DUALITY

Associated with LPP there is always a corresponding LPP called the dual problem of the given LPP. The two problems i.e. primal and dual are remarkably symmetric.

1. The cost vector associated with the objective function of one is just the RHS vector in the other set of constraints.
2. The constraint coefficient associated with one problem is simply transpose of the constraint coefficient matrix associated with the other. However the two problems differ in one respect one of the problem is a maximization problem while the other is minimization.

DEFINITIONS:

1. PRIMAL PROBLEM: A LPP of determining x1,x2,…….,xn,

so as to maximize

$Z=\sum_{j=1}^{n} c_j x_j$

such that $\sum_{j=1}^{n}$ aijXj $\leq$ bi , i=1,2,…..,m

x1,x2,……xn ≥ 0

where A is a matrix of order m×n is called primal problem.

2. DUAL PROBLEM:

A LPP of determining w1,w2,w3,…, wm

so as to minimize z*=$\sum_{i=1}^{m}$ biwi ,

such that $\sum_{i=1}^{n}$ ajiwi $\geq$ cj , j=1,2,…….,n.

w1,w2,w3,…, wm $\geq$0.

Here a matrix A^T is a transpose of matrix A in primal problem this is called as a dual problem of the primal.

The variables w1,w2,……, wm are called dual variables and constraints of dual problem are called dual constraints.

Note that, Dual of Dual is Primal.

The characteristics of dual problem:

1. No. of variables in the dual problem is equal to no. of constraints in primal and vice versa.
2. The elements of requirement vector (bi) in one problem are the respective prices in the objective function of the other problem.
3. One of the problems is of maximization type while the other is of minimization type.
4. In general the primal maximization problem has less than or equal to type constraints and dual minimization problem has greater than or equal to type constraints,
5. The variables in both the problems are non-negative.

Steps in the construction of the dual problem:

i. Formulate the primal problem.

ii. The maximization problem in the primal is transferred to minimization problem in the dual.

iii. For a primal with 'n' variables and 'm' constraints the dual will have 'm' variables and 'n' constraints.

iv. The $\leq$ sign of the primal constraint become $\geq$ sign in the dual constraints.

v. The prices say c1,c2,…..cn. with 'n' variables in the objective function of the primal are replaced by b1,b2,…..bm with 'm' variables of the objective function in the dual.

Note that

1) if i^{th} primal constraint is of equal to type then corresponding variable in dual (i.e. i^{th} dual variable) is unrestricted in sign and vice versa.

2) if i^{th} primal variable is unrestricted in sign then corresponding i^{th} dual constraint will be of equal to type.

How to read the solution to the dual from the final simplex table of the primal and conversely:

The final simplex table giving optimum solution of the primal, also contains optimum solution of its dual and itself and conversely. This is based on the fundamental theorem of duality which states:

i. If either the primal or dual problem has a finite optimum solution, then the other problem also has a finite optimum solution further more optimum values of objective functions in both the problems are the same.

ii. If either problem has an unbounded solution then the other problem has no feasible solution at all.

iii. Both the problems may be infeasible.

The following rules will help to write the solution of the dual from the final simplex table of the primal and vice versa.

Rules:

i. If the primal (dual) variable correspond to a slack and surplus variable in the dual (primal) problem, its optimum value can be directly obtain from the net evaluation row of the optimum dual (primal) simplex table, as the net evaluation corresponding to the slack and/or surplus variable.

ii. If the primal variable correspond to an artificial starting variable in the dual its optimum value can be directly obtain from the net evaluation row of the optimum dual simplex table, as the net evaluation corresponding to this artificial variable, after deleting the constant 'M'.

Primal Dual Relationship :

If Primal	Then Dual
1. Objective is to maximize	1. Objective is to minimize
2. Variable Xi	2. ith constraint
3. ith constraint	3. Variable Wi
4. Variable Xi is unrestricted in sign	4. ith constraint of equal to type
5. ith constraint of equal to type	5. Variable Wi is unrestricted in sign
6. $\leq$ type constraint	6. $\geq$ type constraint

Problem A

Maximise $z = 6x_1 + 10x_2$

subject to
$$3x_1 + 4x_2 \leq 18$$
$$2x_1 + x_2 \leq 8$$
$$x_1 + 3x_2 \leq 20$$
$$x_1, x_2 \geq 0.$$

Problem B

Minimize $w = 18y_1 + 8y_2 + 20y_3$

subject to
$$3y_1 + 2y_2 + y_3 \geq 6$$
$$4y_1 + y_2 + 3y_3 \geq 10$$
$$y_1, y_2, y_3 \geq 0$$

Problem A

6	10	
x_1	x_2	
3	4	18
2	1	8
1	3	20

Problem B

18	8	20	
y_1	y_2	y_3	
3	2	1	6
4	1	3	10

Example 1 : Write the dual of the following L.P.P.

Maximise $z = 2x_1 - x_2 + 4x_3$

subject to
$$x_1 + 2x_2 - x_3 \leq 5$$
$$2x_1 - x_2 + x_3 \leq 6$$
$$x_1 + x_2 + 3x_3 \leq 10$$
$$4x_1 + x_3 \leq 12$$
$$x_1, x_2, x_3 \geq 0.$$

Sol. : Since in the given L.P.P. there are $m = 3$ variables and $n = 4$ constraints the dual will have $m = 4$ variables and $n = 3$ constraints.

We can write the given problem as

Maximise $z = \boldsymbol{c}\,\boldsymbol{x}$

subject to $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ where $c = [\,2, -1, 4\,]$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\quad A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 3 \\ 4 & 0 & 1 \end{bmatrix},\quad \boldsymbol{b} = \begin{bmatrix} 5 \\ 6 \\ 10 \\ 12 \end{bmatrix}$$

In terms of the matrices A, b, c of the primal the dual becomes

Minimise $W = bY$

subject to $A^T Y = c$

$$\text{Minimise} \quad W = [\,5, 6, 10, 12\,] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\text{subject to} \quad \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -1 & 1 & 0 \\ -1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Example : Obtain the dual of the following L.P.P.

Minimise $z = 3x_1 - 2x_2 + x_3$

subject to $2x_1 - 3x_2 + x_3 \le 5$

$4x_1 - 2x_2 \ge 9$

$-8x_1 + 4x_2 + 3x_3 = 8$

$x_1, x_2 \ge 0$, x_3 unrestricted.

Thus, the given problem becomes

Minimise $z = 3x_1 - 2x_2 + x_3' - x_3''$

subject to $-2x_1 + 3x_2 - x_3' + x_3'' \ge -5$

$4x_1 - 2x_2 + 0x_3' - 0x_3'' \ge 9$

$-8x_1 + 4x_2 + 3x_3' - 3x_3'' \ge 8$

$8x_1 - 4x_2 - 3x_3' + 3x_3'' \ge -8$

$x_1', x_2', x_3', x_3'' \ge 0.$

then the dual of the problem is

Maximise $w = -5y_1 + 9y_2 + 8y_3' - 8y_3''$

subject to $-2y_1 + 4y_2 - 8y_3' + 8y_3'' \le 3$

$3y_1 - 2y_2 + 4y_3' - 4y_3'' \le -2$

$-y_1 + 0y_2 + 3y_3' - 3y_3'' \le 1$

$y_1 + 0y_2 \quad 3y_3' + 3y_3'' \le -1$

Maximise $w = -5y_1 + 9y_2 + 8y_3$

subject to $-2y_1 + 4y_2 - 8y_3 \le 3$

$3y_1 - 2y_2 + 4y_3 \le -2$

$-y_1 + 3y_3 = 1$

$y_1, y_2 \ge 0$, y_3 unrestricted.

Example : Find the dual fo the following L.P.P.

Maximise $z = x_1 - 2x_2 + 3x_3$

subject to $-2x_1 + x_2 + 3x_2 = 2$

$2x_1 + 3x_2 + 4x_3 = 1$

$x_1, x_2, x_3 \ge 0.$

Hence, the given problem becomes,

Maximise $z = x_1 - 2x_2 + 3x_3$

subject to $-2x_1 + x_3 + 3x_3 \le 2$

$2x_1 - x_2 - 3x_3 \le -2$

$2x_1 + 3x_2 + 4x_3 \le 1$

$-2x_1 - 3x_2 - 4x_3 \le -1$

$x_1, x_2, x_3 \ge 0.$

then the dual of the problem is

Minimise $w = 2y_1 + y_2$

subject to $-2y_1 + 2y_2 \ge 1$

$y_1 + 3y_2 \ge -2$

$3y_1 + 4y_2 \ge 3.$

Example 1 : Solve the L.P.P. from its primal as well as from its dual.

Minimise $z = 0{\cdot}7x_1 + 0{\cdot}5x_2$

subject to $x_1 \geq 4,\ x_2 \geq 6$

$x_1 + 2x_2 \geq 20$

$2x_1 + x_2 \geq 18$

$x_1,\ x_2 \geq 0$

Sol. : We first write the given L.P.P. in standard form. Since the given problem is of minimisation type, we convert it into maximisation type.

Maximise $z' = -z = -0{\cdot}7x_1 - 0{\cdot}5x_2 - 0s_1 - 0s_2 - 0s_3 - 0s_4 - MA_1 - MA_2 - MA_3 - MA_4$ .... (1)

subject to $x_1 + 0x_2 - s_1 + 0s_2 + 0s_3 + 0s_4 + A_1 + 0A_2 + 0A_3 + 0A_4 = 4$ .... (2)

$0x_1 + x_2 + 0s_1 - s_2 + 0s_3 + 0s_4 + 0A_1 + A_2 + 0A_3 + 0A_4 = 6$ .... (3)

$x_1 + 2x_2 + 0s_1 + 0s_2 - s_3 + 0s_4 + 0A_1 + 0A_2 + A_3 + 0A_4 = 20$ .... (4)

$2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3 - s_4 + 0A_1 + 0A_2 + 0A_3 + A_4 = 18$ .... (5)

Multiply (2), (3), (4) and (5) by M and add to (1)

$$\therefore\ z' = (-0{\cdot}7 + 4M)\,x_1 + (-0{\cdot}5 + 4M)\,x_2 - Ms_1 - Ms_2 - Ms_3 - Ms_4 + 0A_1 + 0A_2 + 0A_3 + 0A_4 - 48M$$

$$\therefore\ z' + (0{\cdot}7 - 4M)\,x_1 + (0{\cdot}5 - 4M)\,x_2 + Ms_1 + Ms_2 + Ms_3 + Ms_4 + 0A_1 + 0A_2 + 0A_3 + 0A_4 = -48M$$

Simplex Table (Primal)

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	s_3	s_4	A_1	A_2	A_3	A_4	R.H.S. Sol.	Ratio
0	z	$0{\cdot}7-4M$	$0{\cdot}5-4M$	M	M	M	M	0	0	0	0	$-48M$	
A_2 leaves	A_1	1	0	−1	0	0	0	1	0	0	0	4	—
x_2 enters	A_2	0	1*	0	−1	0	0	0	1	0	0	6	6 ←
	A_3	1	2	0	0	−1	0	0	0	1	0	20	10
	A_4	2	1	0	0	0	−1	0	0	0	1	18	18
	z	$0{\cdot}7-4M$	0	M	$0{\cdot}5-3M$	M	M	0		0	0	$-3-24M$	
A_1 leaves	A_1	1*	0	−1	0	0	0	1		0	0	4	4 ←
x_1 enters	x_2	0	1	0	−1	0	0	0		0	0	6	—
	A_3	1	0	0	2	−1	0	0		1	0	8	8
	A_4	2	0	0	1	0	−1	0		0	0	12	6
	z	0	0	$0{\cdot}7-3M$	$0{\cdot}5-3M$	M	M			0	0	$-5{\cdot}8-8M$	
A_3 leaves	x_1	1	0	−1	0	0	0			0	0	4	—
s_2 enters	x_2	0	1	0	−1	0	0			0	0	6	—
	A_3	0	0	1	2*	−1	0			1	0	4	2 ←
	A_4	0	0	2	1	0	−1			0	0	4	4

Simplex Table continued on the next page

Simplex Table (For Ex. 1) continued from the previous page.

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	s_3	s_4	A_1	A_2	A_3	A_4	R.H.S. Sol.	Ratio
	z	0	0	$0{\cdot}45-\frac{3}{2}M$	0	$\frac{0{\cdot}5}{2}-\frac{M}{2}$	M				0	$-6{\cdot}8-2M$	
A_4 leaves	x_1	1	0	−1	0	0	0				0	4	—
s_1 enters	x_2	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0				0	8	16
	s_2	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0				0	2	4
	A_4	0	0	$\frac{3}{2}$*	0	$\frac{1}{2}$	−1				0	2	$\frac{4}{3}$ ←
	z	0	0	0	0	0·1	0·3					$-7{\cdot}4$	
	x_1	1	0	0	0	$\frac{1}{3}$	$-\frac{2}{3}$					16/3	
	x_2	0	1	0	0	$-\frac{2}{3}$	$\frac{1}{3}$					22/3	
	s_2	0	0	0	1	$-\frac{2}{3}$	$\frac{1}{3}$					4/3	
	s_1	0	0	1	0	$\frac{1}{3}$	$-\frac{2}{3}$					4/3	

$$x_1 = \frac{16}{3}, \quad x_2 = \frac{22}{3}, \quad z'_{Max} = -7{\cdot}4 \quad \therefore \; z_{Min} = 7{\cdot}4$$

The dual of the given problem is

Maximise $w = 4y_1 + 6y_2 + 20y_3 + 18y_4$

subject to $y_1 + 0y_2 + y_3 + 2y_4 \le 0{\cdot}7$

$0y_1 + y_2 + 2y_3 + y_4 \le 0{\cdot}5$

Introducing slack variables the problem becomes :

Maximise $w = 4y_1 + 6y_2 + 20y_3 + 18y_4 + 0s_1 + 0s_2$

$\therefore \quad w - 4y - 6y_2 - 20y_3 - 18y - 0s_1 - 0s_2 = 0$

subject to $y_1 + 0y_2 + y_3 + 2y_4 + s_1 + 0s_2 = 0{\cdot}7$

$0y_1 + y_2 + 2y_3 + y_4 + 0s_1 + s_2 = 0{\cdot}5$

$y_1, y_2, y_3, y_4 \ge 0.$

Iteration Number	Basic Var.	Coefficients of						R.H.S. Sol.	Ratio
		y_1	y_2	y_3	y_4	s_1	s_2		
0	w	-4	-6	-20	-18	0	0	0	
s_2 leaves	s_1	1	0	1	2	1	0	0·7	0·7
y_3 enters	s_2	0	1	2*	1	0	1	0·5	0·25 ←
1	w	-4	4	0	-8	0	10	5	
s_1 leaves	s_1	1	$-\frac{1}{2}$	0	$\frac{3}{2}$*	1	$-\frac{1}{2}$	0·45	0·3 ←
y_4 enters	y_3	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0·25	0·5
2	w	$\frac{4}{3}$	$\frac{4}{3}$	0	0	$\frac{16}{3}$	$\frac{22}{3}$	7·4	
	y_4	$\frac{2}{3}$	$-\frac{1}{3}$	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0·3	
	y_3	$-\frac{1}{3}$	$\frac{2}{3}$	1	0	$-\frac{1}{2}$	$\frac{2}{3}$	0·1	

In the second iteration of the simplex table we see that in the row of w, the coefficient of s_1 is 16/3 and that of s_2 is 22/3. These are the values of x_1 and x_2 in the primal.

Hence, $x_1 = \frac{16}{3}, \quad x_2 = \frac{22}{3}$

and $w_{Max} = 7{\cdot}4$ *i.e.* $z_{Min} = 7{\cdot}4$.

Example _ : Solve the following L.P.P. from its primal as well as from its dual.

Maximise $z = 2x_1 + x_2$

subject to $-x_1 + 2x_2 \le 2$

$x_1 + x_2 \le 4$

$x_1 \le 3$ and $x_1, x_2 \ge 0$.

: To solve the problem from its primal we write it in the standard form.

Maximise $z = 2x_1 + x_2 - 0s_1 - 0s_2 - 0s_3$

i.e. $z - 2x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to $-x_1 + 2x_2 + s_1 + 0s_2 + 0s_3 = 2$

$x_1 + x_2 + 0s_1 + s_2 + 0s_3 = 4$

$x_1 + 0x_2 + 0s_1 + 0s_2 + s_3 = 3$

Iteration Number	Basic Var.	Coefficients of					R.H.S. Sol.	Ratio
		x_1	x_2	s_1	s_2	s_3		
0	z	−2	−1	0	0	0	0	
s_3 leaves	s_1	−1	2	1	0	0	2	—
x_1 enters	s_2	1	1	0	1	0	4	4
	s_3	1*	0	0	0	1	3	3 ←
1	z	0	−1	0	0	2	6	
s_2 leaves	s_1	0	2	1	0	1	5	2·5
x_2 enters	s_2	0	1*	0	1	−1	1	2 ←
	x_1	1	0	0	0	1	3	—
2	z	0	0	0	1	3	7	
	s_1	0	0	1	−2	3	7	
	x_2	0	1	0	1	−1	1	
	x_1	1	0	0	0	1	3	

$\therefore$ $x_1 = 3$, $x_2 = 1$, $z_{max} = 7$.

The dual of the above problem clearly is

Minimise $w = 2y_1 + 4y_2 + 3y_3$

subject to $-y_1 + y_2 + y_3 \ge 2$

$2y_1 + y_2 + 0y_3 \ge 1$

$y_1, y_2, y_3 \ge 0$.

The dual in the standard form will be

Maximise $w' = -w = -2y_1 - 4y_2 - 3y_3 - 0s_1 - 0s_2 - MA_1 - MA_2$ .... (1)

subject to $-y_1 + y_2 + y_3 - s_1 + 0s_2 + A_1 + 0A_2 = 2$ .... (2)

$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$ .... (3)

Multiply (2) and (3) M and add to (1).

Maximise $w' = -2y_1 - 4y_2 - 3y_3 + My_1 + 2My_2 + My_3 - Ms_1 - Ms_2 - 0A_1 - 0A_2 - 3M$

i.e. $w' + (2 - M)\,y_1 + (4 - 2M)\,y_2 + (3 - M)\,y_3 + Ms_1 + Ms_2 + 0A_1 + 0A_2 = -3M$

subject to $-y_1 + y_2 + y_3 - s_1 + 0s_2 + A_1 + 0A_2 = 2$

$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$

Simplex Table (Dual)

Iteration Number	Basic Var.	Coefficients of y_1	y_2	y_3	s_1	s_2	A_1	A_2	R.H.S. Sol.	Ratio
0	w'	$2-M$	$4-2M$	$3-M$	M	M	0	0	$-3M$	
A_2 leaves	A_1	-1	1	1	-1	0	1	0	2	2
y_2 enters	A_2	2	1^*	0	0	-1	0	1	1	1 ←
1	w'	$-6+3M$	0	$3-M$	M	$4-M$	0	$-4+2M$	$-4-M$	
A_1 leaves	A_1	-3	0	1^*	-1	1	1	-1	1	1 ←
y_3 enters	y_2	2	1	0	0	-1	0	1	1	—
2	w'	3	0	0	3	1	$-3+M$	$-1+M$	-7	
	y_3	-3	0	1	-1	1	1	-1	1	
	y_2	2	1	0	0	-1	0	1	1	

Deleting M from the coefficients of A_1 and A_2 in the final table and then changing the signs of these coefficients we see that A_1 and A_2 correspond to 3 and 1.

These are the values of x_1 and x_2 in the primal.

Hence, $x_1 = 3,\ x_2 = 1$ and $w'_{Max} = -7$ $\therefore\ w_{Min} = 7$

Example _ : When the primal L.P.P is infeasible, its dual is unbounded. Verify if this is true for the following problem using the simplex method.

Maximise $z = 8x_1 + 6x_2$

subject to $x_1 - x_2 \le 0{\cdot}6$

$x_1 - x_2 \ge 2$

$x_1, x_2 \ge 0.$

Example 4 : Using Duality solve the following L.P.P.

Minimise $z = 4x_1 + 3x_2 + 6x_3$

subject to $x_1 + x_3 \geq 2$

$x_2 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0.$

Sol. : We first write the given problem as :

Minimise $z = 4x_1 + 3x_2 + 6x_3$

subject to $x_1 + 0x_2 + x_3 \geq 2$

$0x_1 + x_2 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0.$

The dual of this problem is :

Maximise $w = 2y_1 + 5y_2$

subject to $y_1 + 0y_2 \leq 4$

$0y_1 + y_2 \leq 3$

$y_1 + y_2 \leq 6$

$y_1, y_2 \geq 0$

Maximise $w = 2y_1 + 5y_2 - 0s_1 - 0s_2 - 0s_3$

i.e. $w - 2y_1 - 5y_2 + 0s_1 + 0s_2 + 0s_3 = 0$

subject to $y_1 + 0y_2 + s_1 + 0s_2 + 0s_3 = 4$

$0y_1 + y_2 + 0s_1 + s_2 + 0s_3 = 3$

$y_1 + y_2 + 0s_1 + 0s_2 + s_3 = 6$

Simplex Table

Iteration Number	Basic Var.	Coefficients of y_1	y_2	s_1	s_2	s_3	R.H.S. Sol.	Ratio
0	w	−2	−5	0	0	0	0	
s_2 leaves	s_1	1	0	1	0	0	4	—
y_2 enters	s_2	0	1*	0	1	0	3	3 ←
	s_3	1	1	0	0	1	6	6
			↑					
1	w	−2	0	0	5	0	15	
s_3 leaves	s_1	1	0	1	0	0	4	4
y_1 enters	y_2	0	1	0	1	0	3	—
	s_3	1*	0	0	−1	1	3	3 ←
		↑						
2	w	0	0	0	3	2	21	
	s_1	0	0	1	1	−1	1	
	y_2	0	1	0	1	0	3	
	y_1	1	0	0	−1	1	3	

$s_1 = 0,\ s_2 = 3,\ s_3 = 2,\ w_{Max} = 21$

$\therefore\ x_1 = 0,\ x_2 = 3,\ x_3 = 2,\ z_{Min} = 21.$