CAUCHY'S EQUATION:

Definition: An equation of the form

$$x^n \frac{d^n y}{d\,x^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \;\dots + p_{n-1} x \frac{dy}{dx} + p_n y = X$$

where $p_1,\ p_2 \dots..,p_n$ are constants and X is a function of x is called **homogeneous linear differential equation of order n**. The equation is also known as **Cauchy's equation.**

METHOD OF SOLUTION :

The equation can be transformed into an equation with constant coefficients by the substitution

$$z = logx\ \boldsymbol{or}\ x = e^z$$

Now, $\because z = \log x, \frac{dz}{dx} = \frac{1}{x}\ and$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}.\frac{dy}{dz} \quad \text{.................(i)}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}.\frac{dy}{dz} + \frac{1}{x}.\frac{d^2y}{dz^2}.\frac{dz}{dx} = -\frac{1}{x^2}.\frac{dy}{dz} + \frac{1}{x^2}.\frac{d^2y}{dz^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) \quad \text{.................(ii)}$$

Similarly, it can be shown that $\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right)$ ....(iii)

and so on.

If we put $D = \frac{d}{dz}$ then we get, from (i), (ii), (iii),

$$x\frac{dy}{dx} = \frac{dy}{dz} = Dy$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D^2y - Dy = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz} = D^3y - 3D^2y + 2Dy = D(D-1)(D-2)y$$

and so on.

Further the r.h.s. X by the substitution of $x = e^z$ changes to a function of z only say Z.

Thus, the given equation by the substitution $x = e^z$ changes to a linear differential equation with constant coefficients of the form $f(D)y = Z$ and can be solved by the methods studied in the previous exercise.

EXERCISE

Solve the following differential equations (1 to 20)

1. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^{-1}$

2. $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

3. $x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 \log x$

4. $x^3\frac{d^2y}{dx^2} + 3x^2\frac{dy}{dx} + xy = \sin\log x$

5. $x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{x} = 4\log x$

6. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = \sin(\log x)$

7. $x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = x^3 + 3x.$

8. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2\sin\log x$

9. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = coslog\ x + xsin\log x$

10. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \frac{(\sin\log x)+1}{x}$

11. $(x^2D^2 + 5xD + 3)y = \left(1 + \frac{1}{x}\right)^2 log\ x$

12. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x.\sin(\log x)$

13. $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=x\log x$

14. $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-12y=x^3\log x$

15. $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=-x^4\sin x$

16. $u=r\frac{d}{dr}\left(r\frac{du}{dr}\right)+ar^3$

17. $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{(1-x)^2}$

18. $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+3y=\frac{\log x\cdot\cos\log x}{x}$

19. $\left(\frac{d}{dx}+\frac{1}{x}\right)^2y=\frac{1}{x^4}$

20. $\left(\frac{d^2}{dx^2}-\frac{2}{x^2}\right)^2y=x^2$

21. Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=\frac{x^3}{x^2+1}$ by the method of variation of parameters

22. Solve $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=x^3\sec^2 x$ by the methods of variation of parameters

23. The radial displacement 'u' in a rotating disc at a distance 'r' from the axis is given by $\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}+kr=0$. Find the displacement if $u = 0$ when $r = 0$ and $r = a$.

24. Find the equation of the curve which satisfies the differential equation $4x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+y=0$ and crosses the x-axis at an angle of 60^0 at x = 1.

ANSWERS

1. $y=c_1x+c_2x^2+\frac{1}{6x}$

2. $y=\frac{c_1}{x}+x\big(c_2cos(log\ x)+c_3sin\ (log\ x)\big)+5x+\frac{2}{x}log\ x$

3. $y=c_1+c_2log\ x+c_3(logx)^2+\frac{1}{27}x^3(log\ x-1)$

4. $y=\frac{1}{x}[c_1+c_2log\ x-sin\ (log\ x)]$

5. $y=\frac{c_1}{x}+\sqrt{x}\left[c_2cos\left(\frac{\sqrt{3}}{2}log\ x\right)+c_3sin\left(\frac{\sqrt{3}}{2}log\ x\right)+x(2log\ x-3)\right]$

6. $y=x^2(c_1\cos\log x+c_2\sin\log x)+\frac{1}{8}(\cos\log x+\sin\log x)$

7. $y=(c_1+c_2log\ x)x+c_3x^2+\frac{x^3}{4}-\frac{3x}{2}(log\ x)^2$

8. $y=x^2(c_1coslogx+c_2sin\ logx)-\frac{1}{2}x^2logx\ (cos\ log\ x)$

9. $y=x\big[c_1cos\big(\sqrt{3}log\ x\big)+c_2\sin\big(\sqrt{3}log\ x\big)\big]+\frac{1}{13}\big(3\ cos(log\ x)-2sin(logx)\big)+\frac{x}{2}sin\ (log\ x)$

10. $y=x^2\big[c_1cos\ h\big(\sqrt{3}log\ x\big)+c_2sin\ h\big(\sqrt{3}log\ x\big)\big]+\frac{1}{6x}+\frac{1}{61x}[5sin(log\ x)+6cos(log\ x)]$

Or $y=\left[c_1x^{(2+\sqrt{3})}+c_2x^{(2-\sqrt{3})}\right]+\frac{1}{6x}+\frac{1}{61x}[5sin(log\ x)+6cos(log\ x)]$

11. $y=\frac{c_1}{x}+\frac{C_2}{x^3}+\frac{log\ x}{3}-\frac{4}{9}+\frac{1}{x}\left[\frac{(logx)^2}{2}-\frac{(log\ x)}{2}\right]-\frac{1}{x^2}log\ x$

12. $y=c_1cos(log\ x)+c_2sin(log\ x)-\frac{(log\ x)^2}{4}cos(log\ x)+\frac{(logx)}{4}sin\ (log\ x)$

13. $y=x(c_1\cos\log x+c_2\sin\log x)+x\cdot\log x$

14. $y=c_1x^3+c_2x^{-4}+\frac{x^3\log x}{98}(7\log x-2)$

15. $y=c_1x^2+c_2x^3+x^2\sin x$

16. $u=c_1r+\frac{c_2}{r}+\frac{a}{10}r^3$

17. $y=(c_1+c_2\log x)\cdot\frac{1}{x}-\frac{1}{x}\log\left(\frac{1-x}{x}\right)$

18. $y = \frac{1}{x}\left[c_1 \cos\left(\sqrt{2}\log x\right) + c_2 \sin\left(\sqrt{2}\log x\right)\right] + \frac{1}{x}\left[\log x \cdot \cos(\log x) + 2\sin(\log x)\right]$

19. $y = c_1 + \frac{c_2}{x} + \frac{1}{2x^2}$

20. $y = \frac{a}{x} + bx + cx^2 + dx^4 + \frac{1}{280}x^6$

21. $y = c_1 x + \frac{c_2}{x} + \frac{x}{4}\log(x^2+1) + \frac{1}{4x}\log(x^2+1) - \frac{x}{4}$

22. $y = c_1 e^z + c_2 e^{2z} + e^z \log(\sec e^z)$

23. $u = \frac{k}{8} r(a^2 - r^2)$

24. $y = x^{(2+\sqrt{3})/2} - x^{(2-\sqrt{3})/2}$

SOME SOLVED EXAMPLES:

6. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = \sin(\log x)$

Solution: Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$ we get,

$[D(D-1) - 3D + 5]y = \sin z$ $\therefore (D^2 - 4D + 5)y = \sin z$

$\therefore$ The A.E. is $(D^2 - 4D + 5) = 0$ $\therefore D = \frac{4 \pm 2i}{2} = 2 \pm i$

$\therefore$ The C.F. is $y = e^{2z}(c_1 \cos z + c_2 \sin z)$

$\therefore P.I. = \frac{1}{D^2 - 4D + 5}\sin z = \frac{1}{-4D+4} \cdot \sin z$

$= \frac{1}{-4} \cdot \frac{D+1}{D^2-1} \cdot \sin z = \frac{1}{8}(D+1)\sin z = \frac{1}{8}(\cos z + \sin z)$

$\therefore$ The complete solution is $y = e^{2z}(c_1 \cos z + c_2 \sin z) + \frac{1}{8}(\cos z + \sin z)$

Resubstituting in terms of x, we get, $y = x^2(c_1 \cos\log x + c_2 \sin\log x) + \frac{1}{8}(\cos\log x + \sin\log x)$

7. $x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = x^3 + 3x.$

Solution: Putting $z = \log x$ and $x = e^{2z}$, $\frac{d}{dz} = D$ we get,

$[D(D-1)(D-2) - D(D-1) + 2D - 2]y = e^{3z} + 3e^z$

$\therefore (D^3 - 3D^2 + 2D - D^2 + D + 2D - 2)y = e^{3z} + 3e^z$

$\therefore (D^3 - 4D^2 + 5D - 2)y = e^{3z} + 3e^z$

$\therefore$ The A.E. is $D^3 - 4D^2 + 5D - 2 = 0$

$\therefore (D-1)(D^2 - 3D + 2) = 0$ $\therefore (D-1)(D-1)(D-2) = 0$

$\therefore D = 1, 1, 2$

$\therefore$ The C.F. is $y = (c_1 + c_2 z)e^z + c_3 e^{2z}$

$\therefore P.I. = \frac{1}{(D-1)^2(D-2)}e^{3z} + \frac{1}{(D-1)^2(D-2)}3e^z$

$= \frac{1}{(3-1)^2(3-2)}e^{3z} + \frac{z^2}{2} \cdot \frac{1}{(1-2)}3e^z$

$= \frac{e^{3z}}{4} - \frac{z^2}{2}3e^z$

$\therefore$ The complete solution is $y = (c_1 + c_2 z)e^z + c_3 e^{2z} + \frac{e^{3z}}{4} - \frac{3z^2}{2}e^z$

$\therefore y = (c_1 + c_2 \log x)x + c_3 x^2 + \frac{x^3}{4} - \frac{3x}{2}(\log x)^2$

8. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \sin\log x$

Solution: C.F. $= e^{2z}(c_1\cos z + c_2 \sin z)$ (See example no. 6)

$\therefore P.I. = \frac{1}{D^2-4D+5} e^{2z} \cdot \sin z$

$= e^{2z} \cdot \frac{1}{(D+2)^2-4(D+2)+5} \sin z$

$= e^{2z} \cdot \frac{1}{D^2+1} \sin z$

$= e^{2z} \cdot \left(\frac{-z}{2}\right) \cos z$

$\therefore$ The complete solution is $y = e^{2z}(c_1\cos z + c_2 \sin z) - \frac{1}{2} e^{2z} \cdot z \cos z$

$\therefore y = x^2(c_1 \cos\log x + c_2 \sin\log x) - \frac{1}{2} x^2 \log x \cos\log x$

9. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = coslog\ x + xsin \log x$

Solution: Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$ we get,

$[D(D-1) - D + 4]y = \cos z + e^z \sin z$

$\therefore (D^2 - 2D + 4)y = \cos z + e^z \sin z$

$\therefore$ The auxiliary equation is $D^2 - 2D + 4 = 0$

$\therefore D = 1 \pm \sqrt{3} \cdot i$

$\therefore$ The C.F. is $y = e^z\left(c_1 \cos\sqrt{3}\cdot z + c_2 \sin\sqrt{3}\cdot z\right)$

$P.I.$ for $\cos z = \frac{1}{D^2-2D+4}\cos z = \frac{1}{3-2D}\cos z$

$= \frac{1}{9-4D^2}(3+2D)\cos z = \frac{1}{13}(3+2D)\cos z$

$= \frac{1}{13}(3\cos z - 2\sin z)$

$P.I.$ for $e^z \sin z = \frac{1}{D^2-2D+4} e^z \sin z = e^z \frac{1}{(D+1)^2-2(D+1)+4} \cdot \sin z = e^z \frac{1}{D^2+3} \cdot \sin z = e^z \cdot \frac{1}{2} \sin z$

$\therefore$ The complete solution is $y = e^z\left(c_1 \cos\sqrt{3}\cdot z + c_2 \sin\sqrt{3}\cdot z\right) + \frac{1}{13}(3\cos z - 2\sin z) + e^z \cdot \frac{1}{2}\sin z$

i.e. $y = x\left[c_1\cos\left(\sqrt{3}\log x\right) + c_2 \sin\left(\sqrt{3}\log x\right)\right] + \frac{1}{13}(3\cos\log x - 2\sin\log x) + \frac{x}{2}\sin(\log x)$

10. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \frac{(\sin\log x)+1}{x}$

Solution: Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$ we get,

$[D(D-1) - 3D + 1]y = (\sin z + 1) \cdot e^{-z}$

$\therefore (D^2 - 4D + 1)y = e^{-z}\sin z + e^{-z}$

$\therefore$ The A.E. is $D^2 - 4D + 1 = 0$ $\quad\therefore D = 2 \pm \sqrt{3}$

$\therefore$ The C.F. is $y = Ae^{(2+\sqrt{3})z} + Be^{(2-\sqrt{3})z}$

$\therefore y = e^{2z}\left(Ae^{\sqrt{3}\cdot z} + Be^{-\sqrt{3}\cdot z}\right)$ which can be expressed as

$y = e^{2z}\left(c_1\cosh\sqrt{3}\cdot z + c_2 \sinh\sqrt{3}\cdot z\right)$ by putting $A = \frac{c_1+c_2}{2}$, $B = \frac{c_1-c_2}{2}$

$P.I.$ for $e^{-z} = \frac{1}{D^2-4D+1} e^{-z} = \frac{1}{6} e^{-z}$

$$P.I. \text{ for } e^{-z}\sin z \quad = e^{-z}\cdot\frac{1}{(D-1)^2-4(D-1)+1}\sin z$$
$$= e^{-z}\cdot\frac{1}{D^2-6D+6}\sin z = e^{-z}\cdot\frac{1}{5-6D}\sin z$$
$$= e^{-z}\cdot\frac{5+6D}{25-36D^2}\sin z = e^{-z}\frac{(5\sin z+6\cos z)}{61}$$

$\therefore$ The complete solution is $y = e^{2z}\left(c_1\cosh\sqrt{3}\cdot z + c_2\sinh\sqrt{3}\cdot z\right) + \frac{1}{6}e^{-z} + \frac{e^{-z}}{61}(5\sin z + 6\cos z)$

$\therefore\ y = x^2\left[c_1\cosh\left(\sqrt{3}\log x\right) + c_2\sinh\left(\sqrt{3}\log x\right)\right] + \frac{1}{6x} + \frac{1}{61x}[5\sin(\log x) + 6\cos(\log x)]$

11. $(x^2D^2 + 5xD + 3)y = \left(1+\frac{1}{x}\right)^2 log\ x$

Solution: Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$ we get,

$[D(D-1) + 5D + 3]y = (1+e^{-z})^2\cdot z$

$\therefore\ [D^2 + 4D + 3]y = (1+e^{-z})^2 z$

$\therefore$ The auxiliary equation is $D^2 + 4D + 3 = 0$

$\therefore\ (D+1)(D+3) = 0 \qquad \therefore\ D = -1, -3$

$\therefore$ The C.F. is $y = c_1e^{-z} + c_2e^{-3z}$

$P.I. = \frac{1}{D^2+4D+3}(z + 2e^{-z}z + e^{-2z}z)$

Now, $\frac{1}{D^2+4D+3}z = \frac{1}{3}\left[1+\frac{4D+D^2}{3}\right]^{-1}z = \frac{1}{3}\left[1-\frac{4D}{3}\ldots.\right]z = \frac{1}{3}\left[z-\frac{4}{3}\ldots.\right]$

$$\frac{1}{D^2+4D+3}2e^{-z}z \quad = 2e^{-z}\cdot\frac{1}{(D-1)^2+4(D-1)+3}z$$
$$= 2\cdot\frac{e^{-z}}{D^2+2D}z = 2\cdot\frac{e^{-z}}{2D}\left[1+\frac{D}{2}\ldots.\right]^{-1}z$$
$$= \frac{e^{-z}}{D}\left[z-\frac{1}{2}\right] = e^{-z}\int\left[z-\frac{1}{2}\right]dz = e^{-z}\left(\frac{z^2}{2}-\frac{z}{2}\right)$$
$$\frac{1}{D^2+4D+3}e^{-2z}\cdot z \quad = e^{-2z}\cdot\frac{1}{(D-2)^2+4(D-2)+3}z$$
$$= \frac{e^{-2z}}{D^2-1}z = e^{-2z}\cdot(-1)[1-D^2]^{-1}z$$
$$= -e^{-2z}[1+D^2+\cdots]z = -e^{-2z}\cdot z$$

$\therefore\ P.I. = \frac{z}{3} - \frac{4}{9} + e^{-z}\left(\frac{z^2}{2}-\frac{z}{2}\right) - e^{-2z}\cdot z$

$\therefore$ The complete solution is $y = c_1e^{-z} + c_2e^{-3z} + \frac{z}{3} - \frac{4}{9} + e^{-z}\left(\frac{z^2}{2}-\frac{z}{2}\right) - e^{-2z}\cdot z$

$y = \frac{c_1}{x} + \frac{c_2}{x^3} + \frac{\log x}{3} - \frac{4}{9} - \frac{1}{x}\left[\frac{(\log x)^2}{2} - \frac{(\log x)}{2}\right] - \frac{1}{x^2}\cdot\log x$

12. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x.\sin(\log x)$

Solution: Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$, we get,

$[D(D-1) + D + 1]y = z\sin z$

$\therefore\ [D^2+1]y = z\sin z$

$\therefore$ The auxiliary equation is $D^2 + 1 = 0 \qquad \therefore\ D = i, -i$

$\therefore$ The C.F. is $y = c_1\cos z + c_2\sin z$

$\therefore P.I. = \frac{1}{D^2+1} z \sin z = \text{I. P. of } \frac{1}{D^2+1} e^{iz} \cdot z$

$= \text{I. P. of } e^{iz} \frac{1}{(D+i)^2+1} \cdot z = \text{I. P. of } e^{iz} \frac{1}{D^2+2iD} \cdot z$

$= \text{I. P. of } e^{iz} \frac{1}{D^2+2iD} \cdot z$

$= \text{I. P. of } e^{iz} \frac{1}{2iD} \left[1 + \frac{D}{2i}\right]^{-1} \cdot z$

$= \text{I. P. of } e^{iz} \frac{1}{2iD} \left[1 - \frac{D}{2i} + \cdots\right] \cdot z$

$= \text{I. P. of } e^{iz} \frac{1}{2iD} \left[z - \frac{1}{2i}\right]$

$= \text{I. P. of } e^{iz} \cdot \frac{1}{2i} \int \left(z - \frac{1}{2i}\right) dz$

$= \text{I. P. of } e^{iz} \cdot \frac{1}{2i} \left[\frac{z^2}{2} - \frac{z}{2i}\right]$

$= \text{I. P. of } (\cos z + i \sin z) \frac{1}{2i} \left(\frac{z^2}{2} - \frac{z}{2i}\right)$

$= \text{I. P. of } (\cos z + i \sin z) \left(-\frac{i}{2}\right) \left(\frac{z^2}{2} + \frac{zi}{2}\right)$

$= -\frac{z^2}{4} \cos z + \frac{z}{4} \sin z$

$\therefore$ The complete solution is $y = c_1 \cos z + c_2 \sin z - \frac{z^2}{4} \cos z + \frac{z}{4} \sin z$

$\therefore y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{(\log x)^2}{4} \cos(\log x) + \frac{(\log x)}{4} \sin(\log x)$

13. $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + 2y = x \log x$

Solution: Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$, we get,

$[D(D-1) - D + 2] = ze^z$

$\therefore (D^2 - 2D + 2)y = ze^z$

$\therefore$ The auxiliary equation is $D^2 - 2D + 2 = 0$ $\therefore D = 1 \pm i$

$\therefore$ The C.F. is $y = e^z(c_1 \cos z + c_2 \sin z)$

$\therefore P.I. = \frac{1}{D^2-2D+2} e^z \cdot z$

$= e^z \cdot \frac{1}{(D+1)^2-2(D+1)+2} \cdot z$

$= e^z \cdot \frac{1}{D^2+1} z = e^z \cdot [1 + D^2]^{-1} z$

$= e^z [1 - D^2 + \cdots] z = e^z \cdot z$

$\therefore$ The complete solution is $y = e^z(c_1 \cos z + c_2 \sin z) + e^z \cdot z$

$\therefore y = x(c_1 \cos \log x + c_2 \sin \log x) + x \cdot \log x$

14. $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 12y = x^3 \log x$

Solution: Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$, we get,

$[D(D-1) + 2D - 12]y = e^{3z} \cdot z$

$\therefore (D^2 + D - 12)y = e^{3z} \cdot z$

$\therefore$ The auxiliary equation is $D^2 + D - 12 = 0$

$\therefore (D+4)(D-3) = 0 \qquad \therefore D = 3, -4$

$\therefore$ CF is $y = c_1 e^{3z} + c_2 e^{-4z}$

$$\therefore P.I. = \frac{1}{(D^2+D-12)} e^{3z} \cdot z = e^{3z} \cdot \frac{1}{(D+3)^2+(D+3)-12} \cdot z$$

$$= e^{3z} \cdot \frac{1}{D^2+7D} \cdot z = e^{3z} \cdot \frac{1}{7D[1+(D/7)]} \cdot z$$

$$= e^{3z} \cdot \frac{1}{7D}\left[1+\frac{D}{7}\right]^{-1} z = e^{3z} \cdot \frac{1}{7D}\left[1-\frac{D}{7}+\cdots\right] z$$

$$= e^{3z} \cdot \frac{1}{7D}\left[z-\frac{1}{7}\right] = e^{3z} \cdot \frac{1}{7}\left[\int z\,dz - \frac{1}{7}\int dz\right]$$

$$= e^{3z} \cdot \frac{1}{7}\left[\frac{z^2}{2}-\frac{z}{7}\right] = e^{3z} \cdot \frac{z}{98}(7z-2)$$

$\therefore$ The complete solution is $y = c_1 e^{3z} + c_2 e^{-4z} + e^{3z} \cdot \frac{z}{98}(7z-2)$

$$\therefore y = c_1 x^3 + c_2 x^{-4} + \frac{x^3 \log x}{98}(7\log x - 2)$$

15. $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = -x^4 \sin x$

Solution: Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$, we get,

$[D(D-1) - 4D + 6]y = -e^{4z} \cdot \sin e^z$

$\therefore (D^2 - 5D + 6)y = -e^{4z} \cdot \sin e^z$

$\therefore$ The auxiliary equation is $(D^2 - 5D + 6) = 0$

$\therefore (D-2)(D-3) = 0 \qquad \therefore D = 2, 3$

$\therefore$ The CF is $y = c_1 e^{2z} + c_2 e^{3z}$

$$\therefore P.I. = \frac{1}{(D^2-5D+6)}(-e^{4z} \cdot \sin e^z)$$

$$= -e^{4z} \cdot \frac{1}{(D+4)^2-5(D+4)+6} \sin e^z$$

$$= -e^{4z} \cdot \frac{1}{D^2+3D+2} \sin e^z$$

$$= -e^{4z} \cdot \frac{1}{(D+2)(D+1)} \sin e^z$$

$$= -e^{4z} \cdot \frac{1}{D+2} \cdot e^{-z} \cdot \int e^z \sin e^z\,dz \qquad [\text{Put } e^z = t,\ e^z dz = dt]$$

$$= -e^{4z} \cdot \frac{1}{D+2} \cdot e^{-z}(-\cos e^z)$$

$$= e^{4z} \cdot e^{-2z} \cdot \int e^{2z} \cdot e^{-z} \cos e^z\,dz$$

$$= e^{2z} \int e^z \cos e^z\,dz \qquad [\text{Put } e^z = t,\ e^z dz = dt]$$

$$= e^{2z} \cdot \sin e^z$$

$\therefore$ The complete solution is $y = c_1 e^{2z} + c_2 e^{3z} + e^{2z} \cdot \sin e^z$

$$\therefore y = c_1 x^2 + c_2 x^3 + x^2 \sin x$$

16. $u = r\frac{d}{dr}\left(r\frac{du}{dr}\right) + ar^3$

Solution: The equation can be written as $u = r\left[r\frac{d^2u}{dr^2} + 1 \cdot \frac{du}{dr}\right] + ar^3$

$\therefore r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} - u = -ar^3$

Putting $z = \log r$ and $r = e^z$, $\frac{d}{dz} = D$, we get,

$[D(D-1) + D - 1]u = -ae^{3z}$

$\therefore (D^2 - 1)u = -ae^{3z}$

$\therefore$ The auxiliary equation is $D^2 - 1 = 0$

$\therefore (D-1)(D+1) = 0 \qquad \therefore D = 1, -1$

$\therefore$ The C.F. is $u = c_1e^z + c_2e^{-z}$

$\therefore P.I. = \frac{1}{D^2-1}(-ae^{3z}) = \frac{a}{10}e^{3z}$

$\therefore$ The complete solution is $u = c_1e^z + c_2e^{-z} + \frac{a}{10}e^{3z} = c_1r + \frac{c_2}{r} + \frac{a}{10}r^3$

17. $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$

Solution: Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$, we get,

$[D(D-1) + 3D + 1]y = \frac{1}{(1-e^z)^2}$

$\therefore [D^2 + 2D + 1]y = \frac{1}{(1-e^z)^2}$

$\therefore$ The auxiliary equation is $(D+1)^2 = 0 \qquad \therefore D = -1, -1$

$\therefore$ The C.F. is $y = (c_1 + c_2z)e^{-z}$

$\therefore P.I. = \frac{1}{D+1} \cdot \frac{1}{D+1} \cdot \frac{1}{(1-e^z)^2}$

$= \frac{1}{D+1}e^{-z}\int \frac{e^z}{(1-e^z)^2}dz = \frac{1}{D+1}e^{-z} \cdot \frac{1}{(1-e^z)}$

$= e^{-z}\int \frac{e^z \cdot e^{-z}}{1-e^z}dz = e^{-z} \cdot \int \frac{dz}{1-e^z}$

$= e^{-z}\int \frac{e^{-z}}{e^{-z}-1}dx = -e^{-z} \cdot \log(e^{-z} - 1)$

$\therefore$ The complete solution is $y = (c_1 + c_2z)e^{-z} - e^{-z} \cdot \log(e^{-z} - 1)$

$\therefore y = (c_1 + c_2\log x) \cdot \frac{1}{x} - \frac{1}{x}\log\left(\frac{1}{x} - 1\right)$

$\therefore y = (c_1 + c_2\log x) \cdot \frac{1}{x} - \frac{1}{x}\log\left(\frac{1-x}{x}\right)$

18. $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = \frac{\log x \cdot \cos\log x}{x}$

Solution: Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$, we get,

$[D(D-1) + 3D + 3]y = e^{-z} \cdot z \cdot \cos z$

$\therefore (D^2 + 2D + 3)y = e^{-z} \cdot z \cdot \cos z$

$\therefore$ The auxiliary equation is $D^2 + 2D + 3 = 0$

$\therefore D = \frac{-2 \pm 2\sqrt{2} \cdot i}{2} = -1 \pm \sqrt{2} \cdot i$

$\therefore$ The C.F. is $y = e^{-z}\left(c_1\cos\sqrt{2}\,z + c_2\sin\sqrt{2}\,z\right)$

$\therefore P.I. = \frac{1}{D^2+2D+3}e^{-z} \cdot z \cdot \cos z$

$= e^{-z} \cdot \frac{1}{(D-1)^2+2(D-1)+3} \cdot z\cos z$

$= e^{-z} \cdot \frac{1}{D^2+2} \cdot z\cos z$

$= e^{-z} \cdot \left[z - \frac{1}{D^2+2} \cdot 2D\right] \cdot \frac{1}{D^2+2} \cdot \cos z$

$= e^{-z} \cdot \left[z - \frac{1}{D^2+2} \cdot 2D\right] \cos z$

$= e^{-z} \cdot \left[z\cos z + \frac{2}{D^2+2} \cdot \sin z\right]$

$= e^{-z} \cdot [z\cos z + 2\sin z]$

$\therefore$ The complete solution is $y = e^{-z}\left(c_1 \cos\sqrt{2}\, z + c_2 \sin\sqrt{2}\, z\right) + e^{-z} \cdot (z\cos z + 2\sin z)$

$\therefore y = \frac{1}{x}\left[c_1 \cos\left(\sqrt{2}\log x\right) + c_2 \sin\left(\sqrt{2}\log x\right)\right] + \frac{1}{x}[\log x \cdot \cos(\log x) + 2\sin(\log x)]$

19. $\left(\frac{d}{dx} + \frac{1}{x}\right)^2 y = \frac{1}{x^4}$

Solution: We have $\left(\frac{d}{dx} + \frac{1}{x}\right)\left(\frac{dy}{dx} + \frac{y}{x}\right) = \frac{1}{x^4}$

$\therefore \frac{d}{dx}\left(\frac{dy}{dx} + \frac{y}{x}\right) + \frac{1}{x}\left(\frac{dy}{dx} + \frac{y}{x}\right) = \frac{1}{x^4}$

$\therefore \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} + \frac{1}{x}\frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^4}$

$\therefore \frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = \frac{1}{x^4}$

Multiplying by x^2, we get, $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{1}{x^2}$

Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$, we get,

$[D(D-1) + 2D]y = e^{-2z}$

$\therefore$ The auxiliary equation is $D^2 + D = 0$

$\therefore D(D+1) = 0$ $\therefore D = 0, -1$

$\therefore$ The C.F. is $y = c_1 + c_2 e^{-z}$

$P.I. = \frac{1}{D(D+1)} e^{-2z} = \frac{1}{-2(-2+1)} e^{-2z} = \frac{1}{2} e^{-2z}$

$\therefore$ The complete solution is $y = c_1 + c_2 e^{-z} + \frac{1}{2} e^{-2z}$

$\therefore y = c_1 + \frac{c_2}{x} + \frac{1}{2x^2}$

20. $\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)^2 y = x^2$

Solution: Let $\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right) y = v$ ....(1)

$\therefore$ The equation becomes $\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right) v = x^2$

$\therefore \frac{d^2v}{dx^2} - \frac{2v}{x^2} = x^2$ $\therefore x^2\frac{d^2v}{dx^2} - 2v = x^4$

Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$, we get,

$[D(D-1) - 2]v = e^{4z}$

$\therefore (D^2 - D - 2)v = e^{4z}$

$\therefore$ The auxiliary equation is $D^2 - D - 2 = 0$

$\therefore (D-2)(D+1) = 0 \qquad \therefore D = -1, 2$

$\therefore$ The C.F. is $y = c_1 e^{-z} + c_2 e^{2z}$

$\therefore P.I. = \frac{1}{D^2-D-2} \cdot e^{4z} = \frac{1}{16-4-2} \cdot e^{4z} = \frac{1}{10} e^{4z}$

$\therefore$ The complete solution is $v = c_1 e^{-z} + c_2 e^{2z} + \frac{1}{10} e^{4z} = \frac{c_1}{x} + c_2 x^2 + \frac{x^4}{10}$

Putting the value of v in (1), we get, $\frac{d^2y}{dx^2} - \frac{2y}{x^2} = \frac{c_1}{x} + c_2 x^2 + \frac{x^4}{10}$

$\therefore x^2 \frac{d^2y}{dx^2} - 2y = c_1 x + c_2 x^4 + \frac{x^6}{10}$

Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$, we get,

$[D(D-1) - 2]y = c_1 e^z + c_2 e^{4z} + \frac{1}{10} e^{6z}$

$\therefore$ The auxiliary equation is $D^2 - D - 2 = 0$

$\therefore (D+1)(D-2) = 0 \qquad \therefore D = -1, 2$

$\therefore$ The C.F. is $y = c_3 e^{-z} + c_4 e^{2z}$

$\therefore P.I. = \frac{1}{D^2-D-2} \left(c_1 e^z + c_2 e^{4z} + \frac{1}{10} e^{6z}\right)$

$= \frac{c_1}{1-1-2} e^z + \frac{c_2}{16-4-2} e^{4z} + \frac{1}{10} \cdot \frac{1}{36-6-2} e^{6z}$

$= -\frac{c_1}{2} e^z + \frac{c_2}{10} e^{4z} + \frac{1}{280} e^{6z}$

$\therefore$ The complete solutionn is

$y = c_3 e^{-z} + c_4 e^{2z} - \frac{c_1}{2} e^z + \frac{c_2}{10} e^{4z} + \frac{1}{280} e^{6z}$

$\therefore y = \frac{c_3}{x} + c_4 x^2 - \frac{c_1}{2} x + \frac{c_2}{10} x^4 + \frac{1}{280} x^6$

Since the constants are aribitrary we can write the solution as $y = \frac{a}{x} + bx + cx^2 + dx^4 + \frac{1}{280} x^6$

21. Solve $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = \frac{x^3}{x^2+1}$ by the method of variation of parameters

Solution: Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$, we get,

$(D^2 - 1)y = \frac{e^{3z}}{e^{2z}+1}$

$\therefore$ The auxiliary equation is $D^2 - 1 = 0 \qquad \therefore D = 1, -1$

$\therefore$ The C.F. is $y = c_1 e^z + c_2 e^{-z}$

Here $y_1 = e^z, y_2 = e^{-z}, X = \frac{e^{3z}}{e^{2z}+1}$

Let $P.I.$ be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^z & e^{-z} \\ e^z & -e^{-z} \end{vmatrix} = -2$

$\therefore u = -\int \frac{y_2 X}{W} dz = -\int \frac{e^{-z}}{-2} \cdot \frac{e^{3z}}{e^{2z}+1} dz = \frac{1}{2} \int \frac{e^{2z}}{e^{2z}+1} dz$

$= \frac{1}{4} \log(e^{2z} + 1) \qquad$ [Put $e^{2z} = t$, $e^{2z} dz = \frac{1}{2} dt$]

& $v = \int \frac{y_1 X}{W} dz = \int \frac{e^z}{-2} \cdot \frac{e^{3z}}{(e^{2z}+1)} dz = -\frac{1}{2} \int \frac{e^{4z}}{e^{2z}+1} dz \qquad$ [Put $e^{2z} = t$, $e^{2z} dz = \frac{1}{2} dt$]

$= -\frac{1}{2}\int \frac{t}{t+1} \cdot \frac{1}{2} dt = -\frac{1}{4}\int \frac{t+1-1}{t+1} dt = -\frac{1}{4}\int \left(1 - \frac{1}{t+1}\right) dt = -\frac{1}{4}(t - \log(t+1))$

$= -\frac{e^{2z}}{4} + \frac{1}{4}\log(e^{2z}+1)$

$\therefore$ The complete solution is $y = c_1 e^z + c_2 e^{-z} + \frac{e^z}{4}\log(e^{2z}+1) - \frac{e^z}{4} + \frac{e^{-z}}{4}\log(e^{2z}+1)$

$\therefore y = c_1 x + \frac{c_2}{x} + \frac{x}{4}\log(x^2+1) + \frac{1}{4x}\log(x^2+1) - \frac{x}{4}$

22. Solve $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3 \sec^2 x$ by the methods of variation of parameters

Solution: Putting $z = \log x$ and $x = e^z$, $\frac{d}{dz} = D$, we get,

$[D(D-1) - 2D + 2]y = e^{3z}\sec^2(e^z)$

$\therefore [D^2 - 3D + 2]y = e^{3z}\sec^2(e^z)$

$\therefore$ The auxiliary equation is $D^2 - 3D + 2 = 0$

$\therefore (D-1)(D-2) = 0 \qquad \therefore D = 1, 2$

$\therefore$ The C.F. is $y = c_1 e^z + c_2 e^{2z}$

Here, $y_1 = e^z, y_2 = e^{2z}, X = e^{3z}\sec^2(e^z)$

Let $P.I.$ be $y = uy_1 + vy_2$

Now, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^z & e^{2z} \\ e^z & 2e^z \end{vmatrix} = e^{3z}$

$\therefore u = -\int \frac{y_2 X}{W} dz$

$= -\int \frac{e^{2z} \cdot e^{3z}\sec^2(e^z)}{e^{3z}} dz$

$= -\int e^{2z}\sec^2(e^z)\, dz$ [Put $e^z = t,\ e^z dz = dt$]

$= -\int t\sec^2 t\, dt$

$= -[t \cdot \tan t - \int \tan t\, dt]$ ....by parts

$= -[t \cdot \tan t - \log\sec t]$

$= -e^z\tan(e^z) + \log\sec(e^z)$

& $v = \int \frac{y_1 X}{W} dz = \int \frac{e^z \cdot e^{3z}\sec^2(e^z)}{e^{3z}} dz$

$= \int e^z \sec^2(e^z)\, dz$ [Put $e^z = t,\ e^z dz = dt$]

$= \int \sec^2 t\, dt = tant$

$= \tan(e^z)$

$\therefore PI = [-e^z\tan(e^z) + \log\sec(e^z)]e^z + \tan(e^z) \cdot e^{2z} = e^z \log\sec(e^z)$

$\therefore$ The complete solution is $y = c_1 e^z + c_2 e^{2z} + e^z \log(\sec e^z)$

23. The radial displacement 'u' in a rotating disc at a distance 'r' from the axis is given by $\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + kr = 0$. Find the displacement if $u = 0$ when $r = 0$ and $r = a$.

Solution: Given differential equation is $r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = -kr^3$

Put $r = e^z$ i.e. $z = \log r$ and $\frac{d}{dz} = D$, we get,

$\therefore D(D-1)u + Du - u = -ke^{3z}$

Auxiliary equation is $D^2 - 1 = 0$ $\therefore D = \pm 1$

$\therefore CF = u_c = c_1 e^{-z} + c_2 e^{z} = \frac{c_1}{r} + c_2 r$

$PI = u_p = \frac{1}{D^2-1}(-ke^{3z}) = \frac{-ke^{3z}}{8} = \frac{-kr^3}{8}$

$\therefore$ The cmplete solution is $u = u_c + u_p = \frac{c_1}{r} + c_2 r - \frac{kr^3}{8}$

i.e. $ur = c_1 + c_2 r^2 - \frac{kr^4}{8}$

When $r = 0,\ u = 0$ $\therefore c_1 = 0$

and when $r = a,\ u = 0$ $\therefore c_1 + c_2 a^2 - \frac{ka^4}{8} = 0$ $\therefore c_2 = \frac{ka^4}{8a^2} = \frac{ka^2}{8}$

$\therefore u = \frac{ka^2 r}{8} - \frac{kr^3}{8} = \frac{kr}{8}(a^2 - r^2)$

24. Find the equation of the curve which satisfies the differential equation $4x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0$ and crosses the x-axis at an angle of 60^0 at x = 1.

Solution: Given differential equation is $4x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0$

Put $x = e^z$ i.e. $z = \log r$ and $\frac{d}{dz} = D$, we get,

$\therefore 4D(D-1)y - 4Dy + y = 0$

Auxilliary equation is $4D^2 - 4D - 4D + 1 = 0$ i.e. $4D^2 - 8D + 1 = 0$

$\therefore D = 1 \pm \frac{\sqrt{3}}{2}$

$\therefore CF = y_c = c_1 e^{\left(1+\frac{\sqrt{3}}{2}\right)z} - c_2 e^{\left(1-\frac{\sqrt{3}}{2}\right)z}$

$y = c_1 x^{\left(1+\frac{\sqrt{3}}{2}\right)} + c_2 x^{\left(1-\frac{\sqrt{3}}{2}\right)}$

$\frac{dy}{dx} = c_1\left(1+\frac{\sqrt{3}}{2}\right)x^{\sqrt{3}/2} + c_2\left(1-\frac{\sqrt{3}}{2}\right)x^{-\sqrt{3}/2}$

At $x = 1, y = 0$ $\therefore 0 = c_1 + c_2$ i.e. $c_1 = -c_2$

and at $x = 1, \frac{dy}{dx} = \tan 60° = \sqrt{3}$ $\therefore \sqrt{3} = c_1\left(1+\frac{\sqrt{3}}{2}\right) + c_2\left(1-\frac{\sqrt{3}}{2}\right)$,

but $c_2 = -c_1$ $\therefore \sqrt{3} = c_1\sqrt{3}$ $\therefore c_1 = 1$ and $c_2 = -1$

$\therefore y = x^{(1+\sqrt{3}/2)} - x^{(1-\sqrt{3}/2)}$