Prof Gilbert Strang

Lecture 16

Projection Matrices and least Squares

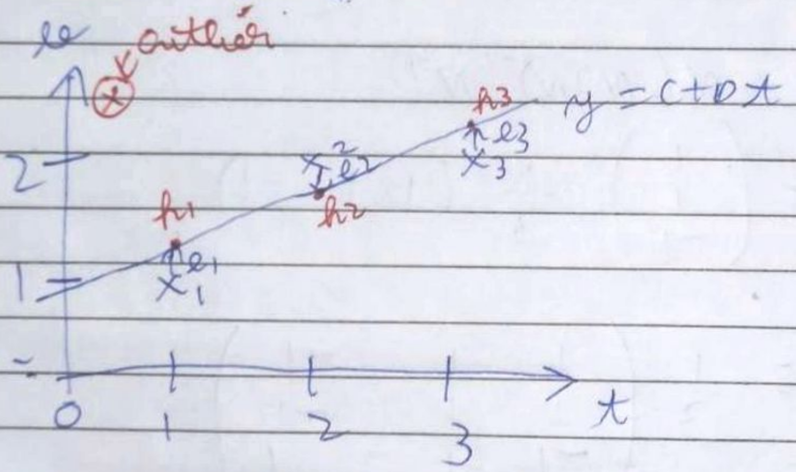
$P = A(A^TA)^{-1}A^T$

If b is in column space, $Pb = b$

If b is $\perp$ column space, $Pb = 0$, i.e. b is in $N(A^T)$

If b is in the column space, it's of the form Ax

* least square



$C + D = 1$

$C + 2D = 2$

$C + 3D = 2$

$Ax = b$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Minimize $||A - bx||^2 = ||e||^2$
$= e_1^2 + e_2^2 + e_3^2$

This is called linear regression.

If any pt is way off the line's path, (outlier), the we shouldn't use least square method ∵ $e^2_{outlier}$ would be extremely large.

p_1, p_2, p_3 lie on the line and they lie in the column space of A. It's the best option we have. It's closest point in column space.

Find $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, p

$A^T A \hat{x} = A^T b$ Normal eqs.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Date
Page

$\therefore\ 3C + 6D = 5$
$6C + 14D = 11$

$$e_1^2 + e_2^2 + e_3^2 = (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$$

To minimize above expression, we take partial derivatives w.r.t 'C' and 'D'
① $C = 0$ then ② $D = 0$
We will get these eqs only.

$$3C + 6D = 5$$
$$6C + 14D = 11$$

$\therefore D = \frac{1}{2}$

$\therefore C = \frac{2}{3}$

$\therefore$ Best line $\frac{2}{3} + \frac{1}{2}t$, $\therefore C = \frac{2}{3}$, $D = \frac{1}{2}$

$y = \frac{2}{3} + \frac{1}{2}t$
or b

At $t=1$, $b_1 = 7/6$

At $t=2$, $b_2 = 5/3$

At $t=3$, $b_3 = 13/6$

$e_1 = \cancel{C+D-1}\ 1-C-D = -1/6$

$e_2 = \cancel{C+2D-2}\ 2-C-2D = +2/6$

$e_3 = \cancel{C+3D-2}\ 2-C-3D = -1/6$

$\therefore p_1 + e_1 = b_1$
$p_2 + e_2 = b_2$
$p_3 + e_3 = b_3$

$\therefore b = p + e$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$$

→ If A has independent columns, then A^TA is invertible

To prove: If $A^TAx = 0$, then ~~xx~~ x must be 0.
i.e. A matrix is invertible if it's nullspace only has the zero vector.

$$A^TAx = 0$$
$$x^TA^TAx = 0$$
$$\therefore (Ax)^T Ax = 0$$
let $Ax = y$
$$\therefore y^T y = 0$$

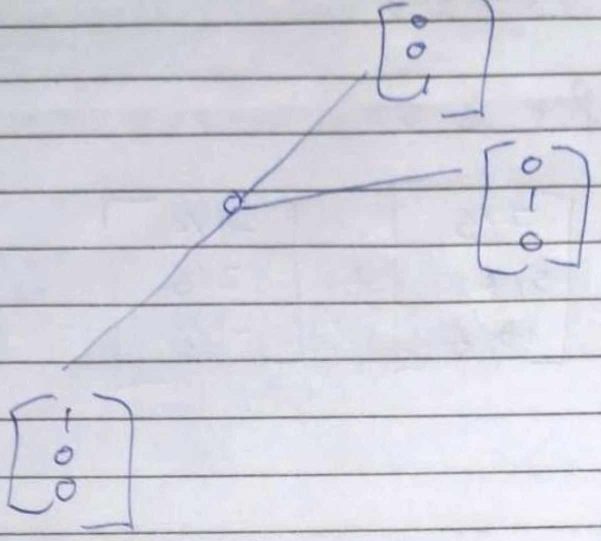
But $y^Ty = \|y\|^2$
$$\therefore \|y\|^2 = 0$$
$$\therefore y = 0$$
$$\therefore Ax = 0$$

$\therefore$ ~~then A is invertible~~ $\therefore$ $|A| \neq 0$, $\therefore$ A can't be null matrix

If A has independent columns and $Ax = 0$, then $x = 0$

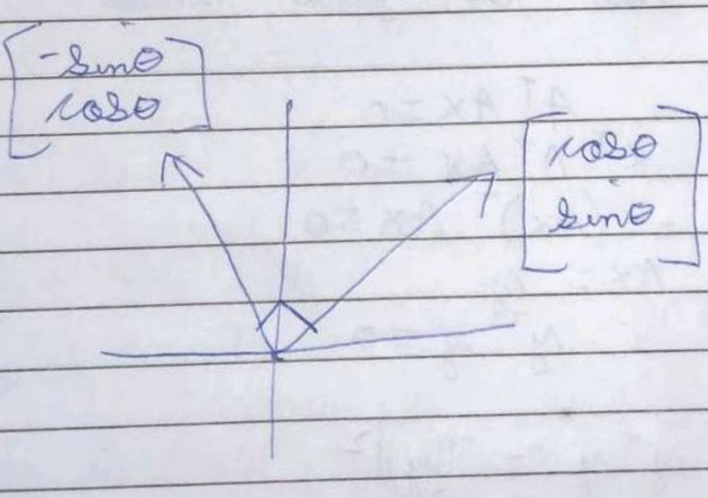
* Columns are definitely independent if they are perpendicular unit vectors.

Ex



Perpendicular unit vectors are called orthonormal vectors.

Another example:



Q. Find the quadratic equation through the origin that is a best fit for the points $(1,1)$, $(2,5)$, $(-1,-2)$

Soln :-

$\because$ Eq passes through origin, constant term is 0.

$\therefore$ Required eq: $ct + dt^2 = y$

$$A = \begin{array}{c} \begin{array}{cc} t & t^2 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \end{array}, \quad \hat{x} = \begin{pmatrix} c \\ d \end{pmatrix}, \quad b = \begin{array}{c} y \\ \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \end{array}$$

Can't solve $A\hat{x} = b$ directly

$\therefore$ We solve, $A^TA\hat{x} = A^Tb$ (Projection eq)

$$A^TA = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 8 & 10 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 13 \\ 19 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 8 \\ 8 & 10 \end{pmatrix} \hat{x} = \begin{pmatrix} 13 \\ 19 \end{pmatrix}$$

$$\sim \begin{pmatrix} 6 & 8 \\ 0 & -2 \end{pmatrix} \hat{x} = \begin{pmatrix} 13 \\ 5 \end{pmatrix}$$

$d = -5/2, \quad c = 11/2$

$\therefore \boxed{y = \frac{11}{2}t - \frac{5}{2}t^2}$