Malkus Waterwheel Simulation

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1 Introduction

The Malkus Waterwheel (MWW) is a mechanical system that exhibits chaotic behaviour. The system consists of a wheel with buckets attached equidistantly around the rim. The bucket at the top of the wheel is filled at a constant rate of water while the remaining buckets are emptied at a constant rate of water. These constants need not be the same. The chaos of the system is exemplified best by its center of mass (COM). If we graph the COM of the system resembles the shape of a 2-dimensional Lorenz attractor. The MWW's motion is governed by the Lorenz equations. The Lorenz equations are a system of three ordinary differential equations.

2 Methods

As mentioned, the below Lorenz equations were used to model the MWW simulation:

$$\frac{dx}{dt} = \sigma(y - x) \tag{1}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{2}$$

$$\frac{dz}{dt} = xy - \beta z \tag{3}$$

The constants σ, ρ, and β are constant system parameters that influence the system evolution over time. The X, Y, and Z values are changing values that represent the state of the system at a given time. X represents the rotational speed, Y represents the flow rate, and Z represents the internal dynamics of the system such as friction and damping. To solve the above system, the 4th Order Runge-Kutta (RK4) method was used.

3 Tests

A method that I discovered that could be used to test the program was whether the limit of the COM approached the center of the wheel if the angular velocity of the wheel was high enough. As the wheel accelerates the amount of liquid getting filled approaches a limit which is proportional to the speed at which the wheel is spinning. If the wheel spins sufficiently fast, the COM should approach the center of the wheel resulting in a COM graph that resembles a spiral. It is hard to compare the results of external simulations with one another as the programmer can add their own contraints on the system such as friction, damping, gravity, etc. However, we can compare general behaviour to compare what the COM should be given general conditions about the system.

3.1 Proof of Chaos

To prove the chaotic nature of the system, the simulation was tested with the same initial conditions with the difference being that simulation two had a θ_0 of 1e-5 more than simulation one. The below figures demonstrate the chaos.

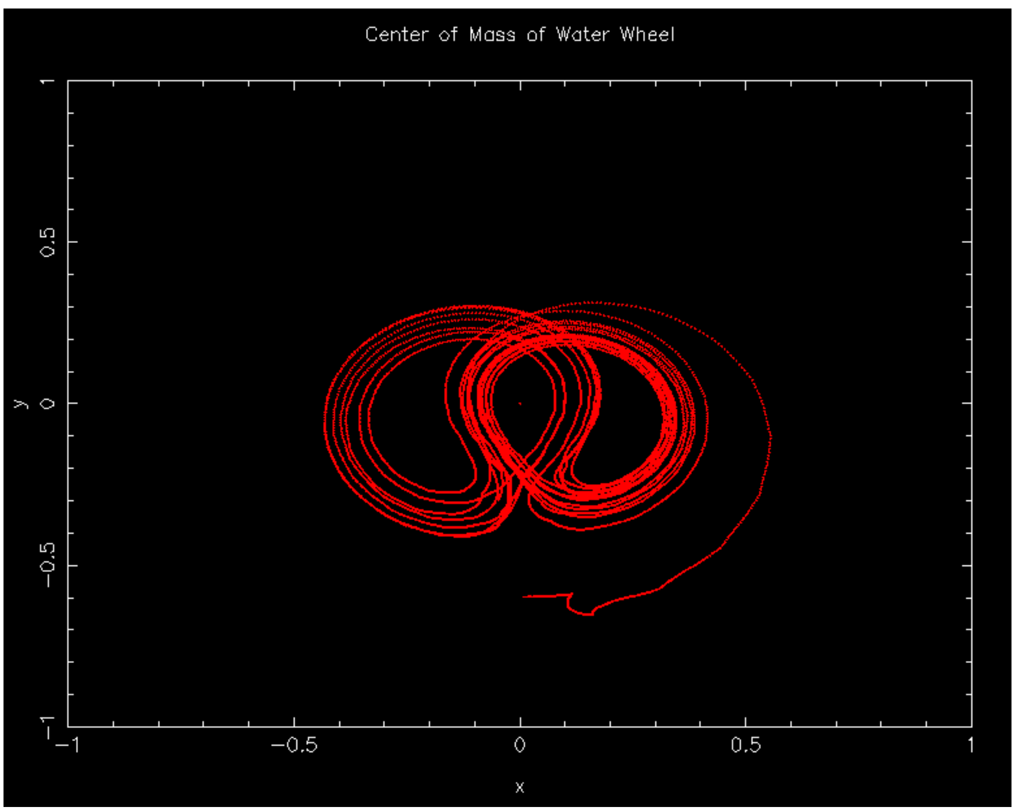

Figure 1: $\theta_0 = \pi$

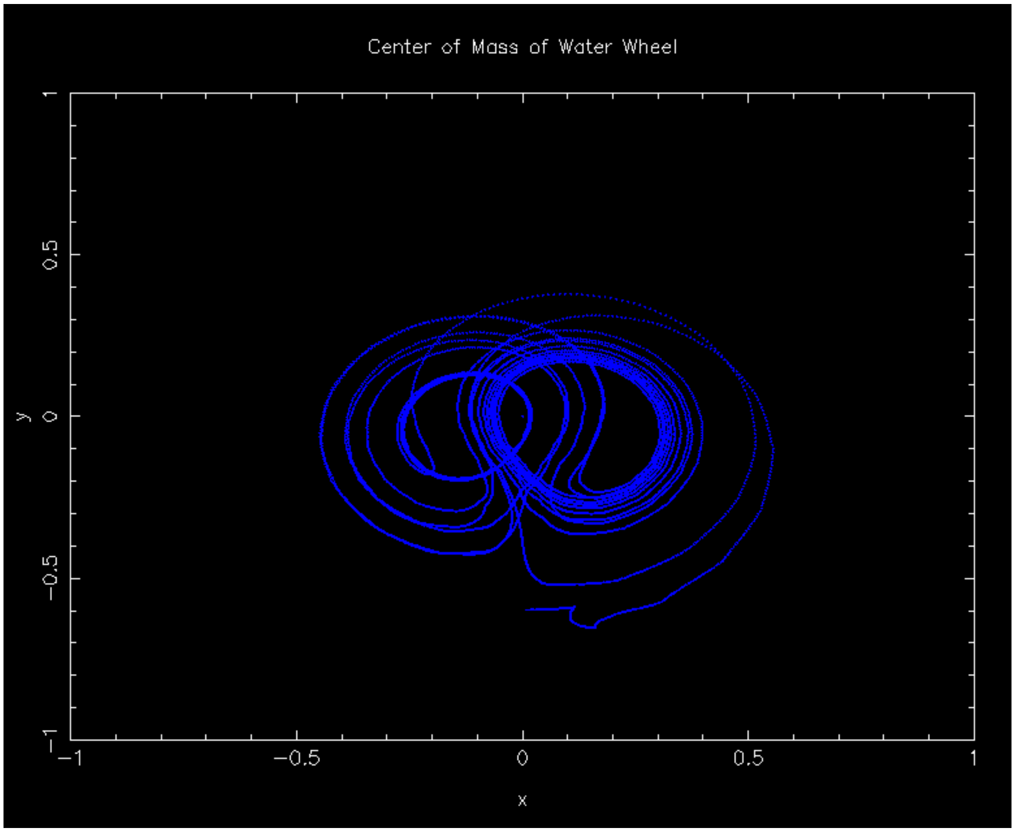

Figure 2: $\theta_0 = \pi + 1e - 5$

We can clearly see that initially both the center of masses followed the same path. However, as t increased the two paths diverged due to the slight difference in their initial angle. This proves the chaotic nature of the MWW is correctly expressed.

3.2 Proof of Correctness

To prove the whether the MWW simulation is correct we can use the spiral test. Mathamatically the center of a non-uniform disk is defined as

$$x = \frac{1}{M}\int_0^{2\pi}\int_0^R r\cos(\theta)\sigma r\ drd\theta \tag{4}$$

$$y = \frac{1}{M} \int_0^{2\pi} \int_0^R r \sin(\theta) \sigma r \ dr d\theta \tag{5}$$

Where M is the mass, R is the radius, r is the distance from the center of the wheel, and θ is the mass density. If the fill rate >> drain rate then the wheel will accelerate. This will cause the buckets to become more and more equally filled. As $\lim_{acceleration \to C}$ then the wheel's center of mass will approach some center of mass based on the remaining conditions of the MWW. We can model this via equations 4 and 5:

Proof of Center of Mass.

$$x = \frac{1}{M} \int_0^{2\pi} \int_0^R r \cos(\theta) \sigma r \ dr d\theta$$

Since σ is a constant on a uniform distribution, we can pull it out of the integral:

$$x = \frac{\sigma}{M} \int_0^{2\pi} \int_0^R r \cos(\theta) r \ dr d\theta$$

We can then solve the integral and see that it evauluates to 0. The same is true for the y value. □

This behaviour is also modeled in the simulation as displayed in figure 3.

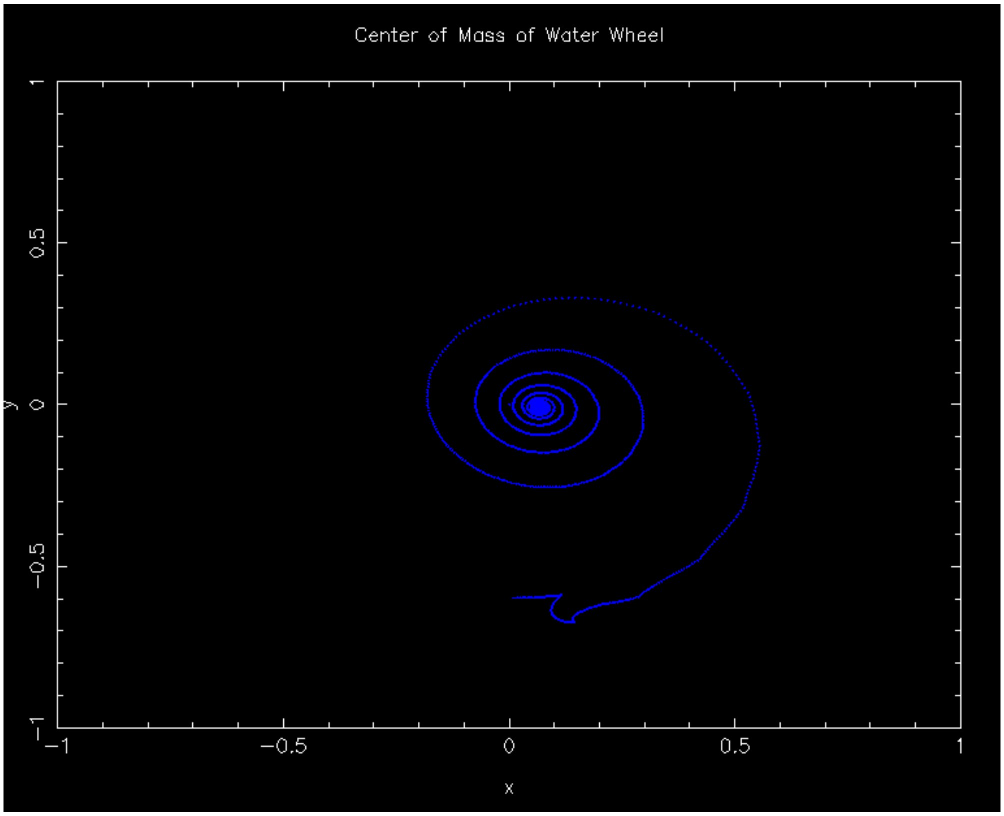

Figure 3: Spiral Test

In this case, the ratio of Fill Rate and Drain Rate was 3 to 1 resulting in an acceleration of the wheel which approached a center of mass approximately approaching $(0.1, 0)$. This proves the correctness of the MWW simulation via the above proof.

4 Results

The primary extension made to the system was the introduction of more variables in the system. The initial implementation of the system only had one variable for both the flow and drain rates. Separating the two rates and allowing them to be different values allowed for better testing of the system and brought about interesting results such as the aforementioned spiral COM function as well as warped versions of the simulation such as crescent shapes.

To add more realism to the simulation, friction and damping were added to the system to add a touch of realism to the simulation as well as add more parameters to tune thus allowing better control and more interesting interactions in the simulation. Some interesting results happen when the drain rate > fill rate. When the drain right is just slightly larger than the fill rate we get a result similar to the spiral with a bit more complexity as seen in Figure 4. When the drain rate is much larger we get a strange jagged spiral shape as seen in Figure 5.

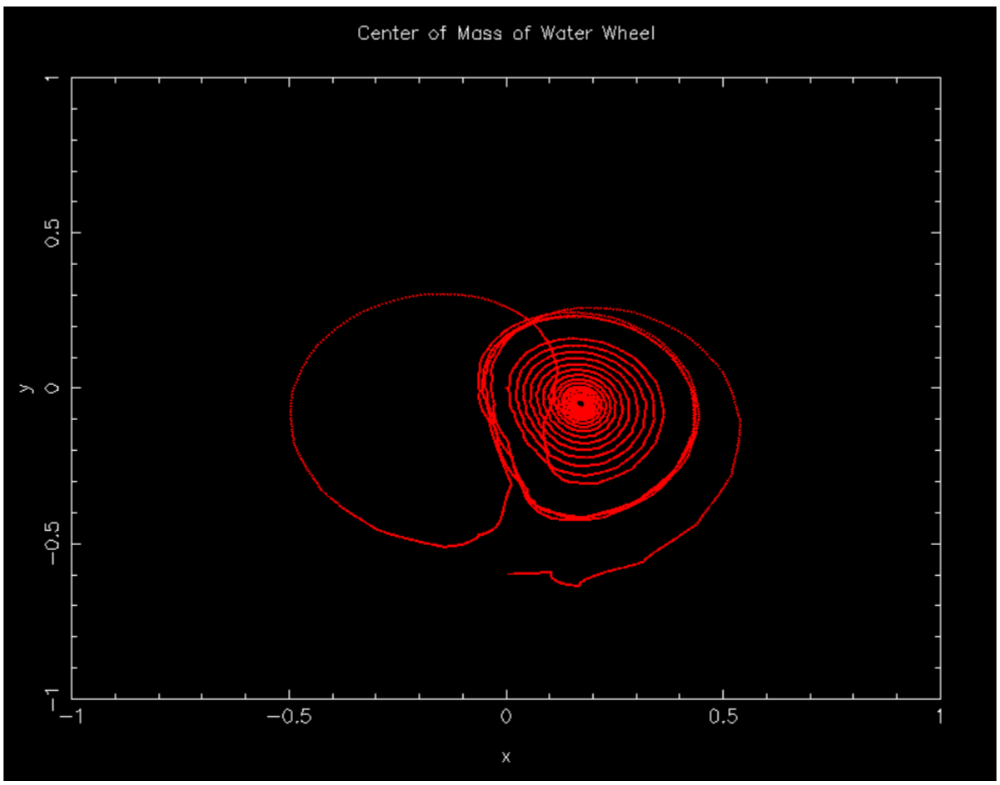

Figure 4: Drain Rate > Fill Rate

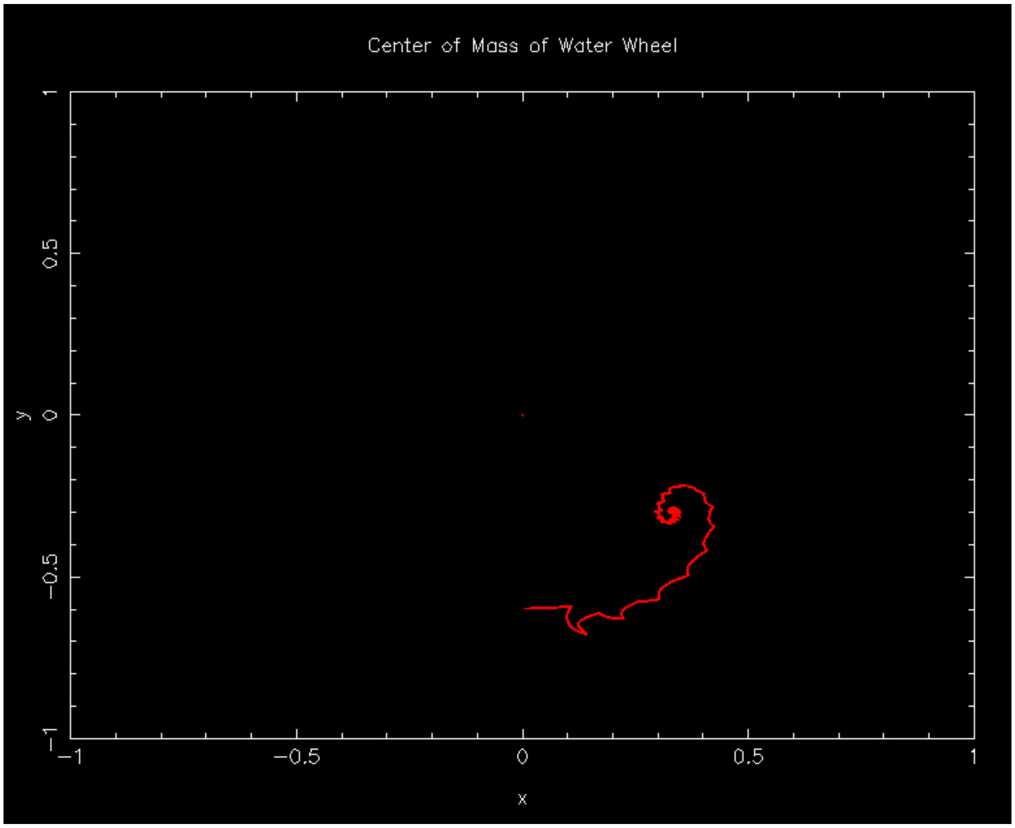

Figure 5: Drain Rate >> Fill Rate

5 Conclusion

The simulation provided insight into the chaos and how it can be modelled and used in real-world scenarios. The simulation also provided insight into the Lorenz equations and how they are used to model chaos. The behaviours of the system supported intended behaviours that are expected based on different initial parameters. The realistic parameters of the system added more nuance to the simulation and allowed for a more realistic and thus complex simulation of the MWW. Possible future directions of the system could be to experiment with non-constant flow rates to see how that can change the simulation. For example, if we have an exponential fill rate and a polynomial drain rate. Another area that could be focused on in the future is if the wheel was a different shape. For example what the chaotic behaviour would look like, given the same initial conditions, if the wheel was a square or an oval rather than a circle? For alternative approaches, the use of different solving techniques for the differential equations can be used to see how each of them differs and what the best approach is for solving the system.

References

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