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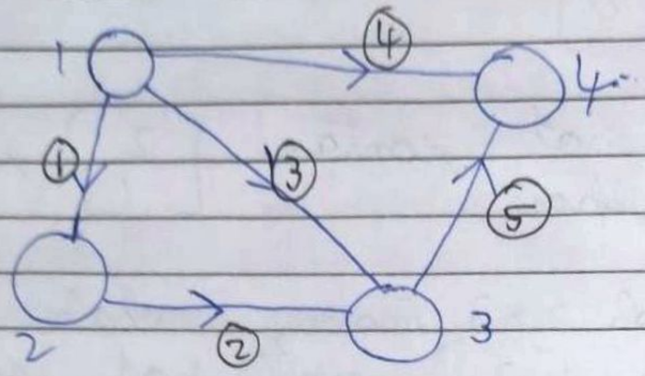
Lecture - 12

Graphs, Networks, ~~Matrix~~ Incidence Matrices

Let this be a circuit



m = 5 edges
n = 4 nodes.

* Incidence matrix

$$A = \begin{array}{cccc|l} \text{Node1} & \text{Node2} & \text{Node3} & \text{Node4} & \\ -1 & 1 & 0 & 0 & \text{edge 1} \\ 0 & -1 & 1 & 0 & \text{edge 2} \\ -1 & 0 & 1 & 0 & \text{edge 3} \\ -1 & 0 & 0 & 1 & \text{edge 4} \\ 0 & 0 & -1 & 1 & \text{edge 5} \end{array}$$

−1 indicates that the edge is going away from the node.
1 indicates that the edge is coming towards the node.

Edges ①, ②, ③ form a loop here.
Matrix only formed by edges 1, 2 and 3.

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

This matrix has linearly dependent rows.
Row 1 + Row 2 = Row 3

* Nullspace

Solve $AX=0$

$$AX = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$X = x_1, x_2, x_3, x_4$
Potentials at node

$\therefore x_2 - x_1, x_3 - x_2, \ldots$ are all potential differences (PD)

For A to have nullspace, all PD'S $= 0$.
i.e. All potentials must be equal..

let all $x_1 = x_2 = x_3 = x_4 = 1$.

$$\therefore X = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \leftarrow \text{Null Space}$$

$\dim N(A) = 1$

$\because$ All PD'S are zero, current won't flow through the circuit.

Now we ground the node x_4.
i.e. $V_4 = 0V$ $(x_4 = 0)$

$\therefore$ The 4^{th} column in A (column corresponding to x_4) becomes 0.

$\therefore$ Rank of $A = 3 = r$

* Left Null Space, $N(A^T)$
$A^T y = 0$

$\dim N(A^T) = m - r = 5 - 3 = 2$

$A^T_{\substack{n \times m \\ 4 \times 5}}$

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$e = Ax$ $\downarrow$ $x_2 - x_1, \ldots$ PD

$y = Ce$ $\longrightarrow$ Ohm's Law; Conductance

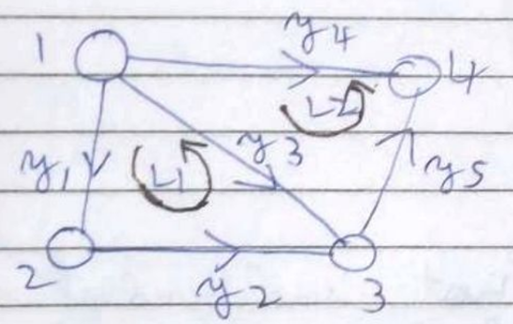
Currents $y_1, y_2, \ldots, y_5$ on edges $\uparrow$ $A^T y = 0$ Kirchhoff's Current Law

From $A^T y = 0$, we get

$-y_1 - y_3 - y_4 = 0$
$y_1 - y_2 = 0$
$y_2 + y_3 - y_5 = 0$
$y_4 + y_5 = 0$

All these eqs when seen in the graph satisfy KCL i.e. Summation of current at a node / Junction is 0.

For basis of $N(A^T)$



Send 1A current through loop 1 satisfying KCL

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$\leftarrow$ Flow in opp directn

Send 1A current through loop 2 satisfying KCL

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$ ← opp directn

If we consider loop 1234 [U],

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

But this is not independent. It's a combinatn of loop 1 and loop 2 solutions.

∴ Basis for $N(A^T)$ will only have 2 vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

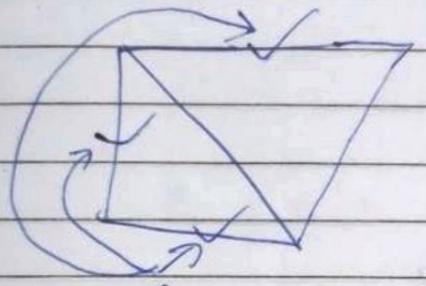
* Row space of A

dim (row space of A) = 3

i.e. No of independent rows of A or no of independent ~~rows~~ columns of A^T.

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

pivot colm of A^T



These are basis of row space of A

These 3 independent rows form a graph without a loop which is a tree.

dim $N(A^T) = m - r$, rank = r

No of loops = No of edges − No of nodes − 1

$$\therefore \boxed{\text{No of nodes} - \text{No of edges} + \text{No of loops} = 1}$$

This is Euler's formula which is true for any graph.

→ If, we add current source to our circuit, $A^Ty = 0$ changes to $A^Ty = f$ where f = Current source

$$\therefore \overset{PD}{e} = Ax$$

But $y = Ce$

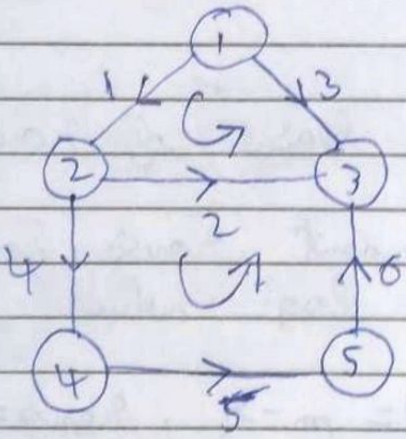
$\therefore y = CAx$

But $A^Ty = f$

$$\therefore \boxed{A^TCAx = f}$$

Note: A^TA is always a symmetric matrix

Q.



Find incidence matrix A

$N(A)$, $N(A^T)$

Trace (A^TA)

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$$A = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{ccccc|l} \text{Node1} & \text{Node2} & \text{Node3} & \text{Node4} & \text{Node5} & \\ -1 & 1 & 0 & 0 & 0 & \text{edge1} \\ 0 & -1 & 1 & 0 & 0 & \text{edge2} \\ -1 & 0 & 1 & 0 & 0 & \text{edge3} \\ 0 & -1 & 0 & 1 & 0 & \text{edge4} \\ 0 & 0 & 0 & -1 & 1 & \text{edge5} \\ 0 & 0 & 1 & 0 & -1 & \text{edge6} \end{array}$$

$Ax = 0$

This means that all potentials are equal.

$$\therefore N(A) = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$A^T y = 0$

$$y = c_1 \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{Loop 1}} + c_2 \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{Loop 2}}$$

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→ Trace (A^TA) is sum of diagonal elements of A^TA.

→ The diagonal enteries of $A^T A$ are the magnitude squared of columns of A.

→ Each entry of A is either -1, 0 or 1.
∴ Square of enteries will be either 0 or 1.

→ This equals to number of non-zero enteries in each column.

→ This tells us that Trace (A^TA) is the sum of degrees of each node.

Degree of ① = 2
Degree of ② = 3
Degree of ③ = 3
Degree of ④ = 2
Degree of ⑤ = 2
~~Degree of ⑥ 4~~

∴ Trace $(A^TA) = 2 + 3 + 3 + 2 + 2$
$= 12$