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Lect - 9

Independence, Basis and Dimension

Suppose A is $m \times n$ with $m < n$

Then there are non-zero solutions to $AX = 0$

Why non-zero solns?

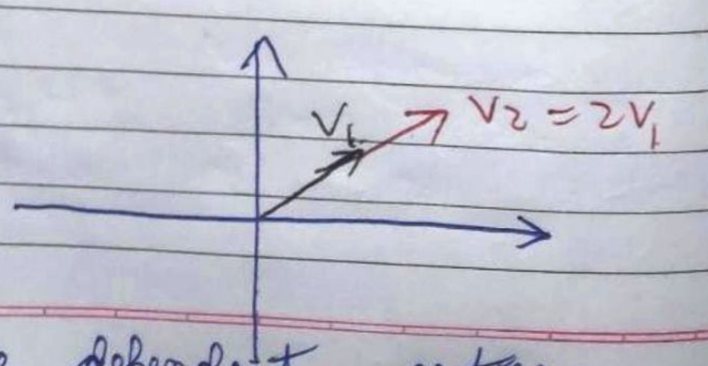
→ Because there are more unknowns than eqs, therefore there will be $n-m$ variables and we can take the values of these free variables as non-zero values.

* Independence

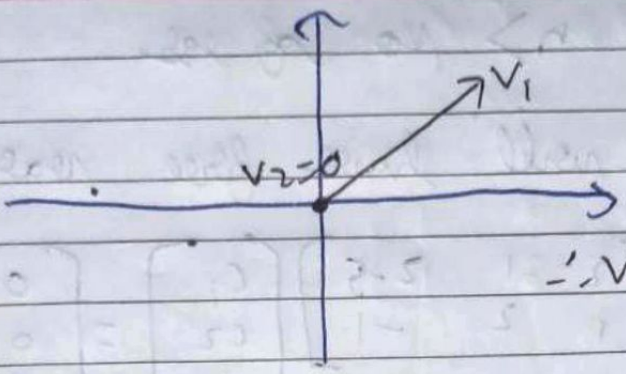
→ Vectors $x_1, x_2, \ldots x_n$ are linearly independent if no combination gives zero vector (except the zero combination all $c_i = 0$)

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \neq 0$$

otherwise they are linearly dependent vectors.



These are dependent vectors



$\therefore V_2 = 0 V_1$

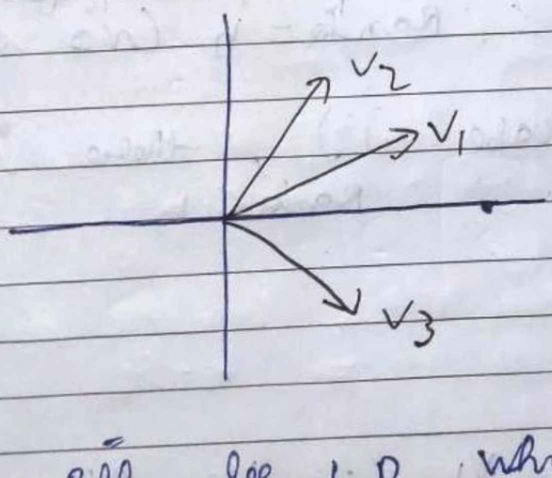
These are dependent vectors.

$\underline{0}\, V_1 + \underline{K}\, V_2 = 0$

K = Any constant

$\therefore$ If a set of vectors has one vector as zero vector, then those set of vectors cannot be linearly Independent.

→



These will be L.D. why?

Let $V_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $V_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $V_3 = \begin{bmatrix} 2.5 \\ -1 \end{bmatrix}$

Because let $A = \begin{array}{c} \begin{matrix} V_1 & V_2 & V_3 \end{matrix} \\ \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \end{array}$

Here No of unknowns (n) = No of columns = 3

No of eqs = No of rows = 2

$\because$ ~~n~~ $n >$ No of eqs.

$\therefore$ We will have free variables

$$\therefore \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

→ When $v_1, \ldots, v_n$ are columns of A.

i) They are independent if nullspace of A is {zero vector}.

ii) They are dependent if $AC = 0$ for some non zero C.

In case i), no of free vars must be 0. $\therefore$ Rank $= n$ (No of unknowns)

In case ii), there must be free vars $\therefore$ Rank $< n$

* Span

Vectors $v_1, \ldots, v_l$ span a space means: The space consists of all combs of those vectors.

* Basis

Basis for a space is a sequence of vectors $v_1, v_2, \ldots, v_d$ with 2 properties

1. They are independent.
2. They span the space.

Example:
Space is R^3

One basis is: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

Another Basis: $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$

R^n n vectors give basis if the $n \times n$ matrix with those columns is invertible.

∴ Basis are not unique for a space.

→ Given a space, every ~~of~~ basis for the space has the same number of vectors. And this number (of vectors) is the dimension of the space.

Space is C(A)

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

~~N(A)~~ 2 = rank(A) = Number of pivot columns = dimension of C(A) → dim of C(A) not of A.

Another Basis for C(A). $\begin{bmatrix} 2 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$

N(A)

X of AX=0 → $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

∵ We hv 2 pivot vars and hence we have 4−2 = 2 free vars,

∴ dim N(A) = Number of free variables = n − r

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Q. Find the dimension of the vector space spanned by the vectors
(1, 1, −2, 0, −1)
(1, 2, 0, −4, 1)
(0, 1, 3, −3, 2)
(2, 3, 0, −2, 0)

and find a basis for that space.

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 1 & 2 & 0 & -4 & 1 \\ 0 & 1 & 3 & -3 & 2 \\ 2 & 3 & 0 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 1 & 2 & -4 & 2 \\ 0 & 1 & 3 & -3 & 2 \\ 0 & 1 & 4 & -2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 1 & 2 & -4 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 1 & 2 & -4 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \Leftarrow \\ \Leftarrow \\ \Leftarrow \\ \\ \end{matrix}$$

∴ Basis : (1, 1, −2, 0, −1)
(0, 1, 2, −4, 2)
(0, 0, 1, 1, 0)

We could also have taken the vectors ~~s~~ in the 1st 3 rows of the initial matrix as basis

i.e. $(1, 1, -2, 0, -1)$,
$(1, 2, 0, -4, 1)$
$(0, 1, 3, -3, 2)$

But while doing this, we must take care that if we switch rows while eliminating final basis will also switch accordingly.

$\therefore$ Dimension of the vector space = 3.

Another method is put the given vectors as columns of a matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 3 \\ -2 & 0 & 3 & 0 \\ 0 & -4 & -3 & -2 \\ -1 & 1 & 2 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Dimension = 3

But here we can't use the first columns of reduced matrix as basis as the column space of A and reduced matrix are different.

$\therefore$ Here, basis vectors have to be taken from the initial matrix.

$$\therefore \text{Basis} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ -3 \\ 2 \end{bmatrix}$$