Pigeonhole Principle & Ramsey   NO. 5

If 3 pigeons are to be put into two compartments, then you will certainly agree that one of the compartments will accomodate atleast 2 pigeons.

↳ lets, $k, n \in \mathbb{Z}^+$

If atleast $(kn+1)$ ~~projects are~~ objects are distributed among n boxes, then one of the boxes must contain atleast $(k+1)$ objects.

In general, if atleast $n+1$ objects are to be put into n boxes, then one of the boxes must contain atleast 2 objects.

PP. → also known as Dirichlet Drawer principle.

Q) Arrange any group of 7 people, there must be atleast 4 of the same sex.

↳ 2 boxes

Male   Female

$7 = kn+1$

by PP.

$7 = 3 \times 2 + 1$

$3+1 = 4$ → in a box for sure

Similarly,

Among any grp of 13 people, there must be atleast 2, whose b'days are at same month.

Among any group of 3000 people, there are at least 9 who have the same birthday.

↳ PP statement → if atleast $(kn+1)$ things are distributed among n boxes, then one of the boxes must contain atleast $(k+1)$ objects / things.

here $kn+p = 3000$

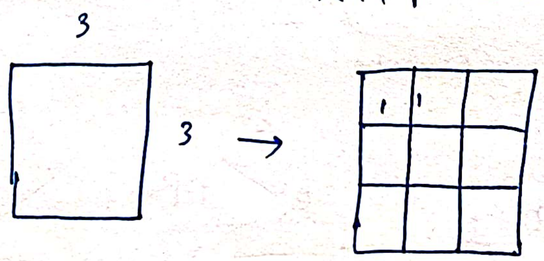
$n = 365$ $\quad (365 \times 8) + 80$

$k=8$

∴ at one of the day should have contain atleast $(8+1) = 9$ bday.

~~3.2~~ # Show that for any set of 10 points chosen within a square whose sides are of 3 units, there are two points in the sets whose distance apart is atmost $\sqrt{2}$.

$kn+p = 10$

↳ 

acording to P.P atleast one of these squares must contain atleast two points.

∴ So, the max^m possible dist b/w them, along diagonal $= \sqrt{1+1} = \sqrt{2}$ Ans

P.P donot tell us about the particular box, in which there're $(k+1)$ objects.

\# $A = \{a_1, a_2, \ldots a_5\}$ be a set of 5 positive integers. Show that for any permutation $a_{i_1}\, a_{i_2}\, a_{i_3}\, a_{i_4}\, a_{i_5}$ of A the product $(a_{i_1} - a_1)(a_{i_2} - a_2) \ldots (a_{i_5} - a_5)$ is always even.

$\hookrightarrow$ $(a_{i_k} - a_k) \rightarrow$ even

if $a_{i_k}, a_k \rightarrow$ both even

$a_{i_k}, a_k \rightarrow$ both odd

$|A| = 5$ by (P.P). two <u>boxes</u>

$k = n = 2$

there exist atleast 3 elements of A (say a_1, a_2, a_3). which are of same parity.

$\{a_{i_1}, a_{i_2}, a_{i_3}\} \cap \{a_1\, a_2\, a_3\} \neq 0$ otherwise, $|A|$ will be 6.

let $a_1 = a_{i_3}$

a_1 and a_3 are of same parity

$a_{i_3} - a_3 = a_1 - a_3.$

$\therefore a_1 - a_3 = \text{even}$

and it completes the proof.

Ten players took part in a round-robin chess ~~tou~~ tournament.

win → 1 point
lose → −1 point
draw → 0 point

it was found that more than 70 % of the games ended in a draw.

Show that there were two players, who had the same total score.

↳ 10 players ≡ 10 objects
total scores ≡ objects.

$$\text{Total games} = {}^{10}C_2 = 45$$

$$\lceil 45 \times 70\% \rceil = 32 \text{ games atleast draw.}$$

13 games → W/L

Let suppose 10 players have diff. scores.
⇒ atmost one player had total score as 'zero'.
& rest 9 players have ⊕, ⊖ ~~some~~ scores.

9 players
⊕ Scores | ⊖ Scores

(by PP. atleast 5 players had positive total scores / atleast 5 had −ve total scores.)

Sum of ⊕ total scores must be atleast = 1+2+3+4+5 = 15

but this implies,
There must be atleast 15 games, which contradicts.

∠ (PROVED)

A be a set of m positive integers. $m \geq 1$. Show that, $\exists$ a nonempty subset B of A, such that the sum, $\Sigma(x \mid x \in B)$ is divisible by m.

↳ let $A = \{3, 9, 14, 18, 23\}$

$m = |A| = 5$, we take, $B = \{3, 14, 18\}$

$\Sigma(x \mid x \in B) = 3 + 14 + 18 = 35$

35 is div by $5 = m$

$A = \{a_1, a_2, \dots a_m\}$

m subsets of A, and their respective sums,

$A_1 = \{a_1\}$, $A_2 = \{a_1, a_2\}$, $A \dots$, $A_m = \{a_1, a_2, \dots a_m\}$

a_1 $\quad a_1 + a_2$ $\quad a_1 + a_2 + \dots + a_m$

$(a_1 + a_2 + a_3)$

$B = A_3 = \{a_1, a_2, a_3\}$

we may assume no sums are divisible by m,
we have,

$a_1 \equiv r_1 \pmod{m}$

$a_1 + a_2 \equiv r_2 \pmod{m}$

$\vdots$

$a_1 + a_2 + \dots + a_m \equiv r_m \pmod{m}$

Now, treat the m sums as m objects.
and, create $(m-1)$ boxes for the $m-1$ residue classes modulo m.

(1) $\boxed{}$ (2) $\boxed{}$ ... (m-1) $\boxed{}$

$x \equiv 1 \bmod m$ $x \equiv 2 \bmod m$ $x \equiv m-1 \pmod{m}$.

By PP, there are 2 sums,

say,

$a_1 + a_2 + \dots + a_i$ and, $a_1 + a_2 + \dots + a_i + \dots + a_j$

$(i < j)$

$$a_1 + a_2 + \dots + a_i + \dots + a_j \equiv a_1 + a_2 + \dots + a_i \pmod{m}$$

$$\therefore m \mid ((a_1 + a_2 + \dots + a_j) - (a_1 + a_2 + \dots + a_i))$$

$$\text{or, } m \mid (a_{i+1} + a_{i+2} + \dots + a_j)$$

& is reqd subset of A. □

#

$x \subseteq \{1, 2, \dots, 99\}$

$|x| = 10$

Show that it is possible to select two disjoint nonempty proper subsets Y, Z of x, such that

$$\sum y \mid y \in Y = \sum z \mid z \in Z.$$

↳ $|x| = 10$, number of non-empty proper subsets of x, is,

(excluding ϕ) $\rightarrow 2^{10} - 2 = 1022$

$$1 \leq \sum a < (91 + 92 + \dots + 99) = 885$$

1022 objects, 855 boxes.

$1022 > 855$, by PP, there are two distinct nonempty subsets B and C of x, which have same sum

as $1022 = km + p$

$k = 1$

atleast $(1+1) = 2$ boxes with same no. of objects.

$\triangle ABC \to$ equilateral triangle; and E be the set of all points contained in the 3 segments AB, BC, CA (including A, B & C). Show that for every partition of E, into 2 disjoint subsets, atleast one of the 2 subsets containg the vertices of a right angled triangle.

↳ we have an equilateral $\triangle ABC$, and E is the set of all points on its three sides (AB, BC, CA) including the vertices A B and C.

we want to prove that if we dvide E into two disjoint subsets, then atleast one of these subsets will contain three points that forms a right angled triangle.

Right-angled $\triangle$s can be formed by?

one of the vertices (A, B or C) as the right angle and the midpoint of the opposite side.

using PP, let's consider E partitioned into two subsets, E_1 and E_2, every point on triangles perimeter must belong to either E_1 / E_2. ⊥.