5. The Dual Simplex Method

We have seen chapter IX how to solve an L.P.P. by simplex method. In this method we start with a basic feasible but not optimal solution, starting with $x_1 = 0$, $x_2 = 0$, ... etc. and work towards the optimal solution. In the dual simplex method we start with the optimal but infeasible solution and work downwards the feasible solution.

The constraints are converted into less than or equal to type ($\leq$) and the objective function is converted into minimisation if not so already.

Working Procedure for Dual Simplex Method

1. Convert the problem into minimisation type.
2. Convert all constraints in less than or equal to type. If any constraint is of greater than or equal to type multiply throughout by (–1) and change the inequality sign.
3. Convert the inequality constraints into equalities by adding slack variables.
4. Put this information in a table.
5. If all the coefficients in the row of z are negative and all the right-hand side constants *i.e.* b's are positive then basic feasible solution is obtained.

   If all the coefficients in the row of z are negative and atleast one 'b' is negative then go to step 6.

   (If all the coefficient in the row of z are positive the method fails.)
6. Look at the last right hand side column. Select the row which contains the smallest (negative) number in the column of b's. Denote it by an arrow-head like $\leftarrow$. This is the key row and the corresponding variable on the left is the outgoing variable.
7. Find the ratios of the elements in **the row of z to the corresponding elements in the key row**. Write these **ratios in another row below the tables**.
8. **Now select the incoming variable :** Select the column which contains the smallest ratio and denote it by an arrow-head like $\uparrow$. This is the key column. The element where the key row and the key column intersect is the key element.
9. Construct another table as in the usual simplex method. Divide each element in the key row by key element, so that key element will be converted into unity. Now, make all other elements in the key column zero by subtracting proper multiples of the elements in the key row from the corresponding elements in the other rows as in the simplex method.
10. Continue the procedure till all the b's are positive. These are the required values of the decision variables.

Comparison between the Regular Simplex Method and the Dual Simplex Method

1. In regular simplex method the objective function is of maximisation type while in the dual simplex method the object function is of minimisation type.
2. In the regular simplex method we start with basic feasible but not optimal solution. In the dual simplex method we start with basic infeasible but optimal solution.
3. In the regular simplex method we arrive at the optimal solution in the end if it exists. In the dual simplex method we arrive at the feasible solution in the end.
4. In the regular simplex method we first decide incoming variable and from the ratio column we then decide the outgoing variable. In the dual simplex method, we first decide the outgoing variable and then from ratio row, we decide the incoming variable.
5. In the regular simplex method artificial variables are required when the constraints are of greater than type. In the dual simplex method artificial variables are not required.
6. The dual simplex method is more convenient than the regular simplex method.

Example 1 : Use the dual simplex method to solve the following L.P.P.

$$\begin{aligned} \text{Minimise} \quad & z = 2x_1 + 2x_2 + 4x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 + 5x_3 \geq 2 \\ & 3x_1 + x_2 + 7x_3 \leq 3 \\ & x_1 + 4x_2 + 6x_3 \leq 5 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

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Sol. : We first express the given problem using $\leq$ in the first constraint.

$$\begin{aligned} \text{Minimise} \quad & z = 2x_1 + 2x_2 + 4x_3 \\ \text{subject to} \quad & -2x_1 - 3x_2 - 5x_3 \leq -2 \\ & 3x_1 + x_2 + 7x_3 \leq 3 \\ & x_1 + 4x_2 + 6x_3 \leq 5 \end{aligned}$$

Introducing the slack variables s_1, s_2, s_3, we have

$$\begin{aligned} \text{Minimise} \quad & z = 2x_1 + 2x_2 + 4x_3 - 0s_1 - 0s_2 - 0s_3 \\ \textit{i.e.} \quad & z - 2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3 = 0 \\ \text{subject to} \quad & -2x_1 - 3x_2 - 5x_3 + s_1 + 0s_2 + 0s_3 = -2 \\ & 3x_1 + x_2 + 7x_3 + 0s_1 + s_2 + 0s_3 = 3 \\ & x_1 + 4x_2 + 6x_3 + 0s_1 + 0s_2 + s_3 = 5 \end{aligned}$$

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	x_3	s_1	s_2	s_3	R.H.S. Solution
0	z	−2	−2	−4	0	0	0	0
s_1 leaves	s_1	−2	−3*	−5	1	0	0	−2 ←
x_2 enters	s_2	3	1	7	0	1	0	3
	s_3	1	4	6	0	0	1	5
Ratio		1	2/3 ↑	4/5				
1	z	−2/3	0	−2/3	−2/3	0	0	4/3
	x_2	2/3	1	5/3	−1/3	0	0	2/3
	s_2	7/3	0	16/3	1/3	1	0	7/3
	s_3	−5/3	0	−2/3	4/3	0	1	7/3

$$\therefore \quad x_1 = 0, \quad x_2 = \frac{2}{3}, \quad x_3 = 0, \quad z_{Min} = \frac{4}{3}.$$

Example 2 : Use the dual simplex method to solve the following L.P.P.

Maximise $z = -3x_1 - 2x_2$

subject to
$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \leq 10$$
$$x_2 \leq 3$$
$$x_1, x_2 \geq 0.$$

$\therefore$ Minimise $z' = -z = 3x_1 + 2x_2$

subject to
$$-x_1 - x_2 \leq -1$$
$$x_1 + x_2 \leq 7$$
$$-x_1 - 2x_2 \leq -10$$
$$x_2 \leq 3$$

Introducing the slack variables s_1, s_2, s_3 and s_4, we have

$$\text{Minimise} \quad z' = 3x_1 + 2x_2 - 0s_1 - 0s_2 - 0s_3 - 0s_4$$

$$\text{i.e.} \quad z' - 3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 = 0$$

$$\text{subject to} \quad -x_1 - x_2 + s_1 + 0s_2 + 0s_3 + 0s_4 = -1$$

$$x_1 + x_2 + 0s_1 + s_2 + 0s_3 + 0s_4 = 7$$

$$-x_1 - 2x_2 + 0s_1 + 0s_2 + s_3 + 0s_4 = -10$$

$$0x_1 + x_2 + 0s_1 + 0s_2 + 0s_3 + s_4 = 3$$

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	s_1	s_2	s_3	s_4	R.H.S. Solution
0	z'	−3	−2	0	0	0	0	0
s_3 leaves	s_1	−1	−1	1	0	0	0	−1
x_2 enters	s_2	1	1	0	1	0	0	7
	s_3	−1	−2*	0	0	1	0	−10
	s_4	0	1	0	0	0	1	3
Ratio		3	1					
1	z'	−2	0	0	0	−1	0	10
s_4 leaves	s_1	−1/2	0	1	0	−1/2	0	4
x_1 enters	s_2	1/2	0	0	1	1/2	0	2
	x_2	1/2	1	0	0	−1/2	0	5
	s_4	−1/2*	0	0	0	1/2	1	−2
Ratio		4						
2	z'	0	0	0	0	−3	−4	18
	s_1	0	0	1	0	−1	−1	6
	s_2	0	0	0	1	1	1	0
	x_2	0	1	0	1	−1	1	3
	x_1	1	0	0	0	−1	−2	4

$\therefore \quad x_1 = 4, \; x_2 = 3, \; z'_{\text{Min}} = 18, \; z_{\text{Max}} = -18.$

Example : Use the dual simplex method to solve the following L.P.P.

Minimise $z = x_1 + x_2$

subject to $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$; $x_1, x_2 \geq 0$.

Minimise $z = x_1 + x_2 - 0s_1 - 0s_2$

i.e. $z - x_1 - x_2 + 0s_1 + 0s_2 = 0$

subject to $-2x_1 - x_2 + s_1 + 0s_2 = -2$

$x_1 + x_2 + 0s_1 + s_2 = -1$

Simplex Table

Iteration Number	Basic Variables	Coefficients of x_1	x_2	s_1	s_2	R.H.S. Solution
0	z	-1	-1	0	0	0
s_1 leaves	s_1	-2^*	-1	1	0	-2 ←
x_1 enters	s_2	1	1	0	1	-1
Ratio		1/2 ↑	1			
1	z	0	$-1/2$	$-1/2$	0	1
	x_1	1	1/2	$-1/2$	0	1
	s_2	0	1/2	1/2	1	-2 ←
Ratio		—	-1	-1	—	

Since s_2 row is negative, s_2 leaves. But since all ratios are negative, the L.P.P. has no feasible solution.