# Non Linear Programming

NLPP

# Introduction

An optimization problem in which either the object function or some or all constraints are non linear is called non linear programming problem.

# Examples for NLPP

(i) Optimise $z = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 3x_2x_3 + 100$

subject to $x_1 + x_2 + x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Here, the object function is non-linear

(ii) Optimise, $z = 3x_1 + 4x_2 + 7x_3$

subject to $x_1^2 + 2x_2^2 \geq 20$

$3x_1 + x_2 + 2x_3 \leq 10$

$x_1, x_2, x_3 \geq 0.$

Here, although the object function is linear, one of the constraints is non-linear.

(iii) Optimise, $z = x_1^2 + x_2^2 - 3x_1 x_2$

subject to $x_1^2 + x_1x_2 + 2x_3 = 35$

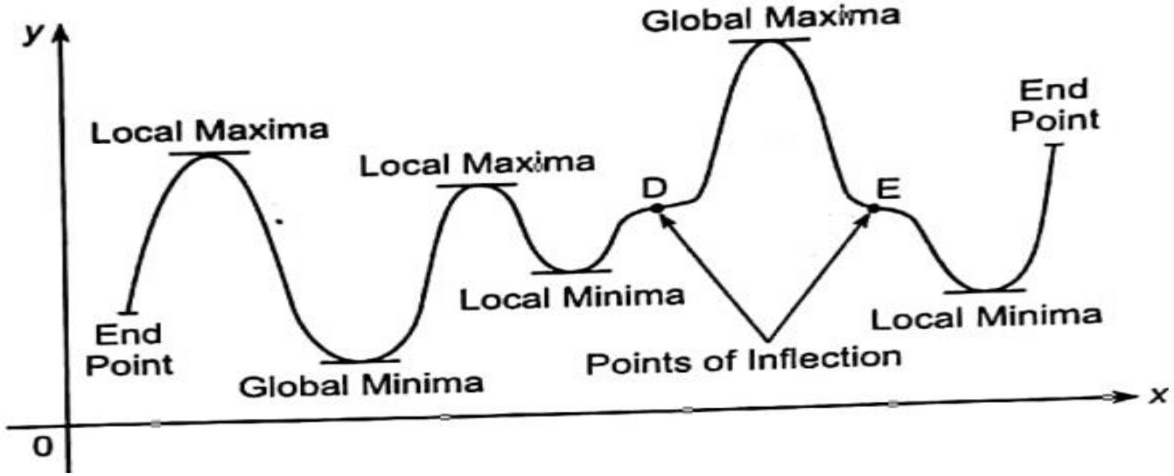
$x_1, x_2 \geq 0$

Here, both the object function as well as the constraints are non-linear.

# Maxima and Minima for one variable

If $y = f(x)$ is differentiable function of real variable $x$ upto second order then $f'(x) = 0$ gives the stationary values of $x$. Suppose $f'(x) = 0$, for $x = x_0$.

Now, $x_0$ is a stationary point, and if $f''(x_0) > 0$ then $x_0$ is a **minima**, if $f''(x_0) < 0$ then $x_0$ is a **maxima**, if $f''(x_0) = 0$, no conclusion can be drawn about the nature of $f(x)$ at $x_0$. It is called the point of **inflexion**.



# Maxima and Minima for two variables

If $z = f(x, y)$ is a function of two variables $x\ and\ y$ such that the second order partial derivatives exist then the stationary points are given by $f_x(x, y) = 0$ and $f_y(x, y) = 0$

(i) If $f_{xx}f_{yy} - {f_{xy}}^2 > 0$, then $X_O(x_0, y_0)$ is stationary point and

$X_O(x_0, y_0)$ is maxima if $f_{xx} < 0$ at $(x_0, y_0)$

$X_O(x_0, y_0)$ is minima if $f_{xx} > 0$ at $(x_0, y_0)$

$(ii)\ If f_{xx}f_{yy} - {f_{xy}}^2 < 0$ , then $X_O(x_0, y_0)$ is saddle point

(iii) $If f_{xx}f_{yy} - {f_{xy}}^2 = 0$ , then no definite conclusion can be drawn about the nature of $f(x, y)$at $(x_0, y_0)$

## Types of Non linear programming problems

1. Quadratic programming problems with no constraints (Using Hassian matrix)

2. Quadratic programming problems  of 2 or 3 variables with 1 linear equality constraints using Lagrange's Multiplier method.

3. . Quadratic programming problems of 2 or 3 variables with 1 linear inequality constraints using Kuhn-Tucker conditions

In Quadratic programming problems object function looks like

$$z = a_{11}{x_1}^2 + a_{22}{x_2}^2 + \cdots + a_{nn}{x_n}^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \cdots + c_1x_1 + \cdots + c_nx_n$$

where $x_1, \ldots x_n$ are decision variables

# 1.Quadratic programming problems with no constraints (Using Hassian matrix)

Hassian matrix

If $f(x_1, \dots x_n)$ is a quadratic function of variables $x_1, \dots x_n$ and if all 2nd order partial derivatives of $f(x_1, \dots x_n)$ exist, then the matrix of 2nd order partial derivatives defined as given below is called Hassian matrix (H).

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

1. If all the principal minor determinants of Hassian Matrix at $X_O$ are positive, $X_O$ is minima.
2. If the principal minor determinants $D_1, D_3, ..$ are negative and $D_2, D_4, ..$ are positive , $X_O$ is maxima.
3. If H is indefinite at $X_O$, $X_O$ is a saddle point ie. no maxima no minima.

Ex 1- Find the maximum or minimum of the function

$$z = {x_1}^2 + {x_2}^2 + {x_3}^2 - 4x_1 - 8x_2 - 12x_3 + 100$$

We have $f(x_1, x_2, x_3) = {x_1}^2 + {x_2}^2 + {x_3}^2 - 4x_1 - 8x_2 - 12x_3 + 100$

The stationary points are given by

$$\frac{\partial f}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} = 0, \frac{\partial f}{\partial x_3} = 0 \Rightarrow 2x_1 - 4 = 0; 2x_2 - 8 = 0; 2x_3 - 12 = 0$$

Thus $X_O = (2,4,6)$ is stationary point.

Now to check maxima or minima, find values of all 2nd order partial derivatives of $f(x_1, x_2, x_3)$ at the point $X_O = (2,4,6)$ and Hassian Matrix

•

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

the principal minor determinants are

$D_1 = |2| = 2$, $D_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$ =4, $D_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$ =8,

Since all the principal minor determinants of Hassian Matrix at $X_O$=(2,4,6) are positive, $X_O$ =(2,4,6) is minima.

$$z_{min} = {x_1}^2 + {x_2}^2 + {x_3}^2 - 4x_1 - 8x_2 - 12x_3 + 100$$

$$= 2^2 + 4^2 + 6^2 - 4 * 2 - 8 * 4 - 12 * 6 + 100 = 44$$

Ex 2- Find the maximum or minimum of the function

$$z = x_1 + 2x_3 + x_2x_3 - x_1{}^2 - x_2{}^2 - x_3{}^2$$

$X_O$=(1/2,2/3,4/3) $H = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

$D_1 = -2$, $D_2$ =4, $D_3 = -6$,

Since the principal minor determinants $D_1, D_2, D_3$ are alternately negative and positive , $X_O$ is maxima.

$z_{max} = 19/12$

# EXERCISE

1. Find the relative maximum or minimum of the function

   $z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 10x_2 - 14x_3 + 103$

   [ **Ans.** : $x_1 = 3$, $x_2 = 5$, $x_3 = 7$. $z_{Min} = 20$ ]

2. Find the relative maximum or minimum of the function

   $z = x_1^2 + x_2^2 + x_3^2 - 8x_1 - 10x_2 - 12x_3 + 100$ [ **Ans.** : $x_1 = 4$, $x_2 = 5$, $x_3 = 6$. $z_{Max} = 23$ ]

# 2.Quadratic programming problems of 2 or 3 variables with 1 linear equality constraints (using Lagrange's Multiplier method).

A general non-linear programming problem in which the object function is non-linear but the constraints are linear and in the form of equalities, takes the following form.

Optimise $z = f(x_1, x_2, \ldots\ldots, x_n)$

subject to
$$g_1(x_1, x_2, \ldots\ldots, x_n) = b_1,$$
$$g_2(x_1, x_2, \ldots\ldots, x_n) = b_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$g_n(x_1, x_2, \ldots\ldots, x_n) = b_m.$$
$$x_1, x_2, \ldots\ldots, x_n \geq 0$$

where, $f(x_1, x_2, \ldots\ldots, x_n)$ is a non-linear function and $g_1(x_1, x_2, \ldots\ldots, x_n)$, $g_2(x_1, x_2, \ldots\ldots, x_n)$ ..... $g_n(x_1, x_2, \ldots\ldots, x_n)$ are linear functions and $m < n$.

The problem of this type is solved by forming what is called **Lagrangian Function with Lagrange's multiplier** $\lambda$.

We first express the constraints with r.h.s. equal to zero and denote it by $h$.

The problem then becomes

Optimize $\quad z = f(x_1, x_2, \ldots\ldots, x_n)$

subject to $\quad h(x_1, x_2, \ldots\ldots, x_n) = g(x_1, x_2, \ldots\ldots, x_n) - b = 0$

$\quad x_1, x_2, \ldots\ldots, x_n \geq 0.$

Now, we construct a new function called **Lagrangian Function** using the multiplier called the **Lagrangian multiplier** as

$$L(x_1, x_2, \ldots\ldots, x_n, \lambda) \equiv f(x_1, x_2, \ldots\ldots, x_n) - \lambda\, h(x_1, x_2, \ldots\ldots, x_n) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The necessary conditions for maxima or minima subject to the constraint $h(x_1, x_2, \ldots\ldots, x_n) = 0$ are

$$\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0, \ldots\ldots \frac{\partial L}{\partial x_n} = 0,\ \frac{\partial L}{\partial \lambda} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now, from (1), we get,

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1}\ ;\ \frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} \ldots\ldots$$

$$\frac{\partial L}{\partial x_n} = \frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n}\ ;\ \frac{\partial L}{\partial \lambda} = -\lambda\, h$$

Using (2), we get the following $(n + 1)$ necessary conditions.

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0\ ;\ \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0 \ldots\ldots$$

$$\frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n} = 0\ ;\ h(x_1, x_2, \ldots\ldots, x_n) = 0$$

Solving these $(n + 1)$ equations we can find $x_1, x_2 \ldots\ldots x_n$ and $\lambda$. Thus, the point of maxima or minima can be obtained.

To determine whether the point obtained above is a maxima or a minima, we find the value of the following determinant of order $(n+1)$ at $\boldsymbol{X_0}$.

$$\Delta_{n+1} = \begin{vmatrix} 0 & \dfrac{\partial h}{\partial x_1} & \dfrac{\partial h}{\partial x_2} & \cdots & \dfrac{\partial h}{\partial x_n} \\ \dfrac{\partial h}{\partial x_1} & \dfrac{\partial^2 f}{\partial x_1^2} - \lambda \dfrac{\partial^2 h}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \dfrac{\partial^2 h}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda \dfrac{\partial^2 h}{\partial x_1 \partial x_n} \\ \dfrac{\partial h}{\partial x_2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \dfrac{\partial^2 h}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} - \lambda \dfrac{\partial^2 h}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} - \lambda \dfrac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial h}{\partial x_n} & \dfrac{\partial^2 f}{\partial x_n \partial x_1} - \lambda \dfrac{\partial^2 h}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} - \lambda \dfrac{\partial^2 h}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} - \lambda \dfrac{\partial^2 h}{\partial x_n^2} \end{vmatrix}$$

If the signs of the principal minors $\Delta_3, \Delta_4, \Delta_5, \ldots$ are **alternately positive and negative** *i.e.* $\Delta_3 > 0, \Delta_4 < 0, \Delta_5 > 0 \ldots$ etc. then the points $\boldsymbol{X_0}$ **is a maxima.**

If all the principal minors $\Delta_3, \Delta_4, \Delta_5, \ldots \Delta_{n+1}$ are **negative** *i.e.* $\Delta_3 < 0, \Delta_4 < 0, \ldots$ etc. then the point $\boldsymbol{X_0}$ **is a minima.**

## Note

- If z is function of 2 variables, we get only one determinant of third order $\Delta_3$.

  If $\Delta_3$ is positive, $X_O$ is maxima.

  If $\Delta_3$ is negative, $X_O$ is minima.

$$\Delta_3 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} \end{vmatrix}$$

- If z is function of 3 variables, we get determinant of fourth order $\Delta_4$.

  If $\Delta_3$ and $\Delta_4$ both are negative, $X_O$ is minima.

  If $\Delta_3$ is positive but $\Delta_4$ is negative, $X_O$ is maxima.

$$\Delta_4 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} - \lambda \frac{\partial^2 h}{\partial x_3^2} \end{vmatrix}$$

## NLPP with 2 variables with 1 linear equality constraints

**Ex1- Using method of Lagranges Multipliers, solve the NLPP**

Optimize $z = 4x_1 + 8x_2 - {x_1}^2 - {x_2}^2$ subject to $x_1 + x_2 = 4$, $x_1, x_2 \geq 0$

**Sol. :** We have the Lagrangian function

$$L(x_1, x_2, \lambda) = (4x_1 + 8x_2 - x_1^2 - x_2^2) - \lambda(x_1 + x_2 - 4)$$

We, now, obtain the following partial derivatives

$$\frac{\partial L}{\partial x_1} = 4 - 2x_1 - \lambda, \quad \frac{\partial L}{\partial x_2} = 8 - 2x_2 - \lambda, \quad \frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 4)$$

Solving the equations $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial \lambda} = 0$, we get

$$\therefore \quad 4 - 2x_1 - \lambda = 0, \quad 8 - 2x_2 - \lambda = 0, \quad x_1 + x_2 = 4$$

Adding the first two, we get

$$12 - 2(x_1 + x_2) - 2\lambda = 0 \qquad \therefore \quad 12 - 8 = 2\lambda \qquad \therefore \quad \lambda = 2.$$

Hence, from the first equation, we get

$$4 - 2x_1 - 2 = 0 \qquad \therefore \quad 2x_1 = 2 \qquad \therefore \quad x_1 = 1$$

And from the second equation, we get

$$8 - 2x_2 - 2 = 0 \qquad \therefore \quad 2x_2 = 6 \qquad \therefore \quad x_2 = 3.$$

Hence, $X_0$ is $(1, 3)$.

Now, $\quad h(x_1, x_2) = x_1 + x_2 - 4 = 0$

$\therefore \quad \frac{\partial h}{\partial x_1} = 1$, $\frac{\partial h}{\partial x_2} = 1$ and all other partial derivatives are zero.

And $\quad f(x_1, x_2) = 4x_1 + 8x_2 - x_1^2 - x_2^2$

$$\therefore \quad \frac{\partial f}{\partial x_1} = 4 - 2x_1, \quad \frac{\partial^2 f}{\partial x_1 x_2} = 0, \quad \frac{\partial^2 f}{\partial x_1^2} = -2,$$

$$\frac{\partial f}{\partial x_2} = 8 - 2x_2, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0, \quad \frac{\partial^2 f}{\partial x_2^2} = -2.$$

.

$$\Delta_3 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = -1 \begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} = 2 + 2 = 4$$

Since $\Delta_3$ is positive, $X_O$ is maxima.

$x_1 = 1, \qquad x_2 = 3, \qquad z_{max} = 18$

# EXERCISE

1. Optimise $z = 3x_1^2 + x_2^2 + 2x_1x_2 + 6x_1 + 2x_2$

   subject to $2x_1 + x_2 = 4$

   $x_1, x_2 \geq 0.$

   [ **Ans. :** $x_1 = 1, x_2 = 2, z_{Min} = 28$ ]

2. Optimise $z = 2x_1 + 6x_2 - x_1^2 - x_2^2 + 14$

   subject to $x_1 + x_2 = 4$

   $x_1, x_2 \geq 0.$

   [ **Ans. :** $x_1 = 1, x_2 = 3, z_{Max} = 24$ ]

## NLPP with 3 variables with 1 linear equality constraints

Ex 1- Using method of Lagranges Multipliers, solve the NLPP

Optimize $z = 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 260$ subject to

$x_1 + x_2 + x_3 = 11,\ x_1, x_2, x_3 \geq 0$

**Sol. :** We have the Lagrangian function

$$L(x_1, x_2, x_3, \lambda) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 260 - \lambda(x_1 + x_2 + x_3 - 11)$$

We, now, obtain the following partial derivatives.

$$\frac{\partial L}{\partial x_1} = 4x_1 - 24 - \lambda, \quad \frac{\partial L}{\partial x_2} = 4x_2 - 8 - \lambda,$$

$$\frac{\partial L}{\partial x_3} = 4x_3 - 12 - \lambda, \quad \frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 11$$

Solving the equations, $\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0,\ \frac{\partial L}{\partial x_3} = 0,\ \frac{\partial L}{\partial \lambda} = 0$, we get

$$\therefore \quad 4x_1 - 24 - \lambda = 0,\ 4x_2 - 8 - \lambda = 0,$$

$$4x_3 - 12 - \lambda = 0,\ x_1 + x_2 + x_3 - 11 = 0$$

Adding the first three equations, we get,

$$4(x_1 + x_2 + x_3) - 44 - 3\lambda = 0. \qquad \text{But } x_1 + x_2 + x_3 = 11$$

$$\therefore \quad 44 - 44 - 3\lambda = 0 \qquad \therefore \quad \lambda = 0.$$

Hence, $4x_1 = 24 \quad \therefore \quad x_1 = 6;\ 4x_2 = 8 \quad \therefore \quad x_2 = 2.$

$\therefore \quad 4x_4 = 12 \quad \therefore \quad x_3 = 3; \quad \therefore \quad X_0$ is $(6, 2, 3)$.

$$\Delta_4 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} - \lambda \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} - \lambda \frac{\partial^2 h}{\partial x_3^2} \end{vmatrix}$$

where $f \equiv 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 260$ and $h \equiv x_1 + x_2 + x_3 - 11.$

Now, $\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix}$

By $C_2 - C_4$, $C_3 - C_4$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 4 & -4 & -4 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 4 \\ 1 & -4 & -4 \end{vmatrix} = -[(16) - 4(-8)] = -48$$

Now, taking the first three rows and the first three columns from $\Delta_4$,

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} = -1(4) + 1(-4) = -8$$

Since, $\Delta_3$, $\Delta_4$ are both negative, $X_0 = (6, 2, 3)$ is the minima.

$\therefore \quad x_1 = 6,\ x_2 = 2,\ x_3 = 3$

and $\quad z_{Min} = 2(36) + 2(4) + 2(9) - 1244 - 16 - 36 + 260 = 162.$

## EXERCISE

Using method of Lagranges Multipliers, solve the following NLPP

1. Optimize $z = 12x_1 + 8x_2 + 6x_3 - {x_1}^2 - {x_2}^2 - {x_3}^2 - 23$ subject to $x_1 + x_2 + x_3 = 10$, $x_{1,} x_2, x_3 \geq 0$

Ans $x_1 = 5, x_2 = 3, x_3 = 2, z_{max} = 35$

2. Optimize $z = 3{x_1}^2 + {x_2}^2 + {x_3}^2$ subject to $x_1 + x_2 + x_3 = 2$, $x_{1,} x_2, x_3 \geq 0$

Ans $x_1 = 0.81, x_2 = 0.35, x_3 = 0.28, z_{min} = 0.84$

# 3.Quadratic programming problems of 2 or 3 variables with 1 linear inequality constraints (using Kuhn-Tucker conditions)

Consider the problem $z = f(x_1, \dots x_n)$

Subject to $\mathrm{g}(x_1, \dots x_n) \leq b$

$$x_1, \dots x_n \geq 0$$

Let $\mathrm{h}(x_1, \dots x_n) = \mathrm{g}(x_1, \dots x_n) - b \leq 0$

The necessary conditions for maxima are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} \lambda \frac{\partial h}{\partial x_2} = 0, \dots \frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n} = 0, \lambda \mathrm{h}(x_1, \dots x_n) = 0$$

$$\mathrm{h}(x_1, \dots x_n) \leq 0 \quad \lambda \geq 0$$

These conditions are called Kuhn-Tucker conditions.

If the problem is of minimisation, then the last condition changes to $\lambda < 0$

Ex 1- Solve the following NLPP
Maximise $z = 10x_1 + 4x_2 - 2{x_1}^2 - {x_2}^2$
subject to $2x_1 + x_2 \leq 5, x_1, x_2 \geq 0$

We rewrite the given problem as
$f(x_1, x_2) = 10x_1 + 4x_2 - 2{x_1}^2 - {x_2}^2$
$h(x_1, x_2) = 2x_1 + x_2 - 5$

Now Kuhn-Tucker conditions are
$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} \lambda \frac{\partial h}{\partial x_2} = 0\ ;\ \lambda \mathrm{h}(x_1, x_2) = 0\ ; \mathrm{h}(x_1, x_2) \leq 0\ ;\ \ \lambda \geq 0$
We get
$10 - 4x_1 - 2\lambda = 0 \dots \dots \dots (i)$
$4 - 2x_2 - \lambda = 0 \dots \dots \dots \dots . (ii)$
$\lambda(2x_1 + x_2 - 5) = 0 \dots \dots . (iii)$
$2x_1 + x_2 - 5 \leq 0 \dots \dots \dots \dots (iv)$
$x_1, x_2, \lambda \geq 0 \dots \dots \dots \dots \dots \dots . (v)$
From $(iii)$, we get either $\lambda = 0 \quad or \quad (2x_1 + x_2 - 5) = 0$

Case 1- If $\lambda = 0$

From $(i)$ $and$ $(ii)$, we get $x_1 = \frac{5}{2}$ $and$ $x_2 = 2$

But these values do not satisfy equation $(iv)$ so we reject these values.

Case 2- If $\lambda \neq 0$ so $2x_1 + x_2 - 5 = 0 \dots \dots \dots \dots (vi)$

From $(i), (ii)$ $and$ $(vi)$ , we get $x_1 = \frac{11}{6}$ $and$ $x_2 = \frac{4}{3}, \lambda = 4/3$

These values satisfy all necessary conditions. The optimal solution is

$x_1 = \frac{11}{6}$ $and$ $x_2 = \frac{4}{3}$ and $z_{max} = 10 * \frac{11}{6} + 4 * \frac{4}{3} - 2\left(\frac{11}{6}\right)^2 - \left(\frac{4}{3}\right)^2 = \frac{91}{6}$

## Ex 2- Solve the following NLPP
## Maximise $z = 2{x_1}^2 - 7{x_2}^2 + 12x_1x_2$
## subject to $2x_1 + 5x_2 \le 98$, $x_1, x_2 \ge 0$

We rewrite the given problem as

$f(x_1, x_2) = 2{x_1}^2 - 7{x_2}^2 + 12x_1x_2$

$h(x_1, x_2) = 2x_1 + 5x_2 - 98$

Now Kuhn-Tucker conditions are

$$\frac{\partial f}{\partial x_1} - \lambda\frac{\partial h}{\partial x_1} = 0, \frac{\partial f}{\partial x_2}\lambda\frac{\partial h}{\partial x_2} = 0\ ;\ \lambda \mathrm{h}(x_1, x_2) = 0\ ; \mathrm{h}(x_1, x_2) \le 0\ ;\ \ \lambda \ge 0$$

We get

$4x_1 + 12x_2 - 2\lambda = 0 \ldots \ldots .. \ldots \ldots \ldots (i)$

$-14x_2 + 12x_1 - 5\lambda = 0 \ldots \ldots \ldots \ldots .(ii)$

$\lambda(2x_1 + 5x_2 - 98) = 0 \ldots \ldots .(iii)$

$2x_1 + x_2 - 98 \le 0 \ldots \ldots \ldots \ldots (iv)$

$x_1, x_2, \lambda \ge 0 \ldots \ldots \ldots \ldots \ldots \ldots .(v)$

From $(iii)$, we get either $\lambda = 0\ \ or\ \ (2x_1 + 5x_2 - 98) = 0$

.

Case 1- If $\lambda = 0$

From $(i)$ $and$ $(ii)$, we get $x_1 = 0$ $and$ $x_2 = 0$

But these values give $z = 0$ which is not feasible solution so we reject these values.

Case 2- If $\lambda \neq 0$ so $2x_1 + 5x_2 - 98 = 0 \dots \dots \dots \dots (vi)$

From $(i), (ii)$ $and$ $(vi)$ , we get $x_1 = 44$ $and$ $x_2 = 2, \lambda = 100$

These values satisfy all necessary conditions. The optimal solution is

$x_1 = 44$ $and$ $x_2$ =2 and $z_{max} = 4900$

## Ex 3- Solve the following NLPP

Minimise $z = {x_1}^3 - 4x_1 - 2x_2$

subject to $x_1 + x_2 \le 1$, $x_1, x_2 \ge 0$

We rewrite the given problem as

$f(x_1, x_2) = {x_1}^3 - 4x_1 - 2x_2$

$h(x_1, x_2) = x_1 + x_2 - 1$

Now Kuhn-Tucker conditions are

$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} \lambda \frac{\partial h}{\partial x_2} = 0\ ;\ \lambda \mathrm{h}(x_1, x_2) = 0\ ;\ \mathrm{h}(x_1, x_2) \le 0\ ;\ \ \lambda \le 0$

We get

$3{x_1}^2 - 4 - \lambda = 0 \ldots \ldots \ldots \ldots (i)$

$-2 - \lambda = 0 \ldots \ldots \ldots \ldots \ldots \ldots . (ii)$

$\lambda(x_1 + x_2 - 1) = 0 \ldots \ldots \ldots . (iii)$

$x_1 + x_2 - 1 \le 0 \ldots \ldots \ldots \ldots (iv)$

$x_1, x_2 \ge 0, \lambda \le 0 \ldots \ldots \ldots \ldots . (v)$

From $(ii)$, we get $\lambda = -2\ \ and\ as\ \lambda \ne 0\ \ from\ (iii), we\ get\ \ x_1 + x_2 - 1 \ldots \ldots . (vi)$

.

From $(i)$ $and$ $(vi)$ , we get $x_1 = \sqrt{\frac{2}{3}}$ $and$ $x_2 = 1 - \sqrt{\frac{2}{3}}$

These values satisfy all necessary conditions. The optimal solution is

$x_1 = \sqrt{\frac{2}{3}}$ $and$ $x_2 = 1 - \sqrt{\frac{2}{3}}$ and $z_{min} = -3.093$

## EXERCISE

Using the Kuhn-Tucker conditions, solve the following N.L.P.P.

1. Maximise $z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

   subject to $3x_1 + 2x_2 \leq 6$

   $x_1, x_2 \geq 0$

   [ **Ans. :** $x_1 = 17/13$, $x_2 = 46/13$, $z_{Max} = 411/13$ ]

2. Maximise $z = 2x_1^2 - 7x_2^2 - 16x_1 + 2x_2 + 12x_1x_2 + 7$

   subject to $2x_1 + 5x_2 \leq 105$

   $x_1, x_2 \geq 0$

   [ **Ans. :** $x_1 = 45$, $x_2 = 3$, $z_{Max} = 4900$ ]

3. Minimise $z = x_1^2 - 2x_1 - x_2$

   subject to $2x_1 + 3x_2 \leq 6$

   $x_1, x_2 \geq 0.$

## Ex 4- Solve the following NLPP
## Maximise $z = -{x_1}^2 - {x_2}^2 - {x_3}^2 + 4x_1 + 6x_2$
## subject to $x_1 + x_2 \leq 2,\ x_1, x_2, x_3 \geq 0$

We rewrite the given problem as

$f(x_1, x_{2,}, x_3) = -{x_1}^2 - {x_2}^2 - {x_3}^2 + 4x_1 + 6x_2$

$h(x_1, x_2, x_3) = x_1 + x_2 - 2$

Now Kuhn-Tucker conditions are

$$\frac{\partial f}{\partial x_1} - \lambda\frac{\partial h}{\partial x_1} = 0, \frac{\partial f}{\partial x_2}\lambda\frac{\partial h}{\partial x_2} = 0\ ; \frac{\partial f}{\partial x_2}\lambda\frac{\partial h}{\partial x_2} = 0\ ;\ \lambda \mathrm{h}(x_1, x_2, x_3) = 0\ ; \mathrm{h}(x_1, x_2, x_3) \leq 0\ ;$$
$$\lambda \geq 0$$

We get

$-2x_1 + 4 - \lambda = 0 \ldots\ldots\ldots (i)$

$-2x_2 + 6 - \lambda = 0 \ldots\ldots\ldots\ldots. (ii)$

$x_3 = \quad 0 \qquad\qquad \ldots\ldots\ldots (iii)$

$\lambda\ (x_1 + x_2 - 2) = 0 \ldots\ldots. (iv)$

$x_1 + x_2 - 2 \leq 0 \ldots\ldots\ldots\ldots (v)$

$x_1, x_2, \lambda \geq 0 \ldots\ldots\ldots\ldots\ldots\ldots. (vi)$

From $(iv)$, we get either $\lambda = 0 \quad or \quad (x_1 + x_2 - 2) = 0$

.

Case 1- If $\lambda = 0$

From $(i)$ , $(ii)$ $and$ $(iii)$, we get $x_1 = 2$ , $x_2 = 3 and$ $x_3 = 0$

But these values do not satisfy equation $(v)$ so we reject these values.

Case 2- If $\lambda \neq 0$ so $x_1 + x_2 - 2 = 0 \ldots \ldots \ldots \ldots (vii)$

From $(i), (ii), (iii)$ $and$ $(vii)$ , we get $x_1 = \frac{1}{2}$ $and$ $x_2 = \frac{3}{2}, x_3 = 0,$
$\lambda$ =3

These values satisfy all necessary conditions. The optimal solution is

$x_1 = \frac{1}{2}$; $x_2 = \frac{3}{2}$ ; $x_3 = 0$ and $z_{max}$ =17/2