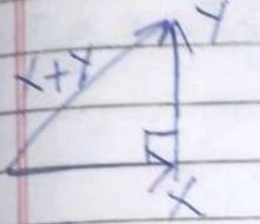
Prof Gilbert Strang

Lecture - 14

Orthogonal Vectors and Subspaces

→ Orthogonal means perpendicular vectors.

→ Test 1 of orthogonality



$\|x\|^2 + \|y\|^2 = \|x+y\|^2$

$\rightarrow = x^T x$

→ Test 2 of orthogonality

$x^T y = 0$ where x, y are two column vectors

Ex:

$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \; x+y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$\|x\|^2 = 14, \; \|y\|^2 = 5, \; \|x+y\|^2 = 19.$

$\therefore \|x\|^2 + \|y\|^2 = \|x+y\|^2$

→ How are Test 1 and Test 2 the same?

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$

$$x^T x + y^T y = (x+y)^T(x+y)$$

$$\therefore x^T x + y^T y = x^T x + y^T y + x^T y + y^T x$$

$$\therefore \quad 2x^T y = 0$$

$$\therefore \quad x^T y = 0$$

→ The zero vector is orthogonal to every other vector.

* Subspace S is orthogonal to subspace T means: Every vector in S is orthogonal to every vector in T.

Two ~~planes~~ spaces can only be orthogonal if they are ⊥ to each other and intersect only at the origin.

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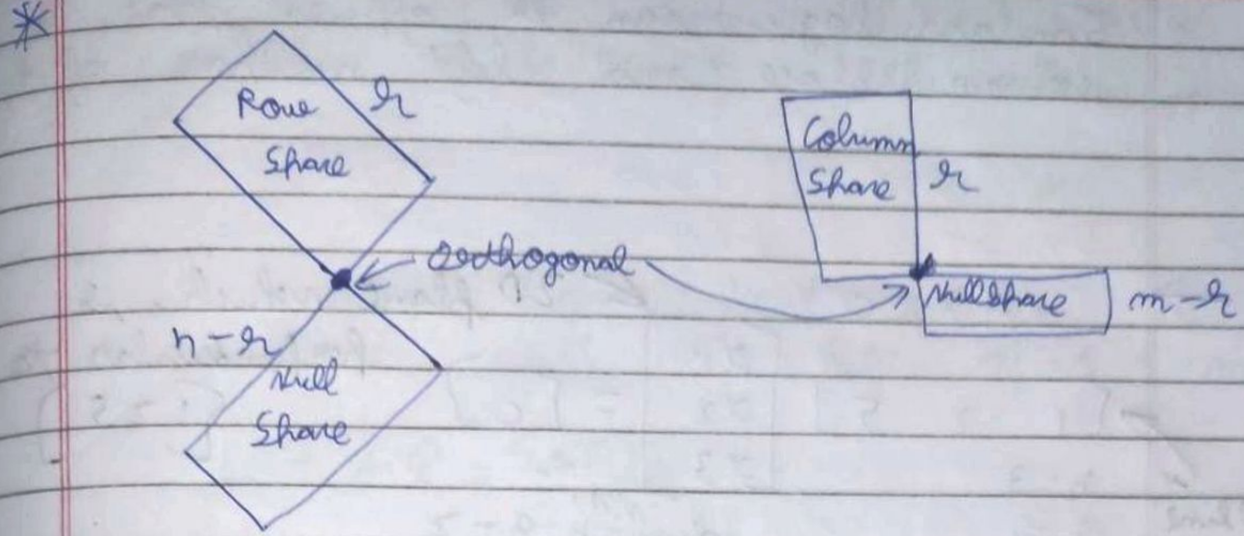

Why is row space orthogonal to null space

Consider $AX = 0$

$$\begin{bmatrix} - \text{row } 1 - \\ - \text{row } 2 - \\ - \vdots - \\ - \vdots - \\ - \text{row } 5 - \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

row1 . X_1, row1 . X_2, - - - .
row2 . X_1, row2 . X_2, - - .

Each row of A when multiplied to each vector in X (which is like a dot product) gives 0.

Also, the combinations of rows when multiplied with the vectors in X give 0.

A is the rowspace and X is the nullspace.
Hence, we say that rowspace and nullspace are orthogonal

Similar logic can be applied for column space and left nullspace of A.

→ $AX = 0$ ← 2D plane which is perpendicular to $[1\ 2\ 5]$

$$\begin{bmatrix} 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0]$$

1D line

$n = 3$

$r = 1$

$\dim = r = 1$

$\dim_A^{N(A)} = n - r = 2$

$[1\ 2\ 5]$ is normal vector to the plane.

→ The dimensions of nullspace and rowspace add to the whole space.

→ Nullspace and row space are orthogonal complements in R^n.

→ the orthogonal complement of a row space contains all the vectors orthogonal to it.

∴ Nullspace contains all vectors ⊥ to row space.

* "Solve" $Ax = b$ when there is no solution (i.e. b isn't in the column space of A) $A_{m\times n}$

We consider our best solution to $Ax=b$ will come with the help of A^TA matrix.

1. $A^TA\hat{x} = A^Tb$

Note: A^TA is always a square symmetric matrix

→ $$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Here we have 3 eqs and 2 unknowns which is generally not solvable. It will only be solvable if b is in column space of A but usually it won't be.

Consider A^TA here,

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

Here A^TA is ~~inverdible~~ invertible. But it always won't be invertible.

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$$

Here A^TA is not invertible. Here A has rank 1 and hence, A^T will have rank 1. The product of two rank 1 matrices is rank 1.

$\therefore$ We can say that,
$N(A^TA) = N(A)$

rank of A^TA = rank of A

$\rightarrow$ A^TA is invertible exactly if nullspace of A only has the zero vector. i.e. columns of A must be independent.

Q. S is spanned by (1 2 2 3) and (1 3 3 2).

i) Find a basis for $S^\perp$

ii) Can every vector in R^4 be written uniquely in terms of S and $S^\perp$?

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Soln :-

1) Any vector x in $S^{\perp}$ must be perpendicular to every vector in S.

∴ Since S is spanned by $(1\ 2\ 2\ 3)$ and $(1\ 3\ 3\ 2)$,

$$(1\ 2\ 2\ 3) \cdot x = 0$$
$$(1\ 3\ 3\ 2) \cdot x = 0$$

We can write this in matrix form as,

$$\begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{pmatrix} x = 0$$

= vectors in

∴ we have to find nullspace of this matrix

$$\sim \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{pmatrix} x = 0$$

A

pivots

rank $(A) = 2$

let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

∴ Rank $(x) = n - r = 4 - 2 = 2$

∴ There must be 2 vectors in the nullspace.

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$x_3 = a \qquad x_4 = b$

Let ~~$x_4 = a$~~ and ~~$x_3 = b$~~

$\therefore x_2 = -x_3 + x_4 = -a + b$

$$\begin{aligned} x_1 &= -2x_2 - 2x_3 - 3x_4 \\ &= -2(-a+b) - 2a - 3b \\ &= -5b \end{aligned}$$

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5b \\ -a+b \\ a \\ b \end{pmatrix} = \underbrace{a\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{S^{\perp}}$$

Take any vector in S and dot it with $S^{\perp}$, ans is zero.

ii) Yes!

$$v = \underbrace{c_1\begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 3 \\ 3 \\ 2 \end{pmatrix}}_{S} + \underbrace{c_3\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c_4\begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{S^{\perp}}$$

i.e. $$\underbrace{\begin{pmatrix} 1 & 1 & 0 & -5 \\ 2 & 3 & -1 & 1 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = v$$

We know that all 4 columns of A are linearly independent $\therefore$ Matrix A is invertible. This means that we can get unique values of c_1, c_2, c_3, c_4 and hence v in $\mathbb{R}^4$ can be written uniquely in terms of S and $S^{\perp}$.