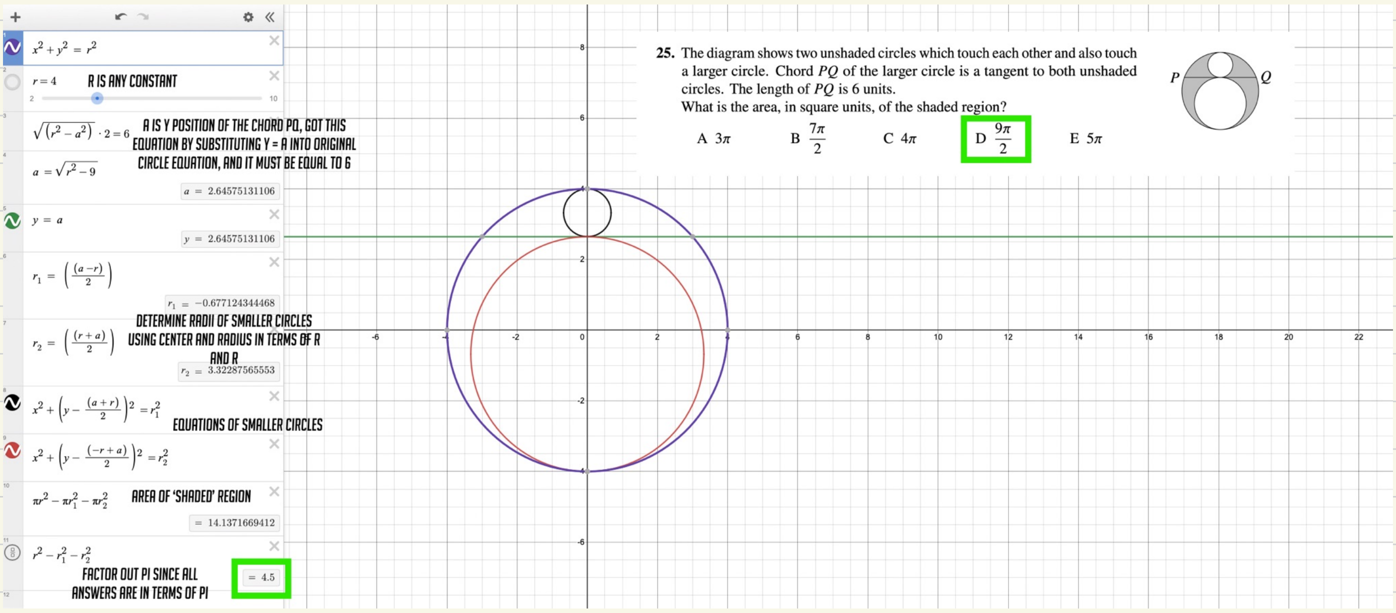
Idea: App that converts geometry problems into algebraic equivalents on the Cartesian plane

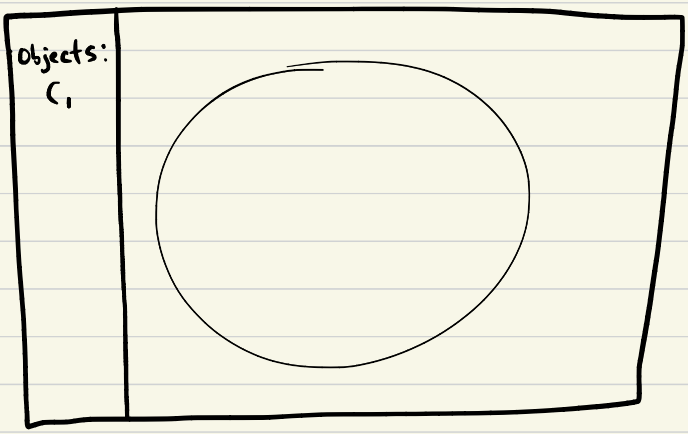
Example (Q25 from 2023 IMC)

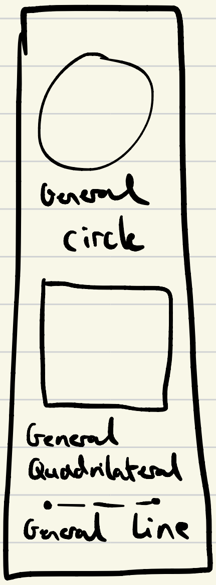


Steps taken:

- Make general circle around origin with general radius r.
- Use variable a to represent y-coordinate of chord PQ
- Then make smaller circles within the larger circles by calculating their centers and radii in terms of r and a.
- Finally find shaded region by subtracting area of bigger circle (πr^2) − smaller circles ($\pi r_1^2 + \pi r_2^2$)

First Sketches





Will need to choose 1 object as the 'base', which will be centered at (0,0)

Constraints System:

After adding all the general shapes, there will need to be some kind of constraints system.

Some examples of possible constaints:
- A circle must have a fixed radius
- A line must be a fixed length
- A shape must pass through certain points

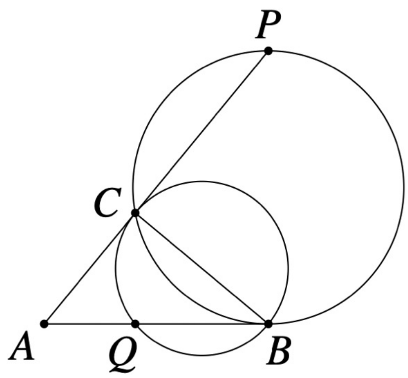
* Fixed can refer to a fixed length or fixed ratio.

Having issues trying to come up with appropriate data structures, don't want to make the program too specific for 1 type of problem but also don't want to make it too generalised (otherwise the problem would be impossible to solve.

Going to solve another example problem to understand the workflow better:

3. The diagram shows a triangle ABC. A circle touching AB at B and passing through C cuts the line AC at P. A second circle touching AC at C and passing through B cuts the line AB at Q.

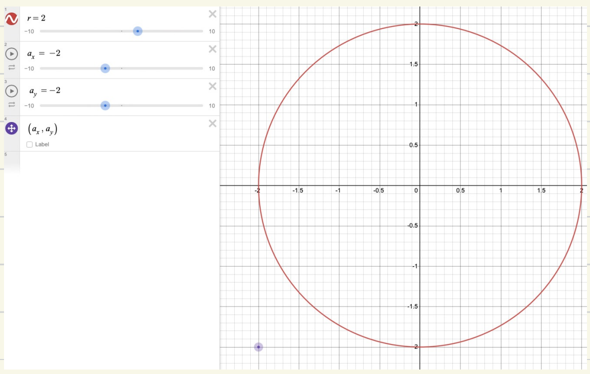
Prove that $\dfrac{AP}{AQ} = \left(\dfrac{AB}{AC}\right)^3$.



(UKMT Maclaurin Olympiad 2021)

First I added in the required generals:
r, a_x and a_y

These were points which didn't seem to be affected by any constraints



New method

Step 1 – Adding variables and creating the scene

8 $c_{1x} = 0$ (−10 to 10)

9 $c_{1y} = 0$ (−10 to 10)

10 $r_1 = 2$ (−10 to 10)

11 $c_{2x} = -1$ (−10 to 10)

12 $c_{2y} = -1$ (−10 to 10)

13 $r_2 = 1$ (−10 to 10)

14 $a_x = -2$ (−10 to 10)

15 $a_y = -2$ (−10 to 10)

16 $b_x = 5$ (−10 to 10)

17 $b_y = -2$ (−10 to 10)

18 $c_x = 0$ (−10 to 10)

19 $c_y = 0$ (−10 to 10)

20 $p_x = 5$ (−10 to 10)

21 $p_y = 10$ (−10 to 10)

22 $q_x = 0$ (−10 to 10)

23 $q_y = -2$ (−10 to 10)

Add in all possible variables which will be used in the scene, for this example there were:
- 2 radii for each circle (r_1, r_2)
- 2 coordinates for the center of each circle, needing 4 variables (C_{1x}, C_{1y}, C_{2x}, C_{2y})
- 5 points: A, B, C, P and Q, needing 10 variables (x and y for each point)

The values I have entered are simply pseudovalues, as the actual problem is meant to be general, these values just help initially to visualise the solution.

Next I used these variables to form the scene, the equations of which you can see below:

Inputting all the points

17 $a = (a_x, a_y)$ Label

18 $b = (b_x, b_y)$ Label

19 $c = (c_x, c_y)$ Label

20 $p = (p_x, p_y)$ Label

21 $q = (q_x, q_y)$ Label

Circles

22 $(x - c_{1x})^2 + (y - c_{1y})^2 = r_1^2$

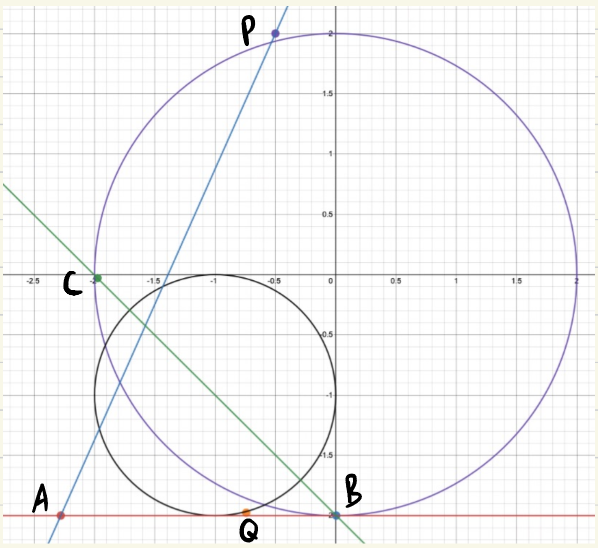
23 $(x - c_{2x})^2 + (y - c_{2y})^2 = r_2^2$

Lines AB, AP and CB respectively

24 $(y - a_y) = \frac{(a_y - b_y)}{(a_x - b_x)}(x - a_x)$

25 $(y - a_y) = \frac{(a_y - p_y)}{(a_x - p_x)}(x - a_x)$

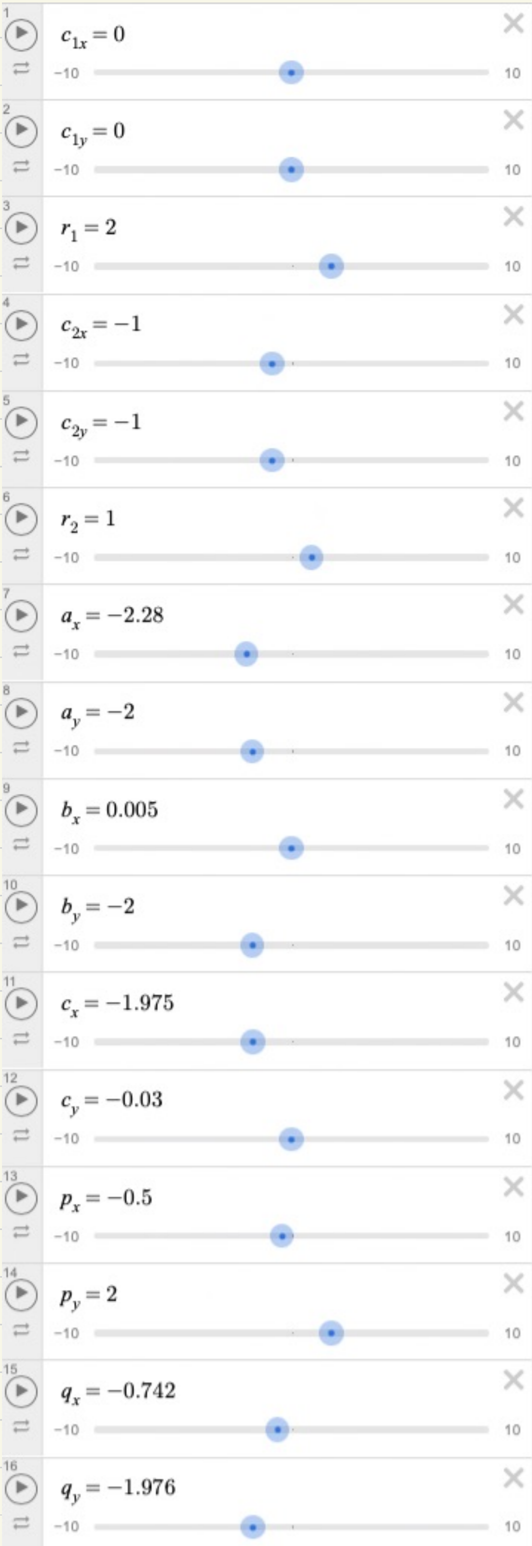
26 $(y - c_y) = \frac{(c_y - b_y)}{(c_x - b_x)}(x - c_x)$



As you can see from the graph above, there is clearly a need for constraints on the points. However the basic 'shape' of the original problem has been constructed.

Step 2 – Adding constraints

In this step I will rewrite almost all variables in terms of a few select variables.



Since C_1 is the base object, we can center it at $(0,0)$

r_1 is a constant, we 2 as its pseudo value

Will come back to

A is not really bound / dependant on other points, so it is a general constant

B passes through C_1, therefore substitute its coordinates in:

$b_x^2 + b_y^2 = r_1^2 \rightarrow b_y = -\sqrt{r_1^2 - b_x^2}$

Similarly for C:

$C_y = -\sqrt{r_1^2 - c_x^2}$

Lies on C_1

$\therefore p_y = \sqrt{r_1^2 - p_x^2}$

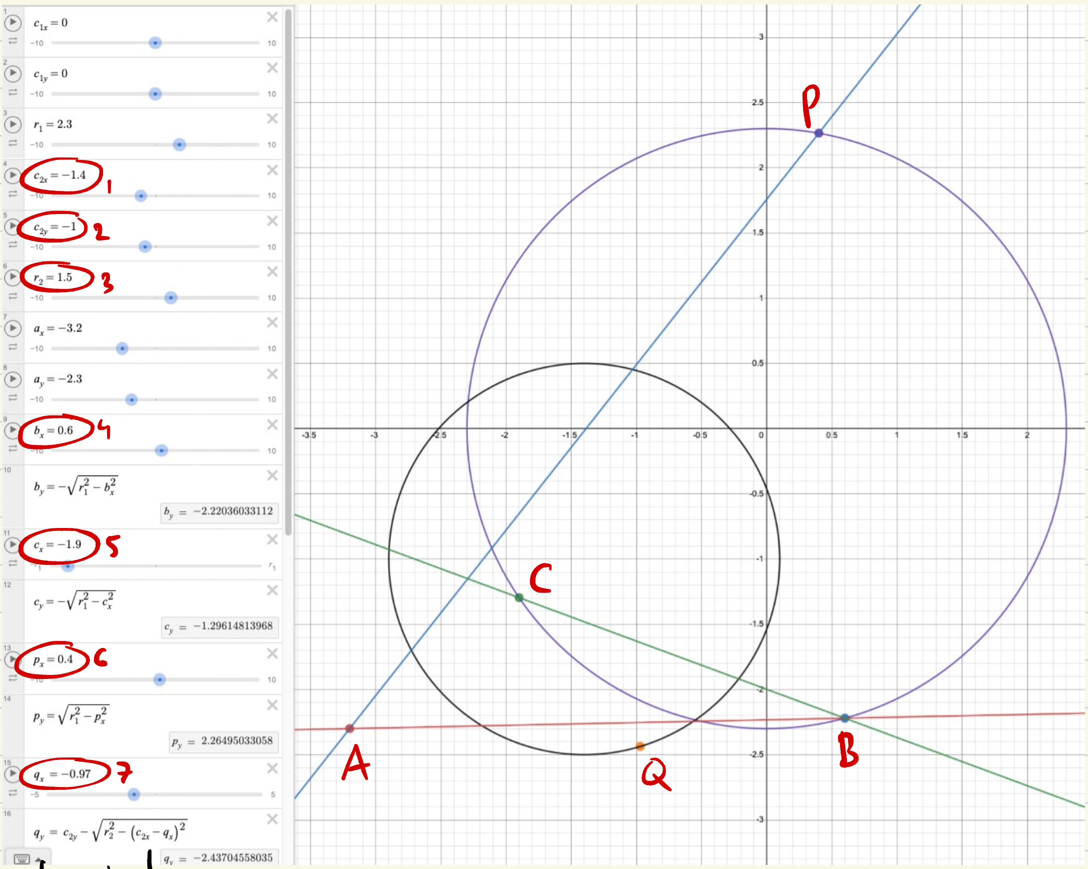
q lies on C_2

$\therefore (C_{2x} - q_x)^2 + (C_{2y} - q_y)^2 = r_2^2$

$\rightarrow C_{2y} - q_y = \sqrt{r_2^2 - (C_{2x} - q_x)^2}$

$q_y = C_{2y} - \sqrt{r_2^2 - (C_{2x} - q_x)^2}$

After applying all of those constrainsts / reductions, we have the shape on the right.



We still need to convert all the constants circled in yellow into terms based on just 1 or 2 others.

To do this we must generate more equations of constraints

1, 2, 3. These are the C_2 constants, and we know that C_2 passes through the points B, C, Q

There are many equations which can do this, but I just want 1 method which can be used universally.

https://web.archive.org/web/20161011113446/http://www.abecedarical.com/zenosamples/zs_circle3pts.html

Matrix determinant method:

$$\begin{bmatrix} C_{2x}^2 + C_{2y}^2 & C_{2x} & C_{2y} & 1 \\ b_x^2 + b_y^2 & b_x & b_y & 1 \\ c_x^2 + c_y^2 & c_x & c_y & 1 \\ q_x^2 + q_y^2 & q_x & q_y & 1 \end{bmatrix} = 0$$

This would involve calculating determinates of a matrix, which would be relatively easy using a library, however I would like a simpler method

This was given by someone on math.stackexchange.

As a programmer this was the best solution for me as there is no division by zero:

For given (x1,y1) , (x2,y2) and (x3,y3) first form A,B,C:

$$A = x_1(y_2 - y_3) - y_1(x_2 - x_3) + x_2y_3 - x_3y_2$$

$$B = (x_1^2 + y_1^2)(y_3 - y_2) + (x_2^2 + y_2^2)(y_1 - y_3) + (x_3^2 + y_3^2)(y_2 - y_1)$$

$$C = (x_1^2 + y_1^2)(x_2 - x_3) + (x_2^2 + y_2^2)(x_3 - x_1) + (x_3^2 + y_3^2)(x_1 - x_2)$$

$$D = (x_1^2 + y_1^2)(x_3y_2 - x_2y_3) + (x_2^2 + y_2^2)(x_1y_3 - x_3y_1) + (x_3^2 + y_3^2)(x_2y_1 - x_1y_2)$$

If A=o then the points are colinear elasewhere using A,B,C you can find center and radius of circle:

$$x_c = -B/2A$$

$$y_c = -C/2A$$

$$r = \sqrt{\frac{B^2 + C^2 - 4AD}{4A^2}}$$

Finally the formula of the circle is:

$$(x - x_c)^2 + (y - y_c)^2 = R^2$$

If we take:

$(x_1, y_1) = (b_x, b_y)$

$(x_2, y_2) = (c_x, c_y)$

$(x_3, y_3) = (q_x, q_y)$

27 $x_1 = b_x$ — $x_1 = 0.6$

28 $y_1 = b_y$ — $y_1 = -2.22036033112$

29 $x_2 = c_x$ — $x_2 = -1.9$

30 $y_2 = c_y$ — $y_2 = -1.29614813968$

31 $x_3 = q_x$ — $x_3 = -0.97$

32 $y_3 = q_y$ — $y_3 = -2.43704558035$

33 $A = x_1(y_2 - y_3) - y_1(x_2 - x_3) + x_2y_3 - x_3y_2$ — $A = 1.99272626363$

34 $B = (x_1^2 + y_1^2)(y_3 - y_2) + (x_2^2 + y_2^2)(y_1 - y_3) + (x_3^2 + y_3^2)(y_2 - y_1)$ — $B = 1.46958163621$

35 $C = (x_1^2 + y_1^2)(x_2 - x_3) + (x_2^2 + y_2^2)(x_3 - x_1) + (x_3^2 + y_3^2)(x_1 - x_2)$ — $C = 3.97522790174$

36 $D = (x_1^2 + y_1^2)(x_3y_2 - x_2y_3) + (x_2^2 + y_2^2)(x_1y_3 - x_3y_1) + (x_3^2 + y_3^2)(x_2y_1 - x_1y_2)$ — $D = -2.59683257616$

4 $c_{2x} = -\frac{B}{2A}$

5 $c_{2y} = -\frac{C}{2A}$

6 $r_2 = \sqrt{\frac{B^2 + C^2 - 4AD}{4A^2}}$

However I had run into an issue: I had defined the point q_y using the fact that it lay on the C_2 circle, but now I am trying to define the circle using this point.

To fix this I will have to use a different constraint for q_y.

Use the fact that it lies on the line AB, with equation:

$$(y - a_y) = \frac{(a_y - b_y)}{(a_x - b_x)}(x - a_x) \quad [(y - y_1) = m(x - x_1)]$$

Substitute q_x and q_y, then rearrange:

$$(q_y - a_y) = \left(\frac{a_y - b_y}{a_x - b_x}\right)(q_x - a_x) \rightarrow q_y = \left(\frac{a_y - b_y}{a_x - b_x}\right)(q_x - a_x) + a_y$$

This also shows an important note for the order of this process:

1. Add in all points / variables
2. Cut down variables – draw lines before circles

$\boxed{\therefore \text{ A point should be constrained to a line wherever possible, not a shape (e.g. circle)}}$

Now we just have to constrain these x coordinates of the points

4. The line AB is meant to be 'touching' C_1.

$\therefore$ Substituting the line AB into C_1, we need $b^2 - 4ac = 0$

① $(y - y_1) = m(x - x_1)$

② $(x-a)^2 + (y-b)^2 = r^2$

Substitute ② into ①, then rearrange into form $ax^2 + bx + c$

① $y = m(x - x_1) + y_1$

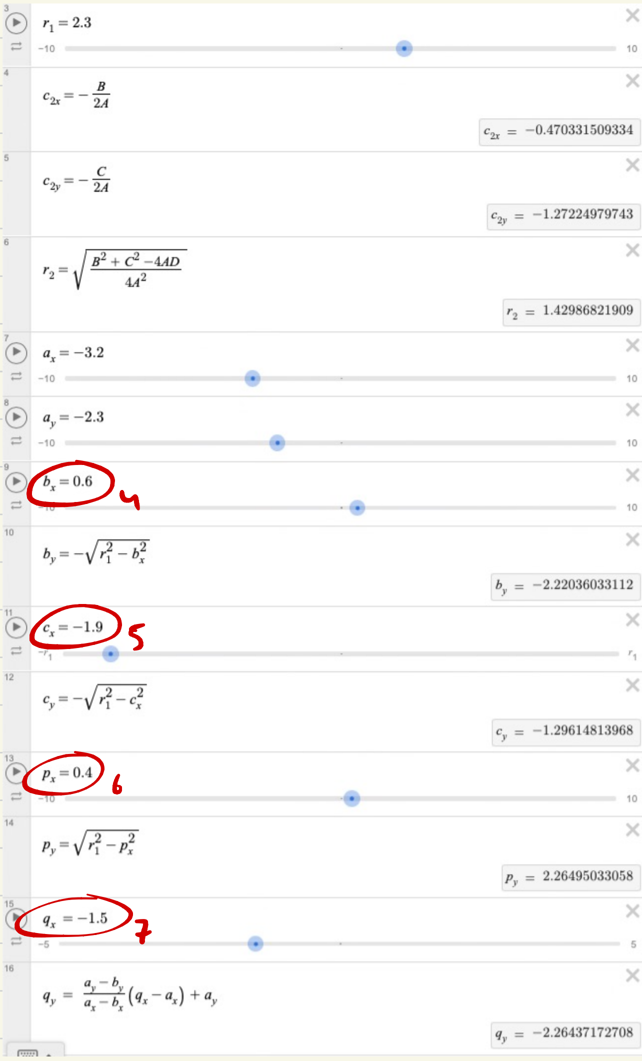
① → ②: $(x-a)^2 + \underbrace{(m(x-x_1)+y_1-b)^2}_{} = r^2$

$\rightarrow mx - mx_1 + y_1 - b \rightarrow mx + (-mx_1 + y_1 - b)$

$x^2 - 2ax + a^2 + m^2x^2 + 2mx(-mx_1+y_1-b) + (-mx_1+y_1-b)^2 = r^2$

$\rightarrow x^2(1+m^2) + x(-2a + 2m(-mx_1+y_1-b)) + (a^2 + (-mx_1+y_1-b)^2 - r^2) = 0$

$\therefore$ $a = 1 + m^2$
$b = -2a + 2m(-mx_1 + y_1 - b)$
$c = a^2 + (-mx_1 + y_1 - b)^2 - r^2$



Using generic formula above: $a = 0$, $b = 0$, $r = r_1$, $y_1 = a_y$, $x_1 = a_x$, $m = \frac{a_y - b_y}{a_x - b_x}$

This approach probably won't work, as the value of b_y is dependent on b_x, however this value of b_x will involve a b_y (from the m value)

We can also use the fact that since AB is tangent to C_1 at B, the gradient of AB will be the negative reciprocal of the gradient from $(C_{1x}, C_{1y}) \rightarrow (b_x, b_y)$

Can have an option when adding line "Is tangent to circle X".

Now we are effectively modifying the position of point a, since b is now a general constant

$$m_{C_1 \rightarrow B} : \left(\frac{C_{1y} - b_y}{C_{1x} - b_x}\right)$$

$$\therefore m_{AB} : -\frac{1}{\left(\frac{C_{1y} - b_y}{C_{1x} - b_x}\right)} \rightarrow$$

Also rewrite line AB in terms of b_x and b_y

AB now tangent

24

$$(y - b_y) = -\frac{1}{\left(\frac{c_{1y} - b_y}{c_{1x} - b_x}\right)}(x - b_x)$$

7 $a_x = -3.2$ −10 10

8 $a_y = -2$ 4 −10 10

We can now rewrite a_y (4) in terms of a_x, using this line equation

$$a_y = -\frac{1}{\left(\frac{C_{1y} - b_y}{C_{1x} - b_x}\right)}(a_x - b_x) + b_y$$

5. We know point c must go through the line AP, so I will have to revert one of my previous constraints of binding C_y to C_1:

I also know that point C is formed by the intersection between C_1 and AP

Continues onto next page $\rightarrow$

① $(y - y_1) = m(x - x_1)$

② $(x-a)^2 + (y-b)^2 = r^2$

Substitute ② into ①, then rearrange into form $ax^2 + bx + c$

① $y = m(x - x_1) + y_1$

①→②: $(x-a)^2 + \underline{(m(x-x_1)+y_1-b)}^2 = r^2$

↪ $mx - mx_1 + y_1 - b \to mx + (-mx_1 + y_1 - b)$

$x^2 - 2ax + a^2 + m^2x^2 + 2mx(-mx_1+y_1-b) + (-mx_1+y_1-b)^2 = r^2$

$\to x^2(1+m^2) + x(-2a + 2m(-mx_1+y_1-b)) + (a^2 + (-mx_1+y_1-b)^2 - r^2) = 0$

$\therefore\ a = 1 + m^2$
$b = -2a + 2m(-mx_1 + y_1 - b)$
$c = a^2 + (-mx_1 + y_1 - b)^2 - r^2$

$a = 0$
$b = 0$
$r = r_1$
$y_1 = a_y$
$x_1 = a_x$

$$m = \left(\frac{a_y - p_y}{a_x - p_x}\right)$$

These are the values for a, b and c (held as u, i and o because I had already used a, b and c)

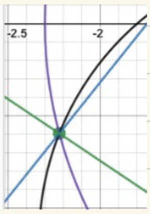
37 $u = 1 + \left(\frac{a_y - p_y}{a_x - p_x}\right)^2$ $u = 2.55362950955$

38 $i = 2\left(\frac{a_y - p_y}{a_x - p_x}\right)\left(-\left(\frac{a_y - p_y}{a_x - p_x}\right)a_x + a_y\right)$ $i = 5.41750726738$

39 $o = \left(-\left(\frac{a_y - p_y}{a_x - p_x}\right)a_x + a_y\right)^2 - r_1^2$ $o = -0.567287021833$

11 $c_x = \frac{-i - \sqrt{i^2 - 4uo}}{2u}$ $c_x = -2.22149305415$

12 $c_y = \frac{(a_y - p_y)}{(a_x - p_x)}(c_x - a_x) + a_y$ $c_y = -0.595792422223$



Then I use the quadratic formula to find c_x to the intersection points of AP and C_1

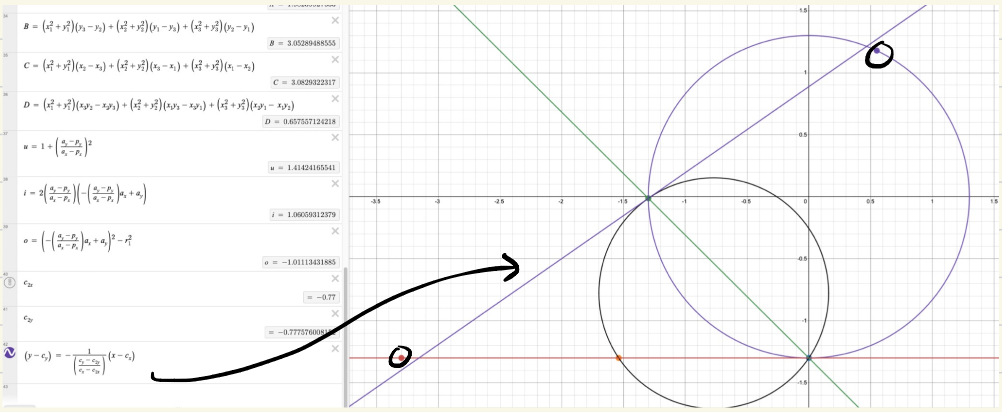
Also changed c_y to be dependant on the line AP rather than to circle C_1 → to follow principle of binding to lines whenever possible

6. To constrain p_x, I must use the fact that AP is meant to be tangent to C_2 at C

$\therefore$ Use similar technique to (4)

$$m_{C_2 \to C} = \frac{C_y - C_{2y}}{C_x - C_{2x}}$$

$$\therefore\ m_{AP \text{ or } AC} = -\frac{1}{\frac{C_y - C_{2y}}{C_x - C_{2x}}}$$



If we use this gradient with the point c, then it is clear how we can eliminate both (a_x) and (b)

a_x is simply the intersection between this line and AB:

$$-\frac{1}{\left(\frac{c_{1y}-b_y}{c_{1x}-b_x}\right)}(a_x-b_x)+b_y=-\frac{1}{\left(\frac{c_y-c_{2y}}{c_x-c_{2x}}\right)}(a_x-c_x)+c_y$$

$$m_{AB} = -\frac{1}{\frac{c_{1y}-b_y}{c_{1x}-b_x}}$$

$$m_{AP} = -\frac{1}{\frac{c_y-c_{2y}}{c_x-c_{2x}}}$$

$$\rightarrow m_{AB}(a_x-b_x)+b_y = m_{AP}(a_x-c_x)+c_y$$

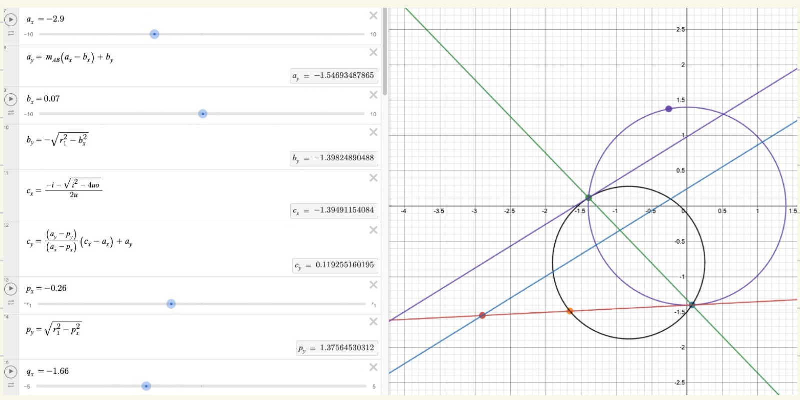
$$m_{AB}a_x - m_{AB}b_x + b_y = m_{AP}a_x - m_{AP}c_x + c_y$$

$$m_{AB}a_x - m_{AP}a_x = m_{AB}b_x - b_y - m_{AP}c_x + c_y$$

$$\therefore a_x = \frac{m_{AB}b_x - b_y - m_{AP}c_x + c_y}{m_{AB}-m_{AP}}$$

Again, this raised an error that a_x could not be defined with m_{AP}, c_x or c_y

I could try and define p_x as the point of intersection between the line from c with the correct gradient, then define a_x as that intersection, however I would probably run into another similar issue.



Overall I managed to reduce it to 4 variables (should ideally be 1).

Will try again, with a better strategy to avoid cycle dependance.

Takeaways:

- Start off by defining all variables required to build the scene
- Then start constraining these, go in order and only build upon what you've already done
- Use principles mentioned previously in this log
- Continue until you only have 1 independant variable (may also be 2 if question is asking about ratios).

Attempt 2

First initialised all points, did not create variables for the circles

I then used the points p, c, b and c, q, b to construct the circles C_1 and C_2 respectively.

Notice how there is no "base shape"

$$A = p_1(c_2 - b_2) - p_2(c_1 - b_1) + c_1 b_2 - b_1 c_2$$
$A = 12.11112$

$$B = (p_1^2 + p_2^2)(b_2 - c_2) + (c_1^2 + c_2^2)(p_2 - \ldots$$
$B = -19.27338288$

$$C = (p_1^2 + p_2^2)(c_1 - b_1) + (c_1^2 + c_2^2)(b_1 - \ldots$$
$C = -16.94724624$

$$D = (p_1^2 + p_2^2)(b_1 c_2 - c_1 b_2) + (c_1^2 + c_2^2)(p \ldots$$
$D = -46.7706476064$

$$C_{1x} = -\frac{B}{2A}$$
$C_{1x} = 0.795689534907$

$$C_{1y} = -\frac{C}{2A}$$
$C_{1y} = 0.699656441353$

$$C_{1r} = \sqrt{\frac{B^2 + C^2 - 4AD}{4A^2}}$$
$C_{1r} = 2.2325847641$

$$(x - C_{1x})^2 + (y - C_{1y})^2 = C_{1r}^2$$

$$E = c_1(q_2 - b_2) - c_2(q_1 - b_1) + q_1 b_2 - b_1 q_2$$
$E = 9.90587$

$$F = (c_1^2 + c_2^2)(b_2 - q_2) + (q_1^2 + q_2^2)(c_2 - \ldots$$
$F = 13.83950887$

$$G = (c_1^2 + c_2^2)(q_1 - b_1) + (q_1^2 + q_2^2)(b_1 - \ldots$$
$G = 6.75633371$

$$H = (c_1^2 + c_2^2)(b_1 q_2 - q_1 b_2) + (q_1^2 + q_2^2)(c \ldots$$
$H = -33.6484645254$

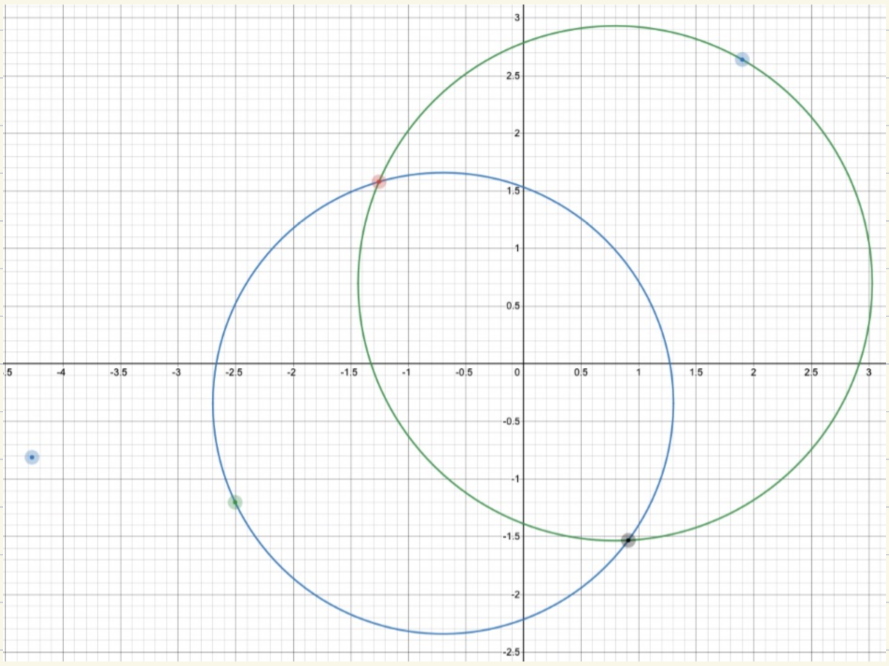
$$C_{2x} = -\frac{F}{2E}$$
$C_{2x} = -0.698550903151$

$$C_{2y} = -\frac{G}{2E}$$
$C_{2y} = -0.341026770491$

$$C_{2r} = \sqrt{\frac{F^2 + G^2 - 4EH}{4E^2}}$$
$C_{2r} = 2.00027331845$

$$(x - C_{2x})^2 + (y - C_{2y})^2 = C_{2r}^2$$

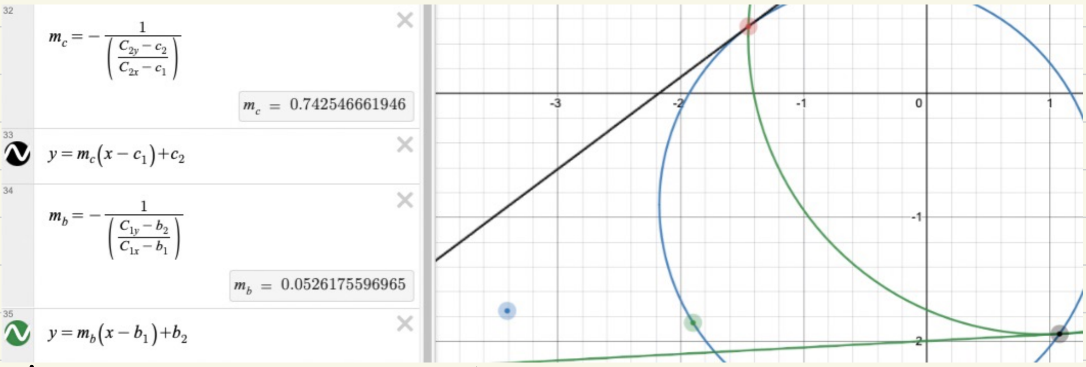
$\boxed{\text{Try and make all shapes in terms of the original points}}$



So now I have correctly bounded these 2 circles.

I have also made sure that C always goes through C_1 and C_2, as it is involved in their formation

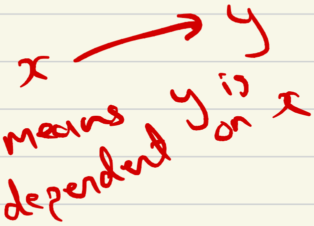
I added the lines AB and AC, except not using the point A.

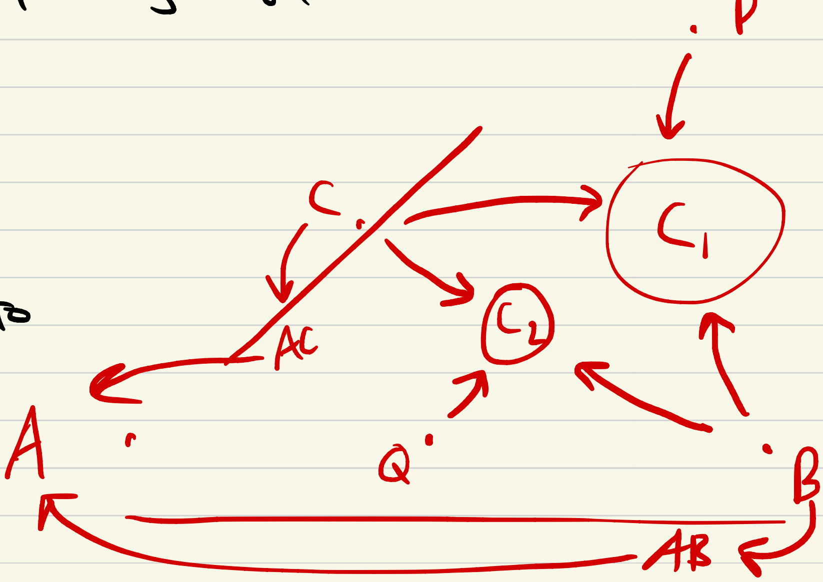


I have constructed them by using the gradient of $-1/z$, where z is the gradient of the radius to B or C.

Now I can lock A to the intersection of these lines, and we a similar technique for Q and P, since these lines will always be tangent to the circles.

To avoid running into cyclical dependancy issues, here is a dependency graph





I just need to make sure that I dont have 2 opposite arrows on 1 connection, e.g. $A \rightleftarrows B$

Intersection of 2 lines

① $(y-y_1) = m_1(x-x_1)$

② $(y-y_2) = m_2(x-x_2)$

Combine:

$m_1(x-x_1)+y_1 = m_2(x-x_2)+y_2$

Isolate x

$m_1x - m_1x_1 + y_1 = m_2x - m_2x_2 + y_2$

$m_1x - m_2x = m_1x_1 - y_1 - m_2x_2 + y_2$

$\therefore x = \dfrac{m_1x_1 - y_1 - m_2x_2 + y_2}{m_1 - m_2}$

$x_1 = b_1$ $\quad y_1 = b_2$

$x_2 = c_1$ $\quad y_2 = c_2$

$m_1 = m_b$ $\quad m_2 = m_c$

6

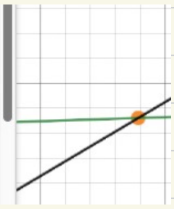
$a_1 = \dfrac{m_b b_1 - b_2 - m_c c_1 + c_2}{m_b - m_c}$

$a_1 = -5.10808755686$

7

$a_2 = m_b(a_1 - b_1) + b_2$

$a_2 = -2.13921537907$



Q and P are just intersection points of a line and a circle:

Q:

① $(y-y_1) = m(x-x_1)$

② $(x-a)^2 + (y-b)^2 = r^2$

Substitute ② into ①, then rearrange into form ax^2+bx+c

① $y = m(x-x_1)+y_1$

①→②: $(x-a)^2 + \underbrace{(m(x-x_1)+y_1-b)^2}_{} = r^2$

$\hookrightarrow mx - mx_1 + y_1 - b \rightarrow mx + (-mx_1+y_1-b)$

$x^2 - 2ax + a^2 + m^2x^2 + 2mx(-mx_1+y_1-b) + (-mx_1+y_1-b)^2 = r^2$

$\rightarrow x^2(1+m^2) + x(-2a+2m(-mx_1+y_1-b)) + (a^2+(-mx_1+y_1-b)^2-r^2) = 0$

$\therefore a = 1+m^2$

$b = -2a + 2m(-mx_1+y_1-b)$

$c = a^2 + (-mx_1+y_1-b)^2 - r^2$

$x_1 = b_1$ $\quad y_1 = b_2$ $\quad m = m_b$

$a = C_{2x}$ $\quad b = C_{2y}$ $\quad r = C_{2r}$

36

$a_q = 1 + m_b^2$

$a_q = 1.00010880603$

37

$b_q = -2C_{2x} + 2m_b(-m_b b_1 + b_2 - C_{2y})$

$b_q = 0.938703550363$

38

$c_q = C_{2x}^2 + (-m_b b_1 + b_2 - C_{2y})^2 - C_{2r}^2$

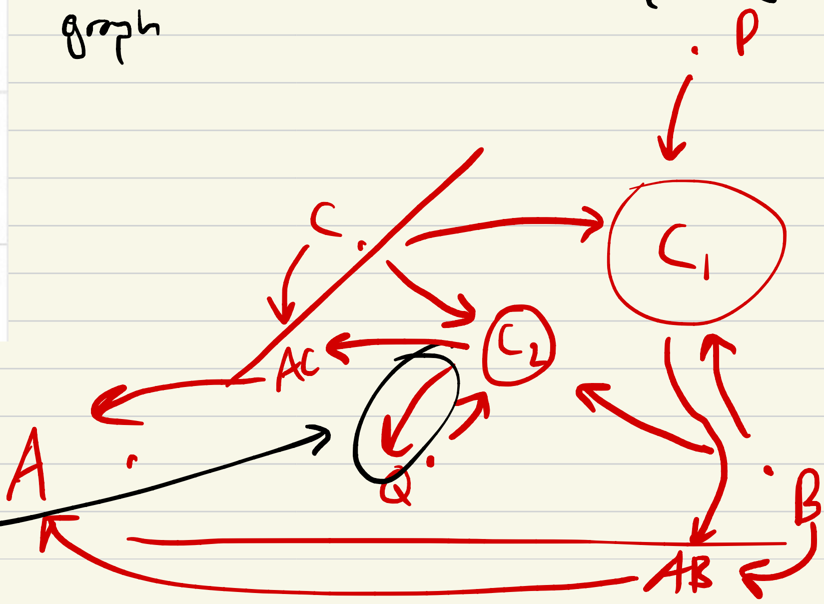
$c_q = -1.76607747835$

39

$y = a_q x^2 + b_q x + c_q$

Solving this quadratic for q_1 does not work, and we can see why if we look at the dependency graph

C_2 is defined using q_1 so we cannot define q_1 as the intersection point



15

$q_2 = m_b(q_1 - b_1) + b_2$

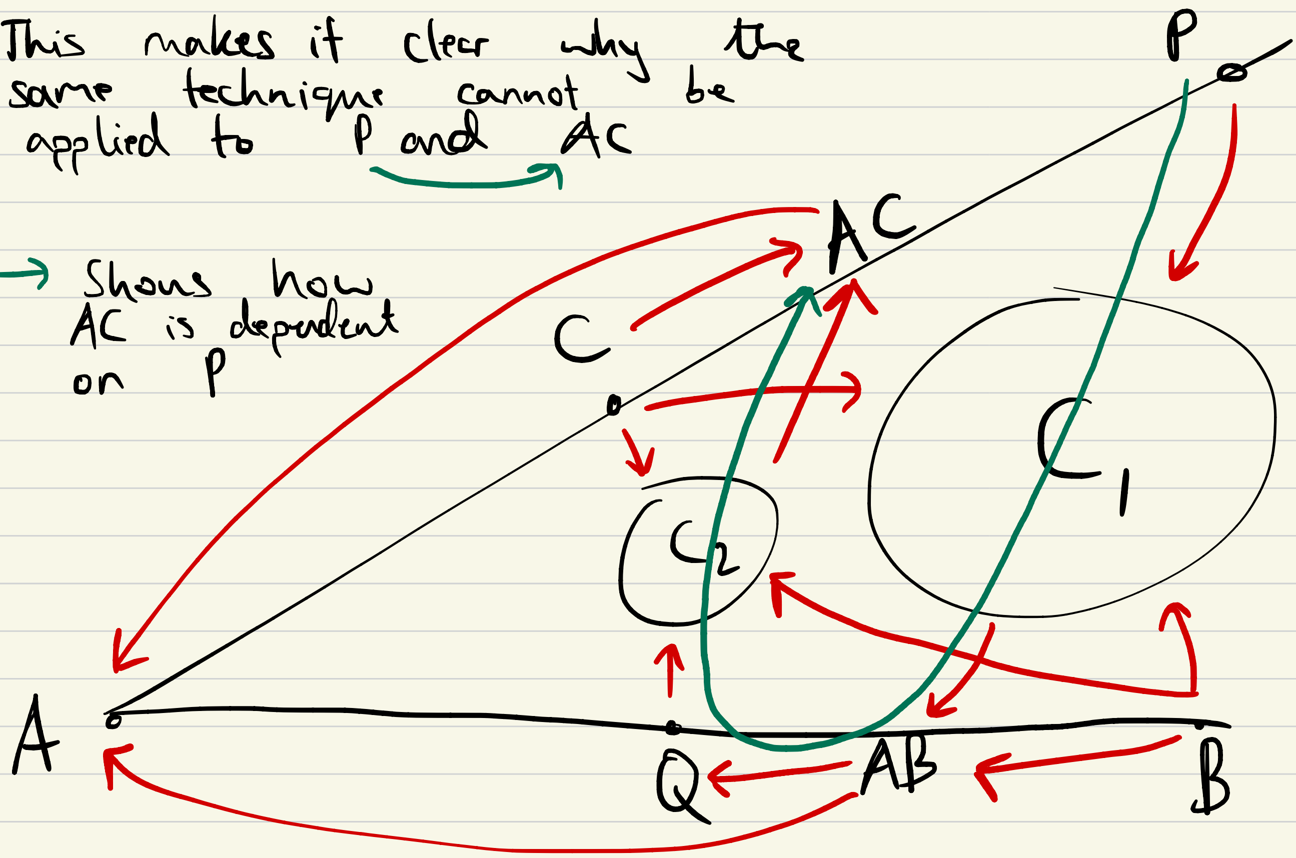
$q_2 = -1.76360307131$

However F can define q_2 to always be on the AB line, since the AB line does not interfere with point q

Dependency map:

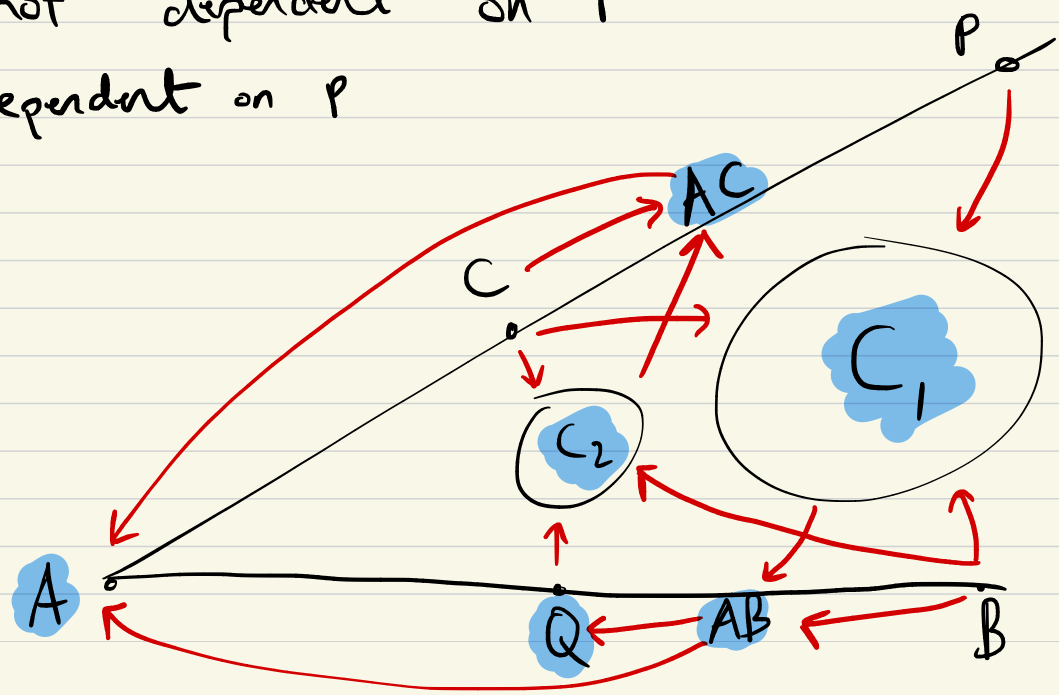
This makes it clear why the same technique cannot be applied to P and AC

→ Shows how AC is dependent on P



It would be useful to know what values are not dependent on P

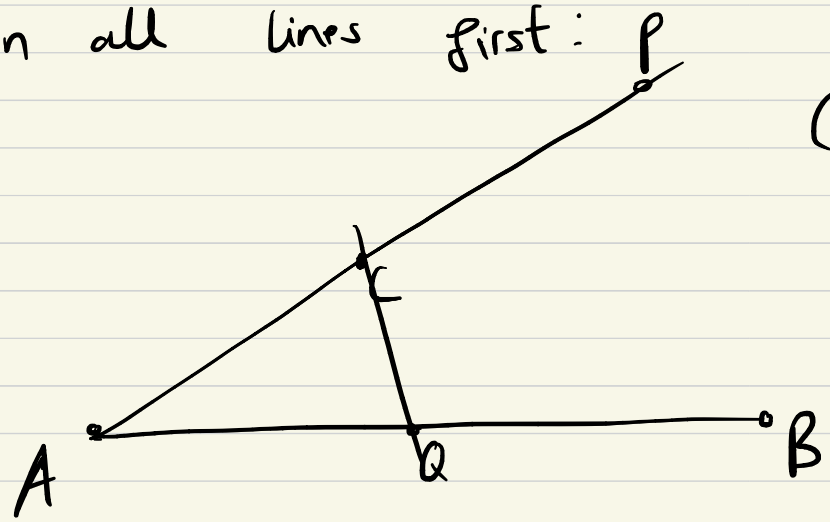
= Dependent on P



∴ Must represent P using points B and C

Possible solution:

Draw in all lines first:



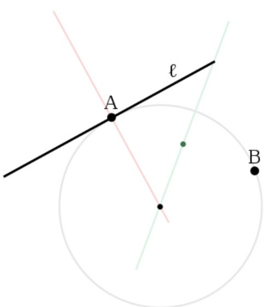
(Constrict C to lie on AP and Q to lie on AB)

Then draw in circles C_1 and C_2:

C_1: Constructed using information that it passes through C and P, and that it is tangent to AB at B

C_2: Constructed using information that it passes through Q and B, and it tangent to C.

Here is a geometric version, not using a single formula. Start with the points A and B and a line ℓ through A (see the figure below).



Construct the perpendicular line to ℓ through A (the red line). Construct the perpendicular bisectors between A and B (the green line, the green dot is the midpoint of A and B). The intersection of both constructed lines is the circle's center. The readius is the distance of the center to A.

You can translate every step into a formula to solve it numerically if necessary.

This solution essentially follows M. Winter's solution algebraically rather than geometrically.

Plugging $(1,7)$ into $(x-a)^2+(y-b)^2=r^2$, we get

$$(1-a)^2+(7-b)^2=r^2.$$

In the same manner, $(-6,0)$ gives

$$(-6-a)^2+b^2=r^2.$$

Eliminating r^2 between these, multiplying out the binomials, then collecting the like powers of a and b, these reduce to

$$a+b=1,$$

the equation of the perpendicular bisector of the segment between the given points.

The slope of the given tangent line is $2/9$, so the slope of the line through the center of the circle and $(1,7)$ is $-9/2$. The equation of this line is

$$(b-7)=(-9/2)(a-1).$$

These two lines intersect at the center of the circle. Taking $b=1-a$, plugging into this last equation, and solving for a, we get

$$a=3$$

and then

$$b=-2.$$

Putting these back into either of the first two displays gives

$$r^2=85.$$

So the equation you requested is

$$(x-3)^2+(y+2)^2=85.$$

Related **A line by y=x is tangent to the circle at point (2,2). What is the equation of the circle?**

A line by y=x is tangent to the circle at point (2,2). What is the equation of the circle?

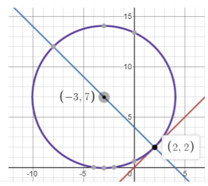
y = 4 - x is the perpendicular to that point, so (h, k) = (h, 4-h) that the center must lie on.

$r^2 = (2 - h)^2 + (2 - h)^2 = 2(2-h)^2$

$(x - h)^2 + (y + h - 4)^2 = 2(2-h)^2$

You choose h

Example: h = -3



https://math.stackexchange.com/questions/2464364/how-can-i-find-the-equation-of-a-circle-given-two-points-and-a-tangent-line-thro

Circle can be made with 1 point and 1 point with tangent

Method Ia: Same thing, new equation

This is very similar to Method I above, except that we use the equation

$$(x-h)^2+(y-k)^2=r^2 \tag{i}$$

where (h, k) is the center of the circle and r is the radius.

As above, if points $A(x_1, y_1)$ and $B(x_2, y_2)$ are on the circle, A and B must satisfy the above equation. Therefore, we can write:

$$(x_1-h)^2+(y_1-k)^2=r^2 \tag{ii}$$

$$(x_2-h)^2+(y_2-k)^2=r^2 \tag{iii}$$

Given the equation of the line tangent to the circle, that is,

$$ax+by+d=0 \tag{iv}$$

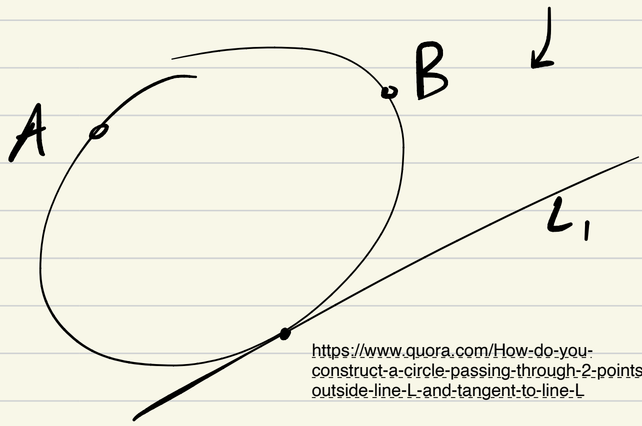
we can solve for the distance between the line and the center:

$$r=\frac{|ax_1+by_1+d|}{\sqrt{a^2+b^2}} \tag{v}$$

As above, you can substitute $A(x_1, y_1)=(1,7), B(x_2, y_2)=(-6,0), a=2, b=-9$, and $d=61$ into equations (ii), (iii), and (v) to solve for h, k, and r, and then plug those values back into equation (i) to get the solution to your circle. In your particular case, however, this formula for r happens to yield 0; for that reason, although it's much messier, you may wish to stick with method I for this particular problem.

Call the points **A** and **B**. And the given line L_1.

1. Draw the line L_2, through **A** and **B**, and assuming L_1 and L_2 are not parallel, call their intersection point **X**.
2. With diameter **AB**, draw circle C_1.
3. Draw tangent **T** to C_1 and through **X** with point of contact **Y**.
4. Construct circle C_2 with centre **X** and through **Y**.
5. Call the intersections of C_2 and L_1, **P** and **Q**.
6. The sought after circle passes through points A, B and P.
7. There is another circle. Find it.



https://www.quora.com/How-do-you-construct-a-circle-passing-through-2-points-outside-line-L-and-tangent-to-line-L

Attempt 3

I ran into the same dependancy issue last time, so this time I will strictly follow the order:

1. Points
2. Lines
3. Shapes (e.g. circles)

1.

Added points into positions which 'seemed right'.

Continues →

Added lines

AB $\quad (y-a_2) = \left(\frac{a_2-b_2}{a_1-b_1}\right)(x-a_1)$

AC $\quad (y-a_2) = \left(\frac{a_2-c_2}{a_1-c_1}\right)(x-a_1)$

BC $\quad (y-b_2) = \left(\frac{b_2-c_2}{b_1-c_1}\right)(x-b_1)$

Line constraints:

For each line there may be more points on the line apart from just the 2 points the line is defined with

$$p_2 = \left(\frac{a_2-c_2}{a_1-c_1}\right)(p_1-a_1)+a_2$$

$p_2 = 2.85517241379$

$q_1 = -0.35$

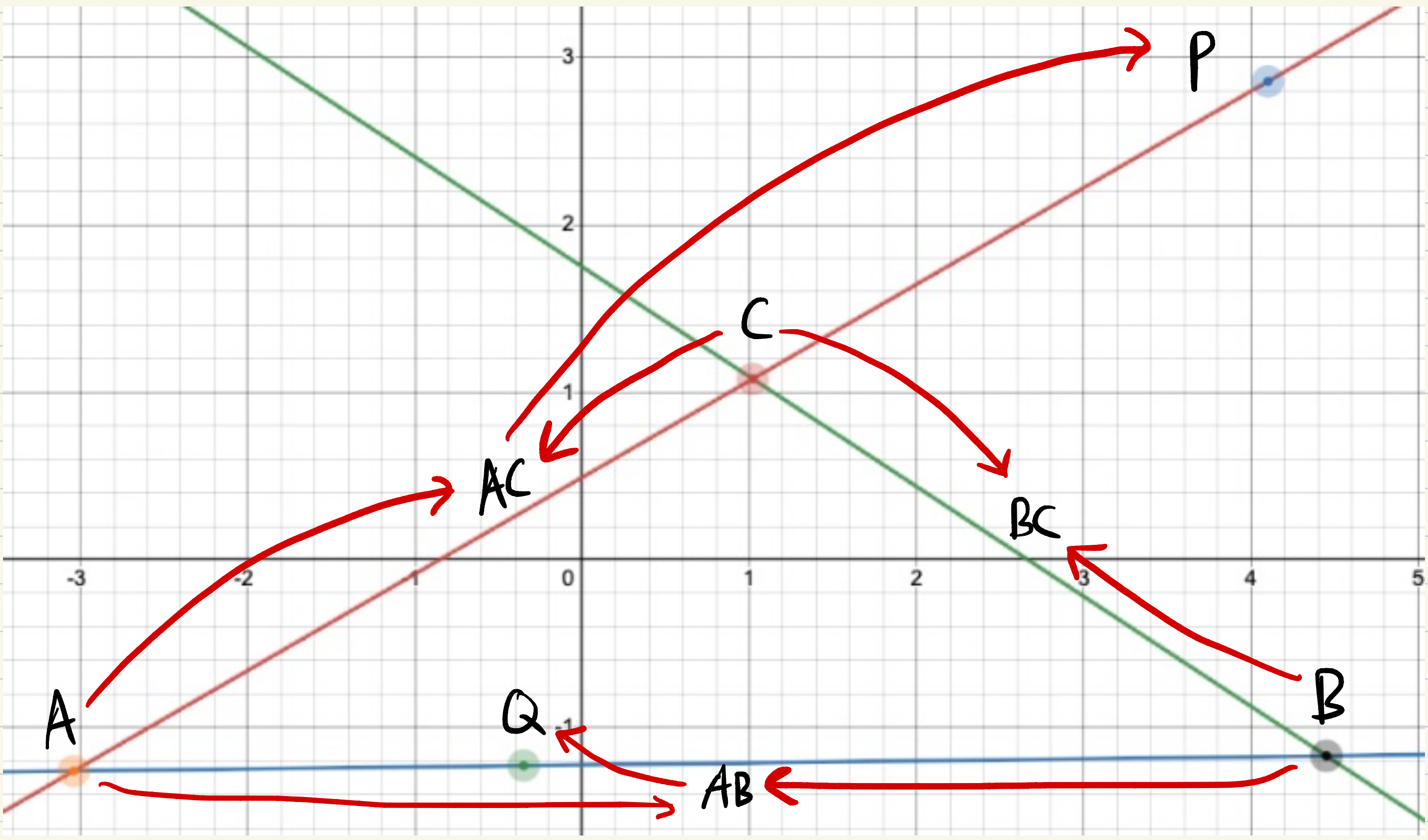
-10 10

$$q_2 = \left(\frac{a_2-b_2}{a_1-b_1}\right)(q_1-a_1)+a_2$$

$q_2 = -1.22767690254$

In this case the points P and Q were on the lines AC and AB respectively, so I rewrote p_2 and q_2 to be in terms of p_1 and q_1 with the lines of AC and AB

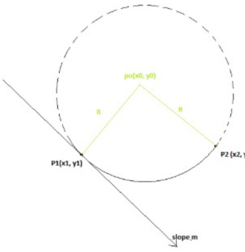
Here is the dependency graph:



Now will create circles C_1 and C_2, using 1 free-standing point and 1 point with a tangent through it.

Two points P1(x1,y1) and P2(x2,y2) are known. In addition, a line slope passing through P1 is known. The aim is to construct a circle (or circular arc) that it passes through both P1 and P2 and it is tangent to the line. How can we find the center of the circle and the radius with those given information? (2 points and 1 tangent)

As it can be seen in the figure, the radius and the center (in green) are unknown and aim is to find the center point and radius (or curvature)



https://math.stackexchange.com/questions/996582/finding-circle-with-two-points-on-it-and-a-tangent-from-one-of-the-points

Point 1 : (x_1, y_1)

Tangent through point 1: $(y-y_1) = m(x-x_1)$

$$y - y_1 = mx - mx_1$$

$$-mx + y = -mx_1 + y_1$$

$$mx - y = mx_1 - y_1$$

$$\therefore a = m, \quad b = -1, \quad c = mx_1 - y_1$$

Point 2 : (x_2, y_2)

The line from the centre to P1 is perpendicular to the given line; and the centre is equidistant from P1 and P2. That gives you two lines that the centre must lie on.

Let the given line be $ax + by = c$. If you just have its slope m, you can take $a = -m$ and $b = 1$. (And $c = ax_1 + by_1$, but we're not going to use that.) A perpendicular line has the form $bx - ay =$ something, and the something can be determined by using the fact that the line is supposed to go through P1. Thus our first line is

$$bx - ay = bx_1 - ay_1 \tag{1}$$

$\rightarrow bx - ay = bx_1 - ay_1$

The line through P1 and P2 is

$$(y_1 - y_2)x - (x_1 - x_2)y = x_2y_1 - x_1y_2$$

A perpendicular line has the form

$$(x_1 - x_2)x + (y_1 - y_2)y = \text{something}$$

$\rightarrow (x_1 - x_2)x + (y_1 - y_2)y = \text{something}$

and the something can be determined by the fact that we want the line to go through the midpoint of P1P2, that is, $(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2))$. Thus our second line is

$$(x_1 - x_2)x + (y_1 - y_2)y = \tfrac{1}{2}(x_1^2 + y_1^2 - x_2^2 - y_2^2) \tag{2}$$

$\rightarrow (x_1 - x_2)x + (y_1 - y_2)y = \frac{1}{2}(x_1^2 + y_1^2 - x_2^2 - y_2^2)$

The equations (1) and (2) together form a linear system which can be solved by standard methods, e.g., Cramer's rule. The solution is the centre of the desired circle, and the radius, if needed, is the distance from that point to P1.

$\therefore$ ① $bx - ay = bx_1 - ay_1$

② $(x_1 - x_2)x + (y_1 - y_2)y = \frac{1}{2}(x_1^2 + y_1^2 - x_2^2 - y_2^2)$

Solve system to find center coordinates of produced circle

②

$$(x_1 - x_2)x + (y_1 - y_2)y = \frac{1}{2} \times (x_1^2 + y_1^2 - x_2^2 - y_2^2)$$

Solve for x

$$x = \frac{x_1^2 + y_1^2 - x_2^2 - y_2^2 - 2yy_1 + 2yy_2}{2x_1 - 2x_2}$$

$$(x_1 - x_2)x + (y_1 - y_2)y = \frac{1}{2} \times (x_1^2 + y_1^2 - x_2^2 - y_2^2)$$

Solve for y

$$y = \frac{x_1^2 + y_1^2 - x_2^2 - y_2^2 - 2xx_1 + 2xx_2}{2y_1 - 2y_2}$$

①

$$x = \frac{bx_1 - ay_1 + ay}{b} = \frac{-x_1 - my_1 + my}{-1}$$

$\therefore$ Solution for y:

$$\frac{x_1^2 + y_1^2 - x_2^2 - y_2^2 - 2yy_1 + 2yy_2}{2x_1 - 2x_2} = \frac{-x_1 - my_1 + my}{-1}$$

① $bx - ay = bx_1 - ay_1$

$$-ay = bx_1 - ay_1 - bx$$

$$y = \frac{bx_1 - ay_1 - bx}{-a} = \frac{-x_1 - my_1 + x}{-m}$$

$\therefore$ Solution for x:

$$\frac{x_1^2 + y_1^2 - x_2^2 - y_2^2 - 2xx_1 + 2xx_2}{2y_1 - 2y_2} = \frac{-x_1 - my_1 + x}{-m}$$

Solution for x:

$$\frac{x_1^2 + y_1^2 - x_2^2 - y_2^2 - 2xx_1 + 2xx_2}{2y_1 - 2y_2} = \frac{-x_1 - my_1 + x}{-m}$$

$$\frac{x_1^2 + y_1^2 - x_2^2 - y_2^2}{2y_1 - 2y_2} - 2x\left(\frac{x_1 - x_2}{2y_1 - 2y_2}\right) = \frac{-x_1 - my_1}{-m} + \frac{x}{-m}$$

$$2x\left(\frac{x_1 - x_2}{2y_1 - 2y_2}\right) + x\left(\frac{1}{-m}\right) = \frac{x_1^2 + y_1^2 - x_2^2 - y_2^2}{2y_1 - 2y_2} + \frac{x_1 + my_1}{-m}$$

$$x\left(\frac{2x_1 - 2x_2}{2y_1 - 2y_2} + \frac{1}{-m}\right) = \ldots$$

$$\therefore\ x = \left(\frac{x_1^2 + y_1^2 - x_2^2 - y_2^2}{2y_1 - 2y_2} + \frac{x_1 + my_1}{-m}\right) \div \left(\frac{2x_1 - 2x_2}{2y_1 - 2y_2} + \frac{1}{-m}\right)$$

Solution for y:

$$\frac{x_1^2+y_1^2-x_2^2-y_2^2-2yy_1+2yy_2}{2x_1-2x_2}=\frac{-x_1-my_1+my}{-1}$$

$$\frac{x_1^2+y_1^2-x_2^2-y_2^2}{2x_1-2x_2}+y\left(\frac{-2y_1+2y_2}{2x_1-2x_2}\right)=x_1+my_1-y(m)$$

$$y\left(\frac{-2y_1+2y_2}{2x_1-2x_2}\right)+y(m)=x_1+my_1-\frac{x_1^2+y_1^2-x_2^2-y_2^2}{2x_1-2x_2}$$

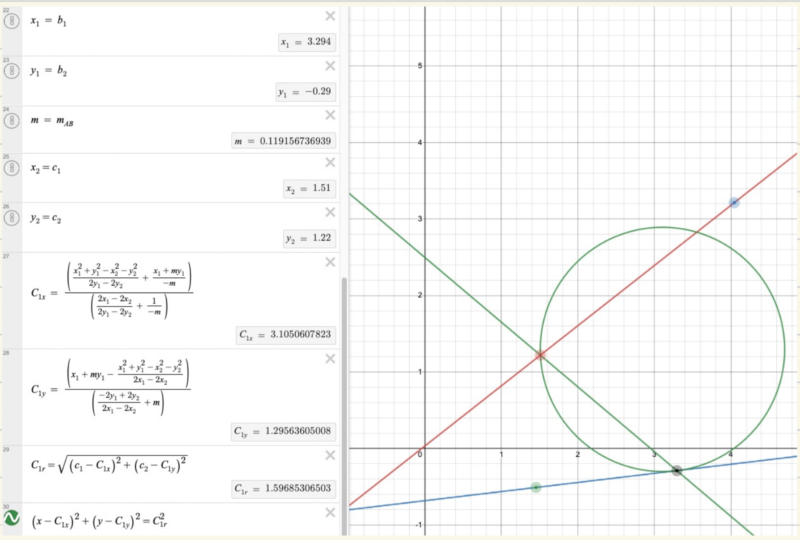
$$y\left(\frac{-2y_1+2y_2}{2x_1-2x_2}+m\right)=\ldots$$

$$y=\left(x_1+my_1-\frac{x_1^2+y_1^2-x_2^2-y_2^2}{2x_1-2x_2}\right)\div\left(\frac{-2y_1+2y_2}{2x_1-2x_2}+m\right)$$

Where (x_1, y_1) and (x_2, y_2) are points on a circle and m is the gradient of the tangent to the circle at (x_1, y_1)

$\therefore$ These are the center coordinates of C_1 if we use (x_1, y_1) as B, (x_2, y_2) as C and m as m_{AB}

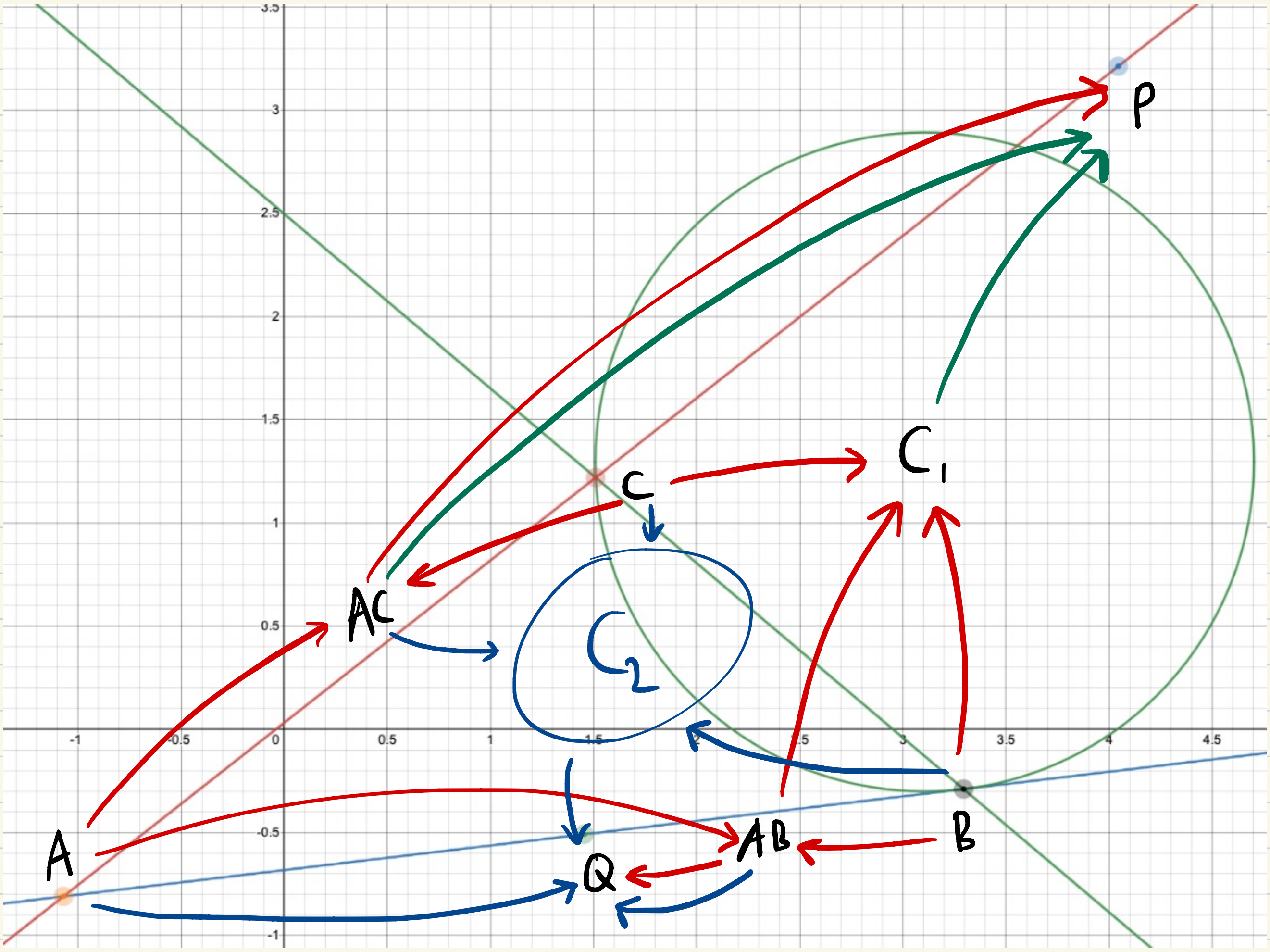
To find the radius just calculate the distance from this center point to one of the points



I have now constructed C_1 using only:

- Point B
- Point C
- Line AB

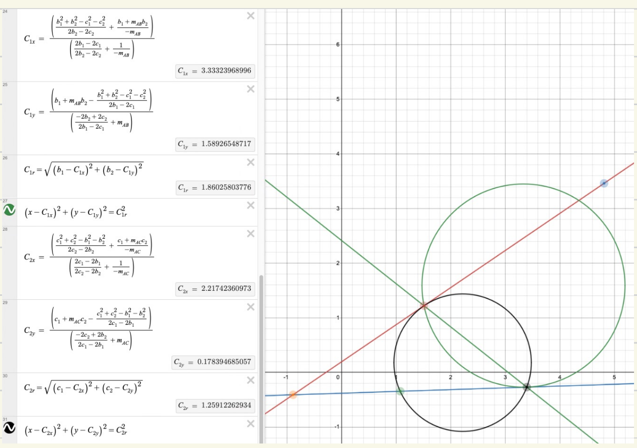
Dependency graph currently:



* Excluded BC as it is simply an 'output object' (does not control anything else)

⟶ = Proposed dependencies after adding C_2:
- Use B, C and m_{AC} to construct C_2
- Then constrain Q to the intersection between C_2 and AB

⟶ = Proposed dependency to constrain P:
- Bound P to the intersection between C_1 and AC



1a. Added in C_2 according to plan

1b. Constrained point Q to lie on the 'left' intersection between AB and C_2.

I used the line-circle quadratic intersection formula which I derived earlier (scroll/go to attempt 2).

$a_q = 1 + m_{AB}^2$

$a_q = 1.0002782929$

$b_q = -2C_{2x} + 2m_{AB}\left(-m_{AB}a_1 + a_2 - C_{2y}\right)$

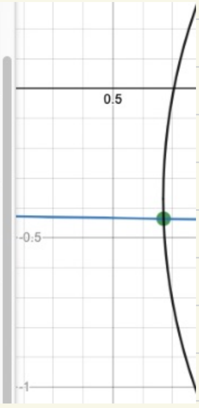
$b_q = -5.17613577859$

$c_q = C_{2x}^2 + \left(-m_{AB}a_1 + a_2 - C_{2y}\right)^2 - C_{2r}^2$

$c_q = 3.01781872125$

$q_1 = \frac{-b_q - \sqrt{b_q^2 - 4a_q c_q}}{2a_q}$

$q_1 = 0.669695697535$



$a_p = 1 + m_{AC}^2$

$a_p = 1.54691184424$

$b_p = -2C_{1x} + 2m_{AC}\left(-m_{AC}a_1 + a_2 - C_{1y}\right)$

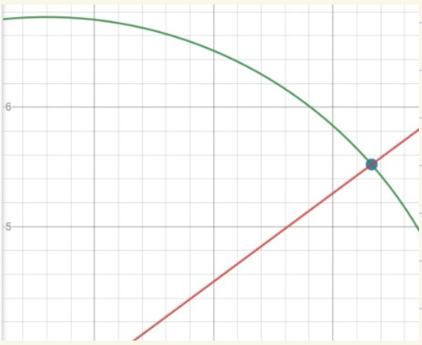
$b_p = -13.6657784208$

$c_p = C_{1x}^2 + \left(-m_{AC}a_1 + a_2 - C_{1y}\right)^2 - C_{1r}^2$

$c_p = 17.1082117193$

$p_1 = \frac{-b_p + \sqrt{b_p^2 - 4a_p c_p}}{2a_p}$

$p_1 = 7.3242321973$



2. Also constrained point P to the right intersection between AC and C_1.

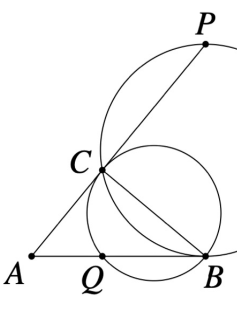
I now believe all constraints have been satisfied. Below are some takeaways from attempt 3:

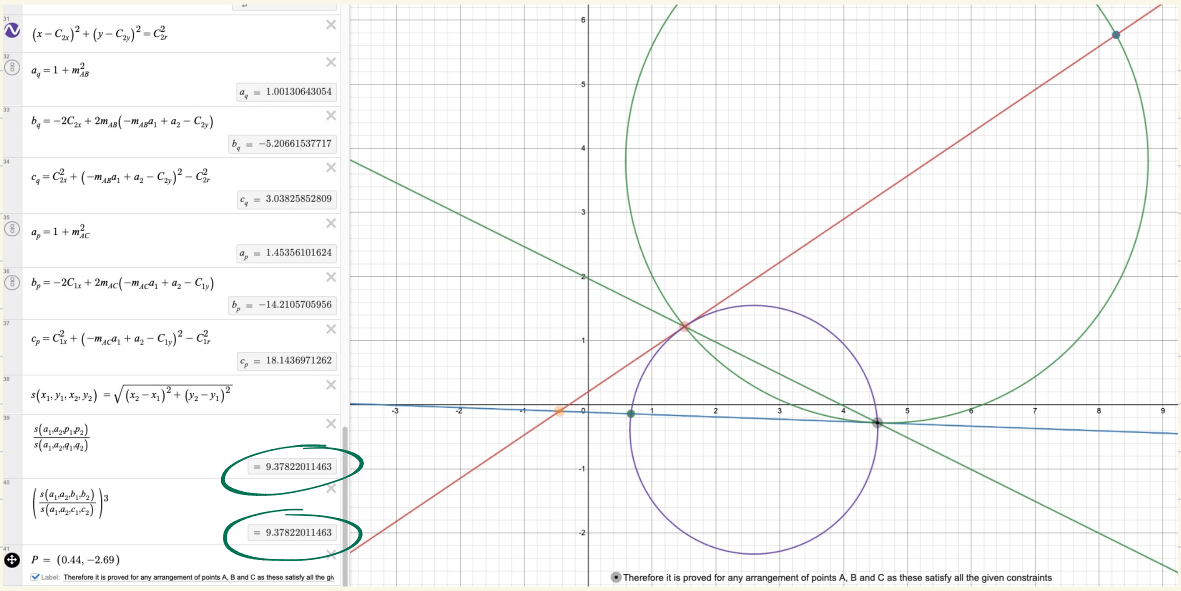
- Points → Lines → Shapes seems to be correct order
- For circles there are numerous ways to form the equation, e.g. with 3 points, or 2 points + tangent, or even 2 points + radius. Will have to offer all of them in the finished app
- When constraining to the intersection between a line and a circle — need to ask user whether they want left/right/top/bottom intersection (this corresponds with the ± from quadratic formula.

Now that we have the model, we can answer the question

3. The diagram shows a triangle ABC. A circle touching AB at B and passing through C cuts the line AC at P. A second circle touching AC at C and passing through B cuts the line AB at Q.

Prove that $\frac{AP}{AQ} = \left(\frac{AB}{AC}\right)^3$.





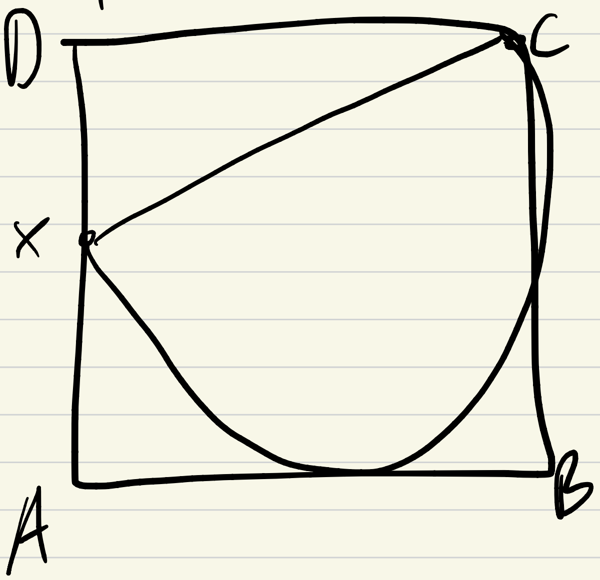
I defined a distance function $s(x_1, y_1, x_2, y_2)$

The distances are always equal no matter however you change points A, B and C

https://www.desmos.com/calculator/eq3vxx6wcs You can try this model here

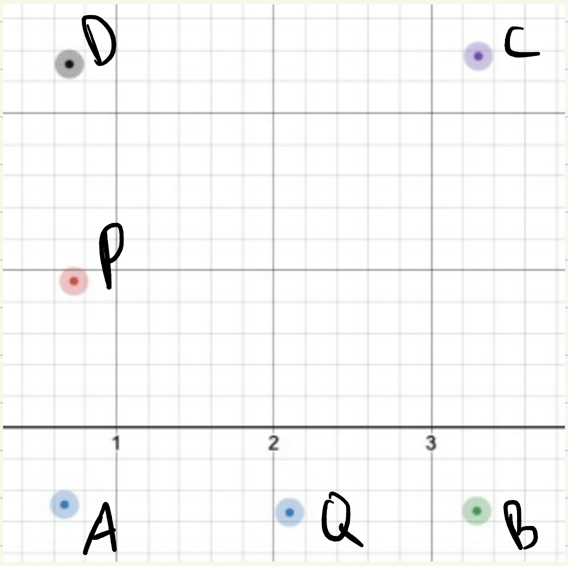
The dependency graph plays a crucial role in the ordering of constraints, so I believe the entire scene should be built around a dependency graph (stored in a graph data structure), at least in the background.

Example 2:



Square ABCD contains a semicircle with diameter CX, where X is a point on AD. This semicircle touches AB, find the ratio AX : XD

The reason I picked this example is due to it having vertical lines, as well as a (semi) circle tangent to a horizontal line, which would not work normally with the current formulas.

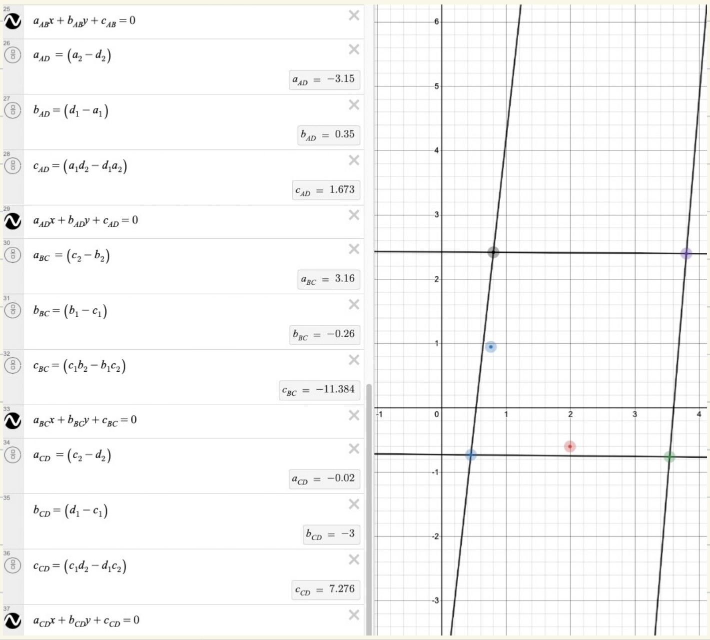


First I initialised the points A, B, C, D, P and Q (tangental point to circle)

I positioned them roughly in the correct position

Then I generated the lines AB, AD, BC and CD.

However instead of using $(y-y_1) = m(x-x_1)$, I opted to use $ax + by + c = 0$, as it handles horizontal and vertical lines much better

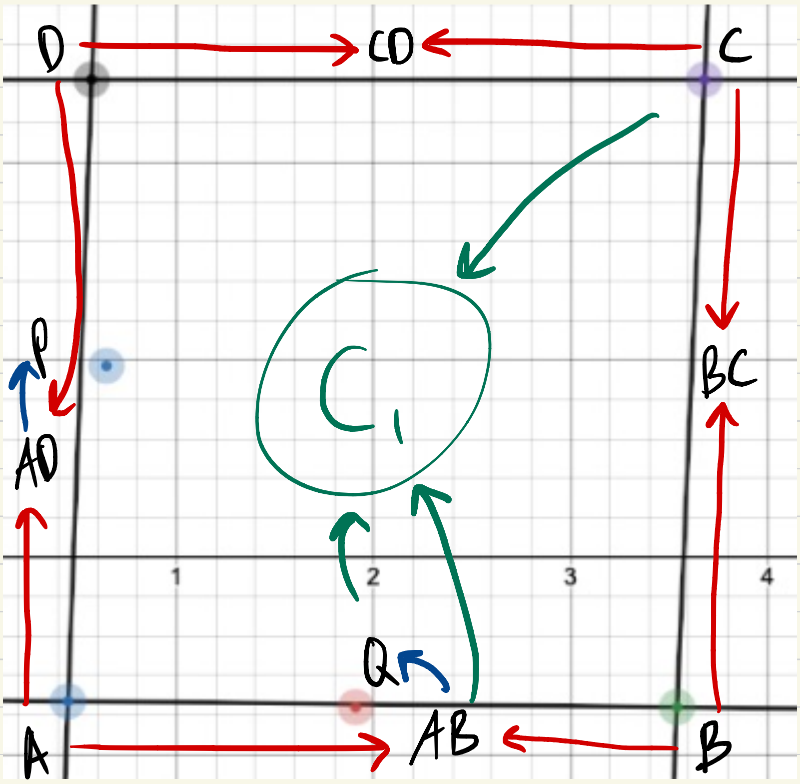


$ax + by + c = 0$
$a = (y_1 - y_2)$ $b = (x_2 - x_1)$
$c = (x_1 y_2 - x_2 y_1)$

Dependency graph →

→ After constraining P and Q to AD and AB respectively

→ Create (semi) circle constrained with point C, Q and tangent at AB. (Not using P as that is the 'dependent variable')



1. Must rearrange $ax + by + c = 0$ to $y = \dots$

$$ax + by = -c \qquad \text{or} \qquad ax + by = -c$$
$$by = -c - ax \qquad\qquad ax = -c - by$$
$$y = \frac{-c - ax}{b} \qquad\qquad x = \frac{-c - by}{a}$$

$\boxed{\text{Use } y \text{ constraining when point lies on a horizontal line, and } x \text{ constraining when point lies on a vertical line}}$

$$p_1 = \frac{-c_{AD} - b_{AD}p_2}{a_{AD}}$$
$p_1 = 0.532587417219$

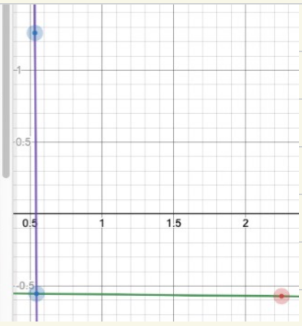
$p_2 = 1.26$

−10 10

$q_1 = 2.25$

−10 10

$$q_2 = \frac{-c_{AB} - a_{AB}q_1}{b_{AB}}$$
$q_2 = -0.571690213869$



Using above principle, constrained p_1 to AD and q_2 to AB.

2.

The line from the centre to P1 is perpendicular to the given line; and the centre is equidistant from P1 and P2. That gives you two lines that the centre must lie on.

Let the given line be $ax + by = c$. If you just have its slope m, you can take $a = -m$ and $b = 1$. (And $c = ax_1 + by_1$, but we're not going to use that.) A perpendicular line has the form $bx - ay =$ something, and the something can be determined by using the fact that the line is supposed to go through P1. Thus our first line is

$$bx - ay = bx_1 - ay_1 \qquad (1)$$

The line through P1 and P2 is

$$(y_1 - y_2)x - (x_1 - x_2)y = x_2y_1 - x_1y_2$$

A perpendicular line has the form

$$(x_1 - x_2)x + (y_1 - y_2)y = \text{something}$$

and the something can be determined by the fact that we want the line to go through the midpoint of P1P2, that is, $(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2))$. Thus our second line is

$$(x_1 - x_2)x + (y_1 - y_2)y = \tfrac{1}{2}(x_1^2 + y_1^2 - x_2^2 - y_2^2) \qquad (2)$$

The equations (1) and (2) together form a linear system which can be solved by standard methods, e.g., Cramer's rule. The solution is the centre of the desired circle, and the radius, if needed, is the distance from that point to P1.

Need to re-derive equation for circle given 2 points and tangent $ax + by + c$ passes through (x_1, y_1)

$\therefore$ System:

1. $bx - ay = bx_1 - ay_1$
2. $(x_1 - x_2)x + (y_1 - y_2)y = \frac{1}{2}(x_1^2 + y_1^2 - x_2^2 - y_2^2)$

Solve for x and y to get center of circle

$$\boxed{\text{Solution for } x \quad x = \left(\frac{x_1^2 + y_1^2 - x_2^2 - y_2^2}{2y_1 - 2y_2} - \frac{-bx_1 + ay_1}{a}\right) \div \left(\frac{b}{a} + \frac{2x_1 - 2x_2}{2y_1 - 2y_2}\right)}$$

$$\boxed{\text{Solution for } y \quad y = \left(\frac{x_1^2 + y_1^2 - x_2^2 - y_2^2}{2x_1 - 2x_2} - \frac{bx_1 - ay_1}{b}\right) \div \left(\frac{a}{b} + \frac{2y_1 - 2y_2}{2x_1 - 2x_2}\right)}$$

$$bx - ay = bx_1 - ay_1 \qquad 1$$

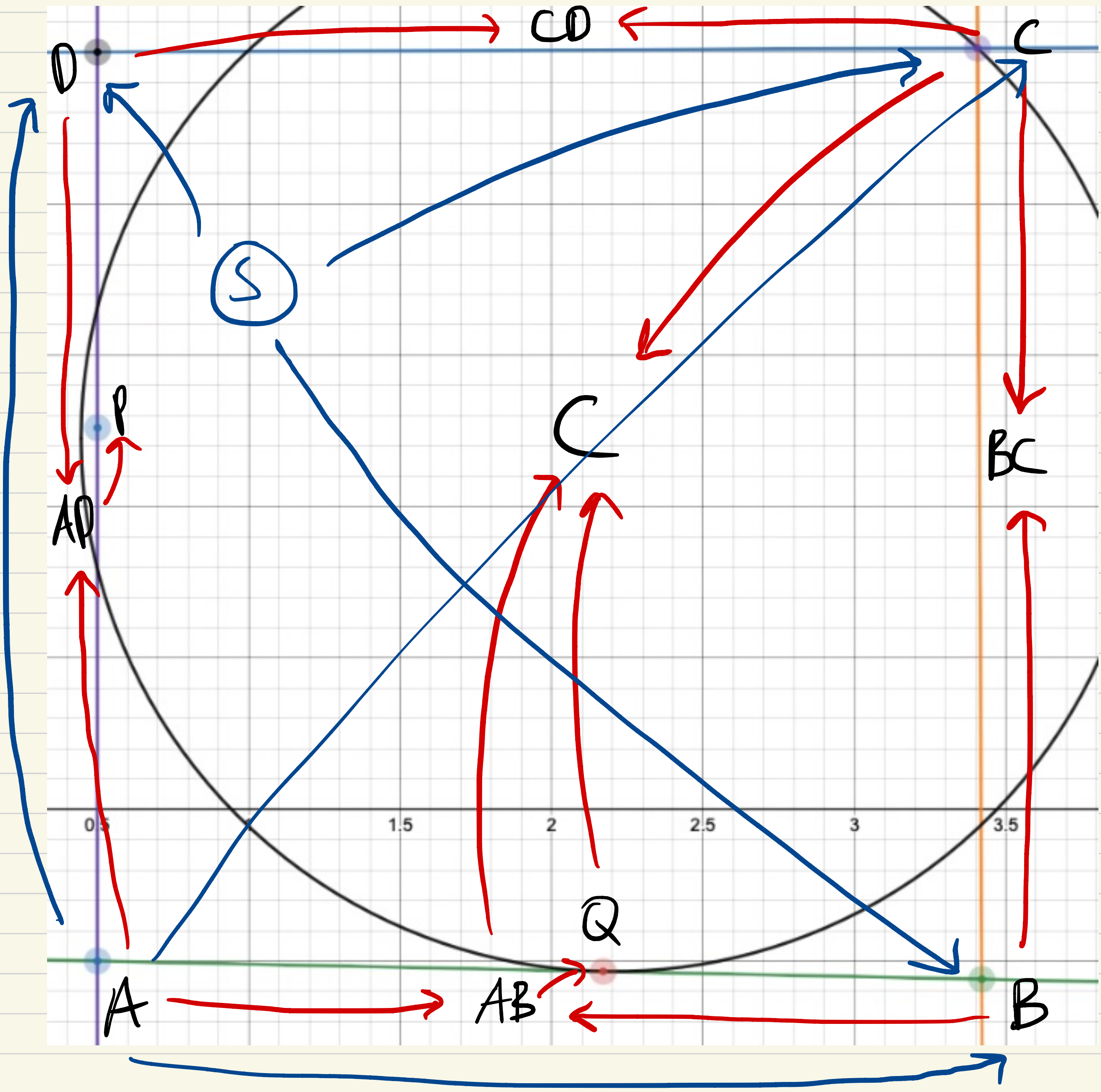
$$(x_1 - x_2)x + (y_1 - y_2)y = \frac{1}{2}\left(x_1^2 + y_1^2 - x_2^2 - y_2^2\right) \qquad 2$$

$$C_x = \left\{ a = 0: x_1, \frac{\left(\frac{x_1^2 + y_1^2 - x_2^2 - y_2^2}{2y_1 - 2y_2} - \frac{-bx_1 + ay_1}{a}\right)}{\left(\frac{b}{a} + \frac{2x_1 - 2x_2}{2y_1 - 2y_2}\right)} \right\}$$
$C_x = 2.25626666667$

$$C_y = \left\{ b = 0: y_1, \frac{\left(\frac{x_1^2 + y_1^2 - x_2^2 - y_2^2}{2x_1 - 2x_2} - \frac{bx_1 - ay_1}{b}\right)}{\left(\frac{a}{b} + \frac{2y_1 - 2y_2}{2x_1 - 2x_2}\right)} \right\}$$
$C_y = 1.26$

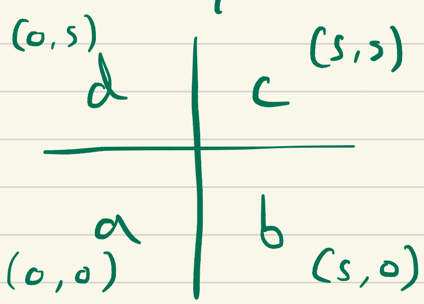
$$C_r = \sqrt{(x_1 - C_x)^2 + (y_1 - C_y)^2}$$
$C_r = 1.75626666667$

I also added the edge cases, when a or $b = 0$, by evaluating what the corresponding x or y value from 1 would be

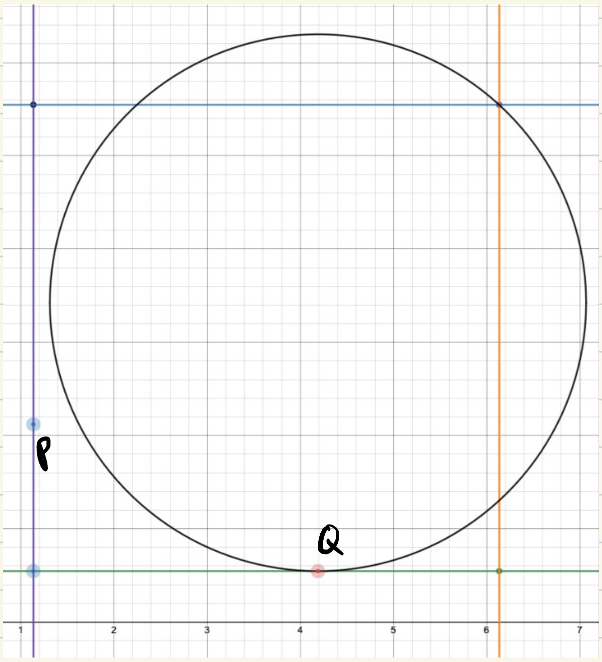


→ Next proposed constraint: Using external variable S to make sure all points form a square, with A as the bottom-left corner

When adding a square in app, either ask user or determine which points lie in which quadrant, e.g. for this example:



I have now run into an issue.



Points P and Q do not move with the rest of the square ABCD.

While I could configure them to work, I believe that would be the wrong method, as it is not generic enough to be used in the final product.

Attempt 2

Same process to create points & lines, however now I will construct C using P → C as the diameter, rather than introducing the 3rd point Q.

Construct circle with diameter $(x_1, y_1) \rightarrow (x_2, y_2)$

$$\text{Center } (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\text{Radius} = \frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

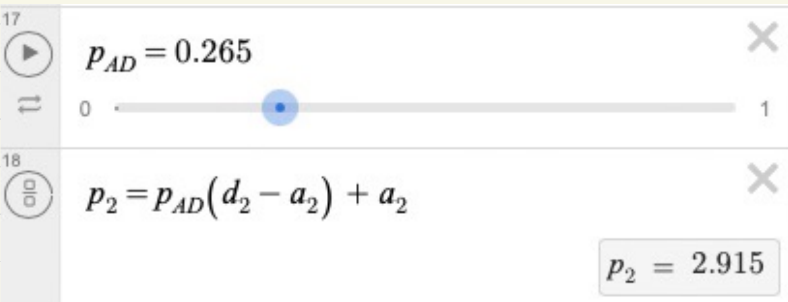
I still need to 'lock' P relation to A, so I thought I could interpolate between point A and D.

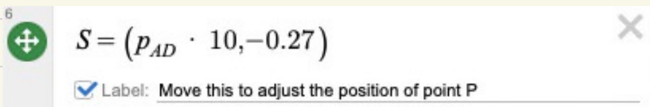
Interpolate values between (x_1, y_1) and (x_2, y_2)

t = Constant of interpolation

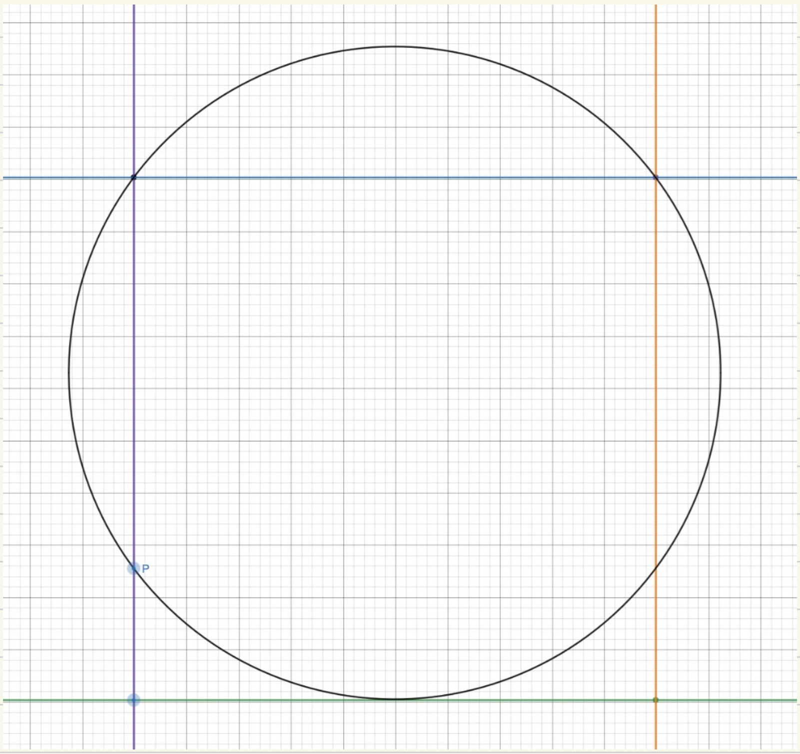
$$x = t(x_2 - x_1) + x_1$$
$$y = t(y_2 - y_1) + y_1$$

I called this parameter PAD, as it is the relative position of P on AD.





A large downside to this is the inability to move P directly, and thus I may not actually use this



Finally this is what the user is left with. Although the answer is not explicitally clear, this provides an algebraic representation of the problem.

In this case, the user could choose a square length (s), position a at (0,0), then adjust p until the circle is more or less tangent to the line AB.

https://www.desmos.com/calculator/rhszpf7njj

Notes

- If user selects square/rectangle, try and position independent point (bottom-left) at (0,0) for convienience.

- Desmos has an API – use it to preview after/while user is setting up constraints.

I now believe I have enough information to start development

Also will need to think of a way to implement the dependency graph, as it is crucial to avoid 'in terms of' errors.