Lecture - 16

* Abstract vector spaces

→ There are many vectorish things. One prominent example is "functions".

$$(f+g)(x) = f(x) + g(x)$$

This is similar to,

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}$$

$$(2f)(x) = 2f(x)$$

This is similar to,

$$2\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

→ Like vectors, functions too, can be said to have linear transformations.

$$L\left(\frac{1}{9}x^3 - x\right) = \frac{1}{3}x^2 - 1$$

L can be a derivative (d/dx). Derivatives too, convert one function too another.

Here, they can also be called as "linear operators".

* How can transformation of functions be linear ?

→ Formal definition of linearity :

A function is linear if it satisfies two properties :

① Additivity : $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$

② Scaling : $L(c\vec{v}) = cL(\vec{v})$

① Additivity means that if you add 2 vectors 'v' and 'w', then apply a transformation to their sum, you get the same result as if you add the transformed versions of 'v' and 'w'.

② Scaling means when you scale a vector 'v' by some number, then apply the transformation, you get the same ultimate vector as if you scaled the transformed version of 'v' by ~~the~~ that same amount.

∴ "Linear transformations preserve addition and scalar multiplication"

* $\left(\frac{d}{dx}\right)$ Derivatives are additive and have scaling property.

$$\frac{d}{dx}(u+V) = \frac{du}{dx} + \frac{dV}{dx}$$

$$\frac{d}{dx}(Ku) = K\frac{du}{dx}$$

→ Describing the derivative with a matrix (Limit to polynomials)

Basis for polynomials:
$1, x, x^2, x^3, - - -$

$\therefore$ $b_0(x) = 1,$
$b_1(x) = x,$
$b_2(x) = x^2,$
$b_3(x) = x^3,$
⋮
∞

$\therefore$ Degree of a polynomial can be infinite

A polynomial $1x^2 + 3x + 5 \cdot 1$ will be described by the co-ordinates,

$$\begin{array}{r} 5\cdot 1 \\ +3x \\ +1x^2 \\ +0x^3 \\ +0x^4 \\ + \vdots \\ \vdots \end{array} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}$$

∞ many zeros

$$4x^7 - 5x^2 = \begin{bmatrix} 0 \\ 0 \\ -5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \\ \vdots \\ \vdots \end{bmatrix}$$

$$\frac{d}{dx}(1x^3 + 5x^2 + 4x + 5) = 3x^2 + 10x + 4$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 5 \\ 1 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 1\cdot 4 \\ 2\cdot 5 \\ 3\cdot 1 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}$$

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this is possible because the derivative is linear

→ How can we construct the derivative matrix ?

Take derivative of each basis ~~vector~~ function and putting the co-ordinate of the results in each column.

$$\frac{d}{dx} b_0(x) = \frac{d}{dx}(1) = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{d}{dx} b_1(x) = \frac{d}{dx}(x) = 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{d}{dx} b_2(x) = \frac{d}{dx}(x^2) = 2x = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{d}{dx} b_3(x) = \frac{d}{dx}(x^3) = 3x^2 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Linear algebra concepts	Alternate names when applied to functions
1) Linear transformations	Linear operations
2) Dot products	Inner products
3) Eigen vectors	Eigen functions

* This set of vectorish things like arrows, functions, etc are called vector spaces.

→ Rules for vectors addition and scaling. There are 8 rules / axioms that any vector space must satisfy.

1. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

2. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

3. There is a vector 0 such that $0 + \vec{v} = \vec{v}$ for all $\vec{v}$

4. For every vector $\vec{v}$ there is a vector $-\vec{v}$ so that $\vec{v} + (-\vec{v}) = 0$.

5. $a(b\vec{v}) = (ab)\vec{v}$

6. $1\vec{v} = \vec{v}$

7. $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$

8. $(a+b)\vec{v} = a\vec{v} + b\vec{v}$

In addition to these 8 axioms, every vectorish thing must obey it's basic rules of additivity and scaling.

→ Axioms are an interface between the person who discovered the results and others who want to apply those results to new sorts of vector spaces. So we form our results abstractly (i.e in terms of axioms).