Spinor Report

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1 Introduction

The spin transformations are used to understand the abstract space of null directions in the Minkowski space.They help us to understand the geometry of spin-vectors and explain many properties of Lorentz Transformations. These spin transformations are nothing but Mobius Transformations.In this report we would like to explore some mathematical concepts and results of mobius transformations(bilinear transformations) and use them to understand Lorentz transformations.

2 What is a Mobius Transformation?

Mobius transformations are defined on the extended complex plane($\hat{\mathbf{C}}$) i.e., the complex plane along with the point at infinity. Mobius transformations are the most general bijective conformal maps from the Reimann sphere to itself. These mappings are of the form:

$$w = f(z) = \frac{az + b}{cz + d}$$

where a,b,c,d are complex numbers such that $ad - bc \neq 0$ and we can normalize these complex numbers so that $ad - bc = 1$.Some of its properties are, composition of two transformations is also a mobius transformation, each transformation has a inverse(these two should point towards some group property!!), and that every transformation is a composition of translation, rotation, inversion, and dilation.Try to prove these results!! Does it help to think the complex numbers as entries of a 2×2 matrix ?

3 Analyzing the Spin Transformations

3.1 Introduction

Before diving into Spin transformations let us understand the connection between null directions and complex numbers.

A Null direction $\equiv$ A point on the Reimann Sphere $\equiv$ A complex number in $\hat{\mathbf{C}}$

Spin transformations are the mobius transformations(uniquely upto a sign [1]) those act on the Reimann spheres S^+ and S^-. These Reimann spheres are mapped to the future-pointing null vectors and past-pointing null vectors respectively. These spin transformations which map the complex number z to w, also maps a null Minkowski tetrad (T,X,Y,Z) to another null Minkowski tetrad $(\tilde{T}, \tilde{X}, \tilde{Y}, \tilde{Z})$ such that it preserves the Lorentz norm. Also we associate a matrix A $= \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), det(A) = 1$ called as the **Spin Matrix**, with a spin transformation. In other words it describes a Lorentz transformation.So, we can understand Lorentz transformations by analyzing these spin transformations. But before that we should familiarize ourselves with some common terms and results associated with Mobius Transformations.

3.1.1 Mobius Transformations Nomenclature

Given a Mobius transformation we should know about its fixed points. A **Fixed point** is a point in $\hat{\mathbf{C}}$ which remains unchanged($w = z$) after the transformation. We can clearly see that we can get atmost two fixed points. Another important concept is that of **Cross ratio**. The **Cross ratio** of four complex numbers is defined as

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

and it is same for the four complex numbers w_1, w_2, w_3, w_4 that z_1, z_2, z_3, z_4 gets mapped into. In other words it means that the cross ratio is invariant under mobius transformations. An astonishing result that is frequently used is that circle and lines are mapped to circles and lines under this transformation. This is due to the fact that we can map any three points to any other three points by using only a unique mobius transformation.
Another very important concept is that of the **Normal Form**. It is very useful form as it is expressed only in terms of the fixed points and a 'multiplier'(observe that we get the same information as before when we were using a,b,c,d and $ab - cd = 1$). Using the cross-ratio we have:

$$[w, \frac{a}{c}, f_1, f_2] = [z, \infty, f_1, f_2]$$

$$\Rightarrow \frac{w - f_1}{w - f_2} = k\frac{z - f_1}{z - f_2}$$

where k is a complex number and it is called the multiplier(try to right this in terms of the trace of a spin matrix). There is also an interesting classification of mobius transformations into parabolic,elliptic, hyperbolic or loxodromic based on this multiplier. These names come from the shape of these mobius transformations on the Reimann sphere. Another interesting result that we can derive

[1]The complex number associated with spin transformations z is actually defined as $z = \frac{\xi}{\eta}$, where both are complex numbers and the linear transformations of these define a spin transformation. So, $w = \frac{az+b}{cz+d} = \frac{a\xi+b\eta}{c\xi+d\eta}$. Now while the set (a,b,c,d) and (-a,-b,-c,-d) produce the same transformation of $z \mapsto w$, they are different linear transformations of ξ and η . Another way of saying it is both spin-matrices A and -A produce the same mobius transformation.

from the multiplier is the 'flow' in the reimann sphere. What I mean is that for any point $z^{(0)}$ on the sphere, we can write after 'n' iterations of the map:

$$\frac{w^{(n)} - f_1}{w^{(n)} - f_2} = k^n \frac{z^{(0)} - f_1}{z^{(0)} - f_2}$$

. From this equation as, n$\to \infty$, we will see that depending upon the $|k|$ value, the flow will vary. For $|k| \neq 1$, all points of the Reimann sphere except one(which one?) will converge towards a fixed point. But for $|k| = 1$, the flow will be neither in or out.

3.1.2 Understanding some Properties of Lorentz Transformations

i)We know that circles in the Reimann sphere gets mapped into circles under spin transformations. This property explains a surprising special relativity effect known as the 'invisibility of Lorentz contraction'. Due to the circle-preserving property if one observer perceives an object of having a circular outline, so will all other inertial observers coincident with him. This creates the illusion that there is no Lorentz contraction taking place!!

ii)Any Lorentz transformation is uniquely the composition of one boost and one proper space rotation.It can be easily proved that every unitary spin transformation corresponds to a unique proper rotation of S^+, and a pure boost corresponds to a positive definite Hermitian spin matrix. This boost is nothing but dilation of the complex plane($w = rz$ where $r \in R$). Now the most general form of any unitary 2×2 matrix is of the form : $= \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, where $|a|^2 + |b|^2 = 1$. Now solving for fixed points for a unitary transformation we get two fixed points of the form f and $-1/\overline{f}$. These points are such that they represent diametrically opposite points on a Reimann sphere. So, the two fixed points can be thought of as two poles of a sphere. This proves that every rotation of the sphere is equivalent to a rotation about a single axis.

iii)Now let us take this idea further and analyze a general Lorentz transformation. Let us think of it as a composition of a proper rotation and then a boost.So, the unitary transformation will have anti-podal fixed points as discussed above. Then a boost of the type $w = rz$ has fixed points $0, \infty$. So, we see that for the Lorentz transformation to have two fixed points the fixed points have to be $0, \infty$. The most general spin transformation of this type is $w = cz$ where c is a finite complex number. Geometrically, this represents a composition of rotation and boost along the a single direction. The resultant 'flow' in the reimann sphere results in something called a **Four Screw**. (Exercise:Try to find k corresponding to rotation and boost and predict the flow)

iv)For Lorentz transformations with coincident null directions or equal fixed points, we can WLOG assume that the fixed point corresponds to $f = \infty$. These are called **Null rotations**. It is not difficult to see that the most general such transformation is $w = z + c$ where c is finite complex number This corresponds to a translation in Argand-plane. This is useful in places where we want to keep a particular null-direction fixed.