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Lect 8

Solving $Ax=b$: Row Reduced form R,

$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$
$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$
$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$

Here Row 1 + Row 2 = Row 3
∴ For the system to have a solution, $b_1 + b_2 = b_3$ is necessary

$$[A : B] = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right]$$

Pivots

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

∴ For solution to exist, $b_3 - b_2 - b_1 = 0$ is necessary condition.

One example is $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$

→ Solvability condition on b

$Ax = b$ is solvable when b is in column space of A (i.e. $C(A)$).
i.e. b is combination of columns of A.

If a combination of rows of A gives zero row, then the same combination of entries of b must give 0.

→ To find complete solution x to $Ax = b$

① $X_{particular}$: Set all free variables to zero. Solve $Ax = b$ for pivot variables

$\therefore x_2 = x_4 = 0$

$\therefore x_1 + 2x_3 = 1$
$2x_3 = 3$

$\therefore x_3 = 3/2, \; x_1 = -2$

$$\therefore X_{particular} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

② $X_{nullspace}$: Find nullspace of A (i.e. Solve $Ax = 0$)

let $x_2 = 0, \; x_4 = 1$

$$\therefore \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

let $x_2 = 1,\ x_4 = 0$

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore X_{nullspace} = c_1 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

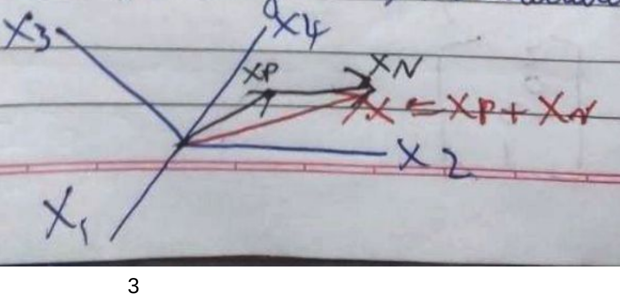
③ $X = X_{particular} + X_{nullspace}$

$$\therefore X = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \underbrace{c_1 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{Nullspace is a 2D subspace } \therefore \text{ we have 2 constants } c_1 \text{ and } c_2}$$

$$A X_{particular} = b$$
$$A X_{nullspace} = 0$$
$$\therefore A(X_{particular} + X_{nullspace}) = b$$

* What does solution x look like?

Ans It looks like a 2D subspace in R^4, But the subspace does not go through the origin because it goes through X particular.



* m by n matrix A of rank r
$(r \leq m, \ r \leq n)$

Full column rank means $r = n$: No free variables

$N(A) = \begin{Bmatrix} \text{zero} \\ \text{vector} \end{Bmatrix}$

Solution to $Ax = b$: $x = x_{particular}$
(Unique solution if it exists)
(0 or 1 solution)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$$

Reduced Row Echelon form:

$r = n < m$ $$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we will only have (1) solution when b is a sum of the combinations of the columns of A

Full row rank means $r = m$
Every row has a pivot.

We can solve $Ax = b$ for every b.

We ~~are~~ have r pivots, $\therefore$ we are left with $n - r$ free variables.
$= n - m$

$r = m < n$

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}^{I} & \begin{matrix} - & - \\ - & - \end{matrix}^{F} \end{bmatrix}$$

If $r = m = n$,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

This would be a square matrix $\because m = n$.

$\therefore r = m = n$, it's an invertible matrix

For a squared invertible matrix,

$R = I$

$N(A) = \left\{ \begin{matrix} \text{zero} \\ \text{vector} \end{matrix} \right\}$

There are no conditions on b to solve $Ax = b$. $Ax = b$ is solvable for every b. ($\because r = m$)

Also, we will have only one solution ($\because r = n$)

* Conclusion

① $r = m = n$

$R = I$

1 solution to $Ax = b$

② $r = n < m$

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

(0 or 1 solution)

The rank tells you everything about the solution of a matrix

③ $r = m < n$

$R = \begin{bmatrix} I & F \end{bmatrix}$

(∞ solution)

④ $r < m$, $r < n$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

(0 or ∞ solution)

$n - r$ free variables

Q: Find all solutions, depending on b_1, b_2, b_3:

$$x - 2y - 2z = b_1$$
$$2x - 5y - 4z = b_2$$
$$4x - 9y - 8z = b_3$$

Soln:

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 2 & -5 & -4 & b_2 \\ 4 & -9 & -8 & b_3 \end{array}\right]$$

$R_2 - 2R_1,\ R_3 - 4R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & -2b_1 + b_2 \\ 0 & -1 & 0 & -4b_1 + b_3 \end{array}\right]$$

$R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & -2b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \end{array}\right]$$

* If $-2b_1 - b_2 + b_3 \neq 0$
→ No solutions

* If $-2b_1 - b_2 + b_3 = 0$

Pivot vars x, y Free var z

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & 2b_1 - b_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

→ Particular soln for
$Ax = b$,
Put $z = 0$

$$\therefore x_p = \begin{bmatrix} 5b_1 - 2b_2 \\ 2b_1 - b_2 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{array}{l} 1x + 0y - 2(0) = 5b_1 - 2b_2 \\ \therefore x = 5b_1 - 2b_2 \\ 0x + 1y + 0z = 2b_1 - b_2 \\ \therefore y = 2b_1 - b_2 \\ z = 0 \end{array} \right\}$$

→ Special Soln for
$Ax = b$,
Set 1 free variable as 1 and others as 0.
Put $z = 1$

$$x_s = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{array}{l} 1x + 0y - 2(1) = 0 \\ \therefore x = 2 \\ 0x + 1y + 0z = 0 \\ y = 0 \\ z = 1 \end{array} \right\}$$

All solns :-

$$\vec{x} = \vec{x}_p + c\vec{x_s}$$

$$= \begin{bmatrix} 5b_1 - 2b_2 \\ 2b_1 - b_2 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$