Lecture - 12
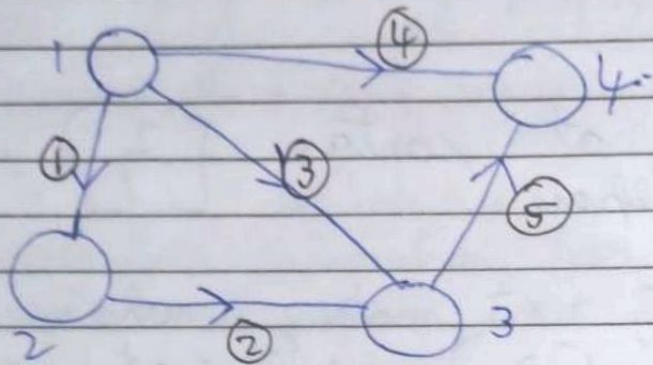
Graphs, Networks, Incidence Matrices

let this
be an
circuit

$m = 5$ edges
$n = 4$ nodes

* Incidence matrix

	Node1	Node2	Node3	Node4	
$A =$	-1	1	0	0	edge 1
	0	-1	1	0	edge 2
	-1	0	1	0	edge 3
	-1	0	0	1	edge 4
	0	0	-1	1	edge 5

-1 indicates that the edge is going away
from the node.
1 indicates that the edge is coming
towards the node.

Edges ①, ②, ③ form a loop here.
Matrix only formed by edges 1, 2 and 3,

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

This matrix has linearly dependent rows.
Row 1 + Row 2 = Row 3

* Nullspace

Solve $AX = 0$

$$AX = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$X = x_1, x_2, x_3, x_4$
Potentials at node

$\therefore x_2 - x_1, x_3 - x_2, \quad \cdots \quad$ are all potential differences (PD)

For *A to have nullspace, all PD'S $= 0$.
i.e. All potentials must be equal.

let all $x_1 = x_2 = x_3 = x_4 = 1$.

$$\therefore X = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \leftarrow \text{Null space}$$

$\dim N(A) = 1$

$\therefore$ All PD'S are zero, current won't flow through the circuit

Now we ground the node x_4.
i.e. $V_4 = 0V$ $(x_4 = 0)$

∴ The 4th column in A (column corresponding to x_4) becomes 0.

∴ Rank of $A = 3 = r$

* left Null space, $N(A^T)$
  $A^T y = 0$

dim $N(A^T) = m - r = 5 - 3 = 2$

A^T $\underset{4 \times 5}{n \times m}$

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{e = Ax}$
↓

$x_2 - x_1, \cdots$
PD

$\boxed{y = Ce}$ →
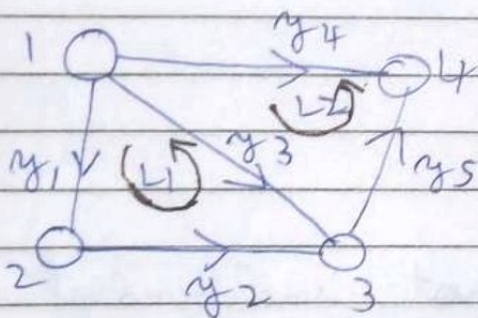Conductance
Ohm's law

$\boxed{A^T y = 0}$ Kirchoff's current Law
↑
Currents $y_1, y_2, \cdots, y_5$ on edges

From $A^T y = 0$, we get

$$-y_1 - y_3 - y_4 = 0$$
$$y_1 - y_2 = 0$$
$$y_2 + y_3 - y_5 = 0$$
$$y_4 + y_5 = 0$$

All these eqs when seen in the graph satisfy KCL i.e. Summation of current at a node / Junction is 0.

For basis of $N(A^T)$

Send 1A current through loop1 satisfying KCL

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{Flow in 0fh direct}^n$$

Send -1A current through loop 2
satisfying KCL

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \longleftarrow \text{oph direct}$$

If we consider loop 1234 $\boxed{1}$, $\cancel{2}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

But this is not independent. It's a
combination of loop 1 and loop 2
solutions.

∴ Basis for $N(A^T)$ will only have
2 vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$
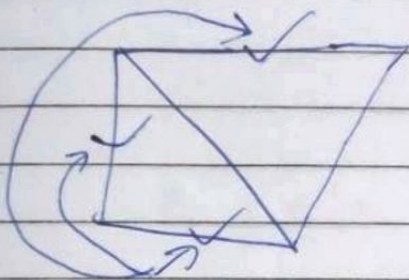
* Row Space of A

dim (row space of A) = 3

i.e. No of independent rows of A or no of independent rows columns of A^T.

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Pivot colm of A^T

These are basis of row space of A

These 3 independent rows form a graph without a loop which is a tree.

dim $N(A^T) = m - r$, rank $= r$

No of loops = No of edges - No of nodes -1.

$\therefore$ No of nodes - No of edges + No of loops $= 1$

This is Euler's formula which is true for any graph.
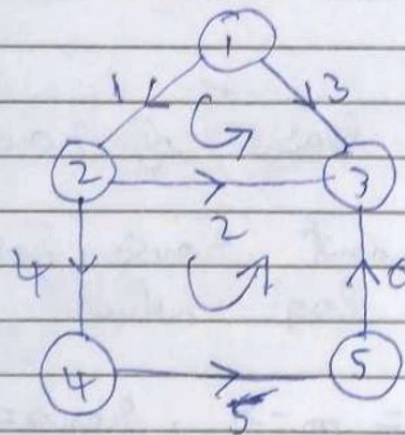
→ If , we add current source to our circuit , $A^T y = 0$ changes to $A^T y = b$ where $b =$ Current source

$\therefore \underset{\underset{PD}{\downarrow}}{e} = Ax$

But $y = Ce$
$\therefore y = CAx$

But $A^T y = b$
$\therefore \boxed{A^T C A x = b}$

Note: $A^T A$ is always a symmetric matrix

Q.

Find incidence matrix A

$N(A)$, $N(A^T)$

Trace $(A^T A)$

	Node1	Node2	Node3	Node4	Node5	
$A =$	-1	1	0	0	0	edge 1
	0	-1	1	0	0	edge 2
	-1	0	1	0	0	edge 3
	0	-1	0	1	0	edge 4
	0	0	0	-1	1	edge 5
	0	0	1	0	-1	edge 6

$$Ax = 0$$

This means that all potentials are equal.

$$\therefore N(A) = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A^T y = 0$$

loop1   loop2

$$y = c_1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

→ Trace (A^TA) is sum of diagonal elements of A^TA.

→ The diagonal entries of A^TA are the magnitude squared of columns of A.

→ Each entry of A is either $-1, 0$ or 1. ∴ Square of entries will be either 0 or 1.

→ This equals to number of non-zero entries in each column.

→ This tells us that Trace (A^TA) is the sum of degrees of each node.

Degree of ① $= 2$
Degree of ② $= 3$
Degree of ③ $= 3$
Degree of ④ $= 2$
Degree of ⑤ $= 2$
Degree of ⑥ ~~$= 1$~~

∴ Trace $(A^TA) = 2+3+3+2+2$ ~~$+$~~

$= 12$.