Chapter -12

* Cramer's rule, explained geometrically

$$3x + 2y = -4$$
$$-x + 2y = -2$$

$$\begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

Here we know the transformation matrix and the output vector. The unknown is the vector with which the dot product is taken (the initial vector)

* Here we are finding which input vector is going to lie on the output vector $-4\hat{i} - 2\hat{j}$ when dotted with the matrix

$$\begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}.$$

$$\therefore x \begin{bmatrix} 3 \\ -1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

$$A\vec{x} = \vec{v}$$

→ If $\det(A) = 0$, then either none of the inputs land on the output or a whole bunch of inputs land on the output.

→ Here we consider $\det(A) \neq 0$, i.e. every input has only one output and vice versa.

→ For linear transformations, dot products before and after transformation are generally different.

→ The transformations whose dot products remain same before and after transformation are called orthonormal ~~dot products~~ transformations.

Imp These leave all basis vectors $\perp$ to each other and still with unit length. We know them as rotation matrices :

If $T(\vec{v}) \cdot T(\vec{w}) = \vec{v} \cdot \vec{w}$ for all $\vec{v}$ and $\vec{w}$, then T is "orthonormal"

∵ Dot products are preserved, ~~taking dot~~

$$\underbrace{\begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{bmatrix}}_{\text{Orthonormal}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Taking dot product between output vector and all the columns of the matrix will be the same as taking the dot product between the mystery input vector and all of the basis vectors. i.e. directly finding 'x' and 'y'.

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \cos(30^\circ) \\ \sin(30^\circ) \end{bmatrix}$$

i.e. $$\boxed{x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \cos(30^\circ) \\ \sin(30^\circ) \end{bmatrix}}$$

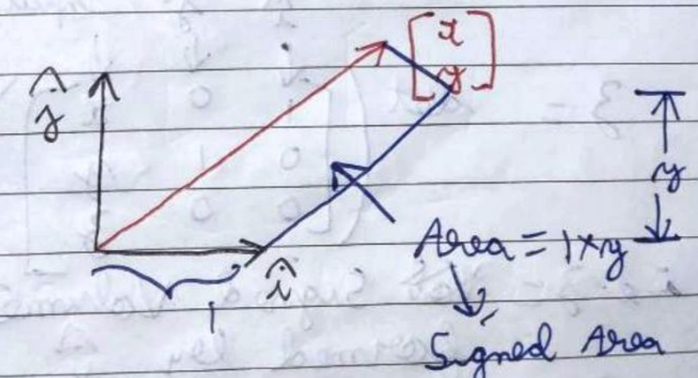
$$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -\sin(30^\circ) \\ \cos(30^\circ) \end{bmatrix}$$

i.e. $$\boxed{y = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \cancel{\cos} -\sin(30^\circ) \\ \cos(30^\circ) \end{bmatrix}}$$

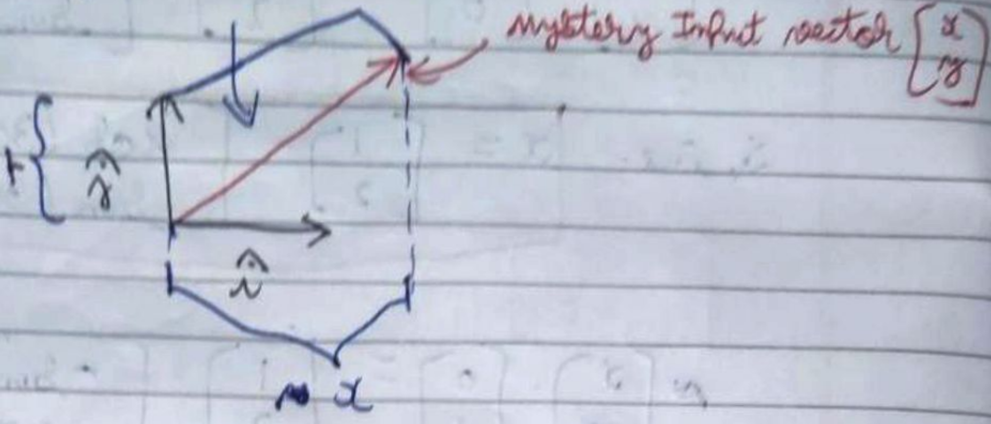
Basis remains $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ after transformation too is because we are using orthonormal transformation

But why such a specific case?

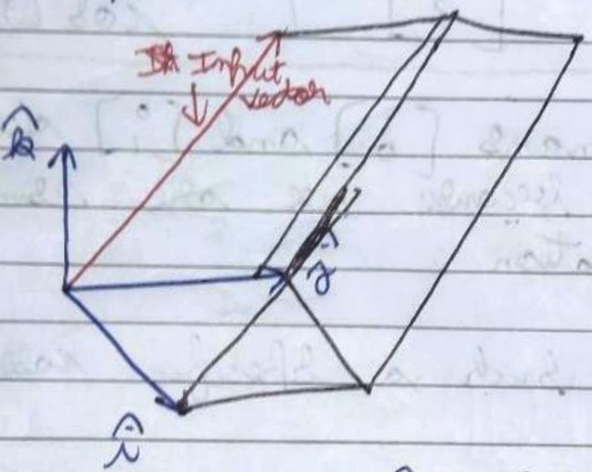
*



Signed Area = $1 \times x$



* For 3D,



$$z = \det\left(\begin{array}{ccc} \hat{i} & \hat{j} & \text{Input vector} \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \left.\begin{matrix} x \\ y \\ z \end{matrix}\right] \end{array} \right)$$

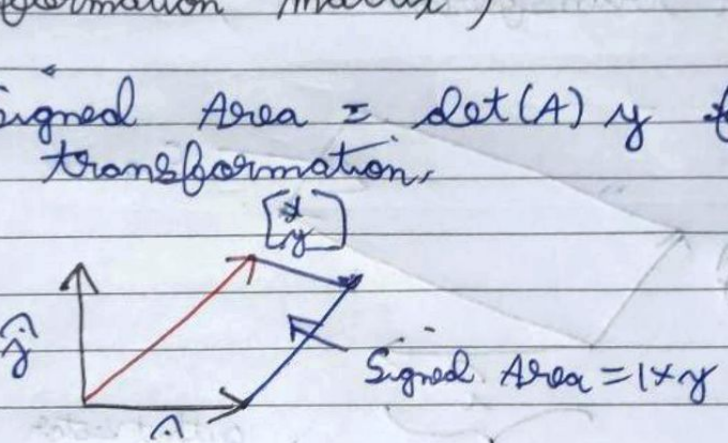
i.e. $z =$ ~~Vol~~ Signed Volume of parallelopiped formed by $\hat{i}$, $\hat{j}$ and Input vec

$$\therefore \cancel{x}\ y = \det\left(\begin{array}{ccc} \hat{i} & \text{IP vec} & \hat{k} \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} x \\ y \\ z \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right] \end{array} \right)$$

$$x = \det\left(\begin{array}{ccc} \text{IP vec} & \hat{j} & \hat{k} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \end{array}\right)$$

Now when we apply a transformation, all these areas and volumes get scaled by the same amount (i.e. determinant of transformation matrix)

∴ Signed Area = det(A) y ~~from~~
Before transformation,

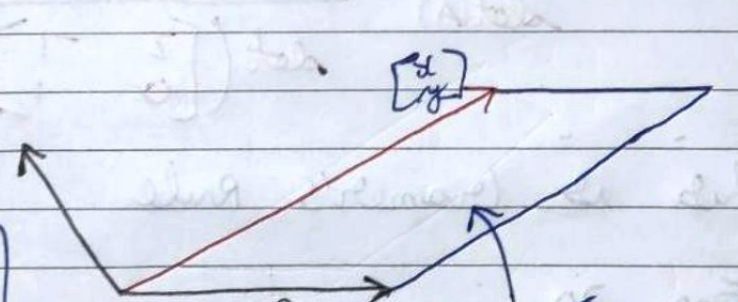


After transformation,

$$\underbrace{\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

where $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

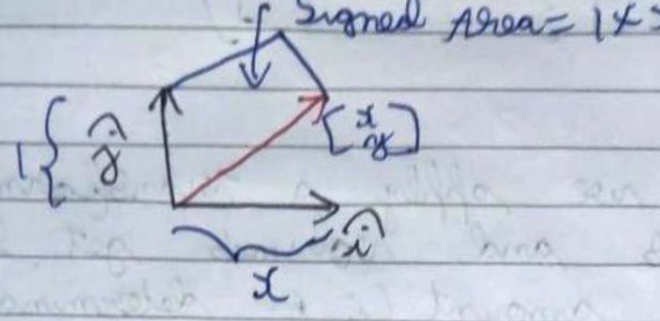
Transformation matrix



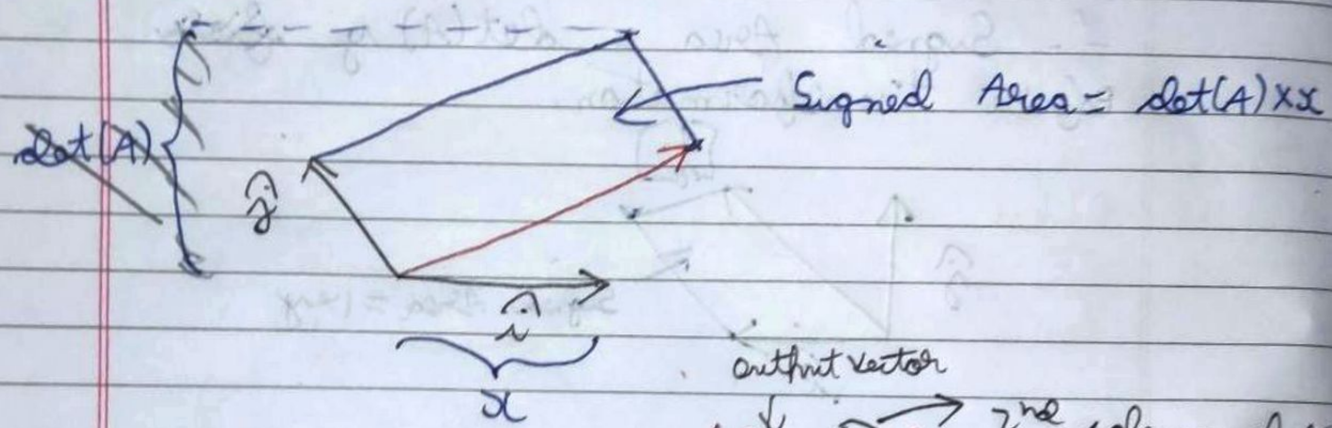
1st column of A → output vector

$$\therefore y = \frac{\text{Area}}{\det(A)} = \frac{\det\left(\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\right)}$$

∴ Before transformation,



After transformation,



output vector → [4 2]; 2nd column of A → [-1 1]

$$\therefore\ x = \frac{\text{Area}}{\det(A)} = \frac{\det\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\right)}$$

This is Cramer's Rule.

* For 3D,

$3x + 2y - 7z = 4,$

$1x + 2y - 4z = 2,$

$4x + 0y + 1z = 5$

$$\begin{bmatrix} 3 & 2 & -7 \\ 1 & 2 & -4 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

We use similar logic as 2D.
For volume, we consider the 2 basis vectors forming volume and the input vector for finding co-ordinates.

$$\therefore x = \frac{\det\left(\begin{array}{c} \text{OP Vec} \\ \begin{bmatrix} 4 & 2 & -7 \\ 2 & 2 & -4 \\ 5 & 0 & 1 \end{bmatrix} \end{array}\right)}{\det\left(\begin{bmatrix} 3 & 2 & -7 \\ 1 & 2 & -4 \\ 4 & 0 & 1 \end{bmatrix}\right)} \rightarrow \text{Volume of } A$$

$$y = \frac{\det\left(\begin{array}{c} \text{OP Vec} \\ \begin{bmatrix} 3 & 4 & -7 \\ 1 & 2 & -4 \\ 4 & 5 & 1 \end{bmatrix} \end{array}\right)}{\det\left(\begin{bmatrix} 3 & 2 & -7 \\ 1 & 2 & -4 \\ 4 & 0 & 1 \end{bmatrix}\right)}$$

$$z = \frac{\det\left(\begin{array}{c} \text{OP Vec} \\ \begin{bmatrix} 3 & 2 & 4 \\ 1 & 2 & 2 \\ 4 & 0 & 5 \end{bmatrix} \end{array}\right)}{\det\left(\begin{bmatrix} 3 & 2 & -7 \\ 1 & 2 & -4 \\ 4 & 0 & 1 \end{bmatrix}\right)}$$