

Lecture - 10

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The Four Fundamental Subspaces

In last lecture, we, in the basis section, wrote $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$ as another basis for 3-D space.

But this is wrong because these vectors are not linearly independent (LI). If we form matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix}$,

$|A| = 0$. \therefore The matrix is not invertible (because row 1 and row 2 are same).

We already know about column space and null space. Now we'll learn about row space.

Here, the row space of A is only 2-D. \therefore $\text{Rank}(A)$ with these columns is 2. \therefore only 2 of these columns can be independent.

If A is $m \times n$

* 4 Subspaces

1) Column Space $C(A)$ in \mathbb{R}^m

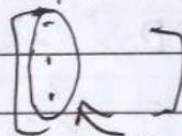
2) Null Space $N(A)$ in \mathbb{R}^n

3) Row Space = all combs of rows
= all combs of columns of A^T
 $C(A^T)$ in \mathbb{R}^n

4) Nullspace of $A^T = N(A^T)$ = Left Nullspace of A in \mathbb{R}^m

Rowspace is the span of rows of A .
The rows form basis for rowspace if they are linearly independent.

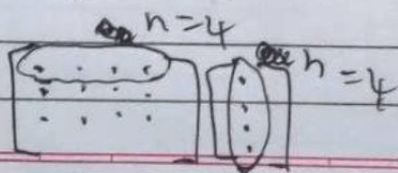
1) Column space is a LC of columns and each column has m components



3 components.

$\therefore C(A)$ is in \mathbb{R}^m

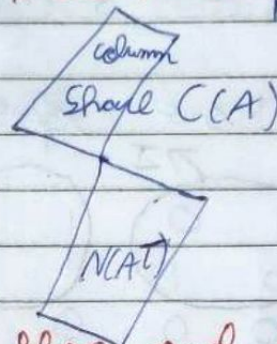
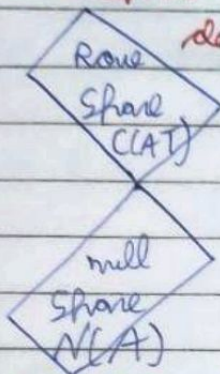
2) Null space is soln to $AX=0$. If A is $m \times n$, X is $n \times 1$. $\therefore X$ has n components.
Hence, it's in \mathbb{R}^n



3) and 4) are transpose of 1) and 2) respectively.

* Basis and dimension of all 4 spaces

Row space and nullspace are said to be orthogonal because the dot product of ^{each vector in} basis of row space with vectors in \mathbb{R}^n nullspace is 0 and the other vectors in row space are dependent on the basis vectors. \mathbb{R}^m



Same reason is for column space and left null space being orthogonal. Here dot product of each basis vector of column space of A with $N(A^T) = 0$ and other vectors are dependent on them.

→ $\dim(C(A)) = r = (\text{No of pivots}) = \text{No of vectors in basis}$

∴ Basis for $C(A)$ are pivot columns

→ $\dim(\text{Row space}) = r$ as seen in Ex 1 today.

→ For finding nullspace, we reduce to RREF and then find special solns by putting free variables 1 and 0. Also, $\text{No of special solns} = \text{No of free variables} = n - r$ ($n = \text{No of columns}$, matrix is $m \times n$)

Then, special solns form basis for nullspace and $\dim(N(A)) = n - r$

→ $\therefore \dim(N(A^T)) = m - r$. \therefore Here matrix is ~~$n \times m$~~ $n \times m$. No. of columns is m .

→ Basis for row space

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

By row operations,

$$\sim \begin{bmatrix} \overset{\nearrow I}{1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \nearrow F \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Due to row operations, column space of A and R are different.

$$C(A) \neq C(R)$$

But the row space of A and R remain same.

The 3rd row of A is useless just as the 3rd row of R .

Whatever you get by taking combinations of first 2 rows of A , will be obtained by taking combinations of first 2 rows of R .

\therefore Here basis for row space (R) = basis for row space (A) = first 2 rows of R .

Generalizing,

Basis for row space of A = Basis for row space of R = first r rows of R where r = rank.

∴ The row space is spanned by all the rows but the basis for row space is formed by the r independent rows.

→ 4th space: $N(A^T)$

$$A^T y = 0$$

$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$(A^T y)^T = 0^T$$

$$\therefore y^T A = 0^T$$

$$\therefore \begin{bmatrix} y^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Row Vector

$$\text{row} \left[\begin{array}{c|c} A & I \end{array} \right] \rightarrow \left[\begin{array}{c|c} R & E \end{array} \right]$$

$m \times n \quad m \times m \quad m \times n \quad m \times m$

E is row because E contains the records of the operations it took to go from A to R

$$\therefore E \left[\begin{array}{c|c} A & I \end{array} \right] \rightarrow \left[\begin{array}{c|c} R & E \end{array} \right]$$

$m \times n \quad m \times m \quad m \times n \quad m \times m$

$$EA = R$$

Rank
↓
 $\rho(A) = 2$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

performing these same operations of I , we get E

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

E

$$\therefore \dim(N(A^T)) = m - \rho = 3 - 2 = 1 \text{ (i.e. only 1 vector which gives } A^T y = 0 \text{)}$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

↑ vector

Row which produces the zero row.

∴ only 1 vec in nullspace

* All 3×3 matrices form a space. let's call it M .

\therefore We can do $A+B$, CA and other's axioms are also satisfied

Subspaces of M

- ① Upper triangular matrices
- ② Symmetric matrices
- ③ ① \cap ② = Diagonal matrices

$$\dim((3)^{\text{rd}} \text{ space}) = 3$$

$$\text{Ex} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix} \right\}$$

Note: For a $m \times n$ matrix, max rank can be $\min(m, n)$.

Q. Suppose L U

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Find a basis for and compute the dimension of each of the 4 fundamental subspaces.

Soln:

For $\dim(C(B))$, we look at no. of pivots in U matrix.

$$\dim(C(B)) = 2$$

Basis for $C(B)$ is first columns in L matrix.

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\dim N(B) = 1 \quad [3 - \dim(C(B)) = 3 - 2 = 1]$$

Basis for $N(B)$ is,

$$\left\{ \begin{pmatrix} -3/5 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Put 1 for free variable in U and solve $UX=0$ for other 2 vars.

$$\dim C(B^T) = \dim C(B) = 2$$

∴ Elimination does n't change row space,
we take the 2 rows in U as basis

$$\left\{ \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\dim N(B^T) = m - r = 3 - 2 = 1$$

For basis,

$$B = LU$$

$$\therefore L^{-1}B = U$$

$$\therefore EB = U \text{ where } E = L^{-1}$$

$$\therefore \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

corresponds to free row

A basis for $N(B^T)$ is,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$