Moule1 - Unit 1.2

Discrete and Continuous Probability Distribution

Definition

Random variable (R.V.):

Consider a random experiment whose sample space is S. A random variable X is a function from the sample space S into the set of real numbers IR

In a particular experiment ,a random variable X would be some function that assigns a real number X(s) to each possible outcomes in the sample space.

Random variables are usually denoted by capital letters such as A, B, X, Y, E1, E2 etc.

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Definition

Random variable (R.V.):

Random variables are of two types

(I) Discrete random variable.

(II) Continuous random variables.

Discrete random variable:

If random variable X is function $X : S \rightarrow IR$. Where X(S) is finite or countably infinite then it is called discrete random variable.

i.e. a Random variable can assume only a discrete set of values (precisely, integral values) or countably infinite number of values,

The range of the function X is called as image set X(S)

Example :

Consider the coin tossing experiment.

If a random variable X represents number of Tail What is the space (image) of this random variable X?

Answer: The sample space of this experiment is given by

S = {Head, Tail}.

X represents number of tail

X(Head) = 0   X(Tail) = 1.

The image set or space of this random variable is

X(S) = {0, 1}.

Example :

Consider two coins tossing experiment.

If a random variable X represents number of Head What is the space (image) of this random variable X?

Answer: The sample space of this experiment is given by

S = {HH,HT,TH,TT}.

X represents number of Head

X(HH) =2 X(HT)=1=X(TH),X(TT)=0

The image set or space of this random variable is

X(S) = {0, 1,2}.

Probability Function or
Probability Mass Function (p.m.f.) or
Probability Density Function (p.d.f.)

Let X be discrete random variable on a sample space S function P(x) (or f(x)) defined on X(S) and satisfies following two properties.

$0 \leq P(x) \leq 1 \ \forall \ x \in X(S)$

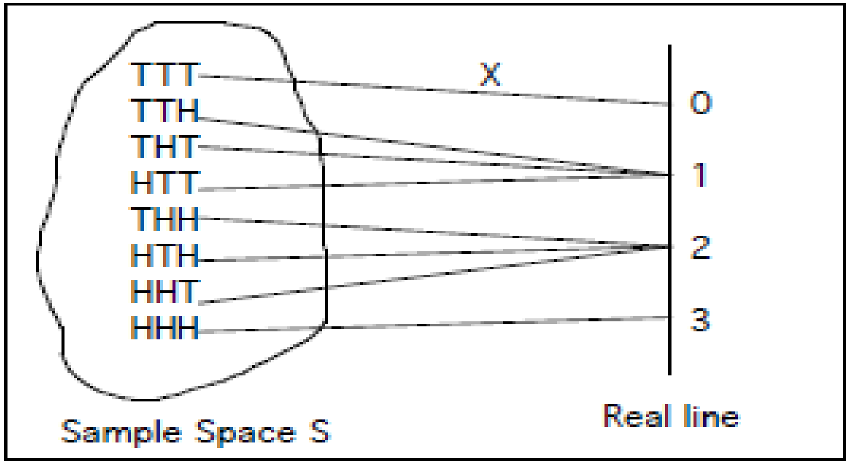
$\sum p(x)$ =1 then function P(x) is called p.d.f.

Note :P(x) denotes the probability that discrete random variable X takes .

Example A fair coin is tossed 3 times. Let the random variable X denote the number of heads in 3 tosses of the coin. Find the sample space, the space of the random variable, and the probability density function of X.

Answer: The sample space S of this experiment consists of all binary sequences of length 3, that is

S = {TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}.



Sample space	TTT	HTT	THT	TTH	THH	HTH	HHT	HHH
R.V X	0	1	1	1	2	2	2	3

R.V X	0	1	2	3
p.d.f.	1/8	3/8	3/8	1/8

Ex: A fair die is tossed 2 times. Let the random variable X denotes the sum of numbers on the die. Find the sample space, the space of the random variable, and the probability density function of X.

Example Consider an experiment in which a coin is tossed ten times . Let x : s → IR be a function from the sample space S into the set of reals IR defined as follows:

X(s) = number of heads in sequences.

then find the space of the random variable.

Answer

This random variable, for example, maps the sequence HHTTTHTTHH to the real number 5, that is X(HHTTTHTTHH) = 5.

The space of this random variable is

RX = {0, 1, 2, ..., 10}

Example : *A random variable X has the following probability distribution :*

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k, (ii) Evaluate $P(X < 6)$, $P(X \geq 6)$, *and* $P(0 < X < 5)$, *(iii) If* $P(X \leq c) > \frac{1}{2}$, *find the minimum value of c, and (iv) Determine the distribution function of X.*

Solution. Since $\sum_{x=0}^{7} p(x) = 1$, we have

$$\Rightarrow \quad k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow \quad 10k^2 + 9k - 1 = 0$$

$$\Rightarrow \quad (10k - 1)(k + 1) = 0 \Rightarrow k = 1/10$$

[$\because k = -1$, is rejected, since probability canot be negative.]

(*ii*) $P(X < 6) = P(X = 0) + P(X = 1) + \ldots + P(X = 5)$

$$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = 1 - P(X < 6) = \frac{19}{100}$$

$P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 8k = 4/5$

(*iii*) $P(X \leq c) > \frac{1}{2}$. By trial, we get $c = 4$.

(*iv*)

X	$F_X(x) = P(X \leq x)$
0	0
1	$k = 1/10$
2	$3k = 3/10$
3	$5k = 5/10$
4	$8k = 4/5$
5	$8k + k^2 = 81/100$
6	$8k + 3k^2 = 83/100$
7	$9k + 10k^2 = 1$

Example If the probability of a random variable X with space RX = {1, 2, 3, ..., 12} is given by f(x) = k (2x − 1), then, what is the value of the constant k?

$$
\begin{aligned}
1 &= \sum_{x \in R_X} f(x) \\
&= \sum_{x \in R_X} k\,(2x-1) \\
&= \sum_{x=1}^{12} k\,(2x-1) \\
&= k\left[2\sum_{x=1}^{12} x - 12\right] \\
&= k\left[2\frac{(12)(13)}{2} - 12\right] \\
&= k\,144.
\end{aligned}
$$

$$k = \frac{1}{144}.$$

Example

Let X be a random variable such that

$$P(X=-2)=P(X=-1),\ P(X=2)=P(X=1) \text{ and}$$

$$P(X>0)=P(X<0)=P(X=0).$$

Obtain the probability mass function of X and its distribution function.

Ans.

X	:	-2	-1	0	1	2
$p(x)$	:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
$F(x)$	:	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

Definition

Cumulative distribution function (c.d.f.)

The cumulative distribution function F(x) of a random variable X is defined as

$F(x) = P(X \leq x)$ for all real numbers x.

Theorem

If X is a discrete random variable sample space S with the image set X(S), then

$F(x) = \sum_{t \leq x} f(t)$ for $x \in X(S)$.

Theorem 3.3. Let X be a random variable with cumulative distribution function $F(x)$. Then the cumulative distribution function satisfies the followings:

(a) $F(-\infty) = 0$,

(b) $F(\infty) = 1$, and

(c) $F(x)$ is an increasing function, that is if $x < y$, then $F(x) \leq F(y)$ for all reals x, y.

Theorem 3.4. If the space R_X of the random variable X is given by $R_X = \{x_1 < x_2 < x_3 < \cdots < x_n\}$, then

$$
\begin{aligned}
f(x_1) &= F(x_1) \\
f(x_2) &= F(x_2) - F(x_1) \\
f(x_3) &= F(x_3) - F(x_2) \\
&\ldots \ldots\ldots \\
&\ldots \ldots\ldots \\
f(x_n) &= F(x_n) - F(x_{n-1}).
\end{aligned}
$$

Example If the probability density function of the random variable X is given by

f(x) = $\frac{1}{144}(2x - 1)$ for x = 1, 2, 3, ..., 12

then find the cumulative distribution function of X.

Answer:

$$F(1) = \sum_{t \leq 1} f(t) = f(1) = \frac{1}{144}$$

$$F(2) = \sum_{t \leq 2} f(t) = f(1) + f(2) = \frac{1}{144} + \frac{3}{144} = \frac{4}{144}$$

$$F(3) = \sum_{t \leq 3} f(t) = f(1) + f(2) + f(3) = \frac{1}{144} + \frac{3}{144} + \frac{5}{144} = \frac{9}{144}$$

.. ..

.. ..

$$F(12) = \sum_{t \leq 12} f(t) = f(1) + f(2) + \cdots + f(12) = 1.$$

Example 3.9. Find the probability density function of the random variable X whose cumulative distribution function is

$$F(x) = \begin{cases} 0.00 & \text{if } x < -1 \\ 0.25 & \text{if } -1 \leq x < 1 \\ 0.50 & \text{if } 1 \leq x < 3 \\ 0.75 & \text{if } 3 \leq x < 5 \\ 1.00 & \text{if } x \geq 5 \ . \end{cases}$$

Also, find (a) $P(X \leq 3)$, (b) $P(X = 3)$, and (c) $P(X < 3)$.

Answer: The space of this random variable is given by

$$R_X = \{-1, 1, 3, 5\}.$$

By the previous theorem, the probability density function of X is given by

$$\begin{aligned} f(-1) &= 0.25 \\ f(1) &= 0.50 - 0.25 = 0.25 \\ f(3) &= 0.75 - 0.50 = 0.25 \\ f(5) &= 1.00 - 0.75 = 0.25. \end{aligned}$$

The probability $P(X \le 3)$ can be computed by using the definition of F. Hence

$$P(X \le 3) = F(3) = 0.75.$$

The probability $P(X = 3)$ can be computed from

$$P(X = 3) = F(3) - F(1) = 0.75 - 0.50 = 0.25.$$

Finally, we get $P(X < 3)$ from

$$P(X < 3) = P(X \le 1) = F(1) = 0.5.$$

Example A random variable X assumes the values –3, –2, –1, 0, 1, 2, 3 such that

$$P(X=-3)=P(X=-2)=P(X=-1),$$

$$P(X=1)=P(X=2)=P(X=3),$$

and $$P(X=0)=P(X>0)=P(X<0),$$

Obtain the probability mass function of X and its distribution function, and find further the probability mass function of $Y = 2X^2 + 3X + 4$.

Ans.

X	:	–3	–2	–1	0	1	2	3
$p(x)$	:	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
Y	:	13	6	3	4	9	18	31
$p(y)$	:	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

Continuous Random Variables

A random variable X is said to be continuous if its space or image set X(s) is either an interval or a union of intervals

OR

A random variable X is said to be continuous if X takes uncountable infinite values.

Let X be a continuous random variable.
Then a probability distribution or
probability density function (p.d.f.) of X is a function f(x)
Which satisfies the following two conditions
$0 \leq f(x) \leq 1 \; \forall \; x$ and $\int_{-\infty}^{\infty} f(t)dt=1$

Note :For any two numbers a and b with $a \leq b$,
$P(a \leq X \leq b)= \int_{a}^{b} f(t)dt$

Example

Is the real valued function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2\,x^{-2} & \text{if } 1 < x < 2 \\ 0 & \text{otherwise,} \end{cases}$$

a probability density function for some random variable X?

$0 \leq f(x) \ \forall \ x \in \mathrm{IR}$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{1}^{2} 2\,x^{-2}\,dx \\ &= -2\,\left[\frac{1}{x}\right]_{1}^{2} \\ &= -2\,\left[\frac{1}{2} - 1\right] \\ &= 1. \end{aligned}$$

Example

For what value of the constant c, the real valued function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} c & \text{if } a \le x \le b \\ 0 & \text{otherwise,} \end{cases}$$

where a, b are real constants, is a probability density function for random variable X?

Answer: Since f is a pdf, k is nonnegative. Further, since the area under f is unity, we get

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)\,dx \\ &= \int_a^b c\,dx \\ &= c\,[x]_a^b \\ &= c\,[b-a] \end{aligned}$$

Hence $c = \frac{1}{b-a}$,

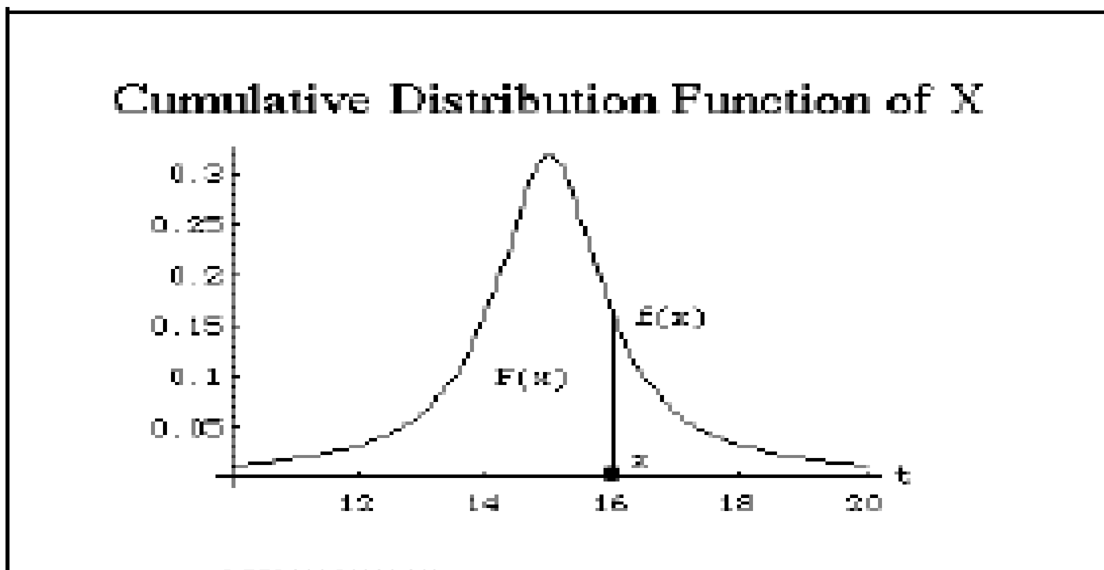
Definition

Let f(x) be the probability density function of a continuous random variable X.

The cumulative distribution function F(x) of X is defined as

F(x) = P(X $\le$ x) = $\int_{-\infty}^{x} f(t)dt$

The cumulative distribution function F(x) represents the area under the probability density function f(x) on the interval $(-\infty, x)$



Theorem

If F(x) is the cumulative distribution function of a continuous random variable X, the probability density function f(x) of X is the derivative of F(x), that is

$\frac{d}{dx}F(x) = f(x)$

Theorem

If F(x) is the cumulative distribution function of a continuous random variable X

(i) $F(x) = P(X \leq x)$

(ii) $1-F(x) = P(X \geq x)$

(iii) $P(a < X < b) = F(b) - F(a)$.

(iv) $P(X = x) = 0$

Example What is the probability density function of the random variable whose cdf is

$$F(x) = \frac{1}{1+e^{-x}}, \qquad -\infty < x < \infty?$$

Answer: The pdf of the random variable is given by

$$\begin{aligned} f(x) &= \frac{d}{dx}F(x) \\ &= \frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right) \\ &= \frac{d}{dx}\left(1+e^{-x}\right)^{-1} \\ &= (-1)\left(1+e^{-x}\right)^{-2}\frac{d}{dx}\left(1+e^{-x}\right) \\ &= \frac{e^{-x}}{(1+e^{-x})^2}. \end{aligned}$$

Example *A continuous random variable X has a p.d.f.*

$f(x) = 3x^2,\ 0 \le x \le 1$. *Find a and b such that*

(i) $P\{X \le a\} = P\{X > a\}$, and

(ii) $P\{X > b\} = 0{\cdot}05$.

Solution. (i) Since $P(X \le a) = P(X > a)$,

each must be equal to 1/2, because total probability is always one.

$$\therefore \quad P(X \le a) = \frac{1}{2} \quad \Rightarrow \quad \int_0^a f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \quad 3\int_0^a x^2\,dx = \frac{1}{2} \quad \Rightarrow \quad 3\left|\frac{x^3}{3}\right|_0^a = \frac{1}{2}$$

$$\Rightarrow \quad a^3 = \frac{1}{2} \quad \Rightarrow \quad a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

$$(ii) \quad P(X > b) = 0{\cdot}05 \quad \Rightarrow \quad \int_b^1 f(x)\,dx = 0{\cdot}05$$

$$\Rightarrow \quad 3\left|\frac{x^3}{3}\right|_b^1 = \frac{1}{20} \quad \Rightarrow \quad 1 - b^3 = \frac{1}{20}$$

$$\Rightarrow \quad b^3 = \frac{19}{20} \quad \Rightarrow \quad b = \left(\frac{19}{20}\right)^{\frac{1}{3}}.$$

Expectation of a discrete random variable:

If discrete random variable X takes values $x_1, x_2, \ldots . x_n$ with corresponding probabilities $p_1, p_2, \ldots . p_n$ where $p_i = P(X = x_i)$ and $\Sigma p_i = 1$, then mathematical expectation or expected value or mean value of random variable X is denoted by E(X) or μ and is defined as

$$E(X) = (x_1 \cdot p_1) + (x_2 \cdot p_2) + \ldots + (x_n \cdot p_n)$$
$$= \sum_{i=1}^{n} x_i \cdot p_i = \sum_{i=1}^{n} x_i \cdot P(X = x_i)$$

Or simply $\mu = E(X) = \Sigma x \cdot p(x)$ = mean or arithmetic average

Mathematical expectation of a function of discrete random variable X say $\phi(X)$ is given by

$$E[\phi(X)] = \Sigma \phi(x) \cdot P(x)$$

$$E(X^n) = \begin{cases} \sum\limits_{x \in R_X} x^n f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^n f(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

Variance

Now $\text{Var}(X) = E(X - \bar{x})^2$

$$= E[X - E(X)]^2$$

$$= E\left[X^2 - 2X\,E(X) + \{E(X)\}^2\right]$$

$$= E(X^2) - 2E(X) \cdot E(X) + [E(X)]^2$$

$$\boxed{\text{Var.}(X) = E(X^2) - [E(X)]^2}$$

Example : A lot of 8 TV sets includes 3 that are defective. If 4 of the sets are chosen at random for shipment to a hotel, find expected no of defective sets?

Answer: Let X be the random variable representing the number of defective TV sets in a shipment of 4. Then the space of the random variable X is

RX = {0, 1, 2, 3}.

Then the probability density function of X is given by

f(x) = P(X = x)

= P(x defective TV sets in a shipment of four)

$=3_{c_x} 5_{c_{4-x}} / 8_{c_4}$, x = 0, 1, 2, 3.

$$f(0) = \frac{\binom{3}{0}\binom{5}{4}}{\binom{8}{4}} = \frac{5}{70}$$

$$f(1) = \frac{\binom{3}{1}\binom{5}{3}}{\binom{8}{4}} = \frac{30}{70}$$

$$f(2) = \frac{\binom{3}{2}\binom{5}{2}}{\binom{8}{4}} = \frac{30}{70}$$

$$f(3) = \frac{\binom{3}{3}\binom{5}{1}}{\binom{8}{4}} = \frac{5}{70}.$$

$$\begin{aligned}
E(X) &= \sum_{x \in Rx} x\, f(x) \\
&= \sum_{0}^{3} x\, f(x) \\
&= f(1) + 2\, f(2) + 3\, f(3) \\
&= \frac{30}{70} + 2\,\frac{30}{70} + 3\,\frac{5}{70} \\
&= \frac{30 + 60 + 15}{70} \\
&= \frac{105}{70} = 1.5.
\end{aligned}$$

Example: Four fair coins are tossed. Find the expectation of number of heads

X denotes the number of heads in 4 tosses of the coin.

X takes values 0,1,2,3,4

$$P(X=0) = \frac{1}{16}, \quad P(X=1) = \frac{4}{16} = \frac{1}{4}, \quad P(X=2) = \frac{6}{16} = \frac{3}{8},$$

$$P(X=3) = \frac{4}{16} = \frac{1}{4} \quad \text{and} \quad P(X=4) = \frac{1}{16}.$$

Thus the probability distribution of X can be summarised as follows :

x :	0	1	2	3	4
$p(x)$:	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

$$E(X) = \sum_{x=0}^{4} x\,p(x) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{16}$$

$$= \frac{1}{4} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4} = 2.$$

Example **Write down the probability distribution of the sum of numbers appearing on the toss of two unbiased dice. Hence find mean of the distribution**

Value of X : x	2	3	4	5	6	7	.....	11	12
Probability	1/36	2/36	3/36	4/36	5/36	6/36	.....	2/36	1/36

$$E(X) = \sum_i p_i x_i$$

$$= 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36} + 7 \times \frac{6}{36}$$

$$+ 8 \times \frac{5}{36} + 9 \times \frac{4}{36} + 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36}$$

$$= \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12)$$

$$= \frac{1}{36} \times 252 = 7$$

Example

A discrete R.V. X has the pdf as given below

X :	**-2**	**-1**	**0**	**1**	**2**	**3**
P(X=x):	**0.2**	**k**	**0.1**	**2k**	**0.1**	**2k**

Find k, mean and variance

Sol. : We must have $\Sigma p_i = 1$.

$$\therefore \quad 5k + 0\cdot4 = 1 \quad \therefore \quad 5k = 0\cdot6 \quad \therefore \quad k = \frac{0\cdot6}{5} = \frac{3}{25}$$

Hence, the probability distribution is

X :	-2	-1	0	1	2	3
$P(X = x)$:	2/10	3/25	1/10	6/25	1/10	6/25

Now, $\quad \text{Mean} = E(X) = \Sigma\, p_i\, x_i$

$$= -\frac{4}{10} - \frac{3}{25} + 0 + \frac{6}{25} + \frac{2}{10} + \frac{18}{25} = \frac{60}{250} = \frac{6}{25}$$

$$E(X^2) = \Sigma\, p_i\, x_i^2$$

$$= \frac{2}{10}(4) + \frac{3}{25}(1) + 0 + \frac{6}{25}(1) + \frac{1}{10}(4) + \frac{6}{25}(9) = \frac{73}{250}$$

$\therefore \quad \text{Variance} = \sigma^2 = E(X^2) - [E(X)]^2$

$$= \frac{73}{250} - \frac{36}{625} = \frac{293}{625}.$$

Example A and B throw a fair dice for a stake of Rs 44, which is won by the player who throws 6 first. If A starts first, find their expectations

To find expectation of A

RV = An amount the plyer A gains =44 or 0

A can win the game, in the first throw or the third throw or the fifth throw …….

P(A winning)

= (1/6)+(5/6) (5/6)(1/6)+ (5/6) (5/6) (5/6) (5/6)(1/6)+…..

=(1/6){1+(5/6)^2+(5/6)^4+……………}

=(1/6){1/[1-25/36]}

=6/11

R.V X	44	0
p.d.f P(x)	P(A wins) =6/11	P(A loses) =5/11

$$E(X) = \sum xp(x)$$

$$=44(6/11) + 0(5/11) = 24$$

R.V X	44	0
p.d.f P(x)	P(B wins) =5/11	P(B loses) =6/11

$$E(X) = \sum xp(x)$$

$$=44(5/11) + 0(6/11) = 20$$

Example- A continuous R.V. X has the pdf defined as f(x)=`k $x^2\ e^{-x}$.x>0. Find k, mean and variance

Sol. : We must have $\int_0^{\infty} k\,x^2 e^{-x} \bullet dx = 1$

$$\therefore \quad k\left[x^2(-e^{-x}) - \int -e^{-x}2x\,dx\right]_0^{\infty} = 1$$

$$\therefore \quad k\left[-x^2e^{-x} + 2x(-e^{-x}) - \int -2e^{-x}dx\right]_0^{\infty} = 1$$

$$k\left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right]_0^{\infty} = 1$$

$$k[0-(-0-0-2)] = 1 \quad \therefore\ 2k = 1 \quad \therefore\ k = \frac{1}{2}..$$

Now, men $\bar{x} = \int_0^{\infty} x\,f(x)\,dx$

$$\therefore \quad \bar{x} = \int_0^{\infty} \frac{1}{2}x^3e^{-x}dx$$

$$= \frac{1}{2}\left[x^3(-e^{-x}) - (3x^2)(e^{-x}) + (6x)(-e^{-x}) - (6)(e^{-x})\right]_0^{\infty}$$

$$\therefore \quad \bar{x} = \frac{1}{2}[0-(-6)] = \frac{1}{2}\cdot 6 = 3$$

Now, $$\mu_2' = \frac{1}{2}\int_0^\infty x^2 f(x)\,dx = \frac{1}{2}\int_0^\infty x^2 \cdot x^2 e^{-x}\,dx$$

$$= \frac{1}{4}\int_0^\infty x^4 \cdot e^{-x} dx$$

$$= \frac{1}{2}\Big[x^4(-e^{-x}) - (4x^3)(e^{-x}) + (12x^2)(-e^{-x})$$

$$- (24x)(e^{-x}) + 24(-e^{-x})\Big]_0^\infty$$

$$= \frac{1}{2}[0-(-24)] = \frac{24}{2} = 12$$

$\therefore$ Variance $= \mu_2' - \mu_1'^2 = 12 - 9 = 3.$

Ex.If a coin is tossed by a player two times , he wins Rs 3 for each head and Rs 2 for each tail. Find the probability distribution table and his expectation

Ex. If a player wins Rs 3 if he draws one white ball and wins Rs 2 if he draws one black ball from a bag containing 5 white and 4 black balls ,then find his expectation