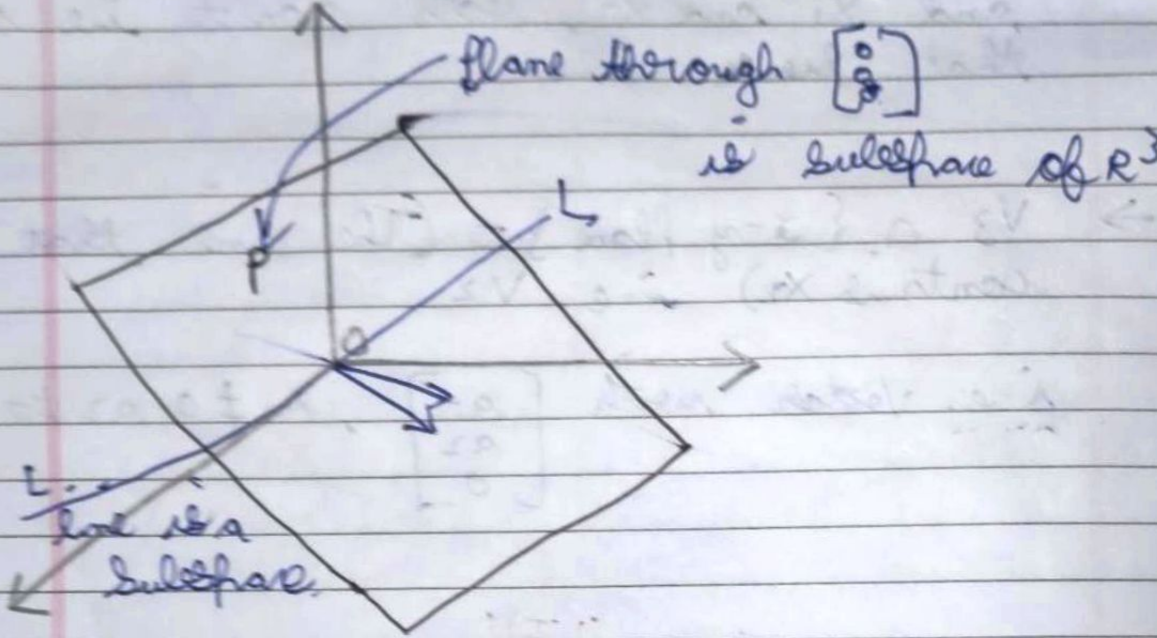
Prof Gilbert Strang

lect 6

Column Space and Nullspace



2 Subspaces: P and L

$P \cup L$ = all vectors in P or L or both
This is not a subspace.

Why?

→ Because if we take a vector from P and other one from L and add them, it does not lie in $P \cup L$.

$P \cap L$ = all vectors in both P and L.
It is a subspace.

This can happen if two vectors which are added are in both P and L.

Date
Page

→ Column Space of A is Subspace of R^4.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

$C(A)$ = all linear combs of columns of A
$C(A)$ is in R^4.

Does $Ax = b$ have a solution for every b?

Ans No ∵ we have 4 eqs and 3 unknowns

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Which b's allow the system to be solved?

① $b = 0$, ∴ $x_1 = x_2 = x_3 = 0$

② $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, ∴ $x_1 = 1$, $x_2 = x_3 = 0$

Basically $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = b$

③ $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, ∴ $x_2 = 1$, $x_1 = x_3 = 0$

④ $b = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, ∴ $x_1 = x_2 = 0$, $x_3 = 1$

Date
Page

∴ We can solve $Ax = b$ when b ~~is~~ is a linear combination of the column i.e. when it's in the column space $C(A)$

→ If we observe the columns of A, the 3^{rd} column is a LC of the first two columns. ∴ All 3 columns are not independent. This first two columns are the pivot columns and the 3^{rd} one is not a pivot column.

* Null Space

Null Space of A = All solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in R^3 to eq $Ax = 0$.

Now $b = 0$ and we are interested in x now.

Nullspace $N(A)$

$$Ax = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 C_1 + x_2 C_2 + x_3 C_3 = 0$

Solns :-

① $x_1 = x_2 = x_3 = 0$

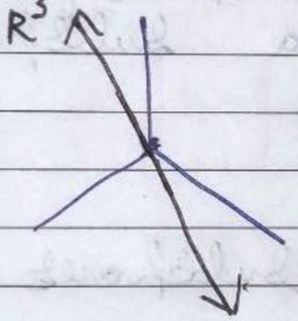
$x_1 = 1,\ x_2 = 1,\ x_3 = -1.$

② ~~$x_1 = x_2 = x_3$~~

It can be $x_1 = C,\ x_2 = C,\ x_3 = -C$

OR $C\begin{bmatrix}1\\1\\-1\end{bmatrix}$.

The nullspace is a line in R^3



But why is nullspace a space?

→ If $AV = 0$ and $AW = 0$, then $A(V+W) = 0$ which is true because $AV + AW = 0$ i.e. $A(V+W) = 0$

→ If $AV = 0$, then $A(12V) = 0$ because $12(AV) = 0$

∴ Nullspace is a space.

But if $b \neq 0$. Suppose

$$b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

In this case,

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \ldots$$

This is like a line / plane that does not go through the origin.
∴ It's not a subspace.

Q:- Which are subspaces of $R^3 = \left\{ \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \right\}$

1) $B_1 + B_2 - B_3 = 0$

2) $B_1 B_2 - B_3 = 0$

3) $\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

4) $\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Date ____
Page ____

Soln :-

1) $(1 \quad 1 \quad -1)\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = 0$

$\therefore \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$ is the nullspace of $(1 \ 1 \ -1)$.

$\therefore$ It's a subspace.

2) Vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a subset of the relatn given.

But $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ is not a subset of the relatn

But any multiple of a vector $2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ~~in~~ a subspace is ~~x~~ & must be in the subspace.

$\therefore$ The relatn is not a subspace.

3) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \left(\frac{1}{2}+C_1\right)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \left(\frac{1}{2}+C_2\right)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ It's a subspace

4) Any vector space must have zero vector in it. But there's no value of C1, C2, C3 such that $\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = 0$

∴ It's not a subspace

Also here, ~~one~~ 1st column can't be expressed as a LC of the other two columns.