UNIT 4.1
Types of solution, Standard and Canonical form of LPP, Basic and feasible solutions, simplex method

Linear Programming Problem

L.P.P. is mostly concerned with business problems. These problems are used so as to yield maximum production or to give the maximum profit.

Linear programming is a technique for determining an optimum schedule of interdependent activities in view of available resources.

Programming is nothing but planning or the process of determining a particular plan of action.

In any industry resources of men, machine and material are always limited and they are to be so used that the output is maximum or the cost of the production is minimum. A businessman for obvious reasons cannot increase the number of workers infinitely if he wants to increase the production. If he wants to maintain the proper supply, he cannot increase his stocks infinitely. The problem is to maintain minimum stocks, employ minimum workers and still earn maximum profit.

There are constraints on men, machines, and material. This constraints and the objective function such as max. Profit can be expressed in linear form. Problems of this type are called L.P.P.

Linear Programming Problem

The word linear stands for indicating that all relationships in the problem are linear.

Non -Linear Programming Problem

The relationships in the problem are not linear.

MATHEMATICAL DEFINITION OF L.P.P.:

The problem of finding non- negative values of $x_1, x_2, x_3 \ldots x_n$ which satisfy the given 'm' linear constraints and maximize or minimize the objective function 'Z' is called LPP. That is we have to find 'n' non-negative values of $x_1, x_2, x_3 \ldots x_n$ which optimize,

Maximise(or Minimise)

$z = c_1\, x_1 + c_2\, x_2 + c_3\, x_3 + \cdots \ldots \ldots \ldots \ldots + c_n\, x_n$

Subject to $\sum_{j=1}^{n} a_{ij}\, x_j \leq (\text{or} \geq)\ b_i$ **, 1 ≤ i≤ m**

Where $x_j \geq 0$ **, 1 ≤ j≤ n**

CANONICAL FORM

Maximise z = $c_1\, x_1 + c_2\, x_2 + c_3\, x_3 + \cdots \ldots \ldots \ldots \ldots + c_n\, x_n$

Subject to

$a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 + \cdots \ldots \ldots \ldots \ldots + c_{1n}\, x_n \le b_1$

$a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 + \cdots \ldots \ldots \ldots \ldots + c_{2n}\, x_n \le b_2$

$a_{m1}\, x_1 + a_{m2}\, x_2 + a_{n3}\, x_3 + \cdots \ldots \ldots \ldots \ldots + c_{mn}\, x_n \le b_m$

$where\;\; x_1, x_2, \ldots \ldots \ldots \ldots x_n \geq 0$

OR

Maximise Z=$\sum_{i=1}^{n} c_i\, x_i$

Subject to $\sum_{j=1}^{n} a_{ij}\, x_j \le b_i$, 1 ≤ i≤ m

Where $x_j \geq 0$, 1 ≤ j≤ n

- **SLACK VARIABLE**

Let the constraints of general LPP be $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ then non-negative variable s_i which will satisfy $\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i$ is called slack variable.

- **SURPLUS VARIABLE**

Let the constraints of general LPP be $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$ then non-negative variable s_i which will satisfy $\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i$ is called surplus variable

Standard form of LPP

1. The objective function is of maximization type,

2. All the constraints are expressed in the form of equations . (By introducing slack or surplus variables) except for non-negative restrictions.

3. The R.H.S of each constraints equation is non-negative.

NOTE: the coefficients of slack or surplus variables in the objective function are always assume to be zero

Standard form of LPP

Maximise $z = c_1 x_1 + c_2 x_2 + c_3 x_3 + \cdots \ldots \ldots \ldots \ldots + c_n x_n + +0s_1 + \cdots . +0s_m$

Subject to

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \cdots \ldots \ldots \ldots \ldots + a_{1n} x_n + s_1 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \cdots \ldots \ldots \ldots \ldots + a_{2n} x_n + s_2 = b_2$$

$$a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \cdots \ldots \ldots \ldots \ldots + a_{mn} x_n + s_m = b_m$$

$$x_1, x_2, \ldots \ldots \ldots . . x_n \geq 0$$

$$s_1, s_2, \ldots \ldots \ldots . . s_m \geq 0$$

Note :Sometimes decision variables are positive, negative or zero values, then they are called unrestricted variables. In all such cases, the decision variables can be expressed as the difference between two non-negative variables

Example 1: Convert given LPP into the standard form

$$\text{Max } Z = 3X_1+2X_2$$

$$\text{s.t.} \quad -X_1+2X_2 \le 4$$

$$3X_1+2X_2 \le 14$$

$$X_1-X_2 \le 3$$

$$X_1, X_2 \ge 0$$

Standard Form

$$\max Z = 3X_1+2X_2+0S1+0S2+0S3$$

$$\begin{aligned} \text{s.t.} \quad -X_1+2X_2+S1 &= 4 \\ 3X_1+2X_2 \quad +S2 &= 14 \\ X_1-X_2 \quad +S3 &= 3 \\ X_1, X_2, S_1, S_2, S_3 &\ge 0 \end{aligned}$$

Example 2

$$\text{Max } Z = X_1 - 3X_2$$

$$\text{s.t.} \quad -X_1 + 2X_2 <= 15$$

$$X_1 + 3X_2 = 10$$

X_1, X_2 unrestricted in size

Standard Form

Let $X_1 = X_1' - X_1''$ and $X_2 = X_2' - X_2''$ where $X_1', X_1'', X_2', X_2'' \geq 0$

$$\max Z = (X_1' - X_1'') - 3(X_2' - X_2'') + 0S_1$$

$$\text{s.t.} \quad -(X_1' - X_1'') + 2(X_2' - X_2'') + S_1 = 15$$

$$(X_1' - X_1'') + 3(X_2' - X_2'') = 10$$

$$X_1', X_1'', X_2', X_2'', S_1 \geq 0$$

Example 3

$$\text{Max } Z = 3X_1 + 4X_2 + 6X_3$$

$$\text{s.t.} \quad 2X_1 + X_2 + 6X_3 \geq 6$$

$$3X_1 + 2X_2 = 8$$

$$7X_1 - 3X_2 + 5X_3 \geq 9$$

$$X_1, X_2 \geq 0, \; X_3 \text{ is unrestricted in sign}$$

Example 4

$$\text{Min } Z = 12X_1 + 5X_2$$

$$\text{s.t.} \quad -5X_1 + 3X_2 \geq 15$$

$$7X_1 + 2X_2 \geq -14$$

$$X_1, X_2 \geq 0$$

Standard Form :

$$\text{Max } (-Z) = \text{Max } (Z^*) = -12X_1 - 5X_2 - 0S_1 + 0S_2$$

$$\text{s.t.} \quad -5X_1 + 3X_2 - S_1 = 15$$

$$-7X_1 - 2X_2 + S_2 = 14$$

$$X_1, X_2, S_1, S_2 \geq 0$$

Example 5

Convert the following L.P.P. in the standard form.

$$\begin{aligned} \text{Minimise} \quad & z = 2x_1 + 3x_2 \\ \text{subject to} \quad & 2x_1 - 3x_2 - x_3 = -4 \\ & 3x_1 + 4x_2 - x_4 = -6 \\ & 2x_1 + 5x_2 + x_5 = 10 \\ & 4x_1 - 3x_2 + x_6 = 18 \\ & x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

Sol. : Carefully observing the given problem we first note that the decision variables x_1, x_2 are unrestricted. The variables x_3, x_4, x_5, x_6 are slack variables.

Hence, we put $x_1 = y_1 - y_2$, $x_2 = y_3 - y_4$, $x_3 = y_5$, $x_4 = y_6$, $x_5 = y_7$ and $x_6 = y_8$.

The problem then becomes :

$$\begin{aligned} \text{Minimise} \quad & z = 2y_1 - 2y_2 + 3y_3 - 3y_4 \\ \text{subject to} \quad & 2y_1 - 2y_2 - 3y_3 + 3y_4 - y_5 = -4 \\ & 3y_1 - 3y_2 + 4y_3 - 4y_4 - y_6 = -6 \\ & 2y_1 - 2y_2 + 5y_3 - 5y_4 + y_7 = 10 \\ & 4y_1 - 4y_2 - 3y_3 + 3y_4 + y_8 = 18 \end{aligned}$$

Multiply the object function, the first and the second constraints by (–1), then the given problem in the standard form becomes,

DEFINITIONS:

1) **SOLUTION: The set of values of decision variables xj (j=1, 2,……,n) which satisfy the constraints of LPP is called a solution to the given LPP.**

2) **FEASIBLE SOLUTION: The set of values of decision variables xj (j=1,2,…..,n) which satisfies all the constraints and non-negative constraints of LPP is called the feasible solution to the LPP.**

3) **INFEASIBLE SOLUTION: The set of values of decision variables xj where (j= 1,2,….,n) which do not satisfy all non-negative conditions of an LPP simultaneously is called infeasible solution.**

4) **BASIC SOLUTION: Given a system of 'm' simultaneous equations in 'n' unknowns AX=b (m<n) where A is m×n matrix of rank m. let B be any m×m sub matrix formed by m linearly independent columns of A. Then a solution obtained by setting n-m variables equal to zero and solving the resulting system of equations is called a basic solution to the given system of equations.**

5) BASIC FEASIBLE SOLUTION (BFS): A feasible solution to LPP which is also the basic solution is called basic feasible solution

There are two types of BFS:

(1) Degenerate BFS: if value of at least one basic variable is zero, the solution is called as degenerate BFS.

(2) Non-degenerate BFS: if values of all basic variables in a BFS are non-zero and positive then that BFS is called non-degenerate BFS.

6) OPTIMUM BFS: A BFS which optimizes the objective function value of given LPP is called an optimum BFS.

7) UNBOUNDED SOLUTION: A solution which can increase or decrease the value of objective function indefinitely is called an unbalanced solution.

Example 1: Find (i) All basic solutions

(ii) degenerate solution for the following L.P.P.

maximise z =$x_1 - 2x_2 + 4x_3$

Subject to $x_1 + 2x_2 + 3x_3$ =7

$3x_1 + 4x_2 + 6x_3$=15

where $x_1, x_2, x_3 \geq 0$

Here m=2 ,n=3

no. of basic solutions =ncm=3c2=3

To get B.F.S. put n-m variables = 0

Example 1 : Determine all basic solutions to the following problem.

Maximise $z = x_1 - 2x_2 + 4x_3$

subject to $x_1 + 2x_2 + 3x_3 = 7$

$3x_1 + 4x_2 + 6x_3 = 15$

No. of basic solutions	Non-basic variables = 0	Basic Variables	Equations And the values of the basic variables	Is the solution feasible $x_i \geq 0$?	Is the solution degenerate ?
1.	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 7$ $3x_1 + 4x_2 = 15$ $x_1 = 1, x_2 = 3$	Yes	No
2.	$x_2 = 0$	x_1, x_3	$x_1 + 3x_3 = 7$ $3x_1 + 6x_3 = 15$ $x_1 = 1, x_3 = 2$	Yes	No
3.	$x_1 = 0$	x_2, x_3	$2x_2 + 3x_3 = 7$ $4x_2 + 6x_3 = 15$ unbounded solution	—	—

Example 2 : Find all basic solutions to the following system of linear equations.

$x + 2y + z = 4$

$2x + y + 5z = 5$

Answer:

A basic solution to the given system of equation is obtained by setting $z = 0$ and solving the system

$X + 2y = 4$ & $2x + y = 5$ we get $x = 2$ and $y = 1$

Therefore, the basic variables are $x = 2$, $y = 1$ and the non basic variable is $z = 0$.

This basic solution is non-degenerate Basic feasible solution (BFS)

A basic solution to the given system of equation is obtained by setting $y = 0$ and solving the system

$x + z = 4$ & $2x + 5z = 5$ we get $x = 5$ and $z = -1$

Therefore, the basic variables are $x = 5$, $z = -1$.

It is infeasible solution.

A basic solution to the given system of equation is obtained by setting $x = 0$ and solving the system

$2y + z = 4$ & $y + 5z = 5$ we get $y = 5/3$ and $z = 2/3$

Therefore, the basic variables are $y = 5/3$, $z = 2/3$.

It is non degenerate Basic feasible solution

Example 3: Find (i) All basic solutions (ii) All feasible basic solutions (iii) Optimal feasible basic solution for the following L.P.P.

maximise z =$2x_1 - 2x_2 + 4x_3 - 5x_4$

Subject to $x_1 + 4x_2 - 2x_3 + 8x_4$ =2

- $x_1 + 2x_2 + 3x_3 + 4x_4$ =1

where $x_1, x_2, x_3, x_4 \geq 0$

Here m=2 ,n=4

no. of basic solutions =ncm=4c2=6

To get B.F.S. put n-m variables = 0

Example 3:

No. of Basic solns.	Non basic Variables	Basic Variables	Eqⁿs & the values of Basic Variables	B.F.S.	Deg.	Value of Z	Optimal soln
1	$x_3=0$, $x_4=0$	x_1, x_2	$x_1+4x_2=2$, $-x_1+2x_2=1$; $x_1=$ [illegible], $x_2=1/2$	Yes	Deg	-1.5	NO
2	$x_2=0$, $x_4=0$	x_1, x_3	$x_1-2x_3=2$, $-x_1+3x_3=1$; $x_1=8$, $x_3=3$	Yes	Non Deg.	28	Yes
3	$x_1=0$, $x_4=0$	x_2, x_3	$4x_2-2x_3=2$, $2x_1+3x_3=1$; $x_2=1/2$, $x_3=$ [illegible]	Yes	Deg.	-1	NO
4	$x_2=0$, $x_3=0$	x_1, x_4	$x_1+8x_4=2$, $-x_1+4x_4=1$; $x_1=$ [illegible], $x_4=1/4$	Yes	Deg	-1.25	NO
5	$x_1=0$, $x_3=0$	x_2, x_4	$4x_2+8x_4=2$, $2x_2+4x_4=1$ (unbdd) ; infinite solns	NO	—	—	—
6	$x_1=0$, $x_2=0$	x_3, x_4	$-2x_3+8x_4=2$, $3x_1+x_4=12$; $x_3=$ [illegible], $x_4=\frac{1}{4}$	Yes	Deg	-12.5	NO