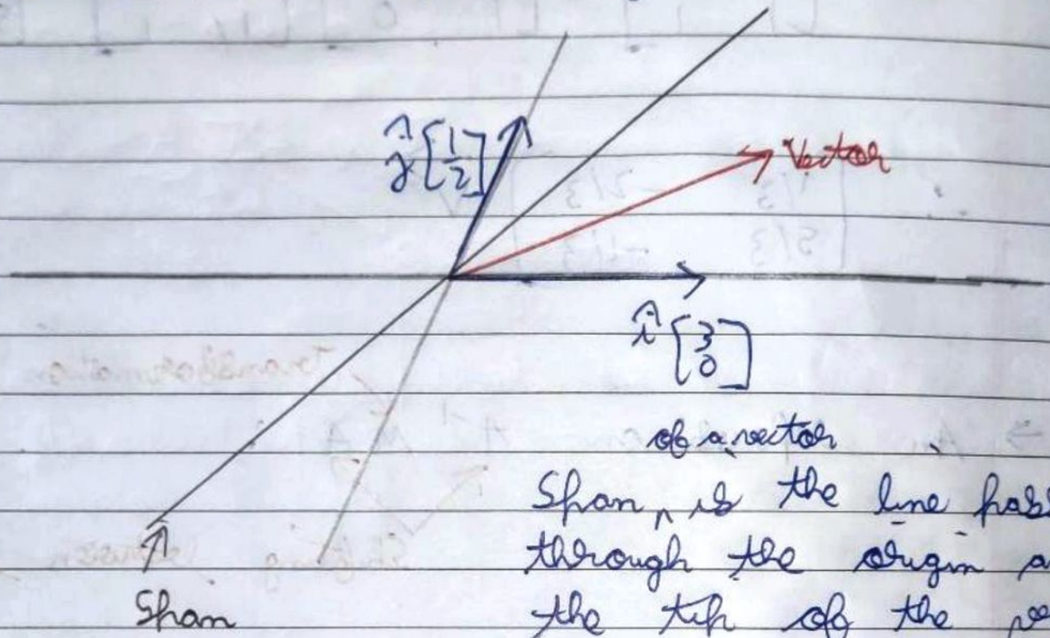
# Lecture -14

* Eigen vectors and eigen values:

→ Consider a transformation $\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}$.

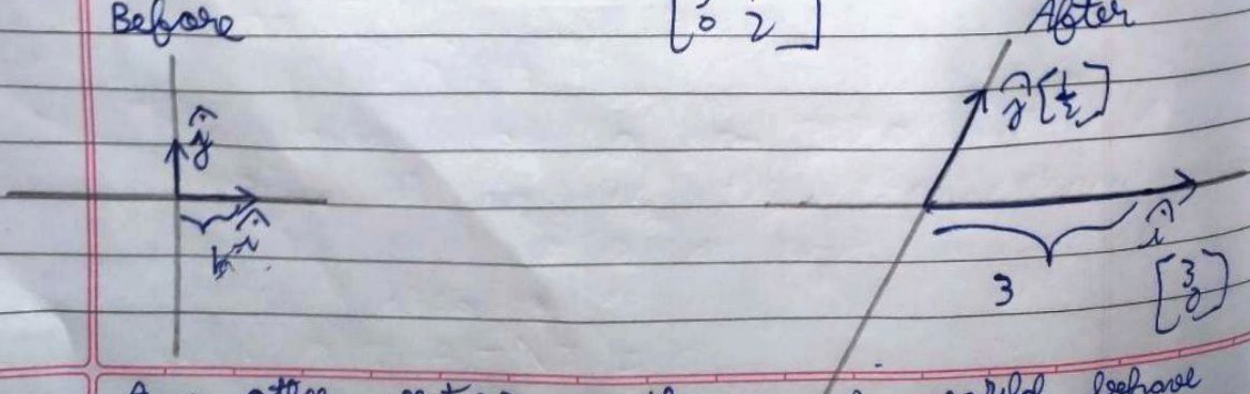
Most of the vectors wouldn't overlap with their span after transformation




Span of a vector is the line passing through the origin and the tip of the vector

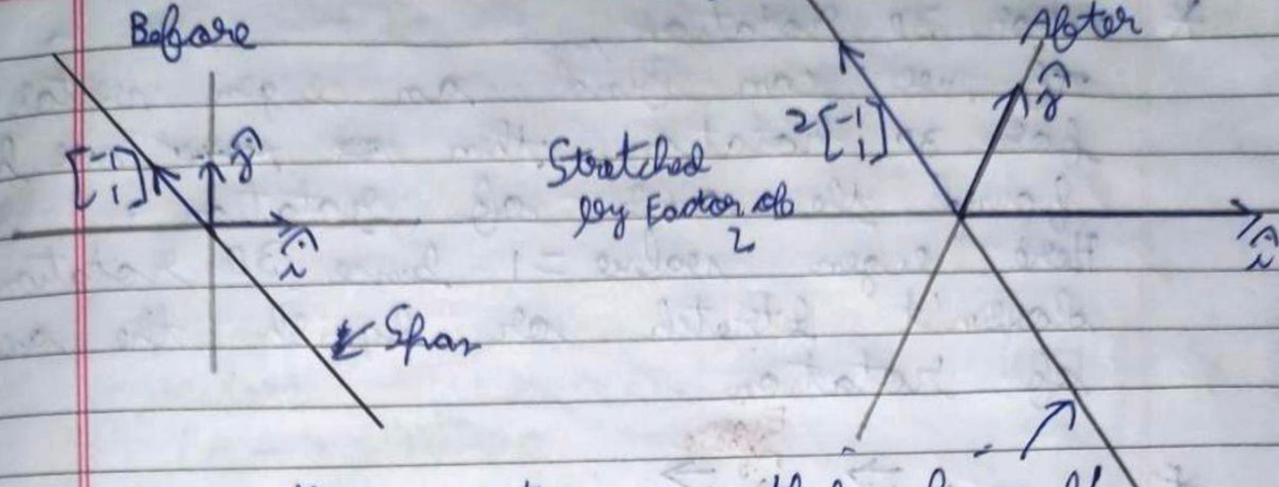
→ But some vectors remain on their span after transformation. The only effect on them is that they get stretched or squished.

Ex: $\hat{i}$




Any other vectors on the x-axis would behave same as $\hat{i}$.

$\begin{bmatrix}3 & -1\\0 & 2\end{bmatrix}$

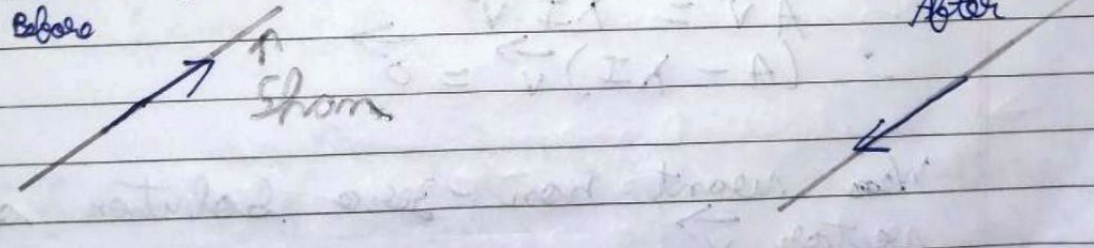



Any other vector on this diagonal/ span will be stretched out by 2

→ These special vectors that remain on their span for a particular transformation are called eigen vectors for that transformation.

→ The factor by which they are stretched or squished during transformation are called eigen values.

→ Eigen values can be -ve too. It just means the vector is flipped but is still on the span.



* For 3D rotation,
If we can find an eigen vector for 3D rotation, then we have found the axis of rotation.
Here eigen value = 1 since 3D rotation doesn't stretch or squish the axis of rotation.

* $$\underbrace{A}_{\text{Transformation matrix}} \vec{V} = \overset{\text{Eigen value}}{\lambda} \vec{V}$$

Eigen vector

It tells us that the matrix vector product $A\vec{V}$ is same as scaling the vector $\vec{V}$ by $\lambda$.

This tells us what happens to each basis vector

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\therefore A\vec{V} = \lambda I \vec{V}$$
$$\therefore (A - \lambda I)\vec{V} = \vec{0}$$

We want non-zero solution of eigen vector $\vec{V}$.

Then only way a non zero matrix $(A-\lambda I)$ becomes 0 is if the transformation squishes it onto a line, (i.e. makes it's area 0).
i.e. $\det(A - \lambda I) = 0$

This gives us the ~~v~~ eigen value $\lambda$ which in turn gives us eigen vector $\vec{v}$ of A which stays on it's span during the transformation A.

* Steps to compute eigen values and eigen vectors

1. $(A - \lambda I)\vec{v} = 0$

But $\det(A - \lambda I) = 0$

Ex: $\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$

$\therefore (3-\lambda)(2-\lambda) = 0$

$\therefore \lambda = 2$ or $\lambda = 3$

2. If $\lambda = 2$

$(A - \lambda I)\vec{v} = \vec{0}$

$\therefore \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ where $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ if A is $2\times 2$ matrix

Compute by Row echelon / Cramers rule

3. Do same for $\lambda = 3$

* If during a transformation, there are no eigen vectors, then value of $\lambda$ comes out to be imaginary.

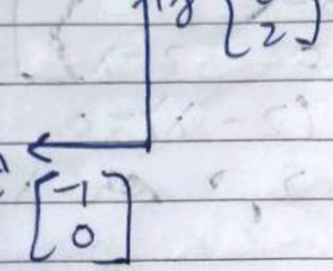
* It is also possible to have one eigen value and multiple eigen vectors. If a vector doesn't change the orientation of the grid at all but just scales

every vector in the plane by a constant K, then we have only one eigen value K but all vectors in that plane become eigen vectors.

## * Eigen bases

What if our basis are eigen vectors?

Ex:




$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Here eigen values sit on the diagonals; Every other element is 0

→ For a diagonal matrix, all the basis vectors are eigen vectors and all the diagonal elements are called eigen values.

→ multiply Diagonal matrices

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & x \\ 2 & y \end{bmatrix} = \begin{bmatrix} 3^2 x \\ 2^2 y \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3^2 & x \\ 2^2 & y \end{bmatrix} = \begin{bmatrix} 3^3 x \\ 2^3 y \end{bmatrix}$$

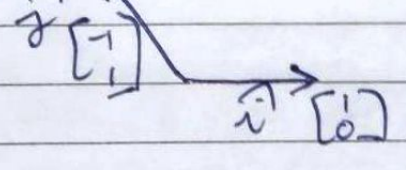
$$\underbrace{\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}}_{m \text{ times}}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3^m & x\\2^m & y\end{bmatrix}$$

This scales each basis vectors $3^m$ and $2^m$ times respectively.

But this is very tough to compute for non diagonal vectors.

* If ~~we~~ our transformation has enough eigen vectors such that we can find a set that spans the full space, then we can change the co-ordinate system such that these eigen vectors are our basis vectors.

∴ We use change of basis




$$\begin{bmatrix}3 & 1\\0 & 2\end{bmatrix}$$

$$\therefore \underbrace{\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}^{-1}\begin{bmatrix}3 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}}_{\text{Change of Basis}} = \underbrace{\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}}_{\text{Diagonal matrix with 3 and 2 as eigen values}}$$

A set of basis vectors which are also eigen vectors are called eigen bases.

∴ If we wanted to compute $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^{100}$, we should change to an eigen basis, compute the 100th power in that system and then convert back to our system.

Note: A transformation (eg: Shear) which does not have enough eigen vectors to span the whole space won't be able to do this.