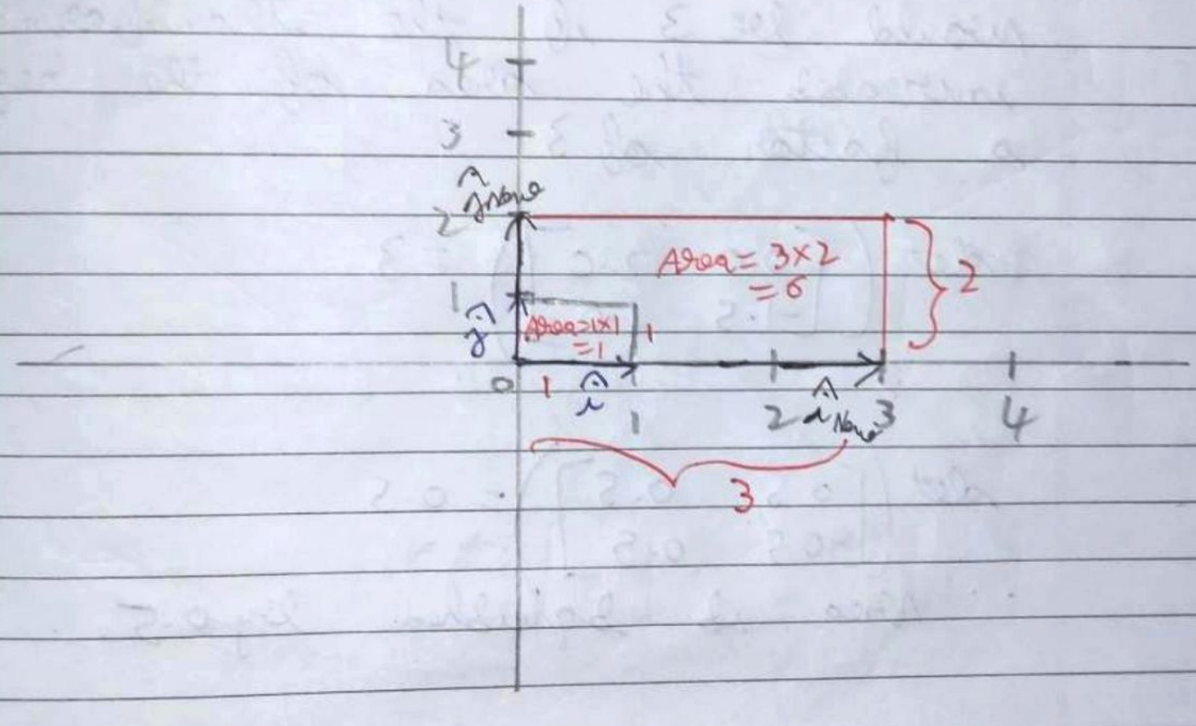
~~Chapter~~ Lecture -6

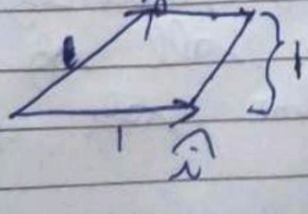
* The Determinant

Determinant of a matrix (i.e. transformation) tells us how much area is scaled/ ~~stretched~~ squished during the transformation

Ex: $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$



Shear matrix : $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



Area = $b \times h = 1 \times 1 = 1$ (Remains Same)

Whatever scaling / squishing happens to one square on the grid must happen to every square on the grid. Since grid lines have to remain parallel and evenly spaced.
Hence we can determine the change for any grid square on the grid.

The determinant of a transformation would be 3 if the transformation increases the area of the region by a factor of 3

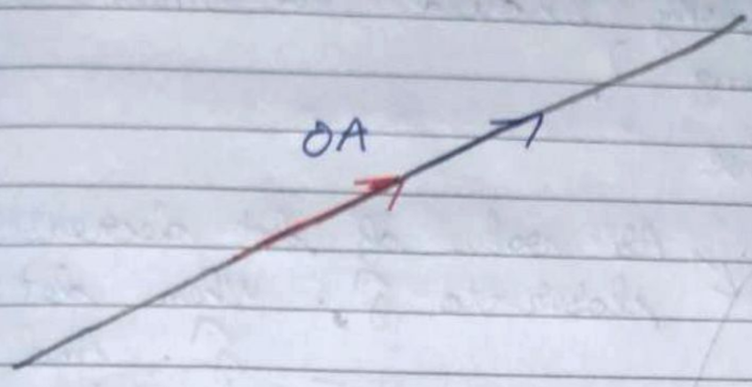
$$\det\left(\begin{bmatrix} 0.0 & 2.0 \\ -1.5 & 1.0 \end{bmatrix}\right) = 3$$

$$\det\left(\begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}\right) = 0.5$$

$\therefore$ Area is squished by 0.5.

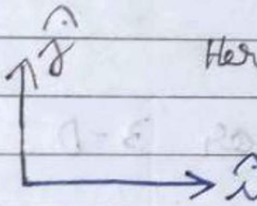
$$\det\left(\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\right) = 0$$

Determinant of a 2D transformation is 0, if it squishes all of area onto a single line or point. ($\because$ Area $= 0$)



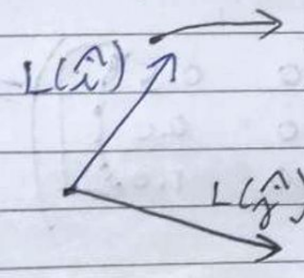
But determinants can be -ve too

In -ve Determinants, the '$-$' sign indicates that the orientation has been reversed.



Here $\hat{j}$ is to the left of $\hat{i}$

After transformation,

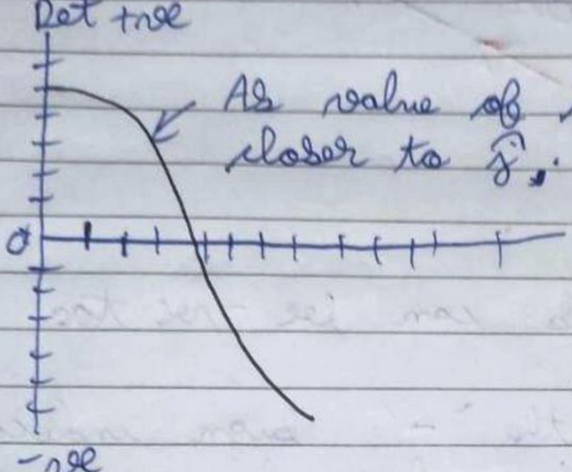


Now $\hat{j}$ is to the right of $\hat{i}$. $\therefore$ orientation has been reversed.

Suppose Determinant $= -5$
Here '$-$' indicates changed orientation and 5 means scaling (increase) of area by factor of 5.

Det $= -0.5$
0.5 still indicates squishing of area by 0.5.

But why should -ve mean orientation flipping ?



As value of det decreases, $\hat{i}$ comes closer to $\hat{j}$. When det = 0, $\hat{i}$ overlaps $\hat{j}$. And then det = -ve, orientation changes, $\hat{i}$ moves away from $\hat{j}$. and det increases in magnitude again

* Determinants for 3-D.

→ In 3-D, determinants give us the volume that is squished or scaled.

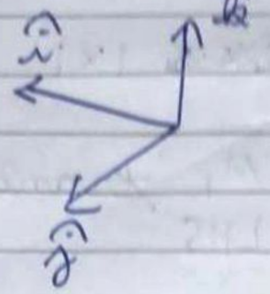
$$\det\left(\begin{bmatrix} 1.0 & 0.0 & 0.5 \\ 0.5 & 1.0 & 0.0 \\ 1.0 & 0.0 & 1.0 \end{bmatrix}\right) = \text{Volume of parallelopiped}$$

→ Det = 0 means all of space is squished to a single plane, line or point.

This means that,

$$\underbrace{\det\left(\begin{bmatrix} 1.0 & 0.0 & 1.0 \\ 0.5 & 1.0 & 1.5 \\ 1.0 & 0.0 & 1.0 \end{bmatrix}\right)}_{\text{Columns must be linearly dependent}} = 0$$

→ If determinant < 0, what does it mean?



Right Hand Thumb Rule (RHTR)

Step 1: Represent the basis vectors with RHTR before transformation.

Step 2: If After transformation, we are still able to do that, then det > 0 and orientation hasn't changed.

Step 3: If we are not able to do that and we now need to represent them using left hand, then det < 0 and orientation has changed.

* Computing determinant

→ $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

→ $\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$

Q. $\det(M_1 M_2) = \det(M_1)\det(M_2)$?

let Initial area = ~~1 ×~~ 1×1

If after M_2 and M_1 transformations,
area = $r \times d$ = LHS

Now on RHS,
$\det(M_1) = r$, $\det(M_2) = d$

$\therefore$ area = $d \times r = rd$ = RHS
$\therefore$ LHS = RHS
$\therefore$ $\det(M_1 M_2) = \det(M_1)\det(M_2)$