Module 1 Unit 1.3

Binomial Distribution, Poisson Distribution

The Binomial Distribution

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Each trial must meet the following requirements:

a. the total number of trials is fixed in advance;

b. there are two mutually exclusive outcomes of each trial appropriately labeled as success and failure;

c. the outcomes of all the trials are statistically independent

For e.g.

i) when a die is thrown 3 times, and the number of uppermost face is odd.

ii) a coin is tossed 10 times and number of heads in the trials is observed.

If any discrete r.v. X follows binomial distribution with parameters n and p it is called a binomial variable and is denoted by, X: B(n,p) or b(x,n,p)

Binomial Distribution

A discrete r.v. X is said to be follow binomial distribution with parameter n,p . If its pdf is given by,

$P(x) = nCx\, p^x q^{(n-x)}$ x=0,1,2.....n

= 0 o.w.

where, n : a number of trials

p : probability of success

q=1-p : probability of failure

x : number of success out of n trials

Mean =np and variance =npq

Additive property of binomial distribution

Let X be a r.v. with X : B (n1,p)

Let Y be a r.v. with Y : B (n2,p)

then Show that X+Y : B (n1 +n2 , p) where X and Y are independent

Example **Ten coins are thrown simultaneously. Find the probability of getting at least seven H**

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Solution. p = Probability of getting a head = $\frac{1}{2}$

q = Probability of not getting a head = $\frac{1}{2}$

The probability of getting x heads in a random throw of 10 coins is

$$p(x) = \binom{10}{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{10-x} = \binom{10}{x}\left(\frac{1}{2}\right)^{10} ; x = 0, , 2, \ldots, 10$$

$\therefore$ Probability of getting at least seven heads is given by

$$P(X \geq 7) = p(7) + p(8) + p(9) + p(10)$$

$$= \left(\frac{1}{2}\right)^{10}\left\{\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right\}$$

$$= \frac{120 + 45 + 10 + 1}{1024} = \frac{176}{1024}.$$

Example

If a six-sided die is thrown 4 times. What is the probability of getting a score of not less than 5 at least 1time

Answer:

Although there are six possible scores {1, 2, 3, 4, 5, 6},

Any score in {1, 2, 3, 4} is a failure and

any score in {5, 6} is a success.

Thus, P(X = 0) = P(failure) =4/6

and P(X = 1) = P(success) =2/6.

Hence, the probability of getting a score of not less than 5 at least 1 time = p(x>=1)

$=1-p(x=0)=1-4_{c_0}\, p^0 q^4$

Example

If a six-sided die is thrown 4 times

What is the probability of getting at least 1 six

Answer:

Although there are six possible scores {1, 2, 3, 4, 5, 6},

Any score in {1, 2, 3, 4, 5} is a failure and

score { 6} is a success.

Thus, $P(X = 0) = P(\text{failure}) = 5/6 = q$

and $P(X = 1) = P(\text{success}) = 1/6 = p$

Hence, the probability of getting at least 1 six

$= p(x>=1)$

$=1-p(x=0)$

$1- 4_{c_0}\ p^0 q^4$

Example

If a six-sided die is thrown 4 times

What is the probability of getting all even numbers

Answer:

Although there are six possible scores {1, 2, 3, 4, 5, 6},

Any score in {1,3,5} is a failure and

Any score in {2, 4 ,6} is a success.

Thus, P(X = 0) = P(failure) =3/6=q

and P(X = 1) = P(success) =3/6=p

Hence, the probability of getting all even numbers

= probability of getting even numbers all time

= p(x=4)

= 4_{c_4} $p^4 q^0$

Example

An irregular six faced die is thrown and the probability of getting five even numbers in 10 throws is twice the probability of getting four even numbers in 10 throws. How many times in 10,000 sets of 10 throws each ,would you expect it to give no even number?

Answer

P= probability of getting even number in one throw

p(X=x)= 10_{c_x} p^x q^{10-x} , x=0,1,2,.....,10

$$P(X = 5) = 2\,P(X = 4)$$

i.e., $$\binom{10}{5} p^5 q^5 = 2 \binom{10}{4} p^4 q^6$$

$\Rightarrow$ $$\frac{10\,!\;p}{5\,!\;5\,!} = 2\,\frac{10\,!\;q}{4\,!\;6\,!}$$

$\Rightarrow$ $$\frac{p}{5} = \frac{2q}{6} = \frac{q}{3}$$

$\therefore\ 3p = 5q = 5(1-p) \quad \Rightarrow \quad 8p = 5 \quad \Rightarrow \quad p = 5/8$ and $q = 3/8$

$\therefore$ $$P(X = x) = \binom{10}{x} \left(\frac{5}{8}\right)^x \left(\frac{3}{8}\right)^{10-x}$$

required number of times that in 10,000 sets of 10 throws each,

$$= 10{,}000 \times P(X = 0) = 10{,}000 \times \left(\frac{3}{8} \right)^{10} = 1$$

Example : If X is Binomial distributed with $E(X) = 2$ and Var $(X) = 4/3$, find the probability distribution of X.

Sol. : We have $E(X) = np = 2$ and Var $(X) = npq = 4/3$.

$$\therefore \quad \frac{npq}{np} = \frac{4/3}{2} \quad \therefore \quad q = \frac{2}{3} \quad \therefore \quad p = 1 - q = \frac{1}{3}$$

But $np = 2$, $\therefore \quad n \cdot \frac{1}{3} = 2 \quad \therefore \quad n = 6$

Hence, the distribution is

$$P(X = x) = {}^nC_x\, p^x\, q^{n-x} = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

Putting $x = 0, 1, 2, \ldots\ldots 6$, we get the following probability distribution of X.

x	:	0	1	2	3	4	5	6
$P(X = x)$	:	$\frac{64}{729}$	$\frac{192}{729}$	$\frac{240}{729}$	$\frac{160}{729}$	$\frac{60}{729}$	$\frac{12}{729}$	$\frac{1}{729}$

Example : The incidence of an occupational disease in an industry is such that the workers have 20% chance of suffering from it. What is the probability that out of 6 workers chosen at random 4 or more will be suffering from the disease ?

Sol. : We have $p = 20\,\% = \dfrac{20}{100} = 0{\cdot}2,\ q = 1 - p = 0{\cdot}8,\ n = 6$

$$\therefore \quad P(X = x) = {}^nC_x\, p^x\, q^{n-x} = {}^6C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{6-x}$$

$$\therefore \quad P(X \ge 4) = P(X = 4) + P(X = 5) + P(X = 6)$$

$$= {}^6C_4 \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^2 + {}^6C_5 \left(\frac{1}{5}\right)^5 \left(\frac{4}{5}\right)^1 + {}^6C_6 \left(\frac{1}{5}\right)^6 \left(\frac{4}{5}\right)^0$$

$$= \frac{1}{5^6}\left[15{\cdot}4^2 + 6{\cdot}4 + 1\right] = \frac{205}{5^6} = \frac{41}{3125}.$$

Example :On a five-question multiple-choice test there are five possible answers, of which one is correct. If a student guesses randomly and independently,
what is the probability that she is correct only on two questions?

Example :**Let X, Y be two independent binomial variates with parameters n_1 =3, p=1/3 and n_2 =5, p=1/3**
What is the probability that (1≤ X+Y)

$$P(X+Y=r) = {}^{8}C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{8-r}$$

$$\begin{aligned} P(X+Y \geq 1) &= 1 - P(X+Y < 1) \\ &= 1 - P(X+Y=0) \\ &= 1 - \left(\frac{2}{3}\right)^8 \end{aligned}$$

Example : Seven coins are tossed and the number of heads obtained is noted. The experiment is repeated 128 times and the following distribution is obtained.

No. of heads	:	0,	1,	2,	3,	4,	5,	6,	7	Total
Frequency	:	7,	6,	19,	35,	30,	23,	7,	1	128

Fit a Binomial distribution if (i) the coins are unbiased, (ii) if the nature of the coins is not known.

Sol. : To fit a distribution to given data means to find the constants of the distribution which will adequately describe the given situation.

(i) *When the coins are unbiased*

$$p = \frac{1}{2},\ q = \frac{1}{2} \text{ and by data } n = 7$$

$$\therefore \quad P(X = x) = {}^7C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{7-x}$$

Putting $x = 0, 1, 2, 3, \ldots\ldots, 7$, we get

$$P(0) = \frac{1}{2^7},\quad P(1) = \frac{7}{2^7},\quad P(2) = \frac{21}{2^7},$$

Expected frequency = Np and $N = 128$.

Multiplying the above probabilities by 128 *i.e.*

we get the expected frequencies as 1, 7, 21, 35, 35, 21, 7, 1

(ii) *When the nature of the coins is not known.*

We have $\bar{X} = \dfrac{\sum f_i x_i}{N} = \dfrac{0 \times 7 + 1 \times 6 + 2 \times 19 + \ldots + 7 \times 1}{128} = \dfrac{433}{128}$

But $\bar{X} = np$

$$\therefore \quad p = \frac{\bar{X}}{n} = \frac{3 \cdot 38}{7} = 0 \cdot 48 \qquad \therefore \; q = 1 - p = 0 \cdot 52$$

$$\therefore \quad P(X = x) = {}^7C_x \, (0 \cdot 48)^x \, (0 \cdot 52)^{7-x}$$

Putting $x = 0, 1, 2, 3, \ldots, 7$ we get

$$P(0) = 0 \cdot 01, \; P(1) = 0 \cdot 066, \; P(2) = 0 \cdot 184, \; \ldots$$

Multiply these probabilities by 128 we get the expected frequencies as

1, 8, 23, 36, 33, 18, 6, 3.

(Last term = 128 – sum of other terms).

Poisson Distribution

If number of independent trials ‘n’ increases indefinitely and probability of getting success ‘p’ in each trial is very small so that ‘np’ remains constant, then Binomial distribution can be approximated to Poisson distribution

Definition.

A random variable X is said to have a Poisson distribution if its probability density function is given by

f(x)= $e^{-m}m^{x}$/x!   X=0,1,2………

Mean =variance =m

Additive property of poisson distribution

Suppose X1, X2, . . . ,Xk are k independent poisson random variables with Xi ~ P(m1).

Then X1 + X2 + . . . + Xk ~ P(m1 + m2 + . . . + mk).

Example

A car hire firm has two cars, which it hires out day by day. The number of demands for a car on each day is distributed as Poisson variate with mean 1.5 Calculate the proportion of days on which i) neither car is used,
ii) some demand is refused

Answer

X= The number of demands for a car on each day

(i) P(neither car is used)=p(x=0)= $e^{-m}m^{0}$/0! =0.2231

(ii)p(some demand is refused)= p(x>2)

= 1-(p(x=0)+p(x=1)+p(x=2))

=0.19126

Exercise : Accidents occur on a particular stretch of highway at an average rate 3 per week. What is the probability that there will be(i) exactly two accidents (ii) atmost two accidents in a given week?

Example . *Suppose that the number of telephone calls coming into a telephone exchange between 10 A.M. and 11 A.M. say, X_1 is a random variable with Poisson distribution with parameter 2. Similarly the number of calls arriving between 11 A.M. and 12 noon say, X_2 has a Poisson distribution with parameter 6. If X_1 and X_2 are independent, what is the probability that more than 5 calls come in between 10 A.M. and 12 noon ?*

Solution. Let $X = X_1 + X_2$. By the additive property of Poisson distribution, X is also a Poisson variate with parameter (say) $\lambda = 2 + 6 = 8$

Hence the probability of x calls in-between 10. A.M. and 12 noon is given by $P(X = x) = \dfrac{e^{-\lambda}\lambda^x}{x!} = \dfrac{e^{-8}8^x}{x!}$; $x = 0, 1, 2, \ldots$

Probability that more than 5 calls come in between 10 A.M. and 12 noon is

$$P(X > 5) = 1 - P(X \leq 5) = 1 - \sum_{x=0}^{5} \frac{e^{-8}\, 8^{x}}{x\,!}$$

$$= 1 - 0 \cdot 1912 = 0 \cdot 8088$$

Example

Let X have a Poisson distribution with parameter $\lambda = 1$.

What is the probability that $X \geq 2$ given that $X \leq 4$?

$$P(X \geq 2 \,/\, X \leq 4) = \frac{P(2 \leq X \leq 4)}{P(X \leq 4)}.$$

$$P(2 \leq X \leq 4) = \sum_{x=2}^{4} \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \frac{1}{e} \sum_{x=2}^{4} \frac{1}{x!}$$

$$= \frac{17}{24\, e}.$$

$$P(X \leq 4) = \frac{1}{e} \sum_{x=0}^{4} \frac{1}{x!}$$

$$= \frac{65}{24\, e}.$$

?

$$P(X \geq 2 \,/\, X \leq 4) = \frac{17}{65}.$$

Example : Fit Poisson Distribution for following data

R.V. X :0,1,2,3,4,5,6

Frequency: 97,67,50,30,16,5,2

Answer m=$\frac{\sum f.x}{\sum f}$=1.3408

f(x)= $e^{-m}m^{x}$/x! X=0,1,2…,6

	x	f	f(x)	N.f(x)	exp.freq
1	0	97	0.26163	69.85521	70
2	1	67	0.35080	93.66360	94
3	2	50	0.23518	62.79306	63
4	3	30	0.10511	28.06437	28
5	4	16	0.03523	9.40641	9
6	5	5	0.00945	2.52315	3
7	6	2	0.00211	0.56337	1

Example: If X_1, X_2, X_3 are three independent Poisson variates with parameters $m_1 = 1$, $m_2 = 2$, $m_3 = 3$ respectively, find (i) $P[(X_1 + X_2 + X_3) \geq 3]$ and (ii) $P[X_1 = 1 / (X_1 + X_2 + X_3) = 3]$.

Answer :

Poisson distribution $Z = X_1 + X_2 + X_3$ is also a Poisson distribution with parameter $m = m_1 + m_2 + m_3 = 6$.

Now, X_1 is a Poisson variate with parameter $m_1 = 1$, $X_2 + X_3$ is a Poisson variate with parameter $m_2 + m_3 = 2 + 3 = 5$,

$$P(Z \geq 3) = 1 - P(Z = \leq 2)$$

$$= 1 - \sum_{Z=0}^{2} \frac{e^{-6} 6^{z}}{z!} = 1 - \left(e^{-6} + 6e^{-6} + \frac{6^{2} e^{-6}}{z!} \right)$$

$$= 1 - 25e^{-6} = 1 - 25(0 \cdot 002478) = 0 \cdot 938$$

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$$P[X_1 = 1/(X_1 + X_2 + X_3) = 3] = \frac{P(X_1 = 1 \text{ and } X_2 + X_3 = 2)}{P[(X_1 + X_2 + X_3) = 3]}$$

$$P[X_1 = 1/(X_1 + X_2 + X_3) = 3] = \frac{\left(e^{-1} \cdot \frac{1}{1}\right)\left(e^{-5} \cdot \frac{5^2}{2!}\right)}{e^{-6} \cdot \frac{6^3}{3!}} = \frac{25}{72}$$

Example . : A hospital switch board receives an average of 4 emergency calls in a 10 minutes interval. What is the probability that (i) there are atleast 2 emergency calls, (ii) there are exactly 3 emergency call in an interval of 10 minutes ?

Example If X, Y are independent Poisson variates with mean 2 and 3, find the variance of $3X-2Y$.

Example : Find the probability that atmost 4 defective bulbs will be found in a box of 200 bulbs if it is known that 2 percent of the bulbs are defective. (Given $e^{-4} = 0{\cdot}0183$).

Uniform Distribution (Discrete)

A discrete random variable X is said to have a uniform distribution if its probability mass function (pmf) is given by P(X=x) = 1/n , x=1,2,⋯,n

Mean = n(n+1)/2 variance = (n^2 -1)/12

Example : A man with n keys in his pocket wants to open the door of his office by trying the keys independently and andomly one by one. Find the mean and the variance of the numbers of trials required to open the door if unsuccessful keys are kept aside.

. Hint: probability at each trial is 1/n

Example

Roll a six faced fair die. Suppose X denote the number appear on the top of a die.

a. Find the probability that an even number appear on the top.

b. Find the probability that the number appear on the top is less than 3.

c. Compute mean and variance of X.