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Lect 7

Pivot Variables, Special Solutions
Solving $AX=0$

In $AX=0$, elimination does not change the solutions to the system (i.e. the null space). The column space changes.

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

Here the 2^{nd} column also becomes 0 along with the 1^{st} column. This tells us that the 2^{nd} column is dependent on the 1^{st} column

These 2 are free columns

Echelon
$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Here we have 2 pivots only. These 2 columns are called pivot columns

$\therefore$ rank of A = Number of pivots $= 2$
$= r$

We call them free columns because we require them to solve: ~~Ax~~ $Ux=0$.

We consider the variables ~~as~~ in vector x corresponding to the free columns in U as 1 and 0.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \end{bmatrix}$$

We can take any values for variables corresponding to free columns and obtain the values ~~of~~ of other variables using eqs obtained by $Ux=0$.

Eqs obtained from $Ux=0$ are,

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \rightarrow (1)$$

$$2x_3 + 4x_4 = 0 \rightarrow (2)$$

From eq (2),

$$2x_3 + 4(0) = 0$$

$$\therefore x_3 = 0$$

From eq (1),

$$x_1 + 2(1) + 2(0) + 2(0) = 0$$

$$\therefore x_1 = -2$$

$$\therefore x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Here we have multiplied by "c" because all multiples of $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ will be in null space.

Now take $x_2 = 0$ and $x_4 = 1$.

$$\therefore X = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 1 \end{bmatrix}$$

From eq ②,

$$2x_3 + 4(1) = 0$$

$$\therefore x_3 = -2$$

From eq ①,

$$x_1 + 2(0) + 2(-2) + 2(1) = 0$$

$$\therefore x_1 = 2$$

$$\therefore X = d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

All multiples of $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ will lie in the null space.

$\therefore$ The null space in total contains all combinations of the two special solutions.

$$\therefore N(A) = N(U) = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Basis for nullspace $N(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

→ There is 1 special solution for each free variable.

→ No of free variables = No of pivots

→ For a "m×n" matrix, we have "n" variables and a rank "r" (i.e. "r" pivot variables),

No of free variables = $n - r$

In our case, it's $4 - 2 = 2$

* The last row in the U matrix being all 0 indicates that the row that was originally there in the "A" matrix was a combination of the other rows.

* R = Reduced Row Echelon form (RREF)

Currently matrix U is in the echelon form. To convert it into RREF, we need to make the element above all pivots as '0'.

$R_1 - R_2$ on U

$$\sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now making the pivot in the 3rd column as "1".

$\frac{R_2}{2}$

$$\approx \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = \underbrace{\text{rref}(A)}_{\text{Matlab command for RREF}}$$

RREF tells us the pivot rows (Here rows 1 and 2), pivot columns (Here columns 1 and 3) and it has an identity matrix formed by the pivot rows and pivot columns.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \text{ in pivot rows/cols}$$

$$I \leftarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix} \rightarrow F$$

Pivot Cols   Free Cols

0 0 0 0

Solutions of $AX=0$, $UX=0$, $RX=0$ are all the same

The Special Solutions

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

turn out to be pivot cols I and negative of the free cols F. The -ve is due to F being taken on other side of "="

* rref Form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$ ← r pivot rows

r pivot cols n-r free cols

$Rx = 0$

null space solution (Columns = Special Solutions)

$RN = 0$

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$$RX = 0$$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} X_{pivot} \\ X_{free} \end{bmatrix} = 0$$

$$\therefore \boxed{X_{pivot} = -F X_{free}}$$

* Another Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

This will be a free column since it's a combination of the first 2 columns.

$R_2 - 2R_1$, $R_3 - 2R_1$, $R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$R_4 - 2R_2$

Pivot Columns

$$\sim \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$$

r = No of pivots $= 2$

Free column $(3 - 2 = 1)$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$ ← Don't set as 0 ∵ Only 1 free column

$$\therefore \quad x_1 + 2x_2 + 3x_3 = 0$$
$$2x_2 + 2x_3 = 0$$

$$N(A) = X = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$ ← Nullspace is a line.

Basis of Nullspace $N(A) = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$$\begin{array}{c} \begin{array}{cc} \quad I & \quad\quad F \end{array} \\ R = \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 1 \end{matrix}} \\ 0 \quad 0 & \boxed{0} \\ 0 \quad 0 & 0 \end{bmatrix} \end{array}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore X = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = c\begin{bmatrix} -F \\ I \end{bmatrix}$$

$-F$ I N

Note: The A in 2nd example was transpose of A in 1st example. Both A, A^T will always have number of pivot columns same.

Q. The set S of points P(x, y, z) S.T. $x - 5y + 2z = 9$ is a plane in R^3. It is parallel to the plane S_0 of P(x, y, z) S.T. $x - 5y + 2z = 0$

All points of S have the form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} \, \\ 0 \\ 0 \end{bmatrix}}_{\text{let it be } P_0} + C_1 \begin{bmatrix} \, \\ 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} \, \\ 0 \\ 1 \end{bmatrix}$$

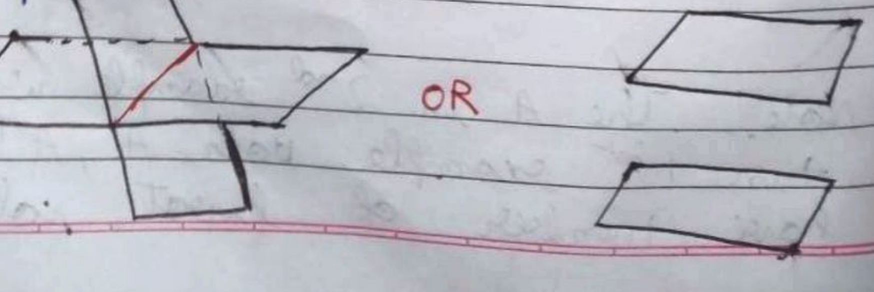
Soln:

1. R^3 is a 3-D space. We apply 1 constraint $x - 5y + 2z = 9$ to it. ∴ We have 2 degrees of freedom left now. ∴ A 2D space (i.e. 2 DOF) can be considered a plane.

   ∴ $x - 5y + 2z = 9$ is a plane in R^3.

2. Similar explanation works for $x - 5y + 2z = 0$.

3. ∴ Now either plane 1 and plane 2 can intersect each other or they can be parallel.



Now the planes $x - 5y + 2z = 9$, and $x - 5y + 2z = 0$ don't have a line of intersection since on solving them we get no solution.
$\therefore$ The two planes are parallel to each other.

(4) P_0 is in S.

(4) Let $c_1 = c_2 = 0$

Now $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a pt in S.

$\because c_1 = c_2 = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore y = z = 0$

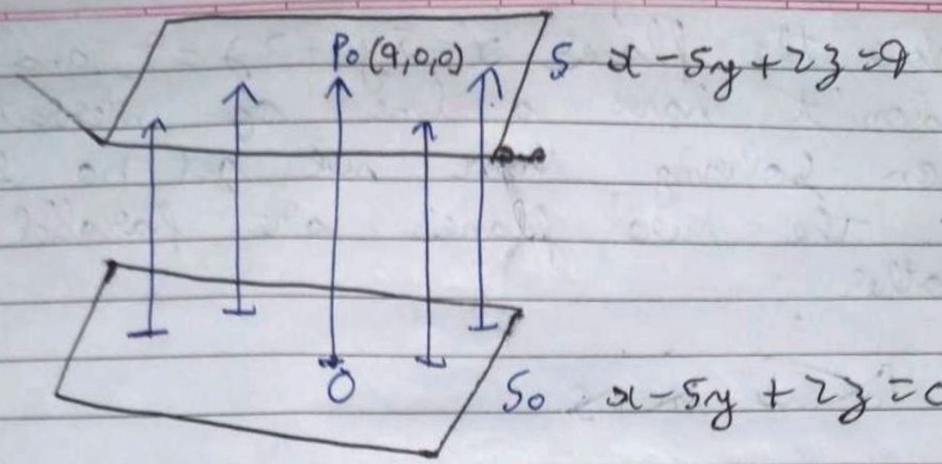
Eq of S: $x - 5y + 2z = 9$

$\therefore x - 0 + 0 = 9$

$\therefore x = 9$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So will have a pt $(0,0,0)$ in it since it's eq is $x-5y+2z=0$



Any point in S = (Go up along) P_0

+ Any point in S_0

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = P_0 + \underbrace{c_1 \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}}_{\text{Any point in } S_0}$$

$x-5y+2z=0$ can be written as:

$$\begin{array}{c} \text{Pivot var } x \\ \downarrow \\ \left[\ \boxed{1} \quad -5 \quad 2\ \right] \\ \text{Free vars } y, z \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

If $y=1,\ z=0$

$x-5+0=0$

$\therefore x=5$

If $y=0, z=1$

$$x - 0 + 2 = 0$$
$$\therefore x = -2$$

We know Null Space solution would be:

$$c_1 \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$