Exploration of All Motion Derivatives

In physics, we usually stop at acceleration.

Displacement, velocity, acceleration ; that's the standard trio we deal with when analyzing motion. Sometimes, if you're in robotics or control theory, you might hear about "jerk" (rate of change of acceleration) or "snap" (derivative of jerk), but rarely beyond that.

But mathematically, there's nothing stopping us from taking more and more time derivatives of position. What do those higher derivatives mean? Can we structure them? Can we sum them, multiply them, or treat them as components of something bigger?

This exploration is an attempt to do exactly that. It starts with a simple idea , labeling each derivative with a symbol ψ_k and follows the thread wherever it leads: summations, infinite products, and more.

Defining ψ_k: The Derivative Notation

We define,

$$\psi_k = \frac{d^k s(t)}{dt^k}$$

Where $s(t)$ is displacement and $k \in N_0$

Each ψ corresponds to a familiar or unfamiliar motion quantity:

k	$\psi_k(t)$	Common Name
0	Displacement $s(t)$	Displacement (Standard)
1	ds/dt	Velocity (Standard)
2	d^2s/dt^2	Acceleration (Standard)
3	d^3s/dt^3	Jerk (known)
4	d^4s/dt^4	Snap (Niche)

5	d^5s/dt^5	Crackle (Informal)
6	d^6s/dt^6	Pop (Informal)
7+	Higher Derivatives	(Lock , Drop, ….) (unnamed)

These names stop meaning anything physical beyond a point, but mathematically they're just derivatives and we can play with them.

Summing Over ψ : Finite and Infinite

Let's see , what happens if we try to add up all these derivatives,

Raw sum:

$$\sum_{k=0}^{n} \psi_k(t)$$

For finite n, this is fine. But if we try infinite :

$$\sum_{k=0}^{\infty} \psi_k(t)$$

We quickly ran into trouble. Even for simple functions like $s(t) = e^t$, all derivatives are just e^t, so the sum becomes infinite:

$$\sum_{k=0}^{\infty} e^t = \infty$$

Let's try a normalized version:

$$\sum_{k=0}^{\infty} \frac{1}{k!} \psi_k(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{d^k s(t)}{dt^k}$$

This is the Taylor expansion of $s(t + 1)$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{d^k s(t)}{dt^k} = s(t + 1)$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \psi_k(t) = s(t + 1)$$

This formula means that if you take all the derivatives of position at a single moment in time starting from position itself, then velocity, acceleration, jerk, and so on and combine them with the correct weights (specifically, dividing each by $k!$), you can exactly reconstruct the position one unit of time later.

In other words, this infinite weighted sum of motion information at time t gives you the position at time $(t + 1)$, assuming the function is smooth and well-behaved.

Product Over ψ

This gives ,

$$\prod_{k=0}^{n} \psi_k(t)$$

If we consider , $s(t) = e^t$

$$\prod_{k=0}^{n} e^t = e^{t(n+1)}$$

But for infinite k

$$\prod_{k=0}^{\infty} \psi_k (t)$$

We obtain an infinite product of terms that do not decay, and so it diverges.

A Unified Sequence of Motion Derivatives in Classical Space-Time

Let $s(t) \in R^3$ denote the position of a particle in space as a function of time t. We define an infinite sequence of time derivatives as follows:

$$\psi_k = \frac{d^k s(t)}{dt^k} \qquad \text{, for } k \in N_0$$

This gives rise to the following hierarchy:

$$\psi_0 = s(t)$$

$\ldots$

$$\psi_2 = \frac{d^2 s(t)}{dt^2}$$

$\ldots$

$$\psi_6 = \frac{d^6 s(t)}{dt^6}$$

.....

$$\psi_\infty = \frac{d^\infty s(t)}{dt^\infty}$$

Each $\psi_k(t) \in R^3$, and is interpreted as the instantaneous rate of change of the previous $\psi_{k-1}(t)$ term.

In theory, so long as $s(t)$ is smooth (infinitely differentiable), the sequence $\{\psi_k(t)\}$ can be continued indefinitely. Each term in the sequence retains clear physical meaning as a geometric vector describing a form of change in the object's motion, whether or not it is given a name in standard physics vocabulary.

To capture the overall behavior of motion at time t, we introduce a **scalar functional** $M(t)$ that accumulates the contribution of all derivatives using a weighted squared-norm sum:

$$M(t) = \sum_{k=0}^{\infty} \omega_k \cdot \|\psi_k(t)\|^2$$

$$M_n(t) = \sum_{k=0}^{n} \omega_k \cdot \|\psi_k(t)\|^2$$

Where:

$\|\cdot\|$ denotes the standard Euclidean norm in R^3

$\omega_k \in R^+$ is a sequence of real, positive weights satisfying decay conditions to ensure convergence

Typical choices for the weights include:

$\omega_k = \frac{1}{k!}$ encourages rapid decay of higher-order contributions, mirroring the structure of Taylor expansions

$\omega_k = \frac{1}{k^2}$ a soft quadratic decay

$\omega_k = e^{-\alpha k}$, for $\alpha > 0$; exponential suppression of higher-order terms

$\omega_k = 1$ commonly used in the finite-order case, where convergence is not a concern and all derivatives are treated equally

$$M_n(t) = \sum_{k=0}^{n} \|\psi_k(t)\|^2$$

The scalar $M(t)$ acts as a **total motion complexity metric** at a specific instant t, blending contributions from all derivative orders. When the object is in uniform motion (constant velocity), higher-order derivatives vanish and $M(t)$ remains small. For more erratic or rapidly changing motion (e.g., sudden jerks or tremors), the higher-order derivatives become active, and $M(t)$ grows accordingly.

The sequence $\psi_k(t)$ represents successive layers of motion, from position and velocity to higher-order changes such as acceleration, jerk, and beyond. Each $\psi_k(t)$ remains a vector in standard three-dimensional space, with time as the only independent variable. Higher-order terms describe increasingly subtle features of motion such as smoothness or abrupt shifts and are especially relevant in precision systems like robotics or trajectory design. The aggregated scalar $M(t)$ provides a compact measure of motion complexity at a given instant, reflecting the overall dynamic behavior of the system.

A Unified Sequence of Motion Derivatives in Infinite Dimensional Space

In classical mechanics, all derivatives of motion such as velocity, acceleration, and jerk are vectors that reside within the same three-dimensional physical space. While this is sufficient for describing physical trajectories, it becomes limiting when we wish to analyze the *structure* of motion itself. To address this, we introduce a mathematical framework where each derivative occupies its own axis in an abstract infinite-dimensional space, allowing us to represent the entire dynamical behavior of motion as a single geometric object.

This formulation is no longer rooted in physical space-time, but rather in an abstract vector space where each dimension corresponds to a specific order of motion change.

Let $s(t) \in R^3$ be a smooth (infinitely differentiable) position function with respect to time t. We define the motion derivative sequence:

$$\rightarrow \Psi(t) \; := \; (\psi_0(t), \psi_1(t), \psi_2(t), \ldots) \; = \; \left(s(t), \frac{d\,s(t)}{dt}, \frac{d^2 s(t)}{dt^2}, \ldots \right)$$

Each $\psi_k(t) \in R^3$ represents the $k - th$ time derivative of position. In this formulation, each derivative is placed along a separate orthogonal axis in the product space:

$$\rightarrow \Psi(t) \; \in \; R^3 \times R^3 \times R^3 \times \ldots. \; \cong \; (R^3)^{\infty}$$

To define a meaningful geometry on this space, we introduce a weighted norm

$$\| \vec{\Psi}(t) \|^2 := \sum_{k=0}^{\infty} \omega_k \cdot \| \psi_k(t) \|^2$$

Typical choices for the weights include:

$$\omega_k = \frac{1}{k!} \;,\; \omega_k = \frac{1}{k^2} \;,\; \omega_k = e^{-\alpha k}$$

The set of all motion vectors $\vec{\Psi}(t)$ with finite norm forms a Hilbert space

$$H_\psi := \{ \vec{\Psi}(t) \mid \sum_{k=0}^{\infty} \omega_k \cdot \| \psi_k(t) \|^2 < \infty \}$$

This space provides a rigorous mathematical foundation for analyzing the global structure of motion through all orders of change.

Conclusion

This study explored motion not just as change in position but as a structured sequence of all its derivatives. By introducing the ψ_k notation and defining both finite and infinite-order functionals, we developed tools to measure and understand motion complexity. The model remains grounded in classical space-time yet extends naturally into infinite-dimensional space, where each order of change acquires its own geometric identity. Though abstract, this perspective offers meaningful insights into the structure of motion, with potential applications in control, analysis, and systems requiring high-order smoothness or dynamic awareness.

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