = 0 \quad\therefore\; c = 0$

$$\therefore\quad I = \pi\alpha \qquad \therefore\quad \int_0^{\pi}\frac{\log(1+\sin\alpha\cos x)}{\cos x}\,dx = \pi\alpha.$$

Example 8 (c) : Prove that $\int_0^{\pi/2}\frac{\log(1+\cos\alpha\cos x)}{\cos x}\,dx = \frac{\pi^2}{8} - \frac{\alpha^2}{2}$. **(M.U. 1995, 2004)**

Sol. : Let $I(\alpha)$ be the given integral. By the rule of differentiation under integral sign.

$$\frac{dI}{d\alpha} = \int_0^{\infty}\frac{\partial f}{\partial\alpha}\,dx = \int_0^{\pi/2}\frac{-\sin\alpha\cos x}{(1+\cos\alpha\cos x)\cos x}\,dx = \int_0^{\pi/2}\frac{-\sin\alpha}{1+\cos\alpha\cos x}\,dx$$

Put $t = \tan\frac{x}{2}$ $\quad\therefore\; dx = \frac{2\,dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

When $x = 0,\ t = 0$; when $x = \pi/2,\ t = 1$.

$$\therefore\quad \frac{dI}{d\alpha} = \int_0^{1}\frac{-\sin\alpha}{1+\cos\alpha\cdot\frac{(1-t^2)}{(1+t^2)}}\cdot\frac{2\,dt}{1+t^2} = -\sin\alpha\int_0^1\frac{2\,dt}{(1+t^2)+\cos\alpha\,(1-t^2)}$$

$$= -\sin\alpha \int_0^1 \frac{2\,dt}{(1+\cos\alpha)+(1-\cos\alpha)\,t^2}$$

$$= \frac{-2\sin\alpha}{(1-\cos x)} \int_0^1 \frac{dt}{[(1+\cos\alpha)/(1-\cos\alpha)]+t^2}$$

$$= \frac{-2\sin\alpha}{(1-\cos x)} \cdot \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}} \left[\tan^{-1}\left\{\frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}\right\} \cdot t\right]_0^1$$

$$= \frac{-2\sin\alpha}{\sqrt{1-\cos\alpha}\,\sqrt{1+\cos\alpha}} \left[\tan^{-1}\left\{\frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}\right\}\right]$$

$$= \frac{-2\sin\alpha}{\sqrt{1-\cos^2\alpha}} \left[\tan^{-1}\sqrt{\frac{2\sin^2(\alpha/2)}{2\cos^2(\alpha/2)}}\right]$$

$$\therefore \quad \frac{dI}{d\alpha} = -2\tan^{-1}\tan\frac{\alpha}{2} = -2\,\frac{\alpha}{2} = -\alpha$$

Integrating both sides w.r.t. α, $\quad I = -\dfrac{\alpha^2}{2} + c$ .... (1)

To find c, we put $\alpha = \pi/2$. **(Note this)**

$$\therefore \quad I\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{8} + c$$

Now, putting $\alpha = \pi/2$ in the given integral,

$$I\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} 0 \cdot dx = 0 \qquad \therefore \ c = \frac{\pi^2}{8}$$

Hence, from (1), $\quad I = -\dfrac{\alpha^2}{2} + \dfrac{\pi^2}{8} = \dfrac{\pi^2}{8} - \dfrac{\alpha^2}{2}$.

Example 9 (c) : Prove that $\displaystyle\int_0^\infty e^{-x^2}\cos ax\,dx = \frac{\sqrt{\pi}}{2}\,e^{-a^2/4}$. (Given $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.)

Hence, evaluate $\displaystyle\int_0^\infty e^{-x^2}\cos 2x\,dx$.

Sol. : Let $I(a) = \displaystyle\int_0^\infty e^{-x^2}\cos ax\,dx$.

By the rule of differentiation under the integral sign.

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\,dx = \int_0^\infty -e^{-x^2}\sin ax \cdot (x)\,dx$$

$$= \int_0^\infty [\sin ax][e^{-x^2} \cdot (-x)]\,dx$$

Integrating by parts,

$$\frac{dI}{da} = \left[\sin ax\left(\frac{e^{-x^2}}{2}\right)\right]_0^\infty - \int_0^\infty \frac{e^{-x^2}}{2} \cdot \cos ax\,a\,dx$$

$$\therefore \quad \frac{dI}{da} = 0 - \frac{a}{2}\int_0^\infty e^{-x^2}\cos ax\,dx = -\frac{a}{2}\,I$$

This is a differential equation of variable separable type.

$$\therefore \quad \frac{dI}{I} = -\frac{a}{2}\,da \qquad \therefore \quad \log I = -\frac{a^2}{4} + c' = -\frac{a^2}{4} + \log c$$

$$\therefore \quad \log\frac{I}{c} = -\frac{a^2}{4} \qquad \therefore \quad I = c\,e^{-a^2/4}$$

To find c, we put $a = 0$. $\quad \therefore \quad I(0) = c$.

But $\quad I(0) = \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ [By data] $\quad \therefore \quad c = \frac{\sqrt{\pi}}{2}$

$$\therefore \quad \int_0^{\infty} e^{-x^2} \cos ax\,dx = \frac{\sqrt{\pi}}{2}\,e^{-a^2/4}.$$

For deduction, putting $a = 2$, we get

$$\int_0^{\infty} e^{-x^2} \cos 2x\,dx = \frac{\sqrt{\pi}}{2e}.$$

Example 10 (c) : Prove that $\int_0^{\infty} \frac{1-\cos mx}{x} \cdot e^{-x}\,dx = \frac{1}{2}\log(m^2+1)$. **(M.U. 1998)**

Hence, deduce that $\int_0^{\infty} \frac{1-\cos x}{x} \cdot e^{-x}\,dx = \log\sqrt{2}$.

Sol. : Let $I = \int_0^{\infty} \frac{1-\cos mx}{x}\,e^{-x} \cdot dx$.

By the rule of differentiation under the integral sign.

$$\frac{dI}{dm} = \int_0^{\infty} \frac{e^{-x}}{x}(x \sin mx)\,dx = \int_0^{\infty} e^{-x} \sin mx\,dx$$

$$= \left[\frac{e^{-x}}{m^2+1}(-\sin x - m\cos mx)\right]_0^{\infty}$$

$$\left[\because \int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)\right]$$

$$\therefore \quad \frac{dI}{dm} = \frac{1}{m^2+1}\left[0-(0-m)\right] = \frac{m}{m^2+1}$$

By Integration, $\quad \therefore \quad I = \frac{1}{2}\log(m^2+1) + c$

To find c, we put $m = 0$. $\quad \therefore \quad I(0) = c$.

But $\quad I(0) = \int_0^{\infty} \frac{1-1}{x}\,dx = \int_0^{\infty} 0\,dx = 0 \qquad \therefore \quad c = 0.$

$$\therefore \quad \int_0^{\infty} \frac{1-\cos mx}{x}\,e^{-x} \cdot dx = \frac{1}{2}\log(m^2+1).$$

For deduction, putting $m = 1$, we get

$$\int_0^{\infty} \frac{1-\cos x}{x} \cdot e^{-x}\,dx = \frac{1}{2}\log 2 = \log\sqrt{2}.$$

Example 11 (c) : Show that $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\,dx = \frac{\pi}{2}\log(1+a)$. **(M.U. 1996, 2003, 04, 12)**

Hence, evaluate $\int_0^\infty \frac{\tan^{-1} x}{x(1+x^2)}\,dx$.

Sol. : Let $I(a)$ be the given integral. Then by the rule of differentiation under integral sign.

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\,dx = \int_0^\infty \frac{1}{x(1+x^2)}\cdot\frac{x}{(1+a^2x^2)}\,dx$$

$$= \frac{1}{1-a^2}\int_0^\infty \left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right]dx \qquad \text{[By partial fractions]}$$

$$= \frac{1}{1-a^2}\left[\tan^{-1} x - a\tan^{-1} ax\right]_0^\infty$$

$$= \frac{1}{1-a^2}\left[\frac{\pi}{2} - a\cdot\frac{\pi}{2}\right] = \frac{\pi}{2}\cdot\frac{1}{1+a}$$

Integrating both sides w.r.t. a, $\quad I = \frac{\pi}{2}\log(1+a) + c$

To find c, we put $a = 0$ $\quad \therefore\ I(0) = \frac{\pi}{2}\log(1) + c = c$

But $\quad I(0) = \int_0^\infty \frac{\tan^{-1} 0}{x(1+x^2)}\,dx = \int_0^\infty 0\,dx = 0 \quad \therefore\ c = 0.$

$$\therefore\ I = \frac{\pi}{2}\log(1+a).$$

For deduction, putting $a = 1$, we get,

$$\int_0^\infty \frac{\tan^{-1} x}{x(1+x^2)}\,dx = \frac{\pi}{2}\log 2.$$

Example 12 (c) : Show that $\int_0^\infty e^{-(x^2+a^2/x^2)}\,dx = \frac{\sqrt{\pi}}{2}e^{-2a}$, $a > 0$. (Given $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.) **(M.U. 1991, 97, 2007, 08)**

Hence, evaluate $\int_0^\infty e^{-(x^2+(1/x^2))}\,dx$.

Sol. : Let $I(a) = \int_0^\infty e^{-(x^2+a^2/x^2)}\,dx$, then by the rule of differentiation under integral sign.

$$\therefore\ \frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\,dx = \int_0^\infty e^{-(x^2+a^2/x^2)}\cdot\left(-\frac{2a}{x^2}\right)dx$$

Putting $\frac{a}{x} = y$, $-\frac{a}{x^2}\,dx = dy$

$$\frac{dI}{da} = \int_\infty^0 e^{-(y^2+a^2/y^2)}\cdot 2\,dy = -2\int_0^\infty e^{-(y^2+a^2/y^2)}\cdot dy = -2I$$

This is a differential equation of variable separable type.

$$\therefore\ \frac{dI}{I} = -2\,da$$

Integrating we get, $\log I(a) = -2a + \log c \quad \therefore \; I = c\,e^{-2a}$.

To find c, put $a = 0$, $\quad \therefore \; I(0) = c$

But $\quad I(0) = \int_0^{\infty} e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2} \qquad$ [By data]

$$\therefore \; c = \frac{\sqrt{\pi}}{2} \qquad \therefore \; I = \frac{\sqrt{\pi}}{2} e^{-2a}.$$

For deduction, putting $a = 1$, we get

$$\int_0^{\infty} e^{-[x^2 + (1/x^2)]} dx = \frac{\sqrt{\pi}}{2e^2}.$$

Example 13 (c) : Prove that

$$\int_0^{\pi/2} \frac{\log(1 + a\sin^2 x)}{\sin^2 x} dx = \pi[\sqrt{a+1} - 1], \quad a > -1.$$

(M.U. 1992, 2002, 03, 15)

Sol. : Let $\quad I(a) = \int_0^{\pi/2} \dfrac{\log(1 + a\sin^2 x)}{\sin^2 x} dx$

By the rule of differentiation under the integral sign.

$$\therefore \; \frac{dI}{da} = \int_0^{\infty} \frac{\partial f}{\partial a} dx = \int_0^{\pi/2} \frac{1}{1 + a\sin^2 x} \cdot \frac{\sin^2 x}{\sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{1}{1 + a\sin^2 x} dx \qquad \text{[Dividing by } \cos^2 x \text{]}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + a\tan^2 x} dx = \int_0^{\pi/2} \frac{\sec^2 x}{1 + (1+a)\tan^2 x} dx$$

$$= \frac{1}{a+1} \int_0^{\pi/2} \frac{\sec^2 x}{[\sqrt{1/(1+a)}\,]^2 + \tan^2 x} dx$$

Putting $t = \tan x$,

$$\frac{dI}{da} = \frac{1}{a+1} \int_0^{\infty} \frac{dt}{[\sqrt{1/(1+a)}\,]^2 + t^2}$$

$$= \frac{1}{a+1} \left[\sqrt{a+1} \cdot \tan^{-1}(t\sqrt{a+1}) \right]_0^{\infty} = \frac{1}{\sqrt{a+1}} \cdot \frac{\pi}{2}$$

Integrating w.r.t. a, we get,

$$I = \frac{\pi}{2} \int \frac{da}{\sqrt{a+1}} = \pi\sqrt{a+1} + c.$$

Putting $a = 0$, we get, $I(0) = \pi + c$

But $\quad I(0) = \int_0^{\pi/2} \dfrac{\log(1)}{\sin^2 x} dx = 0 \qquad \therefore \; c = -\pi$

$$\therefore \; I = \pi\sqrt{a+1} - \pi = \pi[\sqrt{a+1} - 1].$$

Example 14 (c) : Prove that $\int_0^{\infty} \frac{e^{-x} - e^{-\alpha x}}{x \sec x} dx = \frac{1}{2}\log\left(\frac{\alpha^2+1}{2}\right)$. **(M.U. 2005, 16)**

Sol. : Let $I(\alpha) = \int_0^{\infty} \frac{e^{-x} - e^{-\alpha x}}{x \sec x} dx$, then by the rule of differentiation under integral sign.

$$\therefore \quad \frac{dI}{d\alpha} = \int_0^{\infty} \frac{\partial f}{\partial \alpha} dx = \int_0^{\infty} \frac{e^{-\alpha x} \cdot x}{x \sec x} dx = \int_0^{\infty} e^{-\alpha x} \cos x \, dx$$

$$= \frac{1}{\alpha^2+1}\left[e^{-\alpha x}(-\alpha \cos x + \sin x)\right]_0^{\infty} = \frac{\alpha}{\alpha^2+1}$$

$$\therefore \quad dI = \frac{\alpha}{\alpha^2+1} \cdot d\alpha$$

Integrating w.r.t α, we get, $\quad I = \frac{1}{2}\log(\alpha^2+1) + c$

To find c, put $\alpha = 1$ [**Note this**] $\quad \therefore \; I(1) = \frac{1}{2}\log 2 + c$

But $\quad I(1) = \int_0^{\infty} \frac{0}{x \sec x} dx = 0$

$$0 = \frac{1}{2}\log(2) + c \quad \therefore \; c = -\frac{1}{2}\log 2$$

$$\therefore \quad I = \frac{1}{2}\log(\alpha^2+1) - \frac{1}{2}\log 2 = \frac{1}{2}\log\left(\frac{\alpha^2+1}{2}\right).$$

Corollary : Changing α to β, we get

$$\int_0^{\infty} \frac{e^{-x} - e^{-\beta x}}{x \sec x} = \frac{1}{2}\log\left(\frac{\beta^2+1}{2}\right)$$

Then by subtraction, we get

$$\int_0^{\infty} \frac{e^{-\alpha x} + e^{-\beta x}}{x \sec x} dx = \frac{1}{2}\log\left(\frac{\alpha^2+1}{\beta^2+1}\right).$$

Example 15 (c) : Show that **(M.U. 2002)**

$$\int_0^{\pi} \log(1 - a\cos x)\, dx = \pi \log\left[\frac{1+\sqrt{1-a^2}}{2}\right], |a| < 1.$$

Sol. : Let $I(a) = \int_0^{\pi} \log(1 - a\cos x)\, dx$, then by the rule of differentiation under integral sign.

$$\therefore \quad \frac{dI(a)}{da} = \int_0^{\pi} \frac{\partial f}{\partial a} dx = \int_0^{\pi} \frac{-\cos x}{1 - a\cos x} dx$$

$$= \frac{1}{a}\int_0^{\pi} \frac{-a\cos x}{1 - a\cos x} dx = \frac{1}{a}\int_0^{\pi} \frac{(1 - a\cos x) - 1}{1 - a\cos x} dx$$

$$= \frac{1}{a}\int_0^{\pi}\left(1 - \frac{1}{1 - a\cos x}\right) dx = \frac{1}{a}\left[\pi - \int_0^{\pi} \frac{dx}{1 - a\cos x}\right]$$

To evaluate the integral, put $t = \tan\left(\frac{x}{2}\right)$

$$\therefore \int_0^{\pi} \frac{dx}{1-a\cos x} = \int_0^{\infty} \frac{1}{1-a\cdot\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2}$$

$$= \int_0^{\infty} \frac{2\,dt}{(1-a)+(1+a)\,t^2} = \frac{2}{(1+a)} \int_0^{\infty} \frac{dt}{\left(\frac{1-a}{1+a}\right)+t^2}\,dt$$

$$\therefore \int_0^{\pi} \frac{dx}{1-a\cos x} = \frac{2}{(1+a)}\sqrt{\frac{1+a}{1-a}}\left[\tan^{-1}\left(t\cdot\sqrt{\frac{1+a}{1-a}}\right)\right]_0^{\infty} = \frac{\pi}{\sqrt{1-a^2}}$$

$$\therefore \frac{d\,I(a)}{da} = \frac{1}{a}\left[\pi - \frac{\pi}{\sqrt{1-a^2}}\right] = \pi\left[\frac{1}{a} - \frac{1}{a\sqrt{1-a^2}}\right]$$

Integrating w.r.t. a, we get,

$$I(a) = \pi\left[\log a - \int \frac{da}{a\sqrt{1-a^2}}\right] + c$$

To find the integral put $a = \sin\theta$.

$$\therefore \int \frac{da}{a\sqrt{1-a^2}} = \int \frac{\cos\theta}{\sin\theta\cdot\cos\theta}\,d\theta = \int \operatorname{cosec}\theta\,d\theta$$

$$= \log(\operatorname{cosec}\theta - \cot\theta) = \log\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \log\left(\frac{1-\sqrt{1-a^2}}{a}\right) \qquad [\because \sin\theta = a]$$

$$\therefore I(a) = \pi\left[\log a - \log\left(\frac{1-\sqrt{1-a^2}}{a}\right)\right] + c = \pi\cdot\log\left(\frac{a^2}{1-\sqrt{1-a^2}}\right) + c$$

$$= \pi\cdot\log\left(\frac{a^2}{1-\sqrt{1-a^2}}\cdot\frac{1+\sqrt{1-a^2}}{1+\sqrt{1-a^2}}\right) + c$$

$$= \pi\cdot\log(1+\sqrt{1-a^2}) + c$$

To find c put $a = 0$ $\quad\therefore\ I(0) = \pi\log 2 + c$

But $I(0) = \int_0^{\pi} \log 1\,dx = 0$ [By data] $\quad\therefore\ c = -\pi\log 2$

$$\therefore I(a) = \pi\cdot\left[\log(1+\sqrt{1-a^2}) - \log 2\right] = \pi\cdot\log\left(\frac{1+\sqrt{1-a^2}}{2}\right).$$

Corollary : Putting $a = \cos\alpha$, we get

$$\int_0^{\pi} \log(1-\cos\alpha\cos\alpha)\,dx = \pi\log\left[\frac{1+\sin\alpha}{2}\right].$$

Example 16 (c) : Verify the rule of differentiation under the integral sign for $\int_0^\infty e^{-at} \sin bt\, dt$ where a is a parameter. **(M.U. 2002, 07)**

Sol. : If we denote the given integral by I we have to show that

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\, dx \qquad \text{.............. (1)}$$

(a) Now $I = \int_0^\infty e^{-at} \sin bt\, dt = \frac{1}{a^2+b^2}\left[e^{-at}(-a\sin bt - b\cos bt)\right]_0^\infty$

$$\therefore \quad I = \frac{1}{a^2+b^2}[b]$$

$$\therefore \quad \text{l.h.s.} = \frac{dI}{da} = b\left[-\frac{1}{(a^2+b^2)^2}\right] 2a = -\frac{2ab}{(a^2+b^2)^2} \qquad \text{.............. (2)}$$

(b) Now, r.h.s. $= \int_0^\infty \frac{\partial f}{\partial a}\, dt = \int_0^\infty \frac{\partial}{\partial a}(e^{-at}\sin bt)\, dt$

$$= \int_0^\infty e^{-at}(-t)\sin bt\, dt = -\int_0^\infty t(e^{-at}\sin bt)\, dt$$

By integrating by parts,

$$= -\left[\left(t \cdot \frac{e^{-at}}{a^2+b^2}(-a\sin bt - b\cos bt)\right)_0^\infty - \int_0^\infty \frac{e^{-at}}{a^2+b^2}(-a\sin bt - b\cos bt)\cdot(1)\cdot dt\right]$$

$$= 0 - \frac{a}{(a^2+b^2)}\int_0^\infty e^{-at}\sin bt\, dt - \frac{b}{a^2+b^2}\int_0^\infty e^{-at}\cos bt\, dt$$

$$= -\frac{a}{(a^2+b^2)}\left[\frac{e^{-at}}{(a^2+b^2)}(-a\sin bt - b\cos bt)\right]_0^\infty - \frac{b}{(a^2+b^2)}\left[\frac{e^{-at}}{(a^2+b^2)}(-a\cos bt + b\sin bt)\right]_0^\infty$$

$$= -\frac{ab}{(a^2+b^2)^2} - \frac{ab}{(a^2+b^2)^2} = -\frac{2ab}{(a^2+b^2)^2} \qquad \text{.............. (3)}$$

From (2) and (3) the rule is verified.

Example 17 (c) : Evaluate $\int_0^{\pi/2} \frac{dx}{1+a\cos^2 x}$ and hence, deduce that **(M.U. 1998, 2012)**

$$\int_0^{\pi/2} \frac{\cos^2 x}{(3+\cos^2 x)^2}\, dx = \frac{\pi\sqrt{3}}{96}.$$

Sol. : We have by dividing N and D by $\cos^2 x$.

$$\int_0^{\pi/2} \frac{dx}{1+a\cos^2 x} = \int_0^{\pi/2} \frac{\sec^2 x}{a+\sec^2 x}\, dx = \int_0^{\pi/2} \frac{\sec^2 x}{(1+a)+\tan^2 x}\, dx$$

Now put $t = \tan x$ $\quad \therefore \sec^2 x\, dx = dt$

When $x = 0, t = 0$; when $x = \pi/2, t = \infty$.

$$\therefore \int_0^{\pi/2} \frac{dx}{1 + a\cos^2 x} = \int_0^{\infty} \frac{dt}{(1+a) + t^2} = \frac{1}{\sqrt{1+a}}\left[\tan^{-1}\frac{t}{\sqrt{1+a}}\right]_0^{\infty} = \frac{\pi}{2\sqrt{1+a}}$$

This completes the first part.

Now, let I denote the given integral.

$$\therefore I = \frac{\pi}{2\sqrt{1+a}} \quad \therefore \frac{dI}{da} = \frac{\pi}{2}\cdot\left(-\frac{1}{2}\right)(1+a)^{-3/2} = -\frac{\pi}{4}(1+a)^{-3/2} \qquad \text{.............. (1)}$$

But $I = \int_0^{\pi/2} \frac{dx}{1 + a\cos^2 x}$

$\therefore$ By the rule of differentiation under the integral sign.

$$\frac{dI}{da} = \int_0^{\pi/2} \frac{\partial f}{\partial a} dx = \int_0^{\pi/2} -\frac{1}{(1 + a\cos^2 x)^2}\cdot \cos^2 x\, dx$$

$$\therefore \frac{dI}{da} = -\int_0^{\pi/2} \frac{\cos^2 x}{(1 + a\cos^2 x)^2} dx \qquad \text{.............. (2)}$$

From (1) and (2), we get,

$$\int_0^{\pi/2} \frac{\cos^2 x}{(1 + a\cos^2 x)^2} dx = \frac{\pi}{4}(1+a)^{-3/2}$$

Now, put $a = 1/3$.

$$\therefore \int_0^{\pi/2} \frac{\cos^2 x}{\left\{1 + (\cos^2 x)/3\right\}^2} dx = \frac{\pi}{4}\left(1 + \frac{1}{3}\right)^{-3/2}$$

$$\therefore \int_0^{\pi/2} \frac{9\cos^2 x}{(3 + \cos^2 x)^2} dx = \frac{\pi}{4}\left(\frac{3}{4}\right)^{3/2} \qquad \therefore \int_0^{\pi/2} \frac{\cos^2 x}{(3 + \cos^2 x)^2} dx = \frac{\pi\sqrt{3}}{96}.$$

EXERCISE - I

Assuming the validity of differentiation under the integral sign prove that : Class (c) : 8 Marks

1. $\int_0^{\infty} \frac{e^{-\alpha x}\sin x}{x} dx = \cot^{-1}\alpha$. Deduce that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$. **(M.U. 1997, 2004)**

2. $\int_0^{\infty} e^{-x^2}\cos 2ax\, dx = \frac{\sqrt{\pi}}{2} e^{-a^2}$. (Assume that $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$) **(M.U. 1995)**

3. $\int_0^{\infty} \frac{1 - \cos ax}{x^2} dx = \frac{\pi a}{2}$. (Assume that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$)

4. $\int_0^{\pi} \frac{dx}{a - \cos x} = \frac{\pi}{\sqrt{a^2 - 1}}$, $(a > 0)$. Deduce that

   (i) $\int_0^{\pi} \frac{dx}{(a - \cos x)^2} = \frac{\pi a}{(a^2 - 1)^{3/2}}$ **(M.U. 1994)**

   (ii) $\int_0^{\pi} \frac{dx}{(2 - \cos x)^2} = \frac{2\pi}{3\sqrt{3}}$ **(M.U. 1996, 2005)**

5. $\int_0^\infty \frac{\log(1+ax^2)}{x^2}\,dx = \pi\sqrt{a}, \quad (a>0).$ **(M.U. 1997)**

Also deduce that $\int_0^\infty \frac{\log(1+x^2)}{x^2}\,dx = \pi.$

6. $\int_0^\pi \frac{\log(1+\alpha\cos x)}{\cos x}\,dx = \pi\sin^{-1}\alpha, \quad |\alpha|<1.$

7. $\int_0^x \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)}$

(**Hint :** $\int_0^x \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$. Differentiate both sides w.r.t. a.)

8. $\int_0^\infty x e^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}.$ (**Hint :** See solved Ex. 18.) **(M.U. 2004)**

9. Verify the rule of differentiation under the integral sign for $\int_0^\infty e^{-at}\cos bt\,dt$. **(M.U. 2002)**

10. Evaluate $\int_0^{\pi/2} \frac{dx}{1+a\sin^2 x}$ and deduce that $\int_0^{\pi/2} \frac{\sin^2 x}{(3+\sin^2 x)^2}\,dx = \frac{\pi\sqrt{3}}{96}.$

(B) Solved Examples on two parameters : Class (c) : 8 Marks

Example 1 (c) : Show that $\int_0^\infty e^{-xy} = \frac{1}{y}, \quad y>0$ and hence, deduce that

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \log\left(\frac{b}{a}\right), \quad a>0,\ b>0.$$ **(M.U. 1991)**

Sol. : We have $\int_0^\infty e^{-xy}\,dx = \left[\frac{e^{-xy}}{-y}\right]_0^\infty = \left[\frac{0-1}{-y}\right] = \frac{1}{y}$ .... (1)

Now, let $I(a) = \int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx$

$\therefore \quad \frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\,dx = \int_0^\infty -\frac{e^{-ax}\cdot x}{x}\,dx = -\int_0^\infty e^{-ax}\,dx = -\frac{1}{a}$ [By (1)]

$\therefore \quad dI = -\frac{da}{a}$

Integrating both sides we get, $I(a) = -\log a + c$

To find c, we put $a=b$ $\quad \therefore \quad I(b) = -\log b + c$

But $\quad I(b) = \int_0^\infty \frac{e^{-bx}-e^{-bx}}{x}\,dx = 0 \quad \therefore \quad c = \log b$

$\therefore \quad I = \log b - \log a = \log(b/a).$

Example 2 (c) : Show that $\int_0^\infty e^{-bx^2}\cos 2ax\,dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{b}\right)}\cdot e^{-a^2/b}, \quad b>0.$

(Assume $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$). **(M.U. 1991, 2002, 03)**

Sol. : Let $I(a)$ be the given integral.

By the rule of differentiation under the integral sign, we have,

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\, dx = \int_0^\infty e^{-bx^2} \cdot (-2x) \sin 2ax\, dx$$

We obtain the integral by integration by parts.

Put $-bx^2 = t \qquad \therefore\ -2\,bx\,dx = dt$

$$\therefore\ \int e^{-bx^2}(-2x)\,dx = \int e^t \frac{dt}{b} = \frac{e^t}{b} = \frac{e^{-bx^2}}{b}$$

$$\therefore\ \frac{dI}{da} = \left[\sin 2ax \cdot \frac{e^{-bx^2}}{b}\right]_0^\infty - \int_0^\infty \frac{e^{-bx^2}}{b} 2a\cos 2ax\, dx$$

$$\therefore\ \frac{dI}{da} = [0-0] - \frac{2a}{b}\int_0^\infty e^{-bx^2}\cos 2ax\, dx$$

$$\therefore\ \frac{dI}{da} = -\frac{2a}{b} I \qquad \therefore\ \frac{dI}{I} = -\frac{2a}{b}\, da$$

Integrating both sides $\quad \log I = -\frac{a^2}{b} + c \qquad \therefore\ I = e^{(-a^2/b)+c}$ .... (1)

To find c, we put $a = 0 \qquad \therefore\ I(0) = e^c.$ .... (2)

But $\quad I(0) = \int_0^\infty e^{-bx^2}\, dx.$

To find the integral, put $\sqrt{b} \cdot x = t.$

$$\therefore\ I(0) = \int_0^\infty e^{-t^2} \frac{dt}{\sqrt{b}} = \frac{1}{\sqrt{b}}\int_0^\infty e^{-t^2}\, dt \quad \text{where } \sqrt{b}\cdot x = t$$

$$= \frac{1}{\sqrt{b}} \cdot \frac{\sqrt{\pi}}{2} \qquad \text{[By data]}$$ .... (3)

$\therefore$ From (2) and (3), $\qquad e^c = \frac{1}{2}\sqrt{\frac{\pi}{b}}$

$\therefore$ From (1), $\quad I = e^{(-a^2/b)+c} = e^{-a^2/b} \cdot e^c = e^{-a^2/b} \cdot \frac{1}{2}\sqrt{\frac{\pi}{b}}.$

Example 3 (c) : Show that $\int_0^\infty \frac{\tan^{-1}(x/a) - \tan^{-1}(x/b)}{x}\, dx = \frac{\pi}{2}\log\left(\frac{b}{a}\right)$ where $a > 0,\ b > a.$

(M.U. 1989)

Sol. : Let $I(a) = \int_0^\infty \frac{\tan^{-1}(x/a) - \tan^{-1}(x/b)}{x}\, dx$

$\therefore$ By the rule of differentiation under the integral sign,

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\, dx = \int_0^\infty \frac{1}{1+(x^2/a^2)}\left(-\frac{x}{a^2}\right)\cdot\frac{1}{x}\, dx$$

$$= -\int_0^\infty \frac{dx}{a^2+x^2} = -\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty = -\frac{\pi}{2a} \qquad \therefore\ dI = -\frac{\pi}{2}\cdot\frac{da}{a}$$

Integrating both sides, we get $I = -\frac{\pi}{2}\log a + c$

To find c, we put $a = b$. $\therefore \quad I(b) = -\frac{\pi}{2}\log b + c$

But $\quad I(a) = \int_0^\infty \frac{\tan^{-1}(x/a) - \tan^{-1}(x/b)}{x}\,dx$

$\therefore \quad I(b) = \int_0^\infty 0\,dx = 0 \qquad \therefore \quad 0 = -\frac{\pi}{2}\log b + c \qquad \therefore \quad c = \frac{\pi}{2}\log b$

$\therefore \quad I = \frac{\pi}{2}\log b - \frac{\pi}{2}\log a = \frac{\pi}{2}\log\frac{b}{a}.$

Example 4 (c) : Prove that $\int_0^\infty \frac{\cos\lambda x}{x}(e^{-ax} - e^{-bx})\,dx = \frac{1}{2}\log\left(\frac{b^2+\lambda^2}{a^2+\lambda^2}\right), \quad (a > 0, b > 0).$

(M.U. 2002, 03, 14)

Sol. : Let $I(a)$ be the given integral. By the rule of differentiation under the integral sign

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\,dx = \int_0^\infty \frac{\cos\lambda x}{x}(-xe^{-ax})\,dx = -\int_0^\infty e^{-ax}\cos\lambda x\,dx$$

$$= -\left[\frac{e^{-ax}}{a^2+\lambda^2}(-a\cos\lambda x + \lambda\sin\lambda x)\right]_0^\infty = -\left[0 - \frac{1}{a^2+\lambda^2}(-a+0)\right]$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{1}{a^2+b^2}\cdot e^{ax}(a\cos bx + b\sin bx)\right]$$

$$\therefore \quad \frac{dI}{da} = -\frac{a}{a^2+\lambda^2} \qquad \therefore \quad dI = -\frac{a}{a^2+\lambda^2}\,da$$

Integrating both sides we get,

$$I = -\frac{1}{2}\log(a^2+\lambda^2) + c$$

To find c we put $a = b$.

$$\therefore \quad I(b) = -\frac{1}{2}\log(b^2+\lambda^2) + c$$

But $\quad I(b) = \int_0^\infty \frac{\cos\lambda x}{x}(e^{-bx} - e^{-bx})\,dx = \int_0^\infty 0\,dx = 0$

$$\therefore \quad c = \frac{1}{2}\log(b^2+\lambda^2)$$

$$\therefore \quad I = -\frac{1}{2}\log(a^2+\lambda^2) + \frac{1}{2}\log(b^2+\lambda^2) = \frac{1}{2}\log\left(\frac{b^2+\lambda^2}{a^2+\lambda^2}\right).$$

Example 5 (c) : By differentiating under the integral sign, prove that

$I(b) = \pi\sin^{-1}\left(\frac{b}{a}\right)$ where $I(b) = \int_0^{\pi/2}\log\left[\frac{a+b\sin x}{a-b\sin x}\right]\cdot\frac{dx}{\sin x}.$ **(M.U. 1999, 2008)**

Sol. : We have $\quad I(b) = \int_0^{\pi/2}[\log(a+b\sin x) - \log(a-b\sin x)]\cdot\frac{dx}{\sin x}$ .... (1)

By the rule of differentiation under the integral sign,

$$\therefore \quad \frac{dI}{db} = \int_0^{\pi/2} \left[\frac{\sin x}{a + b\sin x} + \frac{\sin x}{a - b\sin x} \right] \cdot \frac{dx}{\sin x}$$

$$= \int_0^{\pi/2} \left[\frac{1}{a + b\sin x} + \frac{1}{a - b\sin x} \right] dx$$

$$= \int_0^{\pi/2} \frac{2a}{a^2 - b^2 \sin^2 x} dx = \int_0^{\pi/2} \frac{2a \operatorname{cosec}^2 x}{a^2 \operatorname{cosec}^2 x - b^2} dx$$

Put $\cot x = t$ $\quad \therefore \; -\operatorname{cosec}^2 x \, dx = dt$

and $\operatorname{cosec}^2 x = 1 + \cot^2 x = 1 + t^2$.

When $x = 0, t = \infty$; when $x = \pi/2, t = 0$.

$$\therefore \quad \frac{dI}{dt} = \int_\infty^0 \frac{-2a\,dt}{(1+t^2)a^2 - b^2} = \int_0^\infty \frac{2a\,dt}{(1+t^2)a^2 - b^2} = \int_0^\infty \frac{2a\,dt}{a^2 t^2 + (a^2 - b^2)}$$

$$= \frac{2}{a} \int_0^\infty \frac{dt}{t^2 + (a^2 - b^2)/a^2} = \frac{2}{a} \cdot \frac{a}{\sqrt{a^2 - b^2}} \left[\tan^{-1} \frac{ta}{\sqrt{a^2 - b^2}} \right]_0^\infty = \frac{2}{\sqrt{a^2 - b^2}} \left[\frac{\pi}{2} - 0 \right]$$

$$\therefore \quad \frac{dI}{dt} = \frac{2}{\sqrt{a^2 - b^2}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{a^2 - b^2}} \qquad \therefore \quad dI = \frac{\pi}{\sqrt{a^2 - b^2}} \cdot db$$

Integrating, we get,

$$I(b) = \pi \int \frac{db}{\sqrt{a^2 - b^2}} = \pi \sin^{-1} \left(\frac{b}{a} \right) + c \qquad \text{.............. (2)}$$

To find c, we put $b = 0$ in (2), $\quad \therefore \; I(0) = c$.

Now, putting $b = 0$ in $I(b)$ *i.e.*, in (1), we get

$$I(0) = \int_0^{\pi/2} [\log a - \log a] \frac{dx}{\sin x} = \int_0^{\pi/2} 0 \, dx = 0$$

Hence, $c = 0$. $\quad \therefore$ From (2), we get,

$$I(b) = \pi \sin^{-1} \left(\frac{b}{a} \right).$$

Example 6 (c) : Evaluate $\int_0^\infty \frac{e^{-\beta x} \sin \alpha x}{x} dx$ and hence, deduce that $\int_0^\infty \frac{\sin \alpha x}{x} dx = \frac{\pi}{2}$.

(M.U. 2011)

Sol. : Let $I(\alpha) = \int_0^\infty \frac{e^{-\beta x} \sin \alpha x}{x} dx$ .... (1)

By the rule of differentiation under the integral sign,

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{\partial f}{\partial \alpha} dx = \int_0^\infty \frac{e^{-\beta x} \cos \alpha x}{x} \cdot x \cdot dx$$

$$\therefore \quad \frac{dI}{d\alpha} = \int_0^\infty e^{-\beta x} \cos \alpha x \, dx \qquad \text{.............. (2)}$$

$$\therefore \quad \frac{dI}{d\alpha} = \left[\frac{e^{-\beta x}}{\alpha^2 + \beta^2} (-\beta \cos \alpha x + \alpha \sin \alpha x) \right]_0^\infty = \frac{0 - (-\beta)}{\alpha^2 + \beta^2}$$

$$\therefore \frac{dI}{d\alpha} = \frac{\beta}{\alpha^2+\beta^2} \quad \therefore dI = \frac{\beta}{\alpha^2+\beta^2}\, d\alpha$$

Integrating both sides, we get, $I = \tan^{-1}\dfrac{\alpha}{\beta} + c$ .... (3)

To find c, we put $\alpha = 0$ in (3) $\quad \therefore I(0) = \tan^{-1} 0 + c = c.$

Now, from (1), $I(0) = \int_0^\infty 0\, dx = 0 \quad \therefore c = 0 \quad \therefore I = \tan^{-1}\dfrac{\alpha}{\beta}$

$$\therefore \int_0^\infty \frac{e^{-\beta x}\sin \alpha x}{x}\, dx = \tan^{-1}\frac{\alpha}{\beta} \qquad \text{............... (4)}$$

For the second part we put $\beta = 0$ in (4)

$$\therefore \int_0^\infty \frac{\sin(\alpha x)}{x}\, dx = \tan^{-1}\alpha$$

Further if $\beta = 0$ and if $\alpha < 0$ then from (4)

$$\int_0^\infty \frac{\sin \alpha x}{x}\, dx = \tan^{-1}(-\infty) = -\frac{\pi}{2}$$

If $\alpha = 0$ then $\displaystyle\int_0^\infty \frac{\sin \alpha x}{x}\, dx = \tan^{-1}(0) = 0$.

If $\beta = 0$ and $\alpha > 0$ then $\displaystyle\int_0^\infty \frac{\sin \alpha x}{x}\, dx = \tan^{-1}(\infty) = \frac{\pi}{2}$.

Example 7 (c) : Evaluate $\displaystyle\int_0^{\pi/2} \log(a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta$.

Sol. : Let $I = \displaystyle\int_0^{\pi/2} \log(a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta$ .... (1)

By the rule of differentiation under the integral sign.

$$\therefore \frac{dI}{da} = \int_0^{\pi/2} \frac{\partial f}{\partial a}\, d\theta = \int_0^{\pi/2} \frac{2a\cos^2\theta\, d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \int_0^{\pi/2} \frac{2a}{a^2 + b^2\tan^2\theta}\, d\theta$$

Put $t = \tan\theta \quad \therefore dt = \sec^2\theta\, d\theta = (1+\tan^2\theta)\, d\theta = (1+t^2)\, d\theta$

$$\therefore \frac{dI}{da} = \int_0^\infty \frac{2a}{(a^2+b^2t^2)}\cdot\frac{dt}{(1+t^2)}$$

$$= \frac{2a}{a^2-b^2}\int_0^\infty \left[\frac{1}{1+t^2} - \frac{b^2}{a^2+b^2t^2}\right] dt \qquad \text{[By partial fractions]}$$

$$= \frac{2a}{a^2-b^2}\left[\tan^{-1} t - \frac{b}{a}\tan^{-1}\left(t\cdot\frac{b}{a}\right)\right]_0^\infty$$

$$\therefore \frac{dI}{da} = \frac{2a}{a^2-b^2}\left[\frac{\pi}{2} - \frac{b}{a}\cdot\frac{\pi}{2}\right] = \frac{\pi}{a+b} \qquad \therefore dI = \frac{\pi}{a+b}\cdot da \qquad \text{............... (2)}$$

Integrating w.r.t. a, we get, $I = \pi\log(a+b) + c$

To find c, we put $a = b$ in (1),

$$\therefore I(b) = \int_0^{\pi/2} \log b^2\cdot d\theta = \log b^2\cdot[\theta]_0^{\pi/2} = 2\log b\cdot\frac{\pi}{2} = \pi\cdot\log b$$

And from (2), $I(b) = \pi \log 2b + c \qquad \therefore \ \pi \log b = \pi \log 2b + c$

$$\therefore \ c = \pi \log\left(\frac{b}{2b}\right) = \pi \log\left(\frac{1}{2}\right)$$

$$\therefore \ I = \pi \log(a+b) + \pi \log\left(\frac{1}{2}\right) = \pi \log\left(\frac{a+b}{2}\right).$$

Example 8 (c) : Evaluate $\int_0^\pi \frac{dx}{a + b\cos x}, \ a > 0, \ b > 0$

and deduce that $\int_0^\pi \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$

and $\int_0^\pi \frac{\cos x \, dx}{(a+b\cos x)^2} = -\frac{\pi b}{(a^2-b^2)^{3/2}}$. **(M.U. 2003, 07, 08, 11)**

Sol. : Let $I = \int_0^\pi \frac{dx}{a + b\cos x}$

Putting $t = \tan\frac{x}{2}$, $dt = \frac{1}{2}\sec^2\frac{x}{2} dx \quad \therefore \ dt = \frac{1}{2}(1+t^2)\, dx \quad \therefore \ dx = \frac{2\,dt}{1+t^2}$

When $x = 0, t = 0$; when $x = \pi$, $t = \infty$.

$$\therefore \ I = \int_0^\infty \frac{2\,dt/(1+t^2)}{a + b \cdot \frac{1-t^2}{1+t^2}} = 2\int_0^\infty \frac{dt}{(a+b)+(a-b)t^2} = \frac{2}{a-b}\int_0^\infty \frac{dt}{t^2 + \left(\frac{a+b}{a-b}\right)}$$

$$= \frac{2}{(a-b)}\sqrt{\left(\frac{a-b}{a+b}\right)}\left[\tan^{-1}\left(t \cdot \sqrt{\left(\frac{a-b}{a+b}\right)}\right)\right]_0^\infty$$

$$= \frac{2}{\sqrt{a^2-b^2}} \cdot \frac{\pi}{2} = \frac{2}{\sqrt{a^2-b^2}}$$

$$\therefore \ \int_0^\pi \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \qquad \text{.............. (1)}$$

Now we apply the rule of differentiation under the integral sign. Differentiating both sides w.r.t a, we get,

$$\int_0^\pi \left[\frac{\partial}{\partial a}\left(\frac{1}{a+b\cos x}\right)\right] dx = -\frac{\pi}{2} \cdot (a^2-b^2)^{-3/2} \cdot 2a$$

$$\therefore \ \int_0^\pi -\frac{1}{(a+b\cos x)^2}\, dx = -\frac{\pi a}{(a^2-b^2)^{3/2}} \quad \therefore \ \int_0^\pi \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$$

Again differentiating (1) w.r.t b,

$$\int_0^\pi \left[\frac{\partial}{\partial b}\left(\frac{1}{a+b\cos x}\right)\right] dx = -\frac{\pi}{2}(a^2-b^2)^{3/2} \cdot (-2b)$$

$$\therefore \ \int_0^\pi -\frac{1}{(a+b\cos x)^2} \cdot \cos x \cdot dx = \frac{\pi b}{(a^2-b^2)^{3/2}}$$

$$\therefore \ \int_0^\pi \frac{\cos x}{(a+b\cos x)^2}\, dx = -\frac{\pi b}{(a^2-b^2)^{3/2}}.$$

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Example 9 (c) : Evaluate $\int_0^{\pi} \frac{dx}{a+b\cos x}$, $a > 0$, $|b| < a$ and hence, find

$$\int_0^{\pi} \frac{\cos x}{(a+b\cos x)^2}\, dx \text{ and } \int_0^{\pi} \frac{dx}{(5+3\cos x)^2}.$$ **(M.U. 1998)**

Sol. : We have already proved above the last but one results.

Now, in $\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$ put $a = 5$ and $b = 3$,

$$\therefore \int_0^{\pi} \frac{dx}{(5+3\cos x)^2} = \frac{5\pi}{(25-9)^{3/2}} = \frac{5\pi}{64}.$$

Example 10 (c) : Assuming $\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{a^2-b^2}} \cos^{-1}\left(\frac{b}{a}\right)$, $a > b$,

show that $$\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\, dx = \frac{\pi^2}{8} - \frac{\alpha^2}{2}.$$ **(M.U. 1995)**

Sol. : Let $I(\alpha) = \int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\, dx$ .... (1)

By the rule of differentiation under the integral sign,

$$\therefore \frac{dI(\alpha)}{d\alpha} = \int_0^{\infty} \frac{\partial f}{\partial\alpha}\, dx = \int_0^{\pi/2} \frac{-\cos x \cdot \sin\alpha}{(1+\cos\alpha\cos x)\cos x} \cdot dx$$

$$= -\int_0^{\pi/2} \frac{\sin\alpha}{1+\cos\alpha\cos x}\, dx$$

Putting $a = 1$ and $b = \cos\alpha$ in

$$\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{a^2-b^2}} \cdot \cos^{-1}\left(\frac{b}{a}\right), \text{ we get}$$

$$\int_0^{\pi/2} \frac{dx}{1+\cos\alpha\cos x} = \frac{1}{\sqrt{1-\cos^2\alpha}} \cos^{-1}\cos\alpha = \frac{\alpha}{\sin\alpha}$$

$$\therefore \frac{dI(\alpha)}{d\alpha} = -\int_0^{\pi/2} \frac{\sin\alpha}{1+\cos\alpha\cos x}\, dx = -\sin\alpha \cdot \frac{\alpha}{\sin\alpha} = -\alpha.$$

$$\therefore dI(\alpha) = -\alpha\, d\alpha$$ .... (2)

Integrating both sides we get, $I(\alpha) = -\frac{\alpha^2}{2} + c$

[**Note this**]

To find c, we put $\alpha = \pi/2$ in (1)

$$\therefore I\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\log(1)}{\cos x}\, dx = 0$$

Putting $\alpha = \pi/2$ in (2),

$$\therefore I\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{8} + c \quad \therefore c = \frac{\pi^2}{8} \quad \therefore I(\alpha) = \frac{\pi^2}{8} - \frac{\alpha^2}{2}.$$

Example 11 (c) : Evaluate $\int_0^{\pi/2} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$ and show that,

$$\int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4ab}\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$ **(M.U. 1990, 2000, 02, 10, 14)**

Sol. : Dividing the numerator and denominator of the given integral by $\cos^2 x$, we get,

$$\int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \int_0^{\pi/2} \frac{\sec^2 x\, dx}{a^2 \tan^2 x + b^2}$$

Putting $t = \tan x$

$$I = \frac{1}{a^2} \int_0^{\infty} \frac{dt}{t^2 + (b/a)^2} = \frac{1}{a^2} \cdot \frac{a}{b} \left[\tan^{-1} \frac{t\,a}{b} \right]_0^{\infty} = \frac{\pi}{2ab}$$

$$\therefore \int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2ab} \qquad \text{.............. (1)}$$

Now, we apply the rule of D.U.I.S. Differentiating both sides w.r.t. a,

$$\int_0^{\pi/2} \left[\frac{\partial}{\partial a} \left(\frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} \right) \right] dx = \frac{\pi}{2b} \left(-\frac{1}{a^2} \right)$$

$$\therefore \int_0^{\pi/2} - \frac{2a \sin^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2}\, dx = \frac{\pi}{2b} \left(-\frac{1}{a^2} \right)$$

$$\therefore \int_0^{\pi/2} \frac{\sin^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2}\, dx = \frac{\pi}{4ab} \left(\frac{1}{a^2} \right) \qquad \text{.............. (2)}$$

Differentiating both sides of (1) again w.r.t. b, we get,

$$\therefore \int_0^{\pi/2} \frac{-2b \cos^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2}\, dx = \frac{\pi}{2a} \left(-\frac{1}{b^2} \right)$$

$$\therefore \int_0^{\pi/2} \frac{\cos^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2}\, dx = \frac{\pi}{4ab} \left(\frac{1}{b^2} \right) \qquad \text{.............. (3)}$$

Adding (2) and (3) we get,

$$\int_0^{\pi/2} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi}{4ab} \left(\frac{1}{a^2} + \frac{1}{b^2} \right).$$

EXERCISE - II

Assuming the validity of differentiation under the integral sign, prove the following : Class (c) : 8 Marks

1. $\displaystyle\int_0^1 \frac{x^a - x^b}{\log x}\, dx = \log\left(\frac{a+1}{b+1}\right).$ **(M.U. 1988, 93, 97, 2003)**

2. $\displaystyle\int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} bx}{x}\, dx = \frac{\pi}{2} \log\left(\frac{a}{b}\right).$ **(M.U. 1987, 91)**

3. $\displaystyle\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \sin mx\, dx = \tan^{-1}\left(\frac{b}{m}\right) - \tan^{-1}\left(\frac{a}{m}\right).$ **(M.U. 1988, 2006)**

4. Evaluate $\displaystyle\int_0^{\pi} \frac{dx}{a + b\cos x}$, $a > 0$, $|b| < a$. Hence, prove that

$$\int_0^{\pi} \frac{\cos x}{(a + b\cos x)^2}\, dx = -\frac{\pi b}{(a^2 - b^2)^{3/2}} \text{ and } \int_0^{\pi} \frac{\cos x}{(5 + 4\cos x)^2}\, dx = -\frac{4\pi}{27}.$$ **(M.U. 1995)**

5. Evaluate $\int_0^{\pi} \frac{dx}{a + b\cos x}$, $a > 0, |b| < a$. Hence, deduce that

$$\int_0^{\pi} \frac{dx}{(5 + 4\cos x)^3} = \frac{11\pi}{81}.$$

(**Hint :** Differentiate twice.)

6. By differentiating $\int_0^{\infty} \frac{dx}{x^2 + a^2} = \frac{\pi}{2a}$ w.r.t. a under the integral sign successively, prove that

$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)^{n+1}} = \frac{(2n)!\,\pi}{2^{2n+1} \cdot (n!)^2\, a^{2n+1}}.$$

Summary

$$\frac{dI}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha} \cdot dx$$

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CHAPTER

8

Rectification

1. Introduction

In this chapter we shall learn how the technique of integration can be applied to find lengths of given plane curves. It is called **rectification.**

2. Length of the Arc of a Curve given by $y = f(x)$

Let the equation of the curve be $\qquad y = f(x)$

Let A be a fixed point from which the length of the arc is measured. Let $P(x, y)$ and $Q(x+$ $y+\delta y)$ be two neighbouring points on the curve. Let arc $AP = s$ and arc $AQ = s + \delta s$.

Draw PM, QN and PR perpendiculars as shown in the figure. When Q is very close to P *i.e.* when δs is very small, we can consider $\Delta\, PQR$ as a right angled triangle and write

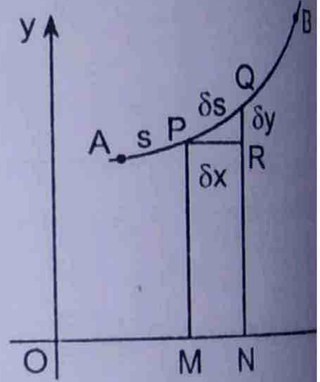

Fig. 8.1

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2 \qquad \therefore \left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

Taking the limits as $\delta x \to 0$, $\qquad \dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

If x coordinates of A and B are respectively x_1 and x_2 and arc $AB = s$ then by integration, we get,

$$\boxed{s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx}$$

Cor. 1 : If the curve is given as $x = f(y)$ then proceeding as above and dividing by $(\delta y$ we get,

$$\boxed{s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy}$$

Cor. 2 : If the curve is given in parametric form as $x = f_1(t)$ and $y = f_2(t)$ then dividing by $(\delta$

$$\left(\frac{\delta s}{\delta t}\right)^2 = \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$$

From this, we get,

$$\boxed{s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt}$$

Solved Examples : Class (b) : 6 Marks

Type I : Using dy/dx

Example 1 (b) : Find the total length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

(M.U. 1988, 92, 2003, 07, 08)

Sol.: The curve is called **four cusped hypocycloid** or **astroid.** Its shape is shown in the figure.

Differentiating the given equation

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \qquad \therefore\ \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

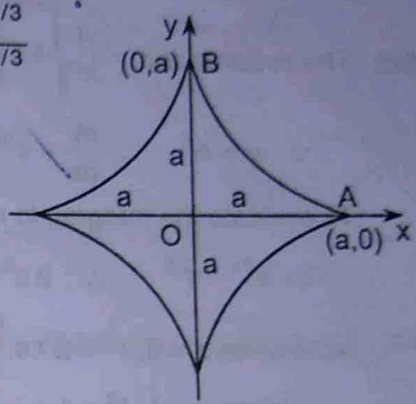

Fig. 8.2

If s is the length of the arc AB from $A\,(a, 0)$ to $B\,(0, a)$ then

$$s = \int_a^0 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx = \int_a^0 \sqrt{1 + \left(\frac{y^{2/3}}{x^{2/3}}\right)} \cdot dx$$

$$= \int_a^0 \sqrt{\left(\frac{x^{2/3} + y^{2/3}}{x^{2/3}}\right)} \cdot dx = \int_a^0 a^{1/3} \cdot x^{-1/3}\, dx$$

$$= a^{1/3} \cdot \frac{3}{2}\left[x^{2/3}\right]_a^0 = -\frac{3}{2}a^{1/3} \cdot a^{2/3} = -\frac{3}{2}a$$

$\therefore$ The total length of the curve $= 4s = 4 \cdot \dfrac{3}{2}a = 6a$.

(For another method, see Ex. 2, page 8-7.)

Example 2 (b) : Show that the length of the arc of the curve $ay^2 = x^3$ from the origin to the point whose abscissa is b is $\dfrac{8a}{27}\left[\left(1 + \dfrac{9b}{4a}\right)^{3/2} - 1\right]$. **(M.U. 1989, 2002, 16)**

Sol.: The curve is shown in the figure.

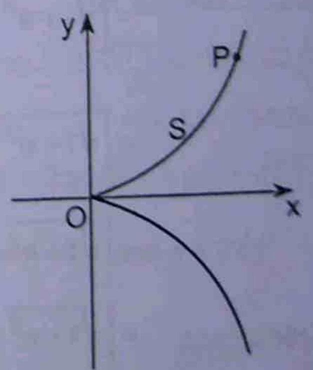

Fig. 8.3

Differentiating w.r.t. x, $\quad 2ay\dfrac{dy}{dx} = 3x^2 \quad \therefore\ \dfrac{dy}{dx} = \dfrac{3x^2}{2ay}$

Now, O is (0, 0) and if the abscissa of P is b then its ordinate $= \sqrt{b^3/a}$.

If s is the length of the arc OP then,

$$s = \int_0^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx = \int_0^b \sqrt{1 + \left(\frac{9x^4}{4a^2y^2}\right)} \cdot dx$$

$$= \int_0^b \sqrt{1 + \frac{9x^4}{4a^2} \cdot \frac{a}{x^3}} \cdot dx = \int_0^b \sqrt{1 + \frac{9}{4} \cdot \frac{x}{a}} \cdot dx$$

To integrate we put $1 + \dfrac{9}{4} \cdot \dfrac{x}{a} = t$. When $x = 0$, $t = 1$ and when $x = b$, $t = 1 + \dfrac{9}{4} \cdot \dfrac{b}{a}$ and $\dfrac{9}{4a}dx = dt$ i.e., $dx = \dfrac{4a}{9}dt$.

$$\therefore \quad s = \int_1^{1+(9b)/4a} \sqrt{t} \cdot \frac{4a}{9} dt = \frac{4a}{9}\left[\frac{t^{3/2}}{3/2}\right]_1^{1+(9b)/4a}$$

$$= \frac{4a}{9} \cdot \frac{2}{3}\left[\left(1+\frac{9b}{4a}\right)^{3/2} - 1\right] = \frac{8a}{27}\left[\left(1+\frac{9b}{4a}\right)^{3/2} - 1\right]$$

Example 3 (b) : Find the length of the arc of $y = e^x$ from (0, 1) to (1, e). **(M.U. 2003)**

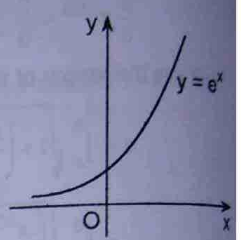

Fig. 8.4

Sol.: The required arc $= \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$

$\because \quad y = e^x \quad \therefore \quad \frac{dy}{dx} = e^x \quad \therefore \quad s = \int_0^1 \sqrt{1+e^{2x}} \cdot dx$

To evaluate the integral, to remove the square root, we put

$1 + e^{2x} = t^2 \quad \therefore \quad 2e^{2x} dx = 2t\, dt \quad \therefore \quad dx = \frac{t}{t^2-1} \cdot dt$

Now, when $x = 0$, $t^2 = 2$ *i.e.*, $t = \sqrt{2}$, and

when $x = 1$, $t^2 = 1 + e^2$ *i.e.*, $t = \sqrt{1+e^2}$.

$$\therefore \quad s = \int_{\sqrt{2}}^{\sqrt{1+e^2}} t \cdot \frac{t}{t^2-1} dt = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{(t^2-1)+1}{t^2-1} dt$$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} dt + \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{dt}{t^2-1}$$

$$= [t]_{\sqrt{2}}^{\sqrt{1+e^2}} + \left[\frac{1}{2}\log\frac{(t-1)}{(t+1)}\right]_{\sqrt{2}}^{\sqrt{1+e^2}}$$

$$= \left[\sqrt{1+e^2} - \sqrt{2}\right] + \frac{1}{2}\left[\log\left(\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right) - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

$$= \left[\sqrt{1+e^2} - \sqrt{2}\right] + \frac{1}{2}\left[\log\frac{(\sqrt{1+e^2}-1)^2}{(1+e^2)-1} - \log\frac{(\sqrt{2}-1)^2}{2-1}\right] \quad \text{[By rationalisation]}$$

$$= \left[\sqrt{1+e^2} - \sqrt{2}\right] + \log\left(\sqrt{1+e^2}-1\right) - \frac{1}{2}\log e^2 - \log(\sqrt{2}-1)$$

$$= \left[\sqrt{1+e^2} - \sqrt{2}\right] + \log\left(\sqrt{1+e^2}-1\right) - 1 - \log(\sqrt{2}-1) \quad [\because \log e^2 = 2\log e = 2]$$

Example 4 (b) : Find the length of the parabola $x^2 = 4y$ which lies inside the circle $x^2 + y^2 = 6y$. **(M.U. 1998, 2012)**

Sol.: The circle can be written as

$x^2 + y^2 - 6y + 9 = 9 \quad$ *i.e.* $\quad x^2 + (y-3)^2 = 3^2$.

Hence, its centre is (0, 3) and radius 3. The parabola $x^2 = 4y$ is symmetrical about the y-axis.

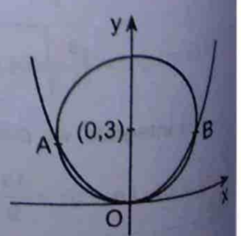

Fig. 8.5

The two curves intersect where $4y + y^2 = 6y \quad [\because x^2 = 4y]$

$\therefore\ y^2 - 2y = 0 \quad \therefore\ y(y-2) = 0 \quad \therefore\ y = 0$ or 2.

When $y = 0$, $x = 0$ and when $y = 2$, $x = \pm 2\sqrt{2}$ $\quad [\because x = \pm 2\sqrt{y}]$

Since, $y = \dfrac{x^2}{4}$, $\dfrac{dy}{dx} = \dfrac{x}{2}$.

$$\therefore \text{ Required length } = 2\int_0^{2\sqrt{2}} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx = 2\int_0^{2\sqrt{2}} \left[1+\frac{x^2}{4}\right] dx$$

$$= \int_0^{2\sqrt{2}} \sqrt{(x^2+4)}\, dx = \left[\frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\log\left(x+\sqrt{x^2+4}\right)\right]_0^{2\sqrt{2}}$$

$$= \sqrt{2}\cdot\sqrt{12} + 2\log\left(2\sqrt{2}+\sqrt{12}\right) - 2\log 2$$

$$= 2\sqrt{6} + 2\log\left(\frac{2\sqrt{2}+2\sqrt{3}}{2}\right) = 2\left[\sqrt{6} + \log(\sqrt{2}+\sqrt{3})\right]$$

EXERCISE - I

Solve the following examples : Class (b) : 6 Marks

1. Find the lengths of the following curves as stated.

   (1) $y = \dfrac{1}{3}(x^2+2)^{3/2}$ from $x = 0$ to $x = 3$. **[Ans. : 12]**

   (2) $y = 2x^{3/2}$ from $x = 0$ to 7. **[Ans. : $\dfrac{1022}{27}$]**

   (3) $y = \dfrac{3}{4}x^{4/3} - \dfrac{3}{8}x^{2/3} + 5$ from $x = 1$ to $x = 8$. **[Ans. : $\dfrac{99}{8}$]**

2. Find the length of the arc of the parabola $y^2 = 8x$ cut off by the latus rectum. **(M.U. 1995, 2205)**

   (See Fig. 8.6, page 8-5 with $a = 2$) **[Ans. : $2\left[\sqrt{2} + \log(1+\sqrt{2})\right]$**

3. Find the length of the parabola $x^2 = 4by$ cut-off by its latus rectum. **(M.U. 23004)**

   (See Fig. 15.10(a), page 15-4) **[Ans. : $2b[\sqrt{2} + \log(1+\sqrt{2})]$]**

4. Prove that the length of the arc of the curve $y = \log\left(\dfrac{e^x-1}{e^x+1}\right)$ from $x = 1$ and $x = 2$ is $\log\left(e+\dfrac{1}{e}\right)$. **(M.U. 2001)**

5. Find the length of the arc of the curve $y = \log\left(\tan h\dfrac{x}{2}\right)$ from $x = 1$ to $x = 2$. **(M.U. 2011)**

   [Note : Example 4 and 5 are similar.] **[Ans. : $\log\left(e+\dfrac{1}{e}\right)$**

6. Find the length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \pi/3$. **[Ans. : $\log(2+\sqrt{3})$]**

Type II : Using dx/dy

Example 1 (b) : Show that the length of the parabola $y^2 = 4ax$ from the vertex to the end of th latus rectum is $a\left[\sqrt{2} + \log(1+\sqrt{2})\right]$. **(M.U. 1996, 98, 99, 2006**

Sol.: Let arc $OP = s$.

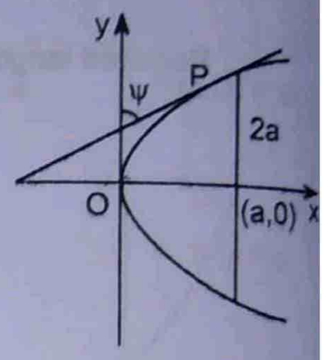

Fig. 8.6

$$\because \quad x = \frac{y^2}{4a} \quad \therefore \quad \frac{dx}{dy} = \frac{y}{2a}$$

$$\therefore \quad s = \int_0^{2a} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy$$

$$= \int_0^{2a} \sqrt{1 + \frac{y^2}{4a^2}} \cdot dy \qquad \text{[By (2), page 8-1]}$$

$$= \frac{1}{2a} \int_0^{2a} \sqrt{y^2 + 4a^2} \cdot dy$$

$$= \frac{1}{2a} \left[\frac{y}{2} \cdot \sqrt{y^2 + 4a^2} + \frac{4a^2}{2} \log\left(y + \sqrt{y^2 + 4a^2}\right)\right]_0^{2a}$$

$$= \frac{1}{2a} \left[a \cdot 2\sqrt{2} \cdot a + 2a^2 \left\{\log(2a + 2\sqrt{2} \cdot a) - \log 2a\right\}\right]$$

$$\therefore \quad s = a\left[\sqrt{2} + \log\left(\frac{2a + 2\sqrt{2} \cdot a}{2a}\right)\right] = a\left[\sqrt{2} + \log(1+\sqrt{2})\right]$$

Example 2 (b) : Show that the length of the arc of the parabola $y^2 = 4ax$ cut-off by the line $3y = 8x$ is $a\left(\log 2 + \frac{15}{16}\right)$. **(M.U. 2013**

Sol. : The parabola and the line intersect at A where

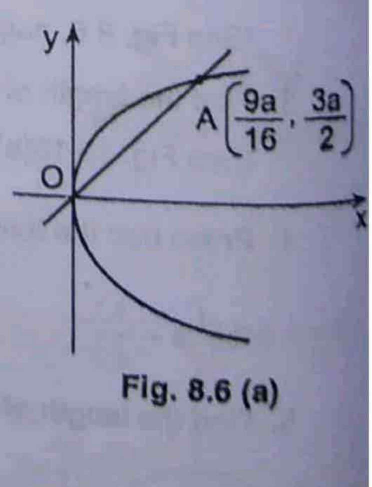

Fig. 8.6 (a)

$$\frac{64}{9} x^2 = 4ax \quad \therefore \quad x = \frac{9a}{16}$$

$$\therefore \quad y^2 = 4a \cdot x = 4a \cdot \frac{9a}{16} = \frac{9}{4} a^2 \quad \therefore \quad y = \frac{3a}{2}$$

i.e., $A\left(\frac{9a}{16}, \frac{3a}{2}\right)$.

$$\text{Now,} \quad x = \frac{y^2}{4a} \quad \therefore \quad \frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}.$$

$$\therefore \quad s = \int_0^{3a/2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy = \int_0^{3a/2} \sqrt{1 + \frac{y^2}{4a^2}} \cdot dy$$

$$= \frac{1}{2a} \int_0^{3a/2} \sqrt{y^2 + 4a^2} \cdot dy$$

$$= \frac{1}{2a} \left[\frac{y}{2} \cdot \sqrt{y^2 + 4a^2} + \frac{4a^2}{2} \log\left(y + \sqrt{y^2 + 4a^2}\right)\right]_0^{3a/2}$$

$$\therefore \quad s = \frac{1}{2}\left[\frac{3a}{4}\sqrt{\frac{9a^2}{4}+4a^2}+\frac{4a^2}{2}\log\left(\frac{3a}{2}+\sqrt{\frac{9a^2}{4}+4a^2}\right)-\frac{4a^2}{2}\log 2a\right]$$

$$= \frac{1}{2a}\left[\frac{3a}{8}\sqrt{25\,a^2}+2a^2\log\left(\frac{3a+\sqrt{25\,a^2}}{2}\right)-2a^2\log 2a\right]$$

$$= \frac{1}{2a}\left[\frac{3a}{8}\cdot 5a+2a^2\log\left(\frac{3a+5a}{4a}\right)\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8}+2a^2\log 2\right] = a\left(\frac{15}{16}+\log 2\right).$$

Example 3 (b) : Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Sol. : We have, $y = (x/2)^{2/3}$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{3}\cdot\left(\frac{x}{2}\right)^{-1/3}\cdot\frac{1}{2} = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

$$\therefore \quad s = \int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx = \int_0^2 \sqrt{1+\frac{4}{9}\cdot x^{-2/3}}\,dx$$

Since dy/dx is not defined at the lower limit $x = 0$, we cannot use the above formula. To overcome the difficulty we consider the same curve in the form $x = \Phi(y)$ and use the other formula (2), page 8-1.

$$\because \quad \frac{x}{2} = y^{3/2} \quad \therefore \quad x = 2(y)^{3/2} \quad \therefore \quad \frac{dx}{dy} = 2\cdot\frac{3}{2}(y)^{1/2} = 3\sqrt{y}$$

This derivative is defined at $x = 0$ *i.e.* $y = 0$.

Now, when $x = 0$, $y = 0$ and when $x = 2$, $y = 1$.

We now use the other formula (2), page 8-1.

$$\therefore \quad s = \int_0^1 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy = \int_0^1 \sqrt{1+9y}\cdot dy$$

$$= \frac{2}{3}\cdot\frac{1}{9}\left[(1+9y)^{3/2}\right]_0^1 = \frac{2}{27}\left[10\sqrt{10}-1\right]$$

Example 4 (b) : Find the length of the curve $x = \dfrac{y^4}{4}+\dfrac{1}{8y^2}$ from $y = 1$ to $y = 2$.

Sol. : We have $\dfrac{dx}{dy} = y^3 - \dfrac{1}{4y^3}$

$$1+\left(\frac{dx}{dy}\right)^2 = 1+\left(y^3-\frac{1}{4y^3}\right)^2 = 1+y^6-\frac{1}{2}+\frac{1}{16\,y^6}$$

$$= y^6+\frac{1}{2}+\frac{1}{16y^6} = \left(y^3+\frac{1}{4y^3}\right)^2$$

$$\therefore \quad s = \int_1^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy = \int_1^2\left(y^3+\frac{1}{4y^3}\right)dy = \left[\frac{y^4}{4}-\frac{1}{8y^2}\right]_1^2 = \frac{123}{32}.$$

Solve the following examples : Class (b) : 6 Marks

1. Find the length of the curve $x = \dfrac{y^3}{3} + \dfrac{1}{4y}$ from $y = 1$ to $y = 2$. **[Ans. :** $\dfrac{59}{24}$

2. Show that the length of the arc of the curve $4\,ax = y^2 - 2a^2 \log \cdot \dfrac{y}{a} - a^2$ from (0, a) to an

point $P(x, y)$ is $\dfrac{y^2}{2a} - \dfrac{a}{2} - x$.

Type III : To find the length of a curve in parametric form

Example 1 (b) : Find the circumference of a circle.

Sol. : It is convenient to use parametric equations of the circle. Let the circle be $x = a \cos$ $y = a \sin t$. Then the circumference by the formula (3), page 8-1.

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} \cdot dt$$

$$= a\int_0^{2\pi} dt = a\left[t\right]_0^{2\pi} = 2a\pi.$$

Example 2 (b) : Find the length of the astroid $x = a \cos^3 t$, $y = a \sin^3 t$.

Sol. : We shall find the length of the curve in the first quadrant only.

$$\because \quad x = a\cos^3 t, \qquad \frac{dx}{dt} = -3a\cos^2 t \sin t$$

$$\because \quad y = a\sin^3 t, \qquad \frac{dy}{dt} = 3a\sin^2 t \cos t$$

$$\therefore \quad s = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

$$\therefore \quad s = \int_0^{\pi/2} \sqrt{9a^2 \cos^4 t \cdot \sin^2 t + 9a^2 \sin^4 t \cdot \cos^2 t} \cdot dt$$

$$= 3a\int_0^{\pi/2} \sqrt{\sin^2 t \cos^2 t\,(\cos^2 t + \sin^2 t)} \cdot dt$$

$$= 3a\int_0^{\pi/2} \sin t \cos t\, dt = \frac{3a}{2}\int_0^{\pi/2} \sin 2t\, dt$$

$$= \frac{3a}{2}\left[-\frac{\cos 2t}{2}\right]_0^{\pi/2} = -\frac{3a}{4}[-1-1] = \frac{3a}{2}.$$

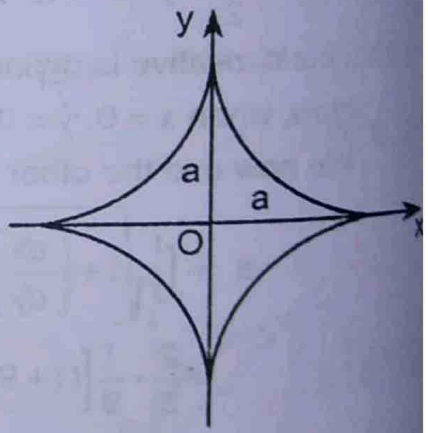

Fig. 8.7

$\therefore$ The length of the astroid $= 4 \cdot \dfrac{3a}{2} = 6a$.

(For another method, see Ex. 1, page 8.2)

Example 3 (b) : Find the total length of the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$. Hence, deduce the total length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

(M.U. 1997, 99, 2002)

Also show that the line $\theta = \pi/3$ divides the length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant in the ratio 1 : 3.

(M.U. 2007, 08)

Sol.: The curve known as **four cusped hypocycloid** or **astroid** is shown in the figure.

(i) To rectify the curve it is convenient to use the parametric equations of the curve. They are

$$x = a\cos^3\theta, \quad y = b\sin^3\theta.$$

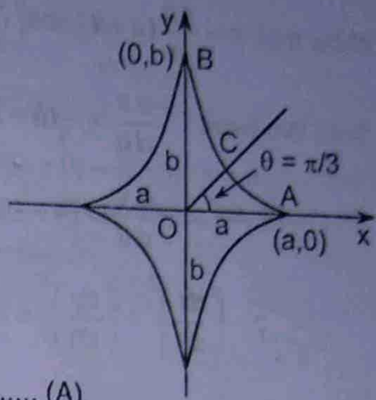

Fig. 8.8

If s is the total length then

$$s = 4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\cdot d\theta$$

$$= 4\int_0^{\pi/2}\sqrt{(3a\cos^2\theta\sin\theta)^2 + (3b\sin^2\theta\cos\theta)^2}\cdot d\theta$$

$$= 4\int_0^{\pi/2} 3\cdot\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\cdot\sin\theta\cos\theta\, d\theta \quad \text{......... (A)}$$

To find the integral, we put $\quad a^2\cos^2\theta + b^2\sin^2\theta = t^2$

$$\therefore \quad 2(b^2 - a^2)\sin\theta\cos\theta\, d\theta = 2t\, dt$$

$$\therefore \quad \sin\theta\cos\theta\, d\theta = \frac{t}{(b^2 - a^2)}dt$$

When $\theta = 0$, $t = a$ and when $\theta = \pi/2$, $t = b$

$$\therefore \quad s = 12\int_a^b t\cdot\frac{t}{(b^2 - a^2)}dt = \frac{12}{(b^2 - a^2)}\int_a^b t^2\, dt = \frac{12}{b^2 - a^2}\left[\frac{t^3}{3}\right]_a^b$$

$$= \frac{4}{b^2 - a^2}\left[b^3 - a^3\right] = 4\frac{(a^2 + ab + b^2)}{(a + b)}$$

(ii) For deduction, we put $b = a$.

Hence, the total length of the second astroid

$$s = 4\cdot\frac{3a^2}{2a} = 6a.$$

(iii) For the third part, the length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant is

$$AB = s = \frac{1}{4}\cdot 6a = \frac{3}{2}a$$

Now, the length cut off by $\theta = \pi/3$,

[Putting $b = a$ in (A)]

$$\text{arc } AC = \int_0^{\pi/3} 3a\sin\theta\cos\theta\, d\theta$$

$$\therefore \quad \text{arc } AC = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/3} = \frac{3a}{2}\cdot\frac{3}{4} = \frac{9a}{8}$$

But arc $AB = 3a/2$,

$$\therefore \quad \text{arc } BC = \frac{3a}{2} - \frac{9a}{8} = \frac{3a}{8}. \qquad \therefore \quad \frac{\text{arc } BC}{\text{arc } AC} = \frac{1}{3}.$$

EXERCISE - II

Solve the following examples : Class (b) : 6 Marks

1. Find the length of the curve $x = \dfrac{y^3}{3} + \dfrac{1}{4y}$ from $y = 1$ to $y = 2$. [Ans. : $\dfrac{59}{24}$]

2. Show that the length of the arc of the curve $4\,ax = y^2 - 2a^2 \log \cdot \dfrac{y}{a} - a^2$ from (0, a) to any point $P(x, y)$ is $\dfrac{y^2}{2a} - \dfrac{a}{2} - x$.

Type III : To find the length of a curve in parametric form

Example 1 (b) : Find the circumference of a circle.

Sol. : It is convenient to use parametric equations of the circle. Let the circle be $x = a\cos t$, $y = a\sin t$. Then the circumference by the formula (3), page 8-1.

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} \cdot dt$$

$$= a\int_0^{2\pi} dt = a\left[\,t\,\right]_0^{2\pi} = 2a\pi.$$

Example 2 (b) : Find the length of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$.

Sol. : We shall find the length of the curve in the first quadrant only.

$$\because\ x = a\cos^3 t, \qquad \frac{dx}{dt} = -3a\cos^2 t \sin t$$

$$\because\ y = a\sin^3 t, \qquad \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\therefore\ s = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

$$\therefore\ s = \int_0^{\pi/2} \sqrt{9a^2\cos^4 t \cdot \sin^2 t + 9a^2 \sin^4 t \cdot \cos^2 t} \cdot dt$$

$$= 3a\int_0^{\pi/2} \sqrt{\sin^2 t\cos^2 t\,(\cos^2 t + \sin^2 t)} \cdot dt$$

$$= 3a\int_0^{\pi/2} \sin t\cos t\, dt = \frac{3a}{2}\int_0^{\pi/2} \sin 2t\, dt$$

$$= \frac{3a}{2}\left[-\frac{\cos 2t}{2}\right]_0^{\pi/2} = -\frac{3a}{4}[-1-1] = \frac{3a}{2}.$$

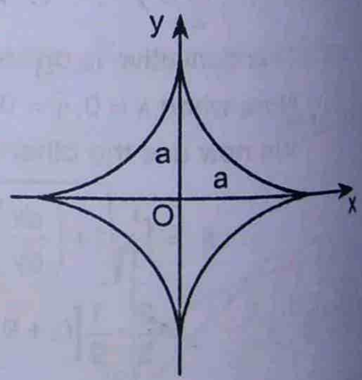

Fig. 8.7

$\therefore$ The length of the astroid $= 4 \cdot \dfrac{3a}{2} = 6a$.

(For another method, see Ex. 1, page 8.2)

Example 3 (b) : Find the total length of the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$. Hence, deduce the total length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

(M.U. 1997, 99, 2002)

Also show that the line $\theta = \pi/3$ divides the length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant in the ratio 1 : 3. **(M.U. 2007, 08)**

Sol.: The curve known as **four cusped hypocycloid** or **astroid** is shown in the figure.

(i) To rectify the curve it is convenient to use the parametric equations of the curve. They are

$$x = a\cos^3\theta, \quad y = b\sin^3\theta.$$

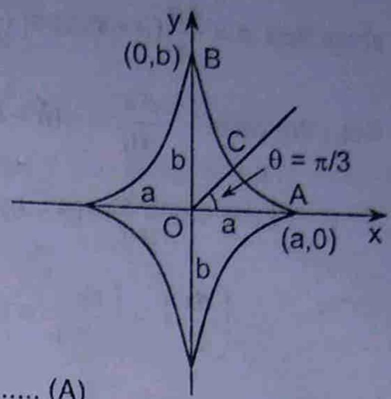

Fig. 8.8

If s is the total length then

$$s = 4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\cdot d\theta$$

$$= 4\int_0^{\pi/2}\sqrt{(3a\cos^2\theta\sin\theta)^2 + (3b\sin^2\theta\cos\theta)^2}\cdot d\theta$$

$$= 4\int_0^{\pi/2} 3\cdot\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\cdot\sin\theta\cos\theta\; d\theta \quad \text{......... (A)}$$

To find the integral, we put $a^2\cos^2\theta + b^2\sin^2\theta = t^2$

$$\therefore\; 2(b^2 - a^2)\sin\theta\cos\theta\; d\theta = 2t\; dt$$

$$\therefore\; \sin\theta\cos\theta\; d\theta = \frac{t}{(b^2 - a^2)}dt$$

When $\theta = 0$, $t = a$ and when $\theta = \pi/2$, $t = b$

$$\therefore\; s = 12\int_a^b t\cdot\frac{t}{(b^2 - a^2)}dt = \frac{12}{(b^2 - a^2)}\int_a^b t^2\,dt = \frac{12}{b^2 - a^2}\left[\frac{t^3}{3}\right]_a^b$$

$$= \frac{4}{b^2 - a^2}\left[b^3 - a^3\right] = 4\,\frac{(a^2 + ab + b^2)}{(a + b)}$$

(ii) For deduction, we put $b = a$.

Hence, the total length of the second astroid

$$s = 4\cdot\frac{3a^2}{2a} = 6a.$$

(iii) For the third part, the length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant is

$$AB = s = \frac{1}{4}\cdot 6a = \frac{3}{2}a$$

Now, the length cut off by $\theta = \pi/3$,

$$\text{arc } AC = \int_0^{\pi/3} 3a\sin\theta\cos\theta\; d\theta \qquad [\text{ Putting } b = a \text{ in (A) }]$$

$$\therefore\; \text{arc } AC = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/3} = \frac{3a}{2}\cdot\frac{3}{4} = \frac{9a}{8}$$

But arc $AB = 3a/2$,

$$\therefore\; \text{arc } BC = \frac{3a}{2} - \frac{9a}{8} = \frac{3a}{8}. \qquad \therefore\; \frac{\text{arc } BC}{\text{arc } AC} = \frac{1}{3}.$$

Example 4 (b) : For the curve

$$x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\cdot\theta\right),\quad y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\cdot\theta\right),$$

show that $s = \frac{4b}{a}(a+b)\cos\left(\frac{a\theta}{2b}\right)$ where s is measured from $\theta = \pi\, b/a$ to θ. **(M.U. 2005)**

Sol.: We have $\frac{dx}{d\theta} = -(a+b)\sin\theta + (a+b)\sin\left(\frac{a+b}{b}\cdot\theta\right)$

$$\frac{dy}{d\theta} = (a+b)\cos\theta - (a+b)\cos\left(\frac{a+b}{b}\cdot\theta\right)$$

$$\therefore\ \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (a+b)^2\left[2 - 2\sin\theta\,\sin\left(\frac{a+b}{b}\cdot\theta\right) - 2\cos\theta\,\cos\left(\frac{a+b}{b}\cdot\theta\right)\right]$$

$$= (a+b)^2\left[2 - 2\cos\left[\theta - \left(\frac{a+b}{b}\right)\cdot\theta\right]\right]$$

$$= 2(a+b)^2\left[1 - \cos\left(-\frac{a\theta}{b}\right)\right]$$

$$= 2(a+b)^2\left[1 - \cos\frac{a\theta}{b}\right] = 4(a+b)^2\sin^2\left(\frac{a\theta}{2b}\right).$$

$$\therefore\ s = \int_{\pi b/a}^{\theta}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\cdot d\theta = \int_{\pi b/a}^{\theta} 2(a+b)\sin\left(\frac{a\theta}{2b}\right)\cdot d\theta$$

$$= 2(a+b)\left[-\frac{2b}{a}\cos\left(\frac{a\theta}{2b}\right)\right]_{\pi b/a}^{\theta}$$

$$= -\frac{4b}{a}(a+b)\left[\cos\left(\frac{a\theta}{2b}\right) - \cos\frac{\pi}{2}\right]$$

$$\therefore\ s = \frac{4b}{a}(a+b)\cos\left(\frac{a\theta}{2b}\right) \qquad [\text{ Numerically }]$$

Example 5 (b) : For the curve $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$, find the length of the arc of the curve measured from $t = 0$ to any point.

Sol.: We have $\frac{dx}{dt} = a(-2\sin t + 2\sin 2t)$, $\frac{dy}{dt} = a(2\cos t - 2\cos 2t)$

$$\therefore\ \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$= 4a^2\left[(\sin 2t - \sin t)^2 + (\cos t - \cos 2t)^2\right]$$

$$= 4a^2\left[2 - 2(\sin 2t\sin t + \cos t\cos 2t)\right]$$

$$= 8a^2\left[1 - \cos(2t - t)\right] = 8a^2\left[1 - \cos t\right]$$

$$= 16a^2\sin^2\frac{t}{2}.$$

$\therefore \quad \dfrac{ds}{dt} = 4a \sin\dfrac{t}{2}.$

$\therefore \quad s = \int_0^t 4a \sin\dfrac{t}{2} = 8a\left[-\cos\dfrac{t}{2}\right]_0^t$

$= 8a\left[1 - \cos\dfrac{t}{2}\right] = 16a \sin^2\dfrac{t}{4}.$

Example 6 (b) : Find the length of one arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$.

(M.U. 1993, 98, 2002, 09, 14)

Sol.: The curve is shown on the right. For A, $\theta = 0$ and for B, $\theta = 2\pi$.

Now, $\dfrac{dx}{d\theta} = a(1 - \cos\theta)$ and $\dfrac{dy}{d\theta} = -a\sin\theta$

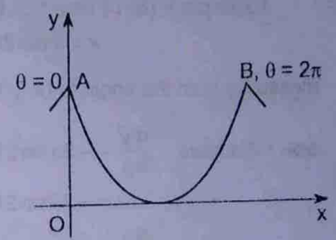

Fig. 8.9

$\therefore \quad s = \int_0^{2\pi} \sqrt{\left[\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2\right]} \cdot d\theta$

$= \int_0^{2\pi} \sqrt{\left[a^2(1 - 2\cos\theta + \cos^2\theta) + a^2\sin^2\theta\right]} \cdot d\theta$

$= \int_0^{2\pi} a\sqrt{[2 - 2\cos\theta]} \cdot d\theta$

$= a\int_0^{2\pi} \sqrt{2 \cdot 2\sin^2(\theta/2)} \cdot d\theta$

$= 2a\int_0^{2\pi} \sin\dfrac{\theta}{2} \cdot d\theta = 2a\left[-2\cos\dfrac{\theta}{2}\right]_0^{2\pi}$

$= -4a[\cos\pi - \cos 0] = 8a.$

Example 7 (b) : Find the length of the cycloid from one cusp to the next cusp $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. **(M.U. 1990, 93, 97, 2003)**

Sol.: The curve is shown on the next page. Let the arc be measured from the origin O. For A, $\theta = -\pi$ and for B, $\theta = \pi$, for O, $\theta = 0$.

Hence, the length of the arc $AB = 2$ arc $OB = 2\int_0^{\pi} \sqrt{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2} \cdot d\theta.$

But, $\dfrac{dx}{d\theta} = a(1 + \cos\theta)$, $\dfrac{dy}{d\theta} = a\sin\theta.$

$\therefore \quad s = 2\int_0^{\pi} \sqrt{a^2(1 + \cos\theta)^2 + a^2\sin^2\theta} \cdot d\theta$

$= 2a\int_0^{\pi} \sqrt{2(1 + \cos\theta)} \cdot d\theta$

$= 4a\int_0^{\pi} \cos\left(\dfrac{\theta}{2}\right) \cdot d\theta = 4a\left[2\sin\dfrac{\theta}{2}\right]_0^{\pi} = 8a.$

Example 8 (b) : Find the length of the above cycloid from one cusp to another cusp. If s is the length of the arc from the origin to a point $P(x, y)$ show that $s^2 = 8ay$. **(M.U. 1995, 2003)**

Sol. : We have proved the first part already.

For the second part, we have

$$s = \int_0^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta$$

$$= 2a \cdot \left[2 \sin\left(\frac{\theta}{2}\right)\right]_0^{\theta} \qquad \text{[As above]}$$

$$= 4a \sin\left(\frac{\theta}{2}\right)$$

$$\therefore \quad s^2 = 16\, a^2 \sin^2\left(\frac{\theta}{2}\right) = 8a\left[a(1-\cos\theta)\right] = 8ay.$$

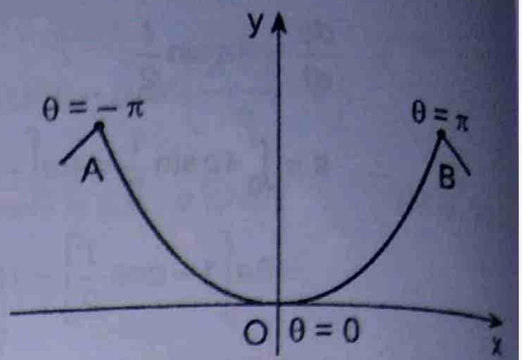

Fig. 8.10

Example 9 (b) : Prove that the length of the arc of the curve

$$x = a \sin 2\theta\,(1 + \cos 2\theta), \quad y = a \cos 2\theta\,(1 - \cos 2\theta)$$

measured from the origin to (x, y) is $\frac{4}{3} a \sin 3\theta$. **(M.U. 2005)**

Sol. : We have $\dfrac{dy}{d\theta} = -2a \sin 2\theta (1 - \cos 2\theta) + 2a \cos 2\theta \sin\ 2\theta$

$$= -2a \sin 2\theta + 2a \sin 4\theta$$

$$\frac{dx}{d\theta} = 2a \cos 2\theta (1 + \cos 2\theta) - 2a \sin^2 2\theta$$

$$= 2a \cos 2\theta + 2a \cos\ 4\theta$$

$$\therefore \quad s = \int_0^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta$$

$$= \int_0^{\theta} \sqrt{\left[4a^2 \cos^2 2\theta + 4a^2 \cos^2 4\theta + \underline{8a^2 \cos 2\theta \cos 4\theta}\right.}$$

$$\left. + 4a^2 \sin^2 2\theta + 4a^2 \sin^2 4\theta - \underline{8a^2 \sin 2\theta \sin 4\theta}\right] \cdot d\theta$$

$$= \int_0^{\theta} \sqrt{\left[4a^2 + 4a^2 + 8a^2 \cos\ 6\theta\right]} \cdot d\theta \qquad ?$$

$$= \int_0^{\theta} 2\sqrt{2}\, a \sqrt{1 + \cos 6\theta} \cdot d\theta = \int_0^{\theta} 2\sqrt{2}\, a \cdot \sqrt{2} \cos 3\theta\ d\theta$$

$$= 4a \int_0^{\theta} \cos 3\theta\ d\theta = 4a\left[\frac{\sin 3\theta}{3}\right]_0^{\theta} = \frac{4a}{3} \sin\theta.$$

Example 10 (b) : Show that the length of the tractrix $x = a\,[\,\cos t + \log \tan (t/2)\,]$, $y = a \sin t$ from $t = \pi/2$ to any point t is $a \log \sin t$.

Sol. : We have $\dfrac{dx}{dt} = a\left(-\sin t + \dfrac{1}{\tan(t/2)} \sec^2\left(\dfrac{t}{2}\right) \cdot \dfrac{1}{2}\right)$

$$= a\left(-\sin t + \frac{1}{2 \sin(t/2) \cos(t/2)}\right)$$

$$\therefore \quad \frac{dx}{dt} = a\left(-\sin t + \frac{1}{\sin t}\right) = a\frac{(1-\sin^2 t)}{\sin t} = a\frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dx} = a\cos t$$

Now, $$s = \int_{\pi/2}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt = \int_{\pi/2}^{t} \sqrt{a^2 \frac{\cos^4 t}{\sin^2 t} + a^2 \cos^2 t} \cdot dt$$

$$= a\int_{\pi/2}^{t} \frac{\cos t}{\sin t}\, dt = a[\log \sin t]_{\pi/2}^{t} = a \log \sin t.$$

Example 11 (b) : Prove that the length of the curve

$$x = e^{\theta}\left[\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right], \quad y = e^{\theta}\left[\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right]$$

measured from $\theta = 0$ to $\theta = \pi$ is $\frac{5}{2}\left[e^{\pi} - 1\right]$. **(M.U. 1999)**

Sol. : We have $x = e^{\theta}\left[\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right]$

$$\frac{dx}{d\theta} = e^{\theta}\left[\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right] + e^{\theta}\left[\frac{1}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right] = \frac{5}{2}e^{\theta}\cos\frac{\theta}{2}$$

$$\frac{dy}{d\theta} = e^{\theta}\left[\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right] + e^{\theta}\left[-\frac{1}{2}\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right] = -\frac{5}{2}e^{\theta}\sin\frac{\theta}{2}.$$

$$\therefore \quad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{25}{4}e^{2\theta}\cos^2\frac{\theta}{2} + \frac{25}{4}e^{2\theta}\sin^2\frac{\theta}{2} = \frac{25}{4}e^{2\theta}$$

$$\therefore \quad s = \int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta = \int_0^{\pi}\frac{5}{2}e^{\theta}d\theta = \frac{5}{2}\left[e^{\theta}\right]_0^{\pi} = \frac{5}{2}\left[e^{\pi} - 1\right].$$

EXERCISE - III

Find the lengths of the following curves : Class (b) : 6 Marks

1. $x = a(2\cos\theta + \cos 2\theta)$, $y = a(2\sin\theta + \sin 2\theta)$ from $\theta = 0$ to any point θ. [**Ans. :** $8a\sin(\theta/2)$] **(M.U. 1997)**
2. $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$. [**Ans. :** $8a$] **(M.U. 1992)**
   (See Fig. 15.53, page 15-18)
3. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$. [**Ans. :** $2\pi^2 a$]
4. $x = a\,e^{\theta}\sin\theta$, $y = a\,e^{\theta}\cos\theta$ from $\theta = 0$ to $\theta = \pi/2$. [**Ans. :** $\sqrt{2}(e^{\pi/2} - 1)a$]
5. $x = a(3\cos\theta - \cos 3\theta)$, $y = a(3\sin\theta - \sin 3\theta)$ from $\theta = \pi/2$ to any point θ. [**Ans. :** $6a\cos\theta$ numerically]
6. $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ between two consecutive cusps. [**Ans. :** $8a$]
   (See Fig. 15.50, page 15-17)

$$\therefore \quad \frac{dx}{dt} = a\left(-\sin t + \frac{1}{\sin t}\right) = a\frac{(1-\sin^2 t)}{\sin t} = a\frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dx} = a\cos t$$

Now,
$$s = \int_{\pi/2}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt = \int_{\pi/2}^{t} \sqrt{a^2\frac{\cos^4 t}{\sin^2 t} + a^2\cos^2 t} \cdot dt$$

$$= a\int_{\pi/2}^{t} \frac{\cos t}{\sin t}\, dt = a\left[\log \sin t\right]_{\pi/2}^{t} = a \log \sin t.$$

Example 11 (b) : Prove that the length of the curve

$$x = e^{\theta}\left[\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right], \quad y = e^{\theta}\left[\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right]$$

measured from $\theta = 0$ to $\theta = \pi$ is $\frac{5}{2}\left[e^{\pi} - 1\right]$. **(M.U. 1999)**

Sol. : We have $\quad x = e^{\theta}\left[\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right]$

$$\frac{dx}{d\theta} = e^{\theta}\left[\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right] + e^{\theta}\left[\frac{1}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right] = \frac{5}{2}e^{\theta}\cos\frac{\theta}{2}$$

$$\frac{dy}{d\theta} = e^{\theta}\left[\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right] + e^{\theta}\left[-\frac{1}{2}\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right] = -\frac{5}{2}e^{\theta}\sin\frac{\theta}{2}.$$

$$\therefore \quad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{25}{4}e^{2\theta}\cos^2\frac{\theta}{2} + \frac{25}{4}e^{2\theta}\sin^2\frac{\theta}{2} = \frac{25}{4}e^{2\theta}$$

$$\therefore \quad s = \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta = \int_0^{\pi} \frac{5}{2}e^{\theta}d\theta = \frac{5}{2}\left[e^{\theta}\right]_0^{\pi} = \frac{5}{2}\left[e^{\pi} - 1\right].$$

EXERCISE - III

Find the lengths of the following curves : Class (b) : 6 Marks

1. $x = a\,(2\cos\theta + \cos 2\theta)$, $y = a\,(2\sin\theta + \sin 2\theta)$ from $\theta = 0$ to any point θ. [**Ans. :** $8a\sin(\theta/2)$]
2. $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$. **(M.U. 1997)**
   (See Fig. 15.53, page 15-18) [**Ans. :** $8a$]
3. $x = a\,(\cos\theta + \theta\sin\theta)$, $y = a\,(\sin\theta - \theta\cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$. **(M.U. 1992)** [**Ans. :** $2\pi^2 a$]
4. $x = a\,e^{\theta}\sin\theta$, $y = a\,e^{\theta}\cos\theta$ from $\theta = 0$ to $\theta = \pi/2$. [**Ans. :** $\sqrt{2}\,(e^{\pi/2} - 1)\,a$]
5. $x = a\,(3\cos\theta - \cos 3\theta)$, $y = a\,(3\sin\theta - \sin 3\theta)$ from $\theta = \pi/2$ to any point θ. [**Ans. :** $6a\cos\theta$ numerically]
6. $x = a\,(\theta + \sin\theta)$, $y = a\,(1 + \cos\theta)$ between two consecutive cusps.
   (See Fig. 15.50, page 15-17) [**Ans. :** $8a$]

Type IV : To find the length of a loop

Example 1 (b) : Show that if s is the arc of the curve $9y^2 = x(3-x)^2$ measured from the origin to the point $P(x, y)$ then $3s^2 = 3y^2 + 4x^2$. **(M.U. 1997)**

Sol.: Differentiating the given equation w.r.t. x,

$$18y\frac{dy}{dx} = (3-x)^2 + x \cdot 2(3-x)(-1)$$

$$= (3-x)(3-x-2x) = (3-x)(3-3x)$$

$$= 3(3-x)(1-x)$$

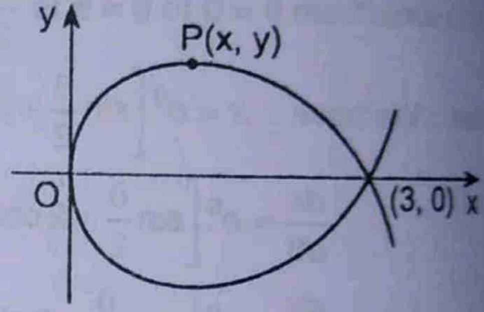

Fig. 8.11

$$\therefore \quad \frac{dy}{dx} = \frac{1}{6}\cdot\frac{1}{y}(3-x)(1-x)$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{1}{36}\cdot\frac{1}{y^2}(3-x)^2(1-x)^2$$

$$= \frac{1}{36}\cdot\frac{9}{x(3-x)^2}\cdot(3-x)^2(1-x)^2$$

$$= \frac{1}{4}\cdot\frac{(1-x)^2}{x}$$

$$\therefore \quad s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx = \int_0^x \sqrt{1+\frac{(1-x)^2}{4x}}\cdot dx$$

$$= \frac{1}{2}\int_0^x \sqrt{\frac{4x+1-2x+x^2}{x}}\cdot dx = \frac{1}{2}\int_0^x \sqrt{\frac{(1+x)^2}{x}}\,dx = \frac{1}{2}\int_0^x \frac{1+x}{\sqrt{x}}\,dx$$

$$= \frac{1}{2}\int_0^x \left(x^{-1/2} + x^{1/2}\right)dx = \frac{1}{2}\left[2x^{1/2} + \frac{2}{3}x^{2/3}\right]_0^x$$

$$= x^{1/2} + \frac{1}{3}x^{3/2} = x^{1/2}\left(1+\frac{x}{3}\right) = \frac{x^{1/2}}{3}(3+x)$$

$$\therefore \quad s^2 = \frac{x}{9}(3+x)^2$$

Now, $3y^2 + 4x^2 = \dfrac{x}{3}(3-x)^2 + 4x^2 = \dfrac{1}{3}\left[x(3-x)^2 + 12x^2\right]$

$$\therefore \quad 3y^2 + 4x^2 = \frac{1}{3}x\left[(3-x)^2 + 12x\right] = \frac{1}{3}x\left[9-6x+x^2+12x\right]$$

$$= \frac{1}{3}x\left[9+6x+x^2\right] = \frac{x}{3}(3+x)^2 = 3s^2$$

Example 2 (b) : Find the perimeter of the loop of the curve $9ay^2 = (x-2a)(x-5a)^2$.

Sol.: Differentiating the given equation w.r.t. x,

$$18\,ay\frac{dy}{dx} = (x-2a)\cdot 2(x-5a) + (x-5a)^2$$
$$= (x-5a)(2x-4a+x-5a)$$
$$= 3(x-5a)(x-3a)$$
$$= (x-5a)(3x-9a)$$

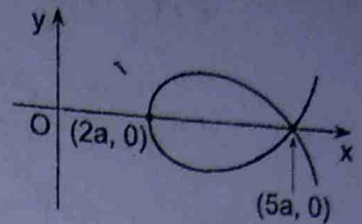

Fig. 8.12

$$\therefore \quad \frac{dy}{dx} = \frac{(x-5a)(x-3a)}{6ay}$$

$$\therefore \quad 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(x-5a)^2(x-3a)^2}{36a^2y^2} = 1+\frac{(x-5a)^2(x-3a)^2}{4a(x-2a)(x-5a)^2}$$
$$= 1+\frac{(x-3a)^2}{4a(x-2a)} = \frac{(x-a)^2}{4a(x-2a)}$$

$\therefore$ The perimeter of the loop of the curve

$$= 2\int_{2a}^{5a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx = 2\int_{2a}^{5a}\frac{x-a}{2\sqrt{a}\cdot\sqrt{x-2a}}\,dx$$
$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}\frac{(x-2a)+a}{\sqrt{x-2a}}\,dx = \frac{1}{\sqrt{a}}\int_{2a}^{5a}\left[\sqrt{x-2a}+a(x-2a)^{-1/2}\right]dx$$
$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}(x-2a)^{3/2}+a\cdot 2\cdot(x-2a)^{1/2}\right]_{2a}^{5a}$$
$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}(3a)^{3/2}+2a(3a)^{1/2}\right]$$
$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}\cdot 3\sqrt{3}\cdot a\sqrt{a}+2a\sqrt{3}\cdot\sqrt{a}\right]$$
$$= 2\sqrt{3}\,a+2a\cdot\sqrt{3} = 4\sqrt{3}\cdot a.$$

Example 3 (b) : Prove that the length of the arc of the curve $y^2 = x\left(1-\frac{1}{3}x\right)^2$ from the origin to the point $P(x, y)$ is given by $s^2 = y^2+\frac{4}{3}x^2$. Hence, rectify the loop. **(M.U. 1995, 97, 2002, 04)**

Sol.: The length of the arc OP is given by

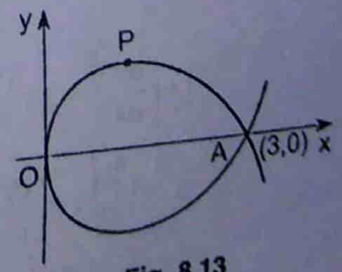

Fig. 8.13

$$s = \int_0^x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$$

$$y = x^{1/2}\left(1-\frac{1}{3}x\right) = x^{1/2}-\frac{1}{3}x^{3/2}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}-\frac{1}{3}\cdot\frac{3}{2}x^{1/2} = \frac{1}{2}\frac{(1-x)}{\sqrt{x}}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(1-x)^2}{4x} = \frac{(1+x)^2}{4x}$$

$$\therefore \quad s=\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx=\int_0^x \frac{1+x}{2\sqrt{x}}\,dx=\frac{1}{2}\int_0^x\left(x^{-1/2}+x^{1/2}\right)dx$$

$$=\frac{1}{2}\left[\frac{x^{1/2}}{1/2}+\frac{x^{3/2}}{3/2}\right]_0^x=x^{1/2}\left(1+\frac{x}{3}\right)$$

$$\therefore \quad s^2=x\left(1+\frac{x}{3}\right)^2=x\left(1-\frac{x}{3}\right)^2+\frac{4}{3}x^2=y^2+\frac{4}{3}x^2.$$

The length of half the loop i.e. the arc OA is obtained by putting $x=3$ and $y=0$ in the above result.

$$\therefore \quad s^2=\frac{4}{3}\cdot 9=12 \qquad \therefore \quad s=2\sqrt{3}$$

$\therefore$ The length of the complete loop $=4\sqrt{3}$.

Example 4 (b) : Find the length of the loop of the curve $x=t^2,\ y=t\left(1-\dfrac{t^2}{3}\right)$.

(M.U. 2002, 06)

Sol.: Eliminating t, we get $\quad y^2=t^2\left(1-\dfrac{t^2}{3}\right)^2=x\left(1-\dfrac{x}{3}\right)^2$

We get the same curve as above.

Example 5 (b) : Find the length of the loop of the curve $3ay^2=x(x-a)^2$.

(M.U. 1987, 91, 2003, 07, 11)

Sol.: The length of the loop is given by $s=2\displaystyle\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$

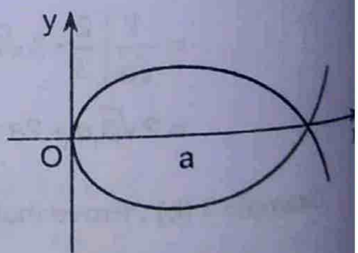

Fig. 8.14

Now differentiating the given function

$$6ay\frac{dy}{dx}=(x-a)^2+x\cdot 2(x-a)=(x-a)(3x-a)$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2=\frac{(x-a)^2(3x-a)^2}{36a^2y^2}$$

$$=\frac{(x-a)^2(3x-a)^2}{36a^2}\cdot\frac{3a}{x(x-a)^2}=\frac{(3x-a)^2}{12ax}$$

$$\therefore \quad 1+\left(\frac{dy}{dx}\right)^2=1+\frac{(3x-a)^2}{12ax}=\frac{(3x+a)^2}{12ax}$$

$$\therefore \quad s=2\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx=2\int_0^a\frac{3x+a}{2\sqrt{3a}\cdot\sqrt{x}}\cdot dx=\frac{1}{\sqrt{3a}}\int_0^a\left(3\sqrt{x}+\frac{a}{\sqrt{x}}\right)dx$$

$$=\frac{1}{\sqrt{3a}}\left[3\cdot\frac{x^{3/2}}{3/2}+a\cdot\frac{x^{1/2}}{1/2}\right]_0^a=\frac{1}{\sqrt{3a}}\left[2a^{3/2}+2a^{3/2}\right]=\frac{4}{\sqrt{3}}a.$$

Example 6 (b) : Find the total length of the loop of the curve $9y^2 = (x+7)(x+4)^2$.

(M.U. 1997, 99, 2003, 14)

Sol.: If $y = 0$, $x = -7$ or $x = -4$, the loop intersects the x-axis at $x = -7$ and at $x = -4$.

If s is the total length of the loop

$$s = 2\int_{-7}^{-4} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$$

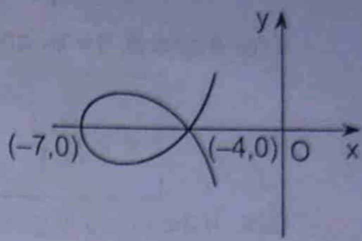

Fig. 8.15

Now, $18y\dfrac{dy}{dx} = (x+7)\cdot 2(x+4) + (x+4)^2$

$$\therefore \quad \frac{dy}{dx} = \frac{(x+4)[2x+14+x+4]}{18y} = \frac{(x+4)(x+6)}{6y}$$

$$\therefore \quad 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(x+4)^2(x+6)^2}{36y^2} = 1+\frac{(x+4)^2(x+6)^2}{4(x+7)(x+4)^2}$$

$$= \frac{4x+28+x^2+12x+36}{4(x+7)} = \frac{(x+8)^2}{4(x+7)}$$

$$\therefore \quad s = 2\int_{-7}^{-4} \frac{x+8}{2\sqrt{x+7}}\, dx = 2\int_0^{\sqrt{3}} \frac{t^2+1}{2t}\cdot 2t\, dt \qquad [\text{ Put } x+7 = t^2 \,]$$

$$= 2\int_0^{\sqrt{3}} (t^2+1)\, dt = 2\left[\frac{t^3}{3}+t\right]_0^{\sqrt{3}} = 2\left(\sqrt{3}+\sqrt{3}\right) = 4\sqrt{3}.$$

EXERCISE - IV

Show that the length of the loop : Class (b) : 6 Marks

1. $3ay^2 = x^2(a-x)$ is $4a/\sqrt{3}$. (See Fig. 15.22, page 15-8)
2. $9ay^2 = x(x-3a)^2$ is $4\sqrt{3}\cdot a$. **(M.U. 1992)** (See Fig. 15.24, page 15-9)

3. Length of the Arc of a Curve given by $r = f(\theta)$

Let $\qquad r = f(\theta) \qquad$ .... (1)

be the polar equation of the given curve. Let A be the fixed point from which the length of the arc is measured. Let $P(r, \theta)$, $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points on the curve. Let arc $AP = s$ and arc $AQ = s + \delta s$.

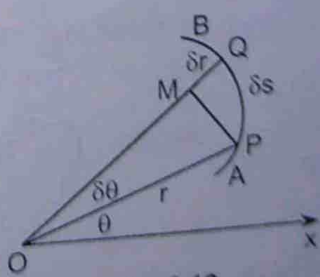

Fig. 8.16

Draw PM perpendicular to OQ. When $\delta\theta$ is small we can write $PM = r\delta\theta$, $MQ = \delta r$, and chord PQ = arc $PQ = \delta s$.

Hence, from the right angled $\Delta\, PQR$, we have

$$(\text{arc } PQ)^2 = (PM)^2 + (MQ)^2 \qquad \text{..................... (2)}$$

$$\therefore \quad (\delta s)^2 = (r\delta\theta)^2 + (\delta r)^2$$

$$\therefore \quad \left(\frac{\delta s}{\delta\theta}\right)^2 = r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2$$

Taking the limit as $\delta\theta \to 0$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

If for A and B, $\theta = \theta_1$ and $\theta = \theta_2$ respectively and arc $AB = s$ then

$$\boxed{s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta} \qquad \text{.................... (3)}$$

Cor. If the curve is given by $\theta = f(r)$, then dividing the equation (2) by $(\delta r)^2$ and proceeding as above, we get

$$\boxed{s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \cdot dr} \qquad \text{.................... (4)}$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Find the perimeter of the cardioid $r = a(1 - \cos\theta)$ and prove that the line $\theta = 2\pi/3$ bisects the upper half of the cardioid. **(M.U. 1995, 98, 2002, 13)**

Sol.: The shape of the curve is shown in the figure.

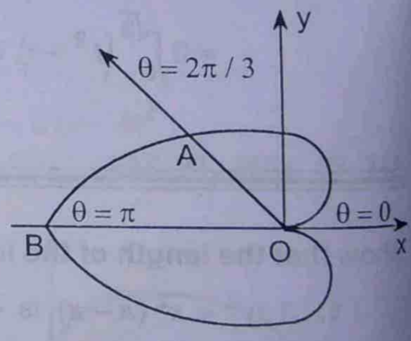

Fig. 8.17

We have, $O(0, 0)$ and $B(2a, \pi)$

$$\text{Arc } OB = s = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$\therefore \quad s = \int_0^{\pi} \sqrt{r^2 + a^2 \sin^2\theta} \cdot d\theta$$

$$= \int_0^{\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta} \cdot d\theta$$

$$= a\int_0^{\pi} \sqrt{2(1 - \cos\theta)} \cdot d\theta = \int_0^{\pi} 2a \sin\left(\frac{\theta}{2}\right) d\theta$$

$$= 2a\left[-2\cos\left(\frac{\theta}{2}\right)\right]_0^{\pi} = 4a$$

$\therefore$ Perimeter of the cardioid $= 2s = 8a$.

Now, the arc where the line $\theta = 2\pi/3$, divides the cardioid is given by

$$\text{Arc } OA = \int_0^{2\pi/3} 2a \sin\frac{\theta}{2} \cdot d\theta = 2a\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi/3}$$

$$= -4a\left[\frac{1}{2} - 1\right] = 2a.$$

Hence, the line $\theta = 2\pi/3$ bisects the upper half of the cardioid.

Example 1'(b) : Find the perimeter of the cycloid $r = a(1 + \cos\theta)$. **(M.U. 2015)**

Sol. : Left to you. **[Ans. : 8a]**

Example 2 (b) : Find the length of the cardioid $r = a(1 - \cos\theta)$ lying outside the circle $r = a\cos\theta$. **(M.U. 1997, 2000, 04, 08, 09, 10, 11, 13, 15)**

Sol.: The circle and the cardioid are shown in the figure. They intersect where $a(1 - \cos\theta) = a\cos\theta$ *i.e.* where $1 - \cos\theta = \cos\theta$ *i.e.* $1 = 2\cos\theta$ *i.e.* $\cos\theta = 1/2$ *i.e.* $\theta = 60^\circ = \pi/3$.

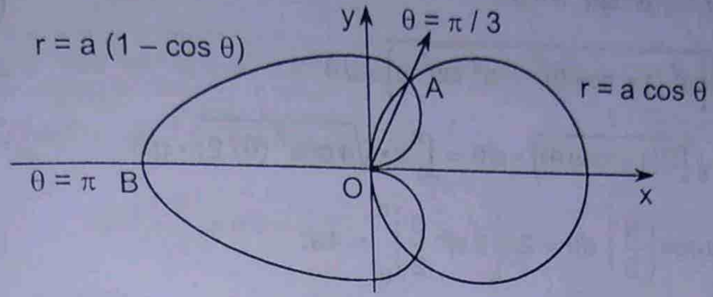

Fig. 8.18

The length of the cardioid outside the circle is 2 arc AB where for B, $\theta = \pi$ and for A, $\theta = \pi/3$.

$$\therefore \quad s = 2\int_{\pi/3}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

Now, as proved in the above Example No. 1,

$$s = 2\int_{\pi/3}^{\pi} 2a\sin\frac{\theta}{2} \cdot d\theta = 4a\left[-2\cos\frac{\theta}{2}\right]_{\pi/3}^{\pi}$$

$$= -8a\left[0 - \frac{\sqrt{3}}{2}\right] = 4a\sqrt{3}.$$

Example 3 (b) : Find the length of the cardioid $r = a(1 + \cos\theta)$ which lies outside the circle $r + a\cos\theta = 0$. **(M.U. 1999, 2004, 12)**

Sol.: The circle and the cardioid are shown in the figure.

They intersect where $a(1 + \cos\theta) = -a\cos\theta$ *i.e.* where $2\cos\theta = -1$ *i.e.* where $\cos\theta = -1/2$ *i.e.* $\theta = 2\pi/3$.

The length of the cardioid outside the circle is 2 arc BA where for B, $\theta = 0$ and for A, $\theta = 2\pi/3$.

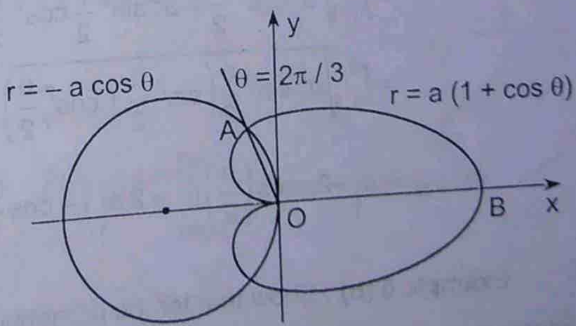

Fig. 8.19

Now, as in the above example,

$$\therefore \quad s = 2\int_{0}^{2\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$= 2\int_{0}^{2\pi/3} 2a\cos\left(\frac{\theta}{2}\right) \cdot d\theta$$

$$= 4a \cdot 2\left[\sin\left(\frac{\theta}{2}\right)\right]_{0}^{2\pi/3} = 4\sqrt{3}\,a.$$

Example 4 (b) : Find the perimeter of the cardioid $r = a(1 + \cos\theta)$. **(M.U. 2015)**

And show that the line $\theta = \pi/3$ divides the upper half of the cardioid into two equal to parts. **(M.U. 1993, 2002, 06)**

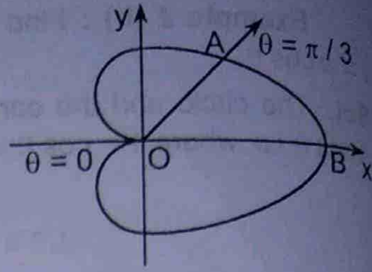

Fig. 8.20

Sol.: Arc $OB = s = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$

$$\therefore \quad s = \int_0^{\pi} \sqrt{r^2 + a^2 \sin^2 \theta} \cdot d\theta$$

$$= \int_0^{\pi} \sqrt{\left[a^2 (1 + \cos \theta)^2 + a^2 \sin^2 \theta\right]} \cdot d\theta$$

$$= \int_0^{\pi} a \sqrt{[2(1 + \cos \theta)]} \cdot d\theta = \int_0^{\pi} a \cdot \sqrt{4 \cos^2 (\theta / 2)} \cdot d\theta$$

$$= \int_0^{\pi} 2a \cos\left(\frac{\theta}{2}\right) d\theta = 2a \left[2 \sin \frac{\theta}{2}\right]_0^{\pi} = 4a.$$

$\therefore$ Perimeter of the cardioid = $2s = 8a$.

Now, the arc where the line $\theta = \pi / 3$ divides the cardioid is given by,

$$\text{Arc } OA = \int_0^{\pi/3} 2a \cdot \cos \frac{\theta}{2} \cdot d\theta = 2a \left[2 \sin \frac{\theta}{2}\right]_0^{\pi/3}$$

$$= 4a \sin \frac{\pi}{6} = 4a \cdot \frac{1}{2} = 2a.$$

$$\therefore \quad \text{Arc } OA = \frac{1}{2} \text{ Arc } OB.$$

Example 5 (b) : Find the length of the arc of the curve $r = a \sin^2 \left(\frac{\theta}{2}\right)$ from $\theta = 0$ to any point $P(\theta)$.

Sol.: The required arc is given by

$$s = \int_0^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta = \int_0^{\theta} \sqrt{r^2 + \left(a \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right)^2} \cdot d\theta$$

$$= \int_0^{\theta} \sqrt{a^2 \sin^4 \frac{\theta}{2} + a^2 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} \cdot d\theta$$

$$= \int_0^{\theta} \sqrt{a^2 \sin^2 \frac{\theta}{2} \left(\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}\right)} \cdot d\theta = a \int_0^{\theta} \sin \frac{\theta}{2} \, d\theta$$

$$\therefore \quad s = a \left[-2a \cos\left(\frac{\theta}{2}\right)\right]_0^{\theta} = 2a \left(1 - \cos \frac{\theta}{2}\right) = 4a \sin^2 \left(\frac{\theta}{4}\right).$$

Example 6 (b) : Show that for the parabola $\frac{2a}{r} = 1 + \cos \theta$, the arc intercepted between the vertex and the extremity of the latus rectum is $a[\sqrt{2} + \log(1 + \sqrt{2})]$.

(M.U. 2002, 12)

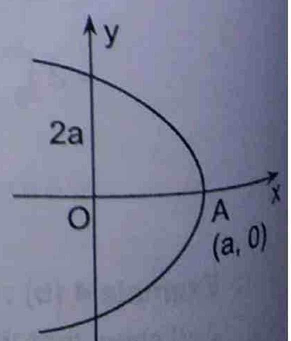

Fig. 8.21

Sol.: Since $r = \frac{2a}{1 + \cos \theta} = \frac{2a}{2 \cos^2 (\theta / 2)} = a \sec^2 \left(\frac{\theta}{2}\right)$.

$$\therefore \quad \frac{dr}{d\theta} = a \sec^2 \left(\frac{\theta}{2}\right) \cdot \tan\left(\frac{\theta}{2}\right)$$

At the extremity L of the latus rectum $\theta = \dfrac{\pi}{2}$.

$$\therefore\ s = \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$= \int_0^{\pi/2} \sqrt{a^2 \sec^4(\theta/2) + a^2 \sec^4(\theta/2) \cdot \tan^2(\theta/2)} \cdot d\theta$$

$$= \int_0^{\pi/2} a \sec^2(\theta/2) \sqrt{1 + \tan^2(\theta/2)} \cdot d\theta$$

Put $\tan\left(\dfrac{\theta}{2}\right) = t \quad \therefore\ \sec^2\left(\dfrac{\theta}{2}\right) \cdot d\theta = 2\,dt$

$$\therefore\ s = \int_0^1 2a\sqrt{1+t^2} \cdot dt = 2a\left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left(t + \sqrt{t^2+1}\right)\right]_0^1$$

$$= 2a\left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\log(1+\sqrt{2})\right] = a\left[\sqrt{2} + \log(1+\sqrt{2})\right]$$

Note ....

Putting $r\cos\theta = x$ and $r = \sqrt{x^2+y^2}$, we get

$2a = r + r\cos\theta \qquad \therefore\ (2a - x)^2 = x^2 + y^2$

$\therefore\ y^2 = 4a^2 - 4ax = -4a\,(x - a)$

This is a parabola with vertex at $(a, 0)$ and opening on the left.

Example 7 (b) : Show that the length of the arc of that part of cardioid $r = a\,(1+\cos\theta)$ which lies on the side of the line $4\,r = 3a\sec\theta$ away from the pole is $4a$. **(M.U. 1999, 2005, 10)**

Sol.: The cardioid is shown in the figure.

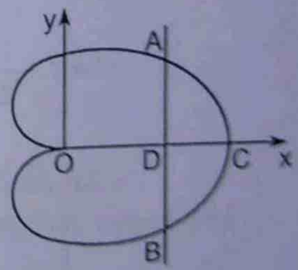

Fig. 8.22

Now, $\quad 4r = 3a\sec\theta \quad$ *i.e.* $\quad 4r\cos\theta = 3a$.

i.e. $\quad 4x = 3a$ *i.e.* $x = \dfrac{3a}{4}$ is a line parallel to the y-axis.

Now, at the point of intersection A,

$a(1+\cos\theta) = \dfrac{3a}{4}\sec\theta \quad$ *i.e.* $\quad 4\,(1+\cos\theta)\cos\theta = 3$

i.e. $\quad 4\cos^2\theta + 4\cos\theta - 3 = 0 \quad \therefore\ (2\cos\theta + 3)\,(2\cos\theta - 1) = 0$

$\therefore\ \cos\theta = -\dfrac{3}{2}$ which is impossible, or $\cos\theta = \dfrac{1}{2} \quad \therefore\ \theta = \dfrac{\pi}{3}$.

$$\therefore\ \text{Required arc } ACB = 2\,\text{Arc } AC = 2\int_0^{\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$= 2\int_0^{\pi/3} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta} \cdot d\theta$$

$$= 2\int_0^{\pi/3} a \cdot \sqrt{2(1+\cos\theta)} \cdot d\theta = 2a\int_0^{\pi/3} 2\cos\frac{\theta}{2}\,d\theta$$

$$= 4a\left[2\sin\frac{\theta}{2}\right]_0^{\pi/3} = 4a.$$

Example 8 (b) : Find the length of the Cissoid $r = 2a \tan\theta \sin\theta$ from $\theta = 0$ to $\theta = \pi/4$. **(M.U. 2002, 0**

Sol. : We have $\because \; r = 2a\tan\theta\sin\theta$

$$\therefore \; \frac{dr}{d\theta} = 2a\left[\sec^2\theta\sin\theta + \tan\theta\cos\theta\right] = 2a\left[\sec^2\theta\sin\theta + \sin\theta\right]$$

$$= 2a\sin\theta(\sec^2\theta + 1)$$

$$\therefore \; r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4a^2\tan^2\theta\sin^2\theta + 4a^2\sin^2\theta(\sec^2\theta+1)^2$$

$$= 4a^2\sin^2\theta(\sec^2\theta - 1 + \sec^4\theta + 2\sec^2\theta + 1)$$

$$= 4a^2\sin^2\theta(\sec^4\theta + 3\sec^2\theta)$$

$$= 4a^2\sin^2\theta\sec^2\theta(\sec^2\theta + 3)$$

$$= 4a^2\tan^2\theta(\sec^2\theta + 3)$$

$$= 4a^2\tan^2\theta(\tan^2\theta + 4)$$

$$\therefore \; s = \int_0^{\pi/4}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\cdot d\theta = \int_0^{\pi/4} 2a\tan\theta\sqrt{4+\tan^2\theta}\cdot d\theta$$

For integration put $4 + \tan^2\theta = t^2$ $\quad \therefore \; 2\tan\theta\sec^2\theta\, d\theta = 2t\, dt.$

$$\therefore \; \tan\theta\, d\theta = \frac{t\,dt}{\sec^2\theta} = \frac{t\,dt}{1+\tan^2\theta} = \frac{t\,dt}{t^2-3}$$

When $\theta = 0,\; t = 2$; when $\theta = \frac{\pi}{4},\; t = \sqrt{5}$.

$$\therefore \; s = \int_2^{\sqrt5} 2a\cdot\frac{t^2}{t^2-3}\cdot dt = \int_2^{\sqrt5} 2a\cdot\left[1 + \frac{3}{t^2-3}\right]dt$$

$$= 2a\left[t + \frac{3}{2\sqrt3}\log\frac{t-\sqrt3}{t+\sqrt3}\right]_2^{\sqrt5}$$

$$= 2a\left[\sqrt5 + \frac{\sqrt3}{2}\log\frac{\sqrt5-\sqrt3}{\sqrt5+\sqrt3} - 2 - \frac{\sqrt3}{2}\log\frac{2-\sqrt3}{2+\sqrt3}\right]$$

$$= 2a\left[\sqrt5 - 2 + \frac{\sqrt3}{2}\left\{\log\frac{(\sqrt5-\sqrt3)^2}{(5-3)} - \log\frac{(2-\sqrt3)^2}{4-3}\right\}\right]$$

$$= 2a\left[\sqrt5 - 2 + \frac{\sqrt3}{2}\left\{\log(4-\sqrt{15}) - \log(7-2\sqrt3)\right\}\right]$$

Example 9 (b) : Find the total length of the curve $r = a\sin^3(\theta/3)$. **(M.U. 1997, 2003, 06)**

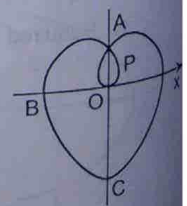

Fig. 8.23

Sol.: The curve is shown in the figure. For half the arc $O\,P\,A\,B\,C$, θ varies from 0 to $3\pi/2$.

Since, $\quad r = a\sin^3\left(\frac{\theta}{3}\right)$

$$\frac{dr}{d\theta} = a \cdot 3\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}\cdot\frac{1}{3} = a\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}$$

$$\therefore\ r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\sin^6\frac{\theta}{3} + a^2\sin^4\frac{\theta}{3}\cos^2\frac{\theta}{3} = a^2\sin^4\frac{\theta}{3}.$$

$$\therefore\ s = 2\int_0^{3\pi/2} a\sin^2\frac{\theta}{3}\,d\theta = a\int_0^{3\pi/2}\left(1-\cos\frac{2\theta}{3}\right)d\theta$$

$$= a\left[\theta - \frac{3}{2}\sin\frac{2\theta}{3}\right]_0^{3\pi/2} = a\cdot\frac{3\pi}{2} = \frac{3}{2}\pi a.$$

Example 10 (b) : Find the length of the upper arc of one loop of Lemniscate $r^2 = a^2\cos 2\theta$.
(M.U. 1990, 2002, 05, 07, 08)

Sol.: The curve is shown in the figure. It is clear that for upper half of one loop θ varies from 0 to $\pi/4$.

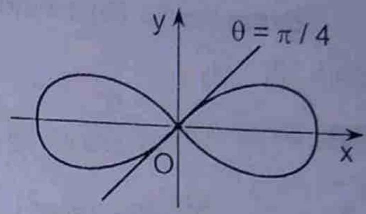

Fig. 8.24

$$\therefore\ s = \int_0^{\pi/4}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\cdot d\theta$$

But $r = a\sqrt{\cos 2\theta}$

$$\therefore\ \frac{dr}{d\theta} = a\cdot\frac{(-2\sin 2\theta)}{2\sqrt{\cos 2\theta}}$$

$$\therefore\ r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\cos 2\theta + \frac{a^2\sin 2\theta}{\cos 2\theta} = \frac{a^2}{\cos 2\theta}$$

$$\therefore\ s = \int_0^{\pi/4}\frac{a}{\sqrt{\cos 2\theta}}\,d\theta.$$

Put $t = 2\theta$ $\quad\therefore\ dt = 2\,d\theta$.

When $\theta = 0$, $t = 0$; when $\theta = \dfrac{\pi}{4}$, $t = \dfrac{\pi}{2}$

$$\therefore\ s = \frac{a}{2}\int_0^{\pi/2}\frac{dt}{\sqrt{\cos t}} = \frac{a}{2}\int_0^{\pi/2}\sin^0 t\cdot\cos^{-1/2}t\,dt$$

$$= \frac{a}{2}\cdot\frac{\overline{|(-1/2+1)/2}\cdot\overline{|1/2}}{\overline{|(-1/2+2)/2}} = \frac{a}{2}\cdot\frac{\overline{|1/4}\cdot\overline{|1/2}}{\overline{|3/4}}$$ [By (18), page 6-28]

But $\overline{|1/4}\cdot\overline{|3/4} = \pi\sqrt{2}$ [By (3), page 6-53]

$$\therefore\ s = \frac{a}{4}\cdot\frac{\sqrt{\pi}\,\overline{|1/4}}{1}\cdot\frac{\overline{|1/4}}{\pi\sqrt{2}} = \frac{a}{4\sqrt{2}}\cdot\frac{\left(\overline{|1/4}\right)^2}{\sqrt{\pi}}.$$

Example 11 (b) : Show that the total perimeter of $r^2 = a^2\cos 2\theta$ is $\dfrac{a}{\sqrt{2\pi}}\left(\overline{|1/4}\right)^2$.
(M.U. 1996, 2000)

Sol.: The perimeter $= 4s$ where s is as given above.

Example 12 (b) : Prove that the length of the spiral $r = ae^{\theta\cot\alpha}$ as r increases from r_1 to r_2 is given by $(r_2 - r_1)\sec\alpha$. **(M.U. 1997)**

Sol.: Since, $r = a e^{\theta \cot \alpha}$. $\quad \therefore \quad \dfrac{dr}{d\theta} = a e^{\theta \cot \alpha} \cdot \cot \alpha = r \cot \alpha$

Since, the limits of integration are in terms of r we shall use the formula (4) given in the corollary. (See page 8-17)

$$\therefore \quad s = \int_{r_1}^{r_2} \sqrt{\left[1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right]} \cdot dr = \int_{r_1}^{r_2} \sqrt{\left[1 + r^2 \cdot \frac{1}{r^2 \cot^2 \alpha}\right]} \cdot dr$$

$$= \int_{r_1}^{r_2} \sqrt{\left[1 + \frac{1}{\cot^2 \alpha}\right]} \cdot dr = \int_{r_1}^{r_2} \sqrt{\left[1 + \tan^2 \alpha\right]} \cdot dr$$

$$= \sec \alpha \left[r \right]_{r_1}^{r_2} = (r_2 - r_1) \sec \alpha .$$

Example 13 (b) : Find the length of the arc of the parabola $r = \dfrac{6}{1 + \cos \theta}$ from $\theta = 0$ to $\theta = \pi / 2$.

Sol.: Since $r = \dfrac{6}{1 + \cos \theta} = \dfrac{6}{2 \cos^2 (\theta/2)} = 3 \sec^2 \left(\dfrac{\theta}{2}\right)$

$$\frac{dr}{d\theta} = 3 \cdot 2 \sec^2 \left(\frac{\theta}{2}\right) \tan \left(\frac{\theta}{2}\right) \cdot \frac{1}{2} = 3 \sec^2 \left(\frac{\theta}{2}\right) \tan \left(\frac{\theta}{2}\right)$$

$$\therefore \quad r^2 + \left(\frac{dr}{d\theta}\right)^2 = 9 \sec^4 \left(\frac{\theta}{2}\right) + 9 \sec^4 \left(\frac{\theta}{2}\right) \cdot \tan^2 \left(\frac{\theta}{2}\right)$$

$$= 9 \sec^4 \left(\frac{\theta}{2}\right) \times \left[1 + \tan^2 \left(\frac{\theta}{2}\right)\right] = 9 \sec^6 \left(\frac{\theta}{2}\right)$$

$$\therefore \quad r = \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta = \int_0^{\pi/2} 3 \cdot \sec^3 \left(\frac{\theta}{2}\right) d\theta$$

We shall find the integral by the method of integration by parts as follows.

$$\int \sec^3 x \, dx = \int \sec^2 x \cdot \sec x \cdot dx$$

[Note this]

$$= \sec x \tan x - \int \tan x \sec x \tan x \cdot dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int \sec^3 x \cdot dx + \int \sec x \cdot dx$$

$$\therefore \quad 2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \cdot dx$$

$$= \sec x \tan x + \log (\sec x + \tan x)$$

$$\therefore \quad r = \frac{3}{2} \left[\sec \left(\frac{\theta}{2}\right) \tan \left(\frac{\theta}{2}\right) + \log \left[\sec \left(\frac{\theta}{2}\right) + \tan \left(\frac{\theta}{2}\right) \right] \right]_0^{\pi/2} \cdot 2$$

$$= 3 \left[\sqrt{2} + \log \left(\sqrt{2} + 1 \right) \right].$$

EXERCISE - V

Solve the following examples : Class (b) : 6 Marks

1. Find the circumferences of a circle of radius a by using the polar equation of the circle $r = a$. [**Ans. :** $2\pi a$]
2. Find the length of the curve $r = \cos^3(\theta/3)$ from $\theta = 0$ to $\theta = \pi/4$. [**Ans. :** $(\pi + 3)/8$]
3. Find the length of the curve $r = \sqrt{1 + \cos 2\theta}$ from $\theta = 0$ to $\theta = 2\pi\sqrt{2}$. [**Ans. :** 4π]
4. Find the length of the curve $r = \theta^2$ from $\theta = 0$ to $\theta = 2\sqrt{3}$. [**Ans. :** $56/3$]
5. Find the length of the cardioid $r = a(1 - \cos\theta)$ lying inside the circle $r = a\cos\theta$.
   (See Fig. 8.18, page 8-18)
   (**Hint :** Point of intersection is given by $a(1 - \cos\theta) = a\cos\theta$
   $$\therefore\ \theta = \frac{\pi}{3} \quad \therefore\ s = 2\int_0^{\pi/3} 2a\sin\left(\frac{\theta}{2}\right) d\theta = 8a\left[1 - \frac{\sqrt{3}}{2}\right]\ )$$
6. Find the length of the spiral $r = a^{m\theta}$ lying inside the circle $r = a$.
   (**Hint :** The circle intersects the spiral at $(a, 0)$. The limits for r are 0 and a *i.e.* for θ are $-\infty$ and 0.) [**Ans. :** $\frac{a}{m}\sqrt{1 + m^2}$]
7. Taking $s = 0$ at $\theta = 0$, find the length of the arc OP of the spiral $r = ae^{\theta\cot\alpha}$ from 0 to $P(\theta)$. [**Ans. :** $a\sec\alpha\,(e^{\theta\cot\alpha} - 1)$]
8. Find the length of the cardiod $r = a(1 + \sin\theta)$. [**Ans. :** $8a$]
   (**Hint :** Integrate from $-\pi/2$ to $\pi/2$) (See Fig. 15.43(a), page 15-15)

EXERCISE - VI

Solve the following examples : Class (a) : 3 or 4 Marks

1. Find the length of the following curve using integration $y = 2x + 5$ from $x = 1$ to 3. [**Ans. :** $2\sqrt{5}$]
2. Find the length of the parabola $2y = x^2$ from $x = 0$ to $x = 1$. [**Ans. :** $\sqrt{2} + \frac{1}{2}\log(1 + \sqrt{2})$]
3. Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 5$. [**Ans. :** $335/27$]
4. Find the length of the curve $y = \log\cos x$ from $x = 0$ to $x = \pi/3$. [**Ans. :** $\log(2 + \sqrt{3})$]
5. Find the length of the curve $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$. [**Ans. :** π]
6. Find the length of the curve $y = x^2 - \frac{1}{8}\cdot\log x$ from $x = 1$ to $x = 2$. [**Ans. :** $3 + \frac{1}{8}\log 2$]
7. Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y = 1$ to $y = 3$. [**Ans. :** $\frac{53}{6}$]
8. Find the length of the curve $x = \frac{y^{3/2}}{3} - y^{1/2}$ from $y = 1$ to $y = 9$. [**Ans. :** $\frac{32}{3}$]

9. Find the length of the curve $x = t^3$, $y = \dfrac{3t^2}{2}$ from $t = 0$ to $t = \sqrt{3}$. **[Ans. : 7]**

10. Find the circumference of the circle $x^2 + y^2 = a^2$. **[Ans. : $2\pi a$]**

Summary

1. $s = \int_{x_1}^{x_2} \sqrt{1+\left(\dfrac{dy}{dx}\right)^2} \cdot dx$; $\quad s = \int_{y_1}^{y_2} \sqrt{1+\left(\dfrac{dx}{dy}\right)^2} \cdot dy$;

   $s = \int_{t_1}^{t_2} \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} \cdot dt$.

2. $s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2} \cdot d\theta$; $\quad s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2} \cdot dr$

❋ ❋ ❋

CHAPTER 9

Double Integrals

1. Introduction

You know that an integral $\int_a^b f(x)\,dx$ has been defined as the limit of the sum as

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{r=1}^{n} f(t_r)\,\delta_r$$

The idea can be extended further to define integral of functions of two independent variables as follows.

(a) Double Integral : Definition

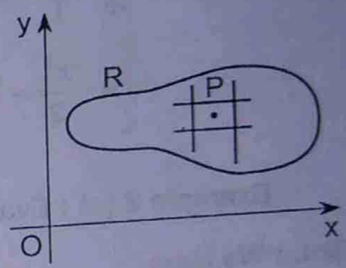

Fig. 9.1

Let $f(x, y)$ be a continuous and single-valued function of two independent variables x, y defined on the region R of area A bounded by a closed simple curve C. Let the region be divided into n sub-intervals in any manner (*e.g.* by drawing horizontal and vertical lines) into sub-regions $R_1, R_2, \ldots.. R_n$ of areas $\delta A_1, \delta A_2, \ldots.. \delta A_n$. Let $P(x_r, y_r)$ be any point inside the r-th sub-region of area δA_r. We now form the sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots.. + f(x_n, y_n)\,\delta A_n$$

i.e. $$\sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

We now increase the number of sub-regions such that the area of each sub-region becomes smaller and smaller. The limit of the sum (1), when it exits, as n tends to infinity and the area of each sub-interval tends to zero is called *the double integral of* $f(x, y)$ *over the region* A and is denoted by .... (2)

$$\iint_A f(x, y)\,dA$$

Thus, $$\iint_A f(x, y)\,dA = \lim_{\substack{n \to \infty \\ \delta A \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

(b) Evaluation of Double Integral

The double integral as defined above can be evaluated by successive single integrations as follows :

If A is a region bounded by the curves $y = f_1(x)$, $y = f_2(x)$, $x = a$, $x = b$, then

$$\boxed{\iint_A f(x, y)\,dA = \int_a^b \left\{ \int_{f_1(x)}^{f_2(x)} f(x, y)\,dy \right\} dx}$$

where the integration w.r.t. y is performed first by treating x as constant, and then the second integral is evaluated w.r.t. x.

Consider the area bounded by two simple curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates at $x = a$ and $x = b$.

Now, consider a strip parallel to the y-axis. On this strip y varies from $y = f_1(x)$ to $y = f_2(x)$. If the strip is moved parallel to itself so that it will sweep the shown area then x varies from a to b.

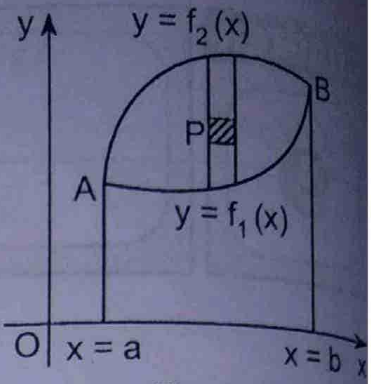

Fig. 9.2

Now, it can be shown that

$$\iint_A f(x, y)\, dA = \int_a^b \left\{ \int_{f_1(x)}^{f_2(x)} f(x, y)\, dy \right\} dx$$

Note ....

The order of integration can be understood from context.

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Evaluate $\int_0^1 dx \int_0^x e^{y/x}\, dy$.

Sol. : We have

$$\therefore\ I = \int_0^1 dx \left[x \cdot e^{y/x} \right]_0^x = \int_0^1 [x \cdot e - x \cdot 1]\, dx$$

$$= \left[e \cdot \frac{x^2}{2} - \frac{x^2}{2} \right]_0^1 = \frac{1}{2}(e-1).$$

Example 2 (a) : Evaluate $\int_0^1 \int_0^x e^{x+y}\, dy\, dx$. **(M.U. 2014)**

Sol. : We have

$$I = \int_0^1 \int_0^x e^{x+y}\, dy\, dx = \int_0^1 \int_0^x e^x \cdot e^y\, dy\, dx = \int_0^1 e^x \left[e^y \right]_0^x dx$$

$$= \int_0^1 e^x (e^x - 1)\, dx = \int_0^1 (e^{2x} - e^x)\, dx$$

$$= \left[\frac{e^{2x}}{2} - e^x \right]_0^1 = \left(\frac{e^2}{2} - e \right) - \left(\frac{1}{2} - 1 \right)$$

$$= \frac{e^2}{2} - e + \frac{1}{2} = \frac{1}{2}(e^2 - 2e + 1) = \frac{1}{2}(e-1)^2.$$

Example 3 (a) : Evaluate $\int_0^1 \int_0^x xy\, dy\, dx$. **(M.U. 2015)**

Sol. : We have

$$I = \int_0^1 \int_0^x xy\, dy\, dx = \int_0^1 x \left[\frac{y^2}{2} \right]_0^x dx = \frac{1}{2} \int_0^1 x [x^2 - 0]\, dx$$

$$= \frac{1}{2} \int_0^1 x^3\, dx = \frac{1}{2} \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{2} \left[\frac{1}{4} \right] = \frac{1}{8}.$$

Example 4 (a) : Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$.

Sol. : When the limits of integration for x and y are constants and specified or equal and when the integrand can be separated as $f(x, y) = f_1(x) \cdot f_2(y)$, we can evaluate the integral as follows.

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^1 \frac{dy}{\sqrt{1-y^2}} = \left[\sin^{-1} x\right]_0^1 \left[\sin^{-1} y\right]_0^1 = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}.$$

Example 5 (a) : Evaluate $\int_1^2 \int_0^x \frac{dy\,dx}{x^2+y^2}$.

Sol. : We have

$$I = \int_1^2 \int_0^x \frac{dy\,dx}{x^2+y^2} = \int_1^2 \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx = \int_1^2 \frac{1}{x} \cdot \left[\frac{\pi}{4} - 0\right] dx$$

$$= \frac{\pi}{4}\int_1^2 \frac{dx}{x} = \frac{\pi}{4}\left[\log x\right]_1^2 = \frac{\pi}{4}\log 2.$$

Example 6 (a) : Evaluate $\int_0^1 \int_0^1 (x+2)\,dy\,dx$.

Sol. : We have

$$I = \int_0^1 \int_0^1 (x+2)\,dy\,dx = \int_0^1 \left[xy + 2y\right]_0^1 dx = \int_0^1 (x+2)\,dx$$

$$= \left[\frac{x^2}{2} + 2x\right]_0^1 = \frac{1}{2} + 2 = \frac{5}{2}.$$

Example 7 (a) : Evaluate $\int_0^2 \int_0^{\sqrt{2x}} xy\,dy\,dx$.

Sol. : We have

$$I = \int_0^2 \left[x \cdot \frac{y^2}{2}\right]_0^{\sqrt{2x}} dx = \int_0^2 \left(\frac{x \cdot 2x}{2}\right) dx = \int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}.$$

(M.U. 2002, 06)

Example 8 (a) : Find $\int_0^1 \int_0^y xy\,e^{-x^2}\,dx\,dy$.

Sol. : We have

$$I = \int_0^1 y\left[\frac{e^{-x^2}}{-2}\right]_0^y dy = -\frac{1}{2}\int_0^1 \left(ye^{-y^2} - y\right) dy = -\frac{1}{2}\left[\frac{e^{-y^2}}{-2} - \frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{4}\left[(e^{-1}+1) - (1)\right] = \frac{1}{4e}.$$

(M.U. 1997, 2004)

Example 9 (a) : Evaluate $\int_0^1 \int_0^{x^2} e^{y/x}\,dy\,dx$.

Sol. : We have $I = \int_0^1 \left[\int_0^{x^2} e^{y/x}\,dy\right] dx = \int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx = \int_0^1 \frac{(e^x - 1)}{1/x}\,dx$

$$\therefore \quad I = \int_0^1 x e^x \, dx - \int_0^1 x \, dx = \left[x e^x - e^x \right]_0^1 - \left[\frac{x^2}{2} \right]_0^1$$

$$= e^1 - e^1 + 1 - \frac{1}{2} = \frac{1}{2}.$$

Example 10 (a) : Evaluate $\int_0^1 \int_0^x (x^2 + y^2)\, x \, dy \, dx$. **(M.U. 2016)**

Sol. : We have

$$I = \int_0^1 \int_0^x (x^3 + xy^2) \, dy \, dx = \int_0^1 \left[x^3 y + \frac{xy^3}{3} \right]_0^x dx$$

$$= \int_0^1 \left(x^4 + \frac{x^4}{3} \right) dx = \int_0^1 \frac{4x^4}{3} \, dx = \left[\frac{4}{3} \cdot \frac{x^5}{5} \right]_0^1 = \frac{4}{15}.$$

Example 11 (a) : Evaluate $\int_0^1 \int_{x^2}^x xy\,(x + y) \, dy \, dx$. **(M.U. 1996)**

Sol. : We have

$$I = \int_0^1 \int_{x^2}^x (x^2 y + xy^2) \, dy \, dx = \int_0^1 \left[\frac{x^2 y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^x dx$$

$$= \int_0^1 \left(\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx = \left[\frac{5}{6} \cdot \frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{3}{56}.$$

Example 12 (a) : Find $\int_0^a \int_0^{\sqrt{a^2 - y^2}} \sqrt{a^2 - x^2 - y^2} \, dx \, dy$. **(M.U. 2013)**

Sol. : We have

$$I = \int_0^a \int_0^{\sqrt{a^2 - y^2}} \sqrt{(a^2 - y^2) - x^2} \, dx \, dy$$

$$= \int_0^a \left[\frac{x}{2} \sqrt{(a^2 - y^2) - x^2} + \left(\frac{a^2 - y^2}{2} \right) \sin^{-1} \frac{x}{\sqrt{a^2 - y^2}} \right]_0^{\sqrt{a^2 - y^2}} dy$$

$$= \int_0^a \left(\frac{a^2 - y^2}{2} \right) \cdot \frac{\pi}{2} \, dy = \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi a^3}{6}.$$

Example 13 (a) : Find $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1 + x^2 + y^2} \, dy \, dx$. **(M.U. 1999, 2002, 09, 13, 14, 15)**

Sol. : We have

$$I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{y^2 + (1 + x^2)} \, dy \, dx$$

$$\therefore \quad I = \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_0^{\sqrt{1+x^2}} dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}} = \frac{\pi}{4} \left[\log\left(x + \sqrt{1+x^2} \right) \right]_0^1 = \frac{\pi}{4} \log(1+\sqrt{2}).$$

Example 14 (a) : Evaluate $\int_0^1 \int_0^{\sqrt{(1-y^2)/2}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$. **(M.U. 1999)**

Sol. : Integrating w.r.t. x, treating y constant *i.e.* $1-y^2 = a^2$ say

$$I = \int_0^1 \int_0^{\sqrt{(1-y^2)/2}} \frac{dx\,dy}{\sqrt{(1-y^2)-x^2}} = \int_0^1 \left[\sin^{-1}\left(\frac{x}{\sqrt{1-y^2}} \right) \right]_0^{\sqrt{(1-y^2)/2}} dy$$

$$= \int_0^1 \left[\sin^{-1}\left(\frac{1}{\sqrt{2}} \right) - \sin^{-1} 0 \right] dy = \int_0^1 \frac{\pi}{4}\, dy = \frac{\pi}{4}[\,y\,]_0^1 = \frac{\pi}{4}.$$

Example 15 (a) : Evaluate $\int_0^{a\sqrt{3}} \int_0^{\sqrt{x^2+a^2}} \frac{x\,dy\,dx}{y^2+x^2+a^2}$. **(M.U. 1999, 2012)**

Sol. : Treating x constant

$$I = \int_0^{a\sqrt{3}} \int_0^{\sqrt{x^2+a^2}} \frac{x}{y^2+(x^2+a^2)}\, dy\, dx$$

$$= \int_0^{a\sqrt{3}} \left[\frac{1}{\sqrt{x^2+a^2}} \tan^{-1}\left(\frac{y}{\sqrt{x^2+a^2}} \right) \right]_0^{\sqrt{x^2+a^2}} x\, dx$$

$$= \int_0^{a\sqrt{3}} \frac{1}{\sqrt{x^2+a^2}} \cdot \left[\frac{\pi}{4} - 0 \right] \cdot x\, dx$$

$$= \frac{\pi}{4} \int_0^{a\sqrt{3}} \frac{x}{\sqrt{x^2+a^2}}\, dx = \frac{\pi}{4} \left[\sqrt{x^2+a^2} \right]_0^{a\sqrt{3}}$$

$$= \frac{\pi}{4}[2a - a] = \frac{\pi}{4} a.$$

Example 16 (a) : Evaluate $\int_0^\infty \int_0^\infty e^{-x^2(1+y^2)}\, x\, dx\, dy$. **(M.U. 2003)**

Sol. : We put $x^2(1+y^2) = t \quad \therefore \; 2x(1+y^2)\, dx = dt$

When $x = 0$, $t = 0$; when $x = \infty$, $t = \infty$.

$$\therefore \quad I = \int_0^\infty \int_0^\infty e^{-t} \cdot \frac{dt}{2(1+y^2)}\, dy = \int_0^\infty \frac{1}{2(1+y^2)} \left[-e^{-t} \right]_0^\infty dy$$

$$= -\int_0^\infty \frac{1}{2(1+y^2)} [0-1]\, dy = \frac{1}{2} \int_0^\infty \frac{dy}{1+y^2} = \frac{1}{2} \left[\tan^{-1} y \right]_0^\infty$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

Example 17 (a) : Evaluate $\int_0^\infty dx \int_0^1 e^{-x^a y}\, dy.$ **(M.U. 1997)**

Sol. : Since the limits are constants and integration with respect to y first and then with respect to x leads to a complicated integral, we reverse the order of integration.

$$\therefore \quad I = \int_0^1 dy \int_0^\infty e^{-x^a y}\, dx$$

Now, put $x^a y = t \quad \therefore \; x = \left(\dfrac{t}{y}\right)^{1/a} \quad \therefore \; dx = \dfrac{1}{a} \cdot \dfrac{t^{(1/a)-1}}{y^{1/a}}\, dt$

When $x = 0,\ t = 0$; when $x = \infty,\ t = \infty$.

$$\therefore \quad I = \int_0^1 dy \int_0^\infty e^{-t} \cdot \frac{1}{a \cdot y^{1/a}} \cdot t^{(1/a)-1}\, dt$$

$$= \frac{1}{a}\int_0^1 y^{-1/a} dy \cdot \int_0^\infty e^{-t} \cdot t^{(1/a)-1}\, dt$$

$$= \frac{1}{a}\left[\frac{y^{-(1/a)+1}}{(-1/a)+1}\right]_0^1 \cdot \left\lceil \frac{1}{a} = \frac{\lceil 1/a}{a-1} \qquad \text{[By definition of } \lceil n \text{]}$$

Example 18 (a) : Evaluate $\int_0^1 \int_y^{\sqrt{y}} \dfrac{x}{(1-y)\sqrt{y-x^2}}\, dx\, dy.$ **(M.U. 2013)**

Sol. : We have

$$I = \int_0^1 \int_y^{\sqrt{y}} \frac{1}{1-y} \cdot \frac{x}{\sqrt{y-x^2}}\, dx\, dy$$

(Putting $y - x^2 = t,\ -2x\, dx = dt$)

$$\therefore \quad I = \int_0^1 \frac{1}{1-y}\left[-\sqrt{y-x^2}\right]_y^{\sqrt{y}} dy = -\int_0^1 \frac{1}{1-y}\left(0 - \sqrt{y-y^2}\right) dy$$

$$= \int_0^1 \frac{1}{1-y}\sqrt{y} \cdot \sqrt{1-y}\, dy = \int_0^1 \sqrt{\frac{y}{1-y}}\, dy$$

Now, put $y = \sin^2\theta,\ dy = 2\sin\theta\cos\theta\, d\theta$.

When $y = 0,\ \theta = 0$; when $y = 1,\ \theta = \pi/2$.

$$\therefore \quad I = \int_0^{\pi/2} \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \cdot \sin\theta \cdot \cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^2\theta\, d\theta = 2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{2} \qquad \text{[By (20), page 6-28]}$$

[Or put $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$.]

EXERCISE - I

Evaluate the following integrals : Class (a) : 3 Marks

1. $\int_0^{\pi/2}\int_0^{3(1-\cos t)} x^2 \sin t \, dx\, dt$

2. $\int_0^1\int_0^x xy\,(x^2y + xy^2)\, dy\, dx$

3. $\int_0^{\pi/2}\int_{\pi/2}^{x} \cos(x+y)\, dy\, dx$

4. $\int_0^4\int_0^{\sqrt{\frac{1}{2}(1-y^2)}} \frac{dx\, dy}{\sqrt{1-x^2-y^2}}$

5. $\int_1^2\int_0^x \frac{1}{x^2+y^2}\, dy\, dx$

6. $\int_0^2\int_0^{\sqrt{2x-x^2}} xy\, dy\, dx$

7. $\int_0^1\int_0^x (x^2+y^2)\, x\, dy\, dx$ **(M.U. 2002, 16)**

8. $\int_0^1\int_0^{x^2} x\,(x^2+y^2)\, dy\, dx$

9. $\int_0^a\int_0^{\sqrt{a^2-x^2}} x^2\, y\, dy\, dx$

10. $\int_0^{a\sqrt{3}}\int_0^{\sqrt{x^2+a^2}} \frac{x\, dy\, dx}{x^2+y^2+a^2}$

11. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} xy\, dy\, dx$

12. $\int_0^5\int_{2-x}^{2+x} dy\, dx$ **(M.U. 2002)**

13. $\int_1^2\int_{-(2-y)}^{(2-y)} 2x^2y^2\, dx\, dy$ **(M.U. 2002)**

14. $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}$ **(M.U. 2002)**

15. $\int_0^{\infty}\int_0^{\infty} \frac{e^{-x^3}}{\sqrt{x}} \cdot e^{-y^6} \cdot y^4\, dx\, dy$

[**Ans. : (1)** $9/36$, **(2)** $1/12$, **(3)** 2, **(4)** π, **(5)** $(\pi \log 2)/4$, **(6)** $2/3$, **(7)** $4/15$, **(8)** $32/3$, **(9)** $a^2/15$, **(10)** $\pi\, a/4$, **(11)** $2\,a^4/3$, **(12)** 25, **(13)** $22/45$, **(14)** $(\pi/4)\log(1+\sqrt{2})$ **(15)** $\pi/9$.]

Sketch The Region Of Integration And Then Evaluate

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Sketch the region of integration and evaluate

$$\int_1^4\int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy\, dx.$$

Sol. : The region is bounded by $y = 0$, *i.e.* the x-axis; $y = \sqrt{x}$ *i.e.* $y^2 = x$ an arc of the parabola; the lines $x = 1$ *i.e.*, CD and $x = 4$ *i.e.*, AB.

$\therefore$ The region of integration is $ABCD$.

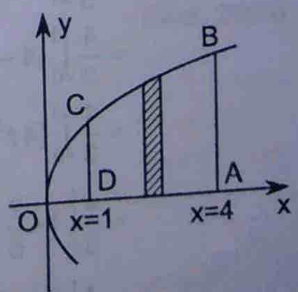

Fig. 9.3

$$I = \int_1^4\int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy\, dx = \frac{3}{2}\int_1^4 \left[\sqrt{x}\, e^{y/\sqrt{x}}\right]_0^{\sqrt{x}} dx$$

$$= \frac{3}{2}\int_0^4 \sqrt{x}\left[e^1 - e^0\right] dx = \frac{3}{2}(e-1)\left[\frac{x^{3/2}}{3/2}\right]_0^4$$

$$= (e-1)[8] = 8(e-1).$$

Example 2 (a) : Sketch the region of integration and evaluate

$$\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy.$$

Sol. : The region is bounded by $x = 0$ *i.e.*, the y-axis, $x = y^2$ *i.e.*, the parabola and the line $y = 0$ *i.e.*, the x-axis and $y = 1$ *i.e.*, the line BA.

$\therefore$ The region of integration is OAB.

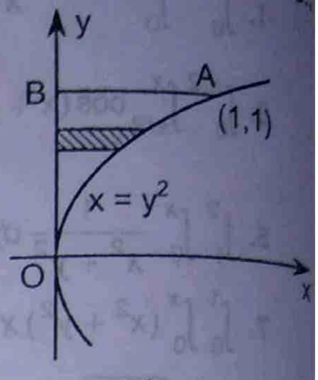

Fig. 9.4

$$\text{Now,} \quad I = \int_0^1 \int_0^{y^2} 3y^3 \cdot e^{xy}\, dx\, dy = \int_0^1 \left[3y^3 \left(\frac{e^{xy}}{y} \right) \right]_0^{y^2} dy$$

$$= \int_0^1 3y^2 \left[e^{xy} \right]_0^{y^2} dy = \int_0^1 3y^2 [e^{y^3} - 1]\, dy$$

$$= \int_0^1 3y^2 e^{y^3} dy - \int_0^1 3y^2\, dy$$

$$= \left[e^{y^3} \right]_0^1 - \left[y^3 \right]_0^1 \qquad [\text{ Put } y^3 = t\,]$$

$$= (e - 1) - (1 - 0) = e - 2.$$

Example 3 (a) : Sketch the area of integration and evaluate $\int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2 y^2\, dx\, dy.$

(M.U. 1987, 91, 2003)

Sol. : We have $x = \pm \sqrt{2 - y} \qquad \therefore \quad x^2 = 2 - y \;\; i.e. \;\; y - 2 = -x^2.$

The curve is a parabola with vertex at (0, 2) as shown in the figure. And y varies from 1 to 2. $y = 1$ is the line AB and $y = 2$ is a line parallel to the x-axis through C. The region of integration is ABC.

$$\therefore \quad I = \int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2 y^2\, dx\, dy = 2\int_1^2 \int_0^{\sqrt{2-y}} 2x^2 y^2\, dx\, dy$$

$$= 2\int_1^2 2y^2 \left[\frac{x^3}{3} \right]_0^{\sqrt{2-y}} dy = \frac{4}{3} \int_1^2 y^2 (2 - y)^{3/2}\, dy$$

Putting $2 - y = t$, $dy = -\,dt$.

When $y = 1$, $t = 1$; when $y = 2$, $t = 0$.

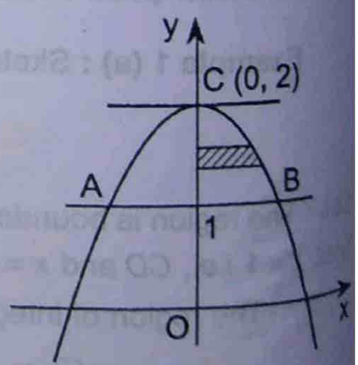

Fig. 9.5

$$\therefore \quad I = \frac{4}{3} \int_1^0 -(2 - t)^2 \cdot t^{3/2}\, dt$$

$$= \frac{4}{3} \int_0^1 (4 - 4t + t^2)\, t^{3/2}\, dt$$

$$= \frac{4}{3} \int_0^1 (4t^{3/2} - 4t^{5/2} + t^{7/2})\, dt$$

$$= \frac{4}{3} \left[4 \cdot \frac{2}{5} t^{5/2} - 4 \cdot \frac{2}{7} t^{7/2} + \frac{2}{9} t^{9/2} \right]_0^1$$

$$= \frac{4}{3} \left[\frac{8}{5} - \frac{8}{7} + \frac{2}{9} \right] = \frac{856}{945}.$$

EXERCISE - II

Sketch the region of integration and then evaluate each of the following integrals : Class (a) : 3 Marks

1. $\int_0^1 \int_0^x (x^2 + y^2)\, x\, dy\, dx$ 2. $\int_0^2 \int_0^{x^2} x(x^2 + y^2)\, dy\, dx$ 3. $\int_0^1 \int_0^x e^{x+y}\, dy\, dx$

4. $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\, dy\, dx$ 5. $\int_0^1 \int_0^x (x^2 + y^2)\, dy\, dx$ 6. $\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx$

7. $\int_0^{\infty} \int_x^{\infty} e^{-y}\, dy\, dx$ 8. $\int_0^1 \int_0^1 (x+2)\, dy\, dx$

[Ans. : (1) $\frac{4}{15}$, (2) $\frac{128}{9}$, (3) $\frac{(e-1)^2}{2}$, (4) $\frac{a^5}{15}$, (5) $\frac{1}{3}$, (6) $\frac{a^4}{8}$, (7) 1, (8) $\frac{5}{2}$.]

2. Double Integral in Polar Co-ordinates

Let the region be defined by the curves $r = f_1(\theta)$, $r = f_2(\theta)$, $\theta = \alpha$, $\theta = \beta$.

Then the double integral can be evaluated as follows

$$\boxed{\iint_A f(r,\theta)\, dA = \int_\alpha^\beta \left\{ \int_{f_1(\theta)}^{f_2(\theta)} f(r,\theta)\, dr \right\} d\theta}$$

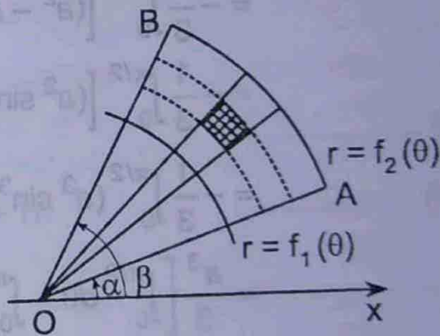

Fig. 9.6

where integration with respect to r is performed first by treating θ as constant and then the second integral evaluated w.r.t. θ.

Let the region be bounded by the two curves $r = f_1(\theta)$ and $f_2(\theta)$ and the radii OA and OB making angles α and β with the xis. Consider a radial strip in the region of integration. On this r varies from $r = f_1(\theta)$ to $r = f_2(\theta)$ and then θ varies from α to $\theta = \beta$.

Thus, we get

$$\iint_A f(r,\theta)\, dA = \int_\alpha^\beta \left\{ \int_{f_1(\theta)}^{f_2(\theta)} f(r,\theta)\, dr \right\} d\theta$$

Solved Examples : Class (a) : 3 Marks

(M.U. 2003)

Example 1 (a) : Evaluate $\int_0^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\, d\theta$.

Sol. : We have

$$I = \int_0^{\pi/4} \frac{1}{2} \left[-\frac{1}{1+r^2} \right]_0^{\sqrt{\cos 2\theta}} d\theta = \frac{1}{2} \int_0^{\pi/4} \left[1 - \frac{1}{1+\cos 2\theta} \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} \left(1 - \frac{1}{2} \sec^2 \theta \right) d\theta = \frac{1}{2} \left[\theta - \frac{1}{2} \tan \theta \right]_0^{\pi/4}$$

$$= \frac{1}{2} \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{1}{8} (\pi - 2).$$

Example 2 (a) : Evaluate the integral $\int_0^{\pi/2}\int_0^{1-\sin\theta} r^2\cos\theta\, dr\, d\theta$.

Sol. : We have

$$I=\int_0^{\pi/2}\int_0^{1-\sin\theta} r^2\cos\theta\, dr\, d\theta=\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{1-\sin\theta}\cos\theta\, d\theta$$

$$=\frac{1}{3}\int_0^{\pi/2}(1-\sin\theta)^3\cos\theta\, d\theta$$

Put $1-\sin\theta=t$ $\quad\therefore\ -\cos\theta\, d\theta=dt$

When $\theta=0,\ t=1$; when $\theta=\pi/2,\ t=0$.

$$\therefore\ I=-\frac{1}{3}\int_1^0 t^3\, dt=-\frac{1}{3}\left[\frac{t^4}{4}\right]_1^0=-\frac{1}{12}[0-1]=\frac{1}{12}$$

Example 3 (a) : Evaluate $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\, dr\cdot d\theta$.

Sol. : We have

$$I=\int_0^{\pi/2}-\frac{1}{3}\left[(a^2-r^2)^{3/2}\right]_0^{a\cos\theta}d\theta$$

$$=-\frac{1}{3}\int_0^{\pi/2}\left[(a^2-a^2\cos^2\theta)^{3/2}-(a^2)^{3/2}\right]d\theta$$

$$=-\frac{1}{3}\int_0^{\pi/2}\left[(a^2\sin^2\theta)^{3/2}-(a^2)^{3/2}\right]d\theta$$

$$=-\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta-a^3)\, d\theta=\frac{a^3}{3}\int_0^{\pi/2}(1-\sin^3\theta)\, d\theta$$

$$=\frac{a^3}{3}\left[\int_0^{\pi/2}d\theta-\int_0^{\pi/2}\sin^3\theta\, d\theta\right]$$

$$=\frac{a^3}{3}\left\{[\theta]_0^{\pi/2}-\frac{2}{3}\cdot 1\right\}\qquad\text{[By (21), page 6-29]}$$

$$=\frac{a^3}{3}\left[\frac{\pi}{2}-\frac{2}{3}\right]$$

(a) To prove that $B(m,n)=\dfrac{\overline{|m}\ \overline{|n}}{\overline{|m+n}}$

Proof : To prove the property we assume a property of double integral. **If the limits of integration are constants and specified for x and y and $F(x)$ and $\Phi(y)$ are respectively functions of x and y only then the double integral can be expressed as the product of two integrals** *i.e.*

$$\int_{x=a}^{b}\int_{y=c}^{d}F(x)\cdot\Phi(y)\, dx\, dy=\int_{x=a}^{b}F(x)\, dx\int_{y=c}^{d}\Phi(y)\, dy.$$

Now consider second form of Gamma function [(4), page 6-3]. We have

$$\overline{|m}\cdot\overline{|n}=2\cdot\int_0^\infty e^{-x^2}\cdot x^{2m-1}\cdot dx\cdot 2\int_0^\infty e^{-y^2}\cdot y^{2n-1}\, dy$$

$$=4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\cdot x^{2m-1}\cdot y^{2n-1}\, dy.$$

We now change the integral on r.h.s. to polar form by putting $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,d\theta\,dr$.

Since, x and y change from 0 to ∞ the region of integration is the entire first quadrant. To span this region in polar coordinates r must vary from 0 to ∞ and θ from 0 to $\pi/2$.

$$\therefore\ \overline{|m} \cdot \overline{|n} = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} \cdot r^{2m-1}\cos\theta^{2m-1} \cdot r^{2n-1} \cdot \sin\theta^{2n-1} \cdot r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r^{2(m+n)-1}\cos\theta^{2m-1} \cdot \sin\theta^{2n-1}\,dr\,d\theta$$

$$= 2\int_0^{\infty} e^{-r^2}\, r^{2(m+n)-1}\,dr \cdot 2\int_0^{\pi/2}\cos\theta^{2m-1} \cdot \sin\theta^{2n-1}\,d\theta$$

$$= \overline{|m+n} \cdot B(m, n) \qquad \text{[By (4), page 6-3 and (4), page 6-17]}$$

Now, the first integral is a Gamma function and the second integral is a Beta function.

$$\therefore\ B(m, n) = \frac{\overline{|m}\ \overline{|n}}{\overline{|m+n}}.$$

Solved Example : Class (a) : 3 Marks

Example : Prove that $\int_0^1 x^m(1-x)^n\,dx = \dfrac{m!\,n!}{(m+n+1)!}$, if m, n are integers.

Sol. : By definition of the Beta function,

$$\int_0^1 x^m(1-x)^n\,dx = B(m+1, n+1) \qquad \text{.................... (i)}$$

Now, by the above result

$$\beta(m+1, n+1) = \frac{\overline{|m+1} \cdot \overline{|n+1}}{\overline{|m+n+2}} \qquad \text{.................. (ii)}$$

But $\overline{|m+1} = m!$, $\overline{|n+1} = n!$ and $\overline{|m+n+2} = (m+n+1)!$ .... (iii)

Combining (i), (ii) and (iii), we get,

$$\int_0^1 x^m(1-x)^n\,dx = \frac{m!\,n!}{(m+n+1)!}.$$

EXERCISE - III

Evaluate the following polar integrals : Class (a) : 3 Marks

1. $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$
2. $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\,dr\,d\theta$
3. $\int_0^{\pi/2}\int_0^{a(1+\sin\theta)} r^2\cos\theta\,dr\,d\theta$
4. $\int_0^{\pi/2}\int_0^{a\cos\theta} r^2\,dr\,d\theta$
5. $\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \dfrac{r}{(1+r^2)^2}\,dr\,d\theta$ **(M.U. 2008)**
6. $\int_0^{\pi/2}\int_0^{a\sin\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$
7. $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$
8. $\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\sin\theta\,dr\,d\theta$

[**Ans. :** (1) $\dfrac{\pi a^2}{4}$ (2) $\dfrac{a^2}{6}$ (3) $\dfrac{5a^3}{4}$ (4) $\dfrac{2a^3}{9}$ (5) $\dfrac{(\pi-2)}{8}$ (6) $\dfrac{a^3}{3}\left[\dfrac{\pi}{2}-\dfrac{2}{3}\right]$ (7) $\dfrac{3}{4}\pi a^2$ (8) $\dfrac{2a^3}{3}$]

3. Evaluation of Integrals by Change of Order of Integration

It is sometimes more easy to evaluate the double integral after changing the order of integration. Sometimes to evaluate integral w.r.t. one variable is easier than that w.r.t. the other variable.

Now, in the region given on page 9-2 (Fig. 9.2), if we consider a strip parallel to the x-axis then on this strip x changes from $\Phi_1(y)$ to $\Phi_2(y)$. [See Fig. 9.7]

When the strip moves parallel to the x-axis y varies from c to d.

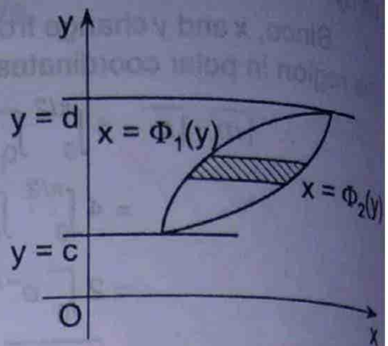

Fig. 9.7

$$\therefore \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\, dy\, dx = \int_c^d \int_{\Phi_1(y)}^{\Phi_2(y)} f(x, y)\, dx\, dy$$

Note ....

If the limits of integration are constants *i.e.* the region of integration is a rectangle then the change in the order of integration does not change the limits of integration.

We shall now solve a few examples of evaluating the given integral by change of order of integration.

To Change the Order of Integration

Suppose we have to evaluate $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\, dy\, dx$.

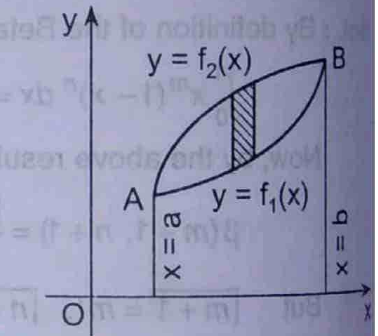

Fig. 9.7 (a)

1. We first note that we have to carry out the integral w.r.t. y first from $y = f_1(x)$ to $y = f_2(x)$ *i.e.*, along the vertical strip.
2. The points of intersection of the two curves are A, $x = a$, B, $x = b$.

   Now, we carry out the integration w.r.t. x from $x = a$ to $x = b$ as the strip moves from $x = a$ to $x = b$.

   If the integration w.r.t. y, first is difficult or impossible we change the order or integration.
3. Now, we first integrate $f(x, y)$ w.r.t. x from $x = \Phi_1(y)$ to $x = \Phi_2(y)$ *i.e.*, along the horizontal strip.
4. The points of intersection of the two curves are A, $y = c$, B, $y = d$.

   Now, we carry out the integration w.r.t. y from $y = c$ to $y = d$ as the strip moves from $y = c$ to $y = d$. Then

$$I = \int_c^d \Phi(y)\, dy$$

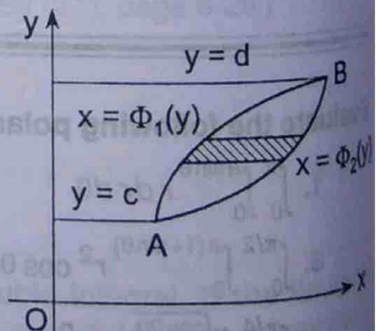

Fig. 9.7 (b)

(a) If we are given the integral as

$$\int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\, dy\, dx$$

it shows that the integral into be carried w.r.t. y, first from $y = f_1(x)$ to $y = f_2(x)$. Then the integration is to be carried w.r.t. x from $x = a$ to $x = b$. To change the order of this integration we consider a strip **parallel to the x-axis** and find the limits of the first integration in terms of $x = \Phi_1(y)$ and $x = \Phi_2(y)$. Then we find the limits of y as $y = c$ and $y = d$ and integrate.

(b) If we are given the integral as

$$\int_c^d \int_{f_1(y)}^{f_2(y)} f(x, y)\, dx\, dy$$

it shows that the integral is to be carried w.r.t. x first from $x = f_1(y)$ to $x = f_2(y)$. Then the integration is to be carried w.r.t. y from $y = c$ to $y = d$. To change the order of this integration, we consider a strip parallel to the y-axis and find the limits of the first integration in terms of $y = \Phi_1(x)$ and $y = \Phi_2(x)$. Then we find the limits of x as $x = a$ and $x = b$ and integrate.

Procedure to Change of Order of Integration

1. Find out the given order and the limits of integration and from these limits, find **the region of integration**, find the points of intersection of all curves given.
2. Consider a strip by changing the order and find the limits of x and y.
3. Now, integrate w.r.t. the new limits.

Solved Examples : Class (b) : 6 Marks

Type I : Integral Changes to a Single Integral

Example 1 (b) : Change the order of integration and evaluate the integral

$$\int_0^2 \int_0^{4-x^2} \frac{x\, e^{2y}}{4-y}\, dy\, dx.$$

Sol. : 1. Given order and given limits : Given order is : first w.r.t. y and then w.r.t. x *i.e.*, a strip parallel to the y-axis. y changes from $y = 0$ to $y = 4 - x^2$ and then x changes from $x = 0$ to $x = 2$.

2. Region of integration : $y = 0$ is the x-axis and $y = 4 - x^2$ *i.e.*, $(y - 4) = -x^2$ is a parabola symmetrical about the y-axis with vertex at (0, 4) and opening downwards. It intersects the x-axis where $4 - x^2 = 0$ *i.e.*, at $x = \pm 2$ *i.e.*, in $(-2, 0)$, $(2, 0)$. $x = 0$ is the y-axis and $x = 2$ is a line parallel to the y-axis. Thus, the region of integration is the part OAB of the parabola.

3. Change of the order of integration : Now, to change the order of integration, consider a strip parallel to the x-axis in the region of integration. On this strip x varies from $x = 0$ to $x = \sqrt{4-y}$. Then the strip moves from $y = 0$ to $y = 4$.

$$\therefore \quad I = \int_{y=0}^{4} \int_{x=0}^{\sqrt{4-y}} \frac{x\, e^{2y}}{4-y}\, dx\, dy$$

$$= \int_0^4 \left[\frac{x^2}{2}\right]_0^{\sqrt{4-y}} \cdot \frac{e^{2y}}{4-y}\, dy$$

$$= \frac{1}{2}\int_0^4 (4-y) \cdot \frac{e^{2y}}{(4-y)}\, dy = \frac{1}{2}\int_0^4 e^{2y}\, dy$$

$$= \frac{1}{2}\left[\frac{e^{2y}}{2}\right]_0^4 = \frac{e^8 - 1}{4}.$$

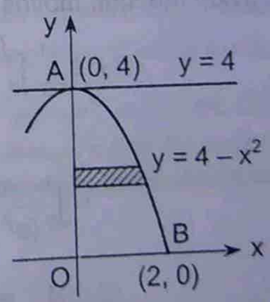

Fig. 9.8

Example 2 (b) : Change the order of integration and evaluate **(M.U. 1999)**

$$\int_0^1 \int_{4y}^{4} e^{x^2}\, dx\, dy.$$

Sol. : 1. Given order and given limits : Given order is : first w.r.t. x and then w.r.t. y *i.e.*, a strip parallel to the x-axis. x changes from $x = 4y$ to $x = 4$ and then y changes from $y = 0$ to $y = 1$.

2. Region of integration : $x = 4y$ *i.e.*, $y = x/4$ is a line passing through the origin. $x = 4$ is a line parallel to the y-axis. $y = 0$ is the x-axis and $y = 1$ is a line parallel to the x-axis. The points of intersection are $A\,(4, 0)$ and $B\,(4, 1)$.

Thus, the region of integration is the triangle OAB.

3. Change of the order of integration : Consider a strip parallel to the y-axis in the region of integration OAB. On this strip y varies from $y = 0$ to $y = x/4$ and then strip moves from $x = 0$ to $x = 4$.

$$\therefore \; I = \int_0^4 \int_0^{x/4} e^{x^2}\, dy\, dx = \int_0^4 e^{x^2}\left[\, y \,\right]_0^{x/4} dx$$

$$= \int_0^4 e^{x^2} \cdot \frac{x}{4}\, dx \qquad [\text{ Put } x^2 = t\,]$$

$$= \frac{1}{4}\left[\frac{e^{x^2}}{2}\right]_0^4 = \frac{1}{8}\left[e^{16} - 1\right]$$

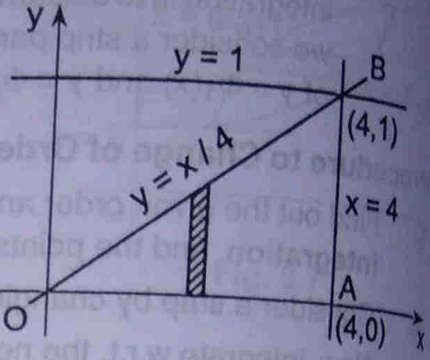

Fig. 9.9

Example 3 (b) : Change the order of integration and evaluate.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y + 1)\sqrt{1 - x^2 - y^2}}\, dy\, dx\,.$$

(M.U 1989, 93, 96, 2002, 16)

Sol. : 1. Given order and given limits : Given order is : first w.r.t. y and then w.r.t. x *i.e.* a strip parallel to the y-axis, y varies from $y = 0$ to $y = \sqrt{1 - x^2}$. x varies from $x = 0$ to $x = 1$.

2. Region of integration : $y = 0$ is the x-axis; $y = \sqrt{1 - x^2}$ *i.e.*, $x^2 + y^2 = 1$ is a circle with centre at the origin and radius unity. $x = 0$ is the y-axis and $x = 1$ is a line parallel to the y-axis. The points of intersection of the circle and the axes are $A\,(1, 0)$ and $B\,(0, 1)$. The region of integration is the quarter of the circle OAB.

3. Change of the order of integration : To change the order of integration, consider a strip parallel to the x-axis in the region of integration OAB. On this strip x varies from $x = 0$ to $x = \sqrt{1 - y^2}$. Then the strip moves from $y = 0$ to $y = 1$.

$$\therefore \; I = \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{e^y}{(e^y + 1)\sqrt{(1 - y^2) - x^2}}\, dx\, dy$$

$$= \int_0^1 \frac{e^y}{(e^y + 1)}\left[\sin^{-1} \frac{x}{\sqrt{1 - y^2}}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \frac{e^y}{e^y + 1} \cdot \frac{\pi}{2} \cdot dy = \frac{\pi}{2}\left[\log(e^y + 1)\right]_0^1 = \frac{\pi}{2} \log\left(\frac{e + 1}{2}\right).$$

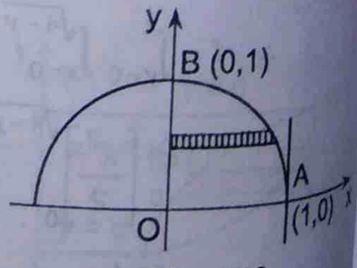

Fig. 9.10

Example 4 (b) : Change the order of integration and evaluate $\int_0^a \int_y^{\sqrt{ay}} \frac{x}{x^2 + y^2}\, dx\, dy$.

(M.U. 1998)

Sol. : 1. Given order and given limits : Give order is : first w.r.t. x and then w.r.t. y *i.e.*, a strip parallel to the x-axis. x varies from $x = y$ to $x = \sqrt{ay}$ (*i.e.*, $x^2 = \ldots$ ... 0 to $y = a$

2. Region of integration : $x = y$ is a line through the origin equally inclined to the axes. $x^2 = ay$ is a parabola, symmetrical about the y-axis with vertex at the origin and opening upwards. $y = 0$ is the x-axis and $y = a$ is a line parallel to the x-axis. The region of integration is the part of the parabola OAB.

3. Change of the order of integration : To change the order of integration, consider a strip parallel to the y-axis in the region of integration. On this strip y varies from $y = x^2/a$ to $y = x$. Then the strip moves from $x = 0$ to $x = a$.

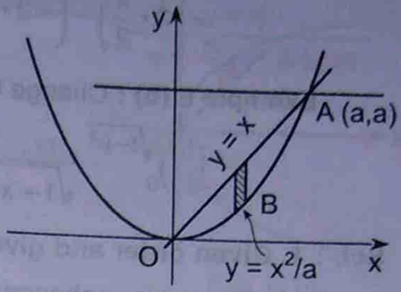

Fig. 9.11

$$\therefore \quad I = \int_0^a \int_{x^2/a}^{x} \frac{x}{x^2 + y^2}\, dy\, dx$$

$$= \int_0^a \left[\frac{x}{x} \tan^{-1} \frac{y}{x} \right]_{x^2/a}^{x} dx$$

$$= \int_0^a \left[\tan^{-1} 1 - \tan^{-1} \frac{x}{a} \right] dx = \int_0^a \left(\frac{\pi}{4} - \tan^{-1} \frac{x}{a} \right) dx$$

$$= \frac{\pi}{4} \cdot [\, x \,]_0^a - \left[x \cdot \tan^{-1} \frac{x}{a} - \int x \cdot \frac{1}{1 + (x^2/a^2)} \cdot \frac{1}{a}\, dx \right]_0^a \qquad \text{[By parts]}$$

$$= \frac{\pi a}{4} - \left[x \tan^{-1} \frac{x}{a} - \int \frac{ax}{x^2 + a^2}\, dx \right]_0^a$$

$$= \frac{\pi a}{4} - \left[x \tan^{-1} \frac{x}{a} - \frac{a}{2} \log (x^2 + a^2) \right]_0^a$$

$$\therefore \quad I = \frac{\pi a}{4} - \left[a \frac{\pi}{4} - \frac{a}{2} \log (2a^2) - 0 + \frac{a}{2} \log a^2 \right] = \frac{a}{2} \log 2. \qquad [\because \log 2a^2 = \log 2 + \log a^2\,]$$

Example 5 (b) : Change the order of integration and evaluate $\int_0^2 \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx\, dy$.

(M.U. 2002)

Sol. : 1. Given order and given limits : Given order is : first w.r.t. x and then w.r.t. y *i.e.*, a strip parallel to the x-axis. x varies from $x = 2 - \sqrt{4 - y^2}$ to $x = 2 + \sqrt{4 - y^2}$. y varies from $y = 0$ to $y = 2$.

2. Region of integration : $x = 2 - \sqrt{4 - y^2}$ is the arc OA and $x = 2 + \sqrt{4 - y^2}$ is the arc OB of the circle $(x - 2)^2 + y^2 = 4$ with centre at $(2, 0)$ and radius $= 2$ above the x-axis. $y = 0$ is the x-axis and $y = 2$ is the line parallel to the x-axis through $A(2, 2)$. The region of integration is the semi-circle OAB above the x-axis. The points of intersection of the circle and the x-axis are $O\,(0, 0)$ and $B\,(4, 0)$.

3. Change of order of integration : To change the order of integration, consider a strip parallel to the y-axis in the region of integration. On this strip y varies from $y = 0$ to $y = \sqrt{4 - (x - 2)^2}$ and then the strip moves from $x = 0$ to $x = 4$.

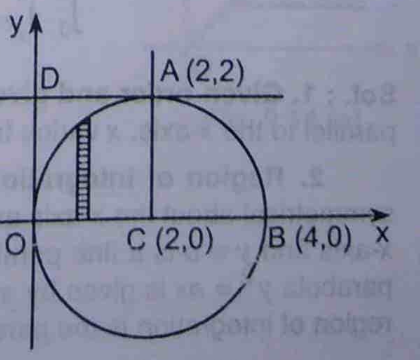

Fig. 9.12

$$\therefore \quad I = \int_0^4 \int_0^{\sqrt{4-(x-2)^2}} dy\, dx$$

$$= \int_0^4 [\, y \,]_0^{\sqrt{4-(x-2)^2}} dx = \int_0^4 \sqrt{4 - (x - 2)^2} \cdot dx$$

$$\therefore\ I = \left[\frac{(x-2)}{2}\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\frac{(x-2)}{2}\right]_0^4$$

$$= \left(2 \cdot \frac{\pi}{2}\right) - \left(-2 \cdot \frac{\pi}{2}\right) = 2\pi.$$

Example 6 (b) : Change the order of integration and evaluate.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{\cos^{-1} x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\, dx\, dy.$$ **(M.U. 1992, 99, 2007)**

Sol. : 1. Given order and given limits : Given order is : first w.r.t. x and then w.r.t. y *i.e.*, a strip parallel to the x-axis. x changes from $x = 0$ to $x = \sqrt{1-y^2}$ and then y changes from $y = 0$ to $y = 1$.

2. Region of integration : $x = 0$ is the y-axis. $x = \sqrt{1-y^2}$ *i.e.*, $x^2 + y^2 = 1$ is a circle with centre at the origin and radius 1. $y = 0$ is the x-axis and $y = 1$ is a line parallel to the x-axis. Thus, the region of integration is the part OAB of the unit circle.

3. Change of order of integration : Now, to change the order of integration consider a strip parallel to the y-axis in the region of integration. On this strip y varies from $y = 0$ to $y = \sqrt{1-x^2}$. Then the strip moves from $x = 0$ to $x = 1$.

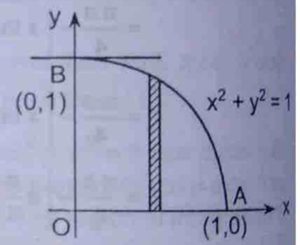

Fig. 9.13

$$\therefore\ I = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\cos^{-1} x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\, dy\, dx$$

$$= \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}} \left[\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2}\int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}}.$$

Put $\cos^{-1} x = t$, $\frac{dx}{\sqrt{1-x^2}} = -\,dt$. When $x = 0$, $t = \frac{\pi}{2}$ and when $x = 1$, $t = 0$.

$$\therefore\ I = -\frac{\pi}{2}\int_{\pi/2}^0 t\, dt = \frac{\pi}{2}\int_0^{\pi/2} t\, dt = \frac{\pi}{2}\left[\frac{t^2}{2}\right]_0^{\pi/2} = \frac{\pi^3}{16}.$$

Example 7 (b) : Change the order of integration and evaluate.

$$\int_0^a \int_{y^2/a}^{y} \frac{y\, dx\, dy}{(a-x)\sqrt{ax-y^2}}.$$ **(M.U. 1991, 98, 14)**

Sol. : 1. Given order and given limits : Given order is : first w.r.t. x and then w.r.t. y. *i.e.*, a strip parallel to the x-axis. x varies from $x = y^2/a$ to $x = y$ and y varies from $y = 0$ to $y = a$.

2. Region of integration : $x = y^2/a$ *i.e.*, $y^2 = ax$ is a parabola with vertex at the origin, symmetrical about the x-axis and opening on the right. $x = y$ is a line through the origin. $y = 0$ is the x-axis and $y = a$ is a line parallel to the x-axis. The point of intersection of the line $y = x$ and the parabola $y^2 = ax$ is given by $x^2 = ax$. Therefore, $x = a$ and $y = a$ *i.e.*, the point $A\ (a, a)$. Thus, the region of integration is the part OAB of the parabola.

3. Change of order of integration : To change the order of integration consider a strip parallel to the y-axis in the region of integration. On this strip y varies from $y = x$ to $y = \sqrt{ax}$ and then the strip moves from $x = 0$ to $x = a$.

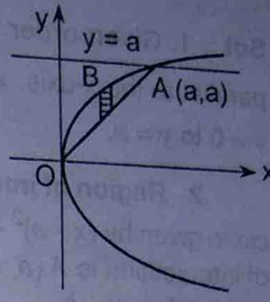

Fig. 9.14

$$\therefore \; I = \int_0^a \int_x^{\sqrt{ax}} \frac{y\, dy\, dx}{(a-x)\sqrt{ax - y^2}}$$

$$= -\int_0^a \frac{1}{(a-x)} \left[\sqrt{ax - y^2} \right]_x^{\sqrt{ax}} dx \qquad [\text{ Put } ax - y^2 = t]$$

$$\therefore \; I = \int_0^a \frac{1}{(a-x)} \cdot \sqrt{x} \cdot \sqrt{a-x}\, dx = \int_0^a \sqrt{\left(\frac{x}{a-x} \right)}\, dx$$

Now, put $x = a\cos^2\theta \quad \therefore \; dx = -2a\cos\theta \sin\theta \, d\theta$

$$\therefore \; I = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta} \cdot 2a \sin\theta \cos\theta \, d\theta$$

$$= a\int_0^{\pi/2} 2\cos^2\theta \, d\theta$$

$$= 2a\left[\frac{1}{2} \cdot \frac{\pi}{2} \right] = \frac{\pi}{2} \cdot a. \qquad [\text{ By (23), page 6-32 }]$$

Example 8 (b) : Change the order of integration and evaluate

$$\int_0^\infty \int_0^x x\, e^{-x^2/y}\, dy\, dx$$ **(M.U. 2010, 11, 14)**

Sol. : 1. Given order and given limits : Given order of integration is : first w.r.t. y and then w.r.t. x. y changes from $y = 0$ to $y = x$ and x changes from $x = 0$ to $x = \infty$.

2. Region of Integration : The region of integration is given by $y = 0$ *i.e.,* the x-axis and $y = x$ *i.e.,* the line through the origin equally inclined to both the aixs. $x = 0$ *i.e.,* the y-axis and x goes to ∞ means the region is infinite to the right of the above line.

3. Change of order of integration : Now, consider a strip parallel to the x-axis. On the strip x varies from y to ∞ and then strip moves from $y = 0$ to $y = \infty$.

$$\therefore \; I = \int_0^\infty \int_y^\infty x \cdot e^{x^2/y}\, dx\, dy$$

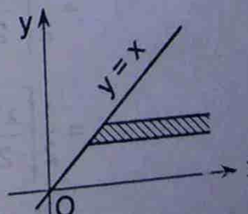

Fig. 9.14 (a)

To find the first integral put $x^2/y = t$.

$$\therefore \; 2x\frac{dx}{y} = dt \quad i.e., \quad x\,dx = \frac{y}{2}\,dt$$

$$\therefore \; \int x \cdot e^{-x^2/y}\, dx = \int \frac{y}{2} e^{-t} dt = \frac{y}{2}[-e^{-t}] = -\frac{y}{2} e^{-x^2/y}$$

$$\therefore \; I = \int_0^\infty \left[-\frac{y}{2} \cdot e^{-x^2/y} \right]_y^\infty dy = -\frac{1}{2}\int_0^\infty [0 - y e^{-y}]\, dx$$

$$= \frac{1}{2}\int_0^\infty y \cdot e^{-y}\, dy \qquad \text{(Now integrate by parts)}$$

$$= \frac{1}{2}\left[y(-e^{-y}) - e^{-y} \right]_0^\infty = -\frac{1}{2}[0 - 1] = \frac{1}{2}.$$

Example 9 (b) : Evaluate $\int_0^a dy \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx$. **(M.U. 1994, 2000, 09)**

Sol. : 1. Given order and given limits : Given order is : first w.r.t. x and then w.r.t. y. *i.e.*, a strip parallel to the x-axis. x goes from $x = 0$ to $x = a - \sqrt{a^2 - y^2}$ *i.e.*, $(x-a)^2 + y^2 = a^2$. y varies from $y = 0$ to $y = a$.

2. Region of integration : $x = 0$ is the y-axis and $x = a - \sqrt{a^2 - y^2}$ is the left half of the semi-circle given by $(x-a)^2 + y^2 = a^2$. $y = 0$ is the x-axis and $y = a$ is a line parallel to the x-axis. The point of intersection is $A\,(a, a)$. Thus, the region is area OAB bounded by the arc of the circle, the y-axis and the line $y = a$.

3. Change of order of integration : To change the order of integration consider a strip parallel to the y-axis. On this strip y varies from $y = \sqrt{a^2-(x-a)^2} = \sqrt{2ax - x^2}$ to $y = a$. Then the strip moves from $x = 0$ to $x = a$.

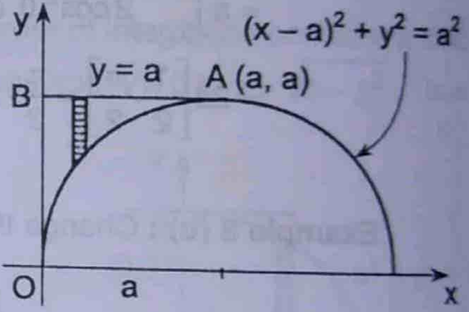

Fig. 9.15

$$\therefore\ I = \int_0^a dx \int_{\sqrt{2ax-x^2}}^{a} \frac{xy\log(x+a)}{(x-a)^2}\,dy$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left[\frac{y^2}{2}\right]_{\sqrt{2ax-x^2}}^{a} dx$$

$$= \int_0^a \frac{x\log(x+a)}{2(x-a)^2}\left[a^2 - 2ax + x^2\right] dx$$

$$= \frac{1}{2}\int_0^a x\log(x+a)\,dx$$

$$= \frac{1}{2}\left[\log(x+a)\cdot\frac{x^2}{2} - \int \frac{x^2}{2(x+a)}\,dx\right]_0^a \qquad \text{[By parts]}$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\int \frac{(x^2-a^2)+a^2}{(x+a)}\,dx\right]_0^a \qquad \textbf{[Note this]}$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\int (x-a) - \frac{a^2}{2}\int \frac{dx}{x+a}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\left(\frac{x^2}{2} - ax\right) - \frac{a^2}{2}\log(x+a)\right]_0^a$$

$$\therefore\ I = \frac{1}{2}\left[\frac{a^2}{2}\log(a+a) - \frac{1}{2}\left(\frac{a^2}{2} - a^2\right) - \frac{a^2}{2}\log(a+a) + \frac{a^2}{2}\log a\right] = \frac{a^2}{8}\left[1 + 2\log a\right].$$

Example 10 (b) : Evaluate $\int_0^a \int_0^x \frac{e^y}{\sqrt{(a-x)(x-y)}}\,dy\,dx$. **(M.U. 1995, 2005, 10)**

Sol. : 1. Given order and given limits : Given order is : first w.r.t. y and then w.r.t. x *i.e.*, a strip parallel to the y-axis. y varies from $y = 0$ to $y = x$ and x varies from $x = 0$ to $x = a$.

2. Region of integration : $y = 0$ is the x-axis and $y = x$ is a line passing through the origin. $x = 0$ is the y-axis and $x = a$ is a line parallel to the y-axis. The points of intersection are $A(a, 0)$ and $B(a, a)$. The region of integration is the triangle OAB.

3. Change of order of integration : To change the order of integration consider a strip parallel to the x-axis in the region of integration. On this strip x varies from $x = y$ to $x = a$ and the strip moves from $y = 0$ to $y = a$.

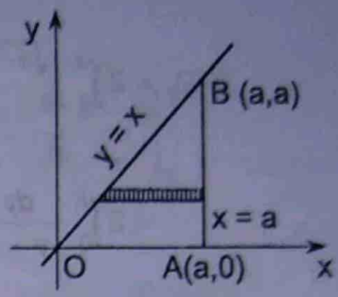

Fig. 9.16

$$\therefore \quad I = \int_0^a \int_y^a \frac{e^y}{\sqrt{(a-x)(x-y)}}\, dx\, dy$$

$$= \int_0^a \int_y^a \frac{e^y}{\sqrt{-ay - [x^2 - (a+y)x]}}\, dx\, dy$$

$$= \int_0^a \int_y^a \frac{e^y}{\sqrt{\left(\frac{a-y}{2}\right)^2 - \left(x - \frac{a+y}{2}\right)^2}}\, dx\, dy \qquad \text{[Complete the square]}$$

$$= \int_0^a e^y \left[\sin^{-1}\left\{\frac{x - (a+y)/2}{(a-y)/2}\right\}\right]_y^a dy = \int_0^a e^y \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] dy$$

$$= \pi \int_0^a e^y\, dy = \pi \left[e^y\right]_0^a = \pi\,(e^a - 1).$$

Aliter : The integral can also be evaluated by putting $x - y = t^2$

i.e. $x = y + t^2 \quad \therefore \; dx = 2t\, dt.$

When $x = y$, $t = 0$; when $x = a$, $t = \sqrt{a - y}$.

$$\therefore \quad I = \int_0^a \int_0^{\sqrt{a-y}} e^y \cdot \frac{1}{\sqrt{[(a-y) - t^2]} \cdot t} \cdot 2t\, dt\, dy$$

$$= \int_0^a e^y dy \int_0^{\sqrt{a-y}} \frac{2\, dt}{\sqrt{(a-y) - t^2}}$$

$$= 2\int_0^a e^y \cdot \left[\sin^{-1}\left(\frac{t}{\sqrt{a-y}}\right)\right]_0^{\sqrt{a-y}} dy = 2\int_0^a e^y \cdot \left[\frac{\pi}{2} - 0\right] dy$$

$$= \pi \int_0^a e^y\, dy = \pi\left[e^a - 1\right].$$

Example 11 (b) : Change the order of integration and evaluate

$$\int_0^a \int_0^x \frac{dy\, dx}{(y+a)\sqrt{(a-x)(x-y)}}.$$

(M.U. 1995, 99, 2003, 12)

Sol. : The given region of integration and the change of order is the same as above.

$$\therefore \quad I = \int_0^a \int_y^a \frac{dx\, dy}{(y+a)\sqrt{(a-x)(x-y)}}$$

Putting $x - y = t^2 \qquad \therefore \; dx = 2t\, dt$

When $x = y$, $t = 0$; when $x = a$, $t = \sqrt{a-y}$.

$$\therefore \quad I = \int_0^a \int_0^{\sqrt{a-y}} \frac{dy}{(y+a)} \cdot \frac{2t\,dt}{\sqrt{(a-y-t^2)}\cdot t}$$

$$= 2\int_0^a \int_0^{\sqrt{a-y}} \frac{dy}{(y+a)} \cdot \frac{dt}{\sqrt{(a-y)-t^2}}$$

$$= 2\int_0^a \frac{dy}{(y+a)} \left[\sin^{-1} \frac{t}{\sqrt{a-y}}\right]_0^{\sqrt{a-y}} = 2\int_0^a \frac{dy}{y+a} \cdot \frac{\pi}{2}$$

$$\therefore \quad I = \pi \int_0^a \frac{dy}{y+a} = \pi\left[\log(y+a)\right]_0^a = \pi\left[\log 2a - \log a\right] = \pi \log 2.$$

(Alternatively as in the above example, you may complete the square on x and then integrate.)

Example 12 (b) : Change the order of integration and evaluate

$$\int_0^{\pi} \int_0^{x} \frac{\sin y\,dy\,dx}{\sqrt{(\pi - x)(x-y)}}. \qquad \textbf{(M.U. 1997)}$$

Sol. : The region of integration and the change of order is the same as above after putting $a = \pi$. Hence,

$$\therefore \quad I = \int_0^{\pi} \int_y^{\pi} \frac{\sin y\,dx\,dy}{\sqrt{(\pi - x)(x-y)}}$$

Put $x - y = t^2$ $\quad\therefore\quad dx = 2t\,dt$

When $x = y$, $t = 0$; when $x = \pi$, $t = \sqrt{\pi - y}$.

$$\therefore \quad I = \int_0^{\pi} \int_0^{\sqrt{\pi-y}} \sin y \cdot \frac{2t\,dt}{\sqrt{(\pi-y)-t^2}\cdot t} = 2\int_0^{\pi} \int_0^{\sqrt{\pi-y}} \sin y \cdot \frac{dt}{\sqrt{(\pi-y)-t^2}}\,dy$$

$$= 2\int_0^{\pi} \sin y \left[\sin^{-1}\left(\frac{t}{\sqrt{\pi-y}}\right)\right]_0^{\sqrt{\pi-y}} dy = 2\int_0^{\pi} \sin y\,\frac{\pi}{2}\,dy = \pi\left[-\cos y\right]_0^{\pi} = 2\pi.$$

Example 13 (b) : Change the order of integration and evaluate

$$\int_0^a \int_0^y \frac{x\,dx\,dy}{\sqrt{(a^2-x^2)(a-y)(y-x)}}. \qquad \textbf{(M.U. 1997, 2002, 04)}$$

Sol. : 1. Given order and given limits : Given order is : first w.r.t. x and then w.r.t. y *i.e.*, a strip parallel to the x-axis. x varies from $x = 0$ to $x = y$ and y varies from $y = 0$ to $y = a$.

2. Region of integration : $x = 0$ is the y-axis, $x = y$ is a line passing through the origin. $y = 0$ is the x-axis and $y = a$ is a line parallel to the x-axis. The points of intersection are A (a, a) and $B\,(0, a)$. Thus, the region of integration is the triangle OAB.

3. Change of order of integration : To change the order of integration, consider a strip in the region of integration parallel to the y-axis. On this strip y varies from $y = x$ to $y = a$ and the strip moves from $x = 0$ to $x = a$.

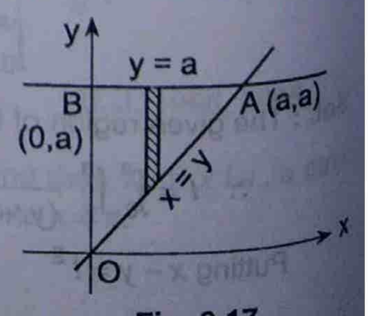

Fig. 9.17

$$\therefore \quad I = \int_0^a \int_x^a \frac{x\, dy\, dx}{\sqrt{a^2 - x^2}\, \sqrt{(a-y)(y-x)}}$$

Put $y - x = t^2 \quad \therefore \quad dy = 2t\, dt$

When $y = x$, $t = 0$; when $y = a$, $t = \sqrt{a - x}$.

$$\therefore \quad I = \int_0^a \int_0^{\sqrt{a-x}} \frac{x\, dx}{\sqrt{a^2 - x^2}} \frac{2t\, dt}{\sqrt{(a-x) - t^2} \cdot t} = 2\int_0^a \frac{x\, dx}{\sqrt{a^2 - x^2}} \left[\sin^{-1}\left(\frac{t}{\sqrt{a-x}}\right)\right]_0^{\sqrt{a-x}}$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2 - x^2}} \frac{\pi}{2}\, dx = \pi\left[-\sqrt{a^2 - x^2}\right]_0^a = \pi[-0 + a] = \pi a.$$

Example 14 (b) : Change the order of integration and evaluate

$$\int_0^a \int_0^x \frac{\sin y\, dy\, dx}{\sqrt{[(a-x)(x-y)]}\,(4 - 5\cos y)}. \qquad \textbf{(M.U. 1996, 2002)}$$

Sol. : The region of integration is the same as in Fig. 9.16. Consider a strip as in Ex. 10.

$$\therefore \quad I = \int_0^a \int_y^a \frac{\sin y}{(4 - 5\cos y)} \cdot \frac{dx}{\sqrt{[(a-x)(x-y)]}}\, dy$$

Put $x - y = t^2 \quad \therefore \quad dx = 2t\, dt$

When $x = y$, $t = 0$; when $x = a$, $t = \sqrt{a - y}$.

$$\therefore \quad I = \int_0^a \int_0^{\sqrt{a-y}} \frac{\sin y\, dy}{(4 - 5\cos y)} \cdot \frac{2t\, dt}{\sqrt{(a-y) - t^2} \cdot t}$$

$$= 2\int_0^a \frac{\sin y}{4 - 5\cos y} \cdot \left[\sin^{-1}\left(\frac{t}{\sqrt{a-y}}\right)\right]_0^{\sqrt{a-y}} dy = \pi \int_0^a \frac{\sin y}{4 - 5\cos y}\, dy$$

$$\therefore \quad I = \pi\left[-\frac{1}{5}\log(4 - 5\cos y)\right]_0^a = \frac{\pi}{5}[-\log(4 - 5\cos a) + 0] = -\frac{\pi}{5}\log(4 - 5\cos a).$$

Example 15 (b) : Change the order of integration and evaluate

$$\int_0^2 \int_{\sqrt{2x}}^2 \frac{y^2\, dy\, dx}{\sqrt{y^4 - 4x^2}}. \qquad \textbf{(M.U. 1995, 2015)}$$

Sol. : 1. Given order and given limits : Given order is : first w.r.t. y and then w.r.t. x, a strip parallel to the y-axis. y varies from $y = \sqrt{2x}$ to $y = 2$ and then x varies from $x = 0$ to $x = 2$.

2. Region of integration : $y = \sqrt{2x}$ *i.e.*, $y^2 = 2x$ is a parabola with vertex at the origin and opening on the right. $y = 2$ is a line parallel to the x-axis. $x = 0$ is the y-axis and $x = 2$ is the line parallel to the y-axis. The points of intersection are $A\,(2, 2)$, $B\,(0, 2)$. The region of integration is OAB.

3. Change of order of integration : To change the order of integration, consider a strip parallel to the x-axis in the region of integration. On this strip x varies from $x = 0$ to $x = y^2/2$ and then the strip moves from $y = 0$ to $y = 2$.

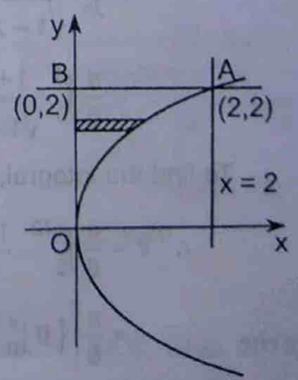

Fig. 9.18

$$\therefore \quad I = \int_0^2 \int_0^{y^2/2} \frac{y^2 dx\, dy}{\sqrt{y^4 - 4x^2}} = \frac{1}{2}\int_0^2 \int_0^{y^2/2} \frac{y^2}{\sqrt{(y^2/2)^2 - x^2}} dx\, dy$$

$$= \frac{1}{2}\int_0^2 y^2 \left[\sin^{-1}\left(\frac{x}{y^2/2}\right)\right]_0^{y^2/2} dy = \frac{1}{2}\int_0^2 y^2 \left[\sin^{-1} 1 - \sin^{-1} 0\right] dy$$

$$\therefore \quad I = \frac{1}{2}\int_0^2 \frac{\pi}{2} \cdot y^2\, dy = \frac{\pi}{4}\left[\frac{y^3}{3}\right]_0^2 = \frac{2\pi}{3}.$$

Example 16 (b) : Change the order of integration and evaluate

$$\int_0^{1/2} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\,\sqrt{1-x^2-y^2}} dx\, dy.$$ **(M.U. 1998, 2001)**

Sol. : 1. Given order and given limits : Given order is : first w.r.t. x and then w.r.t. y *i.e.,* a strip parallel to the x-axis. x changes from $x = 0$ to $x = \sqrt{1-y^2}$ and y changes from $y = 0$ to $y = 1/2$.

2. Region of integration : $x = 0$ is the y-axis, $x = \sqrt{1-4y^2}$ *i.e.,* $x^2 + 4y^2 = 1$ *i.e.,* $x^2 + \dfrac{y^2}{(1/2)^2} = 1$ is an ellipse with centre at the origin and semi-major axis and semi-minor axis equal to 1 and 1/2 respectively. $y = 0$ is the x-axis and $y = 1/2$ is a line parallel to the x-axis. Thus, the region of integration is the area of the ellipse in the first quadrant.

3. Change of order of integration : To change the order of integration, consider a strip parallel to the y-axis. On this strip y varies from $y = 0$ to $y = \dfrac{\sqrt{1-x^2}}{2}$. Then the strip moves from $x = 0$ to $x = 1$.

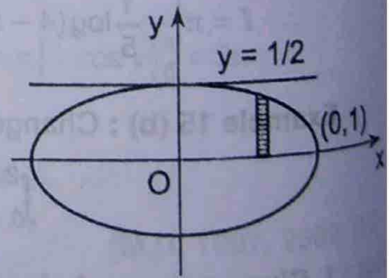

Fig. 9.19

$$\therefore \quad I = \int_0^1 \int_0^{\sqrt{1-x^2}/2} \frac{1+x^2}{\sqrt{1-x^2}} \cdot \frac{dy}{\sqrt{(1-x^2)-y^2}} dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}} \cdot \left[\sin^{-1}\left(\frac{y}{\sqrt{1-x^2}}\right)\right]_0^{\sqrt{1-x^2}/2} dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}} \cdot \left[\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right] dx$$

$$= \frac{\pi}{6}\int_0^1 \frac{1+x^2}{\sqrt{1-x^2}} dx$$

To find the integral, put $x = \sin\theta$, $dx = \cos\theta\, d\theta$.

$$\therefore \quad I = \frac{\pi}{6}\int_0^{\pi/2} \frac{1+\sin^2\theta}{\cos\theta} \cdot \cos\theta\, d\theta = \frac{\pi}{6}\int_0^{\pi/2} (1+\sin^2\theta)\, d\theta$$

$$= \frac{\pi}{6}\left[\{\theta\}_0^{\pi/2} + \frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{\pi}{6}\left[\frac{\pi}{2} + \frac{\pi}{4}\right] = \frac{\pi^2}{8}.$$ [By (20), page 6-28]

Example 17 (b) : Change the order of integration and evaluate

$$\int_{x=0}^{1} dx \int_{y=1}^{\infty} e^{-y}\, y^{x} \log y \, dy .$$

(M.U. 1997, 2005)

Sol. : Since all the limits of integration are constants, the order can be changed without taking the help of a diagram. The limit of x and y remain the same.

$$\therefore \quad I = \int_{y=1}^{\infty} e^{-y} \log y \, dy \cdot \int_{x=0}^{1} y^{x} \, dx$$

$$= \int_{y=1}^{\infty} e^{-y} \log y \left[\frac{y^{x}}{\log y} \right]_0^1 dy = \int_{y=1}^{\infty} e^{-y} (y-1) \, dy$$

$$= \int_{1}^{\infty} (ye^{-y} - e^{-y}) \, dy = \left[-y\, e^{-y} - e^{-y} + e^{-y} \right]_1^{\infty} \qquad \text{[By parts]}$$

$$= \left[-y\, e^{-y} \right]_1^{\infty} = e^{-1} = \frac{1}{e}.$$

Type II : Integral Changes to Two Integrals

Example 1 (b) : Change the order of integration and evaluate.

$$\int_0^5 \int_{2-x}^{2+x} dy\, dx$$

(M.U. 1997, 2008, 13)

Sol. : 1. Given order and given limits : Given order is : first w.r.t. y and then w.r.t. x *i.e.*, a strip parallel to the y-axis. y changes from $y = 2 - x$ to $y = 2 + x$ and then x changes from $x = 0$ to $x = 5$.

2. Region of integration : $y = 2 - x$ is a straight line $x + y = 2$ and $y = 2 + x$ is also a straight line. $x = 0$ is the y-axis and $x = 5$ is a line parallel to the y-axis. The points of intersection are A (0, 2), B (5, – 3) and C (5, 7). The region of integration is the triangle ABC.

3. Change of the order of integration : To change the order of integration, consider a strip parallel to the x-ais in the region of integration. When this strip moves parallel to itself, its base moves on two different straight lines AB and AC. Thus, the region of integration is split into two parts, ADC and ADB. So we consider two strips in the two regions. In the region ABD on the strip x varies from $x = 2 - x$ to $x = 5$ and then the strip moves from $y = -3$ to $y = 2$. In the region ADC, on the strip x varies from $x = y - 2$ to $x = 5$ and then the strip moves from $y = 2$ to $y = 7$.

$$\therefore \quad I = \int_{-3}^{2} \int_{2-y}^{5} dx\, dy + \int_{2}^{7} \int_{y-2}^{5} dx\, dy$$

$$= \int_{-3}^{2} [x]_{2-y}^{5} \, dy + \int_{2}^{7} [x]_{y-2}^{5} \, dy$$

$$= \int_{-3}^{2} (3 + y) \, dy + \int_{2}^{7} (7 - y) \, dy$$

$$= \left[3y + \frac{y^2}{2} \right]_{-3}^{2} + \left[7y - \frac{y^2}{2} \right]_{2}^{7}$$

$$= \left(17 - \frac{9}{2} \right) + \left(37 - \frac{49}{2} \right) = 25.$$

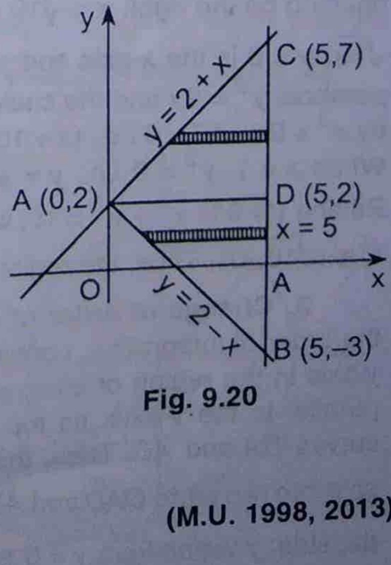

Fig. 9.20

Example 2 : Change the order of integration and evaluate

$$\int_0^1 \int_x^{2-x} \frac{x}{y} \, dy\, dx$$

(M.U. 1998, 2013)

Sol. : 1. Given order and given limits : Given order is : first w.r.t. y and then w.r.t. x. *i.e.*, a strip parallel to the y-axis. Given limits are $y = x$ and $y = 2 - x$ and $x = 0$ to $x = 1$.

2. Region of integration : $y = x$ a line through the origin. $y = 2 - x$ *i.e.*, $x + y = 2$ is also a line. $x = 0$ is the y-axis and $x = 1$ is a line parallel to the y-axis. The points of intersection are $A(1, 1)$ and $B(0, 2)$. The region of integration is the triangle OAB.

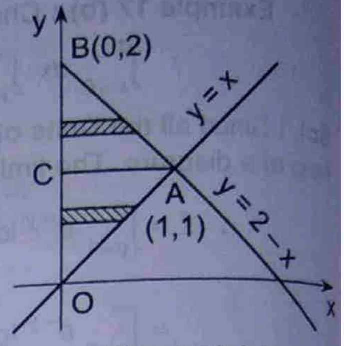

Fig. 9.20 (a)

3. Change of order of integration : To change the order of integration consider a strip parallel to x-axis in the region of integration. As the strip moves parallel itself its top moves on two different lines OA and BA. Thus, the region of integration is split into two parts OAC and CAB, so we consider two strips in the two regions. In the region OAC, on the strip x varies from $x = 0$ to $x = y$ and then the strip moves from $y = 0$ to $y = 1$. In the region CAB, on the strip x varies from $x = 0$ to $x = 2 - y$ and then the strip moves from $y = 1$ to $y = 2$.

$$\therefore \quad I = \int_0^1 \int_0^y \frac{x}{y}\,dx\,dy + \int_1^2 \int_0^{2-y} \frac{x}{y}\,dx\,dy = \int_0^1 \frac{1}{y}\left[\frac{x^2}{2}\right]_0^y dy + \int_1^2 \frac{1}{y}\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 \frac{y^2}{y}\,dy + \frac{1}{2}\int_1^2 \frac{(2-y)^2}{y}\,dy = \frac{1}{2}\int_0^1 y\,dy + \frac{1}{2}\int_1^2 \left(\frac{4}{y} - 4 + y\right) dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2}\right]_0^1 + \frac{1}{2}\left[4\log y - 4y + \frac{y^2}{2}\right]_1^2 = \frac{1}{4} + \frac{1}{2}\left[(4\log 2 - 8 + 2) - \left(0 - 4 + \frac{1}{2}\right)\right]$$

$$= \frac{1}{4} + 2\log 2 - 3 + 2 - \frac{1}{4} = -1 + 2\log 2 = -\log e + \log 4 = \log\left(\frac{4}{e}\right).$$

Example 3 (b) : Change the order of integration and evaluate $\int_0^3 \int_{y^2/9}^{\sqrt{10-y^2}} dx\,dy$.

Sol. : 1. Given order and given limits : Given order is : first w.r.t. x and then w.r.t. y *i.e.*, a strip parallel to the x-axis. x changes from $x = y^2/9$ to $x = \sqrt{10 - y^2}$ and then y changes from $y = 0$ to $y = 3$.

2. Region of integration : $x = y^2/9$ *i.e.*, $y^2 = 9x$ is a parabola with vertex at the origin and opening on the right. $x = \sqrt{10 - y^2}$ *i.e.* $x^2 + y^2 = 10$ is a circle with centre at the origin and radius $\sqrt{10}$. $y = 0$ is the x-axis and $y = 3$ is a line parallel to the x-axis. The points of intersection of the parabola $y^2 = 9x$ and the circle $x^2 + y^2 = 10$ are given by $x^2 + 9x - 10 = 0$ *i.e.*, $(x + 10)(x - 1) = 0$ $\therefore$ $x = 1, 10$. When $x = 1$, $y^2 = 9$ *i.e.*, $y = \pm 3$. The point A is $(1, 3)$. Putting $y = 0$ in $x^2 + y^2 = 10$, we get $x = \pm\sqrt{10}$. Hence, C is $(\sqrt{10}, 0)$. Thus, the region of integration is $OACD$.

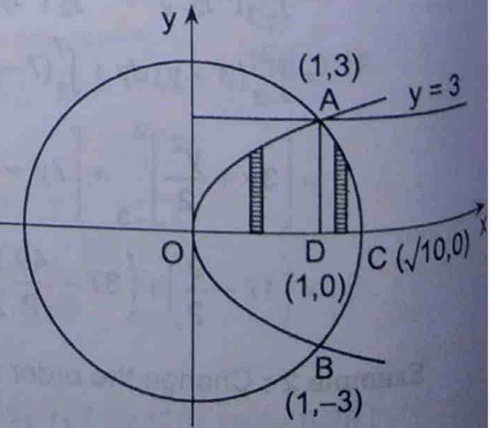

Fig. 9.21

3. Change of order of integration : To change the order of integration, consider a strip parallel to the y-axis in the region of integration. As the strip moves parallel to the y-axis, its top moves on two different curves OA and AC. Thus, the region of integration is split into two parts OAD and ADC. In the region OAD on the strip, y varies from $y = 0$ to $y = 3\sqrt{x}$ and then strip moves from $x = 0$ to $x = 1$. In the region ADC, on the strip y varies from $y = 0$ to $y = \sqrt{10 - x^2}$ and then strip moves from $x = 1$ to $x = \sqrt{10}$.

$$\therefore \quad I = \int_0^1 \int_0^{3\sqrt{x}} dy\, dx + \int_1^{\sqrt{10}} \int_0^{\sqrt{10 - x^2}} dy\, dx$$

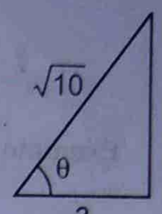

Fig. 9.22

$$\text{Now,} \quad I_1 = \int_0^1 [\, y\,]_0^{3\sqrt{x}}\, dx = \int_0^1 3\sqrt{x}\, dx = 3\left[\frac{2}{3} x^{3/2}\right]_0^1 = 2$$

$$I_2 = \int_1^{\sqrt{10}} [\, y\,]_0^{\sqrt{10 - x^2}}\, dx = \int_1^{\sqrt{10}} \sqrt{10 - x^2} \cdot dx$$

$$= \left[\frac{x}{2}\sqrt{10 - x^2} + \frac{10}{2}\sin^{-1}\frac{x}{\sqrt{10}}\right]_1^{\sqrt{10}} = 5 \cdot \frac{\pi}{2} - \frac{3}{2} - 5\sin^{-1}\frac{1}{\sqrt{10}}$$

$$\therefore \quad I = I_1 + I_2$$

$$= 2 - \frac{3}{2} + 5\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{\sqrt{10}}\right) = \frac{1}{2} + 5\sin^{-1}\left(\frac{3}{\sqrt{10}}\right).$$

Example 4 (b) : Change the order of integration and evaluate.

$$\int_0^1 dx \int_x^{\sqrt{2 - x^2}} \frac{x}{\sqrt{x^2 + y^2}}\, dy\, dx. \qquad \textbf{(M.U. 2003, 07, 08)}$$

Sol. : 1. Given order and given limits : Given order is : first w.r.t. y and then w.r.t. x *i.e.*, a strip parallel to the y-axis. Given limits are, $y = x$ to $y = \sqrt{2 - x^2}$ and $x = 0$ to $x = 1$.

2. Region of integration : $y = x$ is a line through the origin. $y = \sqrt{2 - x^2}$ *i.e.*, $x^2 + y^2 = 2$ is a circle with centre at the origin and radius $\sqrt{2}$. $x = 0$ is the y-axis and $x = 1$ is the line parallel to the y-axis. The points of intersection of the line $y = x$ and the circle $x^2 + y^2 = 2$ are given by $2x^2 = 2$ *i.e.*, $x = 1, -1$ and $y = 1, -1$ *i.e.*, A is $(1, 1)$ and B is $(-1, -1)$. The region of integration is $OACD$.

3. Change of order of integration : To change the order of integration, consider a strip parallel to the x-axis in the region of integration. As the strip moves parallel to itself its top moves on two different curves CA and OA. Thus, the region of integration is split into two parts OAD and DAC. In the region OAD, x varies from $x = 0$ to $x = y$ and then the strip moves from $y = 0$ to $y = 1$. In the region DAC, x varies from $x = 0$ to $x = \sqrt{2 - y^2}$ and then the strip moves from $y = 1$ to $y = \sqrt{2}$.

$$\therefore \quad I = \int_0^1 dy \int_0^y \frac{x}{\sqrt{x^2 + y^2}}\, dx + \int_1^{\sqrt{2}} dy \int_0^{\sqrt{2 - y^2}} \frac{x}{\sqrt{x^2 + y^2}}\, dx$$

$$\text{Now,} \quad I_1 = \int_0^1 dy \left[\sqrt{x^2 + y^2}\,\right]_0^y = \int_0^1 (\sqrt{2} \cdot y - y)\, dy$$

$$= (\sqrt{2} - 1)\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}(\sqrt{2} - 1)$$

$$\text{and} \quad I_2 = \int_1^{\sqrt{2}} dy \left[\sqrt{x^2 + y^2}\,\right]_0^{\sqrt{2 - y^2}} = \int_1^{\sqrt{2}} (\sqrt{2} - y)\, dy$$

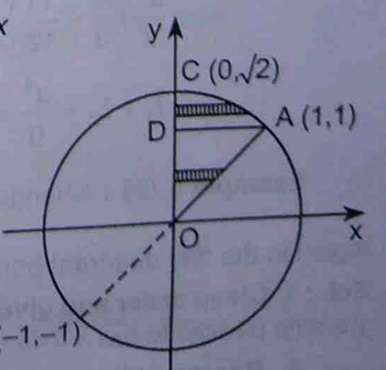

Fig. 9.23

$$\therefore \quad I_2 = \left[\sqrt{2}\,y - \frac{y^2}{2}\right]_1^{\sqrt{2}} = \frac{3}{2} - \sqrt{2}$$

$$\therefore \quad I = I_1 + I_2 = 1 - \frac{1}{\sqrt{2}}.$$

Example 5 (b) : Change the order of integration and evaluate

$$\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx.$$

(M.U. 1992)

Sol. : 1. Given order and given limits : Given order is : first w.r.t. y and then w.r.t. x *i.e.*, a strip parallel to the y-axis. Given limits are $y = x^2/a$ to $y = 2a - x$ and $x = 0$ to $x = a$.

2. Region of integration : $y = x^2/a$ *i.e.*, $x^2 = ay$ is a parabola with vertex at the origin and opening upwards. $y = 2a - x$ *i.e.*, $x + y = 2a$ is a line. $x = 0$ is the y-axis and $x = a$ is a line parallel to the y-axis. The points of intersection of the parabola and the line are obtained by solving $x^2 = ay$ and $x + y = 2a$. They are $A\,(a, a)$ and $D\,(-2a, 4a)$. Thus, the region of integration is $OACB$.

3. To change the order of integration : To change the order of integration, consider a strip parallel to the x-axis. As the strip moves parallel to itself, its top moves on two different curves CA and OA. Thus, the region of integration is split into two parts OAB and BAC. In the region OAB, on the strip x varies from $x = 0$ to $x = \sqrt{ay}$ and then the strip moves from $y = 0$ to $y = a$. In the region BAC, on the strip x varies from $x = 0$ to $x = 2a - y$ and then the strip moves from $y = a$ to $y = 2a$.

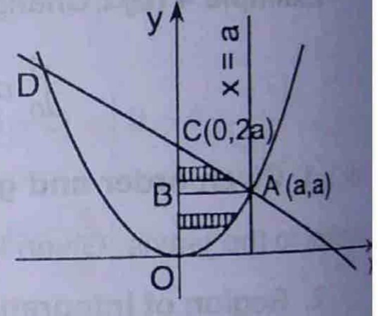

Fig. 9.24

$$\therefore \quad I = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy + \int_a^{2a} \int_0^{2a-y} xy\,dx\,dy$$

$$\text{Now, } I_1 = \int_0^a y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy = \int_0^a \frac{a}{2} y^2\,dy = \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a = \frac{a^4}{6}$$

$$\text{and } I_2 = \int_a^{2a} y\left[\frac{x^2}{2}\right]_0^{2a-y} dy = \int_a^{2a} \frac{1}{2}(4a^2y - 4ay^2 + y^3)\,dy$$

$$= \frac{1}{2}\left[2a^2y^2 - \frac{4a}{3}y^3 + \frac{y^4}{4}\right]_a^{2a} = \frac{a^4}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{a^4}{2}\left(\frac{4}{3} - \frac{11}{12}\right) = \frac{a^4}{2} \cdot \frac{5}{12} = \frac{5a^4}{24}.$$

$$\therefore \quad I = I_1 + I_2 = \frac{a^4}{6} + \frac{5a^4}{24} = \frac{3}{8}a^4.$$

Example 6 (b) : Change the order of integration and evaluate $\iint_R x^2\,dx\,dy$ where R is the region in the first quadrant bounded by $xy = a^2$, $x = 2a$, $y = 0$ and $y = x$. (M.U. 1994, 2013, 15)

Sol. : 1. Given order and given limits : $dx\,dy$ denotes the given order as first w.r.t. x and then w.r.t. y a strip parallel to the x-axis.

2. Region of integration : $xy = a^2$ is a rectangular hyperbola; $x = 2a$ is a line parallel to the y-axis; $y = 0$ is the x-axis, and $y = x$ is a line through the origin. The region of integration is the region $OADCB$. The points of intersection are $A\,(a, 0)$, $D\,(2a, 0)$, $B\,(a, a)$.

3. Change of order : To change the order of integration, consider a strip parallel to the y-axis. As the strip moves parallel to the y-axis, its top moves on two different curves OB and BC. Thus, the region of integration is split into two parts OAB and $ABCD$. In the region OAB on the strip, y varies from $y = 0$ to $y = x$ and the strip moves from $x = 0$ to $x = a$. In the region $ABCD$ on the strip, y varies from $y = 0$ to $y = a^2/x$ and then the strip moves from $x = a$ to $x = 2a$.

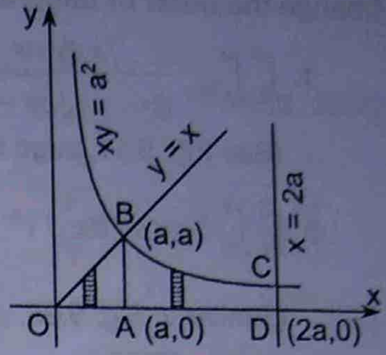

Fig. 9.25

$$\therefore \quad I = \int_0^a \int_0^x x^2 \, dy \, dx + \int_a^{2a} \int_0^{a^2/x} x^2 \, dy \, dx$$

$$= \int_0^a x^2 [\, y \,]_0^x \, dx + \int_a^{2a} x^2 [\, y \,]_0^{a^2/x} \, dx$$

$$= \int_0^a x^3 \, dx + \int_a^{2a} a^2 x \, dx = \left[\frac{x^4}{4}\right]_0^a + \left[\frac{a^2 x^2}{2}\right]_a^{2a}$$

$$= \frac{a^4}{4} + 2a^4 - \frac{a^4}{2} = \frac{7a^4}{4}.$$

Example 7 (b) : Express as a single integral and evaluate

$$I = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dy + \int_1^3 dy \int_{-1}^1 dx. \qquad \textbf{(M.U. 2000)}$$

Sol. : Let $I = I_1 + I_2$.

In I_1, x varies from $x = -\sqrt{y}$ (arc OB) to $x = \sqrt{y}$ (arc OA) and then y varies from $y = 0$ to $y = 1$. $x^2 = y$ is a parabola with vertex at the origin and opening upwards. $y = 0$ is the x-axis and $y = 1$ is a line parallel to the x-axis. This line intersects the parabola in A (1, 1) and B (– 1, 1). **The region of integration is *OAB*.**

In I_2, x varies from $x = -1$ (line BC) to $x = 1$ (line AD) and y varies from $y = 1$ to $y = 3$. $x = -1$, $x = 1$ are the line BC and AD and $y = 1$, $y = 3$ are the lines BA and CD. **The region of integration is *ABCD*.**

The total region of integration is *OADCBO*.

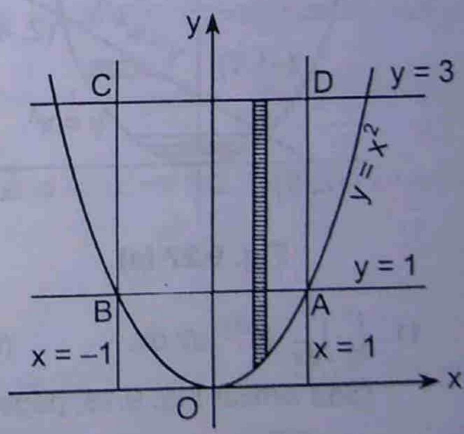

Fig. 9.26

Change of order : To change the order of integration, consider a strip parallel to the y-axis. On this strip y varies from $y = x^2$ to $y = 3$ and then the strip moves from $x = -1$ to $x = 1$.

(**Note** that in this example, two areas are amalgamated into one area.)

$$\therefore \quad I = \int_{-1}^1 \int_{x^2}^3 dy \, dx = \int_{-1}^1 [\, y \,]_{x^2}^3 \, dx$$

$$= \int_{-1}^1 \left[3 - x^2\right] dx = 2\int_0^1 (3 - x^2) \, dx \qquad [\because \text{Even function}]$$

$$= 2\left[3x - \frac{x^3}{3}\right]_0^1 = 2\left[3 - \frac{1}{3}\right] = \frac{16}{3}.$$

EXERCISE - IV

Change the order of integration and evaluate : Class (b) : 6 Marks

1. $\int_0^b \int_{x^2/b}^{x} \frac{x\,dy\,dx}{(b-y)\sqrt{by-y^2}}$.

(See Fig. 9.11, page 9-15, $a = b$)

2. $\int_1^2 \int_1^{x^2} \frac{x^2}{y}\,dy\,dx$.

(Similar to Fig. 9.27 below)

3. $\int_3^5 \int_0^{4/x} xy\,dy\,dx$.

(Similar to Fig. 9.25, page 9-27)

4. $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{y\,dy\,dx}{(1+y^2)\sqrt{1-x^2-y^2}}$.

(See Fig. 9.10, page 9-14)

5. $\int_0^{a/\sqrt{2}} \int_x^{\sqrt{a^2-x^2}} y^2\,dy\,dx$.

(See Fig. 9.41, page 9-38, $a^2 = 2$)

6. $\int_0^1 \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} dx\,dy$.

(See similar Fig. 9.78, page 9-61)

7. $\int_{-1}^2 \int_{x^2}^{x+2} dy\,dx$.

[See Fig. 9.27(a) given below]

8. $\int_0^1 \int_1^{\sqrt{2-y^2}} \frac{y\,dx\,dy}{\sqrt{(2-x^2)(1-x^2y^2)}}$.

[See Fig. 9.41, page 9-38, $a^2 = 2$]

9. $\int_0^2 \int_{\sqrt{2y}}^{2} \frac{x^2\,dx\,dy}{\sqrt{x^4-4y^2}}$.

[See Fig. 9.27(b) given below]

(M.U. 1995)

10. $\int_0^a \int_0^y \frac{dx\,dy}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$.

[See Fig. 9.17, page 9-20]

(M.U. 1997, 2003, 04)

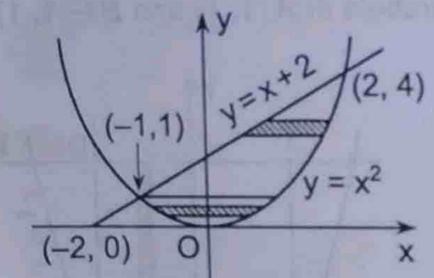

Fig. 9.27 (a)

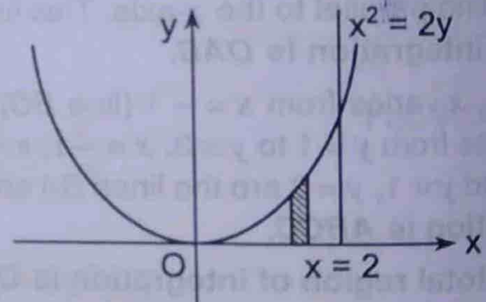

Fig. 9.27 (b)

11. $\int_0^1 \int_{\sqrt{x}}^{1} e^{x/y}\,dy\,dx$. **(M.U. 2003)**

(See similar Fig. 9.18, page 9-21)

12. $\int_0^{\pi/2} \int_0^{y} \cos 2y \cdot \sqrt{1-a^2\sin^2 x} \cdot dx\,dy$.

(See similar Fig. 9.17, page 9-20)

13. $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx$.

(See similar Fig. 9.14, page 9-17)

14. $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$.

(See Fig. 9.16, page 9-19)

15. $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$.

(See Fig. 9.24, page 9-26, $a = 1$)

16. $\int_0^a \int_0^x x\sqrt{(a^2-y^2)(x^2-y^2)}\,dy\,dx$.

(See Fig. 9.16, page 9-19)

17. $\int_0^\infty \int_0^\infty e^{-x^2(1+t^2)}\,x\,dx\,dt$.

(First quadrant)

18. $\int_0^a \int_0^{2\sqrt{ax}} x^2\,dy\,dx$.

(See similar Fig. 9.14, page 9-17)

19. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx.$ (See Fig. 9.14 (a), page 9-17)

20. Express as a single integral and them evaluate

$$\int_0^1 \int_0^y (x^2 + y^2)\, dx\, dy + \int_1^2 \int_0^{2-y} (x^2 + y^2)\, dx\, dy.$$

(See Fig. 9.20 (a), page 9-24) (M.U. 2004)

[**Ans. :** (1) $\frac{\pi b}{2}$, (2) $\frac{2}{9}(24 \log 2 - 7)$, (3) $8 \log \frac{5}{3}$, (4) $\frac{\pi}{4} \log 2$, (5) $\frac{1}{32} a^4 (\pi + 2)$, (6) $\frac{4}{3}$, (7) $\frac{9}{2}$, (8) $1 - \frac{\pi}{4}$, (9) $\frac{2\pi}{3}$, (10) $\pi \cdot \log(1 + \sqrt{2})$, (11) $\frac{1}{2}$, (12) $\frac{1}{3a^2}\left[(1 - a^2)^{3/2} - 1\right]$, (13) $\frac{a}{4}\left[\frac{a^2}{7} + \frac{1}{5}\right]$, (14) $\frac{a^3}{3} \log(1 + \sqrt{2})$, (15) $\frac{3}{8}$, (16) $\frac{8}{45} a^5$, (17) $\frac{\pi}{4}$, (18) $\frac{4}{7} a^4$, (19) 1, (20) $\frac{4}{3}$.]

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4. Cartesian to Polar Coordinates

Sometimes, to evaluate a double integral we change the system from cartesian to polar. Particularly, when the function or the limit contains the terms x^2 and y^2, we put $x = r \cos \theta$, $y = r \sin \theta$. The method is illustrated by the following examples.

Procedure to Change the Integral from Catesian to Polar

1. **Region of integration :** From the given limits for x and y, find the region of integration.
2. **Change to r, θ :** Putting $x = r \cos \theta$, $y = r \sin \theta$, find the equations of the boundary curves en in x, y coordinates in the r, θ coordinates.
3. **Limits of r, θ :** Considering a radial strip in the region of integration, find the limits of r and θ.
4. **Integrand :** Put $x = r \cos \theta$ and $y = r \sin \theta$ in the integrand and find the integrand in terms of r and θ. Also replace $dy\, dx$ by $r\, dr\, d\theta$.

Solved Examples : Class (a) : 4 Marks

Example 1 (a) : Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy$.

Sol. : 1. Region of integration : $x = 0$ is the y-axis, $x = \sqrt{1 - y^2}$ *i.e.*, $x^2 + y^2 = 1$ is the unit circle. $y = 0$ is the x-axis and $y = 1$ is a line parallel to the x-axis. Thus, the region of integration is the quarter OAB of the circle.

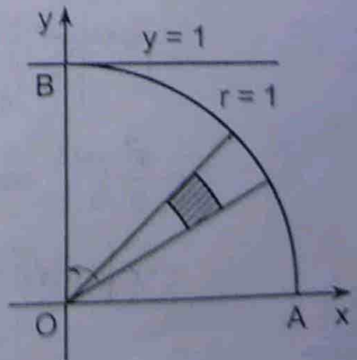

Fig. 9.28

2. Change to r, θ : Putting $x = r \cos \theta$ and $y = r \sin \theta$ in $x^2 + y^2 = 1$, we get $r^2 \cos^2 \theta + r^2 \sin^2 \theta = 1$ *i.e.*, $r = 1$. The x-axis is given by $\theta = 0$ and the y-axis is given by $\theta = \pi / 2$.

3. Limits of r, θ : Considering a radial strip in the region of integration, we see that r varies from $r = 0$ to $r = 1$ and θ varies from $\theta = 0$ to $\theta = \pi / 2$.

4. Integrand : Putting $x = r \cos \theta$ and $y = r \sin \theta$ in $x^2 + y^2$, we get r^2 and $dy\, dx$ is replaced by $r\, dr\, d\theta$.

$$\therefore \quad I = \int_0^{\pi/2}\int_0^1 r^2 \cdot r\,dr\,d\theta = \int_0^{\pi/2} d\theta \int_0^1 r^3\,dr$$

$$= [\theta]_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 = \frac{\pi}{2}\cdot\frac{1}{4} = \frac{\pi}{8}.$$

Example 2 (a) : Change to polar coordinates and evaluate $\int_0^a \int_y^a x\,dx\,dy$.

Sol. : 1. Region of integration : $x = y$ is a line through the origin, $x = a$ is a line parallel to the y-axis, $y = 0$ is the x-axis and $y = a$ is a line parallel to the x-axis. Thus, the region of integration is OAB.

2. Change to r, θ : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $x = y$, we get $r\cos\theta = r\sin\theta$ *i.e.*, $\tan\theta = 1$ *i.e.*, $\theta = \pi/4$. Thus, the line $x = y$ *i.e.*, OB become $\theta = \pi/4$. The line $x = a$ *i.e.*, AB becomes $r\cos\theta = a$ *i.e.*, $r = a/\cos\theta$. The x-axis is given by $\theta = 0$.

3. Limits of r, θ : Considering a radial strip, we see that r varies from $r = 0$ to $r = a/\cos\theta$ and θ varies from $\theta = 0$ to $\theta = \pi/4$.

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in x, we get $r\cos\theta$ and $dy\,dx$ is replaced by $r\,dr\,d\theta$.

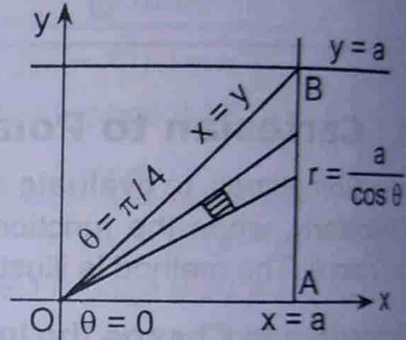

Fig. 9.29

$$\therefore \quad I = \int_0^{\pi/4}\int_0^{a/\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4}\cos\theta\left[\frac{r^3}{3}\right]_0^{a/\cos\theta} d\theta = \frac{a^3}{3}\int_0^{\pi/4}\sec^2\theta\,d\theta$$

$$= \frac{a^3}{3}[\tan\theta]_0^{\pi/4} = \frac{a^3}{3}.$$

Example 3 (a) : Change to polar co-ordinates and evaluate

$$\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy.$$

(M.U. 1997, 2006)

Sol. : Region of integration and limits of r, θ are as above.

Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $\frac{x^2}{\sqrt{x^2+y^2}}$, we get $\frac{r^2\cos^2\theta}{r} = r\cos^2\theta$ and $dy\,dx$ is replaced by $r\,dr\,d\theta$.

$$\therefore \quad I = \int_0^{\pi/4}\int_0^{a/\cos\theta}\frac{r^2\cos^2\theta}{r}\cdot r\,dr\,d\theta = \int_0^{\pi/4}\int_0^{a/\cos\theta} r^2\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4}\left[\frac{r^3}{3}\right]_0^{a/\cos\theta}\cdot\cos^2\theta\,d\theta = \int_0^{\pi/4}\frac{a^3}{3}\cdot\frac{1}{\cos^3\theta}\cdot\cos^2\theta\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4}\sec\theta\,d\theta = \frac{a^3}{3}\left[\log(\sec\theta+\tan\theta)\right]_0^{\pi/4}$$

$$= \frac{a^3}{3}\left[\log(\sqrt{2}+1) - \log 1\right] = \frac{a^3}{3}\log\left(1+\sqrt{2}\right).$$

Example 4 (a) : Evaluate by changing to polar coordinates $\int_0^{a/\sqrt{2}} \int_x^{\sqrt{a^2-x^2}} \frac{x\,dy\,dx}{\sqrt{(x^2+y^2)}}$.

Sol. : 1. Region of integration : $y = x$ is a line passing through the origin, $y = \sqrt{a^2 - x^2}$ *i.e.,* $x^2 + y^2 = a^2$, is a circle with radius a, $x = 0$ is the y-axis and $x = a/\sqrt{2}$ is a line parallel to the y-axis. The line $x = y$ intersects the circle in A where $x^2 + y^2 = a^2$ *i.e.,* $x = a/\sqrt{2}$. Thus, the region of integration is the sector OAB.

2. Change to r, θ : Putting $x = r\cos\theta$ and $y = r\sin\theta$, $y = x$ becomes $r\cos\theta = r\sin\theta$ *i.e.,* $\tan\theta = 1$ *i.e.,* $\theta = \pi/4$. $x^2 + y^2 = a^2$ becomes $r = a$. y-axis is given by $\theta = \pi/2$.

3. Limits of r, θ : Considering a radial strip in the region of integration OAB, we see that r varies from $r = 0$ to $r = a$ and θ varies from $\theta = \pi/4$ to $\theta = \pi/2$.

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in

$$\frac{x}{\sqrt{x^2+y^2}}, \text{ we get } \frac{r\cos\theta}{r}$$

and $dy\,dx$ is replaced by $r\,dr\,d\theta$.

$$\therefore \quad I = \int_{\pi/4}^{\pi/2}\int_0^a \frac{r\cos\theta}{r}\, r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\int_0^a r\cos\theta\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_0^a d\theta = \frac{a^2}{2}\int_{\pi/4}^{\pi/2}\cos\theta\,d\theta$$

$$= \frac{a^2}{2}\left[\sin\theta\right]_{\pi/4}^{\pi/2} = \frac{a^2}{2}\left[1 - \frac{1}{\sqrt{2}}\right].$$

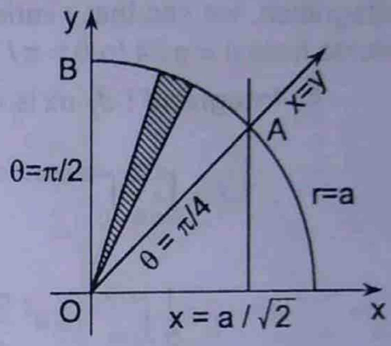

Fig. 9.30

Example 5 (a) : Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\,dy\,dx$ by changing to polar coordinates.

Sol. : 1. Region of integration : $x = 0$ is the y-axis, $x = \sqrt{a^2 - y^2}$ *i.e.,* $x^2 + y^2 = a^2$ is a circle with radius a, $y = 0$ is the x-axis and $y = a$ is a line parallel to the x-axis. Thus, the region of integration is the sector OAB.

2. Change to r, θ : Putting $x = r\cos\theta$ and $y = r\sin\theta$, in $x = \sqrt{a^2 - y^2}$ *i.e.,* $x^2 + y^2 = a^2$, we get $r = a$. The x-axis is given by $\theta = 0$ and the y-axis is given by $\theta = \pi/2$.

3. Limits of r, θ : Considering a radial strip in the region of integration, we see that r varies from $r = 0$ to $r = a$ and θ varies from $\theta = 0$ to $\theta = \pi/2$.

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$, the integrand $y^2 \cdot \sqrt{x^2+y^2}$ becomes $r\sin^2\theta \cdot r$ and $dy\,dx$ is replaced by $r\,dr\,d\theta$.

$$\therefore \quad I = \int_0^{\pi/2}\int_0^a r^2\sin^2\theta \cdot r \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^a r^4\sin^2\theta \cdot dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^5}{5}\right]_0^a \sin^2\theta\,d\theta$$

$$= \frac{a^5}{5}\int_0^{\pi/2}\sin^2\theta\,d\theta$$

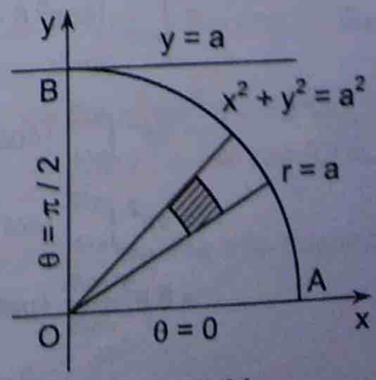

Fig. 9.31

$$\therefore \quad I = \frac{a^5}{5} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^5}{20}$$ [By (20), page 6-28]

Example 6 (a) : Evaluate by changing to polar co-ordinates $\int_0^{4a} \int_{y^2/4a}^{y} dx\, dy$.

Sol. : 1. Region of integration : $x = y^2 / 4a$ *i.e.*, $y^2 = 4ax$ is a parabola with vertex at the origin and opening on the right. $x = y$ is a line through the origin. $y = 0$ is the x-axis and $y = 4a$ is a line parallel to the x-axis. Thus, the region of integration is the part OAB of the parabola.

2. Change to r, θ : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $x = y^2/4a$ *i.e.*, in $y^2 = 4ax$, we get $r^2 \sin^2\theta = 4a\, r\cos\theta$ *i.e.*, $r = 4a\cos\theta / \sin^2\theta$. The line $x = y$ becomes $r\cos\theta = r\sin\theta$ *i.e.*, $\tan\theta = 1$ *i.e.*, $\theta = \pi/4$. The y-axis is given by $\theta = \pi/2$.

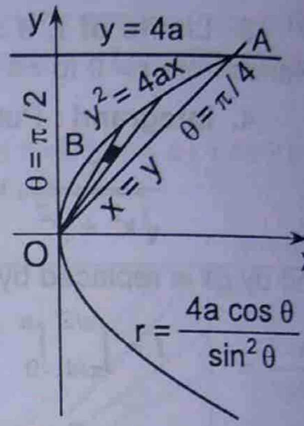

Fig. 9.32

3. Limits of r, θ : Considering a radial strip in the region of integration, we see that r varies from $r = 0$ to $r = 4a\cos\theta/\sin^2\theta$ and θ varies from $\theta = \pi/4$ to $\theta = \pi/2$.

4. Integrand : $dy\,dx$ is replaced by $r\,dr\,d\theta$.

$$\therefore \quad I = \int_{\pi/4}^{\pi/2} \int_0^{4a\cos\theta/\sin^2\theta} r\,dr\,d\theta = \int_{\pi/4}^{\pi/2} \left[\frac{r^2}{2}\right]_0^{4a\cos\theta/\sin^2\theta} d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} 16a^2 \frac{\cos^2\theta}{\sin^4\theta} \cdot d\theta = 8a^2 \int_{\pi/4}^{\pi/2} \cot^2\theta \operatorname{cosec}^2\theta\, d\theta$$

$$= 8a^2 \left[-\frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2} = -\frac{8a^2}{3}[0-1] = \frac{8a^2}{3}. \quad [\text{ Put } \cot\theta = t\,]$$

Example 7 (a) : Express in polar coordinates and evaluate

$$\int_0^{4a} \int_{y^2/4a}^{y} \left(\frac{x^2 - y^2}{x^2 + y^2}\right) dx\ dy.$$ **(M.U. 1994, 96, 99, 2006)**

Sol. : Region of integration and limits of r, θ are as above.

Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $(x^2 - y^2)/(x^2 + y^2)$, we get $\cos^2\theta - \sin^2\theta$ and $dy\,dx$ replaced by $r\,dr\,d\theta$.

$$\therefore \quad I = \int_{\pi/4}^{\pi/2} \int_0^{4a\cos\theta/\sin^2\theta} \left(\cos^2\theta - \sin^2\theta\right) r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(\cos^2\theta - \sin^2\theta\right) \left[\frac{r^2}{2}\right]_0^{4a\cos\theta/\sin^2\theta} d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \left(\cos^2\theta - \sin^2\theta\right) \frac{\cos^2\theta}{\sin^4\theta}\, d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \left(\cot^4\theta - \cot^2\theta\right) d\theta$$

$$= 8\,a^2 \int_{\pi/4}^{\pi/2} \left(\cot^2\theta \operatorname{cosec}^2\theta - 2\operatorname{cosec}^2\theta + 2\right) d\theta$$

$$= 8\,a^2 \left[-\frac{\cot^3\theta}{3} + 2\cot\theta + 2\theta\right]_{\pi/4}^{\pi/2} = 8\,a^2\left[\frac{\pi}{2} - \frac{5}{3}\right].$$

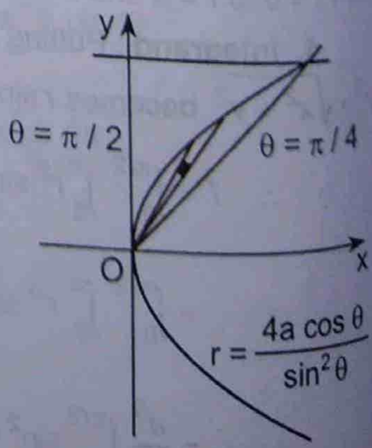

Fig. 9.33

Example 8 (a) : Express the following integral in polar coordinates and evaluate

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dy\,dx}{\sqrt{a^2-x^2-y^2}}.$$ **(M.U. 1989, 92, 2000, 04, 13)** A

Sol. : 1. Region of integration : $y=\sqrt{ax-x^2}$ *i.e.*, $y^2 = ax - x^2$ *i.e.*, $x^2 - ax + y^2 = 0$ *i.e.*, $x^2 - ax + (a^2/4) + y^2 = a^2/4$ *i.e.*, $[x-(a/2)] + y^2 = (a/2)^2$ is a circle with center at $(a/2, 0)$ and radius $a/2$. $y=\sqrt{a^2-x^2}$ *i.e.*, $x^2+y^2=a^2$ is a circle with centre at the origin and radius a. $x=0$ is the y-axis and $x=a$ is a line parallel to the y-axis. The region of integration is the area between the two circle in the first quadrant $OABCD$. ?

2. Change to r, θ : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $x^2+y^2=ax$, we get $r^2 = ar\cos\theta$ *i.e.*, $r = a\cos\theta$ and in $x^2+y^2=a^2$, we get $r=a$. The y-axis is given by $\theta=\pi/2$. The x-axis is given by $\theta = 0$.

3. Limits of r, θ : Considering a radial strip in the region of integration, we see that r varies from $r = a\cos\theta$ to $r=a$ and θ varies from $\theta = 0$ to $\theta = \pi/2$.

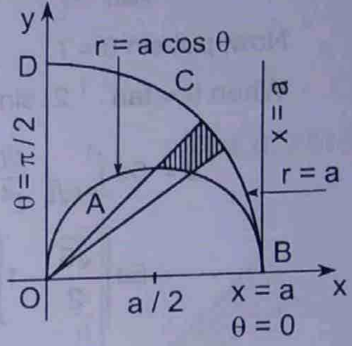

Fig. 9.34

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in

$$\frac{1}{\sqrt{a^2-x^2-y^2}}, \text{ we get } \frac{1}{\sqrt{a^2-r^2}}$$

and $dy\,dx$ replaced by $r\,dr\,d\theta$.

$$\therefore \quad I = \int_0^{\pi/2}\int_{a\cos\theta}^{a} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}} = \int_0^{\pi/2}\left[-\sqrt{a^2-r^2}\right]_{a\cos\theta}^{a} d\theta$$

$$= \int_0^{\pi/2} a\sin\theta\; d\theta = \left[-a\cos\theta\right]_0^{\pi/2} = a.$$

Example 9 (a) : Evaluate $\int_0^a \int_{2\sqrt{ax}}^{\sqrt{5ax-x^2}} \frac{\sqrt{x^2+y^2}}{y^2}\,dy\,dx.$ **(M.U. 2004)**

Sol. : 1. Region of integration : $y = 2\sqrt{ax}$ *i.e.*, $y^2 = 4ax$ is a parabola with vertex at the origin and opening on the right. (See the figure on the next page.)

$$y = \sqrt{5ax - x^2} \quad i.e., \; x^2 - 5ax + y^2 = 0$$

$$i.e., \quad x^2 - 5ax + \frac{25a^2}{4} + y^2 = \frac{25a^2}{4} \qquad i.e., \quad \left(x - \frac{5a}{2}\right)^2 + y^2 = \left(\frac{5a}{2}\right)^2$$

is a circle with centre at $[(5a/2), 0]$ and radius $5a/2$. $x=0$ is the y-axis and $x=a$ is a line parallel to the y-axis. The circle and the parabola intersect where $y^2 = 4ax = 5ax - x^2$ *i.e.*, $x=a$ and $y=2a$. The region of integration is the area OAB between the parabola and the circle.

2. Change to r, θ : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $y^2 = 4ax$, we get $r^2\sin^2\theta = 4ar\cos\theta$ *i.e.*, $r = 4a\cos\theta/\sin^2\theta$ and in $y^2 = 5ax - x^2$ *i.e.*, in $x^2+y^2=5ax$, we get, $r^2 = 5ar\cos\theta$ *i.e.*, $r = 5a\cos\theta$. x-axis is $\theta=0$ and y-axis is $\theta = \pi/2$.

3. Limits of r, θ : Considering a radial strip in the region of integration, we see that r varies from $r = 4a\cos\theta/\sin^2\theta$. Now, at $A(a, 2a)$, $x = r\cos\theta = a$ and $y = r\sin\theta = 2a$.

$$\therefore \quad \frac{r\sin\theta}{r\cos\theta} = \frac{2a}{a} \quad i.e., \quad \tan\theta = 2 \quad i.e., \quad \theta = \tan^{-1} 2.$$

Thus, the line OA makes an angle of $\tan^{-1} 2$ with the x-axis. On the y-axis $\theta = \pi/2$. Hence, θ varies from $\theta = \tan^{-1} 2$ to $\theta = \pi/2$.

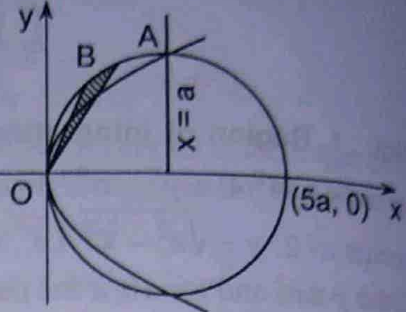

Fig. 9.35

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $\dfrac{\sqrt{x^2+y^2}}{y^2}$, we get $\dfrac{r}{r^2\sin^2\theta}$ and $dy\,dx$ is replaced by $r\,dr\,d\theta$.

$$\therefore\ I = \int_{\tan^{-1}2}^{\pi/2}\int_{4a\cos\theta/\sin^2\theta}^{5a\cos\theta}\frac{r}{r^2\sin^2\theta}\,r\,dr\,d\theta$$

$$= \int_{\tan^{-1}2}^{\pi/2}\left[r\right]_{4a\cos\theta/\sin^2\theta}^{5a\cos\theta}\frac{d\theta}{\sin^2\theta}$$

$$= \int_{\tan^{-1}2}^{\pi/2}\left[5a\cos\theta - \frac{4a\cos\theta}{\sin^2\theta}\right]\frac{d\theta}{\sin^2\theta} = \int_{\tan^{-1}2}^{\pi/2}\left[5a\frac{\cos\theta}{\sin^2\theta} - \frac{4a\cos\theta}{\sin^4\theta}\right]d\theta$$

Now, put $\sin\theta = t$ $\quad\therefore\ \cos\theta\,d\theta = dt$.

When $\theta = \tan^{-1} 2$, $\sin\theta = 2/\sqrt{5}$ and when $\theta = \pi/2$, $\sin\theta = 1$.

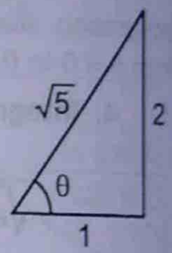

Fig. 9.36

$$\therefore\ I = 5a\int_{2/\sqrt{5}}^{1}\frac{dt}{t^2} - 4a\int_{2/\sqrt{5}}^{1}\frac{dt}{t^4} = 5a\left[-\frac{1}{t}\right]_{2/\sqrt{5}}^{1} + \frac{4a}{3}\left[\frac{1}{t^3}\right]_{2/\sqrt{5}}^{1}$$

$$= 5a\left[\frac{\sqrt{5}}{2} - 1\right] + \frac{4a}{3}\left[1 - \frac{5\sqrt{5}}{8}\right] = \frac{a}{3}\left(5\sqrt{5} - 11\right)$$

Example 10 (a) : Change to polar coordinates and evaluate

$$\int_0^a\int_0^{\sqrt{a^2-x^2}}(x^2+y^2)\,dy\,dx.$$

Sol. : 1. Region of integration : The region of integration is given by $y = 0$ *i.e.*, the x-axis; $y = \sqrt{a^2 - x^2}$ *i.e.*, $x^2 + y^2 = a^2$, a circle with centre at the origin and radius a; $x = 0$ *i.e.*, the y-axis; and $x = a$, *i.e.*, a line parallel to the x-axis at a distance a. Thus, the region of integration is the part of the unit circle in the first quadrant. [See Fig. 9.31, page 9-31]

2. Change to r, θ : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $y = \sqrt{a^2 - x^2}$ *i.e.*, $y^2 = a^2 - x^2$ *i.e.*, $x^2 + y^2 = a^2$ becomes $r^2\cos^2\theta + r^2\sin^2\theta = a^2$ *i.e.*, $r = a$. Thus, the circle $x^2 + y^2 = a^2$ becomes the circle $r = a$. The x-axis is given by $\theta = 0$, y-axis is given by $\theta = \pi/2$.

3. Limits of r, θ : Considering a radial strip in the region of integration, r varies from $r = 0$ to $r = a$ and θ varies from $\theta = 0$ to $\theta = \pi/2$.

4. Integrand : The integrand $x^2 + y^2$ changes to r^2 and $dy\,dx$ is replaced by $r\,dr\,d\theta$.

$$\therefore\ I = \int_0^{\pi/2}\int_0^a r^2\cdot r\,dr\,d\theta = \int_0^{\pi/2}\int_0^a r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a d\theta = \int_0^{\pi/2}\frac{a^4}{4}\,d\theta$$

$$= \frac{a^4}{4}\left[\theta\right]_0^{\pi/2} = \frac{a^4}{4}\cdot\frac{\pi}{2} = \frac{\pi a^2}{8}.$$

the y-axis and $x = 2a$ i.e., a line parallel to the y-axis at $x = 2a$. The region of integration is the upper half of the circle shown in the figure.

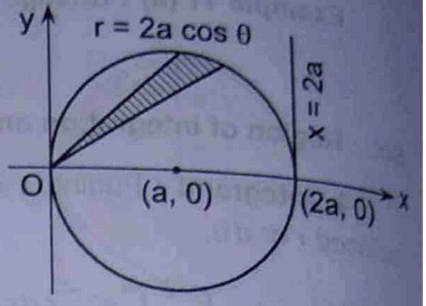

Fig. 9.37 (a)

2. Change to r, θ : Putting $x = r\cos\theta$ and $y = r\sin\theta$, the circle $x^2 + y^2 - 2ax = 0$ becomes

$$r^2\cos^2 + r^2\sin^2\theta - 2ar\cos\theta = 0 \text{ i.e., } r^2 = 2ar\cos\theta$$

i.e., $r = 2a\cos\theta$.

The x-axis is $\theta = 0$ and the y-axis is $\theta = \pi/2$.

3. Limits of r, θ : Considering a radial strip in the region of integration we see that, r varies from $r = 0$ to $r = 2a\cos\theta$ and θ varies from $\theta = 0$ to $\theta = \pi/2$.

4. Integrand : The integrand $\dfrac{x}{\sqrt{x^2+y^2}}$ becomes $\dfrac{r\cos\theta}{r} = \cos\theta$ and $dy\,dx$ is replaced by $r\,dr\,d\theta$.

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{2a\cos\theta} \cos\theta\, r\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\left[\frac{r^2}{2}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 4a^2\cos^2\theta\cos\theta\,d\theta = 2a^2\int_0^{\pi/2}\cos^3\theta\,d\theta$$

$$= 2a^2 \cdot \frac{2}{3}\cdot 1 = \frac{4a^2}{3}. \qquad \text{[By (24), page 6-32]}$$

Example 14 (a) : Evaluate by changing to polar coordinates

$$\int_0^1\int_x^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx.$$

(M.U. 1998, 2005, 12)

Sol. : 1. Region of Integration : The region of integration is given by $y = x$, a line OA through the origin; $y = \sqrt{2x - x^2}$ i.e., $x^2 - 2x + y^2 = 0$ i.e., $(x-1)^2 + y^2 = 1$ a circle with centre at $(1, 0)$ and radius $= 1$; $x = 0$, the y-axis, $x = 1$ a line parallel to the y-axis. Thus, the region of integration is the sector OAB.

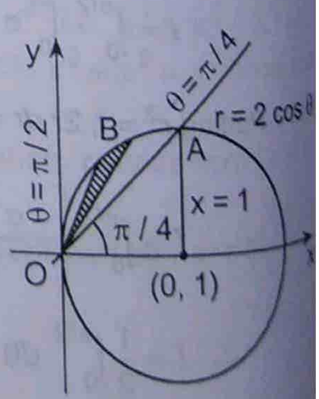

Fig. 9.38

2. Change to r, θ : Putting $x = r\cos\theta$, $y = r\sin\theta$ in $y = \sqrt{2x - x^2}$ i.e., $x^2 + y^2 = 2x$, we get $r^2 = 2r\cos\theta$ i.e., $r = 2\cos\theta$. The line $y = x$ becomes $r\sin\theta = r\cos\theta$ i.e., $\tan\theta = 1$, i.e., $\theta = \pi/4$.

3. Limits of r, θ : Considering a strip in the region of integration OAB, we see that r varies from $r = 0$ to $r = 2\cos\theta$ and θ varies from $\theta = \pi/4$ to $\theta = \pi/2$.

4. Integrand : Putting $x = r\cos\theta$, $y = r\sin\theta$, the integrand $x^2 + y^2$ becomes r^2 and $dx\,dy$ is to be replaced by $r\,dr\,d\theta$.

$$\therefore \quad I = \int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r^2\, r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = \frac{1}{4}\int_{\pi/4}^{\pi/2} 2^4\cos^4\theta\,d\theta = 4\int_{\pi/4}^{\pi/2}\cos^4\theta\,d\theta$$

$$= 4\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \int_{\pi/4}^{\pi/2}(1 + 2\cos 2\theta + \cos^2 2\theta)\,d\theta$$

$$\therefore \quad I = \int_{\pi/4}^{\pi/2} \left(1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\right) d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(\frac{3}{2} + 2\cos 2\theta + \frac{\cos 4\theta}{2}\right) d\theta$$

$$= \left[\frac{3}{2}\theta + \sin 2\theta + \frac{\sin 4\theta}{8}\right]_{\pi/4}^{\pi/2} = \frac{3}{2}\cdot\frac{\pi}{2} - \frac{3}{2}\cdot\frac{\pi}{4} - 1 = \frac{3\pi}{8} - 1.$$

Example 15 (a) : Evaluate $\int_0^a \int_y^{a+\sqrt{a^2-y^2}} \frac{dx\,dy}{(4a^2 + x^2 + y^2)^2}$. **(M.U. 1992, 2002)**

Sol. : 1. Region of integration : $x = y$ is a line OA through the origin; $x = a + \sqrt{a^2 - y^2}$ *i.e.*, $(x-a)^2 + y^2 = a^2$ is a circle with centre at $(a, 0)$ and radius a; $y = 0$ is the x-axis and $y = a$ is a line parallel to the x-axis. Thus, the region of integration is the sector OAB of the circle.

2. Change to r, θ : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $(x-a)^2 + y^2 = a^2$ *i.e.*, in $x^2 + y^2 - 2ax = 0$, we get $r^2\cos^2\theta + r^2\sin^2\theta - 2ar\cos\theta = 0$ *i.e.*, $r^2 = 2ar\cos\theta$ *i.e.*, $r = 2a\cos\theta$. The line $y = x$ becomes $r\sin\theta = r\cos\theta$ *i.e.*, $\tan\theta = 1$ *i.e.*, $\theta = \pi/4$. The x-axis is given by $\theta = 0$.

3. Limits of r, θ : Considering a radial strip in the region of integration OAB, we see that r varies from $r = 0$ to $r = 2a\cos\theta$ and θ varies from $\theta = 0$ to $\theta = \pi/4$.

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $\frac{1}{(4a^2 + x^2 + y^2)^2}$, we get $\frac{1}{(4a^2 + r^2)^2}$ and $dy\,dx$ is replaced by $r\,dr\,d\theta$.

$$I = \int_0^{\pi/4} \int_0^{2a\cos\theta} \frac{r\,d\theta\,dr}{(4a^2 + r^2)^2}$$

$$= -\frac{1}{2}\int_0^{\pi/4} \left[\frac{1}{4a^2 + r^2}\right]_0^{2a\cos\theta} d\theta \qquad [\text{ Put } 4a^2 + r^2 = t]$$

$$= -\frac{1}{2}\int_0^{\pi/4} \left[\frac{1}{4a^2 + 4a^2\cos^2\theta} - \frac{1}{4a^2}\right] d\theta$$

$$= \frac{1}{8a^2}\int_0^{\pi/4} \left[1 - \frac{1}{1 + \cos^2\theta}\right] d\theta$$

$$= \frac{1}{8a^2}\int_0^{\pi/4} \left[1 - \frac{\sec^2\theta}{\sec^2\theta + 1}\right] d\theta$$

$$= \frac{1}{8a^2}\left[\int_0^{\pi/4} d\theta - \int_0^1 \frac{dt}{2 + t^2}\right] \qquad [\because \sec^2\theta + 1 = 1 + \tan^2\theta + 1; \text{ put } t = \tan\theta]$$

$$\therefore \quad I = \frac{1}{8a^2}\left[\{\theta\}_0^{\pi/4} - \left\{\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}\right\}_0^1\right] = \frac{1}{8a^2}\left[\frac{\pi}{4} - \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\right].$$

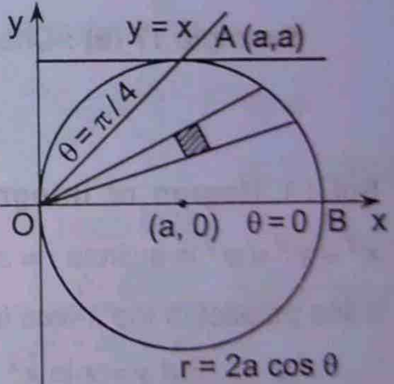

Fig. 9.39

Example 16 (a) : Change to polar coordinate and evaluate

$$\int_0^1 \int_0^x (x + y)\,dy\,dx.$$ **(M.U. 197, 99, 2003, 05, 10)**

Sol. : 1. Region of integration : $y = 0$ is the x-axis; $y = x$ is a line OB through the origion; $x = 0$ is the y-axis and $x = 1$ is a line AB parallel to the y-axis. Thus, the region of integration is the triangle OAB.

2. Change to r, θ : Putting $x = r\cos\theta$ and $y = r\sin\theta$, the line $y = x$ becomes $r\sin\theta = r\cos\theta$ *i.e.*, $\tan\theta = 1$ *i.e.*, $\theta = \pi/4$. The *x*-axis is given by $\theta = 0$; the *y*-axis is given by $\theta = \pi/2$ and the line $x = 1$ is given by $r\cos\theta = 1$ *i.e.*, $r = 1/\cos\theta$ *i.e.*, $r = \sec\theta$.

3. Limits of r, θ : Considering a radial strip in the region of integration *OAB*, we see that r varies from $r = 0$ to $r = \sec\theta$ and θ varies from $\theta = 0$ to $\theta = \pi/4$.

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $(x + y)$, we get,

$r\cos\theta + r\sin\theta = r(\cos\theta + \sin\theta)$ and $dy\,dx$ is replaced by $r\,dr\,d\theta$.

$$\therefore\ I = \int_0^{\pi/4}\int_0^{\sec\theta}(r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{\sec\theta}(\cos\theta + \sin\theta)\,r^2\,dr\,d\theta$$

$$= \int_0^{\pi/4}(\cos\theta + \sin\theta)\left[\frac{r^3}{3}\right]_0^{\sec\theta}d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4}(\cos\theta + \sin\theta)\sec^3\theta\,d\theta$$

$$= \frac{1}{3}\left[\int_0^{\pi/4}\sec^2\theta\,d\theta + \int_0^{\pi/4}\frac{1}{\cos^3\theta}\cdot\sin\theta\,d\theta\right]$$

$$\therefore\ I = \frac{1}{3}\left[\{\tan\theta\}_0^{\pi/4} + \left\{\frac{1}{2\cos^2\theta}\right\}_0^{\pi/4}\right]. \qquad [\text{ Put } \cos\theta = t\,]$$

$$= \frac{1}{3}\left[1 + \frac{1}{2}(2 - 1)\right] = \frac{1}{3}\left(1 + \frac{1}{2}\right) = \frac{1}{2}.$$

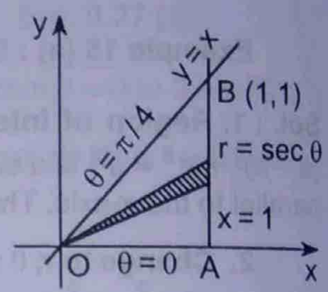

Fig. 9.40

Example 17 (a) : Change to polar coordinates and evaluate

$$\int_0^{a/\sqrt{2}}\int_y^{\sqrt{a^2 - y^2}}\log(x^2 + y^2)\,dx\,dy.$$

(M.U. 2002, 11, 12)

Sol. : 1. Region of integration : $x = y$ is a line *OA* through the origin; $x = \sqrt{a^2 - y^2}$ *i.e.*, $x^2 + y^2 = a^2$ is a circle $r = a$, with centre at the origin and radius a; $y = 0$ is the *x*-axis; $y = a/\sqrt{2}$ is a line parallel to the *x*-axis intersecting the circle in $A(a/\sqrt{2},\ a/\sqrt{2})$.

(Put $y = x$ in $x^2 + y^2 = a^2$ $\quad\therefore\ 2x^2 = a^2 \quad\therefore\ x = a/\sqrt{2}$).

Thus, the region of integration is the sector *OAB*.

2. Change to r, θ : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $y = x$, we get, $r\sin\theta = r\cos\theta$; $\tan\theta = 1$ *i.e.*, $\theta = \pi/4$. The circle $x^2 + y^2 = a^2$ becomes $r = a$. The *x*-axis is given by $\theta = 0$.

3. Limits of r, θ : Considering a radial strip in the region of integration, we see that r varies from $r = 0$ to $r = a$ and θ varies from $\theta = 0$ to $\theta = \pi/4$.

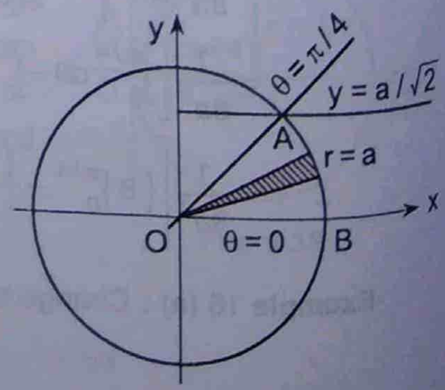

Fig. 9.41

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$, the integrand $\log(x^2 + y^2)$ becomes $\log r^2 = 2\log r$ and $dy\,dx$ is replaced by $r\,dr\,d\theta$.

$$\therefore\ I = \int_0^{\pi/4}\int_0^a \log r^2\cdot r\,dr\,d\theta = \int_0^{\pi/4}\int_0^a 2\log r\cdot r\,dr\,d\theta$$

$$\therefore \quad I = 2\int_0^{\pi/4} \left[\log r \cdot \frac{r^2}{2} - \int \frac{r^2}{2} \cdot \frac{1}{r}\, dr \right]_0^a d\theta \qquad \text{[By parts]}$$

$$= 2\int_0^{\pi/4} \left[\log r \cdot \frac{r^2}{2} - \frac{r^2}{4} \right]_0^a d\theta = 2\int_0^{\pi/4} \left(\frac{a^2}{2}\log a - \frac{a^2}{4} \right) d\theta$$

$$= \left(a^2 \log a - \frac{a^2}{2} \right) \int_0^{\pi/4} d\theta = \left(a^2 \log a - \frac{a^2}{2} \right) [\,\theta\,]_0^{\pi/4}$$

$$= a^2 \left(\log a - \frac{1}{2} \right) \cdot \frac{\pi}{4}.$$

EXERCISE - V

Change to polar coordinates and evaluate : Class (a) : 4 Marks

1. $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{(x^2+y^2)}\, dy\, dx.$

   (See Fig. 9.31, page 9-31)

2. $\int_0^4 \int_y^{4+\sqrt{16-y^2}} \frac{dx\, dy}{(64+x^2+y^2)^2}.$

   (See Fig. 9.39, page 9-37, $a = 4$)

3. $\int_0^a \int_y^a \frac{x\, dx\, dy}{(x^2+y^2)}.$

   (See Fig. 9.40, page 9-38)

4. $\int_0^1 \int_x^{\sqrt{(2x-x^2)}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx.$

   (See Fig. 9.38, page 9-36)

5. $\int_0^a \int_0^x \frac{x^3\, dy\, dx}{\sqrt{(x^2+y^2)}}.$

   (See Fig. 9.40, page 9-38)

6. $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy.$ **(M.U. 1994)**

   (See Fig. 9.40, page 9-38)

7. $\int_0^a \int_y^a \frac{x^2\, dx\, dy}{(x^2+y^2)^{3/2}}.$

   (See Fig. 9.40, page 9-38)

8. $\int_0^1 \int_0^{\sqrt{x-x^2}} \frac{4xy}{x^2+y^2} e^{-(x^2+y^2)}\, dy\, dx.$ **(M.U. 2010)**

   (See Fig. 9.74, page 9-58)

9. $\int_0^4 \int_{\sqrt{4x-x^2}}^{\sqrt{16-x^2}} \frac{dy\, dx}{\sqrt{16-x^2-y^2}}.$ **(M.U. 1989)**

   (See Fig. 9.34, page 9-33, $a = 4$)

10. $\int_0^2 \int_{\sqrt{2x-x^2}}^{\sqrt{4-x^2}} \frac{dy\, dx}{\sqrt{4-x^2-y^2}}.$ **(M.U. 1992)**

    (See Fig. 9.34, page 9-33, $a = 2$)

11. $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{xy\, e^{-(x^2+y^2)}}{x^2+y^2}\, dy\, dx.$ **(M.U. 2001)**

    (See Fig. 9.34, page 9-33)

12. $\int_0^a \int_0^{\sqrt{a^2-x^2}} y \sqrt{(x^2+y^2)}\, dy\, dx.$ **(M.U. 2005)**

    (See Fig. 9.31, page 9-31)

13. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx\, dy}{(a^2+x^2+y^2)^{3/2}}.$ **(M.U. 2002)**

    (Entire x-y plane)

14. $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\, dy\, dx.$

    (See Fig. 9.39, page 9-37)

[**Ans. : (1)** $\frac{\pi a^5}{20}$, **(2)** $\frac{1}{128}\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}} \tan^{-1} \frac{1}{\sqrt{2}} \right)$, **(3)** $\frac{\pi a}{4}$, **(4)** $2-\sqrt{2}$, **(5)** $\frac{a^4}{4}\log\left(1+\sqrt{2}\right)$,

(6) $\dfrac{a^3}{3}\log(1+\sqrt{2})$, (7) $\dfrac{a}{\sqrt{2}}$, (8) $\dfrac{1}{e}$, (9) 4, (10) 2, (11) $\dfrac{1}{4a^2}\left\{(1+a^2)e^{-a^2}-1\right\}$,

(12) $\dfrac{a^4}{4}$, (13) $\dfrac{2\pi}{a}$, (14) $\dfrac{3\pi a^4}{4}$.

5. Evaluation of Integral Over a Given Region : Cartesian Coordinates

Sometimes we are given the region not in terms of limits for x and y but in terms of a curve or area bounded by curves. To evaluate such an integral, it may be noted, that the change of order of integration or changing to polar coordinates is sometimes more convenient.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Find the integral $\iint\limits_R (y-\sqrt{x})\,dA$ where R is the region cut-off by the line $x+y=1$ of the xy-plane in the first quadrant.

Sol. : The line $x+y=1$ is the line AB and the region of integration is the triangle OAB.

Consider a strip parallel to the y-axis. On this strip y varies from 0 to $(1-x)$. Then the strip moves from $x=0$ to $x=1$

$$\therefore\quad I=\int_0^1\int_0^{1-x}(y-\sqrt{x})\,dy\,dx=\int_0^1\left[\frac{y^2}{2}-\sqrt{x}\cdot y\right]_0^{1-x}dx$$

$$=\int_0^1\left[\frac{(1-x)^2}{2}-\sqrt{x}\,(1-x)\right]dx$$

$$=\int_0^1\left[\frac{1}{2}(1-2x+x^2)-\sqrt{x}+x^{3/2}\right]dx$$

$$=\left[\frac{1}{2}\left(x-x^2+\frac{x^3}{3}\right)-\frac{x^{3/2}}{3/2}+\frac{x^{5/2}}{5/2}\right]_0^1$$

$$=\frac{1}{2}\left(1-1+\frac{1}{3}\right)-\frac{2}{3}+\frac{2}{5}=\frac{1}{6}-\frac{2}{3}+\frac{2}{5}=-\frac{1}{10}.$$

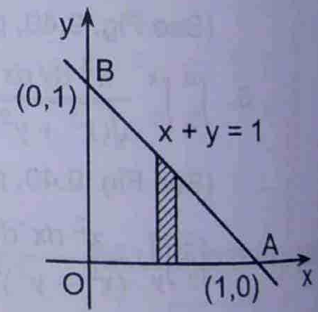

Fig. 9.42

Example 2 (b) : Evaluate $\iint (x^2-y^2)\,x\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$.

Sol. : The region of integration is the quadrant OAB. Consder a strip parallel to the x-axis on the strip x varies from 0 to $\sqrt{a^2-y^2}$ and then y varies from 0 to a.

$$\therefore\quad I=\int_0^a\int_0^{\sqrt{a^2-y^2}}(x^2-y^2)\,x\,dx\,dy$$

$$=\int_0^a\left[\frac{x^4}{4}-\frac{x^2}{2}y^2\right]_0^{\sqrt{a^2-y^2}}dy$$

$$=\int_0^a\left[\frac{(a^2-y^2)^2}{4}-\frac{(a^2-y^2)\,y^2}{2}\right]dy$$

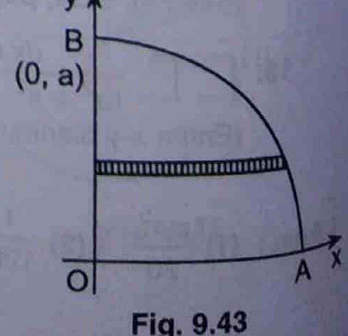

Fig. 9.43

$$\therefore \quad I = \int_0^a \left\{ \frac{1}{4}\left[a^4 - 2a^2y^2 + y^4\right] - \frac{1}{2}\left[a^2y^2 - y^4\right] \right\} dy$$

$$= \left[\frac{1}{4}\left(a^4 y - 2a^2 \frac{y^3}{3} + \frac{y^5}{5} \right) - \frac{1}{2}\left(\frac{a^2y^3}{3} - \frac{y^5}{5} \right) \right]_0^a = \frac{2}{15}a^5 - \frac{1}{15}a^5 = \frac{a^5}{15}.$$

Example 3 (b) : Evaluate $\iint xy\, dx\, dy$ over the region bounded by the x-axis, ordinate at $x = 2a$ and the parabola $x^2 = 4ay$. **(M.U. 2003)**

Sol. : $x = 2a$ is a line AB. $x^2 = 4ay$ is the vertical parabola. The region of integration is OAB. Consider a strip parallel to the y-axis. On the strip y varies from 0 to $x^2/4a$ and then x varies from 0 to $2a$.

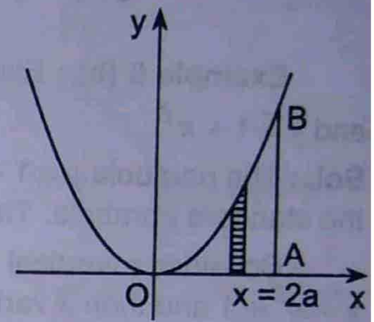

Fig. 9.44

$$\therefore \quad I = \int_0^{2a}\int_0^{x^2/4a} xy\, dy\, dx = \int_0^{2a}\left[x \cdot \frac{y^2}{2} \right]_0^{x^2/4a} dx$$

$$= \int_0^{2a} \frac{x}{2} \cdot \frac{x^4}{16a^2}\, dx = \frac{1}{32a^2}\int_0^{2a} x^5\, dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6} \right]_0^{2a} = \frac{1}{32a^2} \cdot \frac{64a^6}{6} = \frac{a^4}{3}.$$

Example 4 (b) : Evaluate $\iint_R \frac{1}{x^4 + y^2}\, dx\, dy$ where R is the region $x \geq 1$, $y \geq x^2$.

(M.U. 1996, 2002, 07)

Sol. : The boundaries of the region are $y = x^2$ a parabola with vertex at the origin and opening upwards. The line $x = 1$ is a line parallel to the y-axis. The region of integration is the region CAB between the line $x = 1$ and the branch of the parabola in the first quadrant.

In this region consider a strip parallel to the y-axis. On this strip y varies from $y = x^2$ to $y = \infty$. Then x varies from $x = 1$ to $x = \infty$.

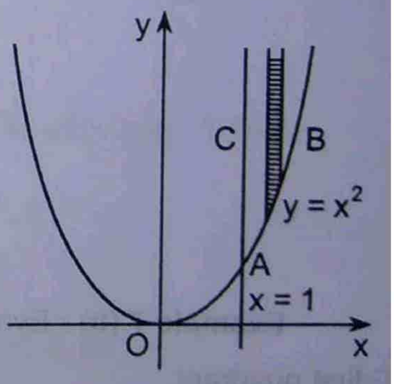

Fig. 9.45

$$\therefore \quad I = \int_1^{\infty}\int_{x^2}^{\infty} \frac{dy}{y^2 + (x^2)^2}\, dx$$

$$= \int_1^{\infty}\left[\frac{1}{x^2}\tan^{-1}\frac{y}{x^2} \right]_{x^2}^{\infty} dx = \int_1^{\infty}\frac{1}{x^2}\left[\frac{\pi}{2} - \frac{\pi}{4} \right] dx$$

$$= \frac{\pi}{4}\int_1^{\infty}\frac{dx}{x^2} = \frac{\pi}{4}\left[-\frac{1}{x} \right]_1^{\infty} = \frac{\pi}{4}[-0 + 1] = \frac{\pi}{4}.$$

Example 5 (b) : Evaluate $\iint xy\, dx\, dy$ over the area bounded by the parabolas $y = x^2$ and $x = -y^2$. **(M.U. 1997, 2005)**

Sol. : The region of integeration is bounded by the two parabolas are as shown in the figure. They intersect in $O(0, 0)$ and $A(-1, 1)$.

In the region OAB, consider a strip parallel to the x-axis. On this strip x varies from $x = -\sqrt{y}$ to $x = -y^2$ and then y varies from $y = 0$ to 1.

$$\therefore \quad I = \frac{1}{2}\int_0^1 (4 - 4x + x^2 - x^4)\,dx$$

$$= \frac{1}{2}\left[4x - 2x^2 + \frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left[4 - 2 + \frac{1}{3} - \frac{1}{5}\right] = \frac{16}{15}.$$

Example 8 (b) : Evaluate $\iint \sqrt{xy(1-x-y)}\,dx\,dy$ over the area bounded by $x = 0$, $y = 0$ and $x + y = 1$.

(M.U. 1996, 97, 2000, 02, 03, 12)

Sol. : The region of integration is the triangle OAB. Consider a strip parallel to the y-axis. On this strip y varies from $y = 0$ to $y = 1 - x$ and this x varies from $x = 0$ to $x = 1$.

$$\therefore \quad I = \int_0^1 \int_0^{1-x} \sqrt{xy(1-x-y)}\,dy\,dx$$

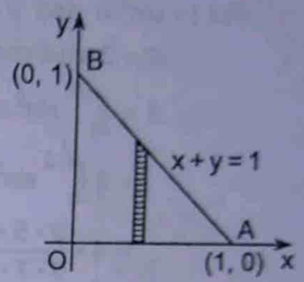

Fig. 9.49

We shall first integrate w.r.t. y. Now y varies from 0 to $1 - x$.

We shall first find

$$\therefore \quad I_1 = \int_0^{1-x} \sqrt{y(1-x-y)}\,dy$$

Suppose $1 - x = a$.

$$= \int_0^a \sqrt{y(a-y)}\,dy. \qquad \text{Put } y = at$$

$$= \int_0^1 a^{1/2}\, t^{1/2}\, a^{1/2}(1-t)^{1/2}\, a\,dt$$

$$= a^2 \int_0^1 t^{1/2}(1-t)^{1/2}\,dt$$

Now put $t = \sin^2\theta \qquad \therefore \quad dt = 2\sin\theta\cos\theta\,d\theta.$

$$\therefore \quad I_1 = a^2 \int_0^{\pi/2} \sin\theta \cdot \cos\theta \cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2a^2 \int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta$$

$$= 2a^2 \cdot \frac{1 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi a^2}{8} \qquad \text{[By (25), page 6-34]}$$

$$\therefore \quad I = \int_0^1 x^{1/2} \cdot a^2 \frac{\pi}{8}\,dx = \frac{\pi}{8}\int_0^1 x^{1/2}(1-x)^2\,dx$$

Again put $x = \sin^2\theta, \qquad dx = 2\sin\theta\cos\theta\,d\theta.$

$$\therefore \quad I = \frac{\pi}{8}\int_0^{\pi/2} \sin\theta \cdot \cos^4\theta \cdot 2\sin\theta\cos\theta\,d\theta = \frac{\pi}{4}\int_0^{\pi/2} \sin^2\theta\cos^5\theta\,d\theta$$

$$= \frac{\pi}{4} \cdot \frac{1 \cdot 4 \cdot 2 \cdot 1}{7 \cdot 5 \cdot 3 \cdot 1} = \frac{2\pi}{105}. \qquad \text{[By (26), page 6-34]}$$

Example 9 (b) : Evaluate $\iint_R xy\sqrt{1-x-y}\,dx\,dy$ over the area of the triangle formed by $x = 0$, $y = 0$, $x + y = 1$.

(M.U. 1996, 2001, 10)

Sol. : The region of integration is the triangle OAB as shown in the figure. Now, consider a strip parallel to the x-axis. On this strip x varies from $x = 0$ to $x = 1 - y$. Then y varies from $y = 0$ to $y = 1$.

$$\therefore \quad I = \int_0^1 \int_0^{1-y} y \cdot x\sqrt{(1-y)-x} \cdot dx\,dy$$

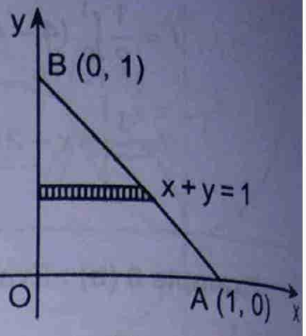

Fig. 9.50

Now, put $x = (1-y)\,t \quad \therefore \quad dx = (1-y)\,dt$

When $x = 0$, $t = 0$; when $x = 1 - y$, $t = 1$.

$$\therefore \quad I = \int_0^1 \int_0^1 y(1-y) \cdot t\sqrt{(1-y)-(1-y)t} \cdot (1-y)\,dt\,dy$$

$$= \int_0^1 \int_0^1 y(1-y)^2 \cdot \sqrt{1-y} \cdot t\sqrt{1-t} \cdot dt\,dy$$

$$= \int_0^1 \int_0^1 y(1-y)^{5/2} \cdot \left[t(1-t)^{1/2}\right] \cdot dt\,dy$$

$$= \int_0^1 y(1-y)^{5/2}\,dy \int_0^1 t(1-t)^{1/2}\,dt$$

Put $t = \sin^2\theta$ and $y = \sin^2\theta$.

$$\therefore \quad dt = 2\sin\theta\cos\theta\,d\theta, \qquad dy = 2\sin\theta\cos\theta\,d\theta.$$

$$\therefore \quad I = \int_0^{\pi/2} \sin^2\theta\cos^5\theta \cdot 2\sin\theta\cos\theta\,d\theta \times \int_0^{\pi/2} \sin^2\theta\cos\theta \cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^3\theta \cdot \cos^6\theta\,d\theta \times 2\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$$

$$= 4 \cdot \frac{2 \cdot 5 \cdot 3 \cdot 1}{9 \cdot 7 \cdot 5 \cdot 3} \cdot \frac{2}{5 \cdot 3 \cdot 1} = \frac{16}{945}. \qquad \text{[By (26), page 6-34]}$$

Example 10 (b) : Evaluate $\iint_R e^{2x-3y}\,dx\,dy$ over the triangle bounded by $x + y = 1$, $x = 1$, $y = 1$.

(M.U. 2013)

Sol. : The region of integration is the triangle ABC as shown in the adjoining figure. $x + y = 1$ is the line AB, $x = 1$ is the line AC and $y = 1$ is the line BC.

Consider a strip parallel to the x-axis in the triangle ABC. On this strip x varies from $x = 1 - y$ to $x = 1$. Then y varies from $y = 0$ to $y = 1$.

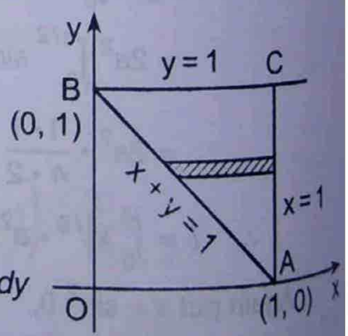

Fig. 9.50 (a)

$$\therefore \quad I = \int_0^1 \int_{1-y}^1 e^{2x-3y}\,dx\,dy = \int_0^1 \int_{1-y}^1 e^{2x} \cdot e^{-3y}\,dx\,dy$$

$$= \int_0^1 e^{-3y}\left[\frac{e^{2x}}{2}\right]_{1-y}^1 dy = \frac{1}{2}\int_0^1 e^{-3y}\,[e^2 - e^{2(1-y)}]\,dy$$

$$= \frac{1}{2}\int_0^1 e^2 \cdot e^{-3y} - [e^2 \cdot e^{-5y}]\,dy = \frac{e^2}{2}\int_0^1 [e^{-3y} - e^{-5y}]\,dy$$

$$= \frac{e^2}{2}\left[\frac{e^{-3y}}{-3} - \frac{e^{-5y}}{-5}\right]_0^1 = \frac{e^2}{2}\left[-\frac{1}{3}(e^{-3} - 1) + \frac{1}{5}(e^{-5} - 1)\right]$$

$$= \frac{e^2}{2}\left[-\frac{e^{-3}}{3} + \frac{e^{-5}}{5} + \frac{2}{15}\right] = -\frac{e^{-1}}{6} + \frac{e^{-3}}{10} + \frac{2e^2}{15}$$

$$= \frac{e^{-3}}{10} - \frac{e^{-1}}{6} + \frac{2}{15}e^2$$

Example 11 (b) : Prove that $\iint_R e^{ax+by}\,dx\,dy = 2R$ where R is area of the triangle whose boundaries are $x = 0$, $y = 0$ and $ax + by = 1$.

(M.U. 1996, 98)

Sol. : The region of integration is the triangle OAB. Consider a strip parallel to the x-axis. On this strip x varies from $x = 0$ to $x = (1 - by)/a$. Then y varies from $y = 0$ to $y = 1/b$.

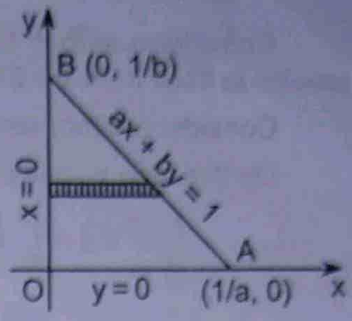

Fig. 9.51

$$\therefore \quad I = \int_0^{1/b} \int_0^{(1-by)/a} e^{ax+by}\, dx\, dy$$

$$= \int_0^{1/b} e^{by} \left[\frac{e^{ax}}{a}\right]_0^{(1-by)/a} dy$$

$$= \int_0^{1/b} \frac{e^{by}}{a}\left[e^{1-by} - 1\right] dy = \frac{1}{a}\int_0^{1/b} (e - e^{by})\, dy$$

$$= \frac{1}{a}\left[ey - \frac{e^{by}}{b}\right]_0^{1/b} = \frac{1}{a}\left[e\left(\frac{1}{b}\right) - \frac{e}{b} + \frac{1}{b}\right] = \frac{1}{ab} = 2\left[\frac{1}{2}\left(\frac{1}{a}\right)\left(\frac{1}{b}\right)\right]$$

$= 2R$ where R is the area of the triangle OAB.

Example 12 (b) : Evaluate $\iint_R (x + y)\, dx\, dy$ where R is the region bounded by $x = 0$, $x = 2$, $y = x$, $y = x + 2$.

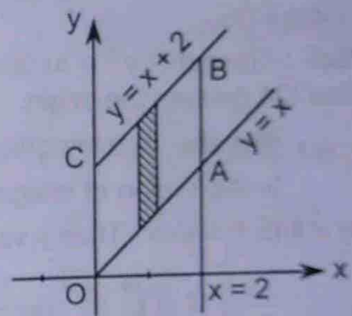

Fig. 9.52

Sol. : The region of integration is bounded by $x = 0$, *i.e.* the y-axis, $x = 2$ *i.e.* the line parallel to the y-axis, $y = x$ *i.e.* the line through the origin, $y = x + 2$ *i.e.* the line parallel to $y = x$ and making intercept 2 on the y-axis.

Thus, the region of integration is $OABC$. Take a strip parallel to the y-axis. On this strip y varies from $y = x$ to $y = x + 2$ and the x varies from $x = 0$ to $x = 2$.

$$\therefore \quad \iint_R (x + y)\, dx\, dy = \int_0^2 \int_x^{x+2} (x + y)\, dy\, dx$$

$$= \int_0^2 \left[xy + \frac{y^2}{2}\right]_x^{x+2} dx = \int_0^2 \left[\left\{x(x+2) + \frac{(x+2)^2}{2}\right\} - \left\{x \cdot x + \frac{x^2}{2}\right\}\right] dx$$

$$= \int_0^2 \left[x^2 + 2x + \frac{x^2}{2} + 2x + 2 - x^2 - \frac{x^2}{2}\right] dx = \int_0^2 (4x + 2)\, dx$$

$$= 2\int_0^2 (2x + 1)\, dx = 2\left[x^2 + x\right]_0^2 = 2(4 + 2) = 12.$$

Example 13 (b) : Evaluate $\iint_R x^2\, dA$ where R is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$ and $x = 8$. **(M.U. 2014)**

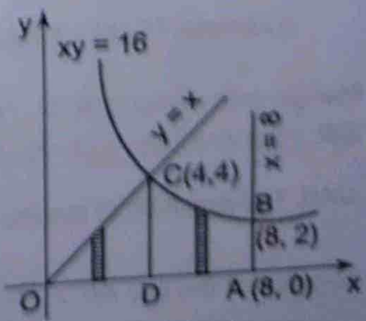

Fig. 9.53

Sol. : The rectangular hyperbola $xy = 16$ and the line $y = x$ intersect at C (4, 4). The line $x = 8$ intersects the hyperbola $xy = 16$ at B (8, 2). And A is (8, 0). $y = x$ is the line OC. Thus, the region of integration is $OABC$.

We have to split it into two regions ODC and $DABC$.

Consider a strip in the region ODC. On this strip y varies from 0 to x and the strip moves parallel to itself from $x = 0$ to $x = 4$.

Consider another strip in the region $CDAB$.

On this strip y varies from 0 to $16/x$. The strip moves from $x = 4$ to $x = 8$.

$$\therefore \iint_R x^2 \, dA = \int_0^4 \int_0^x x^2 \, dy \, dx + \int_4^8 \int_0^{16/x} x^2 \, dy \, dx$$

$$= \int_0^4 \left[x^2 y\right]_0^x dx + \int_4^8 \left[x^2 y\right]_0^{16/x} dx = \int_0^4 x^3 \, dx + \int_4^8 16x \, dx$$

$$= \left[\frac{x^4}{4}\right]_0^4 + \left[8x^2\right]_4^8 = 64 + 8(64 - 16) = 448.$$

Example 14 (b) : Evaluate $\iint_R \frac{y \, dx \, dy}{(a-x)\sqrt{ax - y^2}}$ where R is the region bounded by $y^2 = ax$ and $y = x$. **(M.U. 1998, 2012)**

Sol. : The curve $y^2 = ax$ is a parabola with vertex at the origin and opening on the right ; $y = x$ is a line OA through the origin.

(We shall first integrate with respect to y and then with respect to x.)

In the region of integration consider a strip parallel to the y-axis. On this strip y varies from $y = x$ to $y = \sqrt{ax}$. Then x varies from $x = 0$ to $x = a$.

$$\therefore \; I = \int_0^a \int_x^{\sqrt{ax}} \frac{1}{(a-x)} \cdot \frac{y \, dy}{\sqrt{ax - y^2}} \, dx$$

$$= \int_0^a \frac{1}{(a-x)} \left[-\sqrt{ax - y^2}\right]_x^{\sqrt{ax}} dx$$

$$= \int_0^a \frac{1}{(a-x)} \left[0 + \sqrt{ax - x^2}\right] dx = \int_0^a \frac{\sqrt{x}}{\sqrt{a-x}} \, dx$$

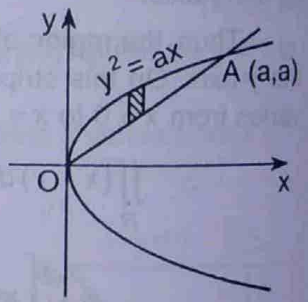

Fig. 9.54

Now, put $x = a \sin^2 \theta \qquad \therefore \; dx = 2a \sin \theta \cos \theta \, d\theta$

$$\therefore \; I = \int_0^{\pi/2} \frac{\sqrt{a} \sin \theta}{\sqrt{a} \cos \theta} \cdot 2a \sin \theta \cos \theta \, d\theta$$

$$= 2a \int_0^{\pi/2} \sin^2 \theta \, d\theta = 2a \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a}{2}. \qquad \text{[By (20), page 6-28]}$$

Example 15 (b) : Evaluate $\iint_R x(x-y) \, dx \, dy$ where R is the triangle with vertices (0, 0), (1, 2), (0, 4). **(M.U. 1997)**

Sol. : Let $O(0, 0)$, $A(1, 2)$, $B(0, 4)$ be the vertices of the triangle OAB. The equation of the line OA is $\frac{y-0}{0-2} = \frac{x-0}{0-1}$ *i.e.* $y = 2x$. The equation of the line AB is $\frac{y-2}{2-4} = \frac{x-1}{1-0}$ *i.e.* $y = -2x + 4$.

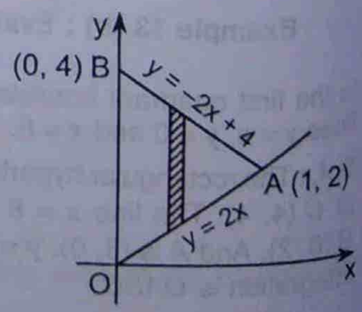

Fig. 9.55

Now, consider a strip parallel to the y-axis in the region of integration *i.e.*, in the triangle OAB. On this strip y varies from $y = 2x$ to $y = -2x + 4$ and then x varies from $x = 0$ to $x = 1$.

$$\therefore \quad I = \int_0^1 \int_{2x}^{-2x+4} (x^2 - xy)\, dy\, dx = \int_0^1 \left[x^2 y - \frac{xy^2}{2} \right]_{2x}^{-2x+4} dx$$

$$= \int_0^1 \left[x^2(-2x+4) - \frac{x}{2}(-2x+4)^2 - x^2 \cdot 2x + \frac{x}{2}(2x)^2 \right] dx$$

$$= \int_0^1 \left[-2x^3 + 4x^2 - 2x^3 + 8x^2 - 8x - 2x^3 + 2x^3 \right] dx$$

$$= \int_0^1 (-4x^3 + 12x^2 - 8x)\, dx$$

$$= \left[-x^4 + 4x^2 - 4x^2 \right]_0^1 = -1.$$

* equation of line must be taken from higher point to lower

Example 16 (b) : Evaluate $\iint (x^2 + y^2)\, dx\, dy$ over the area of the triangle whose vertices are (0, 1) (1, 1) (1, 2). **(M.U. 1997, 2014)**

Sol. : The equation of the line AC is $\dfrac{y-2}{1} = \dfrac{x-1}{1}$ *i.e.*, $y = x + 1$.

Consider a strip parallel to the y-axis in the region of integration *i.e.*, in the triangle ABC. On this strip y varies from 1 to $x + 1$ and then x varies from 0 to 1.

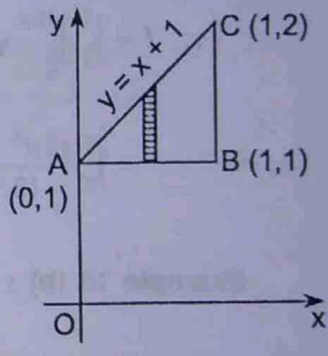

Fig. 9.56

$$\therefore \quad I = \int_0^1 \int_1^{x+1} (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_1^{x+1} dx$$

$$= \int_0^1 \left[\left\{ x^2(x+1) + \frac{(x+1)^3}{3} \right\} - \left\{ x^2 + \frac{1}{3} \right\} \right] dx$$

$$= \frac{1}{3} \int_0^1 (4x^3 + 3x^2 + 3x)\, dx = \frac{1}{3} \left[x^4 + x^3 + \frac{3x^2}{2} \right]_0^1$$

$$= \frac{1}{3} \left[1 + 1 + \frac{3}{2} \right] = \frac{7}{6}.$$

Example 17 (b) : Evaluate $\iint x^{m-1} y^{n-1}\, dx\, dy$ over the region bounded by $x + y = h$, $x = 0$, $y = 0$. **(M.U. 2002)**

Sol. : The region is bounded by the x-axis, y-axis and the line $x + y = h$.

Consider a strip parallel to the y-axis in the region of integration *i.e.*, in the triangle OAB.

On the strip y varies from 0 to $h - x$ and then strip moves from $x = 0$ to $x = h$.

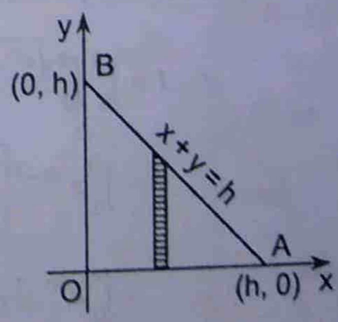

Fig. 9.57

$$\therefore \quad I = \int_0^h \int_0^{h-x} x^{m-1} y^{n-1}\, dy\, dx = \int_0^h x^{m-1} \left[\frac{y^n}{n} \right]_0^{h-x} dx$$

Now, $I = \int_0^h x^{m-1} \cdot \dfrac{1}{n} (h-x)^n\, dx.$ Put $x = ht$

$$\therefore \quad I = \int_0^1 h^{m-1} \cdot t^{m-1} \cdot \frac{1}{n} h^n (1-t)^n \cdot h\, dt$$

$$= \frac{h^{m+n}}{n} \int_0^1 t^{m-1} (1-t)^n\, dt$$

$$= \frac{h^{m+n}}{n} \cdot \frac{\overline{|m}\ \overline{|n+1}}{|m+n+1} = \frac{h^{m+n}\ \overline{|m}\ \overline{|n}}{(m+n)\,|m+n}.$$

Example 18 (b) : Evaluate $\iint_R \sqrt{xy-y^2}\, dx\, dy$ where R is a triangle whose vertices are (0, 0), (10, 1) and (1, 1).

(M.U. 1997, 98, 2002, 04)

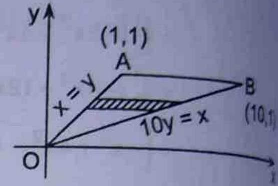

Fig. 9.58

Sol. : The triangle OAB is as shown in the figure. The equation of the line OA is

$$\frac{y-1}{1-0} = \frac{x-1}{1-0} \quad i.e.\ \ y = x.$$

The equation of the line OB is $\quad \dfrac{y-1}{1-0} = \dfrac{x-10}{10-0} \quad i.e., \quad x = 10y.$

Now, consider a strip parallel to the x-axis in the region of integration *i.e.*, in the triangle OAB. On this strip x varies from $x = y$ to $x = 10y$. Then y varies from $y = 0$ to $y = 1$.

$$\therefore \quad I = \int_0^1 \int_y^{10y} \sqrt{xy - y^2}\, dx\, dy = \int_0^1 \left[\frac{(xy-y^2)^{3/2}}{(3/2)\, y} \right]_y^{10y} dy$$

$$= \int_0^1 \frac{(9y^2 - 0)^{3/2}}{(3/2)\, y}\, dy = \frac{2}{3} \int_0^1 27 \cdot y^2\, dy = 18 \left[\frac{y^3}{3} \right]_0^1 = 6.$$

Example 19 (b) : Show that $\iint_R x^{m-1}\, y^{n-1}\, dy\, dx = \dfrac{1}{2n} a^m\, b^n\, B\left(\dfrac{m}{2}, \dfrac{n}{2} + 1\right)$ where R is the positive quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(M.U. 1997)

Sol. : Consider a strip parallel to the y-axis in the region of integration OAB.

On this strip y varies from $y = 0$ to $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$ and then x varies from $x = 0$ to $x = a$.

$$\therefore \quad I = \int_0^a \int_0^{b\sqrt{a^2-x^2}/a} x^{m-1}\, y^{n-1}\, dy\, dx$$

$$= \int_0^a x^{m-1} \left[\frac{y^n}{n} \right]_0^{b\sqrt{a^2-x^2}/a} dx$$

$$= \int_0^a x^{m-1} \cdot \frac{1}{n} \cdot \frac{b^n}{a^n} (a^2 - x^2)^{n/2}\, dx$$

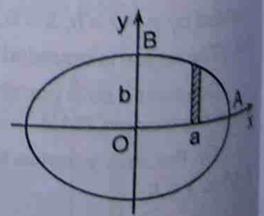

Fig. 9.59

Now, put $x = a \sin\theta \quad \therefore \quad dx = a\cos\theta\, d\theta$

When $x = 0,\ \theta = 0$; when $x = a,\ \theta = \pi/2$.

$$\therefore \quad I = \int_0^{\pi/2} \frac{b^n}{n \cdot a^n}\, a^{m-1} \sin^{m-1}\theta \cdot a^n \cos^n\theta \cdot a\cos\theta\, d\theta$$

$$\therefore \quad I = \frac{1}{n} \cdot \frac{b^n}{a^n} \cdot a^{m-1} \cdot a^n \cdot a \cdot \int_0^{\pi/2} \sin^{m-1}\theta \cdot \cos^{n+1}\theta \, d\theta$$

$$= \frac{a^m \cdot b^n}{n} \cdot \frac{1}{2} B\left(\frac{m}{2}, \frac{n}{2}+1\right) \qquad \text{[By (17), page 6-28]}$$

Example 20 (b) : Evaluate $\iint_R \frac{2x\,y^5}{\sqrt{1+x^2y^2-y^4}}\,dx\,dy$ where R is the region of the triangle whose vertices are (0, 0), (1, 1), (0, 1). **(M.U. 1995, 2003, 05)**

Sol. : Let O (0, 0), A (1, 1), B (0, 1) be the vertices of the triangle OAB. Now, the equation of the line AB is $y = 1$ and the equation of the line OA is

$$\frac{x-0}{0-1} = \frac{y-0}{0-1} \quad \text{i.e.} \quad x = y.$$

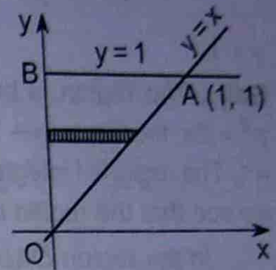

Fig. 9.60

Now, consider a strip parallel to the x-axis in the region of integration i.e., in the triangle OAB. On this strip x varies from $x = 0$ to $x = y$. The strip moves from $y = 0$ to $y = 1$.

$$\therefore \quad I = \int_0^1 \int_0^y \frac{2y^5 \cdot x}{\sqrt{(1-y^4)+x^2y^2}}\,dx\,dy = \int_0^1 \int_0^y \frac{1}{y} \cdot \frac{2y^5 \cdot x}{\sqrt{\dfrac{1-y^4}{y^2}+x^2}}\,dx\,dy$$

$$= \int_0^1 2y^4 \left[\sqrt{\frac{1-y^4}{y^2}+x^2}\right]_0^y dy = \int_0^1 2y^4 \left[\sqrt{\frac{1-y^4}{y^2}+y^2} - \sqrt{\frac{1-y^4}{y^2}}\right] dy$$

$$= \int_0^1 2y^4 \left[\frac{1}{y} - \frac{\sqrt{1-y^4}}{y}\right] dy = 2\int_0^1 \left[y^3 - \sqrt{1-y^4} \cdot y^3\right] dy$$

$$\therefore \quad I = 2\left[\frac{y^4}{4} + \frac{1}{4} \cdot \frac{(1-y^4)^{3/2}}{3/2}\right]_0^1 = 2\left[\frac{1}{4} - \frac{1}{4} \cdot \frac{2}{3}\right] = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Example 21 (b) : Prove that $\iint_R \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \frac{\pi}{4}$ where R is the region of the first quadrant of the ellipse $2x^2 + y^2 = 1$. **(M.U. 1995)**

Sol. : The ellipse $2x^2 + y^2 = 1$ i.e. $\frac{x^2}{1/2} + \frac{y^2}{1} = 1$ has semi-major axis $a = \frac{1}{\sqrt{2}}$ and semi-minor axis $b = 1$.

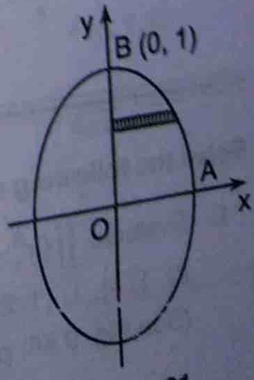

Fig. 9.61

Consider a strip parallel to the x-axis in the region of integration OAB, on this strip x varies from $x = 0$ to $x = \sqrt{1-y^2}\,/\sqrt{2}$. This strip moves from $y = 0$ to $y = 1$.

$$\therefore \quad I = \int_0^1 \int_0^{\sqrt{(1-y^2)/2}} \frac{dx\,dy}{\sqrt{(1-y^2)-x^2}}$$

$$\therefore\ I = \int_0^1 \sin^{-1}\left[\frac{x}{\sqrt{1-y^2}}\right]_0^{\sqrt{(1-y^2)/2}} dy$$

$$= \int_0^1 \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) dy = \int_0^1 \frac{\pi}{4}\, dy = \frac{\pi}{4}\left[\,y\,\right]_0^1 = \frac{\pi}{4}.$$

Example 22 (b) : Evaluate $\iint_R xy\, dx\, dy$ over the region R given by $x^2 + y^2 - 2x = 0$, $y^2 = 2x$, $y = x$. **(M.U. 1994, 97, 2005, 07, 10, 13)**

Sol. : The region is bounded by the line $y = x$, the parabola $y^2 = 2x$, the circle $(x-1)^2 + y^2 = 1$ with centre $(1, 0)$ and radius $= 1$. The region of integration is $OABC$. For evaluating the integral we see that the region is divided into two parts, OAB and ABC.

In the region OAB. Consider a strip parallel to the y-axis. On this strip y varies from $y = \sqrt{2x - x^2}$ to $y = \sqrt{2x}$ and then x varies from $x = 0$ to $x = 1$.

In the region ABC, consider again a strip parallel to the y-axis. On this strip y varies from $y = x$ to $y = \sqrt{2x}$, and then x varies from $x = 1$ to $x = 2$.

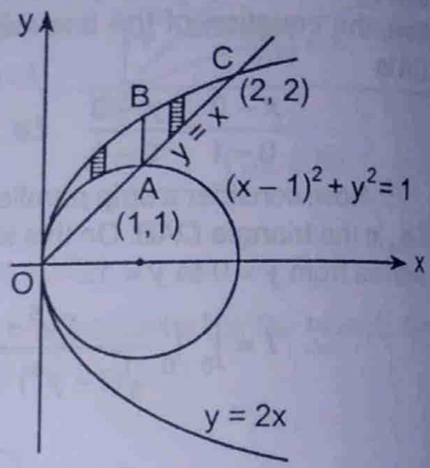

Fig. 9.62

$$\therefore\ I = \int_0^1 \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} xy\, dy\, dx + \int_1^2 \int_x^{\sqrt{2x}} xy\, dy\, dx$$

$$= \int_0^1 x \cdot \left[\frac{y^2}{2}\right]_{\sqrt{2x-x^2}}^{\sqrt{2x}} dx + \int_1^2 x\left[\frac{y^2}{2}\right]_x^{\sqrt{2x}} dx$$

$$= \frac{1}{2}\int_0^1 x \cdot \left[2x - (2x - x^2)\right] dx + \frac{1}{2}\int_1^2 x\left[2x - x^2\right] dx$$

$$= \frac{1}{2}\int_0^1 x^3\, dx + \frac{1}{2}\int_1^2 (2x^2 - x^3)\, dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 + \frac{1}{2}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2 = \frac{1}{8} + \frac{1}{2}\left[\frac{16}{3} - \frac{16}{4} - \frac{2}{3} + \frac{1}{4}\right]$$

$$= \frac{1}{8} + \frac{1}{2}\left[\frac{64 - 48 - 8 + 3}{12}\right] = \frac{1}{8} + \frac{11}{24} = \frac{14}{24} = \frac{7}{12}.$$

EXERCISE - VI

Solve the following examples : Class (b) : 6 Marks

1. Evaluate $\iint (x^2 - y^2)\, dx\, dy$ over the area of the triangle whose vertices are at the points $(0, 1)\ (1, 1)\ (1, 2)$. **(M.U. 2011) [Ans. : $-2/3$]**
   (See Fig. 9.56, page 9-47)

2. Evaluate $\iint x^{n-1} y^{m-1}\, dx\, dy$ over the triangle given by $x \geq 0, y \geq 0, x + y \leq 1$.

(See Fig. 9.50, page 9-44) [**Ans. :** $\dfrac{\Gamma m \cdot \Gamma n}{\Gamma(m+n+1)}$]

3. Evaluate $\iint dx\, dy$ throughout the area bounded by $y = x^2$ and $x + y = 2$.

(See Fig. 9.48, page 9-42) [**Ans. :** $-10/3$]

4. Evaluate $\iint (x^2 + y^2)\, dx\, dy$ over the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(See Fig. 9.59, page 9-48) [**Ans. :** $\dfrac{\pi a b}{4}(a^2 + b^2)$]

5. Evaluate $\iint x(x^2 + y^2)\, dx\, dy$ over the positive quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

[**Ans. :** $\dfrac{1}{15} a^2 b (2a^2 + b^2)$]

6. Evaluate $\iint x^{m-1} y^{n-1}\, dx\, dy$ over the positive quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(M.U.1997) [**Ans. :** $\dfrac{a^m b^n}{n} B\left(\dfrac{m}{2}, \dfrac{n}{2} + 1\right)$]

7. Evaluate $\iint\limits_R \dfrac{y e^{2y}}{\sqrt{(1-x)(x-y)}}\, dx\, dy$ where R is the region of the triangle whose vertices are (0, 0), (1, 0) and (1, 1). **(M.U. 1998)**

(See Fig. 9.16, page 9-19, $a = 1$) [**Ans. :** $\dfrac{\pi}{4}(e^2 + 1)$]

8. Evaluate $\iint (x^2 + y^2)\, dx\, dy$ over the area of the triangle whose vertices are (0, 0) (1, 0), (1, 2).

(Similar to above figure.) **(M.U. 2002)** [**Ans. :** $7/6$]

9. Evaluate $\iint xy\, dx\, dy$ over the area bounded by $y^2 = 4x$ and $y = 2x - 4$. **(M.U. 2002)**

(See Fig. 9.63) [**Ans. :** $45/2$]

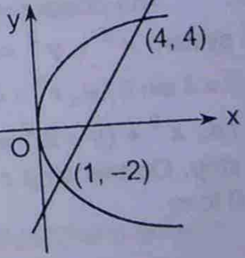

Fig. 9.63

10. Evaluate $\iint\limits_R (x^2 + y^2)\, dx\, dy$ over the area enclosed by the curves $y = 4x$, $x + y = 3$, $y = 0$, $y = 2$. **(M.U. 1989)**

(See similar Fig. 9.20 (a), page 9-24) [**Ans. :** $463/48$]

11. Evaluate $\iint x^2 y^2\, dx\, dy$ over the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. [**Ans. :** $\dfrac{\pi a^3 b^3}{24}$]

(See Fig. 9.59, page 9-48)

12. Evaluate $\iint y\, dx\, dy$ through out the area bounded by $y = x^2$ and $x + y = 2$.

(See Fig. 9.48, page 9-42) [**Ans. :** $16/5$]

13. Evaluate $\iint e^{3x+4y}\, dx\, dy$ over the triangle $x = 0, y = 0, x + y = 1$.

[**Ans. :** $\dfrac{1}{12}(3e^4 - 4e^3 + 1)$]

(See Fig. 9.49, page 9-43)

14. Evaluate $\iint \frac{xy}{\sqrt{1-y^2}}\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

(See Fig. 9.43, page 9-40, $a = 1$) [**Ans. :** 1 / 6]

15. Evaluate $\iint \frac{dy\, dx}{\sqrt{1-2x^2-y^2}}$ over the first quadrant of the ellipse $2x^2 + y^2 = 1$.

(See Fig. 9.61, page 9-49) **(M.U. 1995, 2003)** [**Ans. :** $\pi/2\sqrt{2}$]

16. Evaluate $\iint_R xy(x-1)\, dx\, dy$ where R is the region bounded by $xy = 4$, $y = 0$, $x = 1$, $x = 4$. **(M.U. 1995, 2009)**

(See Fig. 9.64) [**Ans. :** 8 (3 – log 4)]

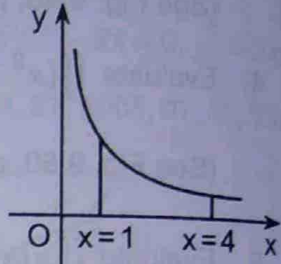

Fig. 9.64

17. Evaluate $\iint_R xy(x+y)\, dx\, dy$ where R is the region bounded by $x^2 = y$ and $x = y$. **(M.U. 1988, 2000, 02, 08)** [**Ans. :** 3 / 56]

(See Fig. 9.11, page 9-15, $a = 1$)

6. Evaluation of Integral Over a Given Region : Polar Coordinates

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Calculate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

(M.U. 1988, 90, 2002, 13)

Sol. : When the region of integration is given in polar coordinates as $r = f(\theta)$, it is better to convert the equation in cartesian coordinates by replacing r^2 by $x^2 + y^2$, $r\cos\theta$ by x and $r\sin\theta$ by y. (Also see § 20, page 15-14)

The circle $r = 2\sin\theta$ *i.e.*, $r^2 = 2r\sin\theta$ becomes in cartesian system $x^2 + y^2 = 2y$ *i.e.*, $x^2 + (y-1)^2 = 1$. Similarly, the circle $r = 4\sin\theta$ *i.e.*, $r^2 = 4r\sin\theta$ becomes in cartesian system $x^2 + y^2 = 4y$ *i.e.*, $x^2 + (y-2)^2 = 4$. In the region of integration, consider a radial strip. On this strip r varies from $2\sin\theta$ to $4\sin\theta$ and θ varies from 0 to π.

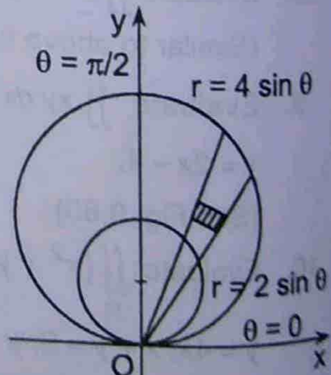

Fig. 9.65

$$\therefore\ I = \int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\, dr\, d\theta = \int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi}\left[4^4\sin^4\theta - 2^4\sin^4\theta\right] d\theta$$

$$= 60\int_0^{\pi}\sin^4\theta\, d\theta = 120\int_0^{\pi/2}\sin^4\theta\, d\theta = 120\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{45}{2}\cdot\pi.$$

Example 2 (b) : Evaluate $\iint r e^{-r^2/a^2}\cos\theta\sin\theta\, d\theta\, dr$ over the upper half of the circle $r = 2a\cos\theta$.

Sol. : The circle $r = 2a\cos\theta$ *i.e.* $r^2 = 2\, ar\cos\theta$ *i.e.* $x^2 + y^2 = 2\, ax$ *i.e.* $(x-a)^2 + y^2 = a^2$ has centre at $(a, 0)$ and radius a.

$r\,dr\,d\theta$ *

In the region of integration, consider a radial strip. On this strip, r varies from 0 to $2a\cos\theta$ and θ varies from 0 to $\pi/2$.

$$\therefore\ I = \int_0^{\pi/2}\int_0^{2a\cos\theta} r\,e^{-r^2/a^2}\sin\theta\cos\theta\,d\theta\,dr$$

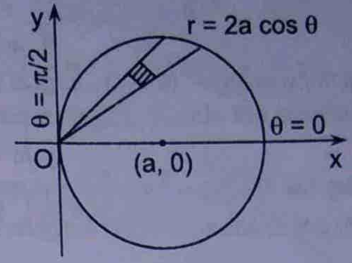

Fig. 9.66

$$= \int_0^{\pi/2}\left[-\frac{a^2}{2}e^{-r^2/a^2}\right]_0^{2a\cos\theta}\sin\theta\cos\theta\,d\theta$$

$$= -\frac{a^2}{2}\int_0^{\pi/2}\left(e^{-4\cos^2\theta}-1\right)\sin\theta\cos\theta\,d\theta$$

$$= -\frac{a^2}{2}\left[\frac{1}{8}e^{-4\cos^2\theta}-\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$$

$$= -\frac{a^2}{2}\left[\frac{1}{8}-\frac{1}{2}-\frac{1}{8}e^{-4}\right]=\frac{a^2}{16}(3+e^{-4}).$$

Example 3 (b) : Evaluate $\iint \dfrac{r\,d\theta\,dr}{\sqrt{r^2+a^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$.

(M.U. 1993, 2000)

Sol. : $r^2 = a^2\cos 2\theta$ is Bernoulli's Lemniscate (See § 22, page 15-15). The region of integration is one loop as the right. In this region of integration, consider a radial strip. On this strip r varies from 0 to $a\sqrt{\cos 2\theta}$ and θ varies from $-\dfrac{\pi}{4}$ to $\dfrac{\pi}{4}$.

$$\therefore\ I = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}}\frac{r\,d\theta\,dr}{\sqrt{a^2+r^2}}$$

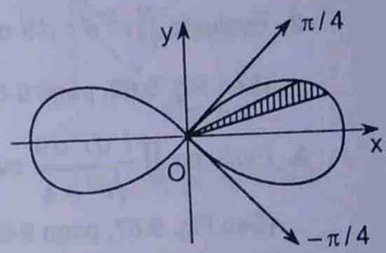

Fig. 9.67

$$= \int_{-\pi/4}^{\pi/4}\left[(r^2+a^2)^{1/2}\right]_0^{a\sqrt{\cos 2\theta}}d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left[\sqrt{a^2+a^2\cos 2\theta}-a\right]d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}\left[\sqrt{2}\cdot\cos\theta-1\right]d\theta = 2a\left[\sqrt{2}\cdot\sin\theta-\theta\right]_0^{\pi/4}$$

$$= 2a\left[1-\frac{\pi}{4}\right]=\frac{a}{2}(4-\pi).$$

Example 4 (b) : Evaluate $\iint_R \sin\theta\,dA$ where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1+\cos\theta)$.

Sol. : The two curves are shown in the figure. The region R outside the circle and inside the cardioid is $ABCDEA$ and in polar coordinates $dA = r\,dr\,d\theta$. Consider a radial strip in the region of integration.

On the strip r varies from E to C *i.e.* from $r = 2$ to $r = 2(1+\cos\theta)$ and the strip rotates from $\theta = 0$ to $\theta = \pi/2$ in the first quadrant.

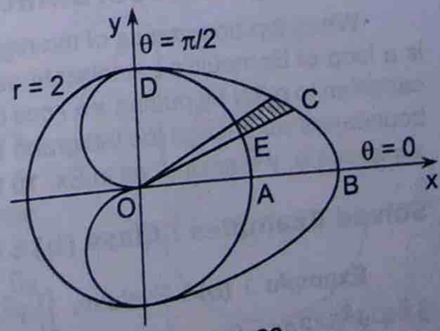

Fig. 9.68

$$\therefore\ \iint_R \sin\theta\,dA = \int_0^{\pi/2}\int_2^{2(1+\cos\theta)}\sin\theta\,r\,dr\,d\theta$$

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$$\therefore \iint_R \sin\theta\, dA = \int_0^{\pi/2} \sin\theta \left[\frac{r^2}{2}\right]_2^{2(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin\theta\,[\,4\,(1+\cos\theta)^2 - 4\,]\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (8\cos\theta + 4\cos^2\theta)\sin\theta\, d\theta$$

$$= \frac{1}{2}\int_1^{0} [\,8t + 4t^2\,](-\,dt) \qquad [\text{ Putting } \cos\theta = t\,]$$

$$= \int_0^1 (4t + 2t^2)\, dt = \left[2t^2 + \frac{2t^3}{3}\right]_0^1 = 2 + \frac{2}{3} = \frac{8}{3}.$$

EXERCISE - VII

Solve the following examples : Class (b) : 6 Marks

1. Evaluate $\iint r\sqrt{a^2 - r^2}\cdot dr\, d\theta$ over upper half of the circle $r = a\cos\theta$.
(See Fig. 9.73, page 9-57) **(M.U. 1988) [Ans. :** $\frac{a^3}{18}(3\pi - 4)$ **]**

2. Evaluate $\iint r^2\, dr\, d\theta$ over the area between the circles $r = a\cos\theta$, $r = 2a\cos\theta$.
(See Fig. 9.69, page 9-55) **(M.U. 1988) [Ans. :** $\frac{28}{9}a^3$ **]**

3. Evaluate $\iint \frac{r\, dr\, d\theta}{\sqrt{r^2 + 4}}$ over one loop of $r^2 = 4\cos 2\theta$. **(M.U. 1989)**
(See Fig. 9.67, page 9-53, $a = 2$.) **[Ans. :** $(4 - \pi)$ **]**

4. Evaluate $\iint \frac{r\, dr\, d\theta}{(a^2 + r^2)^2}$ over one loop of lemniscate $r^2 = a^2\cos 2\theta$.
(See Fig. 9.67, page 9-53.) **[Ans. :** $(\pi - 2)/4a^2$ **]**

5. Evaluate $\iint r\sin\theta\, dA$ over the cardioid $r = a(1 + \cos\theta)$ above the initial line.
(See Fig. 9.68, page 9-53) **(M.U. 2010) [Ans. :** $(4/3)\, a^3$ **]**

7. Change of Coordinate System, Cartesian to Polar

When the boundaries of the region of integration are circles or when the region of integration is a loop of Bernoulli's Lemniscate as in Ex. 1 to 9 below, we change the coordinate system from cartesian to polar by putting $x = r\cos\theta$, $y = r\sin\theta$ and also by putting $dy\, dx = r\, dr\, d\theta$. Along with the boundaries we change the integrand also to polar coordinates. If the boundary is an ellipse, we put $x = ar\cos\theta$, $y = ar\sin\theta$ as in Ex. 10 to 12 below.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Evaluate $\iint y^2\, dx\, dy$ over the area outside $x^2 + y^2 - ax = 0$ and inside $x^2 + y^2 - 2ax = 0$.

(M.U. 1991, 93, 2006)

Sol. : 1. Region in Cartesian Coordinates : We first note that

$$x^2 + y^2 - ax = 0 \quad i.e., \quad \left[x - \left(\frac{a}{2}\right)\right]^2 + y^2 = \left(\frac{a}{2}\right)^2$$

$r^2 = a\cos\theta$ $r^2 = a\sin\theta$ $r^2 = 2a\cos\theta$ circle

is a circle with centre at $(a/2, 0)$ and radian $a/2$ and $x^2 + y^2 - 2ax = 0$ *i.e.*, $(x - a)^2 + y^2 = a^2$ is a circle with centre at $(a, 0)$ and radius a. The region of integration is the region outside the smaller circle and inside the larger circle.

2. Region in Polar Coordinates : Putting $x = r\cos\theta$, $y = r\sin\theta$ in $x^2 + y^2 - ax = 0$, we get $r^2 = ar\cos\theta$ *i.e.*, $r = a\cos\theta$ and in $x^2 + y^2 - 2ax = 0$, we get $r^2 = 2ar\cos\theta$ *i.e.*, $r = 2a\cos\theta$. x-axis is $\theta = 0$ and y-axis is $\theta = \pi/2$.

3. Limits of r, θ : Considering a radial strip in the region of integration, we see that r varies from $r = a\cos\theta$ to $r = 2a\cos\theta$ and θ varies from $\theta = 0$ to $\theta = \pi/2$.

4. Integrand : Putting $x = r\cos\theta$, $y = r\sin\theta$, y^2 becomes $r^2\sin^2\theta$ and $dx\,dy$ is replaced by $r\,dr\,d\theta$.

$$\therefore \quad I = 2\int_0^{\pi/2}\int_{a\cos\theta}^{2a\cos\theta} r^2\sin^2\theta \cdot r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\sin^2\theta\left[\frac{r^4}{4}\right]_{a\cos\theta}^{2a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\sin^2\theta\left[16a^4\cos^4\theta - a^4\cos^4\theta\right]d\theta$$

$$= \frac{15a^4}{2}\int_0^{\pi/2}\cos^4\theta\sin^2\theta\,d\theta$$

$$= \frac{15a^4}{2}\cdot\frac{3\cdot 1\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{15\pi a^4}{64}. \qquad \text{[By (25), page 6-34]}$$

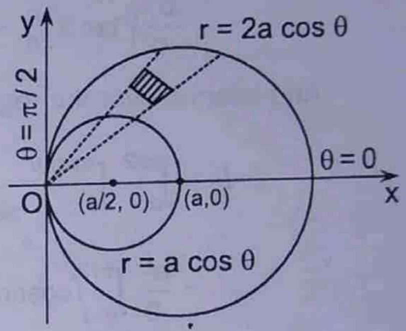

Fig. 9.69

Example 2 (b) : Evaluate $\iint \frac{(x^2 + y^2)^2}{x^2 y^2}\,dx\,dy$ over the area common to $x^2 + y^2 = ax$ and $x^2 + y^2 = by$, $a, b > 0$. **(M.U. 2002, 03, 08)**

Sol. : 1. Region in Cartesian Coordinates : We first note that

$x^2 + y^2 - ax = 0$ *i.e.*, $\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$ is a circle with centre at $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$;

and $x^2 + y^2 - by = 0$ *i.e.*, $x^2 + \left(y - \frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2$ is a circle with centre $\left(0, \frac{b}{2}\right)$ and radius $\frac{b}{2}$.

The region of integration is the region common to the two circles.

2. Region in Polar Coordinates : Putting $x = r\cos\theta$, $y = r\sin\theta$, we see that the equations of the circle become $r = a\cos\theta$, $r = b\sin\theta$. At the point of intersection A, $a\cos\theta = b\sin\theta$ *i.e.*, $\tan\theta = a/b$ or $\theta = \tan^{-1}(a/b) = \alpha$ say. By the radius OA the region of integration is divided into two parts OBA and OCA.

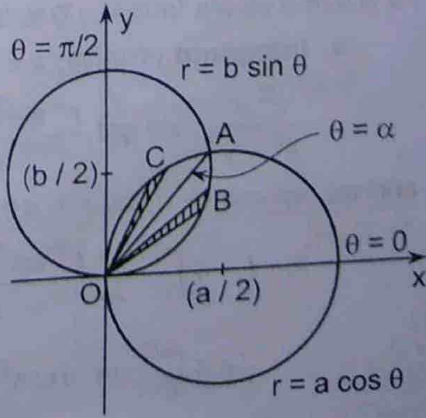

Fig. 9.70

3. Limits of r, θ : Considering a radial strip in the region OBA, we see that r varies from $r=0$ to $r = b \sin\theta$ and θ varies from $\theta = 0$ to $\theta = \alpha$. In the region OCA, r varies from $r = 0$ to $r = a\cos\theta$ and θ varies from $\theta = \alpha$ to $\theta = \pi/2$.

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$, the integrand

$$\frac{(x^2+y^2)^2}{x^2y^2} \text{ becomes } \frac{r^4}{r^4\sin^2\theta\cos^2\theta}$$ and we replace $dx\,dy$ by $r\,dr\,d\theta$.

Now, Integral over the region OBA.

$$I_1 = \int_0^{\alpha}\int_0^{b\sin\theta}\frac{r^4}{r^4\sin^2\theta\cos^2\theta}\,r\,dr\,d\theta = \int_0^{\alpha}\int_0^{b\sin\theta}\frac{1}{\sin^2\theta\cos^2\theta}\,r\,dr\,d\theta$$

$$= \int_0^{\alpha}\frac{1}{\sin^2\theta\cos^2\theta}\left[\frac{r^2}{2}\right]_0^{b\sin\theta} d\theta = \frac{1}{2}b^2\int_0^{\alpha}\sec^2\theta\,d\theta$$

$$= \frac{b^2}{2}\left[\tan\theta\right]_0^{\alpha} = \frac{1}{2}b^2\tan\alpha = \frac{1}{2}b^2\frac{a}{b} = \frac{ab}{2}.$$

And Integral over the region OCA.

$$I_2 = \int_{\alpha}^{\pi/2}\int_0^{a\cos\theta}\frac{1}{\sin^2\theta\cos^2\theta}\,r\,dr\,d\theta = \int_{\alpha}^{\pi/2}\frac{1}{\sin^2\theta\cos^2\theta}\left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{a^2}{2}\int_{\alpha}^{\pi/2}\operatorname{cosec}^2\theta\,d\theta = \frac{a^2}{2}\left[-\cot\theta\right]_{\alpha}^{\pi/2}$$

$$= -\frac{a^2}{2}\left[0-\cot\alpha\right] = \frac{1}{2}a^2\frac{b}{a} = \frac{1}{2}ab.$$

$\therefore$ Required integral $= ab$.

Example 3 (b) : Evaluate $\iint \frac{x^2y^2}{x^2+y^2}$ over the annular region between circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$; $a > b\ (>0)$. **(M.U. 2002)**

Sol. : 1. Region in Cartesian Coordinates : It is easy to see that the region of integration is the region between the two circles.

2. Region in Polar Coordinates : Putting $x = r\cos\theta$ and $y = r\sin\theta$, the two circles become $r = a$ and $r = b\ (a > b)$.

3. Limits of r, θ : Considering a radial strip in the region of integration, r varies from $r = b$ to $r = a$ and θ varies from $\theta = 0$ to $\theta = \pi/2$ in the first quadrant.

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in the integrand

$\frac{x^2y^2}{x^2+y^2}$, we get $\frac{r^4\sin^2\theta\cos^2\theta}{r^2}$

and we replace $dx\,dy$ by $r\,dr\,d\theta$.

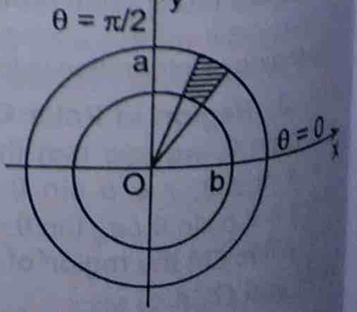

Fig. 9.71

$$\therefore\ I = 4\int_0^{\pi/2}\int_b^a \frac{r^4\sin^2\theta\cos^2\theta}{r^2}\,r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\left[\frac{r^4}{4}\right]_b^a d\theta$$

$$\therefore \quad I = (a^4 - b^4) \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

$$= (a^4 - b^4) \cdot \frac{1 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = (a^4 - b^4) \cdot \frac{\pi}{16} \qquad \text{[By (25), page 6-34]}$$

Example 4 (b) : Evaluate $\iint_R (3x + 4y^2)\, dx\, dy$ where R is the region in the upper half of the area bounded by the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 4$.

Sol. : 1. Region in Cartesian Coordinates : It is easy to see that the region of integration is the region between the two semi-circles.

2. Region in Polar Coordinates : Putting $x = r\cos\theta$ and $y = r\sin\theta$, the two circles becomes $r = 1$ and $r = 2$.

3. Limits of r, θ : Considering a radial strip in the region of integration, r varies from $r = 1$ to $r = 2$ and θ varies from $\theta = 0$ to $\theta = \pi$.

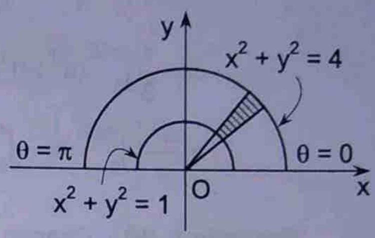

Fig. 9.72

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in the integrand $3x + 4y^2$, we get $3r\cos\theta + 4r^2\sin^2\theta$ and we replace $dx\,dy$ by $r\,dr\,d\theta$.

$$\therefore \quad I = \int_0^{\pi}\int_1^2 (3\,r\cos\theta + 4\,r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi}\int_1^2 (3\,r^2\cos\theta + 4\,r^3\sin^2\theta)\, dr\, d\theta = \int_0^{\pi}\left[r^3\cos\theta + r^4\sin^2\theta\right]_1^2 d\theta$$

$$= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\, d\theta = \int_0^{\pi}\left[7\cos\theta + \frac{15}{2}(1 - \cos 2\theta)\right] d\theta$$

$$= \left[7\sin\theta + \frac{15}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{\pi} = \frac{15}{2}\pi .$$

Example 5 (b) : Evaluate $\iint_R \sqrt{a^2 - x^2 - y^2}\, .\, dx\, dy$ where R is the area of the upper half of the circle $x^2 + y^2 = ax$. **(M.U. 1988, 95, 2008)**

Sol. : 1. Region in Cartesian Coordinates : The circle

$$x^2 + y^2 - ax = 0 \text{ i.e., } \left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2 \text{ is a circle with centre } \left(\frac{a}{2}, 0\right) \text{ and radius } \frac{a}{2}.$$

The region of integration is the upper-half of this circle.

2. Region in Polar Coordinates : Putting $x = r\cos\theta$ and $y = r\sin\theta$, the above circle becomes $r^2 = ar\cos\theta$ *i.e.*, $r = a\cos\theta$. The x-axis is $\theta = 0$ and the y-axis is $\theta = \pi/2$.

3. Limits of r, θ : Considering a radial strip in the region of integration, we see that r varies from $r = 0$ to $r = a\cos\theta$ and θ varies from $\theta = 0$ to $\theta = \pi/2$.

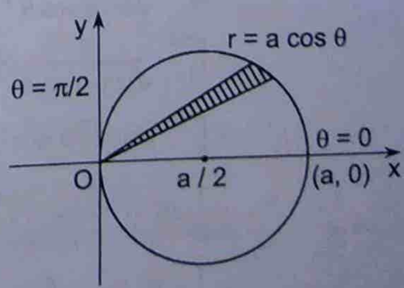

Fig. 9.73

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in the integrand $\sqrt{a^2 - x^2 - y^2}$, we get $\sqrt{a^2 - r^2}$ and we replace $dx\,dy$ by $r\,dr\,d\theta$.

$$I = \int_0^{\pi/2} \int_0^{a\cos\theta} \sqrt{a^2 - r^2} \cdot r\,dr\,d\theta$$

Now, put $a^2 - r^2 = t \quad \therefore \; -2r\,dr = dt$

When $r = a\cos\theta$, $t = a^2 \sin^2\theta$. When $r = 0$, $t = a^2$.

$$\therefore \quad I = \int_0^{\pi/2} \int_{a^2}^{a^2\sin^2\theta} t^{1/2}\left(-\frac{1}{2}\right) dt\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \left[\frac{t^{3/2}}{3/2}\right]_{a^2}^{a^2\sin^2\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2} (a^3\sin^3\theta - a^3)\,d\theta = \frac{a^3}{3}\int_0^{\pi/2}(1 - \sin^3\theta)\,d\theta$$

$$= \frac{a^3}{3}\left[\int_0^{\pi/2} d\theta - \int_0^{\pi/2}\sin^3\theta\,d\theta\right] = \frac{a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3}\cdot 1\right] = \frac{a^3}{18}(3\pi - 4)$$

Example 6 (b) : Change to polar coordinates and evaluate $\iint_R \frac{1}{\sqrt{xy}}\,dx\,dy$ where R is the region of integration bounded by $x^2 + y^2 - x = 0$ and $y \geq 0$. **(M.U. 2001, 06, 07)**

Sol. : 1. Region in Cartesian Coordinates : $x^2 + y^2 - x = 0$ *i.e.*, $\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$ is circle with center at $(a/2, 0)$, radius $1/2$. $y \geq 0$ is the upper half of the x-y plane. Thus, the region is upper half of the above circle.

2. Region in Polar Coordinates : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $x^2 + y^2 - x = 0$, we get $r^2 = r\cos\theta$ *i.e.*, $r = \cos\theta$. The x-axis is $\theta = 0$, the y-axis is $\theta = \pi/2$.

3. Limits of r, θ : Considering a radial strip in the region of integration, we see that r varies from $r = 0$ to $r = \cos\theta$ and θ varies from $\theta = 0$ to $\theta = \pi/2$.

4. Integrand : Putting $x = r\cos\theta$ and $y = r\sin\theta$ in the integrand $\frac{1}{\sqrt{xy}}$, we get $\frac{1}{r\sqrt{\sin\theta\cos\theta}}$ and we replace $dx\,dy$ by $r\,dr\,d\theta$.

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{\cos\theta} \frac{1}{r\sqrt{\sin\theta\cos\theta}}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\cos\theta} \frac{dr\,d\theta}{\sqrt{\sin\theta\cos\theta}}$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta\cos\theta}}\left[r\right]_0^{\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta\cos\theta}} \cdot \cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta\,\cos^{1/2}\theta\,d\theta$$

$$= \frac{1}{2}B\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{2}\cdot\frac{\overline{|1/4}\;\overline{|3/4}}{\overline{|1}}$$

$$= \frac{1}{2}\cdot\sqrt{2}\cdot\pi = \frac{\pi}{\sqrt{2}}.$$

[By (18), page 6-28 and (13), page 6-19]

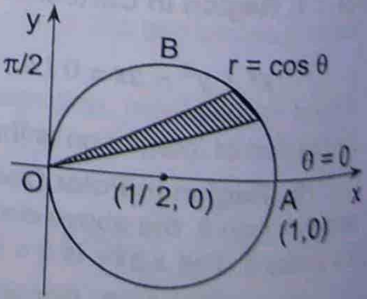

Fig. 9.74

Example 7 (b) : Evaluate $\iint \frac{dx\,dy}{(1+x^2+y^2)^2}$

over one loop of the lemniscate $(x^2+y^2)^2 = x^2 - y^2$. **(M.U. 1985, 2003, 07, 13)**

Sol. : Changing to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$ the equation of the lemniscate becomes $r^4 = r^2(\cos^2\theta - \sin^2\theta)$ *i.e.* $r^2 = \cos 2\theta$. The loop is shown in Ex. 3, Fig. 9.67, page 9-53 ($a = 1$). Then as in the same example.

$$I = \int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{(1+r^2)^2} = \int_{-\pi/4}^{\pi/4} -\frac{1}{2}\left[\frac{1}{(1+r^2)}\right]_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= -\frac{1}{2}\cdot 2\int_0^{\pi/4}\left[\frac{1}{1+\cos 2\theta} - 1\right] d\theta = -\int_0^{\pi/4}\left(\frac{1}{2}\sec^2\theta - 1\right) d\theta$$

$$= -\left[\frac{1}{2}\tan\theta - \theta\right]_0^{\pi/4} = -\left[\frac{1}{2} - \frac{\pi}{4}\right] = \frac{\pi}{4} - \frac{1}{2}.$$

Example 8 (b) : Evaluate $\iint \frac{dx\,dy}{\sqrt{4+x^2+y^2}}$

over one loop of lemniscate $(x^2+y^2) = 4(x^2-y^2)$. **(M.U. 1992)**

Sol. : In Ex. 3, Fig. 9-67, page 9-53, put $a = 2$ $\quad\therefore\; I = 4 - \pi$.

Example 9 (b) : Change to polar coordinates and evaluate $\iint_R \frac{dx\,dy}{(1+x^2+y^2)^2}$ over one loop of the lemniscate $(x^2+y^2)^2 = x^2 - y^2$. **(M.U. 1995, 2007, 13, 14)**

Sol. : If we put $x = r\cos\theta$, $y = r\sin\theta$, $(x^2+y^2)^2 = x^2 - y^2$ becomes

$$r^4 = r^2(\cos^2\theta - \sin^2\theta) \quad i.e. \quad r^2 = \cos 2\theta.$$

$$\frac{1}{(1+x^2+y^2)^2} = \frac{1}{(1+r^2)^2}$$

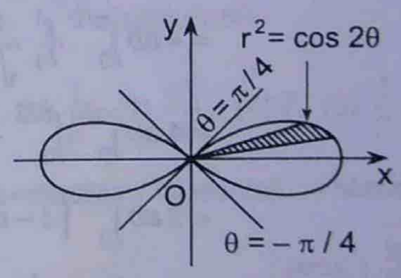

Fig. 9.75

Now, as in Ex. 7 above, on the loop r varies from 0 to $\sqrt{\cos 2\theta}$ and θ varies from $-\pi/4$ to $\pi/4$.

$$\therefore\; I = \int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{(1+r^2)^2}$$

$$= 2\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{(1+r^2)^2} = 2\int_0^{\pi/4}\frac{1}{2}\left[-\frac{1}{1+r^2}\right]_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= -\int_0^{\pi/4}\left[\frac{1}{1+\cos 2\theta} - 1\right] d\theta = \int_0^{\pi/4}\left[1 - \frac{1}{1+\cos 2\theta}\right] d\theta$$

$$= \int_0^{\pi/4}\left[1 - \frac{\sec^2\theta}{2}\right] d\theta = \left[\theta - \frac{\tan\theta}{2}\right]_0^{\pi/4} = \frac{\pi}{4} - \frac{1}{2} = \frac{\pi-2}{4}.$$

Example 10 (b) : Evaluate $\iint_R x^3 y\,dx\,dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. : We use elliptical polar coordinates *i.e.*, we put $x = ar\cos\theta$, $y = br\sin\theta$ and $dx\,dy = a\,b\,r\,dr\,d\theta$.

Now, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ becomes $\dfrac{a^2r^2\cos^2\theta}{a^2}+\dfrac{b^2r^2\sin^2\theta}{b^2}=1.$

$\therefore\ r^2(\cos^2\theta+\sin^2\theta)=1 \qquad \therefore\ r=1$ [See Fig. 9.13, page 9-16]

Thus, the ellipse becomes the unit circle in polar coordinate system. Considering a radial strip, we get

$$I=\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}a^3r^3\cos^3\theta\cdot br\sin\theta\,d\theta\cdot abr\,dr\,d\theta$$

$$=a^4b^2\int_{\theta=0}^{\pi/2}\cos^3\theta\sin\theta\,d\theta\int_{r=0}^{1}r^5\,dr$$

$$=a^4b^2\left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/2}\left[\frac{r^6}{6}\right]_0^1=\frac{a^4b^2}{24}.$$

Example 11 (b) : Evaluate $\iint\sqrt{\left[\dfrac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2+b^2x^2+a^2y^2}\right]}\,dx\,dy$ where R is the region bounded by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$ **(M.U. 1997, 2002, 09)**

Sol. : We use elliptical polar coordinates *i.e.* we put $x=ar\cos\theta$, $y=br\sin\theta$, $dx\,dy=a\,br\,dr\,d\theta$. The ellipse $(x^2/a^2)+(y^2/b^2)=1$ is transformed to circle $r^2=1$ *i.e.* $r=1$.

Considering a radial strip in the first quadrant of the unit circle, we get

$$I=\iint\sqrt{\left[\frac{a^2b^2(1-r^2)}{a^2b^2(1+r^2)}\right]}\,abr\,dr\,d\theta=4\int_0^{\pi/2}\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,abr\,dr\,d\theta$$

$$=4ab\int_0^{\pi/2}\int_0^1\frac{1-r^2}{\sqrt{1-r^4}}\,r\,dr\,d\theta. \qquad \text{Put } r^2=\sin t.$$

$$=4ab\int_0^{\pi/2}\int_0^{\pi/2}\frac{1-\sin t}{\cos t}\cdot\frac{1}{2}\cos t\,dt\,d\theta=2ab\int_0^{\pi/2}\int_0^{\pi/2}(1-\sin t)\,dt\,d\theta$$

$$=2ab\int_0^{\pi/2}\left[t-\cos t\right]_0^{\pi/2}d\theta=2ab\int_0^{\pi/2}\left[\frac{\pi}{2}-1\right]d\theta=2ab\left(\frac{\pi}{2}-1\right)\int_0^{\pi/2}d\theta$$

$$=2ab\left(\frac{\pi}{2}-1\right)\left[\theta\right]_0^{\pi/2}=\pi ab\left(\frac{\pi}{2}-1\right).$$

Example 12 (b) : Evaluate $\iint\sqrt{\left\{\dfrac{1-x^2-y^2}{1+x^2+y^2}\right\}}$ over the area of the positive quadrant of the circle $x^2+y^2=1$ by changing to polar co-ordinates. **(M.U. 1997, 98)**

Sol. : Putting $a=1$, $b=1$ in the above example, we get $\quad I=\pi\left(\dfrac{\pi}{2}-1\right).$

(Standard Circle)

2. $\iint \sin(x^2 + y^2)\, dx\, dy$ over the circle $x^2 + y^2 = a^2$. [**Ans. :** $\pi(1 - \cos a^2)$]

3. $\iint \frac{dx\, dy}{(1 + x^2 + y^2)^{3/2}}$ over the region bounded by $y = 0$, $x = y$, $x = 1$. [**Ans. :** $\frac{\pi}{12}$]

(See Fig. 9.16, page 9-19, $a = 1$) .

4. $\iint \frac{(x^2 + y^2)^2}{x^2 y^2}$ over the area bounded by (i) $x^2 + y^2 = ax$, $ax = by$, (ii) $x^2 + y^2 = by$, $ax = by$.

(See Fig. 9.70, page 9-55) [**Ans. : (i)** $ab/2$, **(ii)** $ab/2$]

5. $\iint xy(x^2 + y^2)^{3/2}\, dx\, dy$ over the first quadrant of the circle $x^2 + y^2 = a^2$.

(M.U. 1995, 2003) [**Ans. :** $a^2/4$]

6. $\iint (x^2 + y^2)\, x\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. [**Ans. :** $a^5/5$]

7. $\iint \frac{dx\, dy}{(a^2 + x^2 + y^2)^{3/2}}$ over the entire xy plane. [**Ans. :** $2\pi/a$]

8. $\iint e^{-(x^2 + y^2)}\, dx\, dy$ over the positive quadrant of the xy plane. [**Ans. :** $\pi/4$]

EXERCISE - IX

(A) Find the following integrals : Class (a) : 3 Marks

1. $\int_0^1 \int_0^1 x^2 y^2\, dx\, dy$

2. $\int_0^{\pi/2} \int_0^1 r^3 \sin\theta \cos\theta\, d\theta$

3. $\int_0^\infty \int_0^\infty \frac{dx\, dy}{(1 + x^2)(1 + y^2)}$

4. $\int_1^2 \int_0^{3y} y\, dx\, dy$

5. $\int_0^1 \int_0^{\pi/2} r \sin\theta\, dr\, d\theta$

[**Ans. : (1)** $\frac{1}{9}$, **(2)** $\frac{1}{8}$, **(3)** $\frac{\pi^2}{4}$, **(4)** 7, **(5)** $\frac{1}{2}$.]

(B) Evaluate the following integrals by changing to polar coordinates : Class (a) : 4 Marks

1. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy$ [See Fig. 9.76] [**Ans. :** $\frac{\pi}{8}$]

2. $\int_0^a \int_y^a x\, dx\, dy$ [See Fig. 9.77] [**Ans. :** $\frac{a^3}{3}$]

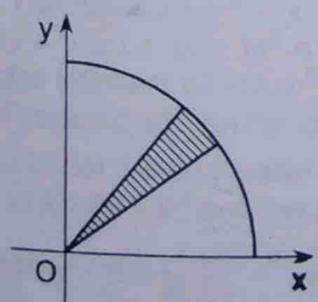

Fig. 9.76

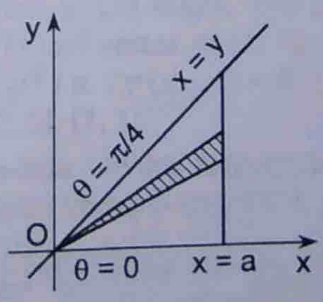

Fig. 9.77

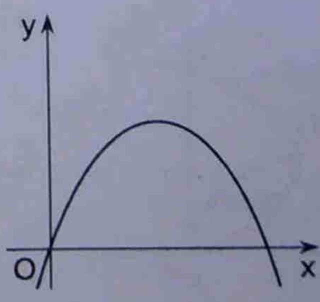

Fig. 9.78

Summary

1. **Double Integral In Cartesian Coordinates**

$$\iint\limits_A f(x, y)\, d\, A = \int_a^b \left\{ \int_{f_1(x)}^{f_2(x)} f(x,\ y)\, d\, y \right\} dx$$

2. **Double Integral In Polar Coordinates**

$$\iint\limits_A f(r, \theta)\, dA = \int_\alpha^\beta \left\{ \int_{f_1(\theta)}^{f_2(\theta)} f(r,\ \theta)\, d\, r \right\} d\,\theta$$

3. **Change of Order of Integration**

$$\int_{x=a}^{b} \int_{y=f_1(x)}^{y=f_2(x)} f(x, y)\, dy\, dx = \int_{y=c}^{d} \int_{x=\Phi_1(x)}^{x=\Phi_2(x)} f(x, y)\, dx\, dy$$

❋ ❋ ❋

CHAPTER 10

Area

1. Introduction

Last year you have learnt how to evaluate the area bounded by a curve, the x-axis and the ordinates at $x = a$ and $x = b$. You have studied double integrals in the previous chapter. We shall now learn how to find the area by double integrals.

2. Area by Double Integration - Cartesian Coordinates

Consider the area enclosed by two plane curves $y = f_1(x)$ and $y = f_2(x)$, intersecting in $A\,(a, c)$ and $B\,(b, d)$.

Consider a strip parallel to the y-axis. On this strip y varies from $y = f_1(x)$ to $y = f_2(x)$. When this strip moves parallel to itself, x varies from $x = a$ to $x = b$. In this way the whole area is obtained.

$$\therefore \quad \boxed{A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\, dx}$$

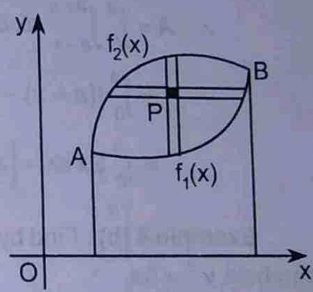

Fig. 10.1

If we consider a strip parallel to the x-axis, then on this strip x varies from $x = \Phi_1(y)$ to $x = \Phi_2(y)$. When the strip moves parallel to itself, y varies from $y = c$ to $y = d$. In this way again the whole area is obtained.

$$\therefore \quad \boxed{A = \int_c^d \int_{\Phi_1(y)}^{\Phi_2(y)} dx\, dy}$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Using double integration find the area bounded by the parabolas $x = y^2$, $x = 2y - y^2$.

Sol. : The parabola $y^2 = x$ has vertex at the origin. The parabola $y^2 - 2y = -x$ *i.e.* $(y-1)^2 = -(x-1)$ has vertex at $(1, 1)$. The two parabolas intersect where $y^2 = 2y - y^2$ *i.e.* $2y\,(y-1) = 0 \;\therefore\; y = 0$, $y = 1$. The points of intersection are $(0, 0)$, $(1, 1)$.

Consider a strip parallel to the x-axis. On this strip x varies from $x = y^2$ to $x = 2y - y^2$ and the strip moves from $y = 0$ to $y = 1$.

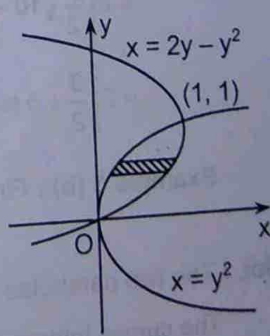

Fig. 10.2

Hence, $A = \int_0^1 \int_{y^2}^{2y-y^2} dx\, dy = \int_0^1 [x]_{y^2}^{2y-y^2} dy$

$$= \int_0^1 (2y - y^2 - y^2)\, dy = 2\int_0^1 (y - y^2)\, dy$$

$$= 2\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

Example 2 (b) : Find by double integration the area enclosed by $y^2 = x^3$ and $y = x$. **(M.U. 2012)**

Sol. : The two curves intersect at the origin $O(0, 0)$ and $A(1, 1)$.

Consider a strip parallel to the y-axis. On this strip y varies from $y = x^{3/2}$ to $y = x$. And the strip moves from $x = 0$ to $x = 1$.

$$\therefore \quad A = \int_0^1 \int_{x^{3/2}}^{x} dy\, dx = \int_0^1 [\, y \,]_{x^{3/2}}^{x}\, dx$$

$$= \int_0^1 [x - x^{3/2}]\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^{5/2}}{5/2}\right]_0^1 = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}.$$

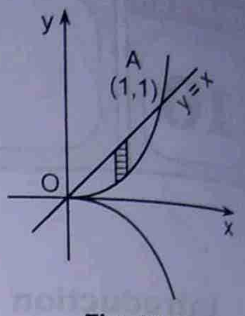

Fig. 10.3

Example 3 (b) : Find by double integration the area bounded by the lines $y = a + x$, $y = a - x$ and $x = a$, $(a > 0)$.

Sol. : Consider a strip parallel to the y-axis. On this strip y varies from $y = a - x$ to $y = a + x$. This strip sweeps the area when it moves parallel to itself between $x = 0$ to $x = a$.

$$\therefore \quad A = \int_0^a \int_{a-x}^{a+x} dx\, dy = \int_0^a [\, y \,]_{a-x}^{a+x}\, dx$$

$$= \int_0^a [(a + x) - (a - x)]\, dx$$

$$= \int_0^a 2x\, dx = \left[x^2\right]_0^a = a^2.$$

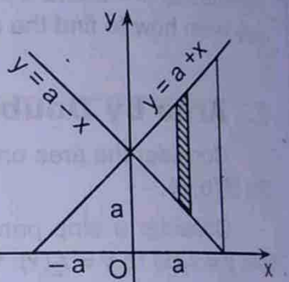

Fig. 10.4

Example 4 (b) : Find by double integration the area common to the circle $x^2 + y^2 = 10$ and the parabola $y^2 = 9x$.

Sol. : Consider a strip parallel to the x-axis. On this strip x varies from $x = y^2/9$ to $x = \sqrt{10 - y^2}$. This strip sweeps the area when it moves parallel to itself from $y = -3$ to $y = 3$.

$$\therefore \quad A = \int_{-3}^{3} \int_{y^2/9}^{\sqrt{10-y^2}} dx\, dy = 2\int_0^3 \int_{y^2/9}^{\sqrt{10-y^2}} dx\, dy$$

$$= 2\int_0^3 [\, x \,]_{y^2/9}^{\sqrt{10-y^2}}\, dy = 2\int_0^3 \left[\sqrt{10 - y^2} - \frac{y^2}{9}\right] dy$$

$$= 2\left[\frac{y}{2}\sqrt{10 - y^2} + \frac{10}{2}\sin^{-1}\frac{y}{\sqrt{10}} - \frac{y^3}{27}\right]_0^3$$

$$= 2\left[\frac{3}{2} + 5\sin^{-1}\frac{3}{\sqrt{10}} - 1\right] = 2\left[\frac{1}{2} + 5\sin^{-1}\frac{3}{\sqrt{10}}\right].$$

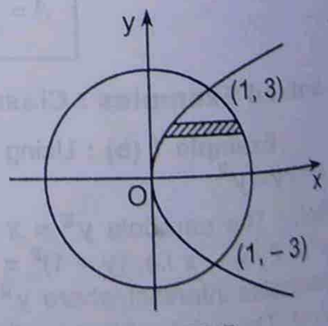

Fig. 10.5

Example 5 (b) : Find by double integration the area included between the curves

$$y^2 = 4a\,(x + a) \quad \text{and} \quad y^2 = 4b\,(b - x).$$

Sol. : The two parabolas are shown in the figure.

The curves intersect at $(b - a, \pm 2\sqrt{ab})$.

Consider a strip parallel to the x-axis.

On this strip x varies from $x = \frac{y^2}{4a} - a$ to $x = b - \frac{y^2}{4b}$.

This strip sweeps the area when it moves parallel to itself from $y = -2\sqrt{ab}$ to $y = 2\sqrt{ab}$.

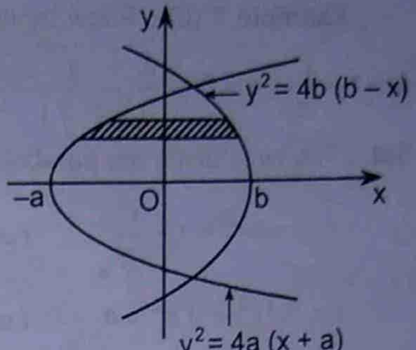

Fig. 10.6

$$\therefore \quad A = \int_{-2\sqrt{ab}}^{+2\sqrt{ab}} \int_{(y^2/4a)-a}^{b-(y^2/4b)} dx\, dy$$

$$= 2\int_0^{+2\sqrt{ab}} \left[x \right]_{(y^2/4a)-a}^{b-(y^2/4b)} dy$$

$$= 2\int_0^{2\sqrt{ab}} \left(b + a - \frac{y^2}{4b} - \frac{y^2}{4a} \right) dy$$

$$= 2\int_0^{2\sqrt{ab}} \left[(b+a) - (b+a)\frac{y^2}{4\,ab} \right] dy$$

$$= 2(b+a)\int_0^{2\sqrt{ab}} \left(1 - \frac{y^2}{4ab} \right) dy = 2(b+a)\left[y - \frac{y^3}{12ab} \right]_0^{2\sqrt{ab}}$$

$$= 2(b+a)\left[2\sqrt{ab} - \frac{4ab \cdot 2\sqrt{ab}}{12\,ab} \right]$$

$$= 2(b+a) \cdot 2\sqrt{ab} \cdot \left(1 - \frac{1}{3} \right) = \frac{8}{3}(a+b)\sqrt{ab}.$$

Example 6 (b) : Find the area bounded between the parabolas $x^2 = 4ay$ and $x^2 = -4a(y - 2a)$. **(M.U. 2015)**

Sol. : The two parabola are as shown in the figure. They intersect where $4ay = -4a(y - 2a)$ *i.e.*, $8ay = 8a^2$ *i.e.*, $y = a$. When $y = a$, $x^2 = 4ay$ gives $x^2 = 4a^2$ *i.e.*, $x = \pm 2a$.

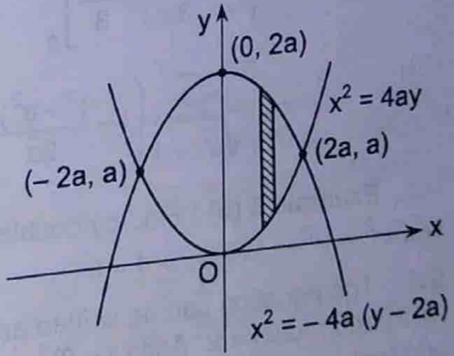

Fig. 10.6 (a)

Now, consider a strip parallel to the y-axis. On this strip y varies from $y = \frac{x^2}{4a}$ to $y = \frac{8a^2 - x^2}{4a}$. Then x varies from $x = -2a$ to $x = 2a$.

$\therefore$ Required area

$$= \int_{-2a}^{2a} \int_{x^2/4a}^{(8a^2-x^2)/4a} dy\, dx = \int_{-2a}^{2a} \left[y \right]_{x^2/4a}^{(8a^2-x^2)/4a} dx$$

$$= \int_{-2a}^{2a} \left[\frac{8a^2 - x^2}{4a} - \frac{x^2}{4a} \right] dx = \frac{2}{4a}\int_0^{2a} (8a^2 - 2x^2)\, dx$$

$$= \frac{1}{2a}\left[8a^2 x - \frac{2x^3}{3} \right]_0^{2a} dx = \frac{1}{2a}\left[16\,a^3 - \frac{16\,a^3}{3} \right]$$

$$= \frac{16\,a^3}{2a}\left[1 - \frac{1}{3} \right] = \frac{16\,a^2}{2} \cdot \frac{2}{3} = \frac{16\,a^2}{3}.$$

Example 7 (b) : Show by double integration that the area between the curves $y = ax^2$ and $y = 1 - \frac{x^2}{a}$, $a > 0$ is $\frac{4}{3}\sqrt{\frac{a}{a^2+1}}$. **(M.U. 1995, 2003)**

Sol. : The two curves are parabolas as shown in the figure. They intersect where

$$ax^2 = 1 - \frac{x^2}{a} \quad \text{i.e.} \quad a^2x^2 = a - x^2$$

$$\text{i.e.} \quad (a^2+1)\,x^2 = a \quad \text{i.e.} \quad x = \pm\sqrt{\frac{a}{a^2+1}}$$

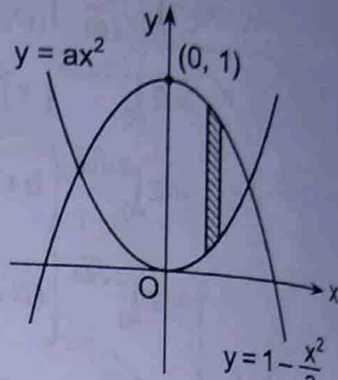

Fig. 10.7

Consider a strip parallel to the y-axis. On this strip y varies from $y = ax^2$ to $y = -\frac{x^2}{a}$. Then x varies from $x = -\sqrt{\frac{a}{a^2+1}}$ to $x = \sqrt{\frac{a}{a^2+1}}$.

area = ∫∫ dy dx or ∫∫ dx dy

$\therefore$ Required area

$$= 2\int_0^{\sqrt{a/(a^2+1)}} \int_{ax^2}^{1-(x^2/a)} dy\,dx$$

$$= 2\int_0^{\sqrt{a/(a^2+1)}} \left[\,y\,\right]_{ax^2}^{1-(x^2/a)} dx = 2\int_0^{\sqrt{a/(a^2+1)}} \left[1 - \frac{x^2}{a} - ax^2\right] dx$$

$$= 2\left[x - \frac{x^3}{3a} - \frac{ax^3}{3}\right]_0^{\sqrt{a/(a^2+1)}} = 2\left[x - \frac{(1+a^2)}{3a}\cdot x^3\right]_0^{\sqrt{a/(a^2+1)}}$$

$$= 2\left[\frac{\sqrt{a}}{\sqrt{a^2+1}}\left(1 - \frac{(1+a^2)}{3a}\cdot\frac{a}{(a^2+1)}\right)\right] = \frac{2\sqrt{a}}{\sqrt{a^2+1}}\left(1 - \frac{1}{3}\right) = \frac{4}{3}\sqrt{\frac{a}{a^2+1}}.$$

Example 8 (b) : Find by double integration the area common to the circles $x^2 + y^2 - 4y = 0$ and $x^2 + y^2 - 4x - 4y + 4 = 0$.

Sol. : The equation can be written as $x^2 + (y-2)^2 = 2^2$. Its centre is (0, 2) and radius = 2. And $(x-2)^2 + (y-2)^2 = 2^2$. Its centre is (2, 2) and radius = 2.

By subtraction, we see that the circles intersect at points where $x = 1$.

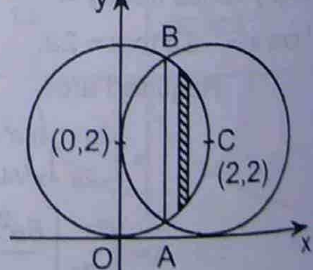

Fig. 10.8

Consider a strip parallel to the y-axis. Then on the circle on the left *i.e.* on $x^2 + y^2 - 4y = 0$ *i.e.* on $y = \frac{4 \pm \sqrt{16 - 4x^2}}{2}$, y varies from $y = 2 - \sqrt{4 - x^2}$ to $y = 2 + \sqrt{4 - x^2}$. Then x varies from $x = 0$ to $x = 2$.

$\therefore$ Required area = 2 area ABC by symmetry.

(For the part ABC, x varies from $x = 1$ to $x = 2$)

$$\therefore \text{ Required area } = 2\int_1^2 \int_{2-\sqrt{4-x^2}}^{2+\sqrt{4-x^2}} dy\,dx = 2\int_1^2 \left[\,y\,\right]_{2-\sqrt{4-x^2}}^{2+\sqrt{4-x^2}} dx$$

$$= 2\int_1^2 2\sqrt{4-x^2}\,dx = 4\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= 4\left[2\cdot\frac{\pi}{2}-\left(\frac{\sqrt{3}}{2}+2\cdot\frac{\pi}{6}\right)\right] = 4\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right).$$

Example 9 (b) : Find by double integration the area between the curve $y^2x = 4a^2(2a-x)$ and its asymptote. **(M.U. 1999)**

Sol. : The curve known as Witch of Agnesi is shown in the figure along with its asymptote which is the y-axis.

Now, consider a strip parallel to the y-axis.

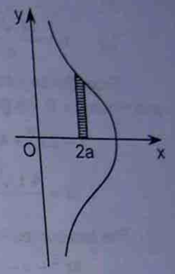

Fig. 10.9

On this strip y varies from $y = 0$ to $y = 2a\sqrt{(2a-x)/x}$. Then x varies from $x = 0$ to $x = 2a$.

$$\therefore\ A = 2\int_0^{2a}\int_0^{2a\sqrt{(2a-x)/x}} dy\,dx = 2\int_0^{2a}[\,y\,]_0^{2a\sqrt{(2a-x)/x}}\,dx$$

$$= 2\int_0^{2a} 2a\sqrt{\frac{2a-x}{x}}\cdot dx = 4a\int_0^{2a}\sqrt{\frac{2a-x}{x}}\,dx$$

Now, put $x = 2a\sin^2\theta \quad \therefore\ dx = 4a\sin\theta\cos\theta\,d\theta$

When $x = 0$, $\theta = 0$; when $x = 2a$, $\theta = \pi/2$.

$$\therefore\ I = 4a\int_0^{\pi/2}\sqrt{\frac{2a\cos^2\theta}{2a\sin^2\theta}}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$\therefore\ I = 16a^2\int_0^{\pi/2}\cos^2\theta\,d\theta = 16a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 4\pi a^2.$$

Example 10 (b) : Find by double integration the area between the parabola $y = x^2-6x+3$ and the line $y = 2x-9$. **(M.U. 1995, 2003, 05)**

Sol. : We have $y = (x-3)^2-6$ *i.e.* $y+6 = (x-3)^2$. It is a parabola with vertex at $(3, -6)$ and opening upwards.

The line intersects the parabola where $x^2-6x+3 = 2x-9$ *i.e.* $x^2-8x+12 = 0$ *i.e.* $(x-6)(x-2)$ *i.e.* when $x = 6$, $x = 2$. Then $y = 12-9 = 3$ and $y = 4-9 = -5$. The points of intersection are $(6, 3)$, $(2, -5)$.

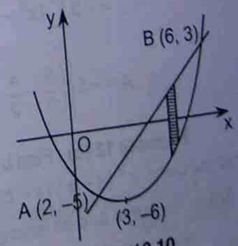

Fig. 10.10

To find the area consider a strip parallel to the y-axis. On this strip y varies from $y = x^2-6x+3$ to $y = 2x-9$. Then x varies from $x = 2$ to $x = 6$.

$$\therefore\ A = \int_{x=2}^{6}\int_{y=x^2-6x+3}^{2x-9} dy\,dx$$

$$= \int_2^6 [\,y\,]_{x^2-6x+3}^{2x-9}\,dx = \int_2^6 (2x-9-x^2+6x-3)\,dx$$

$$= \int_2^6 (8x-12-x^2)\,dx = \left[4x^2-12x-\frac{x^3}{3}\right]_2^6$$

$$= (144-72-72)-\left(16-24-\frac{8}{3}\right) = 8+\frac{8}{3} = \frac{32}{3}.$$

Example 11 (b) : Find by double integration the area included between the curves $y = 3x^2 - x - 3$ and $y = -2x^2 + 4x + 7$. **(M.U. 1997)**

Sol. : By completing the square on x, we have

$$y + \frac{37}{12} = 3\left(x^2 - \frac{1}{3}x + \frac{1}{36}\right) = \left(x - \frac{1}{6}\right)^2$$

a parabola with vertex at (1 / 6, – 37 / 12) and opening upwards and $y - 9 = -2(x-1)^2$ also a parabola with vertex at (1, 9) and opening downwards.

[From the first equation, when $x = 0$, we get $y = -3$ and when $y = 0$, we get $3x^2 - x - 3 = 0$

i.e. $x = \dfrac{1 + \sqrt{37}}{6}$.

From the second equation, when $x = 0$, we get $y = 7$ and when $y = 0$, we get

$$-2x^2 + 4x + 7 = 0 \quad i.e., \quad 2x^2 - 4x - 7 = 0.$$

$$\therefore \quad x = \frac{4 \pm \sqrt{16 + 56}}{4} = \frac{4 \pm 2\sqrt{18}}{4} = \frac{2 \pm \sqrt{18}}{2} \text{]}$$

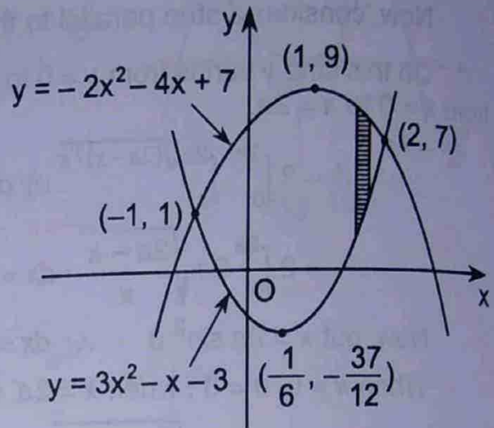

Fig. 10.11

The two curves intersect when

$$3x^2 - x - 3 = -2x^2 + 4x + 7$$

$$\therefore \quad 5x^2 - 5x - 10 = 0 \qquad \therefore \quad x^2 - x - 2 = 0$$

$$\therefore \quad (x-2)(x+1) = 0 \qquad \therefore \quad x = 2 \text{ or } x = -1.$$

When $x = -1$, $y = +1$ and when $x = 2$, $y = 7$. Thus, the two curves intersect in (–1, 1) and (2, 7).

Now, consider a strip parallel to the y-axis. On this trip y varies from $y = 3x^2 - x - 3$ to $y = -2x^2 + 4x + 7$. Then x-varies from $x = -1$ to $x = 2$.

$$\therefore \quad A = \int_{-1}^{2} \int_{3x^2 - x - 3}^{-2x^2 + 4x + 7} dy\, dx = \int_{-1}^{2} \left[y \right]_{3x^2 - x - 3}^{-2x^2 + 4x + 7} dx$$

$$= \int_{-1}^{2} \left[-2x^2 + 4x + 7 - 3x^2 + x + 3\right] dx = \int_{-1}^{2} (-5x^2 + 5x + 10)\, dx$$

$$= -5\int_{-1}^{2} (x^2 - x - 2)\, dx = -5\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2}$$

$$\therefore \quad A = -5\left[\left(\frac{8}{3} - \frac{4}{2} - 4\right) - \left(-\frac{1}{3} - \frac{1}{2} + 2\right)\right] = -5\left(3 - \frac{3}{2} - 6\right) = \frac{45}{2}.$$

Example 12 (b) : Find by double integration the total area included between the two branches of the curve $y^2 (a - x)(x - b) = x^2$, $a, b > 0$, $a > b$ and its asymptotes.

Sol. : The curve and its asymptotes are as shown in the figure (see next page).

Now, consider a strip parallel to the y-axis in the region of integration. On this strip y varies from $y = 0$ to $y = \dfrac{x}{\sqrt{(a-x)(x-b)}}$ and then x varies from $x = b$ to $x = a$.

$$\therefore \quad A = 2\int_{b}^{a} \int_{0}^{x/\sqrt{(a-x)(x-b)}} dy\, dx = 2\int_{b}^{a} \left[y \right]_{0}^{x/\sqrt{(a-x)(x-b)}} dx$$

$$\therefore \quad A = 2\int_b^a \frac{x}{\sqrt{(a-x)(x-b)}}\, dx$$

Now, put $a - x = (a-b)\sin^2\theta$.

When $x = a,\ \theta = 0$; when $x = b,\ \theta = \pi/2$.

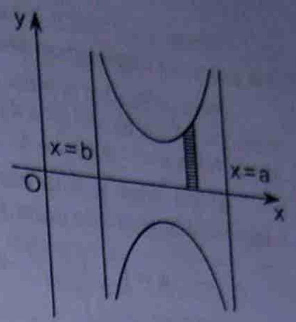

Fig. 10.12

$$\therefore \quad I = 2\int_{\pi/2}^{0} - \frac{[a-(a-b)\sin^2\theta]\cdot 2(a-b)\sin\theta\cos\theta}{\sqrt{(a-b)\sin^2\theta[a-(a-b)\sin^2\theta-b]}}\, d\theta$$

$$= 4(a-b)\int_0^{\pi/2} \frac{(a\cos^2\theta + b\sin^2\theta)\sin\theta\cos\theta}{\sqrt{(a-b)\sin^2\theta\cdot(a-b)\cos^2\theta}}\, d\theta$$

$$= 4(a-b)\int_0^{\pi/2} \frac{(a\cos^2\theta + b\sin^2\theta)\sin\theta\cos\theta}{(a-b)\sin\theta\cos\theta}\, d\theta$$

$$= 4\int_0^{\pi/2} (a\cos^2\theta + b\sin^2\theta)\ d\theta$$

$$= 4a\int_0^{\pi/2} \cos^2\theta\ d\theta + 4b\int_0^{\pi/2} \sin^2\theta\ d\theta$$

$$\therefore \quad I = 4a\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 4b\cdot\frac{1}{2}\cdot\frac{\pi}{2} = (a+b)\pi.$$

Example 13 (b) : Find by double integration the area between the curves $y^2 = 4x$ and $2x - 3y + 4 = 0$. **(M.U. 1991, 98, 2000)**

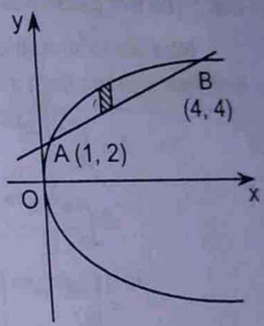

Fig. 10.13

Sol. : We first solve the two equations to find the points of intersection. We get

$$y^2 = 2(3y - 4) \qquad i.e., \qquad y^2 - 6y + 8 = 0$$

$$\therefore \quad (y-4)(y-2) = 0 \qquad \therefore \quad y = 2 \text{ or } 4.$$

When $y = 2$, $x = 1$; when $y = 4$, $x = 4$. Let the points of intersection be $A(1, 2)$ and $B(4, 4)$.

Now, consider a strip parallel to the y-axis. On this strip y varies from $y = \dfrac{2x+4}{3}$ to $y = 2\sqrt{x}$. Then x varies from $x = 1$ to $x = 4$.

$$\therefore \quad A = \int_1^4 \int_{(2x+4)/3}^{2\sqrt{x}} dy\, dx = \int_1^4 [\, y\,]_{(2x+4)/3}^{2\sqrt{x}}\, dx$$

$$= \int_1^4 \left[2\sqrt{x} - \frac{(2x+4)}{3}\right] dx = \left[2\cdot\frac{2}{3}x^{3/2} - \frac{x^2+4x}{3}\right]_1^4$$

$$\therefore \quad A = \left(\frac{32}{3} - \frac{32}{3}\right) - \left(\frac{4}{3} - \frac{5}{3}\right) = \frac{1}{3}.$$

Example 14 (b) : Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$. **(M.U. 2014)**

Sol. : The parabola $y^2 = 4x$ and the line $y = 2x - 4$ intersect where $(2x-4)^2 = 4x$

$$\therefore \quad 4x^2 - 16x + 16 = 4x \qquad \therefore \quad 4x^2 - 20x + 16 = 0$$

$$\therefore \quad x^2 - 5x + 4 = 0 \qquad \therefore \quad (x-4)(x-1) = 0$$

$$\therefore \quad x = 1, 4$$

When $x = 1$, $y = 2 - 4 = -2$; and when $x = 4$, $y = 8 - 4 = 4$. Thus, the points of intersection are $A\,(1, -2)$ and $B\,(4, 4)$.

Now, consider a strip parallel to x-axis. On this strip x varies from $x = y^2/4$ to $x = (y + 4)/2$. The strip then moves parallel to the x-axis from $y = -2$ to $y = 4$.

(If we take a strip parallel to the y-axis, then the region of integration is split into parts.)

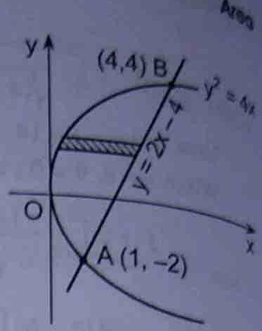

Fig. 10.13 (a)

$$\therefore\ A = \int_{-2}^{4}\int_{y^2/4}^{(y+4)/2} dx\,dy = \int_{-2}^{4}\left[x\right]_{y^2/4}^{(y+4)/2} dy$$

$$= \int_{-2}^{4}\left(\frac{y+4}{2} - \frac{y^2}{4}\right)dy = \frac{1}{4}\int_{-2}^{4}(2y + 8 - y^2)\,dy$$

$$= \frac{1}{4}\left[y^2 + 8y - \frac{y^3}{3}\right]_{-2}^{4} = \frac{1}{4}\left[\left(16 + 32 - \frac{64}{3}\right) - \left(4 - 16 + \frac{8}{3}\right)\right]$$

$$= \frac{1}{4}\left[48 - \frac{64}{3} + 12 - \frac{8}{3}\right] = \frac{1}{4}\left[60 - \frac{72}{3}\right] = \frac{1}{4}(60 - 24) = \frac{1}{4}(36) = 9$$

Example 15 (b) : Find by double integration the area bounded by $y^2 = 4ax$ and $x^2 = 4by$.

Sol. : The two parabolas are shown in the figure. They intersect in $A\,(4a^{1/3}\,b^{2/3},\ 4a^{2/3}\,b^{1/3})$.

Now, consider a strip parallel to the y-axis. On this strip y varies from $y = x^2/4b$ to $y = 2\sqrt{a}\cdot\sqrt{x}$. And then x varies from $x = 0$ to $x = 4a^{1/3}\,b^{2/3}$.

$$\therefore\ A = \int_{0}^{4a^{1/3}b^{2/3}}\int_{x^2/4b}^{2\sqrt{a}\cdot\sqrt{x}} dy\,dx$$

$$= \int_{0}^{4a^{1/3}b^{2/3}}\left[y\right]_{x^2/4b}^{2\sqrt{a}\cdot\sqrt{x}} dx$$

$$= \int_{0}^{4a^{1/3}b^{2/3}}\left[2\sqrt{a}\cdot\sqrt{x} - \frac{x^2}{4b}\right]dx$$

$$= \left[2\sqrt{a}\cdot\frac{x^{3/2}}{3/2} - \frac{1}{4b}\cdot\frac{x^3}{3}\right]_{0}^{4a^{1/3}b^{2/3}}$$

$$= \frac{4}{3}a^{1/2}\cdot 8\,a^{1/2}b - \frac{1}{12b}\cdot 64\,ab^2$$

$$= \frac{32}{3}ab - \frac{16}{3}ab = \frac{16}{3}ab.$$

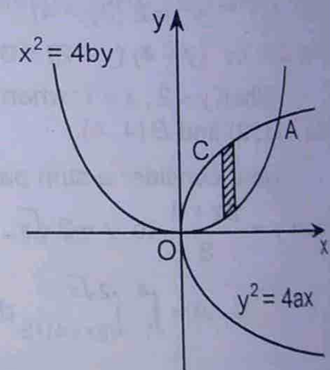

Fig. 10.14

Example 16 (b) : Find by double integration the area enclosed by the curve $9xy = 4$ and the line $2x + y = 2$. **(M.U. 2002, 07, 09, 13, 15)**

Sol. : The curve $9xy = 4$ *i.e.* $xy = 4/9$ is a rectangular hyperbola.

Now, for intersection $\quad 9x\,(2 - 2x) = 4$

$\therefore\ 18x - 18x^2 = 4 \quad \therefore\ 9x^2 - 9x + 2 = 0$

$\therefore\ (3x - 2)\,(3x - 1) = 0 \quad \therefore\ x = 2/3$ or $1/3$.

Hence, the points of intersection of the hyperbola and the line are $\left(\frac{2}{3}, \frac{2}{3}\right)$ and $\left(\frac{1}{3}, \frac{4}{3}\right)$.

Now, consider a strip parallel to the y-axis. On this strip y varies from $y = 4/9x$ to $y = 2 - 2x$. Then x varies from $x = 1/3$ to $x = 2/3$.

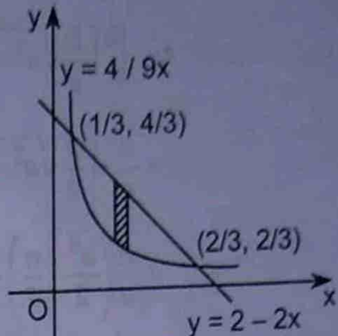

Fig. 10.15

$$\therefore\ A = \int_{1/3}^{2/3} \int_{4/9x}^{2-2x} dy\, dx = \int_{1/3}^{2/3} \left[\, y \,\right]_{4/9x}^{2-2x} dx$$

$$= \int_{1/3}^{2/3} \left(2 - 2x - \frac{4}{9x}\right) dx = \left[2x - x^2 - \frac{4}{9} \log x\right]_{1/3}^{2/3}$$

$$= \left(\frac{4}{3} - \frac{4}{9} - \frac{4}{9} \log \frac{2}{3} - \frac{2}{3} + \frac{1}{9} + \frac{4}{9} \log \frac{1}{3}\right)$$

$$= \frac{1}{3} - \frac{4}{9} \log 2.$$

Example 17 (b) : Find by double integration the area of the smaller region bounded by $x^2 + y^2 = a^2$ and $x + y = a$. **(M.U. 1999)**

Sol. : The circle $x^2 + y^2 = a^2$ and the line $x + y = a$ are shown in the figure.

Now, consider a strip parallel to the y-axis. On this strip y varies from $y = a - x$ to $y = \sqrt{a^2 - x^2}$. Then x varies from $x = 0$ to $x = a$.

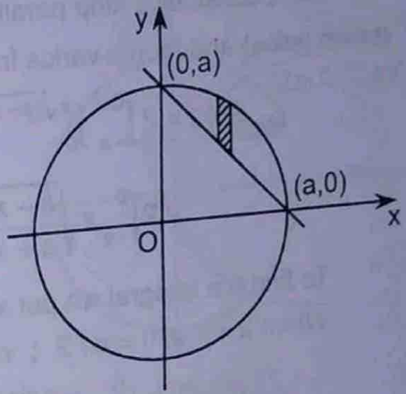

Fig. 10.16

$$\therefore\ A = \int_0^a \int_{y=a-x}^{\sqrt{a^2 - x^2}} dy\, dx = \int_0^1 \left[\, y \,\right]_{a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[\sqrt{a^2 - x^2} - a + x\right] dx$$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} - ax + \frac{x^2}{2}\right]_0^a$$

$$= \left[\left(0 + \frac{a^2}{2} \cdot \frac{\pi}{2} - a^2 + \frac{a^2}{2}\right) - (0)\right]$$

$$= \frac{a^2}{4} (\pi - 2).$$

Example 18 (b) : Find by double integration the area of the smaller region bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

Sol. : The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ are shown in the figure.

The ellipse cuts the axes in $(a, 0)$, $(0, b)$ and the line also cuts the axes in $(a, 0)$, $(0, b)$. Now, consider a strip parallel to the y-axis. On this strip y varies from

$$y = \left(\frac{b}{a}\right)(a - x) \quad \text{to} \quad y = \left(\frac{b}{a}\right)\sqrt{a^2 - x^2}.$$

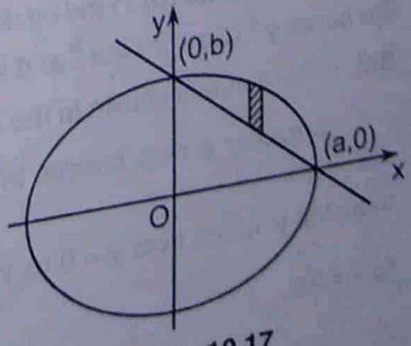

Fig. 10.17

And x varies from $x = 0$ to $x = a$.

$$\therefore\ A = \int_0^a \int_{(b/a)(a-x)}^{(b/a)\sqrt{a^2 - x^2}} dy\, dx = \int_0^a \left[y\right]_{(b/a)(a-x)}^{(b/a)\sqrt{a^2-x^2}} dx$$

$$\therefore \quad A = \int_0^a \left[\frac{b}{a}\sqrt{a^2 - x^2} - \frac{b}{a}(a - x) \right] dx$$

$$= \frac{b}{a} \cdot \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right]_0^a - \frac{b}{a}\left[ax - \frac{x^2}{2} \right]_0^a$$

$$= \frac{b}{a}\left(\frac{a^2}{2} \cdot \frac{\pi}{2} \right) - \frac{b}{a} \cdot \frac{a^2}{2} = ba\left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{ba}{4}(\pi - 2).$$

Example 19 (b) : Find by double integration the area enclosed by the curve $x(x^2 + y^2) = a(x^2 - y^2)$ and its asymptotes. **(M.U. 1996)**

Sol. : See Fig. 10.49, page 10-22. The asymptote is $x = -a$.

We have $x^3 + xy^2 = ax^2 - ay^2 \qquad \therefore \; y^2(a + x) = x^2(a - x)$

$$\therefore \quad y = x\sqrt{\frac{a - x}{a + x}}$$

Now, consider a strip parallel to the y-axis. On this strip y varies from $y = 0$ to $y = x\sqrt{\frac{a-x}{a+x}}$ (taken twice) and then x varies from $x = -a$ to $x = 0$.

$$\therefore \quad \text{Area} = 2\int_{-a}^{0} \int_0^{x\sqrt{(a-x)/(a+x)}} dy\,dx$$

$$= 2\int_{-a}^{0} x\sqrt{\frac{a - x}{a + x}}\,dx = 2\int_{-a}^{0} \frac{x(a - x)}{\sqrt{a^2 - x^2}}\,dx$$

To find the integral we put $x = -a\sin\theta$, $dx = -a\cos\theta\, d\theta$.

When $x = -a$, $\theta = \pi/2$; when $x = 0$, $\theta = 0$.

$$\therefore \quad \text{Area} = 2\int_{\pi/2}^{0} \frac{-a\sin\theta\,(a + a\sin\theta)}{a\cos\theta} \cdot (-a\cos\theta)\, d\theta$$

$$\therefore \quad A = -2\int_0^{\pi/2} a^2(\sin\theta + \sin^2\theta)\, d\theta = -2a^2\int_0^{\pi/2} \sin\theta + \frac{(1 - \cos 2\theta)}{2}\, d\theta$$

$$= -2a^2\left[-\cos\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= -2a^2\left[\frac{\pi}{4} + 1 \right] = 2a^2\left(\frac{\pi}{4} + 1 \right) \qquad \text{[Numerically]}$$

Example 20 (b) : Find by double integration the area bounded by the curve $y^2(2a - x) = x^3$ and its asymptote. **(M.U. 1987, 2002, 03)**

Sol. : The curve is shown in the figure. Its asymptote is $x = 2a$.

Consider a strip parallel to the y-axis (in the first quadrant). On this strip y varies from $y = 0$ to $y = \sqrt{\frac{x^3}{2a - x}}$. Then x varies from $x = 0$ to $x = 2a$.

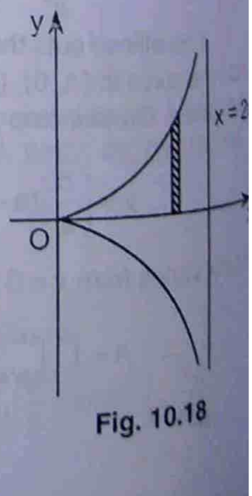

Fig. 10.18

$$\therefore \quad \text{Area} = 2\int_0^{2a} \int_0^{\sqrt{x^3/(2a - x)}} dy\,dx = 2\int_0^{2a} \sqrt{\frac{x^3}{2a - x}}\,dx$$

Put $x = 2a\sin^2\theta$, $\therefore\ dx = 4a\sin\theta\cos\theta\, d\theta$

When $x = 0, \theta = 0$; when $x = 2a, \theta = \pi/2$.

$$\therefore\ \text{Area} = 2\int_0^{\pi/2} \frac{(2a\sin^2\theta)^{3/2}}{\sqrt{(2a - 2a\sin^2\theta)}} \cdot 4a\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{2a\sqrt{2a}\cdot\sin^4\theta\cdot 4a\cos\theta}{\sqrt{2a}\cdot\cos\theta}\, d\theta$$

$$\therefore\ \text{Area} = 16a^2\int_0^{\pi/2}\sin^4\theta\, d\theta = 16a^2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 3\pi a^2.$$

Example 21 (b) : Find by double integration the larger of the two areas into which the circle $x^2 + y^2 = 16a^2$ is divided by the parabola $y^2 = 6ax$.

Sol. : We shall first find the common area *AOBCA*. The points of intersection are given by

$$x^2 + 6ax - 16a^2 = 0 \qquad \therefore\ (x + 8a)(x - 2a) = 0$$

$$\therefore\ x = 2a. \qquad \therefore\ y^2 = 12a^2 \quad \therefore\ y = 2\sqrt{3}\cdot a.$$

Hence, B is $(2a, 2\sqrt{3}\cdot a)$.

Now, consider a strip parallel to the x-axis. On this strip x varies from $x = \dfrac{y^2}{6a}$ to $x = \sqrt{16a^2 - y^2}$. Then y varies from $y = 0$ to $y = 2\sqrt{3}\cdot a$ (in the first quadrant).

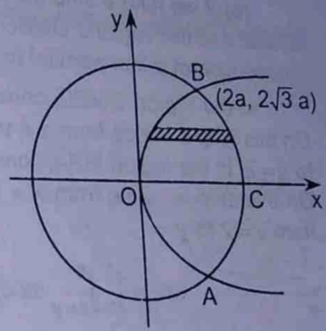

Fig. 10.19

$$\therefore\ \text{Areas} = 2\int_0^{2\sqrt{3}\cdot a}\int_{x = y^2/6a}^{\sqrt{16a^2 - y^2}} dx\, dy$$

$$= 2\int_0^{2\sqrt{3}\cdot a}\left[x\right]_{y^2/6a}^{\sqrt{16a^2 - y^2}} dy$$

$$= 2\int_0^{2\sqrt{3}\cdot a}\left[\sqrt{16a^2 - y^2} - \frac{y^2}{6a}\right] dy$$

$$= 2\left[\frac{y}{2}\sqrt{16a^2 - y^2} + \frac{16a^2}{2}\sin^{-1}\frac{y}{4a} - \frac{y^3}{18a}\right]_0^{2\sqrt{3}\cdot a}$$

$$= \left[y\sqrt{16a^2 - y^2} + 16a^2\sin^{-1}\left(\frac{y}{4a}\right) - \frac{y^3}{9a}\right]_0^{2\sqrt{3}\cdot a}$$

$$= \left[2\sqrt{3}\cdot a\cdot 2a + 16a^2\sin^{-1}\frac{\sqrt{3}}{2} - \frac{24\sqrt{3}}{9a}\cdot a^3\right]$$

$$= 4\sqrt{3}\cdot a^2 + 16a^2\cdot\frac{\pi}{3} - \frac{8}{3}\sqrt{3}\cdot a^2 = \frac{4}{3}(4\pi + \sqrt{3})a^2$$

But area of the circle $= \pi\, 16a^2$.

$$\therefore\ \text{Required area} = \pi 16a^2 - \frac{4}{3}(4\pi + \sqrt{3})a^2 = \frac{4}{3}(8\pi - \sqrt{3})a^2.$$

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Example 22 (b) : Sketch the region bounded by the curves $xy = 16$, $y = x$, $x = 8$ and $y = 0$. Express the area of this region as a double integral in two ways. **(M.U. 2007)**

Sol. : The curve $xy = 16$ is a rectangular hyperbola. $y = x$ is a line OA passing through the origin and equally inclined to the axes. $y = 0$ is the x-axis and $x = 8$ is a line BC parallel to the y-axis. Thus, the region is $OABC$.

The vertices of the figure are $O\,(0, 0)$, $C\,(8, 0)$, $B\,(8, 2)$, $A\,(4, 4)$. If we drop the perpendicular AM, then M is $(4, 0)$.

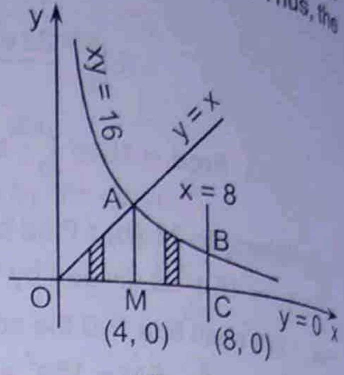


Fig. 10.20

(a) If we take a strip parallel to the y-axis, then the area is divided in two regions OMA and $AMCB$.

In the region OMA, consider a strip parallel to the y-axis. On this strip y varies from $y = 0$ to $y = x$ and then x varies from $x = 0$ to $x = 4$. In the region $AMCB$, consider a strip parallel to the y-axis. On this strip y varies from $y = 0$ to $y = 16 / x$ and then x varies from $x = 4$ to $x = 8$.

$$\therefore \quad \text{Area} = \int_0^4 \int_{y=0}^{x} dx\, dy + \int_4^8 \int_{y=0}^{16/x} dx\, dy$$

(b) If we take a strip parallel to the x-axis, then the area is divided into two regions $OMBC$ and MBA where M is the point of intersection of a line parallel to the x-axis through B.

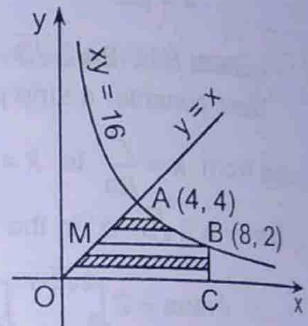


Fig. 10.21

In the region $OMBC$, consider a strip parallel to the x-axis. On this strip x varies from $x = y$ to $x = 8$ then y varies from $y = 0$ to $y = 2$. In the region MBA, consider a strip parallel to the x-axis. On this strip x varies from $x = y$ to $x = y / 16$ and then y varies from $y = 2$ to $y = 4$.

$$\therefore \quad \text{Area} = \int_0^2 \int_{x=y}^{8} dx\, dy + \int_2^4 \int_{x=y}^{y/16} dx\, dy$$

Example 23 (b) : Find by double integration the area of the curvilinear triangle lying in the first quadrant, bounded by the curves $y^2 = 4\,ax$, $x^2 = 4\,ay$, $x^2 + y^2 = 5a^2$. **(M.U. 1991, 2008)**

Sol. : The required triangle is the curvilinear triangle ABC. The vertices are $A\,(4a, 4a)$, $B\,(2a, a)$, $C\,(a, 2a)$. We have to divide the area into two parts ABD and BDC.

(a) In the region ABD, consider a strip parallel to the y-axis. On this strip y varies from $y = x^2 / 4a$ to $y = 2\sqrt{ax}$ and then x varies from $x = 2a$ to $x = 4a$.

$$\text{Area } ABD = \int_{2a}^{4a} \int_{x^2/4a}^{2\sqrt{a}\sqrt{x}} dy\, dx = \int_{2a}^{4a} \left[\, y \,\right]_{x^2/4a}^{2\sqrt{a}\sqrt{x}} dx$$

$$= \int_{2a}^{4a} \left[2\sqrt{a}\,\sqrt{x} - \frac{x^2}{4a} \right] dx = \left[2\sqrt{a} \cdot \frac{2}{3}\, x^{3/2} - \frac{x^3}{12a} \right]_{2a}^{4a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} - \frac{8\sqrt{2} \cdot a^2}{3} + \frac{2}{3}\,a^2 = \frac{18}{3}\,a^2 - \frac{8\sqrt{2}}{3}\,a^2.$$

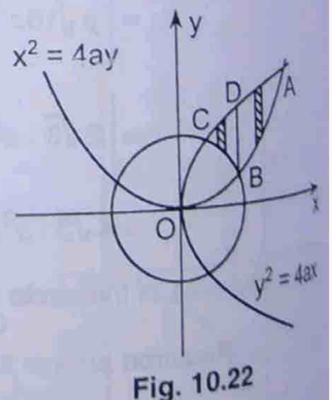


Fig. 10.22

(b) In the region BDC, consider a strip parallel to the y-axis.

On this strip y varies from $y = \sqrt{5a^2 - x^2}$ to $y = 2\sqrt{ax}$. The x varies from $x = a$ to $x = 2a$.

$$\text{Area } BDC = \int_a^{2a} \int_{\sqrt{5a^2-x^2}}^{2\sqrt{a}\sqrt{x}} dy\, dx = \int_a^{2a} [\, y\,]_{\sqrt{5a^2-x^2}}^{2\sqrt{a}\sqrt{x}}\, dx$$

$$= \int_a^{2a} \left[2\sqrt{a}\sqrt{x} - \sqrt{5a^2 - x^2} \right] dx$$

$$= \left[\frac{4}{3}\sqrt{a} \cdot x^{3/2} - \frac{x}{2}\sqrt{5a^2 - x^2} - \frac{5a^2}{2}\sin^{-1}\left(\frac{x}{\sqrt{5} \cdot a}\right) \right]_a^{2a}$$

$$= \frac{8\sqrt{2}}{3}a^2 - \frac{2a^2}{2} - \frac{5a^2}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \frac{4}{3}a^2 + \frac{2a^2}{2} + \frac{5a^2}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$= \frac{8\sqrt{2}}{3}a^2 - \frac{4}{3}a^2 + \frac{5a^2}{2}\left[\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \right]$$

$\therefore$ Required area of the triangle ABC = area ABD + area BDC

$$= \frac{14}{\sqrt{3}}a^2 + \frac{5a^2}{2}\left[\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \right].$$

EXERCISE - I

Find the areas by double integration : Class (b) : 6 Marks

1. The area bounded by $xy = 2$, $4y = x^2$ and $y = 4$.
   (See Fig. 10.23) **(M.U. 2011)** [**Ans. :** $\frac{28}{3} - 4\log 2$]

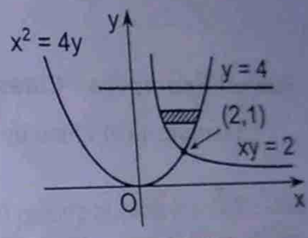


Fig. 10.23

2. The area bounded by the lines $y = x + 2$, $y = -x + 2$, $x = 5$.
   (See Fig. 9.20, page 9-23) **(M.U. 2002)** [**Ans. :** 25]
3. The area between the circle $x^2 + y^2 - 2ax = 0$ and the parabola $y^2 = ax$. [**Ans. :** $2a^2\left(\frac{\pi}{4} - \frac{2}{3}\right)$]
   (See similar Fig. 9.35, page 9-34)
4. The area bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4\,ay$.
   (See similar Fig. 11.3, page 11-3) [**Ans. :** $(16/3)\, a^2$]
5. The area between the parabola $y = 4x - x^2$ and the line $y = x$. (See Fig. 10.24) **(M.U. 1993)** [**Ans. :** 9/2]

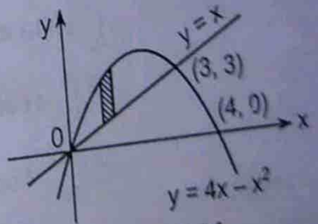


Fig. 10.24

6. The larger of the two areas into which the circle $r^2 = 16^2$ is divided by the parabola $y^2 = 24x$.
   (See similar Fig. 10.19, page 10-11, $a = 4$.)
   (M.U. 1989) [**Ans. :** $\frac{64}{3}(8\pi - \sqrt{3})$]
7. The total area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$. **(M.U. 1997)** [**Ans. :** $3\pi a^2/8$]
   (See Fig. 8.2, page 8-2)
8. The total area included between the two branches of the curve $y^2(4 - x)(x - 2) = x^2$ and its two asymptotes. (See Fig. 10.12, page 10-7 with $a = 4$, $b = 2$) [**Ans. :** 6π]

9. The area between the parabola $y^2 = 4ax$ and the line $2x - 3y + 4a = 0$.
(See Fig. 10.13, page 10-7, $a = 1$) **(M.U. 1991)** [**Ans.** : $a^2/3$]

10. The area of the circle $x^2 + y^2 = 2ax$ which lies outside the circle $x^2 + y^2 = a^2$.
(See Fig. 10.31, page 10-16) **(M.U. 1994)** [**Ans.** : $\frac{a^6}{6}(2\pi + 3\sqrt{3})$]

11. The area between the circles $x^2 + y^2 - 4ax = 0$ and $x^2 + y^2 - 2ax = 0$.
(See similar Fig. 9.69, page 9-55) [**Ans.** : $3\pi a^2$]

3. Area by Double Integration - Polar Coordinates

Consider again the area enclosed by two plane curves $r = f_1(\theta)$ and $r = f_2(\theta)$ intersecting in $A(r_1, \alpha)$ and $A(r_2, \beta)$.

We divide the area into small areas by taking lines θ = constant and circles r = constant.

Then the area of the elementary rectangle (shaded area) is $r\,d\theta\,dr$. The required area then is given by

$$\boxed{A = \int_\alpha^\beta \int_{f_1(0)}^{f_2(0)} r\,dr\,d\theta}$$

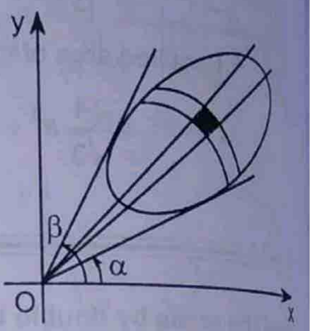


Fig. 10.25

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Find by double integration the area of the cardioid $r = a(1 + \cos\theta)$. **(M.U. 1992)**

Sol. : The cardioid is shown in the Fig. 10.26. Consider a radial strip. On this strip r varies from $r = 0$ to $r = a(1 + \cos\theta)$ and θ varies from $\theta = 0$ to $\theta = \pi$ above the x-axis.

Curve Kaisa banaenge ?

$$\therefore \quad \text{Area} = 2\int_0^\pi \int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^\pi \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^\pi a^2(1 + \cos\theta)^2\,d\theta$$

$$= a^2\int_0^\pi 4\cos^4(\theta/2)\cdot d\theta$$

$$= a^2\int_0^{\pi/2} 4\cos^4\phi \cdot 2\,d\phi \qquad \left[\text{Put } \frac{\theta}{2} = \phi\right]$$

$$= 8a^2 \cdot \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{2}\pi a^2.$$

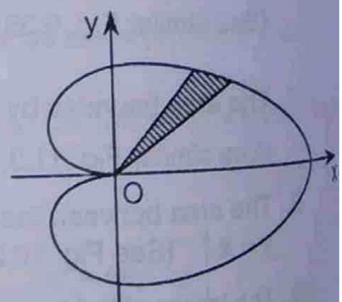


Fig. 10.26

Example 2 (b) : Find by double integration the total area of the curve $r = a\sin 2\theta$. **(M.U. 2000, 1**

Sol. : The curve is a four leaved rose as shown in the figure. Consider a radial strip in the leaf in the first quadrant. On this strip r varies from $r = 0$ to $r = a\sin 2\theta$. Then θ varies from $\theta = 0$ to $\theta = \pi/$

$$\therefore \quad \text{Area} = 4\int_0^{\pi/2}\int_0^{a\sin 2\theta} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{a\sin 2\theta} d\theta$$

$$= 2\int_0^{\pi/2} a^2\sin^2 2\theta\,d\theta$$

$$= 2a^2\int_0^{\pi/2}\frac{(1-\cos 4\theta)}{2}\,d\theta$$

$$= a^2\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{\pi a^2}{2}.$$

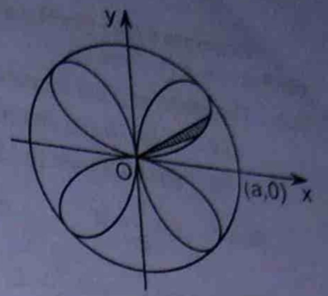


Fig. 10.27

Example 3 (b) : Find by double integration the area enclosed by one loop of $r = \cos 2\theta$.

Sol. : The curve $r = \cos 2\theta$ is a four leaved rose. Consider a radial strip in half of the leaf above the x-axis as shown in the figure. On this strip r varies from $r = 2\cos\theta$ and θ varies from $\theta = 0$ to $\theta = \pi/4$ above the x-axis. The area of the part of one loop above the x-axis is

$$A = \int_0^{\pi/4}\int_{r=0}^{\cos 2\theta} r\,dr\,d\theta$$

The area of one loop is twice of this.

$$\therefore \quad A = 2\int_{\theta=0}^{\pi/4}\int_{r=0}^{\cos 2\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\left[r^2\right]_0^{\cos 2\theta} d\theta$$

$$= \int_0^{\pi/4}\cos^2 2\theta\,d\theta = \int_0^{\pi/4}\left(\frac{1+\cos 4\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/4} = \frac{\pi}{8}.$$

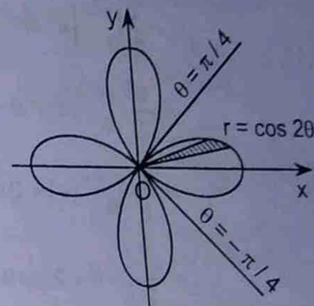


Fig. 10.28

Example 4 (b) : Find by double integration the total area enclosed by the lemniscate of Bernoulli $(x^2+y^2)^2 = a^2(x^2-y^2)$. **(M.U. 2001, 13)**

Sol. : We transform the equation to polar form by putting $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \quad r^4 = a^2r^2\cos 2\theta \quad i.e., \quad r^2 = a^2\cos 2\theta.$$

Now, consider a small radial strip in the upper half in the first quadrant of one loop. On this strip r varies from $r = 0$ to $r = a\sqrt{\cos 2\theta}$ and θ varies from $\theta = 0$ to $\theta = \pi/4$.

$$\therefore \quad A = 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta = 2a^2\int_0^{\pi/4}\cos 2\theta\,d\theta$$

$$= 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = a^2.$$

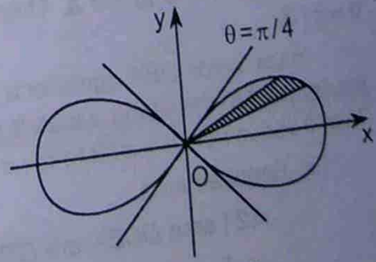


Fig. 10.29

Example 5 (b) : Find by double integration the area inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$. **(M.U. 2001, 02, 06)**

Sol. : The circle and the cardioid intersect where $a \sin \theta = a(1 - \cos \theta)$

i.e. $2 \sin(\theta/2) \cos(\theta/2) = 2 \sin^2(\theta/2)$

i.e. $\sin \theta/2 [\sin(\theta/2) - \cos(\theta/2)] = 0.$

When $\sin\dfrac{\theta}{2} = 0 \quad \therefore \theta = 0$

When $\sin\dfrac{\theta}{2} - \cos\dfrac{\theta}{2} = 0, \ \dfrac{\theta}{2} = \dfrac{\pi}{4} \quad \therefore \theta = \dfrac{\pi}{2}.$

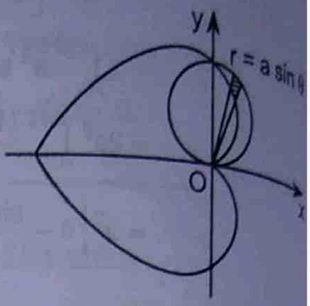


Fig. 10.30

Now, consider a radial strip in the region of integration. On this strip r varies from $r = a(1 - \cos \theta)$ to $r = a \sin \theta$. Then θ varies from $\theta = 0$ to $\theta = \pi/2$.

$$\therefore \quad A = \int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a\sin\theta} r\, dr\, d\theta = \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[a^2 \sin^2\theta - a^2(1-\cos\theta)^2\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} (\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta)\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} (-1 + 2\cos\theta - \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left[-\theta + 2\sin\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{a^2}{2}\left[-\frac{\pi}{2} + 2\right] = \frac{a^2(4-\pi)}{4}.$$

Example 6 (b) : Find by double integration the area common of the circles $r = a$ and $r = 2a \cos \theta$. **(M.U. 1992, 97, 2002, 03, 05, 10)**

Sol. : First we note that $r = a$ *i.e.* $r^2 = a^2$ *i.e.* $x^2 + y^2 = a^2$ is a circle with centre at the origin and radius $= a$ and $r = 2a\cos\theta$ *i.e.* $r^2 = 2ar\cos\theta$ *i.e.* $x^2 + y^2 = 2ax$ *i.e.* $(x-a)^2 + y^2 = a^2$ is the circle with centre at $(a, 0)$ and radius $= a$.

To find the points of intersection, we solve the two equations.

$$\therefore \quad a = 2a\cos\theta \quad \therefore \cos\theta = \frac{1}{2} \quad \therefore \theta = \frac{\pi}{3} \text{ or } -\frac{\pi}{3}.$$

In the region OCB, consider a radial strip. On this strip r varies from $r = 0$ to $r = a$. Then θ varies from $\theta = 0$ to $\theta = \pi/3$.

In the region OBA, consider a radial strip (not shown in the figure). On this strip r varies from $r = 0$ to $r = 2a \cos \theta$. Then θ varies from $\theta = \pi/3$ to $\theta = \pi/2$.

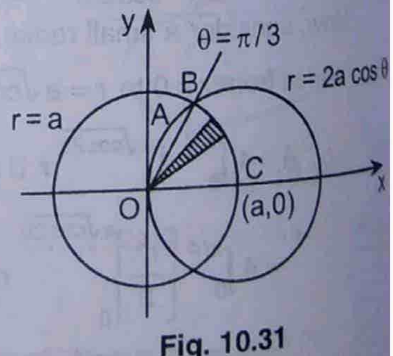


Fig. 10.31

$\therefore$ Required area

$= 2$ [area OCB + are OBA]

$$= 2\left[\int_0^{\pi/3}\int_0^a r\, dr\, d\theta + \int_{\pi/3}^{\pi/2}\int_0^{2a\cos\theta} r\, dr\, d\theta\right]$$

$$= 2\left[\int_0^{\pi/3}\left[\frac{r^2}{2}\right]_0^a d\theta + \int_{\pi/3}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{2a\cos\theta} d\theta\right]$$

$$= 2\left[\int_0^{\pi/3}\frac{a^2}{2}\,d\theta + \int_{\pi/3}^{\pi/2} 2a^2\cos^2\theta\, d\theta\right]$$

$$= a^2\int_0^{\pi/3} d\theta + 2a^2\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)\,d\theta = a^2\left[\theta\right]_0^{\pi/3} + 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$$

$$= a^2\frac{\pi}{3} + 2a^2\left[\frac{\pi}{2} + 0 - \frac{\pi}{3} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right] = \frac{2\pi a^2}{3} - \frac{\sqrt{3}}{2}a^2.$$

Example 7 (b) : Find by double integration the area common to the circles $r = a\cos\theta$, and $r = a\sin\theta$. **(M.U. 2007)**

Sol. : We have $r = a\cos\theta$,

$$\therefore \sqrt{x^2+y^2} = a\cdot\frac{x}{\sqrt{x^2+y^2}}$$

$$\therefore x^2+y^2-ax = 0 \qquad \therefore \left(x-\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

Similarly, $r = a\sin\theta$,

$$\therefore \sqrt{x^2+y^2} = a\cdot\frac{y}{\sqrt{x^2+y^2}}$$

$$\therefore x^2+y^2-ay = 0 \qquad \therefore x^2 + \left(y-\frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2$$

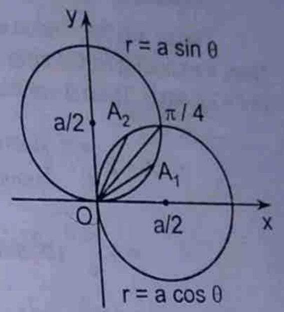
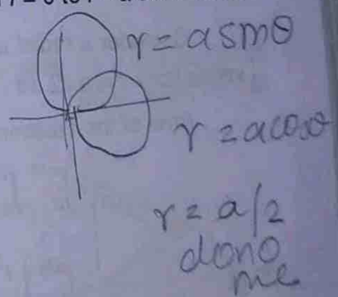


Fig. 10.32

The two circles intersect at $r = a\cos\theta = a\sin\theta$ *i.e.*, $\theta = \pi/4$. We have to find the common area in the two circles separately.

(a) In the region A_1, consider a radial strip. On this strip r varies from $r = 0$ to $r = a\sin\theta$. Then θ varies from $\theta = 0$ to $\theta = \pi/4$.



$$\therefore A_1 = \int_0^{\pi/4}\int_0^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/4}\sin^2\theta\,d\theta = \frac{a^2}{2}\int_0^{\pi/4}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$\therefore A_1 = \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \frac{a^2}{4}\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

(b) In the region A_2, consider a radial strip. On this strip r varies from $r = 0$ to $r = a\cos\theta$. Then θ varies from $\theta = \pi/4$ to $\theta = \pi/2$.

$$\therefore A_2 = \int_{\pi/4}^{\pi/2}\int_0^{a\cos\theta} r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{a^2}{2}\int_{\pi/4}^{\pi/2}\cos^2\theta\,d\theta = \frac{a^2}{2}\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$\therefore \quad A_2 = \frac{a^2}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2} = \frac{a^2}{4}\left[\left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{4}+\frac{1}{2}\right)\right] = \frac{a^2}{4}\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$\therefore \quad A = A_1 + A_2 = \frac{a^2}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right].$$

(Or we can say that $A_1 = A_2$ by symmetry. The two circles have the same radius $r = a/2$.)

Example 8 (b) : Find by double integration the area between the circles $r = 2a \sin\theta$, $r = 2b \sin\theta$ $(b > a)$.

Sol. : We have $r = 2a \sin\theta$

$$\text{i.e.} \quad \sqrt{x^2+y^2} = 2a \cdot \frac{y}{\sqrt{x^2+y^2}}$$

$$\text{i.e.} \quad x^2 + y^2 = 2ay \quad \text{i.e.} \quad x^2 + (y-a)^2 = a^2.$$

Similarly, $r = 2b \sin\theta$ gives $\quad x^2 + (y-b)^2 = b^2.$

These are the circles with centres $(0, a)$, $(0, b)$ and radii a, b.

Now, consider a radial strip. On this strip r varies from $r = 2a \sin\theta$ to $r = 2b \sin\theta$. Then θ varies $\theta = 0$ to $\theta = \pi/2$ in the first quadrant.

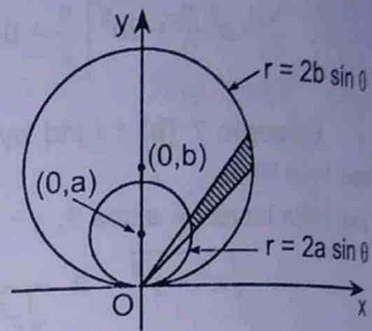


Fig. 10.33

$$\therefore \quad A = 2\int_0^{\pi/2}\int_{a\sin\theta}^{b\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{2a\sin\theta}^{2b\sin\theta} d\theta$$

$$= 4\int_0^{\pi/2} (b^2\sin^2\theta - a^2\sin^2\theta)\,d\theta = 4(b^2 - a^2)\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= 4(b^2 - a^2)\cdot\frac{1}{2}\cdot\frac{\pi}{2} = (b^2 - a^2)\pi.$$

Example 9 (b) : Find by double integration the area of the crescent (the moon in the first or last quarter) bounded by the circles $r = a\sqrt{2}$ and $r = 2a\cos\theta$. **(M.U. 1995, 2004)**

Sol. : The two curves intersect where $a\sqrt{2} = 2a\cos\theta$ i.e., $\cos\theta = \pm 1/\sqrt{2}$, $\theta = \pm\frac{\theta}{4}$.

Consider a radial strip in the region between the two circles above the x-axis. On this strip r varies from $r = a\sqrt{2}$ to $r = 2a\cos\theta$. Then θ varies from $\theta = 0$ to $\theta = \pi/4$.

Area of the crescent

$$= 2\int_0^{\pi/4}\int_{a\sqrt{2}}^{2a\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{a\sqrt{2}}^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/4} (4a^2\cos^2\theta - a^2\cdot 2)\,d\theta$$

$$= 4a^2\int_0^{\pi/4}\left(\frac{1+\cos 2\theta}{2}\right)d\theta - 2a^2\int_0^{\pi/4} d\theta$$

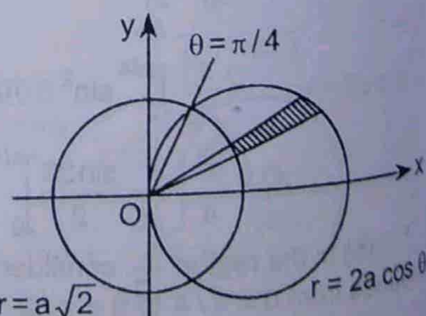


Fig. 10.34

$$= 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} - 2a^2[\theta]_0^{\pi/4} = 2a^2\left[\frac{\pi}{4}+\frac{1}{2}\right] - 2a^2\left[\frac{\pi}{4}\right] = a^2.$$

Example 10 (b) : Find by double integration the area common to the circle $r = a$ and the cardioid $r = a(1 + \cos\theta)$.

Sol. : The two curves are shown in the figure. When they intersect at C,

$$a = a(1 + \cos\theta) \quad \therefore \quad 1 = 1 + \cos\theta \quad \therefore \quad \cos\theta = 0 \quad \therefore \quad \theta = \pi/2.$$

The required area is twice the area $OAPCDO$.

Now, consider a radial strip in the region $OAPC$ bounded by the circle. On this strip r varies from $r = 0$ to $r = a$ and then θ varies from $\theta = 0$ to $\theta = \pi/2$.

Also consider a radial strip in the region $OCDO$ bounded by the cardioid. On this strip r varies from $r = 0$ to $r = a(1 + \cos\theta)$ and θ varies from $\theta = \pi/2$ to $\theta = \pi$.

Area $= 2$ area $OAPC + 2$ area $OCDO$

$$= 2\int_0^{\pi/2}\int_0^{a} r\,dr\,d\theta + 2\int_{\pi/2}^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

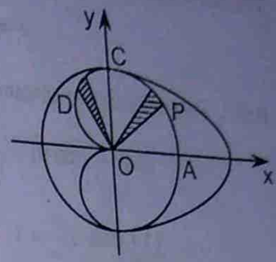


Fig. 10.35

$$= 2\int_0^{\pi/2}\left[\frac{1}{2}r^2\right]_0^{a} d\theta + 2\int_{\pi/2}^{\pi}\left[\frac{1}{2}r^2\right]_0^{a(1+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2} d\theta + \int_{\pi/2}^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= a^2[\theta]_0^{\pi/2} + a^2\int_{\pi/2}^{\pi}(1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \frac{\pi a^2}{2} + a^2\int_{\pi/2}^{\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{\pi a^2}{2} + a^2\int_{\pi/2}^{\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{\pi a^2}{2} + a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi} = \frac{\pi a^2}{2} + a^2\left[\frac{3}{2}\pi - \frac{3\pi}{4} - 2\right]$$

$$= \frac{\pi a^2}{2} + a^2\left(\frac{3\pi}{4} - 2\right) = \left(\frac{5\pi}{4} - 2\right)a^2.$$

Example 11 (b) : Find by double integration the area outside the circle $r = a$ and inside the cardioid $r = a(1 + \cos\theta)$. **(M.U. 2002, 09, 12)**

Sol. : The circle $r = a$ and the cardioid $r = a(1 + \cos\theta)$ are as shown in the figure. When they intersect at C.

$$\therefore \ a = a(1 + \cos\theta) \quad \therefore \ 1 = 1 + \cos\theta$$

$$\therefore \ \cos\theta = 0 \quad \therefore \ \theta = \pm\pi/2.$$

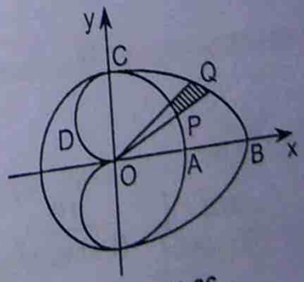


Fig. 10.36

The required area is twice the area $ABQCPA$. Now, consider a radial strip in the region $ABQCPA$. On this strip r varies from $r = a$ to $r = a(1 + \cos\theta)$ and then θ varies from $\theta = 0$ to $\theta = \pi/2$.

$\therefore$ Area $= 2$ area $ABQCPA$

$$= 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{1}{2}r^2\right]_a^{a(1+\cos\theta)} d\theta = \int_0^{\pi/2}\left[a^2(1+\cos\theta)^2 - a^2\right]d\theta$$

$$\therefore\ \text{Area} = a^2 \int_0^{\pi/2} (1 + 2\cos\theta + \cos^2\theta - 1)\ d\theta = a^2 \int_0^{\pi/2} \left[2\cos\theta + \left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} (1 + 4\cos\theta + \cos 2\theta)\ d\theta = \frac{a^2}{2}\left[\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} + 4\right] = \frac{a^2}{4}[\pi + 8].$$

Example 12 (b) : Find by double integration the area inside the cardioid $r = 2a\,(1 + \cos\theta)$ and outside the parabola

$$r = \frac{2a}{1+\cos\theta}.$$

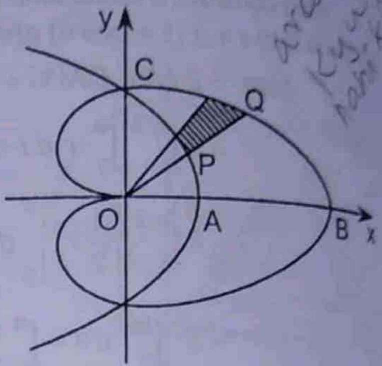


Fig. 10.37

Sol. : The curves are shown in the figure. When they intersect

$$2a(1 + \cos\theta) = \frac{2a}{(1+\cos\theta)}$$

$$\therefore\ (1+\cos\theta)^2 = 1\quad \therefore\ 4\cos^2\frac{\theta}{2} = 1$$

$$\therefore\ \cos\frac{\theta}{2} = \pm\sqrt{\frac{1}{2}}\quad \therefore\ \frac{\theta}{2} = \pm\frac{\pi}{4},\ \theta = \pm\frac{\pi}{2}.$$

The required area is twice the area $ABQCPA$.

Now, consider a radial strip in the region $ABQCPA$. On this strip r varies from $r = \dfrac{2a}{1+\cos\theta}$ to $r = 2a\,(1+\cos\theta)$ and then θ varies from $\theta = 0$ to $\theta = \pi/2$.

$$\text{Area} = 2\ \text{area}\ ABQ\,CPA = 2\int_0^{\pi/2} \int_{2a/(1+\cos\theta)}^{2a(1+\cos\theta)} r\ dr\ d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{1}{2}r^2\right]_{2a/(1+\cos\theta)}^{2a(1+\cos\theta)} d\theta = 4a^2 \int_0^{\pi/2} \left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right] d\theta$$

$$\text{Now,}\ \int_0^{\pi/2} (1+\cos\theta)^2\ d\theta = \int_0^{\pi/2} (1 + 2\cos\theta + \cos^2\theta)\ d\theta$$

$$= \int_0^{\pi/2} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{\pi/2} \left(\frac{3}{2} + 2\cos\theta + \cos 2\theta\right) d\theta$$

$$= \left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \left[\frac{3\pi}{4} + 2\right].$$

$$\text{And}\ \int_0^{\pi/2} \frac{d\theta}{(1+\cos\theta)^2} = \frac{1}{4}\int_0^{\pi/2} \frac{d\theta}{(\cos^2\theta/2)^2}$$

$$= \frac{1}{4}\int_0^{\pi/2} \sec^4\frac{\theta}{2}\,d\theta = \frac{1}{4}\int_0^{\pi/2} \left(1 + \tan^2\frac{\theta}{2}\right)\sec^2\frac{\theta}{2}\ d\theta$$

$$= \frac{1}{4}\int_0^1 (1+t^2)\,2\ dt.\qquad \left(\text{Put}\ \tan\frac{\theta}{2} = t\right)$$

$$\therefore \int_0^{\pi/2} \frac{d\theta}{(1+\cos\theta)^2} = \frac{1}{2}\left[t + \frac{t^3}{3}\right]_0^1 = \frac{1}{2}\left(1+\frac{1}{3}\right) = \frac{2}{3}.$$

Hence, the required area is

$$A = 4a^2\left[\frac{3\pi}{4} + 2 - \frac{2}{3}\right] = 4a^2\left(\frac{3\pi}{4}+\frac{4}{3}\right) = a^2\left(3\pi + \frac{16}{3}\right).$$

Example 13 (b) : Find by double integration the area common to the cardioids $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$. **(M.U. 2002, 14)**

Sol. : The two curves intersect where $a(1+\cos\theta) = a(1-\cos\theta)$.

$\therefore\ 2\cos\theta = 0 \qquad \therefore\ \theta = \pi/2.$

The shaded area is the common area in the first quadrant.

Consider a radial strip in the shaded area. On this strip, r varies from $r = 0$ to $r = a(1-\cos\theta)$ and then θ varies from $\theta = 0$ to $\theta = \pi/2$.

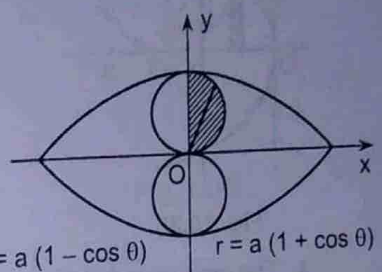


Fig. 10.37 (a)

$$A_1 = \int_0^{\pi/2}\int_0^{a(1-\cos\theta)} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} a^2(1-\cos\theta)^2\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(1-2\cos\theta+\cos^2\theta)\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{a^2}{2}\int_0^{\pi/2}\left(\frac{3}{2}-2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{a^2}{2}\left[\frac{3}{2}\cdot\frac{\pi}{2} - 2\right] = \frac{a^2}{2}\left[\frac{3\pi}{4}-2\right]$$

Total area by symmetry is

$$A = 4A_1 = 4\cdot\frac{a^2}{2}\left(\frac{3\pi}{4}-2\right) = \frac{a^2}{2}(3\pi - 8).$$

EXERCISE - II

Find the areas by double integration : Class (b) : 6 Marks

1. The area of the cardioid $r = a(1-\cos\theta)$. **(M.U. 1992)** [**Ans. :** $3\pi a^2/2$]
   (See Fig. 11.11, page 11-7)
2. The area between the circles $r = 2\sin\theta$, $r = 4\sin\theta$. [**Ans. :** $3\pi/4$]
   (See Fig. 10.33, page 10-18 with $a = 1$, $b = 2$)
3. The area of the crescent bounded by the circles $r = \sqrt{2}$ and $r = 2\cos\theta$. [**Ans. :** 1]
   (See Fig. 10.34, page 10-18, $a = 1$)
4. The area inside the circle $r = 2\sin\theta$ and outside the cardioid $r = 2(1-\cos\theta)$. [**Ans. :** $4-\pi$]
   (See Fig. 10.30, page 10-16 with $a = 2$)

EXERCISE - III

Find the areas by double integration : Class (a) : 3 or 4 Marks

1. The area bounded by $y^2 = 4x$ and the ordinate at $x = 1$. **[Ans. : 8 / 3]**
(See Fig. 10.38)
2. The area bounded by $y = e^x$ and the line $y = x$ from $x = 0$ to $x = 2$. **[Ans. : $e^2 - 3$]**
(See Fig. 10.39)
3. The area bounded by $y = x$ and $y = x^{1/3}$ in the first quadrant. **[Ans. : 1 / 4]**
(See Fig. 10.40)

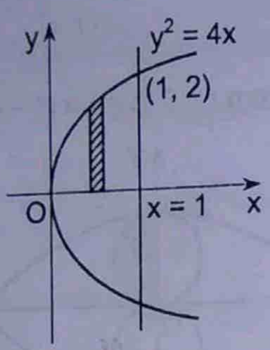


Fig. 10.38

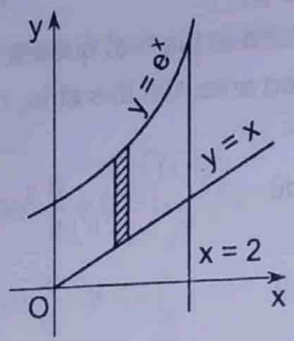


Fig. 10.39

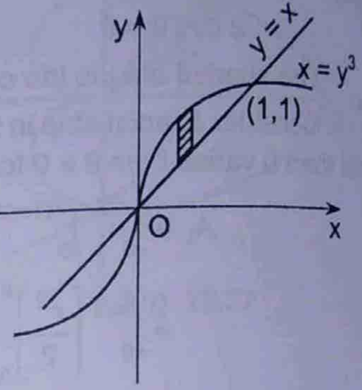


Fig. 10.40

4. The area bounded by $y = x^3$, $y = x^5$ in the first quadrant. **[Ans. : 1 / 12]**
(See Fig. 10.41)
5. The area bounded by the line $y = a + x$ and $y = a - x$ and $x = a$. **[Ans. : a^2]**
(See Fig. 10.42)
6. The area bounded by $y^2 = 4x$ and the line $y = 2x$. **[Ans. : 1 / 3]**
(See Fig. 10.43)

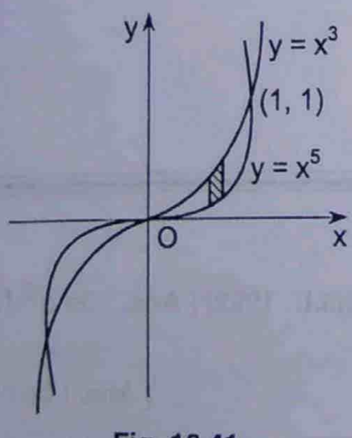


Fig. 10.41

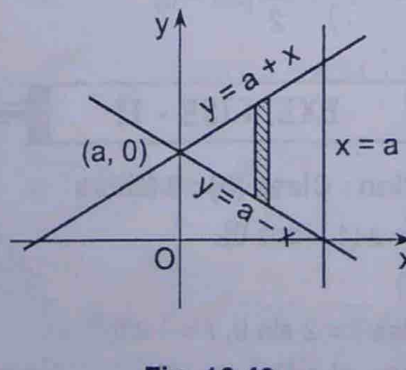


Fig. 10.42

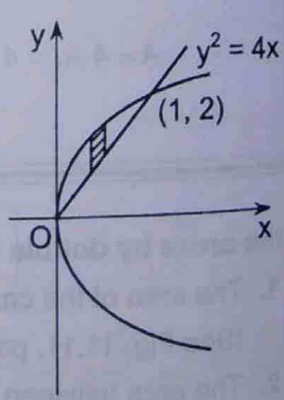


Fig. 10.43

7. The area between the curves $y^2 = x^3$ and $y = x^2$. **[Ans. : 1 / 15]**
(See Fig. 10.44 on the next page)

8. The area between the curve $y = 2x + x^2$, the x-axis and the ordinates at $x = 1$, $x = 3$.
(See Fig. 10.45) [Ans. : 50 / 3]

9. The area bounded by $y = x^2$ and $y^2 = x$.
(See Fig. 10.46) [Ans. : 1 / 3]

10. The area of a circle of radius a (use polar form).
(See Fig. 10.47) [Ans. : πa^2]

11. The area between the circles $r = \sin\theta$ and $r = 2 \sin\theta$.
(See Fig. 10.48) [Ans. : $3\pi / 4$]

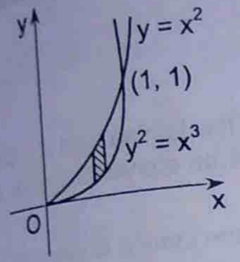


Fig. 10.44

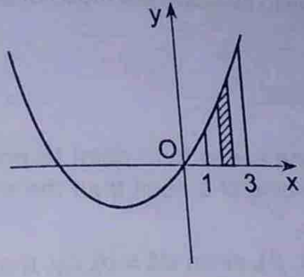


Fig. 10.45

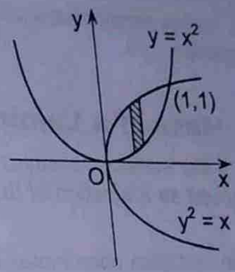


Fig. 10.46

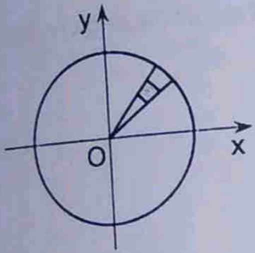


Fig. 10.47

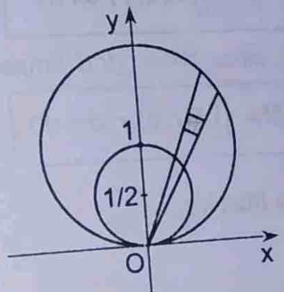


Fig. 10.48

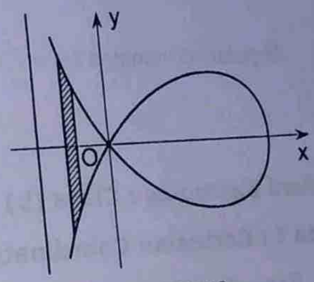


Fig. 10.49

Summary

1. $A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\, dx$; $\quad A = \int_c^d \int_{\Phi_1(y)}^{\Phi_2(y)} dx\, dy$

2. $A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r\, dr\, d\theta$

***

CHAPTER 11

Mass of a Lamina

1. Introduction

In this chapter we shall learn how to evaluate the mass of a lamina as an application of double integration.

2. Mass of a Lamina

If the surface density ρ of a plane varies form point to point of the lamina and if it can be expressed as a function of the coordinates of a point then the mass of an elementary area dA is $\rho \, dA$.

In cartesian coordinates if $\rho = f(x, y)$, since $dA = dx\, dy$, mass of the lamina is given by

$$\boxed{M = \iint f(x, y)\, dx\, dy}$$

In polar coordinates if $\rho = f(r, \theta)$, since $dA = r\, dr\, d\theta$ mass of the lamina is given by

$$\boxed{M = \iint f(r, \theta)\, r\, dr\, d\theta}$$

Solved Examples : Class (b) : 6 Marks

Type I : Cartesian Coordinates

Example 1 (b) : Find the mass of the lamina bounded by the curve $ay^2 = x^3$ and the line $by = x$ if the density at a point varies as the distance of the point from the x-axis.

(M.U. 1997, 2007, 11)

Sol. : The two curves intersect at $A\left(\dfrac{a}{b^2}, \dfrac{a}{b^3}\right)$.

The lamina is the area OBA. On the curve OBA, $y = x^{3/2}/\sqrt{a}$ and on the line OA, $y = x/b$. The surface density is given by $\rho = ky$. Taking the elementary strip parallel to the y-axis, on the strip, y varies from

$y = \left(\dfrac{x^3}{a}\right)^{1/2}$ to $y = \dfrac{x}{b}$ and then x varies from $x = 0$ to $x = \dfrac{a}{b^2}$.

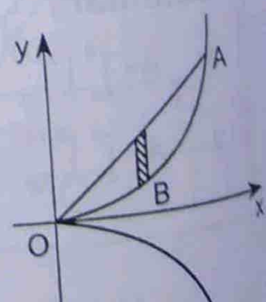


Fig. 11.1

$\therefore$ Mass of the lamina

$$= k\int_0^{a/b^2} \int_{x^{3/2}/\sqrt{a}}^{x/b} y\, dx\, dy = k\int_0^{a/b^2} \left[\frac{y^2}{2}\right]_{x^{3/2}/\sqrt{a}}^{x/b} dx$$

$= \dfrac{k}{2}\int_0 \ldots$

$= \dfrac{k}{2}\Big[\ldots$

Example 2 (b) ... density at a point va... distance of the poin...

Sol. : See Fig. 11.1...

(i) $\rho = ky^3$. As be...

$M = k \ldots$

(ii) $\rho = ky^2$. As ...

$M \ldots$

Exampl... given by (24 ...

Sol. : The c... intersecting ... this parabo... $x = 5$. There... lamina is the... the element... $y = x^2 - 3x$ ...

$\therefore$ Ma...

$$= \frac{k}{2}\int_0^{a/b^2}\left[\frac{x^2}{b^2}-\frac{x^3}{a}\right]dx = \frac{k}{2}\left[\frac{x^3}{3b^2}-\frac{x^4}{4a}\right]_0^{a/b^2}$$

$$= \frac{k}{2}\left[\frac{a^3}{3b^8}-\frac{a^3}{4b^8}\right] = \frac{k}{2}\cdot\frac{a^3}{b^8}\left[\frac{1}{3}-\frac{1}{4}\right] = \frac{k}{24}\cdot\frac{a^3}{b^8}.$$

Example 2 (b) : Find the mass of the lamina bounded by the curves $ay^2 = x^3$ and $by = x$, if the density at a point varies as (i) the cube of distance of the point from the x-axis, (ii) the square of the distance of the point from the x-axis.

Sol. : See Fig. 11.1, on previous page.

(i) $\rho = ky^3$. As before,

$$M = k\int_0^{a/b^2}\int_{x^{3/2}/\sqrt{a}}^{x/b} y^3\,dy\,dx = k\int_0^{a/b^2}\left[\frac{y^4}{4}\right]_{x^{3/2}/\sqrt{a}}^{x/b}dx$$

$$= \frac{k}{4}\int_0^{a/b^2}\left[\frac{x^4}{b^4}-\frac{x^6}{a^2}\right]dx = \frac{k}{4}\left[\frac{x^5}{5b^4}-\frac{x^7}{7a^2}\right]_0^{a/b^2}$$

$$= \frac{k}{4}\left[\frac{1}{5b^4}\cdot\frac{a^5}{b^{10}}-\frac{1}{7a^2}\cdot\frac{a^7}{b^{14}}\right] = \frac{k}{4}\left[\frac{2}{35}\cdot\frac{a^5}{b^{14}}\right] = \frac{k}{70}\cdot\frac{a^5}{b^{14}}.$$

(ii) $\rho = ky^2$. As before,

$$M = k\int_0^{a/b^2}\int_{x^{3/2}/\sqrt{a}}^{x/b} y^2\,dy\,dx = k\int_0^{a/b^2}\left[\frac{y^3}{3}\right]_{x^{3/2}/\sqrt{a}}^{x/b}dx$$

$$= \frac{k}{3}\int_0^{a/b^2}\left[\frac{x^3}{b^3}-\frac{x^{9/2}}{a\sqrt{a}}\right]dx = \frac{k}{3}\left[\frac{x^4}{4b^3}-\frac{2}{11}\cdot\frac{x^{11/2}}{a\sqrt{a}}\right]_0^{a/b^2}$$

$$= \frac{k}{3}\left[\frac{1}{4}\cdot\frac{a^4}{b^8\cdot b^3}-\frac{2}{11}\cdot\frac{a^{11/2}}{b^{11}}\cdot\frac{1}{a\sqrt{a}}\right] = \frac{k}{44}\cdot\frac{a^4}{b^{11}}.$$

Example 3 (b) : A lamina is bounded by $y = x^2 - 3x$ and $y = 2x$. If the density at any point is given by $(24/25)\,xy$. Find the mass of the lamina. **(M.U. 1990, 97, 2001, 02, 12)**

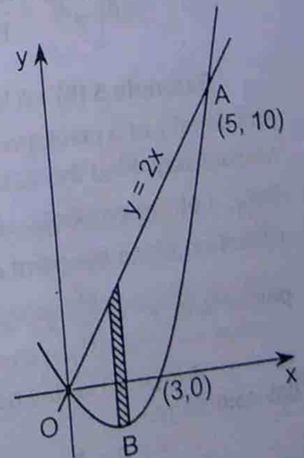


Fig. 11.2

Sol. : The curve $y = x^2 - 3x$ *i.e.* $y + \frac{9}{4} = \left(x - \frac{3}{2}\right)^2$ is a parabola intersecting the x-axis in $x = 0$ and $x = 3$. The line $y = 2x$ intersects this parabola at $x^2 - 3x = 2x$ *i.e.* $x^2 - 5x = 0$ *i.e.* at $x = 0$, $x = 5$. Therefore, points of intersection are (0, 0) and (5, 10). The lamina is the area OAB. The surface density is $\rho = (24/25)\,xy$. Taking the elementary strip parallel to the y-axis, on the strip y varies from $y = x^2 - 3x$ to $y = 2x$ and then x varies from $x = 0$ to $x = 5$.

$\therefore$ Mass of the lamina

$$= \int_0^5\int_{x^2-3x}^{2x}\left(\frac{24}{25}\right)xy\,dx\,dy = \left(\frac{24}{25}\right)\int_0^5 x\left[\frac{y^2}{2}\right]_{x^2-3x}^{2x}dx$$

$$= \left(\frac{24}{50}\right)\int_0^5 \left[4x^3 - x(x^4 - 6x^3 + 9x^2)\right] dx$$

$$= \frac{24}{50}\int_0^5 (-x^5 + 6x^4 - 5x^3)\, dx$$

$$= \frac{24}{50}\left[-\frac{x^6}{6} + \frac{6x^5}{5} - \frac{5x^4}{4}\right]_0^5 = \frac{24}{50}\cdot 5^4\left[-\frac{25}{6} + 6 - \frac{5}{4}\right]$$

$$= \frac{24}{50}\cdot 5^4 \cdot \frac{7}{12} = 175.$$

Example 4 (b) : Find the mass of the lamina bounded by the curves $y^2 = ax$ and $x^2 = ay$ if the density of the lamina at any point varies as the square of its distance from the origin.

(M.U. 1995, 96, 2002, 03, 10, 11, 13)

Sol. : The two curves intersect at $A\,(a, a)$. The lamina is the area $OBACO$. On the curve OCA, $y = \sqrt{ax}$ and on the curve OBA, $y = x^2 / a$. The surface density is given by $\rho = k\,(x^2 + y^2)$. Taking the elementary strip parallel to the y-axis, on the strip y varies from $y = x^2 / a$ to $y = \sqrt{ax}$ and then x varies from $x = 0$ to $x = a$.

$\therefore$ Mass of the lamina

$$= k\int_0^a \int_{x^2/a}^{\sqrt{ax}} (x^2 + y^2)\, dx\, dy = k\int_0^a \left[x^2 y + \frac{y^3}{3}\right]_{x^2/a}^{\sqrt{ax}} dx$$

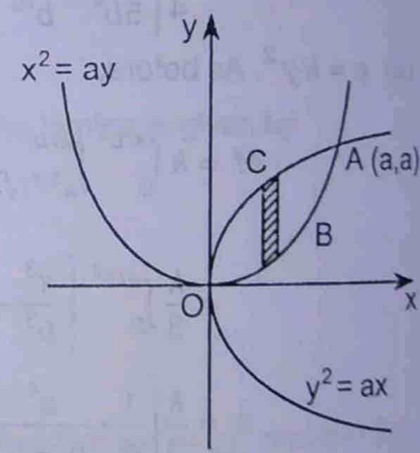


Fig. 11.3

$$= k\int_0^a \left(x^2 \cdot \sqrt{ax} + (ax)\sqrt{ax} - x^2 \cdot \frac{x^2}{a} - \frac{1}{3}\cdot\frac{x^6}{a^3}\right) dx$$

$$= k\int_0^a \left(\sqrt{a}\, x^{5/2} + \frac{a\sqrt{a}}{3}\cdot x^{3/2} - \frac{1}{a}\cdot x^4 - \frac{1}{3a^3}\, x^6\right) dx$$

$$= k\left[\sqrt{a}\,\frac{x^{7/2}}{7/2} + \frac{a\sqrt{a}}{3}\cdot\frac{x^{5/2}}{5/2} - \frac{1}{a}\cdot\frac{x^5}{5} - \frac{1}{3a^3}\cdot\frac{x^7}{7}\right]_0^a$$

$$= k\left[\frac{2}{7}a^4 + \frac{2}{15}a^4 - \frac{a^4}{5} - \frac{a^4}{21}\right] = \frac{6\,ka^4}{35}.$$

Example 5 (b) : A lamina in the form of a parabolic segment of mass M, height h and base $2k$ has density at a point given by $\lambda\, p\, q^3$ per unit area where p, q are distances from the base and axis respectively. Find the value of λ.

(M.U. 1987, 91, 97)

Sol. : Let the parabolic segment be as shown in the figure. Let the equation of the parabola be $y^2 = 4ax$. Since the point $B\,(h, k)$ lies on the parabola $k^2 = 4ah$; $4a = k^2 / h$; the equation of the parabola is $y^2 = \frac{k^2}{h}\, x$.

$\therefore$ If $P\,(x, y)$ is any point on the lamina, then the distances p, q are as shown in the figure on the next page.

$\therefore\ x + p = h$ ...

$\therefore$ The density is $\lambda\, pq^3$ where $p = (h-x)$ and $q = y$. Taking the elementary strip parallel to the y-axis, on the strip y varies from $y = 0$ to $y = k\sqrt{x/h}$ and then x varies from $x = 0$ to $x = h$.

$\therefore$ Mass of the lamina

$$M = 2\int_0^h \int_0^{k\sqrt{x/h}} \lambda\, p\, q^3\, dx\, dy$$

$$= 2\int_0^h \int_0^{k\sqrt{x/h}} \lambda\,(h-x)\,y^3\, dx\, dy$$

$$= 2\lambda \int_0^h (h-x)\left[\frac{y^4}{4}\right]_0^{k\sqrt{x/h}} dx$$

$$= \frac{\lambda}{2}\int_0^h (h-x)\,k^4\,\frac{x^2}{h^2}\,dx = \frac{\lambda k^4}{2h^2}\left[h\frac{x^3}{3} - \frac{x^4}{4}\right]_0^h$$

$$= \frac{\lambda k^4}{2h^2}\left[\frac{h^4}{3} - \frac{h^4}{4}\right] = \frac{\lambda k^4 h^2}{24} \qquad \therefore\ \lambda = \frac{24M}{k^4 h^2}.$$

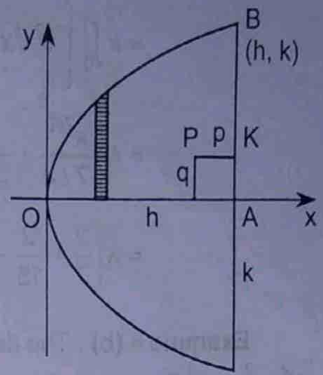


Fig. 11.4

Example 6 (b) : Find the mass of the lamina in the form of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ if the density at any point varies as the product of the distances from the axes of the ellipse.

(M.U. 1993, 2000, 04)

Sol. : The surface density is $\rho = k\,xy$. Consider a strip parallel to the y-axis, y varies from $y = 0$ to $y = b\sqrt{x^2 - a^2}\,/\,a$ and x varies from $x = 0$ to $x = a$ in the first quadrant.

$\therefore$ Mass of the lamina $= 4\iint \rho\, dx\, dy$

$$= 4k\iint xy\, dx\, dy = 4k\int_0^a \int_0^{b\sqrt{a^2-x^2}/a} xy\, dx\, dy$$

$$= 4k\int_0^a x \cdot \left[\frac{y^2}{2}\right]_0^{b\sqrt{a^2-x^2}/a} dx$$

$$= 2k\int_0^a x \cdot \frac{b^2}{a^2}(a^2 - x^2)\, dx = \frac{2kb^2}{a^2}\int_0^a (a^2x - x^3)\, dx$$

$$= \frac{2kb^2}{a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{2kb^2}{a^2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{ka^2b^2}{2}.$$

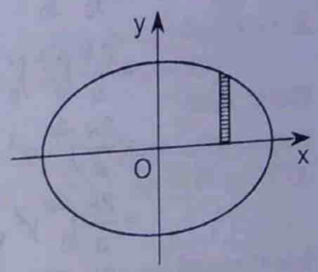


Fig. 11.5

Example 7 (b) : Find by double integration the mass of a thin plate bounded by $y^2 = x$ and $y = x^3$ if the density at any point varies as the square of its distance from the origin. **(M.U. 1997)**

Sol. : The surface density is given by $\rho = k\,r^2 = k(x^2 + y^2)$ at any point. Considering a strip parallel to the y-axis, y varies from $y = x^3$ to $y = \sqrt{x}$ and then x varies from $x = 0$ to $x = h$. Clearly the curves intersect at $A\,(1, 1)$.

$$\therefore\ \text{Mass} = \iint \rho\, dx\, dy = \int_0^1 \int_{x^3}^{\sqrt{x}} k(x^2 + y^2)\, dx\, dy$$

$$\therefore \quad M = k\int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_{x^3}^{\sqrt{x}} dx$$

$$= k\int_0^1 \left[x^2\sqrt{x} + \frac{x^{3/2}}{3} - x^2 \cdot x^3 - \frac{x^9}{3}\right] dx$$

$$= k\left[\frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{3\cdot(5/2)} - \frac{x^6}{6} - \frac{x^{10}}{30}\right]_0^1$$

$$= k\left[\frac{2}{7} + \frac{2}{15} - \frac{1}{6} - \frac{1}{30}\right] = \frac{23}{105}k.$$

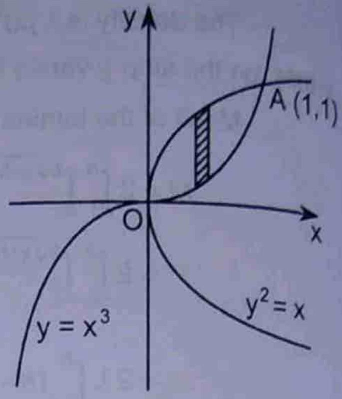


Fig. 11.6

Example 8 (b) : The density at any point of a lamina in the form of area enclosed by the curve $a^2 x^2 = y^3(2a - y)$ varies as k times the square root of the ordinate of the point. Find the mass of lamina.

Sol. : The curve is as shown in the figure.

Density varies as $\sqrt{y}$ *i.e.,* $\rho = k\sqrt{y} = k \cdot \dfrac{y^{3/2}\sqrt{2a-y}}{a}$.

Considering a strip parallel to the x-axis, x varies from $x = 0$ to $x = \dfrac{y^{3/2}\sqrt{2a-y}}{a}$ and then y varies from $y = 0$ to $y = 2a$ in the first quadrant.

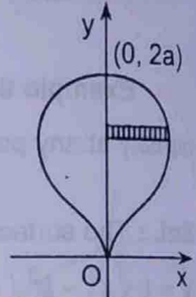


Fig. 11.7

$\therefore$ Mass of the lamina

$$M = 2\int_0^{2a}\int_0^{y^{3/2}\sqrt{2a-y}/a} \frac{k}{a}\cdot y^{3/2}\sqrt{2a-y}\cdot y^{1/2}\,dx\,dy$$

$$= \frac{2k}{a}\int_0^{2a}\int_0^{y^{3/2}\sqrt{2a-y}/a} y^2\sqrt{2a-y}\cdot dx\,dy$$

$$= \frac{2k}{a}\int_0^{2a} y^2\sqrt{2a-y}\cdot[x]_0^{y^{3/2}\sqrt{2a-y}/a}\,dy$$

$$= \frac{2k}{a}\int_0^{2a} y^2\sqrt{2a-y}\cdot y^{3/2}\sqrt{2a-y}\cdot\frac{dy}{a}$$

$$= \frac{2k}{a^2}\int_0^{2a} y^{7/2}(2a-y)\,dy = \frac{2k}{a^2}\int_0^{2a}(2a\cdot y^{7/2} - y^{9/2})\,dy$$

$$= \frac{2k}{a^2}\left[2a\cdot\frac{y^{9/2}}{9/2} - \frac{y^{11/2}}{11/2}\right]_0^{2a}$$

$$= \frac{2k}{a^2}\left[\frac{4}{9}\cdot a(2a)^{9/2} - \frac{2}{11}\cdot(2a)^{11/2}\right]$$

$$= \frac{256}{99}\cdot\frac{k}{a^2}\sqrt{2}\cdot a^{11/2}.$$

Example 9 (b) : A lamina is bounded by the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, the ordinates at the two cusps and the tangent at the vertex. If the density at a point varies as the square of its ordinate, find the mass of the lamina. **(M.U. 1998)**

Sol. : Mass of the lamina by symmetry

$$= 2\int\int_0^y ky^2\,dy\,dx$$

$$= 2\int k\left[\frac{y^3}{3}\right]_0^y dx = \frac{2k}{3}\int y^3\,dx$$

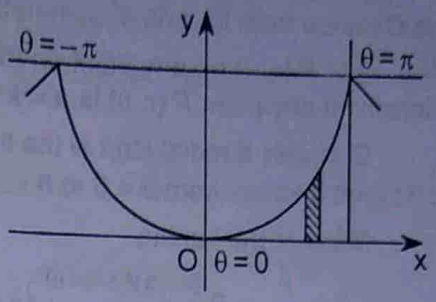


Fig. 11.8

Now, we use parametric equations of the cycloid $y = a(1-\cos\theta)$, $dx = a(1+\cos\theta)\,d\theta$ and in the first quadrant θ varies from $\theta = 0$ to $\theta = \pi$.

$$\therefore \quad \text{Mass} = \frac{2k}{3}\int_0^{\pi} a^3(1-\cos\theta)^3 \cdot a(1+\cos\theta)\,d\theta$$

$$= \frac{2k}{3}a^4\int_0^{\pi} 8\sin^6(\theta/2)\cdot 2\cos^2(\theta/2)\,d\theta$$

Now, put $\theta/2 = t \quad \therefore \quad d\theta = 2\,dt$

$$\therefore \quad \text{Mass} = \frac{32}{3}ka^4\int_0^{\pi/2}\sin^6 t\cos^2 t\cdot 2\,dt$$

$$= \frac{64}{3}ka^4\cdot\frac{1}{2}B\left(\frac{7}{2},\frac{3}{2}\right) = \frac{32}{3}ka^4\cdot\frac{\overline{|7/2}\ \overline{|3/2}}{\overline{|5}}$$

$$= \frac{32}{3}ka^4\cdot\frac{(5/2)(3/2)(1/2)\ \overline{|1/2}\ (1/2)\ \overline{|1/2}}{4\cdot 3\cdot 2\cdot 1}$$

$$= \frac{5}{12}ka^4\pi.$$

Type II : Polar Coordinates

Example 1 (b) : The density of a uniform circular lamina of radius a varies as the square of its distance from a fixed point on the circumference of the circle. Find the mass of the lamina.

(M.U. 2008)

Sol. : Let the fixed point on the circumference of the circle be the origin and the diameter through it be the x-axis. Then the polar equation of the circle is $r = 2a\cos\theta$ [See Ex. 13, page 9-35 and Fig. 9.37 (a), page 9-36]. The density at any point $P(r,\theta)$ is $= kr^2$.

Consider a radial strip in the first quadrant. On this strip r varies from $r = 0$ to $r = 2a\cos\theta$ and then θ varies from $\theta = 0$ to $\theta = \pi/2$.

Mass of the lamina

$$= 2\int_0^{\pi/2}\int_0^{2a\cos\theta}(kr^2)\,r\,dr\,d\theta$$

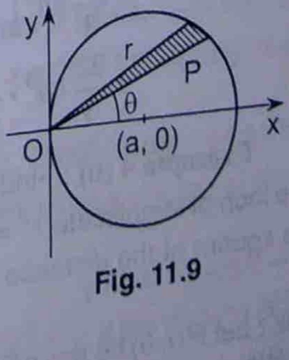


Fig. 11.9

$$= 2k\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{k}{2}\int_0^{\pi/2} 16a^4\cos^4\theta\,d\theta$$

$$= 8ka^4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{2}ka^4\pi.$$

Example 2 (b) : The density at any point of a cardioid $r = a(1 + \cos\theta)$ varies as the square of its distance from its axis of symmetry. Find its mass. **(M.U 1995, 2002, 04, 09)**

Sol. : Let $P(r, \theta)$ be any point on the given cardioid. The distance of P from the axis is $r \sin\theta$. The density at any point $P(r, \theta)$ is $a = k r^2 \sin^2\theta$.

Consider a radial strip in the first quadrant. On this strip r varies from $r = 0$ to $r = a(1 + \cos\theta)$ and then θ varies from $\theta = 0$ to $\theta = \pi$.

Mass of the lamina

$$= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} (k r^2 \sin^2\theta)\, r\, dr\, d\theta$$

$$= 2k\int_0^{\pi} \sin^2\theta \left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{k}{2} a^4 \int_0^{\pi} \sin^2\theta (1 + \cos\theta)^4\, d\theta$$

$$= \frac{k a^4}{2}\int_0^{\pi}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2\left(2\cos^2\frac{\theta}{2}\right)^4 d\theta$$

$$= 32 k a^4 \int_0^{\pi} \sin^2\frac{\theta}{2}\cos^{10}\frac{\theta}{2}\, d\theta$$

$$= 64 k a^4 \int_0^{\pi/2} \sin^2 t \cos^{10} t\, dt \qquad \left[\text{where } \frac{\theta}{2} = t\right]$$

$$= 64 k a^4 \cdot \frac{1 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} \qquad \text{[By (25), page 6-34]}$$

$$= 64 k a^4 \cdot \frac{21}{64 \cdot 32} \cdot \pi = \frac{21}{32} k a^4 \pi.$$

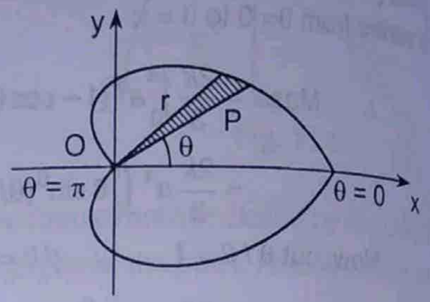


Fig. 11.10

Example 3 (b) : Find the mass of the lamina in the form of a cardioid $r = a(1 + \cos\theta)$ if the density of mass at a point varies as the distance from the pole. **(M.U. 1992, 2000, 08)**

Sol. : As above (See Fig. 10.9)

$$\text{Mass} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} (k r)\, r\, dr\, d\theta = 2k\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{2k}{3}\int_0^{\pi} a^3 (1 + \cos\theta)^3\, d\theta = \frac{16}{3} k a^3 \int_0^{\pi} \cos^6\frac{\theta}{2}\, d\theta \qquad \left[\text{Put } \frac{\theta}{2} = t\right]$$

$$= \frac{16}{3} k a^3 \int_0^{\pi/2} \cos^6 t \cdot 2\, dt$$

$$= \frac{32}{3} k a^3 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5}{3} k a^3 \pi.$$

Example 4 (b) : Find the mass of a plate in the form of one loop of lemniscate $r^2 = a^2 \cos 2\theta$ if the density varies as the square of the distance from the pole.

(M.U. 2002, 05, 09)

Sol. : Let $P(r, \theta)$ be any point on the Lemniscate. The density at P is $\rho = k r^2$. Consider a radial strip in the first quadrant.

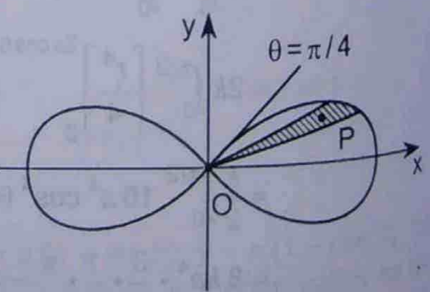


Fig. 11.11

On this strip r varies from $r = 0$ to $r = a\sqrt{\cos 2\theta}$ and θ varies from $\theta = 0$ to $\theta = \pi/4$.

$$\therefore \quad \text{Mass} = 2\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} k r^2 \cdot r \, dr \, d\theta = 2k\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r^3 \cdot dr \, d\theta$$

$$= 2k\int_0^{\pi/4}\left[\frac{r^4}{4}\right]_0^{a\sqrt{\cos 2\theta}} d\theta = \frac{k}{2}\int_0^{\pi/4} a^4 \cos^2 2\theta \cdot d\theta$$

Put $2\theta = t \quad \therefore \quad d\theta = dt/2$. When $\theta = 0$, $t = 0$; when $\theta = \pi/4$, $t = \pi/2$.

$$\therefore \quad I = \frac{ka^4}{2}\int_0^{\pi/2}\cos^2 t \cdot \frac{dt}{2} = \frac{ka^4}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{ka^4\pi}{16}.$$

EXERCISE - I

Solve the following examples : Class (b) : 6 Marks

Type I

1. Find the mass of the lamina bounded by the curve $y^2 = x^3$ and the line $y = x$, if the density at a point varies as the distance of the point from the x-axis.
(See. Fig. 11.1, page 11-1 with $a = 1$, $b = 1$) **(M.U. 2003)** [**Ans. :** $k/24$]

2. Find the mass of the lamina bounded by $4y^2 = x^3$ and $y = x$, if the density at any point varies as the distance of the point from the x-axis.
(See. Fig. 11.1, page 11-1 with $a = 4$, $b = 1$) **(M.U. 2001)** [**Ans. :** $8k/3$]

3. Find the mass of the lamina bounded by $ay^2 = x^3$ and the line $y = x$ if the density at any point varies as the distance of the point from the x-axis.
(See. Fig. 11.1, page 11-1) **(M.U. 2002)** [**Ans. :** $ka^3/24$]

4. Find the mass of the lamina bounded by the curve $16y^2 = x^3$ and the line $2y = x$, if the density at a point varies as the square of the distance of the point from the x-axis.
(See. Fig. 11.1, page 11-1 with $a = 16$, $b = 2$) **(M.U. 1999, 2003, 05, 10)** [**Ans. :** $8k/11$]

5. The boundaries of a plate can be given by $x = 0$, $y = 0$, $x = 1$ and $y = e^x$. If the density at any point varies as the square of its distance from the origin, find the mass of the plate.
[**Ans. :** $k\left(e + \frac{e^2}{9} - \frac{19}{9}\right)$]
(See. Fig. 10.39, page 10-22)

6. A lamina is bounded by the curves $y = x^2 - 3x$ and $y = 2x$. If the density at any point $P(x, y)$ is $k\,xy$, find the mass of the lamina.
(See. Fig. 11.2, page 11-3) **(M. U. 1994)** [**Ans. :** $4375\,k/24$]

7. Find the mass of the lamina bounded by $x^2 + 2y - 4 = 0$ and the x-axis, if the density at any point is k times its distance from the x-axis.
(See similar Fig. 9.5, page 9-8) **(M.U. 1992, 2009)** [**Ans. :** $64\,k/15$]

8. Find the mass of the lamina of the region included between the curves $y = \log x$, $y = 0$, $x = 2$, having uniform density. [**Ans. :** $2\log 2 - 1$]
(See similar Fig. 15.15(b), page 15-6)

9. Find the mass of the lamina bounded by the curves $y^2 = x$ and $x^2 = y$, if the density of the lamina at any point varies as the square of its distance from the origin. [**Ans. :** $\frac{6}{35}k$]
(See Fig. 11.3, page 11-3, $a = 1$)

Type II

1. The density at any point of a uniform circular lamina of radius a varies as its distance from a fixed point on the circumference of the circle. Find the mass of the lamina.

(M.U. 1990) [**Ans. :** $(32/9)\,ka^3$]

(See. Fig. 11.9, page 11-6)

(**Hint :** Circle is $r = 2a\cos\theta$. $M = 2\int_0^{\pi/2}\int_0^{2a\cos\theta} kr \cdot r\,d\theta\,dr$)

2. If the density at any point of the curve $x = a(\theta + \sin\theta)$ and $y = a(1+\cos\theta)$ varies as the distance from the x-axis, find the mass of the lamina bounded by the curve, ordinates at the two cusps and the tangent at the vertex.

(M.U. 1991) [**Ans. :** $3k\pi a^3/2$]

(See. Fig. 15.50, page 15-17)

3. Find the mass of a plate in the form of a cardioid $r = a(1-\cos\theta)$ if the density at any point of the plate varies as its distance from the pole.

(M.U. 2007) [**Ans. :** $5k\pi a^3/3$]

(See. Fig. 11.11, page 11-7)

(**Hint :** See solved Ex. 4, page 11-9, $M = 2\int_0^{\pi}\int_0^{a(1-\cos\theta)} kr \cdot r\,d\theta\,dr$)

4. The density at any point of a non-uniform circular lamina of radius a varies as (i) the distance and (ii) the cube of its distance of from a fixed point on the circumference of the circle. Find the mass of the lamina in each case.

[**Ans. : (i)** $32\,ka^3/9$, **(ii)** $512\,ka^5/75$]

(See. Fig. 11.9, page 11-6)

5. Find the mass of a plate in the form of Lemniscate $r^2 = a^2\sin 2\theta$, whose density varies as the square of its distance from the pole.

(M.U. 2005) [**Ans. :** $ka^4\pi/16$]

(See. Fig. 15.45, page 15-16)

6. Find the mass of the plate which is inside the circle $r = 2a\cos\theta$ and outside the circle $r = a$ if the density varies as the distance from the pole.

[**Ans. :** $\frac{2ka^3}{9}(9\sqrt{3}-\pi)$]

(See. Fig. 10.31, page 10-16)

EXERCISE - II

Solve the following examples : Class (a) : 3 or 4 Marks

1. Find the mass of the lamina bounded by the line $y = ax$, the x-axis and the line $x = a$, if the density at a point varies as the distance of the point from the x-axis.

(See Fig. 11.12) [**Ans. :** $\frac{ka^5}{6}$]

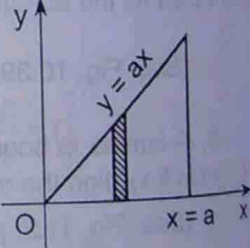


Fig. 11.12

2. Find the mass of the lamina bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$, if the density at a point varies as the product of the distances of the point from the axes.

(See Fig. 11.13) [**Ans. :** $\frac{ka^2b^2}{4}$]

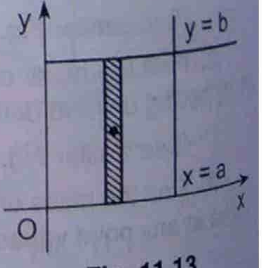


Fig. 11.13

3. Find the mass of the lamina bounded by the parabola $y^2 = ax$, the x-axis and the line $x = a$, if the density varies as the product of the distance of the point from the axes.

(See Fig. 11.4, page 11-3) [**Ans. :** $\frac{k a^4}{6}$]

4. Find the mass of a circular lamina of radius a, if the density at a point varies as its distance from a fixed point on its circumference.

(See Fig. 11.9, page 11-6) [**Ans. :** $\frac{32 k a^3}{9}$]

Summary

1. $M = \iint f(x, y)\, dy\, dx,$ $\quad \rho = f(x, y)$
2. $M = \iint f(r, \theta)\, r\, dr\, d\theta,$ $\quad \rho = f(r, \theta)$

❋ ❋ ❋

CHAPTER 12

Triple Integral

1. Introduction

In this chapter we shall learn how the idea of double integration can be extended to define integrals of functions of three independent variables. But before that we shall learn two more coordinate systems viz. spherical coordinate system and cylindrical coordinate system.

2. Coordinate Systems

(a) Cartesian Coordinate System

Let O be a point in space and consider three mutually perpendicular lines $x'ox$, $y'oy$, $z'oz$ through O. As you already know, we call O the origin and three lines the x-axis, y-axis, and z-axis respectively. You also know that the system is a **right handed system.** The planes determined by these lines are called xy - plane, yz - plane zx - plane as you know. These planes divide the space in eight parts called **octants**.

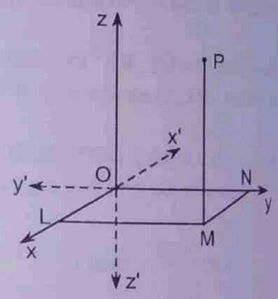


Fig. 12.1

To find the coordinates of a point P we draw perpendicular PM to the xy - plane, from M draw a perpendicular ML to the x-axis and MN perpendicular to the y-axis (It is enough to draw a perpendicular MN to y-axis or ML to x-axis only). Then if x, y, z are the coordinates of the point P, then

$$x = OL = MN; \qquad y = ON = ML; \qquad z = PM.$$

(b) Spherical Polar Coordinate System

If as before PM is perpendicular to the xy plane, OM makes an angle Φ with the x-axis and OP makes an angle θ with the z-axis and if $OP = r$ then (r, θ, Φ) are called **spherical coordinates of P.** Clearly,

$$PM = OP \cos\theta = r\cos\theta$$
$$OM = OP \sin\theta = r\sin\theta$$
$$OL = OM\cos\Phi = r\sin\theta\cos\Phi$$
$$ON = OM\sin\Phi = r\sin\theta\sin\Phi$$

Hence, we have the relations

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\Phi \\ y &= r\sin\theta\sin\Phi \\ z &= r\cos\theta. \end{aligned}}$$

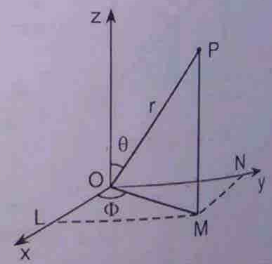


Fig. 12.2

Also, $r^2 = x^2 + y^2 + z^2$, $\dfrac{y}{x} = \tan\Phi$, $\dfrac{\sqrt{x^2+y^2}}{z} = \tan\theta.3$

$$\therefore \quad \Phi = \tan^{-1}\frac{y}{x}, \qquad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right).$$

(c) Cylindrical Coordinate System

If $PM = z$, $OM = \rho$ and OM makes angle Φ with the x-axis then (z, ρ, Φ) are called cylindrical coordinates. Clearly,

$$\boxed{\begin{aligned} x &= \rho \cos \Phi \\ y &= \rho \sin \Phi \\ z &= z \end{aligned}}$$

Also, $\rho = \sqrt{x^2 + y^2}$; $\phi = \tan^{-1}\left(\dfrac{y}{x}\right)$; $z = z$.

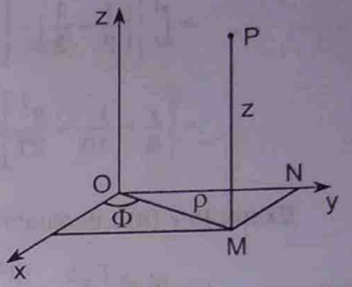


Fig. 12.3

3. Triple Integral

The concept of double integral of a function $f(x, y)$ over a given region in xy - plane can be extended a step further to define triple integral.

Consider, a function $f(x, y, z)$ defined over a finite region V of three dimensional space. Let the region be sub-divided into n sub-intervals δV_1, δV_2, ... δV_n. Let $P(x_r, y_r, z_r)$ be a point in the r-th sub-interval. We now form the sum

$$\sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r$$

The limit of the above, when it exists, as n tends to infinity and the volume of each sub-region tends to zero is called the **triple integral of $f(x, y, z)$ over the region V** and is denoted by

$$\iiint_V (x, y, z)\, dv$$

Thus, $$\iiint_V f(x, y, z)\, dV = \lim_{\substack{n\to\infty \\ \delta v \to 0}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r$$

(a) Evaluation of Triple Integral

The triple integral can be evaluated by successive single integrals as follows.

$$\int_{x=a}^{x=b} \int_{y=\phi_1(x)}^{y=\phi_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

where the integration w.r.t. z is performed first by treating x and y constant. Then the integration w.r.t. y is performed treating x constant and finally the integration w.r.t. x is performed.

Solved Examples : Class (b) : 6 Marks

Type I : Evaluation of Triple Integrals in Cartesian Coordinates

Example 1 (b) : Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$. **(M.U. 2013)**

Sol. : We have

$$I=\int_0^1\int_{y^2}^1\int_0^{1-x} x\,dz\,dx\,dy=\int_0^1\int_{y^2}^1 x\left[z\right]_0^{1-x}dx\,dy$$

$$=\int_0^1\int_{y^2}^1 (x-x^2)\,dx\,dy=\int_0^1\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{y^2}^1 dy$$

$$=\int_0^1\left[\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{y^4}{2}-\frac{y^6}{3}\right)\right]dy=\int_0^1\left(\frac{1}{6}-\frac{y^4}{2}+\frac{y^6}{3}\right)dy$$

$$=\left[\frac{y}{6}-\frac{y^5}{10}+\frac{y^7}{21}\right]_0^1=\left[\frac{1}{6}-\frac{1}{10}+\frac{1}{21}\right]=\frac{4}{35}.$$

Example 2 (b) : Evaluate $\int_0^2\int_1^2\int_0^{yz} xyz\,dx\,dy\,dz.$ **(M.U. 2012)**

Sol. : $$I=\int_0^2\int_1^2\left[\frac{x^2}{2}yz\right]_0^{yz}dy\,dz=\int_0^2\int_1^2\frac{y^3z^3}{2}\,dy\,dz$$

$$=\frac{1}{2}\int_0^2\left\{\left[\frac{y^4}{4}\right]z^3\right\}_1^2 dz=\frac{1}{8}\int_0^2(16-1)\,z^3\,dz$$

$$=\frac{15}{8}\int_0^2 z^3\,dz=\frac{15}{8}\left[\frac{z^4}{4}\right]_0^2=\frac{15}{8}\cdot\frac{16}{4}=\frac{15}{2}.$$

Example 3 (b) : Evaluate $\int_0^a\int_0^a\int_0^a (yz+zx+xy)\,dz\,dy\,dx.$

Sol. : $$I=\int_0^a\int_0^a\left[y\frac{z^2}{2}+x\,\frac{z^2}{2}+xyz\right]_0^a dy\,dx$$

$$=\int_0^a\int_0^a\left(\frac{a^2y}{2}+\frac{a^2x}{2}+axy\right)dy\,dx=\int_0^a\left[\frac{a^2y^2}{4}+\frac{a^2xy}{2}+\frac{axy^2}{2}\right]_0^a dx$$

$$=\int_0^a\left(\frac{a^4}{4}+\frac{a^3}{2}x+\frac{a^3}{2}x\right)dx=\int_0^a\left(\frac{a^4}{4}+a^3x\right)dx$$

$$=\left[\frac{a^4}{4}x+a^3\,\frac{x^2}{2}\right]_0^a=\frac{a^5}{4}+\frac{a^5}{2}=\frac{3}{4}a^5.$$

Example 4 (b) : Find $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{dz\,dy\,dx}{\sqrt{(1-x^2-y^2-z^2)}}.$ **(M.U. 1993, 99, 2011)**

Sol. : $$I=\int_0^1\int_0^{\sqrt{1-x^2}}\left[\sin^{-1}\left\{\frac{z}{\sqrt{1-x^2-y^2}}\right\}\right]_0^{\sqrt{1-x^2-y^2}}dy\,dx$$

$$=\int_0^1\int_0^{\sqrt{1-x^2}}\frac{\pi}{2}\,dy\,dx=\frac{\pi}{2}\int_0^1\left[\,y\,\right]_0^{\sqrt{1-x^2}}dx=\frac{\pi}{2}\int_0^1\sqrt{1-x^2}\;dx$$

$$\therefore \ I = \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{\pi^2}{8}.$$

Example 5 (b) : Evaluate $\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2 \, dz \, dy \, dx.$

Sol. :
$$I = \int_0^a \int_0^{a-x} \left[x^2 z\right]_0^{a-x-y} dy\,dx = \int_0^a \int_0^{a-x} x^2(a-x-y)\,dy\,dx$$

$$= \int_0^a \left[x^2\,(a-x)\,y - x^2\,\frac{y^2}{2}\right]_0^{a-x} dx = \frac{1}{2}\int_0^a x^2(a-x)^2\,dx$$

$$= \frac{1}{2}\int_0^a (a^2x^2 - 2ax^3 + x^4)\,dx = \frac{1}{2}\left[a^2\,\frac{x^3}{3} - 2a\,\frac{x^4}{4} + \frac{x^5}{5}\right]_0^a = \frac{a^5}{60}.$$

Example 6 (b) : Evaluate $\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx\,.$

(M.U. 1997, 99, 2002, 03, 05, 12, 15)

Sol. :
$$I = \int_0^{\log 2} \int_0^{x} e^{x+y}\left[e^z\right]_0^{x+y} dy\,dx = \int_0^{\log 2} \int_0^{x} e^{x+y}(e^{x+y} - 1)\,dy\,dx$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^{2(x+y)} - e^{(x+y)}\right] dy\,dx = \int_0^{\log 2}\left[e^{2x}\cdot\frac{e^{2y}}{2} - e^x\cdot e^y\right]_0^x dx$$

$$= \int_0^{\log 2}\left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right)dx = \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^x\right]_0^{\log 2}$$

$$= \left(\frac{16}{8} - \frac{4}{2} - \frac{4}{4} + 2\right) - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1\right) = 1 - \frac{3}{8} = \frac{5}{8}.$$

Example 7 (b) : Evaluate $\int_0^2 \int_0^x \int_0^{2x+2y} e^{\,x+y+z}\,dz\,dy\,dx.$ **(M.U. 1997, 2003)**

Sol. :
$$I = \int_0^2 \int_0^x \int_0^{2x+2y} e^{\,x+y}\,(e^z)\,dz\,dy\,dx = \int_0^2 \int_0^x e^{\,x+y}\left[e^z\right]_0^{2x+2y} dy\,dx$$

$$= \int_0^2 \int_0^x e^{\,x+y}\left[e^{\,2x+2y} - e^0\right] dy\,dx = \int_0^2 \int_0^x \left[e^{\,3x+3y} - e^{x+y}\right] dy\,dx$$

$$= \int_0^2 \left[e^{3x}\cdot\left(\frac{e^{3y}}{3}\right) - e^x\cdot(e^y)\right]_0^x dx = \int_0^2 \left\{\frac{e^{3x}}{3}\left[e^{3x} - e^0\right] - e^x\left[e^x - e^0\right]\right\} dx$$

$$= \int_0^2 \left[\frac{e^{6x} - e^{3x}}{3} - e^{2x} + e^x\right] dx = \frac{1}{3}\left[\frac{e^{6x}}{6} - \frac{e^{3x}}{3}\right]_0^2 - \left[\frac{e^{2x}}{2}\right]_0^2 + \left[e^x\right]_0^2$$

$$= \frac{1}{3}\left[\frac{e^{12}}{6} - \frac{e^6}{3} - \frac{1}{6} + \frac{1}{3}\right] - \frac{1}{2}\left[e^4 - 1\right] + \left[e^2 - 1\right]$$

$$= \frac{e^{12}}{18} - \frac{e^6}{9} - \frac{e^4}{2} + e^2 - \frac{4}{9}.$$

Example 8 (b) : Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz.$ **(M.U. 2000, 02, 10)**

Sol. :

$$I = \int_{-1}^{1}\int_{0}^{z}\left[(x+z)\,y+\frac{y^2}{2}\right]_{x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[(x+z)^2+\frac{(x+z)^2}{2}-(x^2-z^2)-\frac{(x-z)^2}{2}\right]dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[\frac{3}{2}(x+z)^2-x^2+z^2-\frac{(x-z)^2}{2}\right]dx\,dz$$

$$= \int_{-1}^{1}\left[\frac{3}{2}\cdot\frac{(x+z)^3}{3}-\frac{x^3}{3}+z^2x-\frac{(x-z)^3}{6}\right]_0^z dz$$

$$= \int_{-1}^{1}\left(4z^3-\frac{z^3}{3}+z^3-\frac{z^3}{2}-\frac{z^3}{6}\right)dz = \int_{-1}^{1} 4z^3\,dx = \left[z^4\right]_{-1}^{1} = 0.$$

Examle 9 (b) : Evaluate $\int_{1}^{e}\int_{1}^{\log y}\int_{1}^{e^x}\log z\,dz\,dx\,dy$. **(M.U. 2010, 12, 16)**

Sol. :

$$I = \int_{1}^{e}\int_{1}^{\log y}\int_{1}^{e^x}\log z\cdot 1\cdot dz\,dx\,dy$$

$$= \int_{1}^{e}\int_{1}^{\log y}\left[\log z\cdot z-\int z\cdot\frac{1}{z}\,dz\right]_1^{e^x} dx\,dy \qquad \text{[Integration by parts]}$$

$$= \int_{1}^{e}\int_{1}^{\log y}\left[z\log z-z\right]_1^{e^x} dx\,dy$$

$$= \int_{1}^{e}\int_{1}^{\log y}\left[e^x\log e^x-e^x+1\right]dx\,dy$$

$$= \int_{1}^{e}\int_{1}^{\log y}\left[e^x x-e^x+1\right]dx\,dy \qquad [\because\ \log e^x = x]$$

$$= \int_{1}^{e}\left[x\,e^x-e^x-e^x+x\right]_1^{\log y} dy \qquad \text{[Integration by parts]}$$

$$= \int_{1}^{e}\left[\log y\,e^{\log y}-2\,e^{\log y}+\log y+e-1\right]dy$$

$$= \int_{1}^{e}(y\log y-2y+\log y+e-1)\,dy \qquad [\because\ e^{\log y} = y]$$

$$= \left[\log y\cdot\frac{y^2}{2}-\int\frac{y^2}{2}\cdot\frac{1}{y}\,dy-y^2+\log y\cdot y-\int y\cdot\frac{1}{y}\,dy+ey-y\right]_1^e$$

$$= \left[\log y\cdot\frac{y^2}{2}-\frac{y^2}{4}-y^2+\log y\cdot y-y+ey-y\right]_1^e$$

$$= \left[\left(\log e\cdot\frac{e^2}{2}-\frac{e^2}{4}-e^2+\log e\cdot e-2e+e^2\right)-\left(0-\frac{1}{4}-1+0-2+e\right)\right]$$

$$= \frac{1}{4}(e^2-8e+13). \qquad [\because\ \log e = 1]$$

Example 10 (b) : Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3}\, dz\, dy\, dx.$

(M.U. 1997, 2006, 13, 15)

Sol. :

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3}\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{2(x+y+z+1)^2}\right]_0^{1-x-y} dy\, dx$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[\frac{1}{(x+y+1+1-x-y)^2} - \frac{1}{(x+y+1)^2}\right] dy\, dx$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x+y+1)^2}\right] dy\, dx = -\frac{1}{2}\int_0^1 \left[\frac{y}{4} + \frac{1}{x+y+1}\right]_0^{1-x} dx$$

$$= -\frac{1}{2}\int_0^1 \left[\left(\frac{1-x}{4} + \frac{1}{x+1+1-x}\right) - \left(0 + \frac{1}{x+1}\right)\right] dx$$

$$= -\frac{1}{2}\int_0^1 \left(\frac{3}{4} - \frac{x}{4} - \frac{1}{x+1}\right) dx = -\frac{1}{2}\left[\frac{3}{4}x - \frac{x^2}{8} - \log(x+1)\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{3}{4} - \frac{1}{8} - \log 2\right] = \frac{1}{2}\left(\log 2 - \frac{5}{8}\right).$$

Example 11 (b) : Evaluate $\int_0^{\pi} 2\, d\theta \int_0^{a(1+\cos\theta)} r\, dr \int_0^h \left[1 - \frac{r}{a(1+\cos\theta)}\right] dz.$

$$I = \int_0^{\pi} 2\, d\theta \int_0^{a(1+\cos\theta)} r\, dr \left[\left(1 - \frac{r}{a(1+\cos\theta)}\right) z\right]_0^h$$

$$= \int_0^{\pi} 2\, d\theta \int_0^{a(1+\cos\theta)} h \left[1 - \frac{r}{a(1+\cos\theta)}\right] r\, dr$$

$$= \int_0^{\pi} 2\, d\theta \left[h\left(\frac{r^2}{2} - \frac{r^3}{3a(1+\cos\theta)}\right)\right]_0^{a(1+\cos\theta)}$$

$$= \int_0^{\pi} 2h \left[\frac{a^2(1+\cos\theta)^2}{2} - \frac{a^3(1+\cos\theta)^3}{3a(1+\cos\theta)}\right] d\theta$$

$$= \int_0^{\pi} a^2 \frac{h}{3}(1+\cos\theta)^2\, d\theta = \frac{a^2 h}{3}\int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \frac{a^2 h}{3}\int_0^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2 h}{3}\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{a^2 h}{3}\cdot\frac{3\pi}{2} = \frac{\pi a^2}{2} h.$$

Example 12 (b) : Evaluate $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{(a^2-r^2)/a} r \, dz \, dr \, d\theta$.

Sol. :
$$I = \int_0^{\pi/2} \int_0^{a\sin\theta} \left[r z \right]_0^{(a^2-r^2)/a} dr \, d\theta = \int_0^{\pi/2} \int_0^{a\sin\theta} \frac{r(a^2 - r^2)}{a} \, dr \, d\theta$$
$$= \frac{1}{a}\int_0^{\pi/2} \int_0^{a\sin\theta} (a^2 r - r^3) \, dr \, d\theta = \frac{1}{a}\int_0^{\pi/2} \left[\frac{a^2 r^2}{2} - \frac{r^4}{4} \right]_0^{a\sin\theta} d\theta$$
$$= \frac{1}{a}\int_0^{\pi/2} \left(\frac{a^4 \sin^2\theta}{2} - \frac{a^4 \sin^4\theta}{4} \right) d\theta = \int_0^{\pi/2} \left(\frac{a^3}{2}\sin^2\theta - \frac{a^3}{4}\sin^4\theta \right) d\theta$$
$$= \frac{a^3}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{a^3}{4} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{a^3}{8}\pi - \frac{3a^3}{64}\pi = \frac{5a^3}{64}\pi.$$

Example 13 (b) : Evaluate $\int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz \, dy \, dx$.

Sol. :
$$I = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left[z \right]_{x^2+3y^2}^{8-x^2-y^2} dy \, dx$$
$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left[2(4 - x^2) - 4y^2 \right] dy \, dx$$
$$= \int_{-2}^{2} \left[2(4 - x^2)\, y - 4 \cdot \frac{y^3}{3} \right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} dx = \int_{-2}^{2} \frac{4\sqrt{2}}{3} \cdot (4 - x^2)^{3/2} \, dx$$

Put $x = 2\sin\theta$ $\quad \therefore \; dx = 2\cos\theta \, d\theta$
$$\therefore \; I = \frac{64\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta = \frac{128\sqrt{2}}{3} \int_0^{\pi/2} \cos^4\theta \, d\theta$$
$$= \frac{128\sqrt{2}}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 8\sqrt{2}\,\pi.$$

Example 14 (b) : Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} (x^2 + y^2 + z^2) \, dx \, dy \, dz$.

Sol. : We transform the given integral to spherical polar coordinates *i.e.* we put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $dx\,dy\,dz = r^2 \sin\theta \, dr \, d\theta \, d\phi$.

Now $x^2 + y^2 + z^2$ transforms to r^2.

Since we have to integrate in the first octant only r will vary from 0 to a, ϕ will vary from 0 to $\pi/2$ and θ will vary from 0 to $\pi/2$.

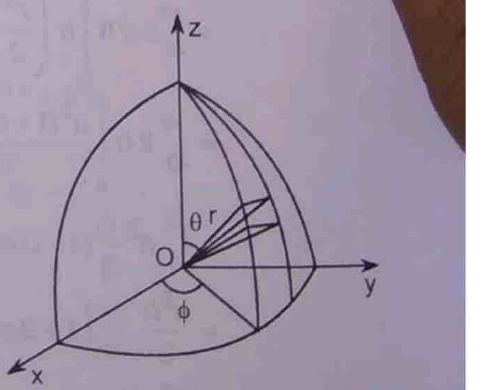


Fig. 12.4

$$\therefore \; I = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a r^2 \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$
$$= \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta \, d\theta \int_0^a r^4 \, dr$$
$$= \left[\phi \right]_0^{\pi/2} \left[-\cos\theta \right]_0^{\pi/2} \left[\frac{r^5}{5} \right]_0^a = \frac{\pi a^5}{10}.$$

Example 15 (b) : Evaluate $\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$.

Sol. : We transform the integral from cartesian to spherical polar coordinates because of the term $x^2+y^2+z^2$.

Since x, y, z all vary from 0 to ∞, the region of integration is the first octant in which θ, ϕ vary from 0 to $\pi/2$ and r varies from 0 to ∞.

$$\therefore \quad I=\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\int_{r=0}^{\infty}\frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{(1+r^2)^2}$$

$$=\int_0^{\pi/2}\sin\theta\,d\theta\int_0^{\pi/2}d\phi\int_0^{\infty}\frac{r^2}{(1+r^2)^2}dr$$

Now, $\int_0^{\pi/2}\sin\theta\,d\theta=[-\cos\theta]_0^{\pi/2}=1$; $\int_0^{\pi/2}d\phi=[\phi]_0^{\pi/2}=\frac{\pi}{2}$

For the last integral we put $r=\tan\theta$, $dr=\sec^2\theta\,d\theta$.

When $r=0$, $\theta=0$; when $r=\infty$, $\theta=\pi/2$.

$$\therefore \quad \int_0^{\infty}\frac{r^2}{(1+r^2)^2}dr=\int_0^{\pi/2}\frac{\tan^2\theta}{\sec^4\theta}\cdot\sec^2\theta\,d\theta$$

$$=\int_0^{\pi/2}\sin^2\theta\,d\theta=\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$$

$$\therefore \quad I=1\cdot\frac{\pi}{2}\cdot\frac{\pi}{4}=\frac{\pi^2}{8}.$$

EXERCISE - I

Evaluate the following integrals : Class (b) : 6 Marks

1. $\int_0^1 dx\int_0^2 dy\int_1^2 x^2yz\,dz$. [**Ans. :** 1]
2. $\int_0^2\int_0^z\int_0^{yz} xyz\,dx\,dy\,dz$. [**Ans. :** 4]
3. $\int_0^2\int_0^x\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz$. **(M.U. 1990)** [**Ans. :** $\frac{e^8}{8}+e^2-\frac{3}{4}e^4-\frac{3}{8}$]
4. $\int_0^{\log 2}\int_0^x\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$. **(M.U. 2002, 15)** [**Ans. :** $\frac{1}{9}(24\log 2-19)$]
5. $\int_0^a\int_0^a\int_0^a(y^2z^2+z^2x^2+x^2y^2)\,dx\,dy\,dz$. [**Ans. :** $\frac{a^7}{3}$]
6. $\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^y z^2\,dx\,dy\,dz$. [**Ans. :** $\frac{a^5}{5}$]
7. $\int_0^a\int_0^{a-x}\int_0^{a-x-y}(x^2+y^2+z^2)\,dz\,dy\,dx$. [**Ans. :** $\frac{a^5}{20}$]
8. $\int_0^1\int_0^{1-x}\int_0^{x+y} e^z\,dx\,dy\,dz$. [**Ans. :** $\frac{1}{2}$]
9. $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dz\,dx\,dy$. [**Ans. :** 8π]

10. $\int_0^a \int_0^x \int_0^{\sqrt{x+y}} z \; dx\,dy\,dz$. [**Ans.** : $\frac{a^3}{4}$]

11. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \; dx\,dy\,dz$. [**Ans.** : $\frac{a^6}{48}$]

Solved Examples : Class (b) : 6 Marks

Type II : When the region of integration is bounded by planes

Example 1 (b) : Evaluate the integral $\iiint\limits_V xyz^2\, dv$ over the region bounded by the planes $x = 0$, $x = 1$, $y = -1$, $y = 2$, $z = 0$, $z = 3$.

Sol. : If we consider a horizontal strip then on this strip y varies form $y = -1$ to $y = 2$, when this strip moves parallel to itself in the xy-plane x varies from $x = 0$ to $x = 1$. Thus we get the area of the bottom. When it moves vertically from $z = 0$ to $z = 3$, we get the required volume.

$$\therefore\ I = \int_{z=0}^{3} \int_{x=0}^{1} \int_{y=-1}^{2} xyz^2\, dy\, dx\, dz$$

$$= \int_{z=0}^{3} \int_{x=0}^{1} xz^2 \left[\frac{y^2}{2}\right]_{-1}^{2} dx\, dz$$

$$= \int_{z=0}^{3} \int_{x=0}^{1} \frac{xz^2}{2}[4-1]\, dx\, dz$$

$$= \frac{3}{2}\int_{z=0}^{3} z^2 \left[\frac{x^2}{2}\right]_0^1 dz = \frac{3}{4}\int_0^3 z^2\, dz$$

$$= \frac{3}{4}\left[\frac{z^3}{3}\right]_0^3 = \frac{3}{4}\cdot 9 = \frac{27}{4}.$$

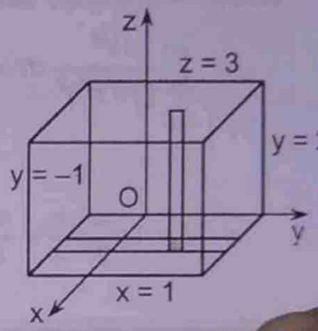


Fig. 12.5

Example 2 (b) : Evaluate $\iiint dx\, dy\, dz$ over the volume of the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Sol. : We first substitute $x = au$, $y = bv$, $z = cw$, $dx = a\, du$, $dy = b\, dv$, $dz = c\, dw$.

$$\therefore\ I = \iiint abc\, du\, dv\, dw$$

and the given planes become $u = 0$, $v = 0$, $w = 0$ and $u + v + w = 1$.

If we consider an elementary cuboid, on this cuboid w varies from $w = 0$ to $w = 1 - u - v$, then v varies from $v = 0$ to $v = 1 - u$, and then u varies from $u = 0$ to $u = 1$.

$$\therefore\ I = \int_{u=0}^{1} \int_{v=0}^{1-u} \int_{w=0}^{1-u-v} abc\, dw\, dv\, du$$

$$= abc \int_{u=0}^{1} \int_{v=0}^{1-u} [w]_0^{1-u-v}\, dv\, du$$

$$= abc \int_{u=0}^{1} \int_{v=0}^{1-u} [1-u-v]\, dv\, du$$

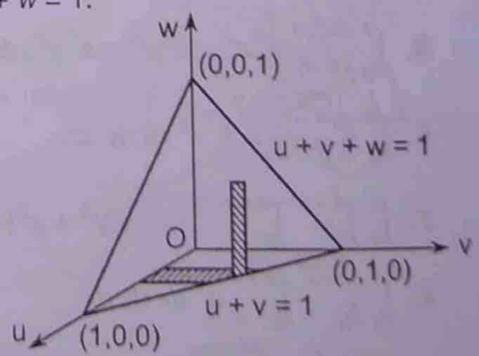


Fig. 12.6

$$\therefore\ I = abc\int_{v=0}^{1}\left[v - uv - \frac{v^2}{2}\right]_0^{1-u} du = abc\int_0^1\left[(1-u) - u(1-u) - \frac{(1-u)^2}{2}\right]du$$

$$= abc\int_0^1\left[1 - u - u + u^2 - \frac{1}{2} + u - \frac{u^2}{2}\right]du$$

$$= abc\int_0^1\left(\frac{1}{2} - u + \frac{u^2}{2}\right)du = abc\left[\frac{1}{2}u - \frac{u^2}{2} + \frac{u^3}{6}\right]_0^1$$

$$= abc\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] = \frac{abc}{6}.$$

Example 3 (b) : Evaluate $\iiint z\,dx\,dy\,dz$ over the volume of the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$, $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

Sol. : As in the above example, we put $x = au$, $y = bv$, $z = cw$, $dx = a\,du$, $dy = b\,dv$, $dz = c\,dw$.

$$\therefore\ I = abc\iiint cw\,du\,dv\,dw$$

As before the limits of integration change and we have

$$I = abc^2\int_{u=0}^{1}\int_{v=0}^{1-u}\int_{w=0}^{1-u-v} w\,dw\,dv\,du$$

$$= abc^2\int_{u=0}^{1}\int_{v=0}^{1-u}\left[\frac{w^2}{2}\right]_0^{1-u-v} dv\,du = \frac{abc^2}{2}\int_{u=0}^{1}\int_{v=0}^{1-u}(1-u-v)^2\,dv\,dx$$

$$= \frac{abc^2}{2}\int_{u=0}^{1}\left[-\frac{(1-u-v)^3}{3}\right]_0^{1-u} du = -\frac{abc^2}{6}\int_0^1[0-(1-u)^3]\,du$$

$$= \frac{abc^2}{6}\int_0^1(1-u)^3\,du = \frac{abc^2}{6}\left[-\frac{(1-u)^4}{4}\right]_0^1 = -\frac{abc^2}{24}[0-1] = \frac{abc^2}{24}.$$

Example 4 (b) : Evaluate $\iiint x^2yz\ dx\,dy\,dz$ throughout the volume bounded by the planes $x = 0$, $y = 0$, $z = 0$, $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$. **(M.U. 2001, 02, 03, 07, 08)**

Sol. : As in the above example, we substitute $x = au$, $y = bv$, $z = cw$.

$$\therefore\ dx = a\,du,\ dy = b\,dv,\ dz = c\,dw.$$

$$\therefore\ I = \iiint a^2bc\ u^2vw\ abc\ du\,dv\,dw$$

and the given planes become $u = 0$, $v = 0$, $w = 0$, $u + v + w = 1$.

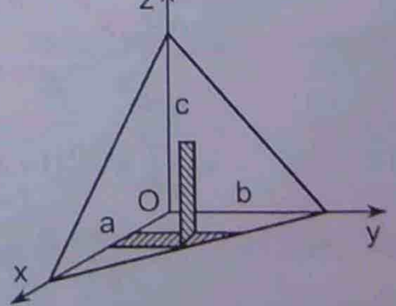


Fig. 12.7

$$\therefore\ I = \int_{u=0}^{1}\int_{v=0}^{1-u}\int_{w=0}^{1-u-v} a^3b^2c\ u^2vw\ dw\,dv\,du$$

$$= \int_0^1\int_0^{1-u} a^3b^2c\ u^2v\left[\frac{w^2}{2}\right]_0^{1-u-v} dv\,du$$

$$= \frac{a^3b^2c}{2}\int_0^1\int_0^{1-u} u^2v(1-u-v)^2\,dv\,du$$

$$\therefore\ I = \frac{a^3b^2c}{2}\int_0^1\int_0^{1-u} u^2 v\left[(1-u)^2 - 2(1-u)v + v^2\right] dv\, du$$

$$= \frac{a^3b^2c}{2}\int_0^1 u^2\left[(1-u)^2\frac{v^2}{2} - 2(1-u)\frac{v^3}{3} + \frac{v^4}{4}\right]_0^{1-u} du\ dv$$

$$= \frac{a^3b^2c}{2}\int_0^1 u^2\left[\frac{(1-u)^4}{2} - \frac{2(1-u)^4}{3} + \frac{(1-u)^4}{4}\right] du\ dv$$

$$= \frac{a^3b^2c}{2}\int_0^1 \frac{u^2(1-u)^4}{12} du = \frac{a^3b^2c}{24} B(3,5) \qquad \text{[By (1-A), page 6-17]}$$

$$= \frac{a^3b^2c}{24}\cdot\frac{\Gamma 3\,\Gamma 5}{\Gamma 8} = \frac{a^3b^2c}{24}\cdot\left(\frac{2!\,4!}{7!}\right) = \frac{a^3b^2c}{2520}.$$

Example 4 (A) (b) : Evaluate $\iiint x^2 yz\, dx\, dy\, dz$ throughtout the volume bounded by the places $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. **(M.U. 2003, 14)**

Sol. : In the above Example 4, put $a = b = c = 1$. [**Ans. :** 1 / 2520]

Example 5 (b) : Evaluate in terms of Gamma function $\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$ throughout the volume of the tetrahedron $x \geq 0$, $y \geq 0$, $z \geq 0$, $x + y + z \leq 1$.

Sol. : As in the earlier examples, we have

$$I = \int_{x=0}^1\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dz\, dy\, dx$$

$$= \int_0^1\int_0^{1-x} x^{l-1} y^{m-1}\left[\frac{z^n}{n}\right]_0^{1-x-y} dy\, dx$$

$$= \frac{1}{n}\int_0^1\int_0^{1-x} x^{l-1} y^{m-1}(1-x-y)^n\, dy\, dx$$

Putting $1 - x = a$ **(Note this step)**

$$I = \frac{1}{n}\int_0^1\int_0^a x^{l-1} y^{m-1}(a-y)^n\, dy\, dx$$

Putting $y = at$; when $y = 0$, $t = 0$; when $y = a$, $t = 1$.

$$I = \frac{1}{n}\int_0^1\int_0^1 x^{l-1} a^{m-1} t^{m-1} a^n (1-t)^n\, a\, dt\, dx$$

$$= \frac{1}{n}\int_0^1\int_0^1 x^{l-1} a^{m+n} t^{m-1}(1-t^n)\, dt\, dx$$

$$= \frac{1}{n}\int_0 x^{l-1} a^{m+n} B(m, n+1)\, dx$$

$$= \frac{1}{n}\int_0^1 x^{l-1}(1-x)^{m+n} B(m, n+1)\, dx$$

$$= \frac{B(m, n+1)}{n} B(l, m+n+1) = \frac{1}{n}\cdot\frac{\Gamma m\,\Gamma(n+1)}{\Gamma(m+n+1)}\cdot\frac{\Gamma l\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{1}{n}\cdot\frac{\Gamma m\cdot n\Gamma n\cdot\Gamma l}{(l+m+n)\Gamma(l+m+n)} = \frac{1}{(l+m+n)}\cdot\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n)}.$$

Example 6 (b) : Evaluate $\iiint \frac{dx\,dy\,dz}{(1+x+y+z)^3}$ over the volume of the tetrahedron $x=0$, $y=0$, $z=0$, $x+y+z=1$. **(M.U. 1990, 93, 97, 98, 2007, 11)**

Sol. :

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(1+x+y+z)^3}\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(1+x+y+z)^{-2}}{-2}\right]_0^{1-x-y} dy\, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} \left[\frac{1}{(1+x+y)^2} - \frac{1}{4}\right] dy\, dx$$

$$= \frac{1}{2}\int_0^1 \left[-\frac{1}{1+x+y} - \frac{y}{4}\right]_0^{1-x} dx = \frac{1}{2}\int_0^1 \left[\frac{1}{1+x} - \frac{1}{2} - \frac{(1-x)}{4}\right] dx$$

$$\therefore\ I = \frac{1}{2}\left[\log(1+x) - \frac{x}{2} + \frac{(1-x)^2}{8}\right]_0^1 = \frac{1}{2}\left[\log 2 - \frac{1}{2} - \frac{1}{8}\right] = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right].$$

Example 7 (b) : Evaluate $\iiint (x+y+z)\, dx\, dy\, dz$ over the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$. **(M.U. 1997, 98)**

Sol. :

$$I = \int_{x=0}^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} (x+y+z)\, dz\, dy\, dx$$

$$= \int_{x=0}^1 \int_{y=0}^{1-x} \left[\frac{(x+y+z)^2}{2}\right]_0^{1-x-y} dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^1 \int_{y=0}^{1-x} \left[1^2 - (x+y)^2\right] dy\, dx = \frac{1}{2}\int_0^1 \left[y - \frac{(x+y)^3}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[(1-x) - \frac{1}{3} + \frac{x^3}{3}\right] dx = \frac{1}{2}\left[\frac{2}{3}x - \frac{x^2}{2} + \frac{x^4}{12}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{2}{3} - \frac{1}{2} + \frac{1}{12}\right] = \frac{1}{2}\cdot\frac{3}{12} = \frac{1}{8}.$$

EXERCISE - II

Solve the following examples : Class (b) : 6 Marks

1. Evaluate $\iiint x^2 yz\ dx\, dy\, dz$
   throughout the volume bounded by $x=0$, $y=0$, $z=0$, $x+y+z=1$. **(M.U. 2003, 14)**

   [**Ans. :** $\frac{1}{2520}$]

2. Evaluate $\iiint xyz \; dx\, dy\, dz$

   throughout the volume bounded $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1.$

   (**Hint** : See solved Ex. 2) [**Ans.** : $\frac{1}{720}$]

3. Evaluate $\iiint x^2 y^2 z^2 \; dx\, dy\, dz$

   throughout the volume bounded $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1.$ [**Ans.** : $\frac{1}{45360}$]

4. Evaluate $\iiint x^2 \; dx\, dy\, dz$ throughout the volume of the tetrahedron $x \geq 0,\ y \geq 0,\ z \geq 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1.$ **(M.U. 1997, 2008)**

   (**Hint** : As in solved Ex. 1, put $x = au,\ y = bv,\ z = cw$) [**Ans.** : $\frac{a^3 bc}{60}$]

5. Evaluate $\iiint dx\, dy\, dz$ over the volume of the tetrahedron bounded by $x = 0,\ y = 0,\ z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$ (**Hint** : Put $x = au,\ y = bv,\ z = cw$) [**Ans.** : $\frac{abc}{6}$]

Type III : When the region of integration in not bounded by planes, but by sphere, ellipsoid etc.

(a) When the region of integration is sphere, we use spherical coordinates. We put

$$\boxed{\begin{aligned} &x = r \sin\theta \cos\phi,\ \ y = r \sin\theta \sin\phi,\ \ z = r\cos\theta \\ &dx\, dy\, dz = r^2 \sin\theta \, dr\, d\theta\, d\phi \end{aligned}}$$

(i) If the region of integration is whole of sphere $x^2 + y^2 + z^2 = a^2$ then clearly r **varies from 0 to a, θ varies from 0 to π and ϕ varies from 0 to 2π.** [See Fig. 12.2, page 12-1]

(ii) If the region of integration is hemisphere of radius a then clearly r **varies form 0 to a, θ varies from 0 to $\pi / 2$ and ϕ varies from 0 to 2π.**

(iii) If the region of integration is the first octant of the sphere of radius a, then clearly r **varies from 0 to a, θ varies from 0 to $\pi / 2$, and ϕ varies from 0 to $\pi / 2$.**

(b) When the region of integration is an ellipsoid, we write $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

and
$$\boxed{\begin{aligned} &x = ar \sin\theta \cos\phi,\ \ y = br \sin\theta \sin\phi,\ \ z = cr\cos\theta \\ &dx\, dy\, dz = abc\, r^2 \sin\theta \, dr\, d\theta\, d\phi \end{aligned}}$$

The limits of integration are as above except that r **varies from 0 to 1** and not from 0 to a.

(c) If the region of integration is a cylinder of base radius a we use cylindrical coordinates

and
$$\boxed{\begin{aligned} &x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z \\ &dx\, dy\, dz = r\, dr\, d\theta\, dz \end{aligned}}$$
[See Fig. 12.3, page 12-2]

Clearly r varies from 0 to a, θ varies from 0 to 2π, and z from $-\infty$ to ∞.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Evaluate $\iiint xyz\ dx\,dy\,dz$ over the positive octant of the sphere

$$x^2 + y^2 + z^2 = a^2.$$ **(M.U. 1985, 90, 2002)**

Sol. : We first transform the integral to spherical coordinates by putting $x = r \sin\theta \cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $dx\,dy\,dz = r^2 \sin\theta\, d r\, d\theta\, d\phi$.

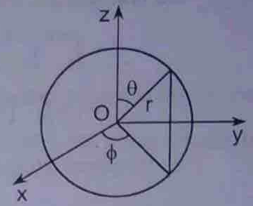


Fig. 12.8

In the first octant, r varies from $r = 0$ to $r = a$, θ varies from $\theta = 0$ to $\theta = \pi/2$, ϕ varies from $\phi = 0$ to $\phi = \pi/2$.

$$\therefore\quad I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^5 \sin^3\theta\cos\theta\sin\phi\cos\phi\ d r\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi \int_0^{\pi/2} \sin^3\theta\cos\theta\, d\theta \int_0^a r^5\, dr$$

$$= \left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2}\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2}\left[\frac{r^6}{6}\right]_0^{a} = \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{a^6}{6} = \frac{a^6}{48}.$$

Alternatively we may use cylindrical coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and $dx\,dy\,dz = r\, d r\, d\theta\, dz$.

Now, $x^2 + y^2 + z^2 = a^2$ transforms to $r^2 + z^2 = a^2$.

$$\therefore\quad I = \int_0^{\pi/2}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} r^3 \sin\theta\cos\theta \cdot z \cdot d r\, d\theta\, dz$$

$$= \int_0^{\pi/2}\int_0^{a} r^3\sin\theta\cos\theta\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-r^2}} d r\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{a} \frac{r^3}{2}(a^2-r^2)\sin\theta\cos\theta\ d r\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left[a^2\frac{r^4}{4} - \frac{r^6}{6}\right]_0^a \sin\theta\cos\theta\ d\theta$$

$$= \frac{a^6}{24}\int_0^{\pi/2}\sin\theta\cos\theta\ d\theta = \frac{a^6}{24}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{a^6}{48}.$$

Example 2 (b) : Evaluate $\iiint (x^2 + y^2 + z^2)\, dx\,dy\,dz$ over the first octant of the sphere

$$x^2 + y^2 + z^2 = a^2.$$ **(M.U. 1997, 2007)**

Sol. : As above we put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

and $dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$.

Now, $x^2 + y^2 + z^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$

$$= r^2\sin^2\theta + r^2\cos^2\theta = r^2$$

$$\therefore\quad I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^2\cdot r^2\sin\theta\ dr\, d\theta\, d\phi = \int_0^{\pi/2} d\phi\int_0^{\pi/2}\sin\theta\ d\theta\int_0^a r^4\ dr$$

$$= [\,\phi\,]_0^{\pi/2}[-\cos\theta\,]_0^{\pi/2}\left[\frac{r^5}{5}\right]_0^a = \frac{\pi}{2}\cdot(1)\cdot\frac{a^5}{5} = \pi\cdot\frac{a^5}{10}.$$

Example 3 (b) : Evaluate $\iiint \dfrac{dx\,dy\,dz}{x^2+y^2+z^2}$ throughout the volume of the sphere

$x^2+y^2+z^2=a^2$. **(M.U. 1987, 91, 2003, 14)**

Sol. : We transform the integral to spherical coordinates by putting

$$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi,\ \ z = r\cos\theta \ \text{ and } \ dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi.$$

We first find the integral I over the first octant.

$$\therefore\ I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a}\frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{r^2} = \int_0^{\pi/2}d\phi\int_0^{\pi/2}\sin\theta\,d\theta\int_0^a dr$$

$$= [\phi]_0^{\pi/2}[-\cos\theta]_0^{\pi/2}[r]_0^a = \frac{\pi}{2}\cdot 1\cdot a = \frac{\pi a}{2}.$$

$\therefore$ Required integral $= 8\cdot\dfrac{\pi}{2}a = 4\pi a$.

Example 4 (b) : Evaluate $\iiint \dfrac{z^2\,dx\,dy\,dz}{x^2+y^2+z^2}$ over the volume of the sphere $x^2+y^2+z^2=2$.

(M.U. 2004, 05, 09)

Sol. : Changing to spherical coordinates we get as in the above example.

$$I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\sqrt{2}}\frac{r^4\cos^2\theta\sin\theta}{r^2}\,dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}d\phi\int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta\cdot\int_0^{\sqrt{2}}r^2\,dr$$

$$= [\phi]_0^{\pi/2}\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{\sqrt{2}} = \frac{\pi}{2}\cdot\frac{1}{3}\cdot\frac{2\sqrt{2}}{3} = \frac{\pi\sqrt{2}}{9}.$$

$\therefore$ Required integral $= 8\cdot\dfrac{\pi\sqrt{2}}{9}$.

Example 5 (b) : Evaluate $\iiint\limits_V \dfrac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$ over the volume of sphere

$x^2+y^2+z^2=a^2$. **(M.U. 1989, 92, 98, 99)**

Sol. : We transform the integral to spherical coordinates by putting

$$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi,\ \ z = r\cos\theta \ \text{ and } \ dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi.$$

We first find the integral I over the first octant.

$$\therefore\ I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a}\frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{a^2-r^2}}$$

$$= \int_0^{\pi/2}d\phi\int_0^{\pi/2}\sin\theta\,d\theta\int_0^a\frac{r^2}{\sqrt{a^2-r^2}}\,dr$$

To find the last integral, we put $r = a\sin t$, $dr = a\cos t\,dt$.

$$\therefore\ \int_0^a\frac{r^2}{\sqrt{a^2-r^2}}\,dr = \int_0^{\pi/2}\frac{a^2\sin^2 t}{a\cos t}\cdot a\cos t\,dt$$

$$= a^2\int_0^{\pi/2}\sin^2 t\cdot dt = a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{a^2\pi}{4}.$$

Now, $\int_0^{\pi/2} d\phi = [\phi]_0^{\pi/2} = \dfrac{\pi}{2}$

and $\int_0^{\pi/2} \sin\theta \, d\theta = [-\cos\theta]_0^{\pi/2} = 1$

$$\therefore \quad I = \frac{\pi}{2} \cdot 1 \cdot \frac{a^2}{4} \cdot \pi = a^2 \frac{\pi^2}{8}$$

$\therefore$ Required integral $= 8 \cdot a^2 \cdot \dfrac{\pi^2}{8} = a^2\pi^2.$

Example 6 (b) : Evaluate $\iiint xyz\,(x^2+y^2+z^2)\,dx\,dy\,dz$ over the first octant of the sphere $x^2+y^2+z^2=a^2$. **(M.U. 1995, 98, 2002, 04, 05, 09)**

Sol. : Using spherical coordinates as above, we have

$$xyz = r^3 \sin^2\theta\cos\theta\sin\phi\cos\phi$$

$$x^2+y^2+z^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$$

$$= r^2\sin^2\theta + r^2\cos\theta = r^2$$

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^3\sin^2\theta\cos\theta\sin\phi\cos\phi\, r^2 \cdot r^2\sin\theta\, dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\sin\phi\cos\phi\,d\phi \int_0^{\pi/2}\sin^3\theta\cos\theta\,d\theta \int_0^a r^7\,dr$$

$$= \left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2}\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2}\left[\frac{r^8}{8}\right]_0^a = \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{a^8}{8} = \frac{a^8}{64}.$$

Example 7 (b) : Evaluate $\iiint (x^2y^2+y^2z^2+z^2x^2)\,dx\,dy\,dz$ over the volume of the sphere $x^2+y^2+z^2=a^2$. **(M.U. 1995, 99, 2002, 03)**

Sol. : As above we put

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$ and $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$.

We first find the integral I over the first octant.

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \left(r^4\sin^4\theta\sin^2\phi\cos^2\phi + r^4\sin^2\theta\cos^2\theta\sin^2\phi + r^4\sin^2\theta\cos^2\theta\cos^2\phi\right)\cdot r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^6\left(\sin^4\theta\sin^2\phi\cos^2\phi + \sin^2\theta\cos^2\theta\right)\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left(\sin^4\theta\sin^2\phi\cos^2\phi + \sin^2\theta\cos^2\theta\right)\cdot\sin\theta\,d\theta\,d\phi\int_0^a r^6\,dr$$

$$= \frac{a^7}{7}\left[\int_0^{\pi/2}\int_0^{\pi/2}\sin^5\theta\sin^2\phi\cos^2\phi\,d\theta\,d\phi + \int_0^{\pi/2}\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta\,d\phi\right]$$

$$= \frac{a^7}{7}\left[\int_0^{\pi/2}\sin^5\theta\,d\theta\int_0^{\pi/2}\sin^2\phi\cos^2\phi\,d\phi + \int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta\int_0^{\pi/2}d\phi\right]$$

$$= \frac{a^7}{7}\left[\frac{4\cdot2}{5\cdot3\cdot1}\cdot\frac{1\cdot1}{4\cdot2}\cdot\frac{\pi}{2} + \frac{2\cdot1}{5\cdot3\cdot1}\cdot\frac{\pi}{2}\right] \quad \text{[By (21), page 6-29; (25), (26), page 6-34]}$$

$$= \frac{a^7}{7}\left[\frac{1}{30}+\frac{1}{15}\right]\pi = \frac{a^7}{7}\cdot\frac{1}{10}\pi.$$

$\therefore$ Required integral $= 8 \cdot \dfrac{a^7}{70}\pi = \dfrac{4a^7\pi}{35}.$

Example 8 (b) : Use spherical coordinates to evaluate $\iiint_V \dfrac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$ where V is the volume bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, $(b>a)$. **(M.U. 1997)**

Sol. : We put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$.

If we consider a radial strip, then r varies from $r=a$ to $r=b$. For complete sphere ϕ varies from $\phi=0$ to $\phi=2\pi$ (See 12.2, page 12-1) and then θ varies from $\theta=0$ to $\theta=\pi$.

Now, $x^2+y^2+z^2=r^2$. [See (a) (i), page 12-13]

$$\therefore\ I = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{r=a}^{b}\frac{r^2}{(r^2)^{3/2}}\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\cdot\int_a^b\frac{dr}{r}$$

$$= [\phi]_0^{2\pi}[-\cos\theta]_0^{\pi}\cdot[\log r]_a^b$$

$$= 2\pi[2]\log\left(\frac{b}{a}\right) = 4\pi\log\left(\frac{b}{a}\right).$$

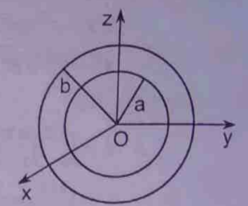


Fig. 12.9

Example 9 (b) : Evaluate $\iiint_V \dfrac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$ where V is the volume in the first octant. **(M.U. 1999)**

Sol. : Using spherical coordinates we easily see that the given integral reduces to (The first octant can be looked upon as (1/8) part of the sphere with infinite radius.)

$$I = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_0^{\infty}\frac{r^2\sin\theta\,d\phi\,d\theta\,dr}{(1+r^2)^2}$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_0^{\infty}\left[\frac{r^2}{(1+r^2)^2}\right]dr\cdot\sin\theta\,d\theta\,d\phi$$

To find the first integral put $r=\tan\alpha$ $\quad\therefore\ dr=\sec^2\alpha\,d\alpha$

$$\therefore\ I = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_0^{\pi/2}\frac{\tan^2\alpha}{\sec^4\alpha}\cdot\sec^2\alpha\,d\alpha\cdot\sin\theta\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}d\phi\int_0^{\pi/2}\sin\theta\,d\theta\int_0^{\pi/2}\sin^2\alpha\,d\alpha = [\phi]_0^{\pi/2}[-\cos\theta]_0^{\pi/2}\int_0^{\pi/2}\sin^2\alpha\,d\alpha$$

$$= \frac{\pi}{2}\cdot 1\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.$$

Example 10 (b) : Evaluate $\iiint_V \dfrac{z^2}{x^2+y^2+z^2}\,dx\,dy\,dz$ where V is the volume bounded by the sphere $x^2+y^2+z^2=z$.

Sol. : We use spherical polar coordinates *i.e.* we put

$$x = r \sin\theta \cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta \text{ and } dx\,dy\,dz = r^2 \sin\theta\, d r\, d\theta\, d\phi.$$

The equation of the sphere now becomes $r^2 = r\cos\theta$ *i.e.* $r = \cos\theta$. Note that the sphere passes through the origin. Its equation can be written as

$$x^2 + y^2 + [z - (1/2)]^2 = (1/2)^2.$$

Its centre is [0, 0, (1/2)] and radius is 1 / 2.

Now, r varies from 0 to $\cos\theta$, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to 2π.

$$\therefore\ I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\cos\theta} \frac{r^2\cos^2\theta}{r^2} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\cos\theta} r^2\cos^2\theta\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{\cos\theta} \cos^2\theta\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{3}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \cos^3\theta\cdot\cos^2\theta\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{3}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \cos^5\theta\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{3}\int_0^{2\pi} d\phi \int_0^{\pi/2}\cos^5\theta\sin\theta\, d\theta$$

$$= \frac{1}{3}[\phi]_0^{2\pi}\left[-\frac{\cos^6\theta}{6}\right]_0^{\pi/2} = \frac{1}{3}\cdot 2\pi\cdot\frac{1}{6} = \frac{\pi}{9}.$$

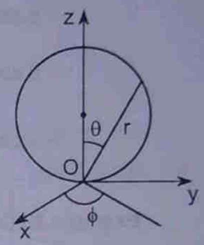


Fig. 12.10

Example 11 (b) : Evaluate $\iiint e^{(x^2+y^2+z^2)^{3/2}}\, dV$ throughout the volume of the unit sphere.

Sol. : Changing to spherical coordinates as above, we get, the integral over the first octant as,

$$I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 e^{(r^2)^{3/2}}\cdot r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 e^{r^3}\cdot r^2\sin\theta\, dr\, d\theta\, d\phi \qquad [\text{ Put } r^3 = t\,]$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{e^{r^3}}{3}\right]_0^1 \sin\theta\, d\theta\, d\phi = \int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{3}(e-1)\sin\theta\, d\theta\, d\phi$$

$$= \frac{(e-1)}{3}\int_0^{\pi/2}[-\cos\theta]_0^{\pi/2}\, d\phi = \frac{(e-1)}{3}\int_0^{\pi/2} 1\cdot d\phi = \frac{(e-1)}{3}\cdot\frac{\pi}{2}.$$

$\therefore$ The required integral $= 8I = \frac{4}{3}\pi(e-1)$.

Example 12 (b) : Evaluate $\iiint\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\, dx\, dy\, dz$ throughout the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.

Sol. : Since the volume is that of ellipsoid, we shall use sperical coordinates.

$$x = ar\sin\theta\cos\phi,\ y = br\sin\theta\sin\phi,\ z = cr\cos\theta$$

$$dx\, dy\, dz = abc\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

By this transformation the ellipsoid transforms to the sphere of radius unity

$$\because \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{a^2r^2\sin^2\theta\cos^2\phi}{a^2}+\frac{b^2r^2\sin^2\theta\sin^2\phi}{b^2}+\frac{c^2r^2\cos^2\theta}{c^2}$$

$$=(r^2\sin^2\theta)\cos^2\phi+(r^2\sin^2\theta)\sin^2\phi+r^2\cos^2\theta$$

$$=r^2\sin^2\theta+r^2\cos^2\theta=r^2$$

$$\therefore \quad I=\iiint\sqrt{1-r^2}\cdot abc\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$=8\,abc\int_0^{\pi/2}\sin\theta\, d\theta\int_0^{\pi/2}d\phi\int_0^1\sqrt{1-r^2}\cdot r^2\, dr$$

$$=8\,abc\left[-\cos\theta\right]_0^{\pi/2}\left[\phi\right]_0^{\pi/2}\int_0^{\pi/2}\cos t\cdot\sin^2 t\cdot\cos t\, dt \qquad [\text{ Put } r=\sin t\ ]$$

$$=8\,abc\,[1]\left(\frac{\pi}{2}\right)\int_0^{\pi/2}\cos^2 t\sin^2 t\, dt$$

$$=4\pi\, abc\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi^2}{4}\, abc.$$

Example 13 (b) : Evaluate $\iiint\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}\, dx\, dy\, dz$ over the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Sol. : Using spherical coordinates as in the above example.

$$I=\iiint 1\cdot abc\cdot r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$=8\,abc\int_0^{\pi/2}\sin\theta\, d\theta\int_0^{\pi/2}d\phi\int_0^1 r^2\, dr$$

$$=8\,abc\left[-\cos\theta\right]_0^{\pi/2}\left[\phi\right]_0^{\pi}\left[\frac{r^3}{3}\right]_0^1$$

$$=8\,abc\,[1]\left(\frac{\pi}{2}\right)\cdot\frac{1}{3}=\frac{4\pi}{3}\cdot abc.$$

EXERCISE - III

Solve the following examples : Class (b) : 6 Marks

1. Evaluate $\iiint\frac{dx\, dy\, dz}{(x^2+y^2+z^2)^{3/2}}$ over the volume bounded by the spheres $x^2+y^2+z^2=16$ and $x^2+y^2+z^2=25$. **(M.U. 2001, 03)** [**Ans. :** $4\pi\log(5/4)$]

2. Evaluate $\iiint\frac{dx\, dy\, dz}{(x^2+y^2+z^2)^{1/2}}$ over the volume bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, $a>b>0$. **(M.U. 2009)** [**Ans. :** $2\pi(a^2-b^2)$]

3. Evaluate $\iiint x^2\, dx\, dy\, dz$ over the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

(**Hint :** Put $x=au$, $y=bv$, $z=cw$ and use spherical polar coordinates.

$$I = a^3\, bc \int_0^1 r^4\, dr \int_0^{2\pi} \cos^2 \phi\, d\phi \int_0^{\pi} \sin^3 \theta\, d\theta\,)$$ [Ans. : $\dfrac{4\pi a^3 bc}{15}$]

4. Evaluate $\iiint \sqrt{\dfrac{x^2}{1^2} + \dfrac{y^2}{2^2} + \dfrac{z^2}{3^2}}\, dx\, dy\, dz$ over the volume of the ellipsoid $\dfrac{x^2}{1^2} + \dfrac{y^2}{2^2} + \dfrac{z^2}{3^2} = 1$.

[Ans. : 8π]

Type IV : When the region of integration is bounded by a cone or a cylinder or a paraboloid.

In such cases we may use cylindrical coordinates.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Evaluate $\iiint \sqrt{x^2 + y^2}\, dx\, dy\, dz$ over the volume bounded by the right circular cone $x^2 + y^2 = z^2$, $z > 0$ and the planes $z = 0$ and $z = 1$. **(M.U. 2002, 04)**

Sol. : We transform the given integral to cylindrical polar coordinates by putting

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$ and $dx\, dy\, dz = r\, dr\, d\theta\, dz$.

Now limits for r are 0 to 1 for θ are 0 to 2π, for z are r to 1.

$$\therefore\ I = \int_0^1 r^2\, dr \int_0^{2\pi} d\theta \int_r^1 dz = \int_0^1 r^2\, dr\, [\,\theta\,]_0^{2\pi}\, [\,z\,]_r^1$$

$$= \int_0^1 r^2 \cdot 2\pi \cdot (1 - r)\, dr = 2\pi \left[\frac{r^3}{3} - \frac{r^4}{4}\right]_0^1 = \frac{2\pi}{12} = \frac{\pi}{6}.$$

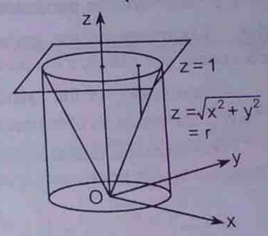


Fig. 12.11

Example 2 (b) : Evaluate $\iiint\limits_V (x^2 + y^2)\, dV$ where V is the solid bounded by the surface $x^2 + y^2 = z^2$ and the planes $z = 0$, $z = 2$.

Sol. : We know that $x^2 + y^2 = z^2$ is a cone with vertex at the origin and the axis along the z-axis.

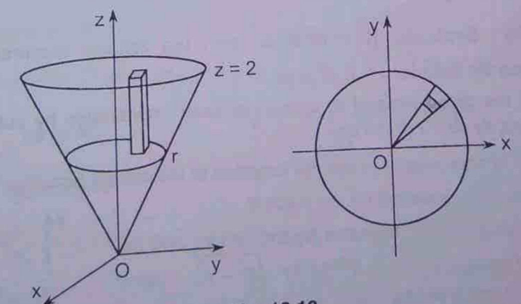


Fig. 12.12

Consider a cuboid as shown in the figure. The section of the cone by a plane parallel to the xy-plane is a circle. We put $x = r\cos\theta$, $y = r\sin\theta$. Then $z^2 = r^2\cos^2\theta + r^2\sin^2\theta \ \therefore\ z = r$. And $dV = dx\, dy\, dz = r\, dr\, d\theta\, dz$, θ varies from 0 to 2π.

Since $z = r$ and at the top $z = 2$, hence r varies from 0 to 2.

$$\therefore \iiint_V (x^2 + y^2)\, dV = \iiint_V (x^2 + y^2)\, dx\, dy\, dz$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=r}^{2} r^2 \cdot r\, dz\, dr\, d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} r^3 \left[z\right]_r^2 dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} r^3 (2 - r)\, dr\, d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (2r^3 - r^4)\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{2r^4}{4} - \frac{r^5}{5}\right]_0^2 d\theta = \int_{\theta=0}^{2\pi} \left[8 - \frac{32}{5}\right] d\theta$$

$$= \int_0^{2\pi} \frac{8}{5}\, d\theta = \frac{8}{5}\left[\theta\right]_0^{2\pi} = \frac{16}{5}\pi.$$

Example 3 (b) : Evaluate $\iiint z^2\, dx\, dy\, dz$ over the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the paraboloid $x^2 + y^2 = z$ and the plane $z = 0$.

Sol. : We transform the cartesian coordinate system to cylindrical polar system *i.e.* we put $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and $dx\, dy\, dz = r\, dr\, d\theta\, dz$.

The equation of the cylinder $x^2 + y^2 = a^2$, now transforms to $r^2 = a^2$. The equation of the paraboloid $x^2 + y^2 = z$ transforms to $r^2 = z$.

The volume is bounded below by the section of the cylinder in xy - plane and by the paraboloid above.

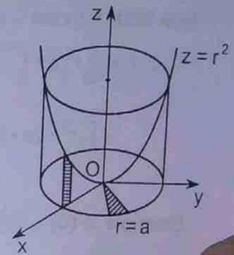


Fig. 12.13

$$\therefore\ I = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \int_{z=0}^{r^2} z^2\, r\, dr\, d\theta\, dz$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \left[\frac{z^3}{3}\right]_0^{r^2} r\, dr\, d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{r^7}{3}\, dr\, d\theta$$

$$= \frac{1}{3} \int_{\theta=0}^{2\pi} \left[\frac{r^8}{8}\right]_0^a d\theta = \frac{a^8}{24} \int_0^{2\pi} d\theta = \frac{2\pi a^8}{24} = \frac{\pi a^8}{12}.$$

Example 4 (b) : Evaluate $\iiint z^2\, dx\, dy\, dz$ over the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$. **(M.U. 1991)**

Sol. : We transform the given integral to cylindrical polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and $dx\, dy\, dz = r\, dr\, d\theta\, dz$.

With the change of coordinate system the equation of the sphere becomes $r^2 + z^2 = a^2$ and of the cylinder becomes $r^2 = ar\cos\theta$ *i.e.* $r = a\cos\theta$.

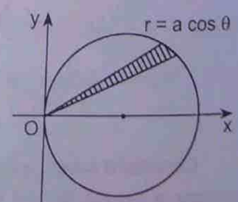


Fig. 12.14

The volume of integration is bounded by the sphere and the cylinder. Thus, z varies from $z = -\sqrt{a^2 - r^2}$ to $z = \sqrt{a^2 - r^2}$, r varies form $r = 0$ to $r = a\cos\theta$ and θ varies from $\theta = -\pi/2$ to $\theta = \pi/2$.

$$\therefore\ I = \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \int_{-\sqrt{a^2 - r^2}}^{\sqrt{a^2 - r^2}} z^2 \cdot r\, dr\, d\theta\, dz$$

$$\therefore \quad I = \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \left[\frac{z^3}{3}\right]_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \frac{2}{3}(a^2-r^2)^{3/2} \cdot r\,dr\,d\theta$$

To evaluate the integral put $a^2 - r^2 = t \quad \therefore \; 2r\,dr = -dt$

$$\therefore \quad I = \int_{-\pi/2}^{\pi/2} \frac{1}{3}\left[-\frac{2}{5}(a^2-r^2)^{5/2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{2a^5}{15}\int_{-\pi/2}^{\pi/2} (1-\sin^5\theta)\,d\theta$$

$$= \frac{2a^5}{15}[\theta]_{-\pi/2}^{\pi/2} \qquad \left[\because \int_{-\pi/2}^{\pi/2} \sin^5\theta\,d\theta = 0 \text{ as } \sin^5\theta \text{ is an odd function.}\right]$$

$$= \frac{2a^5\pi}{15}.$$

EXERCISE - IV

Solve the following examples : Class (b) : 6 Marks

1. Evaluate the integral $\iiint \sqrt{x^2+y^2}\,dV$ throughout the volume bounded by the cylinder $x^2+y^2=1$ over the paraboloid $z = 1-x^2-y^2$ below the plane $z=4$. [**Ans. :** $12\pi/5$]

2. Evaluate $\iiint z^2\,dx\,dy\,dz$ over the volume common to the sphere $x^2+y^2+z^2=1$ and the cylinder $x^2+y^2=x$. [**Ans. :** $2\pi/15$]

EXERCISE - V

Evaluate the following triple integrals : Class (a) : 3 or 4 Marks

1. $\int_0^1\int_0^1\int_0^1 xyz\,dx\,dy\,dz$
2. $\int_0^1\int_0^x\int_0^{xy} dx\,dy\,dz$
3. $\int_0^\infty\int_0^\infty\int_0^\infty e^{-(x+y+z)}\,dx\,dy\,dz$
4. $\int_0^1\int_0^1\int_0^1 e^{(x+y+z)}\,dx\,dy\,dz$
5. $\int_{-1}^1\int_{-2}^2\int_{-3}^3 dx\,dy\,dz$
6. $\int_0^1 dx\int_0^2 dy\int_1^2 x^2\,yz\,dz$
7. $\int_0^1\int_1^z\int_0^{yz} xyz\,dx\,dy\,dz$
8. $\int_0^1\int_{y^2}^1\int_0^{1-x} x\,dz\,dx\,dy$
9. $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 r\,dr\,d\theta\,d\phi$
10. $\int_0^a\int_0^{\pi/2}\int_0^{\pi/2} r\cos\theta\cos\phi\,dr\,d\theta\,d\phi$
11. $\int_0^\pi\int_0^{\pi/3}\int_0^1 r^2\sin\theta\,dr\,d\theta\,d\phi$
12. $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 r^2\sin\theta\,dr\,d\theta\,d\phi$
13. $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 r^4\sin\theta\,dr\,d\theta\,d\phi$
14. $\int_0^\pi\int_0^\pi\int_a^b \frac{\sin\theta}{r}\,dr\,d\theta\,d\phi$

[Ans. : (1) $\frac{1}{8}$, (2) $\frac{1}{8}$, (3) 1, (4) $(e-1)^3$, (5) 48, (6) 1, (7) $-\frac{1}{64}$, (8) $\frac{4}{35}$,
(9) $\frac{\pi^2}{8}$, (10) $\frac{a^2}{2}$, (11) $\frac{\pi}{6}$, (12) $\frac{\pi}{6}$, (13) $\frac{\pi}{10}$, (14) $2\pi \log\left(\frac{b}{a}\right)$]

Summary

1. **Spherical Polar Coordinates**

$$x = r \sin\theta \cos\Phi$$
$$y = r \sin\theta \sin\Phi$$
$$z = r \cos\theta.$$

2. **Cylindrical Coordinates**

$$x = \rho \cos\Phi$$
$$y = \rho \sin\Phi$$
$$z = z$$

3. $dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$dx\, dy\, dz = r\, dr\, d\theta\, dz$

CHAPTER 13 Volumes of Solids

1. Introduction

In this chapter we shall learn how the technique of double and triple integration can be used to evaluate volumes of solids.

2. Volumes of Solids

Let $z = f(x, y)$ be the equation of the surface S. Let R be the orthogonal projection of this surface on the xy - plane. Let the equation of this projection be $f(x, y) = 0$. Consider an elementary parallelepiped with $dx\,dy$ on the xy - plane as the base and bounded by surface S on the top. Its volume is $z\,dx\,dy = f(x, y)\,dx\,dy$.

The summation of all such terms over the region R gives the volume of the cylinder bounded by the surface S and the xy - plane.

$$\therefore \quad \boxed{\text{Volume} = \iint_R f(x, y)\,dx\,dy.}$$

The volume of a solid can also be expressed as a triple integral. If we consider an elementary cuboid then its volume is $dx\,dy\,dz$ and hence, the required volume is

$$\boxed{V = \iiint_R dx\,dy\,dz}$$

where, R is the given region in space.

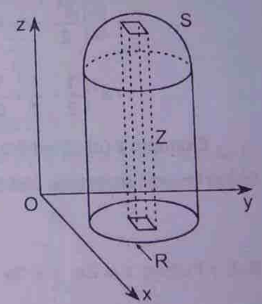


Fig. 13.1

Solved Examples : Class (c) : 8 Marks

Type I : Volume Bounded By Planes

If we have to find a volume bounded by planes, it is convenient to use cartesian coordinates, x, y, z and integrate within proper limits.

Example 1 (c) : Find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$. **(M.U. 2000, 05, 11, 12, 15)**

Sol. : On the elementary cuboid y varies from $y = 0$ to $y = a - x$, x varies from $x = 0$ to $x = a$ and z varies from $z = 0$ to $z = a - x - y$. [See figure on the next page]

$$\therefore \quad V = \int_{x=0}^{a} \int_{y=0}^{a-x} \int_{z=0}^{a-x-y} dz\,dy\,dx = \int_{x=0}^{a} \int_{y=0}^{a-x} \left[z \right]_0^{a-x-y} dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} (a - x - y)\,dy\,dx = \int_{x=0}^{a} \left[ay - xy - \frac{y^2}{2} \right]_0^{a-x} dx$$

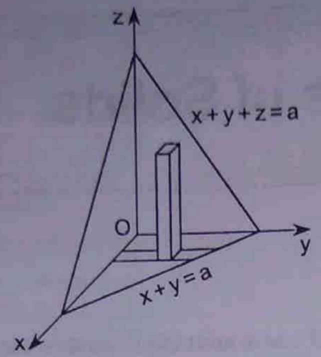


Fig. 13.2 (a)

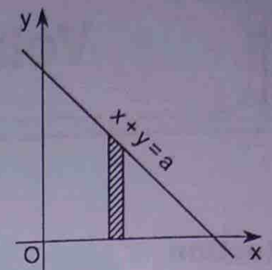


Fig. 13.2 (b)

$$\therefore \quad V = \int_0^a \left[a(a-x) - x(a-x) - \frac{(a-x)^2}{2} \right] dx$$

$$= \int_0^a \left(a^2 - ax - ax + x^2 - \frac{a^2}{2} + ax - \frac{x^2}{2} \right) dx$$

$$= \int_0^a \left(\frac{a^2}{2} - ax + \frac{x^2}{2} \right) dx = \left[\frac{a^2}{2} \cdot x - \frac{ax^2}{2} + \frac{x^3}{6} \right]_0^a$$

$$= a^3 \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) = \frac{a^3}{6}.$$

Example 2 (c) : Find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$.

(M.U. 2014)

Sol. : Putting $x = 2u$, $y = 3v$, $z = 4w$, the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$ becomes $u + v + w = 1$. And $dx\, dy\, dz$ becomes $24\, du\, dv\, dw$.

Considering the elementary cuboid, w varies from $w = 0$ to $w = 1 - u - v$; v varies from $v = 0$ to $v = 1 - u$ and u varies from $u = 0$ to $u = 1$.

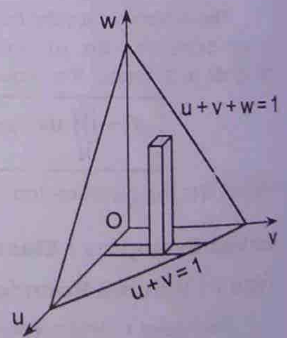


Fig. 13.3

$$\therefore \quad V = \int_{u=0}^{1} \int_{v=0}^{1-u} \int_{w=0}^{1-u-v} 24\, dw\, dv\, du$$

$$= 24 \int_0^1 \int_0^{1-u} [w]_{0}^{1-u-v}\, dv\, du$$

$$= 24 \int_0^1 \int_0^{1-u} (1-u-v)\, dv\, du = 24 \int_0^1 \left[v - uv - \frac{u^2}{2} \right]_0^{1-u} du$$

$$= 24 \int_0^1 \left[(1-u) - u(1-u) - \frac{(1-u)^2}{2} \right] du$$

$$\therefore \quad V = 24\int_0^1 \left[1-2u+u^2-\frac{1}{2}(1-2u+u^2)\right]du = 24\int_0^1\left(\frac{1}{2}-u+\frac{1}{2}u^2\right)du$$

$$= 24\left[\frac{1}{2}u-\frac{u^2}{2}+\frac{u^3}{6}\right]_0^1 = 24\left[\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right]=\frac{24}{6}=4.$$

Example 3 (c) : Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, $z = 0$.

Sol. : The equation of the plane can be written as $\frac{x}{2}+\frac{y}{1}+\frac{z}{2}=1$.

The plane $x = 2y$ intersects this plane in $4y + z = 2$. Putting $z = 0$, we get the point A as $z = 0$, $y = 1/2$ and then $x = 2 - 1 = 1$ *i.e.* $(1, 1/2, 0)$. B is $(0, 1, 0)$ and C is $(0, 0, 2)$.

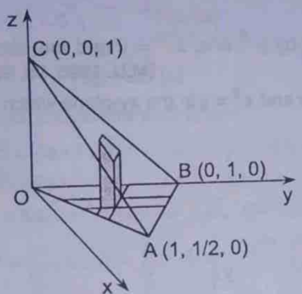


Fig. 13.4

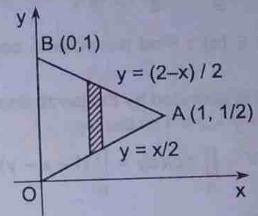


Fig. 13.5

Consider a cuboid as shown in the figure. On this z varies from $z = 0$ to $z = 2 - x - 2y$. As it moves from OA to AB. y varies from $x/2$ to $(2-x)/2$. The equations of OA and AB by two-point formula are $y = x/2$, $y = (2-x)/2$.

Now, x varies from $x = 0$ to $x = 1$.

$$\therefore \quad I = \int_0^1\int_{x/2}^{1-(x/2)}\int_0^{2-x-2y} dz\,dy\,dx$$

$$= \int_0^1\int_{x/2}^{1-(x/2)}[2-x-2y]\,dy\,dx = \int_0^1\left[2y-xy-y^2\right]_{x/2}^{1-(x/2)}dx$$

$$= \int_0^1\left[2\left(1-\frac{x}{2}\right)-x\left(1-\frac{x}{2}\right)-\left(1-\frac{x}{2}\right)^2\right]-\left[2\cdot\frac{x}{2}-x\cdot\frac{x}{2}-\left(\frac{x}{2}\right)^2\right]dx$$

$$= \int_0^1\left[2-x-x+\frac{x^2}{2}-1+x-\frac{x^2}{4}-x+\frac{x^2}{2}+\frac{x^2}{4}\right]dx$$

$$= \int_0^1(1-2x+x^2)\,dx = \left[x-x^2+\frac{x^3}{3}\right]_0^1 = \left(1-1+\frac{1}{3}\right)=\frac{1}{3}.$$

Example 4 (c) : Find the volume of the triangular prism formed by the planes $ay = bx$, $y = 0$, $x = a$ from $z = 0$ to $z = c + xy$. **(M.U. 1997, 2000, 10)**

Sol. : $V = \iiint dz\,dy\,dx = \int_{x=0}^{a} \int_{y=0}^{bx/a} \int_{z=0}^{c+xy} dz\,dy\,dx$

$$= \int_{x=0}^{a} \int_{y=0}^{bx/a} [z]_0^{c+xy}\,dx\,dy = \int_{x=0}^{a} \int_{y=0}^{bx/a} (c+xy)\,dy\,dx$$

$$= \int_{x=0}^{a} \left[cy + \frac{xy^2}{2}\right]_0^{bx/a} dx$$

$$= \int_{x=0}^{a} \left(\frac{cb}{a}\cdot x + x\cdot\frac{b^2x^2}{2a^2}\right) dx = \frac{cb}{a}\left[\frac{x^2}{2}\right]_0^a + \frac{b^2}{2a^2}\left[\frac{x^4}{4}\right]_0^a$$

$$= \frac{abc}{2} + \frac{a^2b^2}{8} = \frac{ab}{8}[4c+ab].$$

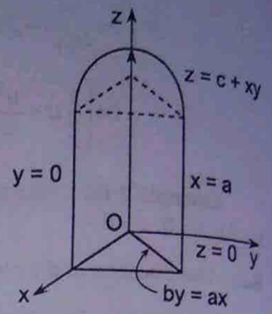


Fig. 13.6

Example 5 (c) : Find the volume bounded by $y^2 = x$, $x^2 = y$ and the planes $z = 0$ and $x + y + z = 1$. **(M.U. 1995, 98, 99, 2007, 09, 14)**

Sol. : The solid is bounded by the parabolas $y^2 = x$ and $x^2 = y$ in the xy-plane which is its base and by the plane $x + y + z = 1$ at the top.

$$\therefore\ V = \iint_R z\,dx\,dy = \iint_R (1-x-y)\,dx\,dy$$

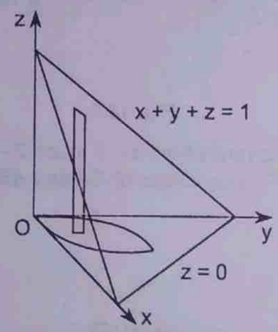


Fig. 13.7 (a)

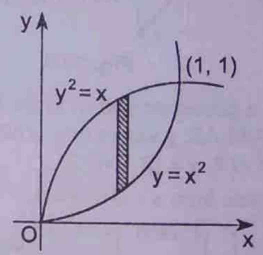


Fig. 13.7 (b)

Now, R is bounded by parabolas $y^2 = x$ and $x^2 = y$ in the xy plane. They intersect at (0, 0) and (1, 1).

$$\therefore\ V = \int_0^1 \int_{x^2}^{\sqrt{x}} (1-x-y)\,dx\,dy = \int_0^1 \left[y - xy - \frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left[\left(\sqrt{x} - x^{3/2} - \frac{x}{2}\right) - \left(x^2 - x^3 - \frac{x^4}{2}\right)\right] dx$$

$$= \left[\frac{2x^{3/2}}{3} - \frac{2x^{5/2}}{5} - \frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10}\right]_0^1$$

$$= \frac{2}{3} - \frac{2}{5} - \frac{1}{4} - \frac{1}{3} + \frac{1}{4} + \frac{1}{10} = \frac{1}{30}.$$

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Example 6 (c) : Find the volume bolunded by $y^2 = 4ax$ and $x^2 = 4ay$ and the planes $z = 0$ and $z = 3$. **(M.U. 2013)**

Sol. : The solid is bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ in the x-y plane and the two planes $z = 0$ and $z = 3$ as shown in the figure.

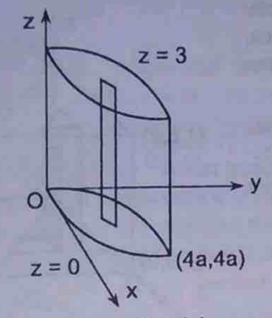


Fig. 13.8 (a)

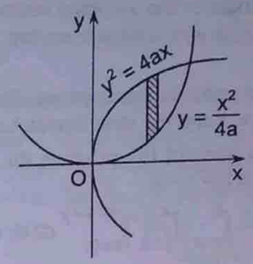


Fig. 13.8 (b)

The two parabolas intersect where

$$y^2 = 4a \cdot 2\sqrt{a}\,y \quad i.e., \quad y^4 = 64a^3 y$$

$$\therefore \quad y^3 = 64a^3 \quad i.e., \quad y = 4a.$$

When $y = 4a$, $x^2 = 4a \cdot 4a$ i.e., $x = 4a$.

$$\therefore \quad V = \iint_R z\,dx\,dy = \iint_R 3\,dy\,dx = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} 3\,dy\,dx = 3\int_0^{4a} [y]_{x^2/4a}^{2\sqrt{ax}} dx$$

$$= 3\int_0^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a}\right] dx = 3\left[2\sqrt{a} \cdot \frac{x^{3/2}}{3/2} - \frac{1}{4a} \cdot \frac{x^3}{3}\right]_0^{4a} dx$$

$$= 3\left[\frac{4}{3} \cdot \sqrt{a} \cdot 8a^{3/2} - \frac{1}{4a} \cdot \frac{64\,a^3}{3}\right] = 32\,a^2 - 16\,a^2 = 16\,a^2.$$

Example 7 (c) : Find the volume of the solid that lies under the plane $3x + 2y + z = 12$ and above the rectangle $R = \{(x, y) \mid 0 \le x \le 1, -2 \le y < 3\}$.

Sol. : Consider an elementary cuboid as shown in the figure.

On this cuboid z varies from $z = 0$ to $z = 12 - 3x - 2y$, y varies from $y = -2$ to $y = 3$ and x varies from $x = 0$ to $x = 1$.

Hence, the volume is given by

$$V = \int_{x=0}^{1} \int_{y=-2}^{3} \int_{z=0}^{12-3x-2y} dz\,dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=-2}^{3} [z]_0^{12-3x-2y} dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=-2}^{3} [12 - 3x - 2y]\,dy\,dx$$

$$= \int_0^1 \left[12y - 3xy - y^2\right]_{-2}^{3} dx$$

$$= \int_{x=0}^{1} [(36 - 9x - 9) - (-24 + 6x - 4)]\,dx$$

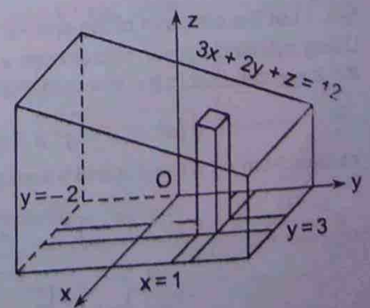


Fig. 13.9

$$\therefore\ V = \int_{x=0}^{1} (55 - 15x)\, dx = \left[55x - \frac{15x^2}{2}\right]_0^1 = 55 - \frac{15}{2} = \frac{95}{2}.$$

Example 8 (c) : Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis, the lines $y = x$, $x = 1$ and whose top lies in the plane $z = 3 - x - y$.

Sol. : Consider a strip in the xy-plane. On this strip y varies from $y = 0$ to $y = x$ and the strip moves from $x = 0$ to $x = 1$.

Since the base of the prism is the xy-plane z varies from $z = 0$ to $z = 3 - x - y$.

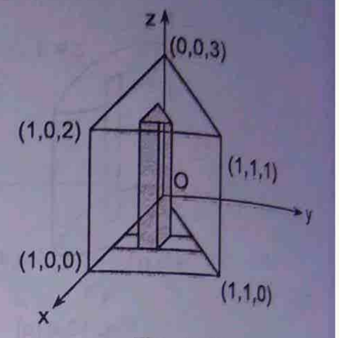


Fig. 13.10

$$\therefore\ V = \int_{x=0}^{1} \int_{y=0}^{x} \int_{z=0}^{3-x-y} dz\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} (3 - x - y)\, dy\, dx$$

$$= \int_{x=0}^{1} \left[3y - xy - \frac{y^2}{2}\right]_0^x dx$$

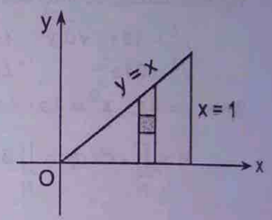


Fig. 13.11

$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2}\right) dx = \int_{x=0}^{1} \left(3x - \frac{3x^2}{2}\right) dx$$

$$= \left[\frac{3x^2}{2} - \frac{1}{2}x^3\right]_0^1 = \frac{3}{2} - \frac{1}{2} = 1.$$

Type II : Volume Bounded by Right Circular Cylinder or a Right Circular Cone

When we are required to find the volume bounded by right circular cylinder or right circular cone. We use cylindrical coordinates z, r, θ.

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : A cylindrical hole of radius b is bored through a sphere of radius a. Find the volume of the remaining solid. **(M.U. 2004)**

Sol. : Let the equation of the sphere be $x^2 + y^2 + z^2 = a^2$. Using cylindrical polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, we see that in the first octant z varies from $z = 0$ to

$$z = \sqrt{[a^2 - (x^2 + y^2)]} = \sqrt{[a^2 - r^2]},$$

r varies from $r = b$ to $r = a$ and θ varies from $\theta = 0$ to $\theta = \pi/2$.

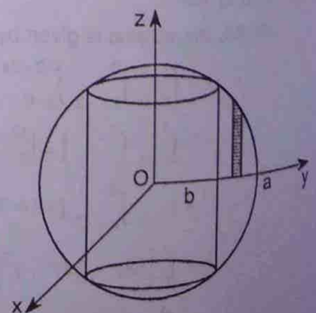


Fig. 13.12

$$\therefore\ V = 8\int_{\theta=0}^{\pi/2} \int_{r=b}^{a} \int_{z=0}^{\sqrt{a^2-r^2}} r\, dr\, d\theta\, dz$$

$$= 8\int_{\theta=0}^{\pi/2} \int_{r=b}^{a} [z]_0^{\sqrt{a^2-r^2}}\, r\, dr\, d\theta$$

$$= 8\int_{\theta=0}^{\pi/2} \int_{r=b}^{a} (a^2 - r^2)^{1/2} \cdot r\, dr\, d\theta$$

$$\therefore \quad V = 8\int_{\theta=0}^{\pi/2}\left[\frac{(a^2-r^2)^{3/2}}{3/2}\cdot\left(-\frac{1}{2}\right)\right]_b^a d\theta \quad \left[\text{Put } a^2-r^2=t\right]$$

$$= -\frac{8}{3}\int_0^{\pi/2} -(a^2-b^2)\, d\theta = \frac{8}{3}(a^2-b^2)\left[\theta\right]_0^{\pi/2} = \frac{4}{3}(a^2-b^2).$$

Example 2 (c) : Show that the volume of the wedge intercepted between the cylinder $x^2+y^2=2ax$ and planes $z=mx$, $z=nx$ is $\pi(n-m)a^3$. **(M.U. 1997, 2000)**

Sol. : We change to cylindrical polar coordinates by putting $x=r\cos\theta$, $y=r\sin\theta$, $z=z$.

The equation $x^2+y^2=2ax$ becomes $r^2=2ar\cos\theta$ *i.e.* $r=2a\cos\theta$. Hence, r varies from $r=0$ to $r=2a\cos\theta$, θ varies from $\theta=-\pi/2$ to $\theta=\pi/2$. Since, volume is to be evaluated we can to take into account symmetry and take the limits of θ from $\theta=0$ to $\theta=\pi/2$, twice. And z varies from $z=nx$ to $z=mx$ *i.e.* from $z=nr\cos\theta$ to $z=mr\cos\theta$.

$$\therefore \quad V = 2\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta}\int_{z=nr\cos\theta}^{mr\cos\theta} r\, dr\, d\theta\, dz$$

$$= 2\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r\left[z\right]_{nr\cos\theta}^{mr\cos\theta} dr\, d\theta$$

$$= 2\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r\cdot(m-n)\, r\cos\theta\, dr\, d\theta$$

$$= 2(m-n)\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r^2\cos\theta\, dr\, d\theta$$

$$= 2(m-n)\int_{\theta=0}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}\cos\theta\, d\theta$$

$$= 2(m-n)\int_{\theta=0}^{\pi/2}\frac{8a^3}{3}\cos^3\theta\cos\theta\, d\theta$$

$$= \frac{16(m-n)}{3}a^3\int_{\theta=0}^{\pi/2}\cos^4\theta\, d\theta \qquad \text{[By (23), page 6-32]}$$

$$= \frac{16(m-n)}{3}\cdot a^3\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = (m-n)\pi a^3.$$

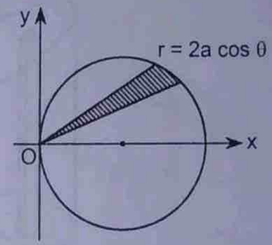


Fig. 13.13

Example 3 (c) : Find the volume bounded by the cone $z^2=x^2+y^2$ and the paraboloid $z=x^2+y^2$. **(M.U. 2008, 13)**

Sol. : If we consider a section by a plane $z=k$ then on the cone we get, a circle $x^2+y^2=k^2$ and on the paraboloid we get, the circle $x^2+y^2=k$.

If we use cylindrical coordinates then at the intersection of the two solids

$$x^2+y^2=r^2=r \quad i.e. \quad r^2-r=0$$

$$\therefore \ r(r-1)=0=0 \qquad \therefore \ r=0 \text{ and } 1.$$

Hence, r varies from 0 to 1, θ varies from $\theta=0$ to $\pi/2$ taken four times by symmetry, z varies from r to r^2 (where $r^2=x^2+y^2$).

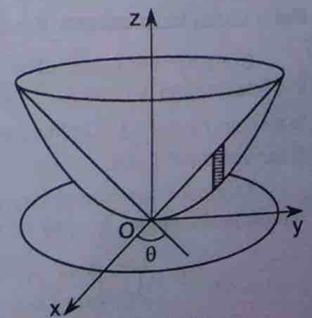


Fig. 13.14

$$\therefore \quad V = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}\int_{z=r}^{r^2} r\, dr\, d\theta\, dz$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r\left[z\right]_{z=r}^{r^2} dr\, d\theta$$

$$\therefore\ V = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r\left[r^2 - r\right]dr\ d\theta = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}(r^3 - r^2)\,dr\ d\theta$$

$$= 4\int_0^{\pi/2}\left[\frac{r^4}{4} - \frac{r^3}{3}\right]_0^1 d\theta = 4\int_0^{\pi/2} -\frac{1}{12}\,d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2} d\theta = -\frac{1}{3}\left[\theta\right]_0^{\pi/2} = \frac{\pi}{6}.$$

Example 4 (c) : Find the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $z = 0$ and $y + z = b$. **(M.U. 2012, 16)**

Sol. : We first note that the solid is **not** symmetrical. Hence, we have to be careful while taking the limits.

Now, z varies from $z = 0$ to $z = b - y = b - r\sin\theta$; θ varies from 0 to 2π and r varies from 0 to a.

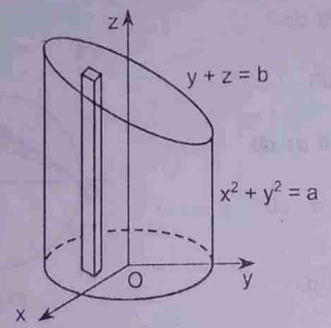


Fig. 13.15

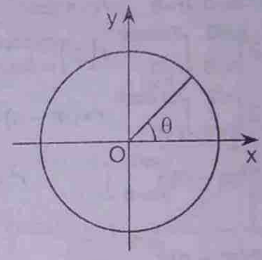


Fig. 13.16

$$\therefore\ V = \int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{b-r\sin\theta} d\theta\, r\,dr\,dz = \int_{r=0}^{a}\int_{\theta=0}^{2\pi}\left[z\right]_0^{b-r\sin\theta} r\,dr\,d\theta$$

$$= \int_{r=0}^{a}\int_{\theta=0}^{2\pi}(b - r\sin\theta)\,r\,dr\,d\theta = \int_{r=a}^{a}\left[b\theta + r\cos\theta\right]_0^{2\pi} r\,dr$$

$$= \int_{r=0}^{a}\left[2\pi b + r(0-0)\right]r\,dr = 2\pi b\int_0^a r\,dr = 2\pi b\left[\frac{r^2}{2}\right]_0^a = \pi a^2 b.$$

Example 5 (c) : Find the volume cut off from the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $x^2 + y^2 = z^2$.

Sol. : Using triple integral $V = \iiint dz\ dx\ dy$.

Consider the intersection of the sphere and the cone. On this intersection we have $x^2 + y^2 = a^2/2$. In polar coordinates it is a circle $r = a/\sqrt{2}$. On this circle r varies from 0 to $a/\sqrt{2}$ and θ varies from 0 to 2π.

Consider, an elementary parallelopiped (as shown in the figure) and change $dx\,dy$ to $r\,d\theta\,dr$.

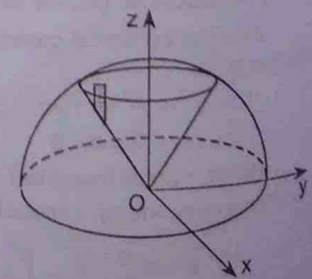


Fig. 13.17

$$\therefore\ V = \int_0^{2\pi}\int_{\theta=0}^{a/\sqrt{2}}\int_{z=r}^{\sqrt{a^2-r^2}} r\ d\theta\ dr\ dz$$

$$\therefore \quad V = \int_{\theta=0}^{2\pi} \int_{r=0}^{a/\sqrt{2}} \left[z \right]_r^{\sqrt{a^2-r^2}} r\, d\theta\, dr$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{a/\sqrt{2}} \left[\sqrt{a^2-r^2} - r \right] r\, d\theta\, dr$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{(a^2-r^2)^{3/2}}{-3} - \frac{r^3}{3} \right]_0^{a/\sqrt{2}} d\theta$$

$$= -\frac{1}{3} \int_0^{2\pi} \left(\frac{a^3}{2\sqrt{2}} + \frac{a^3}{2\sqrt{2}} - a^3 \right) d\theta$$

$$= \frac{1}{3} a^3 \left(1 - \frac{1}{\sqrt{2}} \right) \int_0^{2\pi} d\theta = \frac{2\pi a^3}{3} \left(1 - \frac{1}{\sqrt{2}} \right) = \frac{\pi a^3}{3} (2 - \sqrt{2}).$$

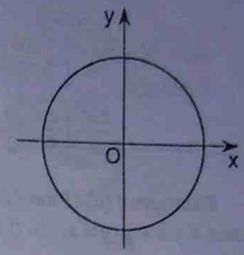


Fig. 13.18

Example 6 (c) : Find the volume cut off from the sphere $x^2 + y^2 + z^2 = a^2$ by the plane $z = 0$ and the cylinder $x^2 + y^2 = ax$.

(M.U. 1994, 97)

Sol. : The base of the cylinder $x^2 + y^2 = ax$ is the circle

$$\left(x - \frac{a}{2} \right)^2 + y^2 = \left(\frac{a}{2} \right)^2.$$

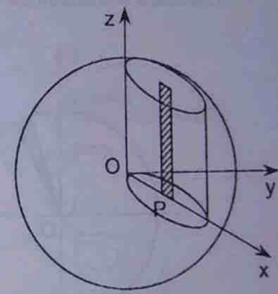


Fig. 13.19

The volume is bounded by this circle in xy - plane in the bottom, by the cylinder on side and by the sphere $x^2 + y^2 + z^2 = a^2$ on the top.

Taking polar coordinates, elementary area at P is $r\, d\theta\, dr$. On the circle $x^2 + y^2 = ax$ *i.e.* on $r = a \cos\theta$, r varies from 0 to $a \cos\theta$ and θ varies from $-\pi/2$ to $\pi/2$. Also z varies from 0 to $\sqrt{a^2 - (x^2 + y^2)}$ *i.e.* from 0 to $\sqrt{a^2 - r^2}$.

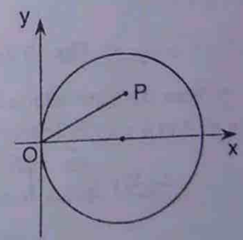


Fig. 13.20

Considering the first quadrant only.

$$\therefore \quad V = 2 \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\cos\theta} \int_{z=0}^{\sqrt{a^2-r^2}} dz\, r\, d\theta\, dr$$

$$= 2 \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\cos\theta} \sqrt{a^2 - r^2}\; r\, d\theta\, dr$$

To find the first integral put $a^2 - r^2 = t$.

$\therefore\; r\, dr = -\dfrac{dt}{2}$ when $r = 0$, $t = a^2$; when $r = a\cos\theta$, $t = a^2 \sin^2\theta$

$$\therefore \quad V = 2 \int_0^{\pi/2} \int_{a^2}^{a^2 \sin^2\theta} -t^{1/2} \frac{dt}{2}\, d\theta = \int_0^{\pi/2} \left[-\frac{t^{3/2}}{3/2} \right]_{a^2}^{a^2 \sin^2\theta} d\theta$$

$$= \frac{2}{3} \int_0^{\pi/2} \left[-a^3 \sin^3\theta + a^3 \right] d\theta = \frac{2a^3}{3} \int_0^{\pi/2} (1 - \sin^3\theta)\, d\theta$$

$$\therefore \quad V = \frac{2a^3}{3}\left[\int_0^{\pi/2} d\theta - \int_0^{\pi/2} \sin^3\theta \, d\theta\right] \qquad \text{[By (21), page 6-29]}$$

$$= \frac{2a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3}\cdot 1\right] = \frac{2a^3}{3}\cdot\frac{(3\pi-4)}{6} = \frac{a^3}{9}(3\pi - 4).$$

Example 7 (c) : Find the volume in the first octant bounded by the cylinder $x^2 + y^2 = 2$ and the planes $z = x + y$, $y = x$, $z = 0$ and $x = 0$. **(M.U. 1995, 98, 2005)**

Sol. : If we take projections on the xy - plane, the area is bounded by the circle $x^2 + y^2 = 2$, the line $y = x$ and the line $x = 0$ *i.e.* the y-axis.

We change the coordinates to cylindrical polar by putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

Then the equation of the cylinder becomes $x^2 + y^2 = 2$ *i.e.* $r = \sqrt{2}$.

The line $y = x$ becomes, $r\sin\theta = r\cos\theta$ $\quad \therefore \ \theta = \pi/4$.

The line $x = 0$ becomes, $r\cos\theta = 0$ $\quad \therefore \ \theta = \pi/2$.

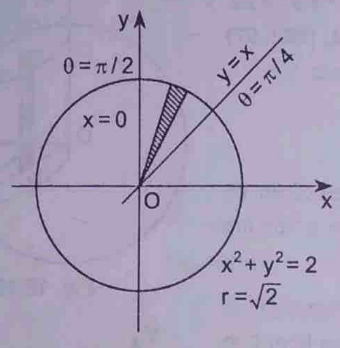


Fig. 13.21

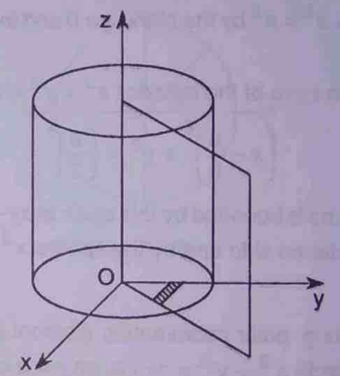


Fig. 13.22

Now, if we consider a radial strip in the projection, r varies from $r = 0$ to $r = \sqrt{2}$, θ varies from $\theta = \pi/4$ to $\theta = \pi/2$. Then z varies from $z = 0$ to $z = x + y = r(\cos\theta + \sin\theta)$.

$$\therefore \quad V = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}}\int_{z=0}^{r(\cos\theta+\sin\theta)} r \, dr \, d\theta \, dz = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}} r\left[z\right]_0^{r(\cos\theta+\sin\theta)} dr \, d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}} r^2(\cos\theta+\sin\theta) \, dr \, d\theta = \int_{\theta=\pi/4}^{\pi/2}(\cos\theta+\sin\theta)\left[\frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$$

$$= \frac{2\sqrt{2}}{3}\int_{\theta=\pi/4}^{\pi/2}(\cos\theta+\sin\theta) \, d\theta = \frac{2\sqrt{2}}{3}\left[\sin\theta - \cos\theta\right]_{\pi/4}^{\pi/2}$$

$$= \frac{2\sqrt{2}}{3}\left[(1-0) - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right] = \frac{2\sqrt{2}}{3}.$$

Example 8 (c) : Find the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$.

Sol. : If we consider volume in the first octant, we see that z varies from $z = 0$ to $z = \sqrt{2ax} = \sqrt{2ar\cos\theta}$, r varies from $r = 0$ to $r = 2a\cos\theta$ and θ varies 0 to $\pi/2$.

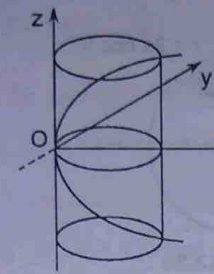


Fig. 13.23

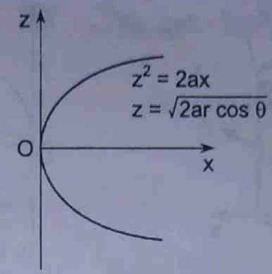


Fig. 13.24

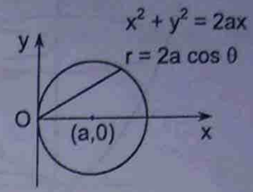


Fig. 13.25

$$\therefore \quad V = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta}\int_{z=0}^{\sqrt{2ar\cos\theta}} dz\, r\, dr\, d\theta$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta}\sqrt{2ar\cos\theta}\cdot r\, dr\, d\theta$$

$$= 4\int_{0}^{\pi/2}\sqrt{2a\cos\theta}\left[\frac{r^{5/2}}{5/2}\right] d\theta = \frac{4\cdot 2}{5}\int_0^{\pi/2}(2a\cos\theta)^3\, d\theta$$

$$= \frac{8a^3\cdot 8}{5}\int_0^{\pi/2}\cos^3\theta\, d\theta = \frac{64a^3}{5}\cdot\frac{2\cdot 1}{3} = \frac{128a^3}{15}. \qquad \text{[By (24), page 6-32]}$$

Example 9 (c) : Find the volume bounded by the paraboloid $x^2 + y^2 = az$ and the cylinder $x^2 + y^2 = a^2$.

(M.U. 2007, 15)

Sol. : The equations of the cylinder and the paraboloid in polar form are $r = a$ and $r^2 = az$.

Now, z varies from $z = 0$ to $z = r^2/a$, r varies from $r = 0$ to $r = a$ and θ varies from $\theta = 0$ to $\theta = \pi/2$ taken 4 times.

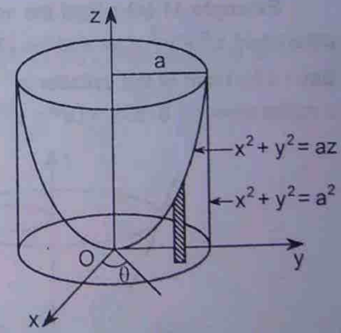


Fig. 13.26

$$\therefore \quad V = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a}\int_{z=0}^{r^2/a} r\, dr\, d\theta\, dz$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\left[z\right]_{z=0}^{r^2/a} dr\, d\theta$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a}\frac{r^3}{a}\, dr\, d\theta = \frac{4}{a}\int_{\theta=0}^{\pi/2}\left[\frac{r^4}{4}\right]_0^a d\theta$$

$$= \frac{4}{a}\int_0^{\pi/2}\left[\frac{a^4}{4}\right] d\theta = a^3\int_0^{\pi/2} d\theta = a^3\left[\theta\right]_0^{\pi/2} = \frac{\pi}{2}a^3.$$

Example 10 (c) : Find the volume of the cylinder $x^2 + y^2 = 2ax$ intercepted between the paraboloid $x^2 + y^2 = 2az$ and the xy - plane.

Sol. : The base of the cylinder $x^2 + y^2 = 2ax$ is the circle $(x-a)^2 + y^2 = a^2$. In polar coordinates the circle becomes $r^2 = 2ar\cos\theta$ *i.e.* $r = 2a\cos\theta$.

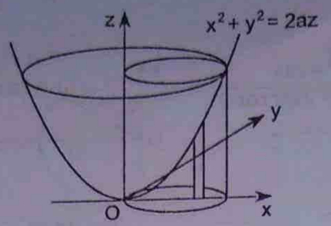


Fig. 13.27

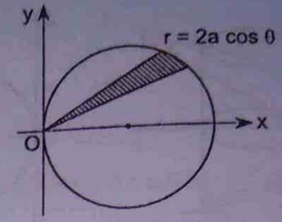


Fig. 13.28

Hence, z varies from $z = 0$ to $z = (x^2 + y^2) / 2a = r^2 / 2a$, r varies from $r = 0$ to $r = 2a \cos \theta$, θ varies from 0 to $\pi / 2$ taken twice.

$$\therefore \quad V = 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} \int_{z=0}^{r^2/2a} r\,dz\,dr\,d\theta = 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r\left[z\right]_0^{r^2/2a} \cdot dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} \frac{r^3}{2a} \cdot dr\,d\theta = \frac{1}{a}\int_{\theta=0}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{4a}\int_0^{\pi/2} 16a^4 \cos^4\theta\,d\theta = 4a^3 \int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$= 4a^3 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi a^3}{4}.$$

[By (23), page 6-32]

Example 11 (c) : Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 2ay$, the paraboloid $x^2 + y^2 = az$ and the plane $z = 0$. **(M.U. 1992, 2008, 09)**

Sol. : The base of the cylinder $x^2 + y^2 = 2ay$ is the circle $x^2 + (y - a)^2 = a^2$ i.e. $r = 2a \sin \theta$. And z varies from $z = 0$ to $z = (x^2 + y^2) / a = r / a$. And θ varies from $\theta = 0$ to $\pi / 2$ taken twice.

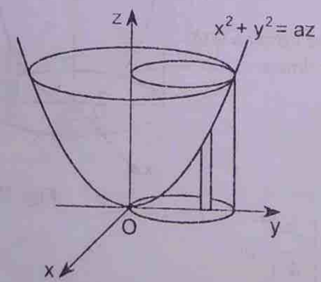


Fig. 13.29

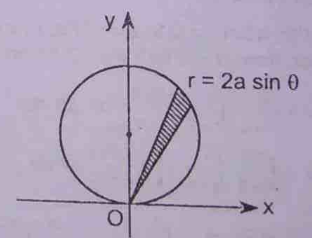


Fig. 13.30

$$\therefore \quad V = 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\sin\theta} \int_{z=0}^{r^2/a} r\,dz\,dr\,d\theta = 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\sin\theta} r\left[z\right]_0^{r^2/a} dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\sin\theta} \frac{r^3}{a} \cdot dr\,d\theta = \frac{2}{a}\int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2a\sin\theta} d\theta$$

$$= 8a^3 \int_0^{\pi/2} \sin^4\theta\,d\theta = 8a^3 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi a^3}{2}$$

[By (20), page 6-28]

Example 12 (c) : Find the volume common to the right circular cylinders

$$x^2 + y^2 = a^2 \text{ and } x^2 + z^2 = a^2.$$ **(M.U. 1988, 92, 2004, 12)**

Sol. : By symmetry the required volume = 8 volume in the first octant.

$$\therefore \quad V = 8 \iiint dx\ dy\ dz$$

In the first octant z varies from 0 to $\sqrt{a^2 - x^2}$.

$$\therefore \quad V = 8 \iint \sqrt{a^2 - x^2} \cdot dx\, dy$$

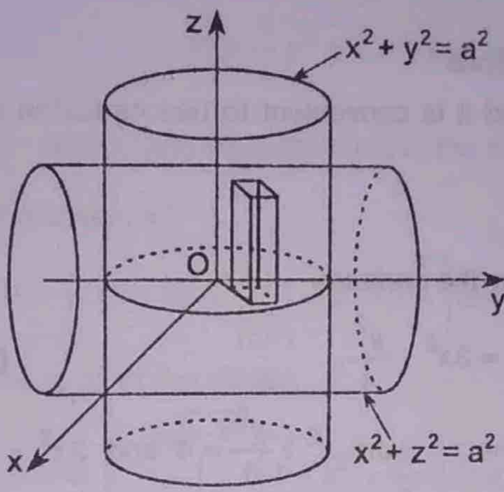


Fig. 13.31

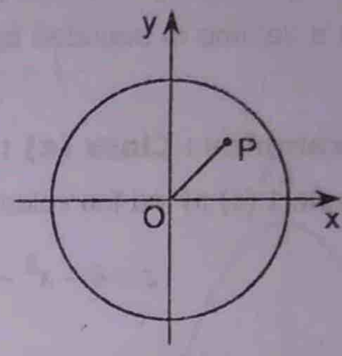


Fig. 13.32

Now in the circle $x^2 + y^2 = a^2$, y varies from 0 to $\sqrt{a^2 - x^2}$ and x varies from 0 to a.

$$\therefore \quad V = 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2} \cdot dx\, dy = 8 \int_0^a \left[\left(\sqrt{a^2 - x^2} \right) \cdot y \right]_0^{\sqrt{a^2 - x^2}} dx$$

$$= 8 \int_0^a (a^2 - x^2)\, dx = 8 \left[a^2 x - \frac{x^3}{3} \right]_0^a = 8 \cdot \frac{2a^3}{3} = \frac{16\, a^3}{3}.$$

Type III : Volume Bounded By The Sphere

When the volume is bounded by a sphere, we use spherical polar coordinates r, θ, ϕ.

Solved Examples : Class (c) : 8 Marks

Example 1 : Find the volume enclosed by the solid $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1.$

Sol. : Let us put $x = au^3$, $y = bv^3$, $z = cw^3$.

$$\therefore \quad dx\, dy\, dz = 27\, abc\, u^2 v^2 w^2\, du\, dv\, dw$$

$$\therefore \quad V = \iiint dx\, dy\, dz = \iiint 27\, abc\, u^2 v^2 w^2\, du\, dv\, dw$$

Now, let us change to spherical polar coordinates

$$u = r \sin\theta \cos\phi, \qquad v = r \sin\theta \sin\phi,$$

$$w = r \cos\theta, \qquad du\, dv\, dw = r^2 \sin\theta\, d r\, d\theta\ \ d\phi$$

In the first positive octant, r varies from 0 to 1, θ varies from 0 to $\pi/2$, ϕ varies from 0 to $\pi/2$.

$$\therefore\ V = 27\,abc \cdot 8 \int_{r=0}^{1} \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} r^2 \sin^2\theta \cos^2\phi \cdot r^2 \sin^2\theta \sin^2\phi \cdot r^2 \cos^2\theta \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 216\,abc \int_{r=0}^{1} r^8\, dr \int_{\phi=0}^{\pi/2} \sin^2\phi \cos^2\phi\, d\phi \int_{\theta=0}^{\pi/2} \sin^5\theta \cos^2\theta\, d\theta$$

$$= 216\,abc \left[\frac{r^9}{9}\right]_0^1 \cdot \frac{1\cdot 1}{4\cdot 2} \cdot \frac{\pi}{2} \cdot \frac{4\cdot 2\cdot 1}{7\cdot 5\cdot 3\cdot 1} \qquad \text{[By (25), (26), page 6-34]}$$

$$= 216\,abc \cdot \frac{1}{9} \cdot \frac{\pi}{16} \cdot \frac{8}{105} = \frac{4}{35}\pi\, abc.$$

Type IV : Volume Bounded By A Paraboloid

When a volume is bounded by a paraboloid it is convenient to use cartesian coordinates again.

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Find the volume bounded by the surfaces

$$z = 4 - x^2 - \frac{1}{4}y^2 \text{ and } z = 3x^2 + \frac{y^2}{4}. \qquad \textbf{(M.U. 2003)}$$

Sol. : The sections of the two surfaces by the plane $z = 0$ are $x^2 + \frac{y^2}{4} = 4$ and $3x^2 + \frac{y^2}{4} = 0$.

The two paraboloids intersect where $4 - x^2 - \frac{1}{4}y^2 = 3x^2 + \frac{y^2}{4}$ *i.e.* $4x^2 + \frac{y^2}{2} = 4$ *i.e.* $x^2 + \frac{y^2}{8} = 1$ which is an ellipse.

Hence, z varies from $z = 3x^2 + y^2/4$ to $z = 4 - x^2 - y^2/4$. The limits of x, y are to be obtained by considering $x^2 + (y^2/8) = 1$. Considering only the first quadrant of the ellipse of intersection.

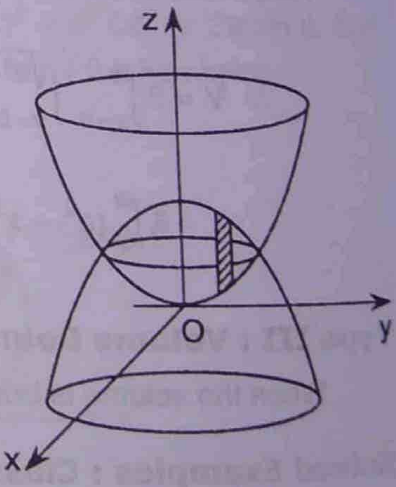


Fig. 13.33

$$V = 4\int_{x=0}^{1} \int_{y=0}^{\sqrt{8(1-x^2)}} \int_{z=3x^2+(y^2/4)}^{4-x^2-(y^2/4)} dz\, dy\, dx$$

$$= 4\int_{x=0}^{1} \int_{y=0}^{\sqrt{8(1-x^2)}} \left(4 - x^2 - \frac{y^2}{4} - 3x^2 - \frac{y^2}{4}\right) dy\, dx$$

$$= 4\int_{x=0}^{1} \int_{y=0}^{\sqrt{8(1-x^2)}} \left(4 - 4x^2 - \frac{y^2}{2}\right) dy\, dx$$

$$= 4\int_{x=0}^{1} \left[4y - 4x^2 y - \frac{y^3}{6}\right]_0^{\sqrt{8(1-x^2)}} dx$$

$$= 4\int_0^1 \left[4(1-x^2)y - \frac{y^3}{6}\right]_0^{\sqrt{8(1-x^2)}} dx \qquad \textbf{[Note this]}$$

$$= 4\int_0^1 \left[4(1-x^2)\cdot\sqrt{8}\cdot\sqrt{1-x^2} - \frac{1}{6}\, 8\sqrt{8}\,(1-x^2)^{3/2}\right] dx$$

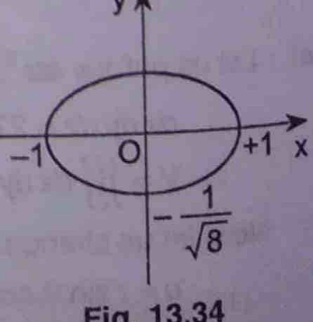


Fig. 13.34

$$V = 4\int_0^1 \left(4\sqrt{8} - \frac{4\sqrt{8}}{3}\right)(1-x^2)^{3/2}\,dx$$

$$= \frac{64\sqrt{2}}{3}\int_0^1 (1-x^2)^{3/2}\,dx \qquad [\text{ Put } x = \sin\theta,\ dx = \cos\theta\, d\theta\,]$$

$$= \frac{64\sqrt{2}}{3}\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{64\sqrt{2}}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 4\sqrt{2}\cdot\pi.$$

Example 2 (c) : Find the volume cut off from the paraboloid

$$x^2 + \frac{1}{4}y^2 + z = 1 \text{ by the plane } z = 0.$$ **(M.U. 1993, 2005)**

Sol. : The xy - plane cuts the paraboloid in the ellipse $x^2 + \dfrac{y^2}{4} = 1$.

Hence, the volume

$$V = \iint_R z\,dx\,dy = \iint_R \left(1 - x^2 - \frac{y^2}{4}\right)dx\,dy$$

where R is the area of the ellipse.

$$\therefore\quad V = 4\int_{x=0}^{1}\int_{y=0}^{2\sqrt{1-x^2}} \left(1 - x^2 - \frac{y^2}{4}\right)dx\,dy$$

$$= 4\int_0^1 \left[(1-x^2)\,y - \frac{y^3}{12}\right]_0^{2\sqrt{1-x^2}} \cdot dx$$

$$= 4\int_0^1 \frac{4}{3}(1-x^2)^{3/2}\,dx \qquad [\text{ Put } x = \sin\theta,\ dx = \cos\theta\, d\theta\,]$$

$$= \frac{16}{3}\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{16}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \pi.$$

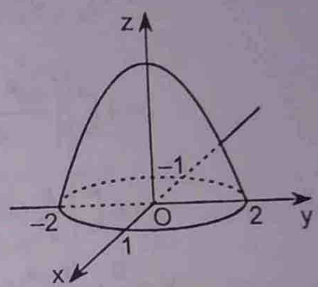


Fig. 13.35

Example 3 (c) : Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = ax$. **(M.U. 2013)**

Sol. : The circle of intersection is $x^2 + y^2 = ax$ *i.e.* $r = a\cos\theta$.

$$\therefore\quad V = \iiint dx\,dy\,dz = \iint_R \int_{z=x^2+y^2}^{ax} dz\,dx\,dy$$

$$= \int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{a\cos\theta} [ax - (x^2+y^2)]\,r\,dr\,d\theta$$

$$= \int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{a\cos\theta} [ar\cos\theta - r^2]\,r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[a\frac{r^3}{3}\cos\theta - \frac{r^4}{4}\right]_0^{a\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{a^4\cos^4\theta}{3} - \frac{a^4\cos^4\theta}{4}\right] d\theta$$

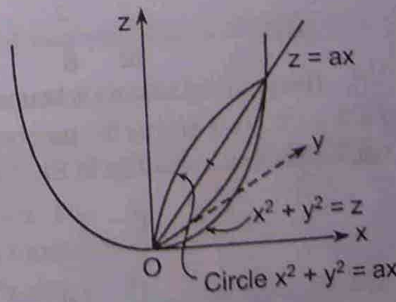


Fig. 13.36

[illegible]

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

$$\boxed{\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}[X + 2R]}$$

[illegible] R = Sum of the Remaining ordinates.

Example: [illegible] by trapezoidal rule by using the following data.

x	: 0	0.2	0.4	0.6	0.8	1
y	: 1	1.0857	1.1448	1.1790	1.1891	1.1755
Ordinate	: y_0	y_1	y_2	y_3	y_4	y_5

Sol.: [illegible] $h = 0.2$

$$X = y_0 + y_5 = 1 + 1.1755 = 2.1755$$

$$R = y_1 + y_2 + y_3 + y_4$$

$$= 1.0857 + 1.1448 + 1.1790 + 1.1891 = 4.5986$$

$$\therefore \int_0^1 f(x)\,dx = \frac{h}{2}[X + 2R] = \frac{0.2}{2}[2.1755 + 9.1972] = 1.13727$$

3. Simpson's (1/3)rd Rule

Let the equation of the parabola passing through the first three points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ be

$$y = a + b(x - x_1) + c(x - x_1)^2$$

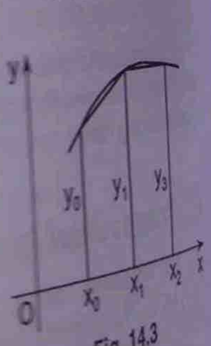


Fig. 14.3

[illegible] when $x = x_0$, when $x = x_1$, when $x = x_2$

[illegible] ... (A)

[illegible] and when $x = x_1 + h,\ u = h$.

[illegible] $\frac{h}{3}(6a + 2ch^2)$

$$\therefore \quad V = 2 \cdot \frac{a^4}{12} \int_0^{\pi/2} \cos^4 \theta \, d\theta = \frac{a^4}{6} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{a^4}{32} \pi.$$

Example 4 (c) : Find the volume bounded by the paraboloid

$$z = 4 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \text{ and the } xy\text{-plane.}$$

Sol. : The intersection of the xy-plane and the paraboloid is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4 \quad i.e. \quad y = \frac{b}{a}\sqrt{4a^2 - x^2}$$

$$\therefore \quad V = \iint z \, dx \, dy = \iint_R \left(4 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx \, dy$$

$$= 4 \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{4a^2 - x^2}} \left[\left(4 - \frac{x^2}{a^2}\right) - \frac{y^2}{b^2}\right] dx \, dy$$

$$= 4 \int_0^a \left[\left(\frac{4a^2 - x^2}{a^2}\right) y - \frac{1}{b^2} \cdot \frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{4a^2 - x^2}} dx$$

$$= 4 \int_0^a \left[\frac{b}{a^3}(4a^2 - x^2)^{3/2} - \frac{1}{3b^2} \cdot \frac{b^3}{a^3}(4a^2 - x^2)^{3/2}\right] dx$$

$$\therefore \quad V = \frac{8}{3} \cdot \frac{b}{a^3} \int_0^a (4a^2 - x^2)^{3/2} \, dx.$$

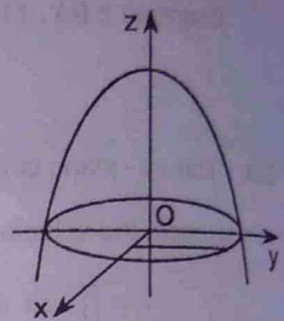


Fig. 13.37

Put $x = 2a \sin \theta$, $dx = 2a \cos \theta \, d\theta$.

$$\therefore \quad V = \frac{8b}{3a^3} \int_0^{\pi/2} 8a^3 \cdot \cos^3 \theta \cdot 2a \cos \theta \, d\theta$$

$$= \frac{8b}{3a^3} \cdot 16 \, a^4 \int_0^{\pi/2} \cos^4 \theta \, d\theta$$

$$= \frac{8b}{3a^3} \cdot 16a^4 \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = 8 \pi ab.$$

Example 5 (c) : Find the volume of the solid bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$ and the planes $x = 2$, $y = 2$ and the three coordinates planes.

Sol. : The section of the paraboloid by the plane $z = 0$ is the ellipse $x^2 + 2y^2 = 16$

$$i.e., \qquad \frac{x^2}{16} + \frac{y^2}{8} = 1.$$

The required volume is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x = 2$, $y = 2$ and by the paraboloid $x^2 + 2y^2 + z = 16$ on the top. Consider a cuboid as in Ex. 1 page 13-1, then

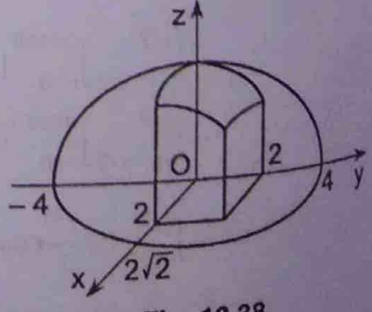


Fig. 13.38

$$V = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{16 - x^2 - 2y^2} dz \, dy \, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} [z]_{z=0}^{16 - x^2 - 2y^2} dy \, dx$$

$$\therefore \quad V = \int_{x=0}^{2} \int_{y=0}^{2} (16 - x^2 - 2y^2)\, dy\, dx$$

$$= \int_{x=0}^{2} \left[16y - x^2 y - \frac{2y^3}{3} \right]_0^2 dx = \int_0^2 \left(32 - 2x^2 - \frac{16}{3} \right) dx$$

$$= \int_0^2 \left(\frac{80}{3} - 2x^2 \right) dx = \left[\frac{80}{3} x - \frac{2x^3}{3} \right]_0^2 = \left[\frac{160}{3} - \frac{16}{3} \right] = 48.$$

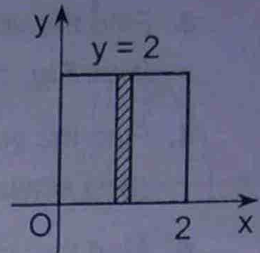


Fig. 13.39

Example 6 (c) : Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$ and $x + y = 2$ in the xy-plane.

Sol. : The base of the required solid is a triangle OAB.

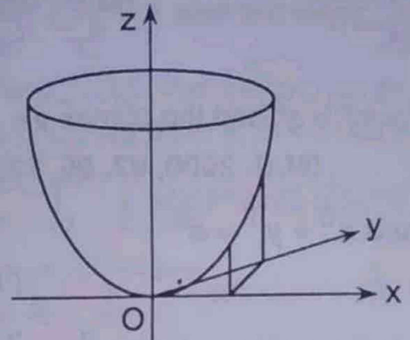


Fig. 13.40

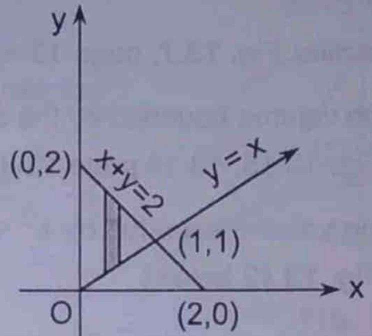


Fig. 13.41

Take a strip parallel to the y-axis. On this strip y-varies from $y = x$ to $y = 2 - x$. The strip moves parallel to itself from $x = 0$ to $x = 1$. z varies from $z = 0$ to $z = x^2 + y^2$.

$$\therefore \quad V = \int_{x=0}^{1} \int_{y=x}^{2-x} \int_{z=0}^{x^2+y^2} dz\, dy\, dx = \int_{x=0}^{1} \int_{y=x}^{2-x} [x^2 + y^2]\, dy\, dx$$

$$= \int_{x=0}^{1} \left[x^2 y + \frac{y^3}{3} \right]_x^{2-x} dx = \int_0^1 \left\{ \left[x^2 (2-x) + \frac{(2-x)^3}{3} \right] - \left[x^3 + \frac{x^3}{3} \right] \right\} dx$$

$$= \int_0^1 \left[2x^2 - \frac{7}{3} x^3 + \frac{(2-x)^3}{3} \right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12} \right]_0^1$$

$$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12} \right) - \left(-\frac{16}{12} \right) = \frac{2}{3} - \frac{8}{12} + \frac{16}{12} = \frac{16}{12} = \frac{4}{3}.$$

3. Find the volume bounded by the coordinate planes and the plane $x + y + z = 1$.
(See Fig. 13.2, page 13-2, $a = 1$) **(M.U. 1995)** [**Ans. :** $1/6$]

4. Find the volume bounded by the coordinates planes and the plane $lx + my + nz = 1$.
(See similar Fig. 13.2, page 13-2) [**Ans. :** $1/(6lmn)$]

5. Find the volume bounded by $y^2 = x$, $x^2 = y$ and the planes $z = 0$ to $z = a$.
(See similar Fig. 13.8, page 13-5) **(M.U. 1999)** [**Ans. :** $a/3$]

6. Find the volume bounded by $y^2 = x$, $x^2 = y$ and the planes $z = 0$ and $z = 3$.
(See similar Fig. 13.8, page 13-5) **(M.U. 1999, 2013)** [**Ans. :** 1]

7. Find the volume bounded by the cylinders $y^2 = x$ and $x^2 = y$ between the planes $z = 0$ and $x + y + z = 2$.
(M.U. 1991, 99, 2002, 12)
(See similar Fig. 13.7, page 13-4) [**Ans. :** $11/30$]

8. Find the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $y + z = 2a$ and $z = 0$.
(See Fig. 13.15, 13.16 page 13-8; $b = 2a$) **(M.U. 2000, 02, 06, 12)** [**Ans. :** $2\pi a^3$]

9. Find the volume bounded by $x^2 + y^2 = bz$ and $x^2 + y^2 = a^2$.
(See Fig. 13.42 below) [**Ans. :** $\pi a^4/2b$]

10. By using triple integration find the volume cut off from the sphere $x^2 + y^2 + z^2 = 16$ by the plane $z = 0$ and the cylinder $x^2 + y^2 = 4x$.
(See Fig. 13.43 below) **(M.U. 1989)** [**Ans. :** $\frac{64}{9}(3\pi - 4)$]

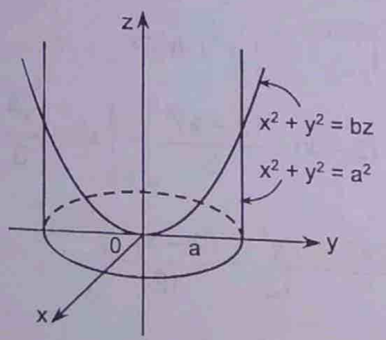


Fig. 13.42

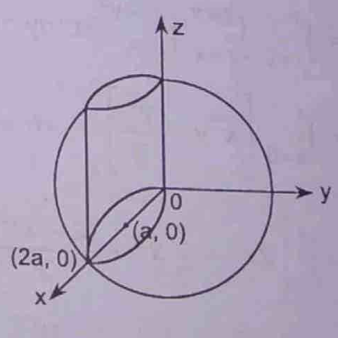


Fig. 13.43

11. Find the volume enclosed by the cylinders $x^2 + y^2 = 4x$ and $z^2 = 4x$ by triple integration.
(See Fig. 13.23, page 13-11. Put $a = 2$) **(M.U. 1989)** [**Ans. :** $1024/15$]

12. Find the volume enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $x + z = 5$ and $z = 0$.
(See Fig. 13.15, page 13-8) **(M.U. 1990)** [**Ans. :** $45\pi - 36$]

13. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ by triple integration.
(See Fig. 15.67, page 15-22)
(**Hint :** $8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2 \sin\theta \; dr \; d\theta \; d\phi$) [**Ans. :** $\frac{4}{3}\pi a^3$]

14. Find the volume enclosed by the sphere $x^2 + y^2 + z^2 = a^2$ and the cone $x^2 + y^2 = z^2 \tan \alpha$. (See Fig. 13.44)

(**Hint :** Use spherical polar coordinates

$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta.$

Sphere is $r = a$, cone is $r^2 \sin^2\theta = r^2\cos^2\theta \tan^2\alpha \quad \therefore \theta = \alpha.$

$$\therefore \quad V = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\alpha} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi .)$$

[**Ans. :** $\dfrac{2\pi a^3}{3}(1-\cos\alpha)$]

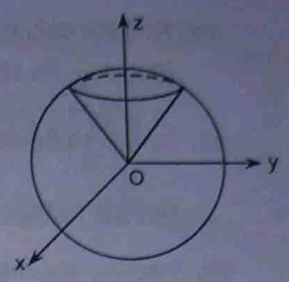


Fig. 13.44

15. Find the volume bounded in the first octant by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ and the plane } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

(**Hint :** $V = V_1 - V_2$. Put $x = au$, $y = bv$, $z = cw$. Then use $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$

$$V_1 = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 abc\, r^2 \sin\theta \, dr\, d\theta\, d\phi = \frac{\pi\, abc}{6}$$

$$V_2 = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} abc\, du\, dv\, dw = \frac{abc}{6})$$

(See Fig. 13.45) [**Ans. :** $\dfrac{abc}{6}(\pi - 1)$]

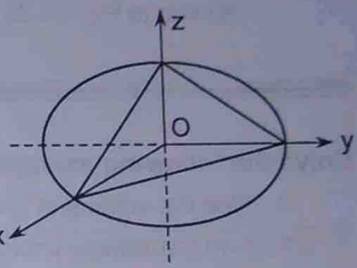


Fig. 13.45

16. Find the volume of region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2x$.

(Similar to Fig. 13.46.) [**Ans. :** $\pi / 2$]

17. Find the volume of region above the xy - plane enclosed by the paraboloid $az = x^2 + y^2$ and the cylinder $x^2 + y^2 = 2ax$.

$$(\textbf{Hint :} \quad V = \iint_R \int_0^{(x^2+y^2)/a} dx\, dy\, dz = \frac{1}{a}\iint_R (x^2+y^2)\, dx\, dy$$

$$= \frac{2}{a}\int_0^{\pi/2}\int_0^{2a\cos\theta} r^3 \, d\theta\, dr)$$

(Similar to Fig. 13.27, page 13-12.) [**Ans. :** $\dfrac{3\pi}{2}a^3$]

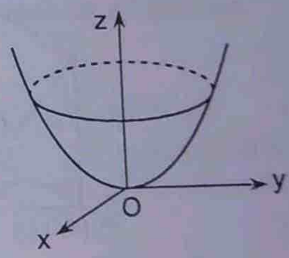


Fig. 13.46

18. Find the volume of the paraboloid $x^2 + y^2 = 4z$ cut off by the plane $z = 4$. (See Fig. 13.46) **(M.U. 1987, 91)**

(**Hint :** $V = 4\displaystyle\int_0^{\pi/2}\int_0^4\int_{r^2/4}^4 r\, dr\, d\theta\, dz$) [**Ans. :** 32π]

19. Find the volume of the solid bounded by the plane $z = 0$, the paraboloid $z = x^2 + y^2 + 2$ and the cylinder $x^2 + y^2 = 4$. (See Fig. 13.47) **(M.U. 1990)**

(**Hint :** $4\displaystyle\int_{\theta=0}^{\pi/2}\int_{r=0}^{2}\int_{z=0}^{r^2+2} r\, dr\, d\theta\, dz$) [**Ans. :** 16π]

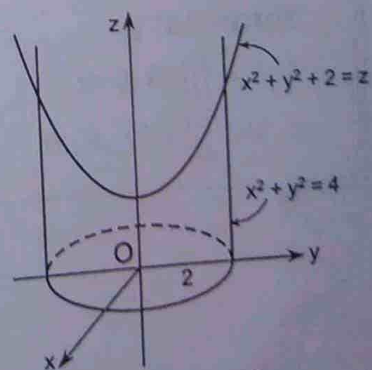


Fig. 13.47

20. Find the volume of the paraboloid $x^2 + y^2 = az$ cut off by the plane $z = a$.
(See Fig. 13.46) **(M.U. 2009)** [**Ans. :** $\pi a^3 / 2$]

21. Find the volume cut off from the paraboloid $x^2 + \frac{y^2}{9} + z = 1$ by the plane $z = 0$.
(Similar to Fig. 13.35, page 13-15) [**Ans. :** $\frac{3}{2}\pi$]

22. Find the volume bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy - plane.
(Similar to Fig. 13.35, page 13-15) [**Ans. :** 8π]

23. Find the volume bounded by they paraboloid $z = 4 - x^2 - \frac{y^2}{4}$ and the plane $z = 0$.
(Similar to Fig. 13.35, page 13-15) [**Ans. :** 16π]

EXERCISE - II

Solve the following examples : Class (a) : 3 or 4 Marks

1. Find the volume of the cuboid of sides a, b, c by triple integration. [**Ans. :** abc]
2. Find the volume enclosed by the planes $y + z = 1$, $z = 0$, $y = 0$, $x = 0$, $x = 1$.
(See Fig. 13.48) [**Ans. :** $1/2$]

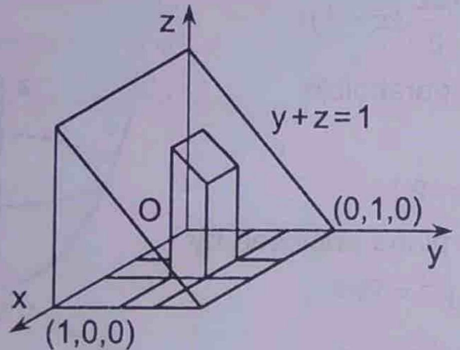


Fig. 13.48

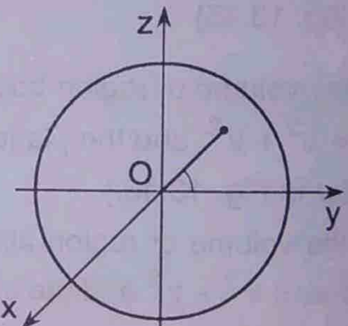


Fig. 13.49

3. Find the volume of the sphere of radius a, using triple integration.
(See Fig. 13.49) [**Ans. :** $\frac{4}{3}\pi a^3$]

Summary

1. $V = \iiint dx\, dy\, dz$
2. $V = \iiint r\, dr\, d\theta\, dz$

CHAPTER 14

Numerical Integration

1. Introduction

Many a time we come across a function $f(x)$ such $\int_a^b f(x)\,dx$ which is highly difficult to find and in some cases impossible, *e.g.*, $\int \sqrt{\sin x}\,dx$, $\int e^{x^2}\,dx$, etc. But we know that $\int_a^b f(x)\,dx =$ the area bounded by the curve $y = f(x)$, the ordinates at $x = 0$, $x = b$ and the x-axis. If we know the values of $y = f(x)$ at $x = x_0, x_1, \ldots, x_n = b$. Then we can find approximately this area by considering trapeziums through two consecutive points, or parabolas passing through three consecutive points or curves passing through four consecutive points. The sum of all such areas will be approximately equal to the required area *i.e.*, $\int_a^b f(x)\,dx$.

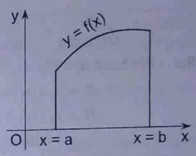


Fig. 14.1

This process of finding the definite integral $\int_a^b f(x)\,dx$ is known as **numerical integration**.

We shall consider here three rules of numerical integration -

(1) Trapezoidal Rule, (2) Simpson's (1/3)rd rule, (3) Simpson's (3/8)th rule.

In trapezoidal rule, we fit straight lines through two consecutive points, in Simpson's (1/3)rd rule we fit parabolas through three consecutive points and in Simpson's (3/8)th rule, we fit cubic curves through three consecutive points.

2. Trapezoidal Rule

Let the curve from $A_0\ (x_0, y_0)$ to $A_n\ (x_n, y_n)$ be given by $y = f(x)$. Let $A_1\ (x_1, y_1)$, ..., $A_{n-1}\ (x_{n-1}, y_{n-1})$ be the points of sub-division as before.

Let these points on the curve be connected by straight lines. Thus, area under the curve from A_0 to A_1 is approximated to the area of the first trapezium, whose area $= \frac{1}{2}(y_0 + y_1) \cdot h$.

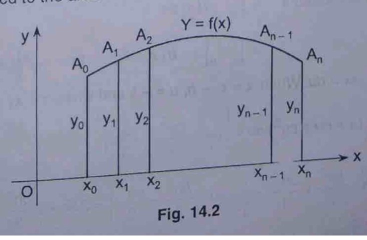


Fig. 14.2

Total area = sum of the areas of all trapeziums.

$$\therefore \quad \int_a^b f(x)\,dx = \frac{1}{2}h\left[(y_0+y_1)+(y_1+y_2)+\ldots..+(y_{n-1}+y_n)\right]$$

$$= \frac{1}{2}h\left[(y_0+y_n)+2(y_1+y_2+\ldots..+y_{n-1})\right]$$

$$\therefore \quad \boxed{\int_a^b f(x)\,dx = \frac{1}{2}h[X+2R]}$$

where X = Sum of the eXtreme ordinates and R = Sum of the Remaining ordinates.

Example : Evaluate $\int_0^1 \sqrt{\sin x + \cos x\,dx}$ by trapezoidal rule by using the following data.

x	:	0	0·2	0·4	0·6	0·8	1
y	:	1	1·0857	1·1448	1·1790	1·1891	1·1755
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5

Sol. : We have $h = 0{\cdot}2$.

$$X = y_0 + y_5 = 1 + 1{\cdot}1755 = 2{\cdot}1755$$

$$R = y_1 + y_2 + y_3 + y_4$$

$$= 1{\cdot}0857 + 1{\cdot}1448 + 1{\cdot}1790 + 1{\cdot}1891 = 4{\cdot}5986$$

$$\therefore \quad \int_a^b f(x)\,dx = \frac{h}{2}[X+2R] = \frac{0{\cdot}2}{2}[2{\cdot}1755 + 9{\cdot}1972] = 1{\cdot}13727$$

3. Simpson's (1/3)rd Rule

Let the equation of the parabola passing through the first three points (x_0, y_0), (x_1, y_1), (x_2, y_2) be

$$y = a + b(x - x_1) + c(x - x_1)^2$$

$$y_0 = a + b(x_0 - x_1) + c(x_0 - x_1)^2 \quad \text{when } x = x_0,$$

$$y_1 = a \quad \text{when } x = x_1,$$

$$y_2 = a + b(x_2 - x_1) + c(x_2 - x_1)^2 \quad \text{when } x = x_2,$$

But $x_0 - x_1 = -h$ and $x_2 - x_1 = h$

$$\therefore \quad y_0 = a - bh + ch^2$$

$$y_1 = a \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

$$y_2 = a + bh + ch^2$$

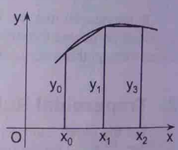


Fig. 14.3

Now, area under this parabola $= \int_{x_1-h}^{x_1+h} f(x)\,dx$

$$= \int_{x_1-h}^{x_1+h}\left[a + b(x-x_1) + c(x-x_1)^2\right]dx$$

Putting $x - x_1 = u$, $dx = du$. When $x = x_1 - h$, $u = -h$ and when $x = x_1 + h$, $u = h$.

$$\therefore \quad \text{Area} = \int_{-h}^{h}(a + bu + cu^2)\,du = \left[au + \frac{bu^2}{2} + \frac{cu^3}{3}\right]_{-h}^{h}$$

$$\therefore \quad \text{Area} = \left(ah + \frac{bh^2}{2} + \frac{ch^3}{3}\right) - \left(-ah + \frac{bh^2}{2} - \frac{ch^3}{3}\right) = 2ah + 2\frac{ch^3}{3} = \frac{h}{3}(6a + 2ch^2)$$

But it can be easily seen from (A) that $y_0 + 4y_1 + y_2 = 6a + 2ch^2$.

Assuming that the number of intervals is even and taking the area under all such parabolas,

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \ldots + (y_{n-2} + 4y_{n-1} + y_n)\right]$$

$$= \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + \ldots + y_{n-2}) + 4(y_1 + y_3 + y_5 + \ldots + y_{n-1})\right]$$

$$\therefore \quad \boxed{\int_a^b f(x)\,dx = \frac{h}{3}[X + 2E + 4O]}$$

where, X = sum of the eXtreme ordinates, E = sum of the Even ordinates,

O = Sum of Odd ordinates.

Note ....

As anywhere else in calculus of finite differences our first point is (x_0, y_0) *i.e.* we start counting from zero and not one. With this convention while using Simpson's rule the order of the last point should be even *i.e.* 2, 4, 6, 8, 10, ... etc. *i.e.*, the number of sub-intervals must be even.

Thomas Simpson (1710 - 1761)

Thomas Simpson was a British mathematician. Now well known Simpson's rule to approximate definite integrals is credited to him, but the rule was first discovered by Johannes Kepler and was in use for 100 years before. He was the son of a weaver and taught himself mathematics. After seeing a solar eclipse he started studying astronomy. He taught mathematics at the Royal Military Academy from 1743. He was a fellow of Royal Society. His published works include Treatise On Fluxions, The Nature and Laws of Chance, A Treatise On Algebra, etc.

Solved Examples : Class (a) : 4 Marks

Example 1 (a) : Obtain by Simpson's rule the area under the curve bounded by the x-axis, ordinates at x = 0 and x = 1, if the ordinates at different points of the curve are :

x	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1
y	1	0·990	0·962	0·917	0·862	0·0800	0·735	0·671	0·610	0·553	0.500
Ordinate	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

Sol. : Simpson's rule can be applied because $n = 10$ is even. Here $h = 0{\cdot}1$,

X = sum of extreme ordinates

$= 1 + 0{\cdot}50 = 1{\cdot}50$

E = sum of even ordered ordinates

$= 0{\cdot}962 + 0{\cdot}862 + 0{\cdot}735 + 0{\cdot}610 = 3{\cdot}169$

O = sum of odd ordered ordinates

$= 0{\cdot}990 + 0{\cdot}917 + 0{\cdot}800 + 0{\cdot}671 + 0{\cdot}553 = 3{\cdot}931$

$$\therefore \quad \text{Area} = \frac{h}{3}[X + 2E + 4O] = \frac{0\cdot1}{3}[1\cdot50 + 6\cdot338 + 15\cdot724]$$

$$= \frac{0\cdot1}{3}(23\cdot562) = 0\cdot7854$$

Example 2 (a) : The water under portion of a water-tank is divided by horizontal planes one metre apart into the following areas : 472, 398, 302, 198, 116, 60, 34, 12 and 4 sq. m. Use the trapezoidal rule to find the volume in cubic metres between the two extreme ends.

Sol. : We prepare the table as follows :

Distance *S* :	1	2	3	4	5	6	7	8	9
Area *A* :	472	398	302	198	116	60	34	12	4

Since, $V = A \times S$.

By Trapezoidal rule, $V = \frac{h}{2}[X + 2R]$

$h = 1, \ X = 472 + 4 = 476$

$R = 398 + 302 + 198 + 116 + 60 + 34 + 12 = 1120$

$\therefore \quad V = \frac{1}{2}[476 + 2240] = 1358$ cubic metres.

Example 3 (a) : The velocity of a train which starts from rest is given by the following table the time being reckoned in minutes from the start and speed in km / hour.

Time :	2	4	6	8	10	12	14	16	18	20
Speed :	10	18	25	29	32	30	11	5	2	0

Find the total distance run in 20 minutes.

Sol. : We know that $\frac{ds}{dt} = v \quad i.e., \quad S = \int v\, dt \quad i.e., \quad S = \int_0^2 v\, dt$

$\therefore$ To use Simpson's Rule we need even number of ordinates. We can take one more ordinate at $t = 0$. By data when $t = 0$, $v = 0$.

Let the ordinates be denoted as

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
0	10	18	25	29	32	30	11	5	2	0

Then by Simpson's (1/3)rd rule

$$S = \frac{h}{3}[X + 2E + 4O]$$

We have $h = \frac{2}{60}$ hrs. $= \frac{1}{30}$ hrs.

$X = 0 + 0, \ E = 18 + 29 + 30 + 5 = 82.$

$O = 10 + 25 + 32 + 11 + 2 = 80$

$\therefore \quad S = \frac{1}{90}[0 + 164 + 320] = 5\cdot378$ kms.

Example 4 (a) : In an experiment a quantity G was measured as follows :

$G(20) = 95\cdot90$, $G(21) = 96\cdot85$, $G(22) = 97\cdot77$, $G(23) = 98\cdot68$,
$G(24) = 99\cdot56$, $G(25) = 100\cdot41$, $G(26) = 101\cdot24$.

Compute $\int_{20}^{26} G(x)\,dx$ by both Simpson's (1/3)rd rule and Trapezoidal rule. **(M.U. 2010)**

Sol. : The table can be prepared as

x	:	20	21	22	23	24	25	26
$G(x)$	:	95·90	96·85	97·77	98·68	99·56	100·41	101·24
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By trapezoidal rule,

$$\int_{20}^{26} G(x)\,dx = \frac{h}{2}[X + 2R]$$

But $h = 1,\ X = 95{\cdot}90 + 101{\cdot}24 = 197{\cdot}14$

$R = [\,96{\cdot}85 + 97{\cdot}77 + 98{\cdot}68 + 99{\cdot}56 + 100{\cdot}41\,] = 493{\cdot}27$

$$\therefore\ \int_{20}^{26} G(x)\,dx = \frac{1}{2}[197{\cdot}14 + 986{\cdot}54] = 591{\cdot}84$$

(ii) By Simpson's (1/3)rd rule,

$$\int_{20}^{26} G(x)\,dx = \frac{h}{3}[X + 2E + 4O]$$

Now $X = 95{\cdot}90 + 101{\cdot}24 = 197{\cdot}14$

$2E = 2\,(97{\cdot}77 + 99{\cdot}56) = 394{\cdot}66$

$4O = 4\,(96{\cdot}85 + 98{\cdot}68 + 100{\cdot}41) = 1183{\cdot}76$

$$\therefore\ \int_{20}^{26} G(x)\,dx = \frac{1}{3}[197{\cdot}14 + 394{\cdot}66 + 1183{\cdot}76] = 591{\cdot}85$$

Example 5 (a) : Find the volume of solid of revolution formed by rotating about the x-axis, the area bounded by the lines $x = 0$, $x = 1$, $y = 0$ and the curve passing through the points given below.

x	:	0	0·25	0·50	0·75	1
y	:	1	0·9896	0·9589	0·9089	0·8415

Sol. : The volume is given by $V = \int_a^b \pi\, y^2\,dx$

We prepare the table of y^2

x	:	0	0·25	0·50	0·75	1
y^2	:	1	0·9793	0·9195	0·8261	0·7081
Ordinate	:	y_0	y_1	y_2	y_3	y_4

(i) By Trapezoidal rule

$$V = \pi \cdot \frac{h}{2}[X + 2R]$$

$$\therefore\ V = 3{\cdot}14156\left(\frac{0{\cdot}25}{2}\right)[(1 + 0{\cdot}7081) + 2(0{\cdot}9793 + 0{\cdot}9195 + 0{\cdot}8261)] = 2{\cdot}8109$$

(ii) By Simpson's (1/3)rd rule

$$V = \pi \cdot \frac{h}{3}[X + 2E + 4O]$$

$$\therefore\ V = 3{\cdot}14159\left(\frac{0{\cdot}25}{3}\right)[(1 + 0{\cdot}7081) + 2(0{\cdot}9195) + 4(0{\cdot}9793 + 0{\cdot}8261)] = 2{\cdot}819$$

Example 6 (a) : Evaluate $\int_0^{\pi} \frac{\sin^2 \theta}{5+4\cos\theta} d\theta$ by taking 5 ordinates by Simpson's (1/3)rd rule.

Sol. : We divide the interval (0, 180°) in 4 equal parts by taking $h = 45^\circ$ as follows :

θ	:	0	45°	90°	135°	180°
y	:	0	0·0639	0·2	0·2302	0
Ordinate	:	y_0	y_1	y_2	y_3	y_4

By Simpson's (1/3)rd rule

$$S = \frac{h}{3}[X + 2E + 4O]$$

We have $h = \pi/4$, $X = 0 + 0 = 0$, $E = 2\,(0·2) = 0·4$

$$4O = 4\,(0·0639 + 0·2302) = 1·1764$$

$$\therefore \quad S = \frac{\pi}{12}[0 + 0·4 + 1·1764] = 0·4127$$

Example 7 (a) : Find the value of the integral $\int_0^1 \frac{x^2}{1+x^3} dx$, using any two methods.

Compare the errors with the exact value of the integral. Also find log 2. **(M.U. 2004, 09, 10)**

Sol. : We shall use Simpson's (1/3)rd rule and trapezoidal rule. Since to use Simpson's rule we need even number of ordinates we shall divide the interval (0, 1) into 4 equal parts by taking $h = 0·25$. We prepare the following table.

x	:	0	0·25	0·50	0·75	1·0
y	:	0	0·06153	0·22222	0·39560	0·5
Ordinate	:	y_0	y_1	y_2	y_3	y_4

(i) By Simpson's (1/3)rd rule

$$S = \frac{h}{3}[X + 2E + 4O]$$

We have $h = 0·25$, $X = 0 + 0·5 = 0·5$, $2E = 2\,(0·22222) = 0·44444$

$$4O = 4\,(0·06153 + 0·39560) = 1·82852$$

$$\therefore \quad I = \frac{0·25}{3}[0·5 + 0·44444 + 1·82852] = 0·23108$$

(ii) Also by Trapezoidal rule

$$I = \frac{h}{2}[X + 2R] = \frac{0·25}{2}[(0·0 + 0·5) + 2(0·06153 + 0·22222 + 0·39560)]$$

$$= 0·23233$$

Now, by putting $1 + x^3 = t \quad \therefore \; x^2\,dx = \frac{dt}{3}$

$$\int_0^1 \frac{x^2}{1+x^3} dx = \left[\frac{1}{3}\log(1+x^3)\right]_0^1 = \frac{1}{3}\log 2 = 0·23105$$

$\therefore$ Error in (1) = 0·23108 − 23105 = − 0·00003

Error in (2) = 0·23233 − 0·23105 = − 0·00128

Thus, Simpson's Rule gives better approximation.

Example 8 (a) : A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the following table.

***t* (sec.)** :	0	10	20	30	40	50	60	70	80
***a* (m/s²)** :	30	31·63	33·34	35·47	37·75	40·33	43·25	46·69	50·67
Ordinate :	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By using Simpson's one-third rule, find the velocity at $t = 80$. **(M.U. 2004)**

Sol. : By Simpson's (1/3)rd rule

$$I = \frac{h}{3}[X + 2E + 4O]$$

We have $X = 30 + 50{\cdot}67 = 80{\cdot}67$

$$E = 33{\cdot}34 + 37{\cdot}75 + 43{\cdot}25 = 114{\cdot}34$$

$$O = 31{\cdot}63 + 35{\cdot}47 + 40{\cdot}33 + 46{\cdot}69 = 154{\cdot}12$$

$$\therefore \quad I = \frac{10}{3}[80{\cdot}67 + 2(114{\cdot}34) + 4(154{\cdot}12)]$$

$$= \frac{10}{3}[925{\cdot}83] = 3086{\cdot}1 \text{ cm/sec.}$$

EXERCISE - I

Solve the following examples : Class (a) : 4 Marks

1. Given the following values of e^x.

x :	0·0	0·5	1·0	1·5	2·0	2·5
e^x :	1·0	1·65	2·72	4·48	7·39	12·18

Evaluate $\int_0^{2\cdot5} e^x \, dx$ using Trapezoidal rule. [**Ans. :** 11·415]

2. Find using the Trapezoidal rule from the following table the area bounded by the curve and x-axis from $x = 7{\cdot}47$ to $x = 7{\cdot}52$.

x :	7·47	7·48	7·49	7·50	7·51	7·52
$f(x)$:	1·93	1·95	1·98	2·01	2·03	2·06

[**Ans. :** 0·0996]

3. A curve is drawn to pass through the points given by

x :	1·0	1·5	2·0	2·5	3·0	3·5	4·0
y :	2·0	2·4	2·7	2·8	3·0	2·6	2·1

Estimate the area bounded by the curve between the x-axis and the lines $x = 1$ and $x = 4$, by Simpson's rule. [**Ans. :** 7·7833]

4. A river is 90 meter wide. The depth d meters from one bank is given by the following table. Find approximately the area of the cross-section.

a :	0	10	20	30	40	50	60	70	80	90
d :	0	4	7	9	12	15	14	8	3	1

[**Ans. :** 725]

5. Calculate upto 5 decimal places $\int_4^{5\cdot2} \log_e x \cdot dx$ by using Trapezoidal rule from the following table.

x	:	4·0	4·2	4·4	4·6	4·8	5·0	5·2
$\log_e x$	:	1·3863	1·4351	1·4816	1·5260	1·5886	1·6094	1·6486

[**Ans. :** 1·83163]

6. Given the following values of sin x

x	:	0	$2\pi/10$	$4\pi/10$	$6\pi/10$	$8\pi/10$	π
sin x	:	0·000	0·5878	0·9511	0·9511	0·5878	0·000

Evaluate $\int_0^{\pi} \sin x\, dx$ by any suitable rule and compare it with its exact value.

[**Ans. :** 0·9669, 1]

7. Apply Simpson's (1/3)rd rule to find $\int_0^6 \frac{1}{1+x}\, dx$. [**Ans. :** 1·9588]

8. Use any rule to evaluate approximately $\int_0^1 \frac{dx}{1+x^2}$ taking five equidistant intervals and find the value of π. **(M.U. 2003)** [**Ans. :** 0·7837 ; $\pi = 3.1348$]

9. Calculate by trapezoidal rule an approximate value of $\int_0^1 e^x\, dx$ in steps of 0·20.

[**Ans. :** 1·72399]

10. Evaluate in two ways $\int_4^{5\cdot2} \log_e x\, dx$ by dividing the interval [4, 5·2] into six equal parts.

[**Ans. :** 1·8279]

11. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by taking suitable sub-intervals and hence, find an approximate value of π by using (i) Trapezoidal rule, (ii) Simpson's (1/3)rd rule. **(M.U. 2003, 05)**

[**Ans. : (i)** 0·7842, **(ii)** 0·7854, **(iii)** 3·1412]

12. Calculate $\int_2^{10} \frac{dx}{1+x}$ upto 4 decimal places by dividing the range into eight equal parts by Simpson's (1/3)rd rule. [**Ans. :** 1·2996]

13. Apply Simpson's (1/3)rd rule to find the value of $\int_0^6 \frac{dx}{1+x^2}$ upto two places of decimals.

(M.U. 2003) [**Ans. :** 1·37]

14. Evaluate $\int_{-3}^{3} x^4\, dx$ by Trapezoidal rule and compare it with the exact value.

[**Ans. :** 115, 97·2]

15. Using Simpson's one third rule obtain $\int_1^2 \frac{dx}{x}$ taking 10 equal intervals. Hence, obtain approximate value of log 2 upto five places of decimals. **(M.U. 2003)** [**Ans. :** 0·69315]

16. Evaluate $\int_{0\cdot5}^{0\cdot7} \sqrt{x}\, e^{-x}\, dx$ using Simpson's (1/3)rd rule. [**Ans. :** 0·08483]

4. Simpson's (3/8)th Rule

Consider a cubic curve passing through the points (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3). Let its equation be

$$y = a + b(x - x_1) + c(x - x_1)^2 + d(x - x_1)^3$$

When $x = x_0$, $y_0 = a + b(x_0 - x_1) + c(x_0 - x_1)^2 + d(x_0 - x_1)^3$

When $x = x_1$, $y_1 = a$

When $x = x_2$, $y_2 = a + b(x_2 - x_1) + c(x_2 - x_1)^2 + d(x_2 - x_1)^3$

When $x = x_3$, $y_3 = a + b(x_3 - x_1) + c(x_3 - x_1)^2 + d(x_3 - x_1)^3$

Since $x_0 - x_1 = -h$, $x_2 - x_1 = h$ and $x_3 - x_1 = -2h$

$$y_0 = a - bh + ch^2 - dh^3$$
$$y_1 = a$$
$$y_2 = a + bh + ch^2 + dh^3$$
$$y_3 = a + 2bh + 4ch^2 + 8dh^3$$

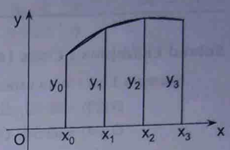


Fig. 14.4

Now, the area under the first cubic curve

$$= \int_{x_1 - h}^{x_1 + 2h} f(x)\,dx$$

$$= \int_{x_1 - h}^{x_1 + 2h} \left[a + b(x - x_1) + c(x - x_1)^2 + d(x - x_1)^3\right] dx$$

Put $x - x_1 = u$, $dx = du$. When $x = x_1 - h$, $-h = u$,

When $x = x_1 + 2h$, $2h = u$

$$\therefore \quad \text{Area} = \int_{-h}^{2h} (a + bu + cu^2 + du^3)\,du$$

$$= \left[au + \frac{bu^2}{2} + \frac{cu^3}{3} + \frac{du^4}{4}\right]_{-h}^{2h}$$

$$= \left[2ah + \frac{4bh^2}{2} + \frac{8ch^3}{3} + \frac{16ch^4}{4}\right] - \left[-ah + \frac{bh^2}{2} - \frac{ch^3}{3} + \frac{dh^4}{4}\right]$$

$$= 3ah + \frac{3bh^2}{2} + 3ch^3 + \frac{15dh^4}{4}$$

$$= \frac{3}{8} h\left[8a + 4bh^2 + 8ch^3 + 10dh^4\right]$$

It can be seen from (A) that this gives

$$\text{Area} = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Considering the area under the consecutive cubic curves and taking the sum.

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \ldots\ldots + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\right]$$

$$= \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots) + 2(y_3 + y_6 + y_9 + \ldots\ldots)\right]$$

$$\therefore \quad \boxed{\int_a^b f(x)\,dx = \frac{3h}{8}\left[X + 2T + 3R\right]}$$

where, X = sum of the e**X**tremes, T = sum of the multiples of **T**hree,

R = sum of the **R**emaining ordinates.

Note ....

As anywhere else where calculus of finite difference is used our first point is (x_0, y_0) *i.e.* we start counting from 0. While using Simpson's (1/3)rd rule, the last ordinate should be even *i.e.* 2, 4, 6, ... etc. and while using Simpson's (3/8)th rule, the last ordinate should be multiple of 3 *i.e.* 3, 6, 9, 12, ... etc.

Solved Examples : Class (a) : 4 Marks

Example 1 (a) : In an experiment a quantity G was measured as follows :

$G(20) = 95{\cdot}90,\ G(21) = 96{\cdot}85,\ G(22) = 97{\cdot}77,\ G(23) = 98{\cdot}68$

$G(24) = 99{\cdot}56,\ G(25) = 100{\cdot}41,\ G(26) = 101{\cdot}24.$

Compute $\int_{20}^{26} G(x)\,dx$ by Simpson's (3/8)th rule.

Sol. : The table can be prepared as

x :	20	21	22	23	24	25	26
$G(x)$:	95·90	96·85	97·77	98·68	99·56	100·41	101·24
Ordinate :	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's (3/8)th rule,

$$\int_{20}^{26} G(x)\,dx = \frac{3h}{8}[X + 2T + 3R]$$

$$\int_{20}^{26} G(x)\,dx = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

Now, $y_0 + y_6 = 95{\cdot}90 + 101{\cdot}24 = 197{\cdot}14$

$y_1 + y_2 + y_4 + y_5 = 96{\cdot}85 + 97{\cdot}77 + 99{\cdot}56 + 100{\cdot}41 = 394{\cdot}59$

$y_3 = 98{\cdot}68.$

$$\therefore \int_{20}^{26} G(x)\,dx = \frac{3}{8}[197{\cdot}14 + 197{\cdot}36 + 1183{\cdot}77] = 591{\cdot}85$$

(See also Ex. 4, page 14-4.)

Example 2 (a) : Evaluate $\int_0^{\pi/2} \frac{\sin x}{x}\,dx$ by Simpson's (3/8)th rule. **(M.U. 2009, 10)**

Sol. : We take $h = \pi/12$ and prepare the following table.

x :	0	$\pi/12$	$2\pi/12$	$3\pi/12$	$4\pi/12$	$5\pi/12$	$6\pi/12$
y :	1	0.9886	0.9579	0.9003	0.8270	0.7379	0.6366
Ordinate :	y_0	y_1	y_2	y_3	y_4	y_4	y_5

Since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, we take $\frac{\sin x}{x} = 1$ at $x = 0$.

By Simpson's (3/8)th rule, $S = \frac{3h}{8}[X + 2T + 3R]$

We have $X = y_0 + y_6 = 1{\cdot}6366,\ T = y_3 = 0{\cdot}9003$

$R = y_1 + y_2 + y_4 + y_5 = 3{\cdot}5086$

$\therefore\ S = \frac{3}{8} \cdot \frac{\pi}{12}[1{\cdot}6366 + 2(0{\cdot}9003)$

Example 3 (a) : Evaluate $\int_0^{\pi} \frac{\sin^2 \theta}{5 + 4\cos\theta} d\theta$ by dividing $(0, \pi)$ into 6 sub-intervals by Simpson's (3/8)th rule.

Sol. : We take $h = \pi / 6$ and prepare the table

x	:	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
y	:	0	0.0295	0.1071	0.2000	0.2500	0.1628	0
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's (3/8)th rule,

$$S = \frac{3h}{8}[X + 2T + 3R]$$

We have $X = y_0 + y_6 = 0, \quad T = y_3 = 0.2.$

$R = y_1 + y_2 + y_4 + y_5 = 0.5494.$

$$\therefore \quad S = \frac{3}{8} \cdot \frac{\pi}{8}[0 + 2(0.2) + 3(0.5494)] = 0.40216$$

Example 4 (a) : Evaluate $\int_0^6 \frac{dx}{1 + x^2}$ by using Simpson's (3/8)th rule. Also find the error.

(M.U. 2003)

Sol. : We divide the interval into 6 equal parts by taking $h = 1$ and prepare the following table.

x	:	0	1	2	3	4	5	6
$y = \frac{1}{1 + x^2}$	:	1	0.5	0.2	0.1	0.0588	0.0385	0.027
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's (3/8)th rule,

$$\int_0^6 \frac{dx}{1 + x^2} = \frac{3h}{8}[X + 2T + 3R] = \frac{3h}{8}[(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)]$$

Now, $X = y_0 + y_6 = 1 + 0.027 = 1.027, \quad T = y_3 = 0.1.$

$R = y_1 + y_2 + y_4 + y_5 = 0.5 + 0.2 + 0.0588 + 0.0385 = 0.7973$

$$\therefore \quad \int_0^6 \frac{dx}{1 + x^2} = \frac{3}{8}[1.027 + 2(0.1) + 3(0.7973)] = 1.3571$$

Now, $\int_0^6 \frac{dx}{1 + x^2} = \left[\tan^{-1} x\right]_0^6 = \tan^{-1} 6 = 1.4056$

$\therefore$ Error in Simpson's (3/8)th rule,

$$= \int_0^6 \frac{dx}{1 + x^2} - \int_0^6 P(x)\, dx = 1.4056 - 1.3735 = 0.0485$$

Example 5 (a) : The velocity of train which starts from rest is given by the following table, the time being reckoned in minutes from the start and speed in km/hour.

Time	:	3	6	9	12	15	18
Velocity	:	22	29	31	20	4	0
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5

Estimate approximately the distance covered in 18 minutes by Simpson's (3/8)th rule.

Sol. : We know that $\frac{ds}{dt} = v \quad \therefore \quad s = \int v\, dt \quad i.e. \quad S = \int_0^{18} v\, dt.$

Since to use Simpson's (3/8) rule, we need 3, 6, 9, ... sub-intervals, we take one more ordinate at $t = 0$.

By data at $t = 0$, $v = 0$. $\therefore$ We have

y_0	y_1	y_2	y_3	y_4	y_5	y_6
0	22	29	31	20	4	0

[**Note this**]

By Simpson's (3/8)th rule

$$S = \frac{3h}{8}[X + 2T + 3R] = \frac{3h}{8}\left[(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)\right]$$

But $h = 3$ min. $= \frac{3}{60} = \frac{1}{20}$ hrs.

$$\therefore \quad S = \frac{3}{8} \cdot \frac{1}{20}\left[(0+0) + 2(31) + 3(22 + 29 + 20 + 4)\right] = 5.38125$$

Example 6 (a) : Evaluate $\int_0^{\pi/2} \sqrt{\sin x + \cos x} \cdot dx$ by Simpson's (3/8)th rule by dividing the interval into six intervals. **(M.U. 2009)**

Sol. : We first prepare the following table, by taking sub-interval

$$h = \frac{(\pi/2) - 0}{6} = \frac{\pi}{12}$$

x :	0	$\pi/12$	$2\pi/12$	$3\pi/12$	$4\pi/12$	$5\pi/12$	$6\pi/12$
y :	1	1.1067	1.1688	1.1892	1.1688	1.1067	1.00
Ordinate :	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now, $X = y_0 + y_6 = 2$, $T = y_3 = 1.1892$

$R = y_1 + y_2 + y_4 + y_5$

$= 1.1067 + 1.1688 + 1.1688 + 1.1067 = 4.551$

$$\therefore \quad S = \frac{3h}{8}[X + 2T + 3R] = \frac{3 \times (\pi/12)}{8}\left[2 + 2(1.1892) + 3(4.551)\right]$$

$= 1.7702$

Example 7 (a) : Evaluate $\int_0^3 e^{\sqrt{x}}$ by Simpson's (3/8)th rule. Take $h = 0.25$.

Sol. : We first prepare the table

x :	0	0.25	0.5	0.75	1.00	1.25	1.50
y :	1	1.6487	2.0281	2.3774	2.7183	3.0588	3.4033
Ordinate :	y_0	y_1	y_2	y_3	y_4	y_5	y_6

1.75	2.00	2.25	2.50	2.75	3.00
3.7542	4.1133	4.4817	4.8605	5.2504	5.6522
y_7	y_8	y_9	y_{10}	y_{11}	y_{12}

By Simpson's (3/8)th rule,

$$S = \frac{3h}{8}[X + 2T + 3R]$$

We have $X = y_0 + y_{12} = 1 + 5{\cdot}6522 = 6{\cdot}6522$

$T = y_3 + y_6 + y_9 = 2{\cdot}3774 + 3{\cdot}4033 + 4{\cdot}4817 = 10{\cdot}2624$

$R = y_1 + y_2 + y_4 + y_5 + y_7 + y_8 + y_{10} + y_{11}$

$= 1{\cdot}6487 + 2{\cdot}0281 + 2{\cdot}7183 + 3{\cdot}0588 + 3{\cdot}7542 + 4{\cdot}1133 + 4{\cdot}8605 + 5{\cdot}2504$

$= 27{\cdot}4323$

$$\therefore \quad S = \frac{3 \times 0{\cdot}25}{8}\left[6{\cdot}6522 + 2(10{\cdot}2624) + 3(27{\cdot}4323)\right] = 10{\cdot}2631$$

Example 8 (a) : A curve is given by the table

x	:	0	1	2	3	4	5	6
y	:	0	2	2·5	2·3	2	1·7	1·5
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5	y_6

The x-coordinate of the C.G. of the area bounded by the curve, the end-ordinates and the x-axis is given by $A\bar{x} = \int_0^6 xy\,dx$ where A is the area. Find $\bar{x}$ by using Simpson's (3/8)th rule.

Sol. : By Simpson's (3/8)th rule,

$$A = \frac{3h}{8}\left[X + 2T + 3R\right]$$

We have $X = y_0 + y_6 = 0 + 1{\cdot}5 = 1{\cdot}5$; $T = y_3 = 2{\cdot}3$

$R = y_1 + y_2 + y_4 + y_5 = 2 + 2{\cdot}5 + 2 + 1{\cdot}7 = 8{\cdot}2.$

$$\therefore \quad A = \frac{3}{8}(1)\left[1{\cdot}5 + 2(2{\cdot}3) + 3(8{\cdot}2)\right] = \frac{3}{8}(30{\cdot}7) = 11{\cdot}5125$$

To find the integral $\int_0^6 xy\,dx$, we have to prepare the table of xy.

x	:	0	1	2	3	4	5	6
y	:	0	4	6·25	5·29	4	2·89	2·25
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's (3/8)th rule,

$$\int y^2 dx = \frac{3h}{8}\left[X + 2T + 3R\right]$$

We have $X = y_0 + y_6 = 0 + 2{\cdot}25 = 2{\cdot}25$

$T = y_3 = 5{\cdot}29,\quad R = y_1 + y_2 + y_4 + y_5$

$\therefore\quad R = 4 + 6{\cdot}25 + 4 + 2{\cdot}89 = 17{\cdot}14$

$$\int y^2 dx = \frac{3\times 1}{8}\left[2{\cdot}25 + 2(5{\cdot}29) + 3(17{\cdot}14)\right] = 24{\cdot}09375$$

$$\therefore\quad \text{Volume} = \pi\int y^2 dx = \pi(24{\cdot}09375) = 75{\cdot}692748$$

Example 10 (a) : Find the volume of solid of revolution formed by rotating about the x-axis the area bounded by the lines $x = 0$, $x = 1{\cdot}5$, $y = 0$ and the curve passing through

x	:	0·00	0·25	0·50	0·75	1·00	1·25	1·50
y	:	1·00	0·9826	0·9589	0·9089	0·8415	0·7624	0·7589

Sol. : Since the volume is given by $\int \pi y^2 dx$, we first prepare the table of y^2.

x	:	0·00	0·25	0·50	0·75	1·00	1·25	1·50
y	:	1·00	0·9655	0·9195	0·8261	0·7081	0·5812	0·5759
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's (3/8)th rule,

$$\int y^2 dx = \frac{3h}{8}\left[X + 2T + 3R\right]$$

We have $X = y_0 + y_6 = 1 + 0{\cdot}5759 = 1{\cdot}5789$

$T = y_3 = 0{\cdot}8261,\quad R = y_1 + y_2 + y_4 + y_5$

$R = 0{\cdot}9655 + 0{\cdot}9155 + 0{\cdot}7081 + 0{\cdot}5812 = 3{\cdot}1703$

$$\therefore\quad \int y^2 dx = \frac{3(0{\cdot}25)}{8}\left[1{\cdot}5759 + 2(0{\cdot}8261) + 3(3{\cdot}1703)\right] = 1{\cdot}19428$$

$$\therefore\quad \text{Volume} = \pi\int y^2 dx = \pi(1{\cdot}19428) = 3{\cdot}7519$$

Miscellaneous Examples

Example 1 (c) : Evaluate $\int_{-1}^{1}\frac{dx}{1+x^2}$ by using (i) Trapezoidal Rule, (ii) Simpson's (1/3)rd rule, and (iii) Simpson's (3/8)th rule. **(M.U. 2013, 14)**

(Also find the exact value.)

Sol. : Our first point is (x_0, y_0). Since for (1/3)rd rule, the last ordinate should be 2, 4, **6**, 8, ..., etc. and for the (3/8)th rule the last ordinate should be 3, **6**, 9, 12, ..., etc; to apply all the three rules we divide the interval − 1 to 1 into **six equal** parts by taking

$$\text{Subinterval} = \frac{1-(-1)}{6} = \frac{1}{3},\quad i.e.,\ x_0 = -1,\ x_1 = -\frac{2}{3},\ x_2 = -\frac{1}{3},\ \text{etc.}$$

We now prepare the following table by calculaing the values of $y_0, y_1, y_2, \ldots\ldots$

x	:	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$y = \frac{1}{1+x^2}$	:	$\frac{1}{2}$	$\frac{9}{13}$	$\frac{9}{10}$	1	$\frac{9}{10}$	$\frac{9}{13}$	$\frac{1}{2}$
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Trapezoidal Rule,

$$I = \frac{h}{2}[X + 2R]$$

Here, $h = \frac{1}{3}$, X = Sum of the extremes $= \frac{1}{2} + \frac{1}{2} = 1$

R = Sum of the remaing

$$= \frac{9}{13} + \frac{9}{10} + 1 + \frac{9}{10} + \frac{9}{13} = \frac{18}{13} + \frac{18}{10} + 1 = \frac{180 + 234 + 130}{130} = \frac{544}{130}$$

$$\therefore\ I = \frac{(1/3)}{2}\left[1 + 2\left(\frac{544}{130}\right)\right] = \frac{1}{6}\left(1 + \frac{544}{65}\right) = \frac{1}{6}\left(\frac{609}{65}\right) = 1{\cdot}5615$$

(ii) By Simpson's (1/3)rd Rule,

$$I = \frac{h}{3}[X + 2E + 4O]$$

Here, $h = \frac{1}{3}$, X = Sum of the extremes $= \frac{1}{2} + \frac{1}{2} = 1$

E = Sum of the even ordinates $= \frac{9}{10} + \frac{9}{10} = \frac{18}{10}$

O = Sum of the odd ordinates $= \frac{9}{13} + 1 + \frac{9}{13} = \frac{18}{13} + 1 = \frac{31}{13}$

$$\therefore\ I = \frac{(1/3)}{3}\left[1 + 2 \cdot \frac{18}{10} + 4 \cdot \frac{31}{13}\right] = \frac{1}{9}\left[1 + \frac{468 + 1240}{130}\right]$$

$$= \frac{1}{9} \cdot \frac{1838}{130} = 1{\cdot}5709$$

(iii) Simpson's (3/8)th rule,

$$I = \frac{3h}{8}[X + 2T + 3R]$$

Here, $h = \frac{1}{3}$, X = Sum of the extremes $= \frac{1}{2} + \frac{1}{2} = 1$

T = Sum of the multiples of three = 1

R = Sum of the remaining $= \frac{9}{13} + \frac{9}{13} + \frac{9}{10} + \frac{9}{10} = \frac{18}{13} + \frac{18}{10} = \frac{414}{130}$

$$\therefore\ I = \frac{3(1/3)}{8}\left[1 + 2(1) + 3\left(\frac{414}{130}\right)\right] = \frac{1}{8}\left[3 + \frac{1240}{130}\right]$$

$$= \frac{1}{8}\left[\frac{390 + 1240}{130}\right] = 1{\cdot}5692$$

Now, the exact value is given by

$$I = \int_{-1}^{1} \frac{dx}{1+x^2} = 2\int_0^1 \frac{dx}{1+x^2} = 2\left[\tan^{-1} x\right]_0^1 = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} = 1{\cdot}57079.$$

Example 2 (c) : Evaluate $\int_0^1 \frac{dx}{1+x}$ by using (i) Trapezoidal Rule, (ii) Simpson's (1/3)rd rule, and (iii) Simpson's (3/8)th rule.

(Also find the exact value.) **(M.U. 2010, 13)**

Sol. : As above we divide the interval into **six** parts by taking each subinterval equal to $\frac{1-0}{6} = \frac{1}{6}$.

x	:	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = \frac{1}{1+x}$	:	1	$\frac{6}{7}$	$\frac{6}{8}$	$\frac{6}{9}$	$\frac{6}{10}$	$\frac{6}{11}$	$\frac{1}{2}$
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Trapezoidal Rule,

$$I = \frac{h}{2}[X + 2R]$$

Here, $h = \frac{1}{6}$, $\quad X$ = Sum of the extremes $= 1 + \frac{1}{2} = \frac{3}{2}$

R = Sum of the remaing

$$= 6\left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11}\right) = 6(0{\cdot}56987) = 3{\cdot}41922$$

$$\therefore\ I = \frac{1/6}{2}[1{\cdot}5 + 2(3{\cdot}41922)] = \frac{1}{12}[1{\cdot}5 + 6{\cdot}83844] = 0{\cdot}6949$$

(ii) By Simpson's (1/3)rd Rule,

$$I = \frac{h}{3}[X + 2E + 4O]$$

Here, $h = \frac{1}{6}$, $\quad X$ = Sum of the extremes $= 1 + \frac{1}{2} = \frac{3}{2}$

E = Sum of the even ordinates $= \frac{6}{8} + \frac{6}{10} = 6 \cdot \frac{18}{80} = \frac{27}{20}$

O = Sum of the odd ordinates $= \frac{6}{7} + \frac{6}{9} + \frac{6}{11} = 6\left(\frac{1}{7} + \frac{1}{9} + \frac{1}{11}\right) = 6\left(\frac{239}{693}\right)$

$$\therefore\ I = \frac{1/6}{3}\left[\frac{3}{2} + 2 \cdot \frac{27}{20} + 4 \cdot 6\left(\frac{239}{693}\right)\right]$$

$$= \frac{1}{18}[1{\cdot}5 + 2{\cdot}7 + 8{\cdot}2771] = \frac{1}{18}[12{\cdot}4771] = 0{\cdot}6932$$

(iii) By Simpson's (3/8)th rule,

$$I = \frac{3h}{8}[X + 2T + 3R]$$

Here, $h = \dfrac{1}{6}$, $\quad X$ = Sum of the extremes $= 1 + \dfrac{1}{2} = \dfrac{3}{2}$

T = Sum of the multiples of three $= \dfrac{6}{9} = \dfrac{2}{3}$

R = Sum of the remaining $= \dfrac{6}{7} + \dfrac{6}{8} + \dfrac{6}{10} + \dfrac{6}{11} = 6(0\cdot4588) = 2\cdot7528$

$$\therefore \; I = \frac{3(1/6)}{8}\left[\frac{3}{2} + 2\cdot\frac{2}{3} + 3(2\cdot7528)\right] = \frac{1}{16}[1\cdot5 + 1\cdot3333 + 8\cdot2584]$$

$$= \frac{1}{16}[11\cdot0917] = 0\cdot6932$$

Now, the exact value is given by

$$I = \int_0^1 \frac{dx}{1+x} = [\log(1+x)]_0^1 = \log 2 = 0\cdot6931.$$

Example 3 : Compute the values of $\int_{0\cdot2}^{1\cdot4} (\sin x - \log_e x + e^x)\, dx$ by using (i) Trapezoidal Rule, (ii) Simpson's (1/3)rd rule, and (iii) Simpson's (3/8)th rule. **(M.U. 2005, 14, 15)**

Sol. : We divide the interval into **six equal** sub-intervals by taking each sub-interval equal to $\dfrac{1\cdot4 - 0\cdot2}{6} = 0\cdot2$.

x	:	0·2	0·4	0·6	0·8	1·0	1·2	1·4
y	:	3·02950	2·79753	2·89759	3·16604	3·55975	4·06984	4·70418
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Trapezoidal Rule,

$$I = \frac{h}{2}[X + 2R]$$

Here, $h = 0\cdot2$,

X = Sum of the extremes $= 3\cdot0295 + 4\cdot70418 = 7\cdot73368$

R = Sum of the remaing

$= 2\cdot79753 + 2\cdot89759 + 3\cdot16604 + 3\cdot55975 + 4\cdot06984 = 16\cdot49075$

$$\therefore \; I = \frac{0\cdot2}{2}[7\cdot73368 + 2(16\cdot49075)] = 4\cdot071518$$

(ii) By Simpson's (1/3)rd Rule,

$$I = \frac{h}{3}[X + 2E + 4O]$$

Here, $h = 0\cdot2$,

X = Sum of the extremes $= 3\cdot0295 + 4\cdot70418 = 7\cdot73368$

E = Sum of the even ordinates

$= 2\cdot89759 + 3\cdot55975 = 6\cdot45734$

O = Sum of the odd ordinates

$= 2\cdot79753 + 3\cdot16604 + 4\cdot06984 = 10\cdot03341$

$$\therefore \; I = \frac{0\cdot2}{3}[7\cdot73368 + 2(6\cdot45734) + 4(10\cdot03341)] = 4\cdot05213$$

(iii) By Simpson's (3/8)th rule,

$$I = \frac{3h}{8}[X + 2T + 3R]$$

Here, $h = 0\cdot2$,

X = Sum of the extremes = $3\cdot0295 + 4\cdot70418 = 7\cdot73368$

T = Sum of the multiples of three = $3\cdot16604$

R = Sum of the remaing

$= 2\cdot79753 + 2\cdot89759 + 3\cdot55975 + 4\cdot06984 = 13\cdot32471$

$$\therefore\ I = \frac{3(0\cdot2)}{8}[7\cdot73368 + 2(3\cdot16604) + 3(13\cdot32471)] = 4\cdot05299$$

Example 4 (c) : Find the approximate value of $\int_0^6 e^x\,dx$ by using (i) Trapezoidal Rule, (ii) Simpson's (1/3)rd rule, and (iii) Simpson's (3/8)th rule. **(M.U. 2015)**

Sol. : We divide the interval into **six equal** sub-intervals by taking $h = (6 - 0) / 6 = 1$.

x	:	0	1	2	3	4	5	6
$y = e^x$	:	1·0000	2·7183	7·3891	20·0855	54·5981	148·4132	403·4288
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Trapezoidal Rule,

$$I = \frac{h}{2}[X + 2R]$$

Here, $h = 1$,

X = Sum of the extremes = $1 + 403\cdot4288 = 404\cdot4288$

R = Sum of the remaing

$= 2\cdot7183 + 7\cdot3891 + 20\cdot0855 + 54\cdot5981 + 148\cdot4132 = 233\cdot2042$

$$\therefore\ I = \frac{1}{2}[404\cdot4288 + 2(233\cdot2042)] = 435\cdot4186$$

(ii) By Simpson's (1/3)rd Rule,

$$I = \frac{h}{3}[X + 2E + 4O]$$

Here, $h = 1$,

X = Sum of the extremes = $1 + 403\cdot4288 = 404\cdot4288$

E = Sum of the even ordinates = $7\cdot3891 + 54\cdot5981 = 61\cdot9872$

O = Sum of the odd ordinates = $2\cdot7183 + 20\cdot0855 + 148\cdot4132 = 171\cdot2170$

$$\therefore\ I = \frac{1}{3}[404\cdot4288 + 2(61\cdot9872) + 4(171\cdot2170)] = 404\cdot4232$$

(iii) By Simpson's (3/8)th rule,

$$I = \frac{3h}{8}[X + 2T + 3R]$$

Here, $h = 1$,

X = Sum of the extremes = $1 + 403\cdot4288 = 404\cdot4288$

T = Sum of the multiples of three = $20\cdot0855$

R = Sum of the remaing

= 2·7183 + 7·3891 + 54·5981 + 148·4132 = 213·1187

$\therefore \; I = \frac{3(1)}{8}$ [404·4288 + 2 (20·0855) + 3 (213·1187)] = 406·4835

Example 5 (c) : Evaluate $\int_0^6 x\,f(x)\,dx$ by using (i) Trapezoidal Rule, (ii) Simpson's (1/3)rd rule using the following table. **(M.U. 2015)**

x	:	0	1	2	3	4	5	6
$f(x)$	:	0·146	0·161	0·176	0·190	0·204	0·217	0·230

Sol. : We first prepare the table for $x \cdot f(x)$.

x	:	0	1	2	3	4	5	6
$x \cdot f(x)$	:	0	0·161	0·352	0·570	0·816	1·085	1·380
Ordinate	:	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Trapezoidal Rule,

$$I = \frac{h}{2}[X + 2R]$$

Here, h = 1,

X = Sum of the extremes = 1 + 1·380 = 1·380

R = Sum of the remaing

= 0·161 + 0·352 + 0·570 + 0·816 + 1·085 = 2·984

$\therefore \; I = \frac{1}{2}$ [1·380 + 2 (2·984)] = 3·674

(ii) By Simpson's (1/3)rd Rule,

$$I = \frac{h}{3}[X + 2E + 4O]$$

Here, h = 1,

X = Sum of the extremes = 1 + 1·380 = 1·380

E = Sum of the even ordinates = 0·352 + 0·816 = 1·168

O = Sum of the odd ordinates = 0·161 + 0·570 + 1·085 = 1·8160

$\therefore \; I = \frac{1}{3}$ [1·380 + 2 (1·168) + 4 (1·8160)] = 3·579

Example 6 (c) : Find the approximate value of $\int_0^4 e^x\,dx$ by using (i) Trapezoidal Rule, (ii) Simpson's (1/3)rd rule. **(M.U. 2016)**

(Compare yours result with the exact value.)

Sol. : We divide the interval into 4 equal parts by taking h = (4 – 0) / 4 = 1.

x	:	0	1	2	3	4
$y = e^x$	:	1	2·7183	7·3891	20·0855	54·5961
Ordinate	:	y_0	y_1	y_2	y_3	y_4

(i) By Trapezoidal Rule,

$$I = \frac{h}{2}[X + 2R]$$

Here, $h = 1$,

X = Sum of the extremes = 1 + 54·5961 = 55·5961

R = Sum of the remaing = 2·7183 + 7·3891 + 20·0855 = 30·1929

$$\therefore \ I = \frac{1}{2}[55{\cdot}5961 + 2\,(30{\cdot}1929)] = 57{\cdot}9905$$

(ii) By Simpson's (1/3)rd Rule,

$$I = \frac{h}{3}[X + 2E + 4O]$$

Here, $h = 1$,

X = Sum of the extremes = 1 + 54·5961 = 55·5961

E = Sum of the even ordinates = 7·3891

O = Sum of the odd ordinates = 2·7183 + 20·0855 = 22·8038

$$\therefore \ I = \frac{1}{3}[55{\cdot}5961 + 2\,(7{\cdot}3891) + 4\,(22{\cdot}8038)] = 53{\cdot}8698$$

(Now, the exact value is given by

$$I = \int_0^4 e^x\,dx = \left[e^x\right]_0^4 = e^4 - e^0 = 54{\cdot}5961 - 1 = 53{\cdot}5961\ )$$

EXERCISE - II

Solve the following examples : Class (a) : 4 Marks

1. A curve is drawn to pass through the points given by

x :	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y :	2.0	2.4	2.7	2.8	3.0	2.6	2.1

Estimate the area bounded by the curve between the x-axis, and the lines $x = 1$ and $x = 4$ by Simpson's (3/8)th rule. [**Ans. :** 7·8375]

2. A river is 90 meter wide. The depth d meters from one bank is given by the following table. Find approximately the area of the cross-section by Simpson's (3/8)th rule.

x :	0	10	20	30	40	50	60	70	80	90
d :	0	4	7	9	12	15	14	8	3	1

[**Ans. :** 727·5]

3. Evaluate $\int_4^{5 \cdot 2} \log x\,dx$ from the following data using Simpson's (3/8)th rule.

x :	4	4.2	4.4	4.6	4.8	5	5.2
log *x* :	1·3863	1·4351	1·4816	1·5261	1·5686	1·6094	1·6487

Also find the error. [**Ans. :** 1·8326475. Actual value = 1·827847409. The error = 0·0045]

4. Evaluate $\int \frac{dx}{1 + x^2}$ between suitable intervals and hence, find an approximate value of π by Simpson's (3/8)th rule. **(M.U. 2003)** [**Ans. :** 0·7854, 3·1416]

5. A body is in the form of a solid of revolution. The diameter d in cms. of its sections at various distances x cms. from one end are as given below.

x	:	0	2·5	5·0	7·5	10·0	12·5	15·0
d	:	5	5·5	6·0	6·75	6·25	5·5	4·0

Estimate the volume of the solid by Simpson's (3/8)th rule.
(**Hint :** Calculate $\pi\, d^2 / 4$) [**Ans. :** 407·58 cub. cms.]

6. Using Simpson's (3/8)th rule, find $\int_0^6 f(x)\,dx$ where

x	:	0	1	2	3	4	5	6
$f(x)$	:	6·9897	7·4036	7·7815	8·1291	8·4510	8·7506	9·0309

[**Ans. :** 48·5396]

7. Use Simpson's (3/8)th rule to find $\int_{20}^{26} y\,dx$. Use the tabulated values of y given below.

x	:	20	21	22	23	24	25	26
y	:	95·20	96·85	97·77	98·68	96·56	100·41	101·24

[**Ans. :** 588·2137]

8. Using Simpson's (3/8)th rule, find $\int_0^{0\cdot 3} \sqrt{1-8x^3}\cdot dx.$ [**Ans. :** 0·2916]

9. Evaluate $\int_{-3}^{3} x^4 dx$ by Simpson's (3/8)th rule and compare it with the exact value.
[**Ans. :** 99. Exact value 97·2]

10. Evaluate $\int_0^1 \frac{dx}{1+x}$ using Simpson's (3/8)th rule. **(M.U. 2004, 11)** [**Ans. :** 0·692]

11. Evaluate $\int_0^6 \frac{dx}{1+x}$ using Simpson's (3/8)th rule. Compare it with the exact value.
[**Ans. :** 1·9661]

Summary

1. Trapezoidal Rule

$$\int_a^b f(x)\,dx = \frac{h}{2}[X+2R]$$

2. Simpson's (1/3)rd Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}[X+2E+4O]$$

(There must be ......, 6, 8, 10 ordinates starting with zero.)

3. Simpson's (3/8)th Rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}[X+2T+3R]$$

(There must be ....., 6, 9, 12, ordinates starting with zero.)

❊ ❊ ❊

CHAPTER 15

Tracing of Curves And Study of Some Solids

1. Introduction

For evaluating areas, volumes of revolution etc. we need to know the general nature of the given curve. In this chapter we shall learn the methods of tracing a curve in general and the properties of some standard curves commonly met in engineering problems.

2. Procedure for Tracing Curves given in Cartesian Equations

1. Symmetry : Find out whether the curve is symmetrical about any line with the help of the following rules :

(i) The curve is symmetrical about the x-axis if the equation of the curve remains unchanged when y is replaced by $-y$ *i.e.* if the equation contains only even powers of y.

(ii) The curve is symmetrical about the y-axis if the equation of the curve remains unchanged when x is replaced by $-x$ *i.e.* if the equation contains only even powers of x.

(iii) The curve is symmetrical in opposite quadrants if the equation of the curve remains unchanged when both x and y are replaced by $-x$ and $-y$.

(iv) The curve is symmetrical about the line $y = x$ if the equation of the curve remains unchanged when x and y are interchanged.

2. Origin : Find out whether the origin lies on the curve. If it does, find the equations of the tangents at the origin by equating to zero the lowest degree terms.

3. Tangent at the origin : Tangents at the origin are obtained by equating to zero, the lowest degree terms.

4. Intersection with the coordinate axes : Find out the points of intersection of the curve with the coordinate axes. Find also the equations of the tangents at these points.

5. Asymptotes : Find out the asymptotes if any.

6. Regions where no part of the curve lies : Find out the regions of the plane where no part of the curve lies.

7. Find out dy / dx : Find out dy / dx and the points where the tangents are parallel to the coordinate axis.

3. Straight Line

General equation of straight line is of the form $ax + by + c = 0$.

To plot a straight line we put $y = 0$ and find x. Also, we put $x = 0$ and find y. We plot these two points and join them to get the required line.

Some particular cases of straight line are shown below.

(i) Lines parallel to the coordinate axes.

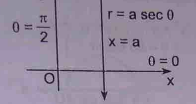


Fig. 15.1 (a)

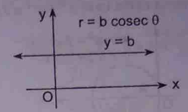


Fig. 15.1 (b)

(ii) Lines passing through origin

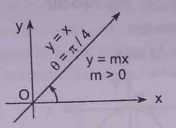


Fig. 15.2 (a)

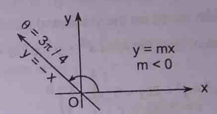


Fig. 15.2 (b)

(iii) Lines making given intercepts on coordinates axes.

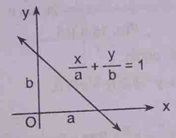


Fig. 15.3 (a)

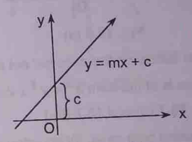


Fig. 15.3 (b)

4. Circle

General equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Its centre is $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

(i) Circle with centre at origin and radius a. $x^2 + y^2 = a^2$ [Fig. 15.4 (a)]

(ii) Circle with centre at $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$. [Fig. 15.4 (b)]

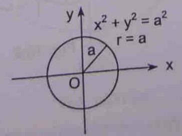


Fig. 15.4 (a)

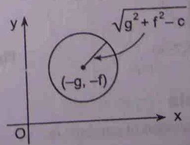


Fig. 15.4 (b)

(iii) Circle with centre on the x-axis and passing through origin.

Its equation is of the form $x^2 + y^2 \pm 2ax = 0$ or $r = \pm 2a \cos\theta$.

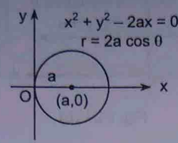


Fig. 15.5 (a)

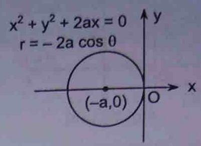


Fig. 15.5 (b)

(iv) Circle with centre on the y-axis and passing through origin.

Its equation is of the form $x^2 + y^2 \pm 2by = 0$ or $r = \pm 2b \sin\theta$.

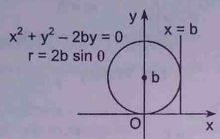


Fig. 15.6 (a)

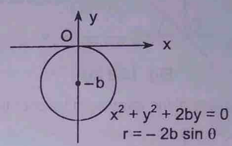


Fig. 15.6 (b)

(v) Circle with centre on the axes but not passing through origin.

Its equation is of the form $x^2 + y^2 \pm 2gx + c = 0$, $x^2 + y^2 \pm 2fy + c = 0$.

[See Fig. 15.7 (a) and 15.7 (b)]

(vi) Circle touching both axes. Its equation is of the form $x^2 + y^2 \pm 2ax \pm 2ay + a^2 = 0$.

[See Fig. 15.8]

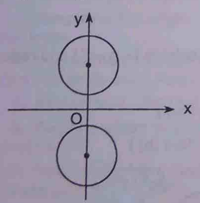


Fig. 15.7 (a)

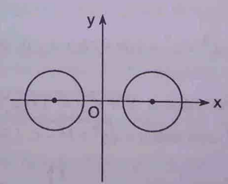


Fig. 15.7 (b)

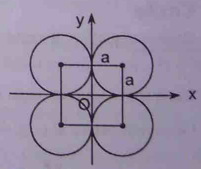


Fig. 15.8

5. Parabola

General equation of parabola is

$$y = ax^2 + bx + c \quad \text{or} \quad x = ay^2 + by + c$$

By completing the square on x or on y and by shifting the origin the equation can be written in standard form as $y^2 = \pm 4ax$ or $x^2 = \pm 4by$.

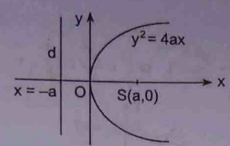


Fig. 15.9 (a)

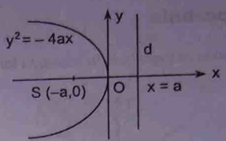


Fig. 15.9 (b)

The equation of the parabola with vertex at (h, k) and

(i) the axis parallel to the x-axis, opening to the right is $(y-k)^2 = 4a(x-h)$

(ii) the axis parallel to the x-axis, opening to the left is $(y-k)^2 = -4a(x-h)$

(iii) the axis parallel to the y-axis, opening upwards is $(x-h)^2 = 4a(y-k)$

(iv) the axis parallel to the y-axis, opening downwards is $(x-h)^2 = -4a(y-k)$

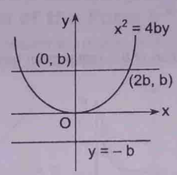


Fig. 15.10 (a)

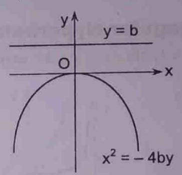


Fig. 15.10 (b)

6. Ellipse

Equation of ellipse in standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a < b)$.

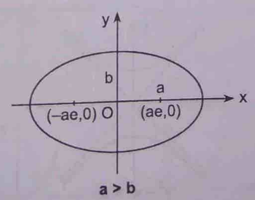


Fig. 15.11 (a)

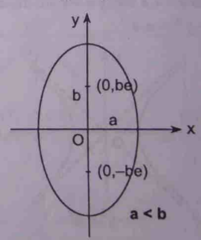


Fig. 15.11 (b)

If the centre of the ellipse is at (h, k) the equation of the ellipse becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

As before if $a > b$, the major axis is parallel to the x-axis and if $a < b$ the major axis is parallel to the y-axis.

7. Hyperbola

Equation of hyperbola in standard form is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or

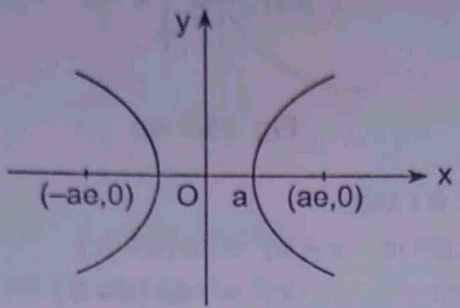


Fig. 15.12 (a)

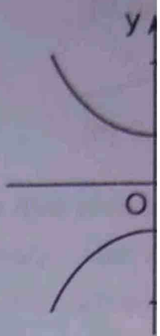
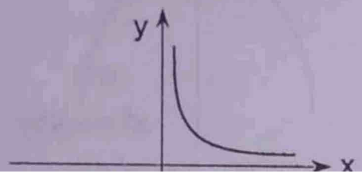


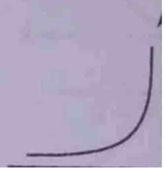
Fig.

8. Rectangular Hyperbola

If the coordinate axes are the asymptotes the equation of the r

or $xy = -k'$.





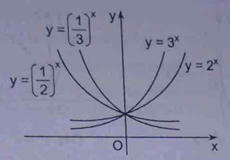


Fig. 15.15 (a)

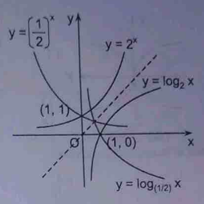


Fig. 15.15 (b)

Note that the graphs of $y = \log_a x$ and $y = a^x$ are symmetrical with respect to the line $y = x$.

10. Graphs of the Form $y^p = x^q$

We first consider the graphs of the form $y^p = x^q$. [Fig. 3.3 (1) to (17)] You are already familiar with some of them and you have come across some more of them in chapters on areas and volumes.

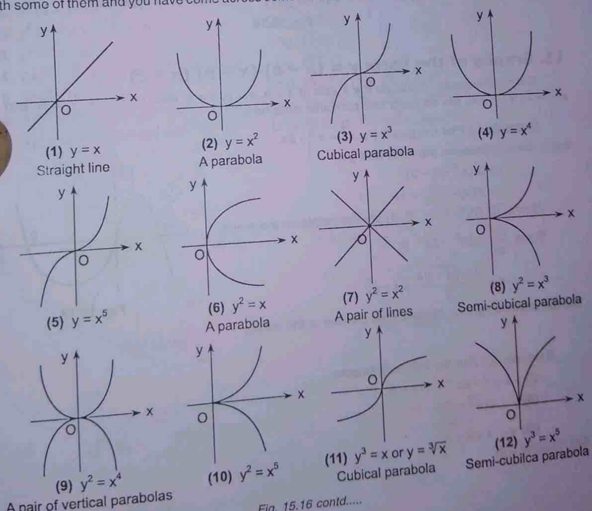
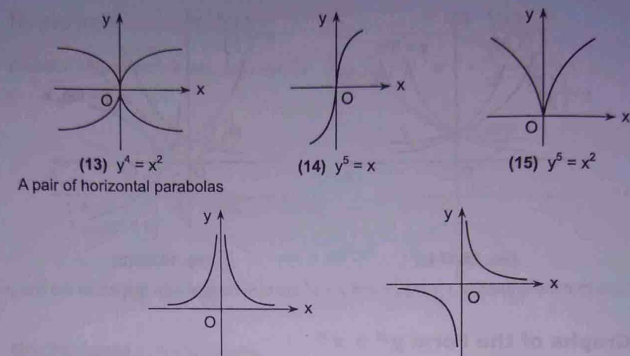


Fig. 15.16 contd.....



(13) $y^4 = x^2$
A pair of horizontal parabolas

(14) $y^5 = x$

(15) $y^5 = x^2$

(16) $y = 1/x^2$

(17) $y = 1/x$ *i.e.* $xy = 1$
A rectangular hyperbola

Fig. 15.16

11. Graphs of the Form $y = (x - a)(x - b)(x - c)$

Clearly the graph intersects the x-axis at $x = a$, $x = b$, $x = c$. Also as $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to -\infty$. We consider one particular case here.

Example 1 : Plot the graph $y = x^3 - x^2 - 2x$.

Sol. : We can factorise the r.h.s. and get

$$y = x(x^2 - x - 2)$$
$$= x(x - 2)(x + 1)$$

Hence, (0, 0), (–1, 0), (2, 0) are the points on the graph.

Since $\dfrac{dy}{dx} = 3x^2 - 2x - 2$

$$\therefore \quad x = \frac{2 \pm \sqrt{4 + 24}}{2} = 1 \pm \sqrt{7}$$

are the points when the tangents are parallel to the x-axis.

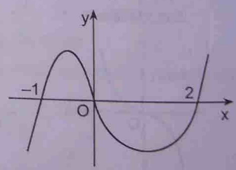


Fig. 15.17

Example 2 : Plot the following graphs

(i) $y = x^3 - 6x^2 + 11x - 6$

(ii) $y = x(x^2 - 9)$

(iii) $y = x\sqrt{9 - x^2}$.

Sol. : Left to you.

12. Strophoid

Example : Trace the curve $y^2(a+x) = x^2(b-x)$.

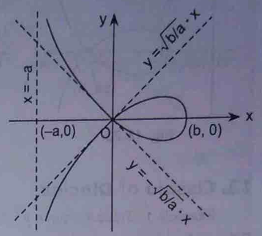


Fig. 15.18

(i) Curve is symmetrical about the x-axis.

(ii) The curve passes through the origin and the equations of the tangents at the origin are

$$y = \pm\sqrt{b/a}\cdot x.$$

(iii) The curve meets the x-axis in (0, 0) and (3a, 0).

(iv) Since, $y^2 = \dfrac{x^2(b-x)}{(a+x)}$

when $x \to -a$, $y \to$ infinity. Hence, $x = -a$ is an asymptote.

(v) When $x > b$ and when $x < -a$, y^2 is negative. Hence, the curve does not exists when $x > b$ and $x < -a$. The curve is shown in the Fig. 15.18.

The curve of the above type is called **a loop**. We show below certain common loops.

1. $y^2(a+x) = x^2(a-x)$ [Fig. 15.19]
2. $y^2(a-x) = x^2(a+x)$ [Fig. 15.20]
3. $y^2 = (x-a)(x-b)^2$, $b > a$ [Fig. 15.21]
4. $3ay^2 = x^2(a-x)$ or $ay^2 = x^2(a-x)$ [Fig. 15.22]
5. $9ay^2 = (x-2a)(x-5a)^2$ [Fig. 15.23 on the next page]
6. $9ay^2 = x(x-3a)^2$ or $3ay^2 = x(x-a)^2$ [Fig. 15.24 on the next page]
7. $xy^2 + (x+a)^2(x+2a) = 0$ [Fig. 15.25 on the next page]

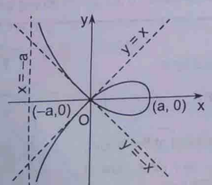


Fig. 15.19

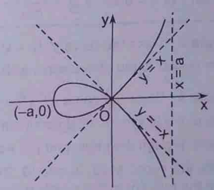


Fig. 15.20

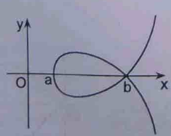


Fig. 15.21

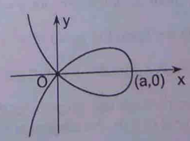


Fig. 15.22

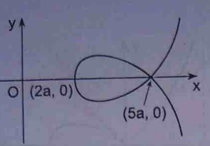


Fig. 15.23

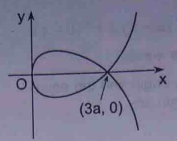


Fig. 15.24

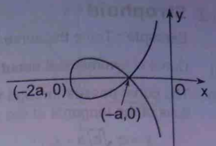


Fig. 15.25

13. Cissoid of Diocles

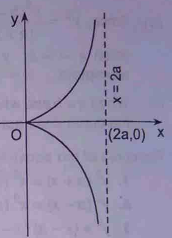


Fig. 15.26

Example 1 : Trace the curve $y^2(2a-x)=x^3$.

Sol. : (i) The curves is symmetrical about the x-axis.

(ii) The curve passes through the origin and the tangents at the origin are $y^2=0$ *i.e.* the x-axis is a double tangent at the origin.

(iii) Since $y^2=\dfrac{x^3}{2a-x}$ as $x\to 2a$, $y\to\infty$, the line $x=2a$ is an asymptote.

(iv) When $x>2a$, y^2 is negative. Hence, the curve does not exist when $x>2a$.

Remark ....

By putting $x=r\cos\theta$, $y=r\sin\theta$, we can show that the polar equation of Cissoid is $r=2a\tan\theta\sin\theta$ or $x=a\sin^2 t$, $y=\dfrac{a\sin^3 t}{\cos t}$ in parametric form.

Example 2 : Trace the curve $y^2(a-x)=x^3$.

Sol. : This is similar to the above curve with $2a$ replaced by a.

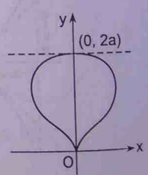


Fig. 15.27

Example 3 : Trace the curve $a^2x^2=y^3(2a-y)$.

Sol. : (i) The curve is symmetrical about the y-axis.

(ii) It passes through the origin and the x-axis is a tangent at the origin.

(iii) It meets the y-axis at (0, 0) and (0, 2a). Further, dy/dx is zero at (0, 2a); the tangent at this points is parallel to the x-axis.

(iv) Since, $x^2=y^3(2a-y)/a^2$ when $y>2a$ and $y<0$, x^2 is negative and the curve does not exist for $y>2a$ and $y<0$.

Example 4 : Trace the curve $a^2y^2=x^3(2a-x)$

Sol. : The curve is symmetrical about the x-axis.

It passes through the origin and the point (2a, 0). Since dy/dx is infinity at (2a, 0), the tangent at this point is parallel to the y-axis.

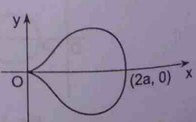


Fig. 15.28

14. Astroid or Four Cusped Hypocycloid

Example : Trace the curve $x^{2/3} + y^{2/3} = a^{2/3}$

or $x = a\cos^3\theta,\ y = a\sin^3\theta.$

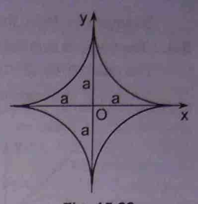


Fig. 15.29

Sol. : (i) The curve is symmetrical about both axes.

(ii) The curve cuts the x-axis in $(\pm\, a, 0)$ and the y-axis in $(0, \pm\, a)$.

(iii) Neither x nor y can be greater than a.

(See Fig. 15.29)

15. Witch of Agnesi

Example 1 : Trace the curve $xy^2 = a^2(a - x)$.

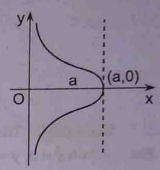


Fig. 15.30

Sol. : (i) The curve is symmetrical about the x-axis.

(ii) The curve passes through the point $(a, 0)$.

(iii) Since $y^2 = a^2\dfrac{(a-x)}{x}$, x cannot be negative. Also, x cannot be greater than a.

(iv) As $x \to 0$, $y \to \infty$, the y-axis is an asymptote.

Example 2 : Trace the curve $xy^2 = 4a^2(2a - x)$.

Sol. : The curve is similar to the curve shown above where a is replaced by $2a$.

Example 3 : Trace the curve $a^2y^2 = x^2(a^2 - x^2)$.

Sol. : (i) The curve is symmetrical about both the axes.

(ii) The points $(0, 0)$, $(a, 0)$ and $(-\,a, 0)$ lie on the curve.

(iii) If $x > a$, y^2 is negative. Hence, there is no curve beyond $x = a$, $x = -\,a$.

(See Fig. 15.31 below)

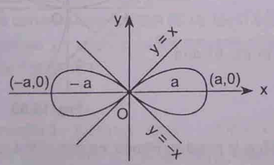


Fig. 15.31

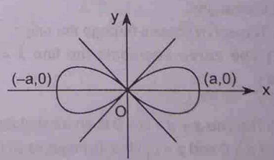


Fig. 15.32

Example 4 : Trace the curve $x^2(a^2 - x^2) = 8\,a^2y^2$.

Sol. : Because the powers of both x and y are even, the curve is symmetrical about both the axes. If $x > a$ or $x < -\,a$, there is no curve.

Equating to zero the lowest degree terms. The tangents at the origin are $y = \pm 2\sqrt{2}\cdot x$.

(See Fig. 15.32 above)

Example 5 : Trace the curve $y^2 = x^2 (4 - x^2)$

Sol. : The curve is symmetrical about both the axes. [See Fig. 15.33]

The point (0, 0), (2, 0), (– 2, 0) lie on the curve.

If $x > 2$, y^2 is negative. Hence, there is no curve beyond $x = 2$, $x = -2$.

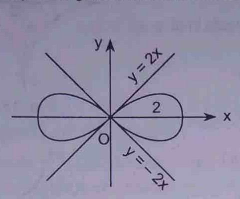


Fig. 15.33

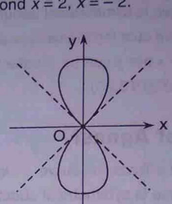


Fig. 15.34

Example 6 : Trace the curve $a^2 x^2 = 4y^2 (a^2 - y^2)$.

Sol. : When $y = a$, $y = -a$, $x = 0$. Then origin lies on the curve. [See Fig. 15.34]

If $y > a$, x is imaginary. Hence, the curve lies between $-a < y < a$.

Equating to zero, the lowest degree terms, the equation of the tangents at the origin are $x = \pm 2y$.

16. Folium of Decartes

Example 1 : Trace the curve $x^3 + y^3 = 3a\, xy$.

Sol. : (i) The curve is symmetrical about the line $x = y$ because when x is replaced by y and if y is replaced by x, the equations remains unchanged.

(ii) The curve passes through the origin.

(iii) The curve intersects the line $x = y$ in (0, 0) and $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

(iv) The line $x + y + a = 0$ is an asymptote.

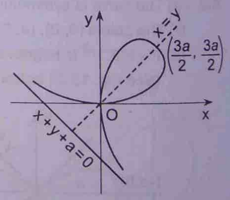


Fig. 15.35

(v) If $x < 0$ and $y < 0$, l.h.s. is negative while r.h.s. is positive. Hence, no part of the curve is in the third quadrant.

(vi) The equation of the tangents at the origin are $x = 0$, $y = 0$.

René Descartes (1596 - 1650)

A well-known philosopher, soldier and mathematician was born in the town of La Haye (now Descartes) Touraine France. At the age of 8 he entered the Jesuit College of Henri IV in La Fléche, where he studied literature, grammar, science and mathematics. Because of poor health he developed the habit of lying in bed thinking until late in the morning; he considered those times most productive. After a short period in military and travel through Europe for 5 years he returned to Paris and

studied Mathematics and Philosophy. He then moved to Holland where he lived for twenty years writing several books on Mathematics and Philosophy. In 1637 he published 'geometry' in which he combined algebra and geometry giving birth to analytical geometry better known as 'Cartesian Geometry'. But more important is his philosophical contribution. **"Cogito Ergo Sum"** (I think, therefore, I am). He wrote three important texts (1) Discourse on the Method of Rightly Conducting the Reason and Seeking Truth in the Science, (2) Meditations on First Philosophy and (3) Principles of Philosophy. In 1649 he moved to Sweden at the invitation of Queen Christina to tutor her in Philosophy. Here he contacted pneumonia and died at the age of 54.

Example 2 : Trace the curve $x^3 - y^3 = 3a\,xy$.

Sol. : The curve is similar to the above one but lies in the third quadrant. (Fig. 15.36)

The curve is symmetric about the line $y = -x$ because the equation of the curve remains unchanged if x is replaced by $-y$ and y is replaced by $-x$.

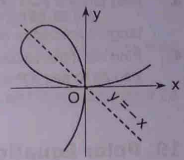


Fig. 15.36

17. Lemniscate of Bernoulli

Example 1 : Trace the curve $y^2(a^2 + x^2) = x^2(a^2 - x^2)$

or $x^2(x^2 + y^2) = a^2(x^2 - y^2)$

or $y^2 = \dfrac{x^2(a^2 - x^2)}{a^2 + x^2}$

Sol. : Since the curve has even powers of both x and y, it is symmetrical about the x and y axes.

The curve passes through (0, 0), (a, 0) and ($-a$, 0).

The lines $y = x$ and $y - x$ are tangents at the origin.

(See Fig. 15.37)

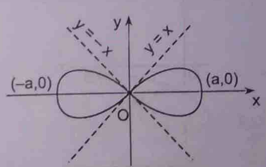


Fig. 15.37

Example 2 : Trace the curve $x^2 y^2 = a^2(y^2 - x^2)$

Solution : Since x and y both appear in even powers the curve is symmetrical about both the axes.

It passes through the origin and the tangents at the origin are $y = \pm x$.

The equation can be written as

$$a^2 x^2 = y^2(a^2 - x^2) \qquad \therefore\ x = \pm a$$

are the asymptotes at the origin.

(See Fig. 15.38)

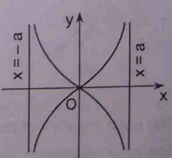


Fig. 15.38

18. Procedure for Tracing Curves given in Polar Equations

1. Find out the symmetry using the following rules.
   (i) If on replacing θ by $-\theta$ the equation of the curve remains unchanged then the curve is symmetrical about the initial line.
   (ii) If on replacing r by $-r$ the equation of the curve remains unchanged *i.e.* if the powers of r are even then the curve is symmetrical about the pole. The pole is then called the centre of the curve.
2. Form the table of values of r for both positive and negative values of θ. Also find the values of θ which give $r = 0$ and $r = \infty$.
3. Find $\tan \Phi = r\dfrac{d\theta}{dr}$. Also find the points where it is zero or infinity *i.e.* find the points at which the tangent coincides with the initial line or is perpendicular to it.
4. Find out if the values of r and θ lie between certain limits *i.e.* find the greatest or least value of r so as to see if the curve lies within or without a certain circle.

19. Polar Equations of Lines

It is easy to see that the polar equation of the x-axis *i.e.* the initial line is $\theta = 0$, the polar equation of the y-axis is $\theta = \pi / 2$.

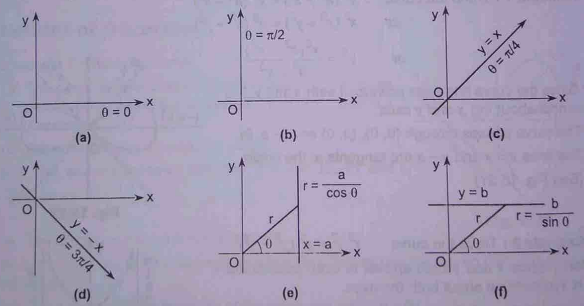


Fig. 15.39

The polar equation of the line $y = x$ is $r \sin \theta = r \cos \theta$ *i.e.* $\tan \theta = 1$ *i.e.* $\theta = \pi / 4$.

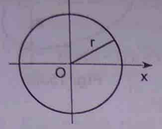
The polar equation of the line $y = -x$ is $r \sin \theta = -r \cos \theta$ *i.e.* $\tan \theta = -1$ *i.e.* $\theta = 3\pi / 4$.

The polar equation of the line $x = a$ *i.e.* of $r \cos \theta = a$ is $r = a / \cos \theta$ *i.e.* $r = a \sec \theta$.

The polar equation of the line $y = b$ *i.e.* of $r \sin \theta = b$ is $r = b / \sin \theta$ *i.e.* $r = b \operatorname{cosec} \theta$.

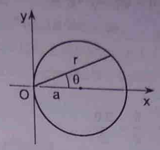
20. Polar Equations of Circles

The polar equation of the circle $x^2 + y^2 = a^2$ is $r^2 \cos^2 \theta + r^2 \sin^2 \theta = a^2$ *i.e.* $r^2 = a^2$ *i.e.* $r = a$.



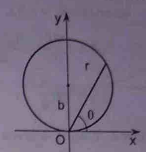
$r = a$;
$x^2 + y^2 = a^2$

Fig. 15.40 (a)



$r = 2a \cos \theta$;
$x^2 + y^2 - 2ax = 0$

Fig. 15.40 (b)



$r = 2b \sin \theta$;
$x^2 + y^2 - 2by = 0$

Fig. 15.40 (c)

The polar equation of the circle $x^2 + y^2 \pm 2ax = 0$ is $r^2 \cos^2 \theta + r^2 \sin^2 \theta = \pm 2a\, r \cos \theta$ *i.e.* $r = \pm 2a \cos \theta$.

The polar equation of the circle $x^2 + y^2 \pm 2by = 0$ is $r^2 \cos^2 \theta + r^2 \sin^2 \theta = \pm 2b\, r \sin \theta$ *i.e.* $r = \pm 2b \sin \theta$.

21. Cardioid

Example 1 : Trace the curves : **(a)** $r = a\,(1 + \cos \theta)$, **(b)** $r = a\,(1 - \cos \theta)$,

Sol. : (a) (i) The curve is symmetrical about the initial line since its equation remains unchanged by replacing θ by $-\theta$.

(ii) When $\theta = 0$, $r = 2a$ and when $\theta = \pi$, $r = 0$.

Also when $\theta = \pi / 2$, $r = a$ and when $\theta = 3\pi\ / 2$, $r = a$.

(iii) Further $\tan \Phi = \dfrac{r\, d\theta}{d\, r} = \dfrac{a(1 + \cos \theta)}{-a \sin \theta}$

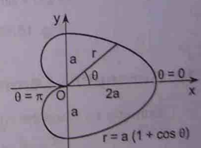


Fig. 15.41

$$\therefore \quad \tan \Phi = -\cot\left(\frac{\theta}{2}\right) = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\therefore \quad \Phi = \frac{\pi}{2} + \frac{\theta}{2}.$$

Hence, $\Psi = \theta + \Phi$ gives

$$\Psi = \theta + \frac{\pi}{2} + \frac{\theta}{2} = \frac{\pi}{2} + \frac{3\theta}{2}$$

When $\theta = 0$, $\Psi = \pi / 2$ *i.e.* the tangent at this point is perpendicular to the initial line. When $\theta = \pi$, $\Psi = 2\pi$ *i.e.* the tangent at this point coincides with initial line.

(iv) The following table gives some values of r and θ.

θ	0	$\pi / 3$	$\pi / 2$	$2\pi / 3$	π
r	$2a$	$3a / 2$	a	$a / 2$	0

(b) (i) The curve is symmetrical about the initial line.

(ii) When $\theta = 0$, $r = 0$, when $\theta = \pi / 2$, $r = a$,

when $\theta = \pi$, $r = 2a$, when $\theta = 3\pi / 2$, $r = a$.

(iii) $\tan \Phi = r \dfrac{d\theta}{dr} = \dfrac{a(1 - \cos\theta)}{a \sin\theta}$

$= \tan \dfrac{\theta}{2} \quad \therefore \; \Phi = \dfrac{\theta}{2}$

Hence, $\Psi = \theta + \Phi$ gives $\Psi = \theta + \dfrac{\theta}{2} = \dfrac{3\theta}{2}$.

When $\theta = 0$, $\Psi = 0$ *i.e.* the tangent at this point coincides with the $\Psi = 3\pi / 2$ *i.e.* the tangent at this point is perpendicular to the initial line.

(iv) The following table gives some values of r and θ.

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	0	$a/2$	a	$3a/2$	$2a$

Two more cardioids $r = a(1 + \sin\theta)$ and $r = a(1 - \sin\theta)$ are shown

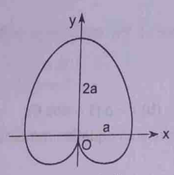


Fig. 15.43 (a)

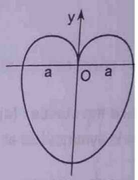


Fig. 15.43 (b)

22. Bernoulli's Lemniscate

Example 1 : Trace the curve

$$r^2 = a^2 \cos 2\theta$$

or $$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

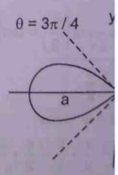


Fig.

So. : (i) Since on changing r by $-r$ the equation remains unchanged, the curve is symmetrical about the initial line.

(ii) Since on changing θ by $-\theta$ the equation remains unchanged the curve is symmetrical about the pole.

(iii) Considering only positive value of r, we get the following table.

θ	0	$\pi/2$	$\pi/4$	$(\pi/4) < \theta < (3\pi/4)$	$3\pi/4$	π
r^2	a^2	$a^2/2$	0	negative	0	a^2

(iv) Considering the equation $(x^2 + y^2)^2 = a^2(x^2 - y^2)$, from $x^2 - y^2 = 0$ *i.e.* $y = \pm x$ *i.e.* $\theta = \pi/4$ and $\theta = 3\pi/4$, we see that $\theta = \pi/4$ and $\theta = 3\pi/4$ are tangents at the origin.

Example 2 : Trace the curve $r^2 = a^2 \sin 2\theta$.

Solution : The curve is symmetrical about the pole.

The curve is also symmetrical about the line $\theta = \pi/4$.

The tangents to the curve at the pole are $\theta = 0$ and $\theta = \pi/2$.

(See Fig. 15.45)

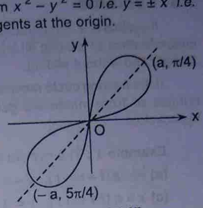


Fig. 15.45

23. Rectangular Hyperbola

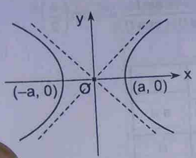


Fig. 15.46

Example : Trace the curve

$$r^2 \cos 2\theta = a^2$$

or $\quad r^2 = a^2 \sec 2\theta$

or $\quad x^2 - y^2 = a^2.$

Sol. : If we put $x = r\cos\theta$, $y = r\sin\theta$, we get

$$r^2(\cos^2\theta - \sin^2\theta) = a^2$$

i.e. $\quad r^2 \cos 2\theta = a^2 \quad$ or $\quad r^2 = a^2 \sec 2\theta.$

The points $(a, 0)$, $(-a, 0)$ are on the curve $\theta = \pi/4$ and $\theta = -\pi/4$ are the asymptotes.

24. Parabola

Example 1 : Trace the curve $\quad r(1 - \sin\theta) = a.$

Sol. : If we put $x = r\cos\theta$, $y = r\sin\theta$, we have

$$r - r\sin\theta = a$$

$\therefore \quad r^2 = (a + r\sin\theta)^2$

$\therefore \quad r^2 = a^2 + 2ar\sin\theta + r^2\sin^2\theta$

$\therefore \quad x^2 + y^2 = a^2 + 2ay + y^2$

$\therefore \quad x^2 = a(a + 2y)$

which is a parabola as shown in Fig. 15.47 (when $y = 0$, $x = \pm a$).

(When $\theta = 3\pi/2$, $r = a/2$)

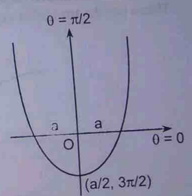


Fig. 15.47

Example 2 : Trace the curve $\quad r(1 + \sin\theta) = a.$

Sol. : If we put $x = r\cos\theta$, $y = r\sin\theta$, we have

$$r + r\sin\theta = a$$

$\therefore \quad r^2 = (a - r\sin\theta)^2$

$\therefore \quad r^2 = a^2 - 2ar\sin\theta + r^2\sin^2\theta$

$\therefore \quad x^2 + y^2 = a^2 - 2ay + y^2$

$\therefore \quad x^2 = a(a - 2y)$

which is a parabola as shown in Fig. 15.48 (when $y = 0$, $x = \pm a$).

(When $\theta = \pi/2$, $r = a/2$)

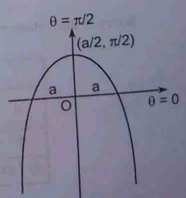


Fig. 15.48

25. Cycloids

A cycloid is a curve traced by a fixed point on the circumference of a circle which moves uniformly on a straight line without sliding.

If the moving circle moves on another circle from outside or from inside we get an epicycloid or an hypocycloid.

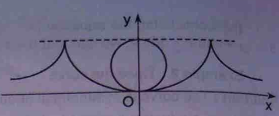


Fig. 15.49

Example 1 : The cycloid is generally given in one of the following forms.

(a) $x = a(t + \sin t),\ y = a(1 + \cos t)$ **(b)** $x = a(t - \sin t),\ y = a(1 + \cos t)$

(c) $x = a(t + \sin t),\ y = a(1 - \cos t)$ **(d)** $x = a(t - \sin t),\ y = a(1 - \cos t)$

Sol. : (a) $\dfrac{dx}{dt} = a(1 + \cos t),\ \dfrac{dy}{dt} = -a\sin t \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-a\sin t}{a(1+\cos t)} = -\tan\left(\dfrac{t}{2}\right)$

Some of the values of x, y and dy/dx are

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	$-a\pi$	$-a\left(\frac{\pi}{2}+1\right)$	0	$a\left(\frac{\pi}{2}+1\right)$	$a\pi$
y	0	a	$2a$	a	0
dy/dx	∞	1	0	-1	$-\infty$

From the above table we see that at $t = -\pi$ and at $t = \pi$, the tangents are parallel to the y-axis. These points are called cusps. Further, at $t = 0$, the tangent is parallel to the x-axis.

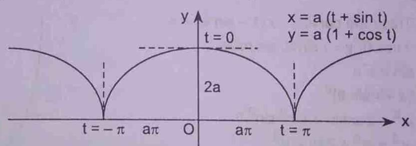


Fig. 15.50

(b) $\dfrac{dx}{dt} = a(1 - \cos t),\ \dfrac{dy}{dt} = -a\sin t \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-a\sin t}{a(1-\cos t)} = -\cot\left(\dfrac{t}{2}\right)$

Some of the values of x, y, $\dfrac{dy}{dx}$ are

t	0	$\pi/2$	π	$3\pi/2$	2π
x	0	$a\left(\frac{\pi}{2}-1\right)$	$a\pi$	$a\left(\frac{3\pi}{2}+1\right)$	$2a\pi$
y	$2a$	a	0	a	$2a$
dy/dx	$-\infty$	-1	0	1	∞

From the above table we see that at $t = 0$ and $t = 2\pi$ the tangents are parallel to the y-axis. These are cusps. Further at $t = \pi$, $y = 0$ and $dy/dx = 0$ *i.e.* the x-axis is the tangent.

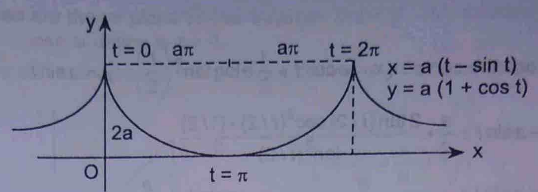


Fig. 15.51

(c) $\frac{dx}{dt} = a(1 + \cos t)$, $\frac{dy}{dt} = a \sin t$ $\quad \therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 + \cos t)} = \tan\left(\frac{t}{2}\right)$

Some of the values of x, y, $\frac{dy}{dx}$ are

t	$-\pi$	$-\pi/2$	π	$\pi/2$	π
x	$-a\pi$	$-a\left(\frac{\pi}{2}+1\right)$	0	$a\left(\frac{\pi}{2}+1\right)$	$a\pi$
y	$2a$	a	0	a	$2a$
dy/dx	$-\infty$	-1	0	1	∞

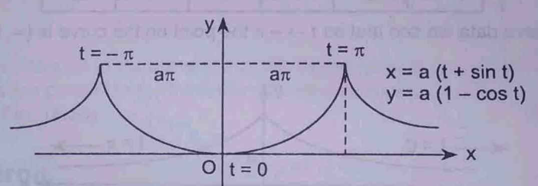


Fig. 15.52

(d) $\frac{dx}{dt} = a(1 - \cos t)$, $\frac{dy}{dt} = a \sin t$ $\quad \therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 - \cos t)} = \cot\left(\frac{t}{2}\right)$

Some of the values of x, y, dy/dx are

t	0	$\pi/2$	π	$3\pi/2$	2π
x	0	$a\left(\frac{\pi}{2}-1\right)$	$a\pi$	$a\left(\frac{3\pi}{2}+1\right)$	$2a\pi$
y	0	a	$2a$	a	0
dy/dx	$-\infty$	1	0	-1	∞

26. Tractrix

Example : Trace the curve : $x = a\cos t + \frac{1}{2} a \log \tan^2\left(\frac{t}{2}\right)$, $y = a \sin t$.

Sol. :
$$\frac{dx}{dt} = -a\sin t + \frac{a}{2} \cdot \frac{2\tan(t/2)\sec^2(t/2)\cdot(1/2)}{\tan^2(t/2)}$$
$$= -a\sin t + \frac{a}{2\sin(t/2)\cos(t/2)} = -a\sin t + \frac{a}{\sin t}$$
$$\frac{dx}{dt} = \frac{a}{\sin t}(1-\sin^2 t) = \frac{a\cos^2 t}{\sin t}; \qquad \frac{dy}{dt} = a\cos t$$
$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = a\cos t \cdot \frac{\sin t}{a\cos^2 t} = \tan t.$$

Some of the values of t, x, y, dy/dx are

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	∞	0	$-\infty$	0	∞
y	0	$-a$	0	a	0
dy/dx	0	$-\infty$	0	∞	0

From the above data we see that as $t \to -\pi$ the point on the curve is $(\infty, 0)$ and as $t \to 0$ the point is $(-\infty, 0)$.

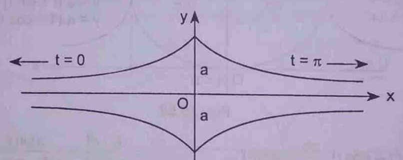


Fig. 15.54

27. Some Solids

You have studied to some extent plane and straight line in three dimensions. We shall here get acquainted with some more three dimensional solids such as sphere, cylinder, cone, paraboloid etc.

28. Plane

The general equation of a plane is linear of the form $a'x + b'y + c'z + d' = 0$ which can be written as
$$\frac{a'}{-d'}x + \frac{b'}{-d'}y + \frac{c'}{-d'}z = 1$$
i.e. $\quad ax + by + cz = 1$

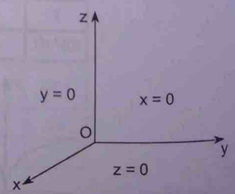


Fig. 15.55

Simplest planes are the yz plane whose equation is $x = 0$; the zx plane whose equation is $y = 0$; the xy plane whose equation is $z = 0$.

The planes parallel to the coordinate planes are $x = \pm a$, $y = \pm b$, $z = \pm c$.

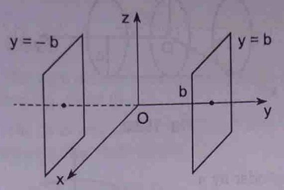


Fig. 15.56

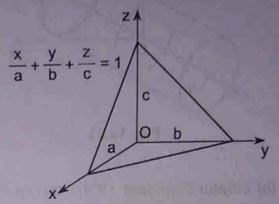


Fig. 15.57

The plane $ax + by + cz = 1$ cuts off intercepts $1 / a$, $1 / b$, $1 / c$ on the coordinate axes. Still more common form of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

This plane cuts off intercepts a, b, c on the coordinate axes.

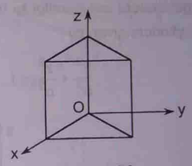


Fig. 15.58

29. Prisms

If three planes intersect in such a way that the intersection of any two of them is a line parallel to the third, then the solid formed is called **prism.** (See Fig. 15.58)

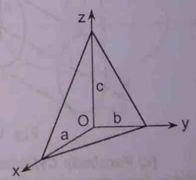


Fig. 15.59

30. Tetrahedron

The solid formed by four planes any three of which intersect in a point is called a **tetrahedron**. The figure shows a tetrahedron formed by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \qquad \text{(See Fig. 15.59)}$$

31. Cylinders

An equation involving only two variables represents a cylinder in three dimensional geometry. Thus, $f(x, y) = 0$, $f(y, z) = 0$, $\Psi(z, x) = 0$ represent cylinders in three dimensions.

(See Fig. 15.60)

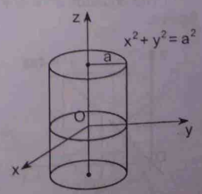


Fig. 15.60

(a) Right Circular Cylinders : $x^2 + y^2 = a^2$ is a right circular cylinder, whose generator is parallel to the z-axis. Similarly, $y^2 + z^2 = b^2$ is a cylinder whose generators are parallel to the x-axis. $z^2 + x^2 = c^2$ is a cylinder whose generators are parallel to the y-axis.

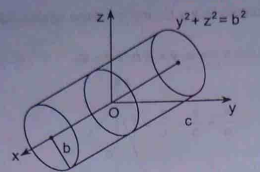


Fig. 15.61

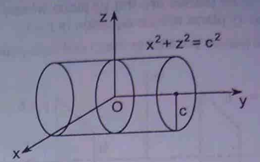


Fig. 15.62

(b) Elliptic Cylinders : If the section of the cylinder by a plane perpendicular to its generator is an ellipse then the cylinder is called elliptic cylinder. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a cylinder whose generators are parallel to the z-axis. Similarly, we have elliptic cylinders given by

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1.$$

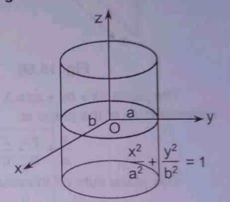


Fig. 15.63

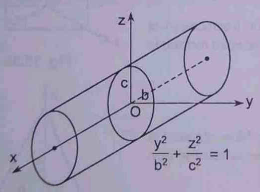


Fig. 15.64

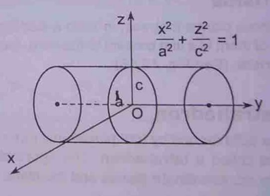


Fig. 15.65

(c) Parabolic Cylinders : The equation $y^2 = 4ax$ represents a parabola in two dimensions. But in three dimensions it represents a parabolic cylinder.

The equation $z^2 = 4by$, $x^2 = 4cz$ also represent parabolic cylinders as shown in the following figures.

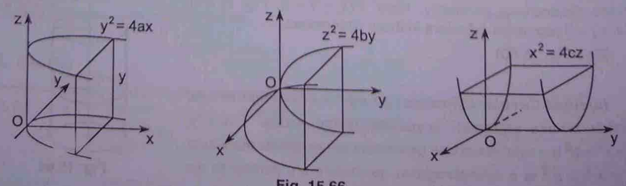


Fig. 15.66

32. Sphere

The equation of sphere in standard form *i.e.* with centre at the origin and radius a is $x^2 + y^2 + z^2 = a^2$. (See Fig. 15.67)

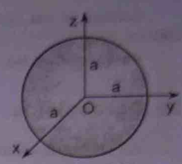


Fig. 15.67

33. Cone

The right circular cone given by $x^2 + y^2 = z^2$ is shown in the following Fig. 15.68 (i)

The other two cones $y^2 + z^2 = x^2$ and $x^2 + z^2 = y^2$ are shown in the following Fig. 15.68 (ii) and (iii).

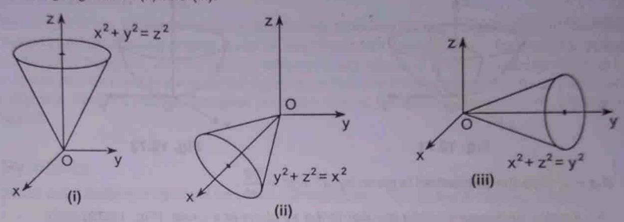


Fig. 15.68 : Cone

34. Ellipsoid

The equation of ellipsoid in standard form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The sections of the ellipsoid by planes parallel to the coordinate planes are ellipses. (See Fig. 15.69)

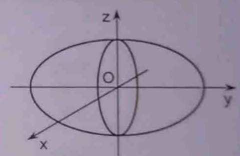


Fig. 17.69

35. Hyperboloid

The equations of hyperboloid of one sheet and two sheet are respectively

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

and $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1.$$

The hyperboloid of one sheet is shown in Fig. 15.70 (a). Sections of the hyperboloid of one sheet parallel to the xy-planes are ellipses and sections parallel to yz-plane or zx-plane are hyperbolas.

The hyperboloid of two sheets is shown in the Fig. 15.70 (b).

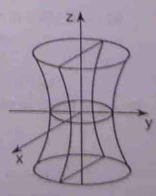


Fig. 15.70 (a)

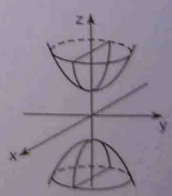


Fig. 15.70 (b)

Sections of this hyperboloid by planes parallel to the xy-plane *i.e.* $z = k$, $|k| > c$ are real ellipses.

Sections of this hyperboloid by planes parallel to the zx-plane *i.e.* $y = k$ are real hyperbolas.

36. Paraboloids

We shall discuss here only one elliptic paraboloid given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$.

Sections of this paraboloid parallel to the xy -planes are ellipses. Sections parallel to the yz-plane or the zx-plane are parabolas. (Fig. 15.71)

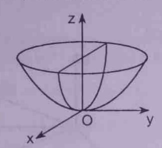


Fig. 15.71

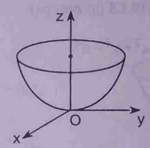


Fig. 15.72

If $a = b$, then the paraboloid is given by $x^2 + y^2 = \frac{2z}{c'}$.

The section of this paraboloid parallel to the xy-plane is a circle. (Fig. 15.72)

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