Moule1
Unit 1.1

BIVARIATE DISTRIBUTION

Discrete bivariate probability distribution

A two dimensional discrete random variable (X,Y) is said to have bivariate probability function or joint probability function p(x,y) if it gives probabilities for all the values of the discrete random variable (X ,Y) in its range .

If p(x,y,) is a joint prob. function it must satisfy the following conditions.

i) $p(x,y) \geq 0$ for all x,y

ii) $\Sigma\Sigma\, p(x,y) = 1$

Marginal probability function :

If we are given the joint probability function the marginal probability function of one variable can be obtained by summing the joint probability function over the range of the remaining random variable ,

Thus, the marginal probability function of the random variable X is given by

$$p(x) = \sum_{y} P(x, y)$$

and marginal probability, function of the random variable Y is given by

$$\mathrm{p}(y) = \sum_{x} P(x, y)$$

Conditional probability function

The conditional probability function of R.V. X for the given value of y of the R.V. Y is given by,

P(X/Y=y) $\frac{\mathrm{p(x,y)}}{p(y)}$; p(y) ≠ 0

The conditional probability function of R.V. Y for the given value of x of the R.V. X is given by,

P(Y/X=x) $\frac{\mathrm{p(x,y)}}{p(x)}$; p(x) ≠ 0

Independent random variables

Two discrete random variables X and Y are said to be independent if P(x,y) =P(x)P(y)

Example : Three balls are drawn at random from a box containing 2 white,3 red and 4 black balls .If X denotes the number of white balls drawn and Y denotes the number of red balls drawn

(I) find joint probability distribution

(II) $p(X \le 1)$, $p(X \le 1, Y \le 2)$, $p(Y \le 2 / X \le 1)$ $p(X+Y \le 2)$

(III) Marginal probability function of X , Marginal probability function of Y

(IV) Conditional probability function of X given Y=1

(V) Conditional probability function of Y given X=2

R.V. X = Number of White balls drawn=0, 1, 2

R.V. Y= Number of Red balls drawn=0, 1, 2, 3

To find P(0,0) ,P(0,1) ,P(0,2) ,P(1,0) ,P(1,1) & so on

Now P(X=1, Y=1) = p(1W,1R), P(X=0, Y=1)= p(0W,1R)

P(X=1, Y=1) = p(1W,1R)=p(1W,1R,1B)

=P (drawing 3balls in which 1 is White and 1 is Red, 1shoulde be Black)

$$=\frac{2_{c_1}\, 3_{c_1}\, 4_{c_1}}{9_{c_3}} = \frac{2}{7}$$

And P(X=0, Y=1)= p(0W,1R)= $\frac{2_{c_0}\, 3_{c_1}\, 4_{c_2}}{9_{c_3}} = \frac{3}{14}$

X	Y				Total
	0	1	2	3	
0	1/21	3/14	1/7	1/84	35/84
1	1/7	2/7	1/14	0	42/84
2	1/21	1/28	0	0	7/84
Total	5/21	15/28	3/14	1/84	1

p(X≤1)= p(X=0) + p(X=1) =(35/84)+(42/84) =77/84

p(X≤1,Y ≤2)=

$$\sum_{j=0}^{2} p\,(\mathbf{X} = \mathbf{0}, \mathbf{Y} = j) + \sum_{j=0}^{2} p\,(\mathbf{X} = \mathbf{1}, \mathbf{Y} = \mathbf{j})$$

$p(X\leq1) = p(X=0) + p(X=1) = (35/84) + (42/84) = 77/84$

$p(X\leq1, Y\leq2) =$

$$\sum_{j=0}^{2} p\,(X = 0, Y = j) + \sum_{j=0}^{2} p\,(X = 1, Y = j)$$

$$p(Y\leq2 / X\leq1) = \frac{p(X\leq1, Y\leq2)}{p(X\leq1)}$$

$p(X+Y\leq2) =$

$$\sum_{j=0}^{2} p\,(X = 0, Y = j) + \sum_{j=0}^{1} p\,(X = 1, Y = j) +$$

$p(X=2, Y=0)$

Marginal probability function of X is

R.V X	0	1	2	Total
marginal probability function of X	35/84	42/84	7/84	1

Marginal probability function of Y is

R.V Y	0	1	2	3	Total
marginal probability function of Y	5/21	15/28	3/14	1/84	1

Conditional probability function of X given Y=1

R.V X	0	1	2	Total
conditional probability function of X	$\frac{p(X=0,Y=1)}{p(Y=1)}$ =2/5	$\frac{p(X=1,Y=1)}{p(Y=1)}$ =8/15	$\frac{p(X=2,Y=1)}{p(Y=1)}$ =1/15	1

Conditional probability function of Y given X=2

R.V Y	0	1	2	3
conditional probability function of Y	$\frac{p(X=2,Y=0)}{p(X=2)}$ =7/5	$\frac{p(X=2,Y=1)}{p(X=2)}$ =3/7	$\frac{p(X=2,Y=2)}{p(X=2)}$ =0	$\frac{p(X=2,Y=3)}{p(X=2)}$ =0

Example : The joint probability distribution function of (X,Y) is given by

X	Y		
	1	2	3
0	3K	6K	9K
1	5K	8K	11K
2	7K	10K	13K

Find value of k , Find all the marginal and conditional probability distributions

Example *For the following bivariate probability distribution of X and Y, fina*

(i) $P(X \leq 1, Y = 2)$, (ii) $P(X \leq 1)$, (iii) $P(Y = 3)$, (iv) $P(Y \leq 3)$ and (v) $P(X < 3, Y \leq 4)$

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution. The marginal distributions are given below :

$X \backslash Y$	1	2	3	4	5	6	$p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$p_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	$\Sigma p(x) = 1$ $\Sigma p(y) = 1$

(i) $$P(X \le 1, Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 2)$$
$$= 0 + \frac{1}{16} = \frac{1}{16}$$

(ii) $$P(X \le 1) = P(X = 0) + P(X = 1)$$
$$= \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$$

(iii) $$P(Y = 3) = \frac{11}{64}$$ (From above table)

(iv) $$P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$$
$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$$

$$
\begin{aligned}
P(X < 3, Y \leq 4) &= P(X = 0, Y \leq 4) + P(X = 1, Y \leq 4) \\
&\quad + P(X = 2, Y \leq 4) \\
&= \left(\frac{1}{32} + \frac{2}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8}\right) \\
&\quad + \left(\frac{1}{32} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64}\right) = \frac{9}{16}
\end{aligned}
$$

Continuous bivariate probability distribution

A two dimensional continuous random variable (X,Y) is said to have bivariate probability function or joint probability function f(x,y) if it gives probabilities for all the values of the continuous random variable (X ,Y) in its range .

If f(x,y,) is a joint prob. function it must satisfy the following conditions.

i) $f(x,y) \geq 0$ for all x,y

ii) $\int_x \int_y f(x,y)dxdy = 1$

Marginal probability function

The marginal probability function of the r.v. X is given by

f(x) = $\int_{y} f(x,y)dx$

The marginal probability function of the r.v. Y is given by,

f(y) = $\int_{x} f(x,y)dy$

Conditional probability function

The conditional probability function of the r.v. X for the given values y of the r.v. Y is given by,

f(X/Y=y) $=f(x,y)/f(y)$;f(y)>0

The conditional probability function of the r.v. Y for the given values x of the r.v. X is given by,

f(Y/X=x) $=f(x,y)/f(x)$;f(x)>0

Two continuous r.vs. X,Y are said to be independent if f(x,y) =f(x)f(y)

Example : *The joint p.d.f. of two random variables X and Y is*

$$f(x,y) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4} \; ; \; \begin{pmatrix} 0 \le x < \infty \\ 0 < y < \infty \end{pmatrix}$$

(i) Find the marginal density functions of X and Y,

(ii) find the conditional density function of Y given X = x

Marginal p.d.f. of X is given by

$$f_X(x) = \int_0^\infty f(x,y)\, dy$$

$$f_X(x) = \int_0^{\infty} f(x, y)\, dy$$

$$= \frac{9}{2(1+x)^4} \int_0^{\infty} \frac{(1+y)+x}{(1+y)^4}\, dy$$

$$= \frac{9}{2(1+x)^4} \cdot \int_0^{\infty} \left[(1+y)^{-3} + x\,(1+y)^{-4} \right] dy$$

$$= \frac{9}{2(1+x)^4} \left[\left| \frac{-1}{2(1+y)^2} \right|_0^{\infty} + x \left| \frac{-1}{3(1+y)^3} \right|_0^{\infty} \right]$$

$$= \frac{9}{2(1+x)^4} \cdot \left[\frac{1}{2} + \frac{x}{3} \right]$$

$$= \frac{3}{4} \cdot \frac{3+2x}{(1+x)^4} \; ; \; 0 < x < \infty$$

Since $f(x, y)$ is symmetric in x and y,

the marginal p.d.f. of Y is given by

$$f_Y(y) = \int_0^\infty f(x, y)\, dx$$

$$= \frac{3}{4} \cdot \frac{3 + 2y}{(1 + y)^4} \; ; \quad 0 < y < \infty$$

The conditional distribution of Y for $X = x$

$$f_{XY}(Y = y \mid X = x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

$$= \frac{f_{XY}(x,y)}{f_X(x)}$$

$$= \frac{9(1+x+y)}{2(1+x)^4(1+y)^4} \cdot \frac{4(1+x)^4}{3(3+2x)}$$

$$= \frac{6(1+x+y)}{(1+y)^4(3+2x)} \; ; \quad 0<y<\infty$$

Example : *The joint p.d.f. of two random variables X and Y is*

$$f(x,y)= e^{-(x+y)} \quad ; \begin{pmatrix} 0\le x< \infty \\ 0< y< \infty \end{pmatrix}$$

$(i)\ P(X>1),\ (ii)\ P(X<Y \mid X<2Y),\ (iii)\ P(1<X+Y<2)$

Solution

$$f(x,y)= e^{-(x+y)} \quad ; \quad 0\le x<\infty,\ 0\le y<\infty$$

$$= \left(e^{-x}\right)\left(e^{-y}\right)$$

$$= f_X(x).f_Y(y)$$

X and Y are independent and

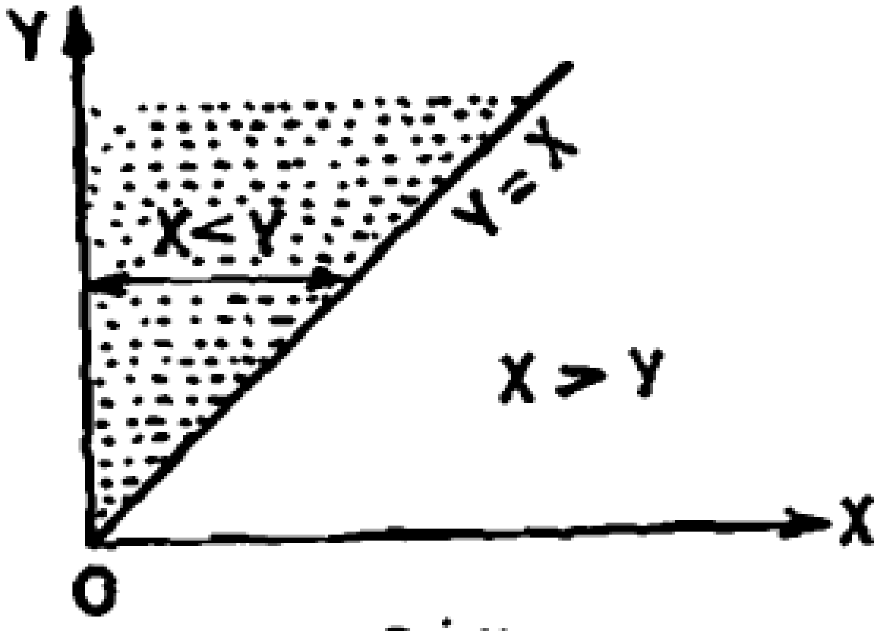
$$f_X(x)= e^{-x}\ ;\ x\ge 0 \quad \text{and} \quad f_Y(y)= e^{-y}\ ;\ y\ge 0$$

$$P(X>1)=\int_{1}^{\infty} f_X(x)\,dx=\int_{1}^{\infty} e^{-x}\,dx$$

$$=\left|\frac{e^{-x}}{-1}\right|_{1}^{\infty}=\frac{1}{e}$$

$$P(X<Y \mid X<2Y)=\frac{P(X<Y \cap X<2Y)}{P(X<2Y)}$$

$$=\frac{P(X<Y)}{P(X<2Y)}$$



Y
X<Y
Y=X
X > Y
X
O

$P(X < Y)$

$$= \int_0^\infty \left[\int_0^y f(x, y)\, dx \right] dy$$

$$= \int_0^\infty \left[e^{-y} \left| \frac{e^{-x}}{-1} \right|_0^y \right] dy = -\int_0^\infty e^{-y}\left(e^{-y} - 1\right) dy$$

$$= -\left| \frac{e^{-2y}}{-2} + e^{-y} \right|_0^\infty = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(X < 2Y)$$

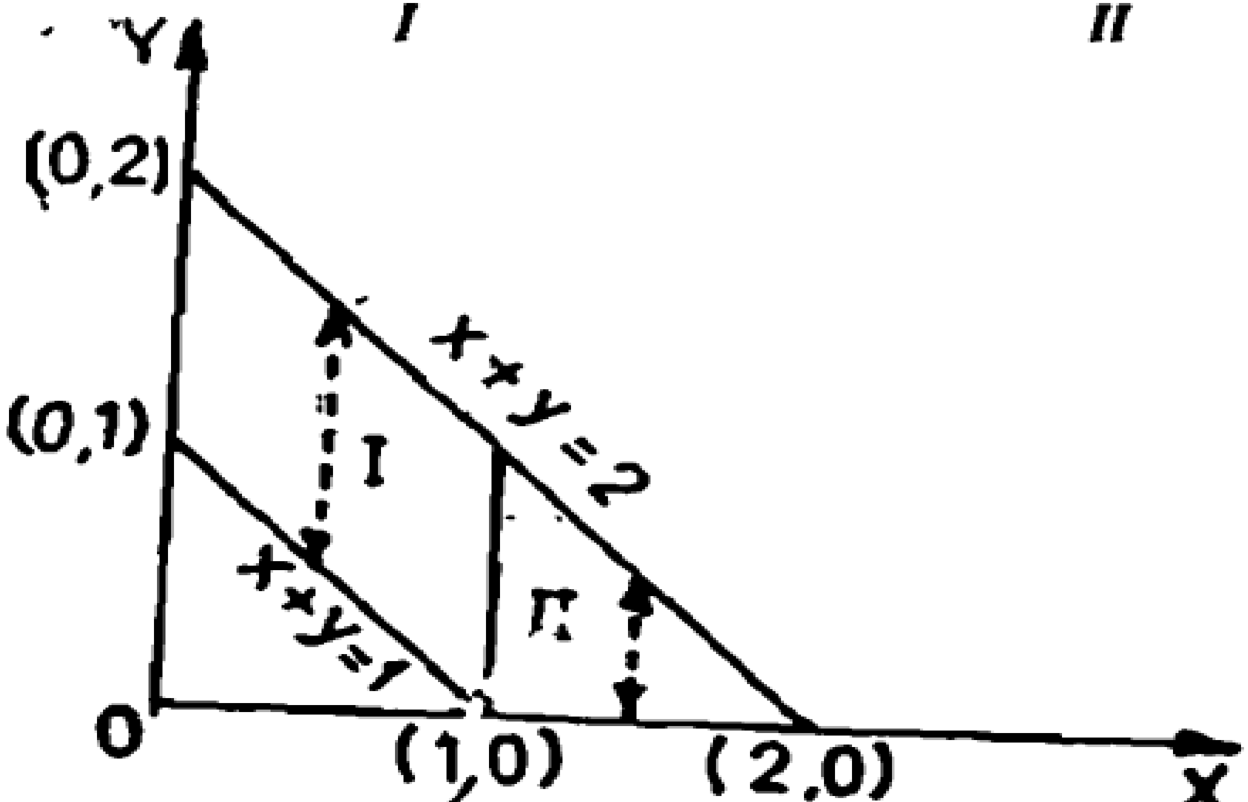
$$= \int_0^\infty \left[\int_0^{2y} f(x, y)\, dx \right] dy = -\int_0^\infty e^{-y}(e^{-2y} - 1)\, dy$$

$$= -\left| \frac{e^{-3y}}{-3} + e^{-y} \right|_0^\infty = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(X < Y \mid X < 2Y) = \frac{1/2}{2/3} = \frac{3}{4}$$

$$P(1 < X + Y < 2) = \iint\limits_{I} f(x, y)\, dx\, dy = \iint\limits_{II} f(x, y)\, dx\, dy$$

I II



$$= \int_0^1 \left(\int_{1-x}^{2-x} f(x,y)\,dy \right) dx + \int_1^2 \left(\int_0^{2-x} f(x,y)\,dy \right) dx$$

$$= \int_0^1 \left(e^{-x} \int_{1-x}^{2-x} e^{-y}\,dy \right) dx + \int_1^2 \left(e^{-x} \int_0^{2-x} e^{-y}\,dy \right) dx$$

$$= \int_0^1 \frac{e^{-x}}{-1} \left(e^{x-2} - e^{x-1} \right) dx + \int_1^2 \frac{e^{-x}}{-1} \left(e^{x-2} - 1 \right) dx$$

$$= -\left(e^{-2} - e^{-1} \right) \int_0^1 1 \cdot dx - \int_1^2 \left(e^{-2} - e^{-x} \right) dx$$

$$= -\left(e^{-2} - e^{-1} \right) \left| x \right|_0^1 - \left| e^{-2} \cdot x + e^{-x} \right|_1^2$$

$$= 2/e - 3/e^2$$

Exercise :

The joint probability distribution function of (X,Y) is given by $f(x, y) = kxye^{-(x^2+y^2)}$ x>0 ,y>0

Find value of k & prove that X & Y are independent