Prof Gilbert Strang

Lect 5

Date______
Page______

Transposes, Permutations, Spaces R^n

→ $PA = LU$ (For any invertible A)
The P (Permutation matrix) is added to ensure that pivots in A won't be 0, L and U turn out to be right.

P = Identity matrix with reordered rows

→ For a n×n ordered matrix the number of reorderings / number of all n×n permutations is $n!$

→ $P^{-1} = P^T$

$\therefore P^T P = I$

* Transpose

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

→ $A_{ji} = (A^T)_{ij}$

Date________
Page________

* Symmetric Matrices

→ $A^T = A$

Ex: $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

→ $R^T R$ is always symmetric.

$$\underset{R}{\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}} \underset{R^T}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}} = \underset{RR^T}{\begin{bmatrix} 10 & 11 & 7 \\ 11 & - & - \\ 7 & - & - \end{bmatrix}}$$

Now $(RR^T)^T = (R^T)^T R^T = RR^T$

* Vector Spaces

Examples: R^2 (all two-D real vectors)

Ex: $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}, \ldots$

"X–Y plane"

→ It is a vector space because it contains every 2-D vector.

→ If we add 2 vectors, multiply a scalar to a vector, etc, the resultant vector formed must be in the vector space for it to qualify as a vector space

→ Every vector space must have the zero vector.

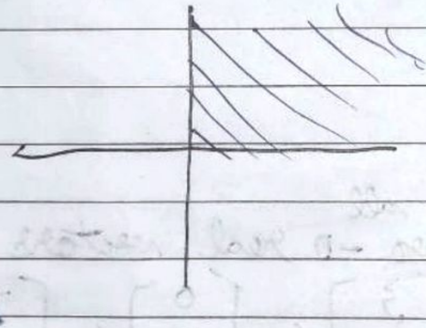
R^3 = all vectors with 3 real components

Ex: $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

R^n = all column vectors with n real components

→ A vector space has to be closed under multiplication and addition of vectors, in other words linear combinations. (Refer the 8 axioms from 3B1B Notes).

→ Example of 'not a vector space'.



← It contains only vector spaces with non-negative components like $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

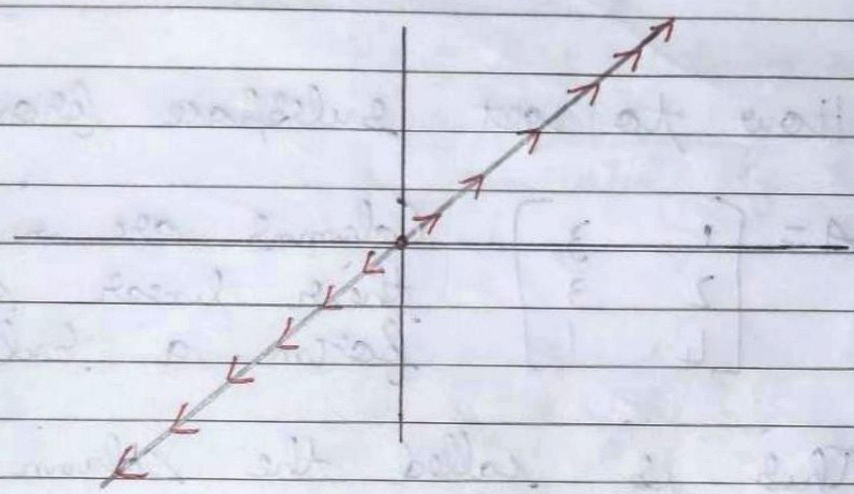
But if we multiply $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ by -5, then we get $\begin{bmatrix} -10 \\ -15 \end{bmatrix}$ as resultant which will not lie in the vector space shown above. ∴ It's not a vector space.

Date ______
Page ______

* Subspaces

A vector space within a vector space is called a subspace.

Example of a subspace in R^2 is the line in R^2 passing through the zero vector.



→ The scalar multiple of any vector on the line lies on the line.

→ The addition / subtraction of ~~two~~ any two vectors on the line lies on the line

→ Subspaces of R^2

① All of R^2

② Any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (One-D subspace)

③ Zero vector only (Z)

Note: Note a one-D subspace here (Pt ②) is not same as R^1 subspace. A R^1 subspace has only 1 component, not 2.

→ Subspaces of R^3

① All of R^3

② Any ~~...~~ plane through the origin $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

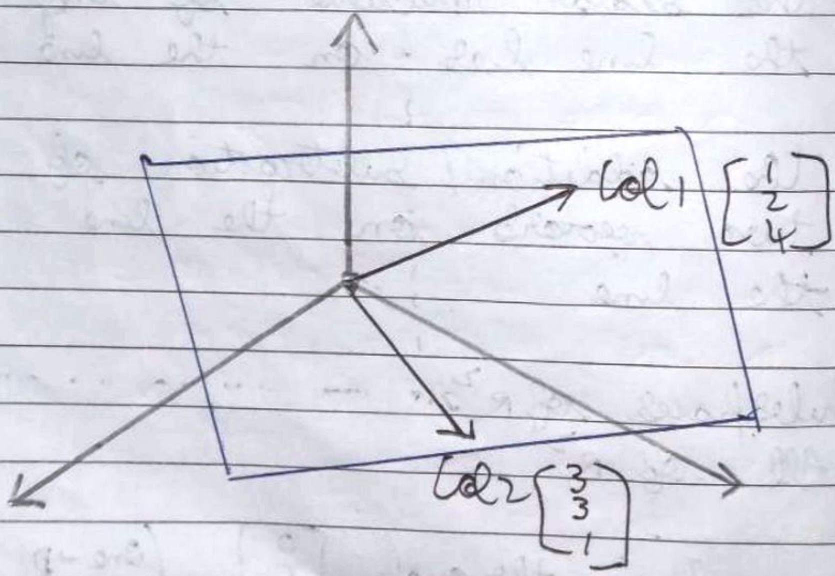
③ Any line through the origin $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

④ Zero Vector only.

* How to get subspace from matrices?

$A = \begin{bmatrix}1 & 3\\2 & 3\\4 & 1\end{bmatrix}$ Columns are in R^3 and all their linear combinations form a subspace

This is called the column space $C(A)$.



The subspace formed by these two column vectors will be a line passing through the origin.

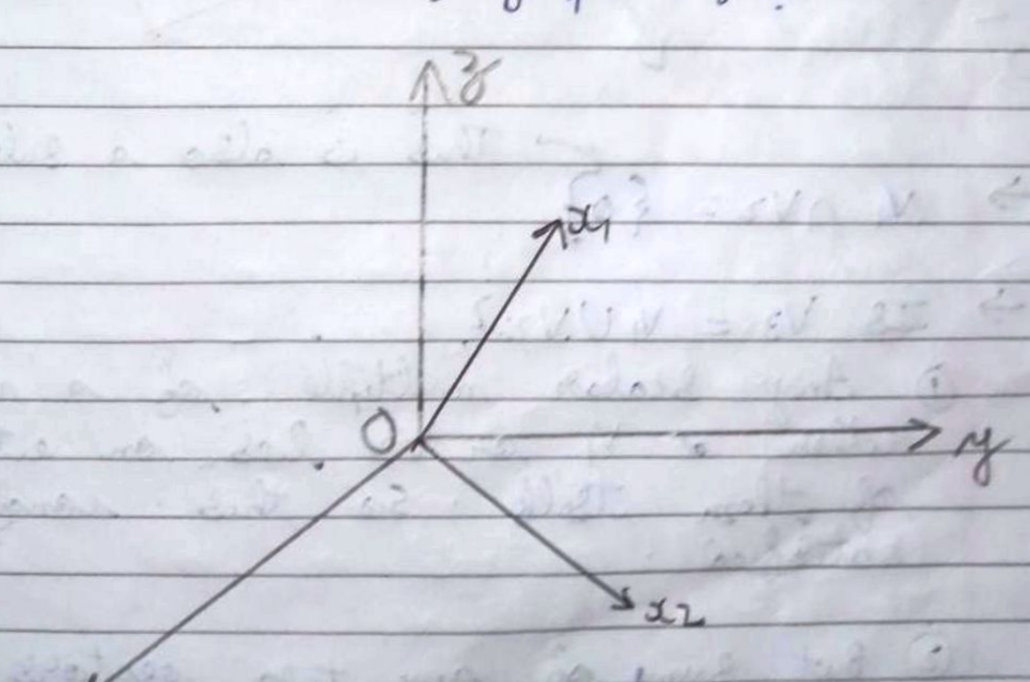
If our two column vectors were on the same line, then our column space would be a line.

Q. $x_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, $x_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$

1. Find V_1 = Subspace generated by x_1 (Smallest subspace containing x_1)
   V_2 = Subspace generated by x_2
   Describe $V_1 \cap V_2$

2. Find V_3 = Subspace generated by $\{x_1, x_2\}$.
   Is V_3 equal to $V_1 \cup V_2$?
   Find a subspace S of V_3 s.t. $x_1 \notin S$, $x_2 \notin S$

3. What is $V_3 \cap \{xy \text{ plane}\}$?



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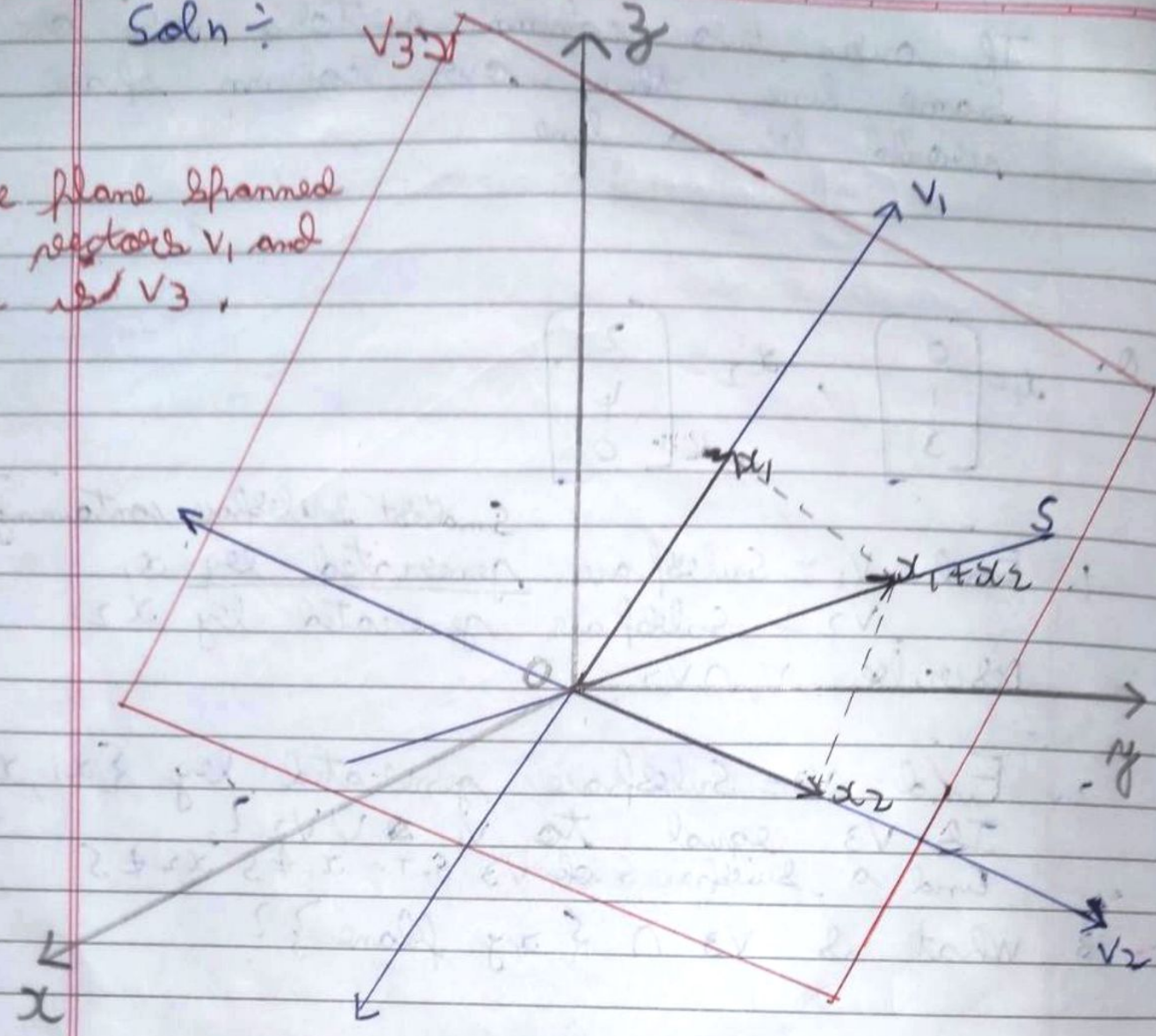

→ $V_1 \cap V_2 = \{0\}$. This is also a subspace of $\mathbb{R}^3$

→ Is $V_3 = V_1 \cup V_2$?

① Any scalar multiple of a vector on either V_1 or V_2 lies on either one of them itself. So this condition is satisfied :

② But sum of any two vectors from V_1 and V_2, ex: $X_1 + X_2 = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} \notin V_1 \cup V_2$

$\therefore V_1 \cup V_2$ is not a subspace

→ Not here S ~~is~~ not unique. We can have $2X_1 + 2X_2$ line which lies within ~~S~~ V_3 and X_1 and X_2, both don't lie on that ~~line~~ line.

→ $V_3 \cap \{x\text{-}y \text{ plane}\}$ = (The line that contains X_2) i.e. V_2

i.e. Vector with $\begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}$, $a_1 \neq 0, a_2 \neq 0$

Note: No of linearly independent (LI) columns = Dimension of column Space ~~(i.e. R^n where n = No of LI columns)~~

Ex: $\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 2 \\ 0 & 3 \end{bmatrix}$ It's Space is R^4 but it's (Column Space) dimension is 2

So, R^n or Space is the number of components of the vector (Eg: $\hat{i} + 2\hat{j} + 3\hat{k}$. Here R^n, n = 3 ∴ It's Space is R^3).

Whereas Dimension = n − r (where n = Number of unknowns and r = Number of non-zero rows) OR
Dimension = Number of LI columns or
Dimension = Number of basis vectors.

∴ For different set of basis vectors for a given vector space, the dimension will always remain same since no of basis vectors remains same.