Module 1, unit 1.4

Uniform Distribution, Normal Distribution, Exponential Distribution

Definition.

Uniform distribution

A random variable X is said to be uniform on the interval [a, b] if its probability density function is of the form

$f(x) = 1/b - a \quad , \quad a \leq x \leq b,$

where a and b are constants.

We denote a random variable X with the uniform distribution on the interval [a, b] as X~ UNIF(a, b).

An important application of uniform distribution lies in random number generation

Mean and variance of Uniform distribution

If X is uniform on the interval [a, b]

then the mean and variance of X are given by

<u>$E(X)=(b+a)/2$, $V(X)=(b-a)^2/12$</u>

Mean

$=E(X)$

$= \int_{-\infty}^{\infty} xf(x)dx$

$= \int_{a}^{b} x(1/\{b-a\})dx$

$= (b+a)/2$

$E(X^2)$

$= \int_{-\infty}^{\infty} x^2 f(x) dx$

$= \int_a^b x^2 (1/\{b-a\}) dx$

$= (1/\{b-a\}) \int_a^b x^2 dx =$

$=(1/3)\ (a^2+ab+b^2)$

Now $V(X)$

$= E(X^2) - \{E(x)\}^2$

$= (1/3)\ (a^2+ab+b^2) - ((b+a)/2)^2$

$=(b-a)^2 /12$

Example. A box to be constructed so that its height is 10 inches and its base is X inches by X inches. If X has a uniform distribution over the interval (2, 8), then what is the expected volume of the box in cubic inches?

Answer: Since X~ UNIF(2, 8),

f(x) =1/(8 – 2)=1/6 on (2, 8).

The volume V of the box is $V = 10X^2$.

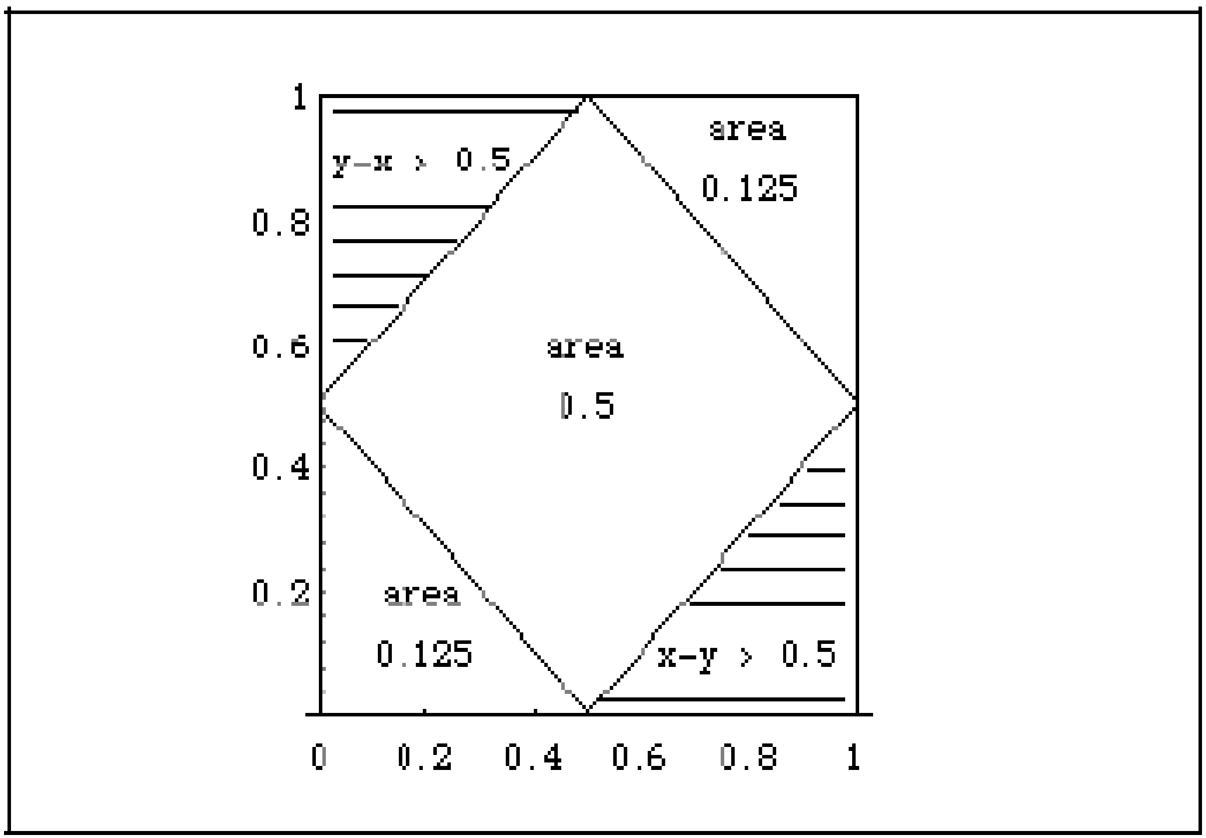
Hence $E(V)=E(10X^2)$

$$=10 \int_2^8 (X^2)(1/6)\,dx$$

= 280 cubic inches

Example : Two numbers are chosen independently and at random from the interval (0, 1). What is the probability that the two numbers differs by more than ½

Ans



Choose x from the x-axis between 0 and 1, and choose y from the y-axis between 0 and 1. The probability that the two numbers differ by more than ½ is equal to the area of the shaded region.

Thus $p(|x-y|>1/2) = p(x-y>1/2)$ or $p(y-x>1/2)$

$$= (1/8) + (1/8) = 0.25$$

Example 3. If X is uniform on the interval from 0 to 3, what is the probability that the quadratic equation $4t^2 + 4tX + X + 2 = 0$ has real solutions?

Answer: Since $X \sim UNIF(0, 3)$,

the probability density function of X is

$f(x) = 1/3$, $0 \leq x \leq 3$

$\quad 0$,otherwise.

The quadratic equation $4t^2 + 4tX + X + 2 = 0$ has real solution if the discriminant of this equation is positive.

That is $16X^2 - 16(X + 2) \geq 0$,

i.e. $X^2 - X - 2 \geq 0$.

i.e. $(X - 2)(X + 1) \geq 0$.

The probability that the quadratic equation

$4t^2 + 4tX + X + 2 = 0$ has real roots is equivalent to

$p((X - 2)(X + 1) \geq 0)$

$= p(X \leq -1) + p(x \geq 2)$

$= \int_{-\infty}^{-1} f(x)dx + \int_{2}^{3} \mathrm{f(x)}dx$

$= \int_{2}^{3} (\frac{1}{3})dx =$

$= 1/3$

Exponential distribution

Definition

A random variable X is said to follow exponential distribution with parameter λ if its probability density function is of the form

f(x) = λ $e^{-\lambda x}$, 0 ≤ x and 0 ≤ λ

=0 , otherwise

We denote exponential distribution X with parameter λ as X~ Exp (λ)

Cumulative distribution function (c.d.f.)

F(X=x)=p(X≤x)=1-$e^{-\lambda x}$, 0 ≤ x

Mean $= E(X)$

$= \int_{-\infty}^{\infty} xf(x)dx$

$= \lambda\int_{0}^{\infty} xe^{-\lambda \mathrm{x}}dx$

$=\lambda\{xe^{-\lambda \mathrm{x}}/(-\lambda) - e^{-\lambda \mathrm{x}} / \lambda^2\}$

$= 1/\lambda$

$E(X^2)$

$= \int_{-\infty}^{\infty} x^2f(x)dx$

$= \lambda\int_{0}^{\infty} x^2e^{-\lambda \mathrm{x}}dx$

$= 2 / \lambda^2$

$V(X)$

$= E(X^2) - \{\mathrm{E(x)}\}^2$

$=2 / \lambda^2 - 1 / \lambda^2$

$=1 / \lambda^2$

Exponential distribution is the only continuous distribution satisfying memoryless property.

Memoryless property

Exponential distribution has remarkable property of forgetfulness. In other words it is immaterial when the event occurred last and how much later we start observing the events. it does not depend upon the past. It forgets what has happened previously and hence the property is referred to as memoryless property

If $X \sim Exp(\lambda)$ then $p(X \geq s+t / X \geq s) = p(X \geq t)$

Example :The amount of time that a watch will run without having to be reset is a random variable having an exponential distribution with mean 120 days. Find the probability that such a watch will (i) have to be set in less than 24 days (ii) not have to reset in at least 180 days.

Answer

$1/\lambda=120$ i.e. $\lambda=1/120$

X= amount of time that a watch will run without having to be reset

P(x<24)

$= \int_0^{24} \lambda e^{-\lambda x} dx$

$=\int_0^{24} (1/120) e^{-\lambda(1/120)} dx$

$=1-e^{-0.2}$

=0.1812

Example : Studies of a single-machine-tool system showed that the time , the machine operates before the breaking down is exponentially distributed with a mean 10 hrs

(i)Find the probability that the machine operates for

(a) at least 12 hrs before the breaking down

(b) at least 14 hrs but fails before20 hrs

(ii) If the machine has already been operating 8hrs find the probability that it will last another 4 hrs

(iii) If the machine has already been operating 6hrs find the probability that it will another 4 hrs

Answer

Let X be life in hours of a machine

X~ Exp (λ)

Mean =1/ λ=10 i.e. λ=1/10

X~ Exp (λ=0.1)

p.d.f.is f(x)=0.1 $e^{-0.1x}$, x ≥ 0

=0 ,o.w.

c.d.f is F(X=x)=p(X≤x)=1-$e^{-0.1x}$, 0 ≤ x

so , p(X ≥x)=$e^{-0.1x}$

(a)

p(the machine operates for at least 12 hrs before the breaking down)

= p(X ≥12)=$e^{-0.1(12)}$

=0.301194

F(X=x)=p(X≤x)=1-$e^{-0.1\text{x}}$, $0 \le x$
p(X ≥x)=$e^{-0.1\text{x}}$

(b)
P(the machine operates for at least 14 hrs but fails before20 hrs)
=p(14 ≤x ≤20)= F(20)-F(14)={1-$e^{-0.1(20)}$}-{1-$e^{-0.1(14)}$}
=0.11126
(ii)
If the machine has already been operating 8hrs the probability that it will last another 4 hrs
Using Memoryless property
Required probability is p(x>12/x>8)=p(x>4)
=$e^{-0.1(4)}$ =0.6703

F(X=x)=p(X≤x)=1-$e^{-0.1\text{x}}$, 0 ≤ x

p(X ≥x)=$e^{-0.1\text{x}}$

(iii)

If the machine has already been operating 6hrs the probability that it will last another 4 hrs

Using Memoryless property

Required probability is p(x>4+6/x>6)=p(x>4)

=$e^{-0.1(4)}$ =0.6703

Definition .

A random variable X is said to have a normal distribution if its probability density function is given by

f(x) = $\frac{1}{\sigma\sqrt{(2\pi)}}\, e^{\frac{-1}{2}((x-m)/\sigma)^2}$, $-\infty < x < \infty$,

where $-\infty < \mu < \infty$ and $0 < \sigma^2 < \infty$ are arbitrary parameters.

NOTE :

If X has a normal distribution with parameters m and σ then we write X~ $N(m,\sigma^2)$.

Definition .

A normal random variable is said to be standard normal, if its mean is zero and variance is one.

We denote a standard normal random variable X by X ~ N(0, 1) OR Z ~ N(0, 1) OR S.N.V. Z

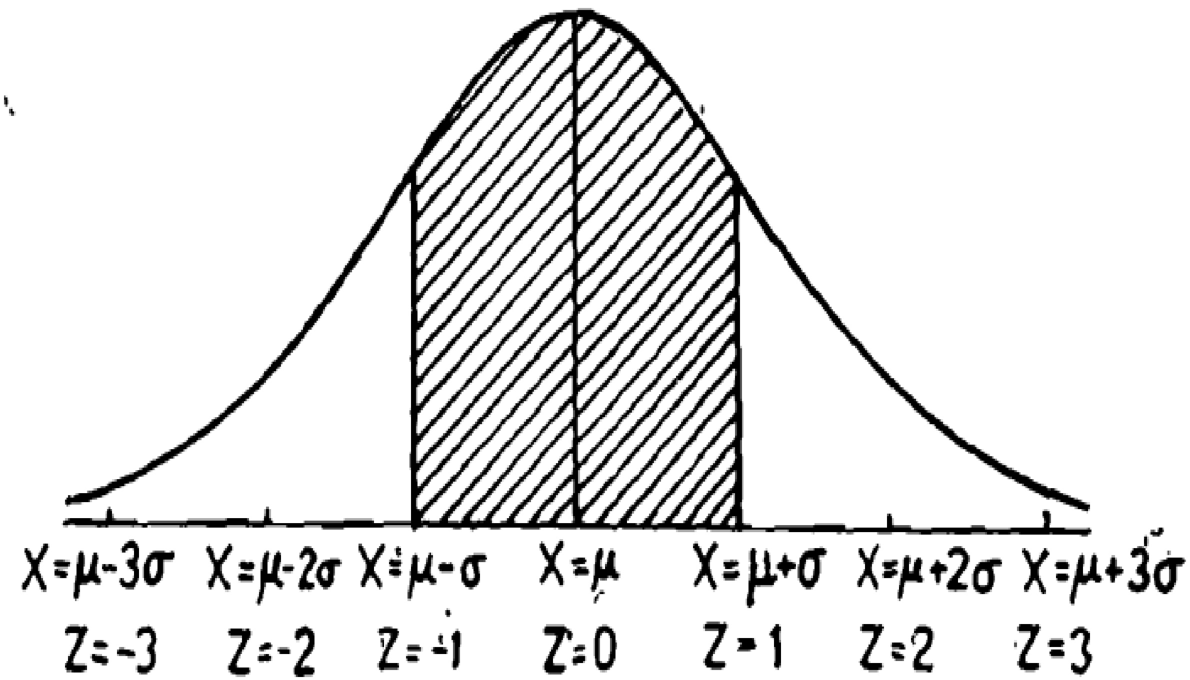
The probability density function of standard normal distribution is the following:

$$f(x) = \frac{1}{\sqrt{(2\pi)}} e^{\frac{-1}{2}(x)^2}$$

OR

$$f(z) = \frac{1}{\sqrt{(2\pi)}} e^{\frac{-1}{2}(z)^2}$$

Graph of normal distribution with parameters m and σ



Note :

(1) X is a discrete random variable

For any two integers a and b with a ≤ b,

P(a ≤ X ≤ b)= $\sum_{a \leq x \leq b} f(t)$

(2) X is a continuous random variable

For any two numbers a and b with a ≤ b,

P(a ≤ X ≤ b)= $\int_a^b f(t)dt$

(3) Z is standard normal random variable with

Mean=0 , S.D.=1 i.e. Z ~ N(0, 1).

P(a ≤ z ≤ b)= $\int_a^b f(z)dz$

=Area under curve f(z) above Z axis between line

Z=a & z=b

(4)

If X is a normal distribution with parameters m and σ i.e. X~ $N(m,\sigma^2)$.

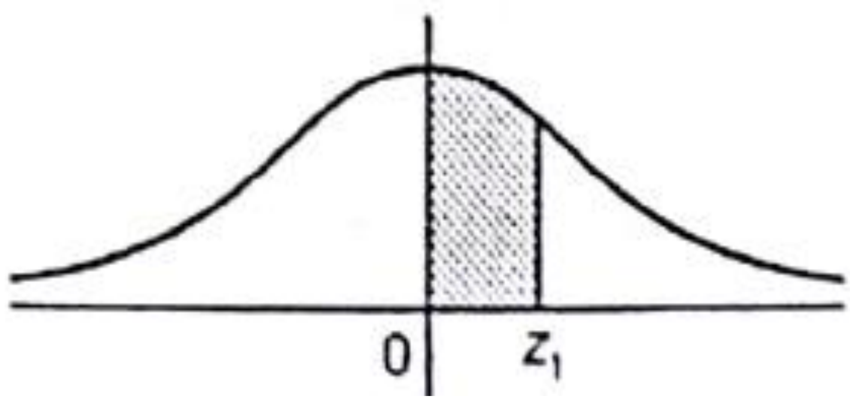
$P(m \leq x \leq x1)$

$=P(\frac{m-m}{\sigma} \leq \frac{x-m}{\sigma} \leq \frac{x1-m}{\sigma})$

$=P(0 \leq z \leq z1)$

$= \int_0^{z1} f(z)dz$

=Area under curve f(z) above Z axis between line Z=0 & z=z1



Example: Type I

If $Z \sim N(0, 1)$

Find $P(0<z < 0.95)$, $P(z > 0.95)$, $P(z <- 0.95)$,

$P(|Z| \leq 0.95)$

Answer

$P(0<z < 0.95)=0.3289$

$P(z < 0.95)= 0.5+0.3289$

$P(z > 0.95) = 0.5-0.3289$

$P(z <- 0.95)=0.5-0.3289$

$P(|Z| \leq 0.95)=2(0.3289)$

Example :Type II

what is the value of the constant c if

(i) $p(0<z<c)=0.2291$

$\therefore c=0.61$

(ii) $p(z<c)= 0.7291=0.5+0.2291$

$\therefore c=0.61$

(iii) $p(z<c)=0.2291$

$\therefore c= -0.74$

(iv) $p(z>c)=0.2291$

$\therefore c= 0.74$

Example Type III

If X ~ N(3, 16), then what is P(4 ≤ X ≤ 8)?

Ans P(4 ≤ X ≤ 8)

$= P(\frac{4-3}{4} \leq \frac{x-3}{4}X \leq \frac{8-3}{4})$

$=P(1/4 \leq z \leq 5/4)$

$=P(Z \leq 1.25) - P(Z \leq 0.25)$

$= 0.3944 - 0.0987$

$= 0.2957$

Example:Type IV

The marks obtained by students in a certain examination follow a normal distribution with a mean 70 and standard deviation 5. If 1000 students appeared at an examination. Calculate the number of students scoring more than 75 marks

R.V. X= marks obtained by students

σ =5 ,m=70

p(student scoring more than 75 marks)=p(x>75)

$=p(\frac{x-m}{\sigma}>\frac{75-70}{5})$

=p(z>1)

=0.5-0.3413=0.1587

The number of students scoring more than 75 marks = 1000(0.1587)=158.7=(159)

Example: The incomes of a group of 10,000 persons were found to be normally distributed with mean Rs.520 and S.D. Rs.60. Find i) the number of persons having incomes between Rs. 400 and Rs.550, ii) the lowest income of the richest 500.

S.N.V. $Z = \frac{x-m}{\sigma} = \frac{x-520}{60}$

When x=400 $z = \frac{400-520}{60} = -2$

When x=550 $z = \frac{550-520}{60} = 0.5$

$P(400 \le X \le 550) = P(-2 \le z \le 0.5)$

= area(from z=-2 to z=0)+ area(from z=0 to z=0.5)

= area(from z=0 to z=2)+ area(from z=0 to z=0.5)

=0.4772+0.1915 =0.6687

the number of persons having incomes between Rs. 400 and Rs.550 =Np=10000 * 0.6687=6687

If we have to consider the richest 500 persons then the probability that a person selected at random is 500/10000=0.05

This is reverse problem We have to find value of z for a given probability

We have to find value of z to the right of which the area is 0.05

area(from z=0 to z=this value)=0.5-0.05=0.45

The required value of z=1.645

But z = $\frac{x-520}{60}$ i.e. 1.645=$\frac{x-520}{60}$

x=520+1.645(60)=618.7 Rs

the lowest income of the richest 500 is 618.7 Rs

Example: Type V

The monthly salary of a company XYZ were found to be normally distributed with mean Rs.3000 and S.D. Rs.250., What should be the minimum salary of the worker in a company XYZ so that the probability that he belongs to top 5%

S.N.V. Z = $\frac{x-m}{\sigma} = \frac{x-3000}{250} = \frac{x-3000}{250}$

We have to find value of z_1 for a given probability 0.05

$p(z > z_1) = 0.05$

$p(0 < z < z_1) = 0.5 - 0.05 = 0.45$

$z_1 = 1.64$

$\frac{x-3000}{250} = 1.64$

x=3000+250 (1.64)= Rs 3410

Example:: Type VI

Q.In a distribution exactly normal 7% are under 35 & 89% are under 63. Assuming a normal distribution, find the mean & standard deviation of the distribution ?

Answer Since 7% are under 35 , 50-7= 43% items between 35 and m

Since 89% are under 63 , 89-50= 39% items between m and 63

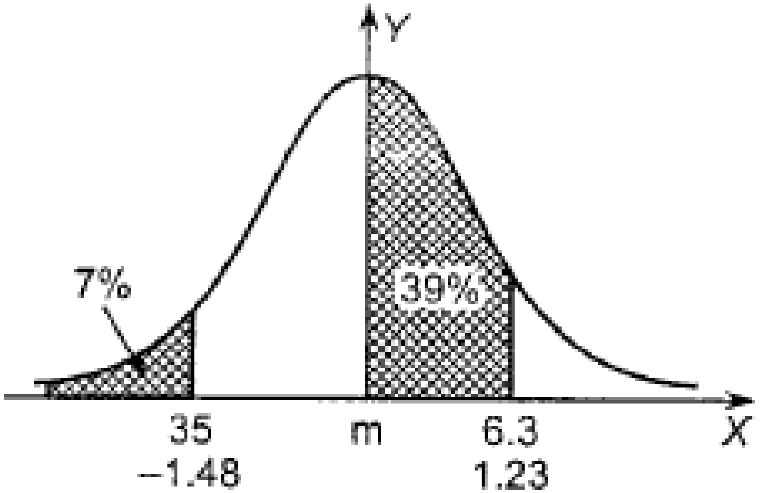
For area 0.43,z=1.48 as35<m

and for area 0.39 ,z=1.23

1.48=$\frac{35-m}{\sigma}$ and 1.23= $\frac{63-m}{\sigma}$

35-m=-1.48σ & 63-m=1.23 σ

σ=10.33 and m=50.3



NOTE

If X_1, X_2 are independent normal variates with mean m_1 & m_2 and Variances $\sigma_1{}^2$ & $\sigma_2{}^2$ and $Y=aX_1-bX_2$ then

Y is also normal variate with mean am_1-bm_2 and variance $a^2\sigma_1{}^2 + b^2\sigma_2{}^2$

Example : Type VII

If X_1, X_2 are independent normal variates with mean 30 &25 and variances16 &12 respectively and $Y=3X_1-2X_2$

Find $p(60<Y<80)$

Answer

X_1, X_2 are independent normal variates with mean 30 &25 and variances16 &12 $Y=3X_1-2X_2$

then Y is also normal variate with

Mean = am_1-bm_2 =3(30)-2(25)= 40 and

Variance = $a^2\sigma_1{}^2 + b^2\sigma_2{}^2 = (3)^2 (16)+(-2)^2 (12) = 192$

$p(60<Y<80)$

$= p\left(\frac{60-40}{\sqrt{192}} < \frac{Y-m}{\sigma} < \frac{80-40}{\sqrt{192}}\right)$

$=P(1.44< Z <2.89)$

$=0.4981-0.4251= 0.0730$

Exercise : In an examination marks obtained by students in Mathematics, Physics and Chemistry are normally distributed with means 51, 53 and 46 with standard deviation 15, 12, 16 respectively. Find the probability of securing total marks (i) 180 0r above, (ii) 90 or below