Prof Gilbert Strang

Lecture - 10

The Four Fundamental Subspaces

In last lecture, we, in the basis section, wrote $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$ as another basis for 3-D space.

But this is wrong because these vectors are not linearly independent (L.I). If we form matrix $A = \begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\2 & 5 & 8\end{bmatrix}$,

$|A| = 0$ ∴ The matrix is not invertible (because row 1 and row 2 are same).

We already know about column space and null space. Now we'll learn about row space.

Here, the row space of A is only 2-D. ∴ Rank(A) with these columns is 2. ∴ only 2 of those columns can be independent.

If A is $m \times n$

* 4 Subspaces

1) Column Space $C(A)$ in R^m

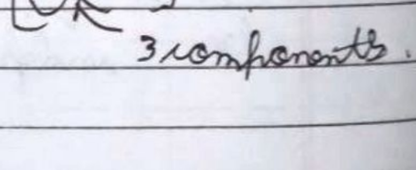
2) Null Space $N(A)$ in R^n

3) Row Space = all combs of rows
= all combs of columns of A^T
$C(A^T)$ in R^n

4) Nullspace of A^T = $N(A^T)$ = left Nullspace of A in R^m

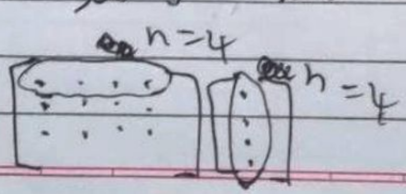
Rowspace is the span of rows of A. The rows form a basis for rowspace if they are linearly independent.

1) Column space is a LC of columns and each column has m components



$\therefore$ $C(A)$ is in R^m

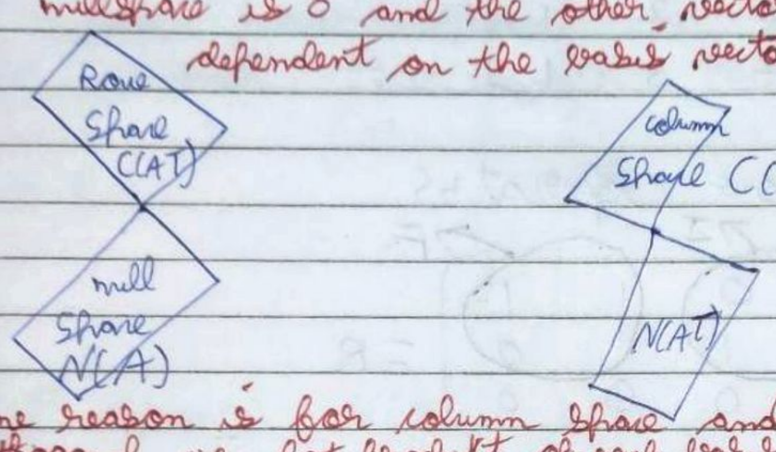
2) Null space is soln to $AX=0$. If A is $m \times n$, X is $\underline{n} \times \underline{1}$ $\therefore$ X has n components. Hence, it's in R^n



3) and 4) are transpose of 1) and 2) respectively.

* Basis and dimension of all 4 spaces

Row space and nullspace are said to be orthogonal because the dot product of each vector in basis of row space with vectors in nullspace is 0 and the other vectors in row space are dependent on the basis vectors.



Same reason is for column space and left null space being orthogonal. Here dot product of each basis vector of column space of A with $N(A^T) = 0$ and other vectors are dependent on them.

→ $\dim(C(A)) = r$ = (No of pivots) = No of vectors in basis

∴ Basis for $C(A)$ are pivot columns

→ $\dim$ (Row space) $= r$ as seen in Sl1 today.

→ For finding nullspace, we reduce to RREF and then find special solns by putting free variables 1 and 0. Also, no of special solns = No of free variables $= n - r$ (n = No of columns, matrix is $m \times n$)

Then, special solns form basis for nullspace and $\dim(N(A)) = n - r$

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→ ∴ $\dim(N(A^T)) = m - r$. ∵ Here matrix is ~~matrix~~ $n \times m$ ∴ No of columns is m.

→ Basis for rowspace

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

By row operatns,

$$\sim \begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{I} & \overbrace{\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}}^{F} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \end{bmatrix} = R$$

Due to row operatns, column space of A and R are different.

$C(A) \neq C(R)$

But the rowspace of A and R remain same.

The 3^{rd} row of A is useless just as the 3^{rd} row of R.

Whatever you get by taking combinatns of first 2 rows of A will be obtained by taking combinatns of first 2 rows of R.

∴ Here basis for rowspace (R) = basis for rowspace (A) = first 2 rows. of R.

Generalizing,

Basis for row space of A = Basis for row space of R = first r rows of R.

where r = rank.

∴ The row space is spanned by all the rows but the basis for row space is formed by the r independent rows.

→ 4th space: $N(A^T)$

$$A^T y = 0$$

$$\begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \end{bmatrix}$$

$(A^T y)^T = 0^T$

$$\therefore y^T A = 0^T$$

$$\therefore \underset{\text{Row Vector}}{\begin{bmatrix} y^T \end{bmatrix}} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

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rref $\left[\underset{m\times n}{A} \quad \underset{m\times m}{I}\right] \longrightarrow \left[\underset{m\times n}{R} \quad \underset{m\times m}{E}\right]$

E is rref because E contains the records of the operations it took to go from A to R

$\therefore E\left[\underset{m\times n}{A} \quad \underset{m\times m}{I}\right] \longrightarrow \left[\underset{m\times n}{R} \quad \underset{m\times m}{E}\right]$

$$EA = R$$

Rank

$\downarrow$

$r(A) = 2$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Performing these same operations of I, we get E

$$\underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \rightarrow \underset{E}{\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}$$

$\therefore \dim(N(A^T)) = m - r = 3 - 2 = 1$ (i.e. Only 1 Vector ~~nullspace~~ which gives $A^T y = 0$.

$\therefore$ Only 1 vec in nullspace)

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$\uparrow$ ~~Row~~ Vector which produces the zero row.

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* All 3×3 matrices for a space. Let's call it m.

∴ We can do $A+B$, cA and other 8 axioms are also satisfied

Subspaces of M

① Upper triangular matrices

② Symmetric matrices

③ ① ∩ ② = Diagonal matrices

dim (③rd space) = 3

$$Ex = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix} \right\}$$

Note: For a $m \times n$ matrix, max rank can be $\min(m, n)$.

Q. Suppose

$$B = \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$$

Find a basis for and compute the dimension of each of the 4 fundamental subspaces.

Soln :-

For dim (C(B)), we look at no of pivots in U matrix.

dim C(B) = 2

Basis for C(B) is pivot columns in L matrix.

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

dim N(B) = 1 [3 - dim C(B) = 3 - 2 = 1]

Basis for N(B) is,

$$\left\{ \begin{pmatrix} -3/5 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Put 1 for free variable in U and solve Ux = 0 for other 2 vars.

$\dim C(B^T) = \dim C(B) = 2$

$\because$ Elimination doesn't change row space, we take the 2 rows in U as basis

$$\left\{ \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\dim N(B^T) = m - r = 3 - 2 = 1$

For basis,

$B = LU$

$\therefore L^{-1}B = U$

$\therefore EB = U$ where $E = L^{-1}$

$$\therefore \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} B = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

corresponds to free row

A basis for $N(B^T)$ is,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$