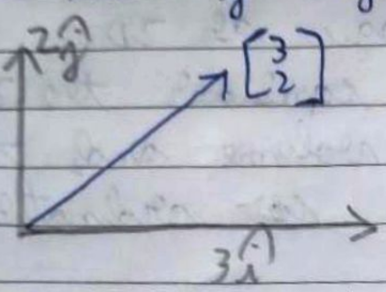
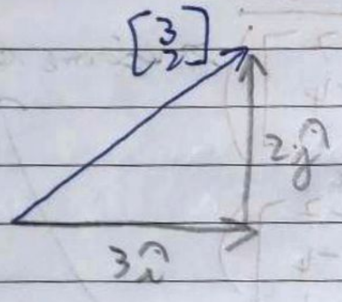
Lecture -13

* Change of basis

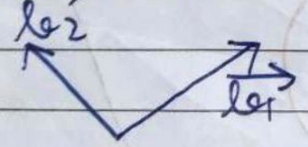
Vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ means first scale the $\hat{i}$ by 3 and $\hat{j}$ by 2



Then the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ will be given by

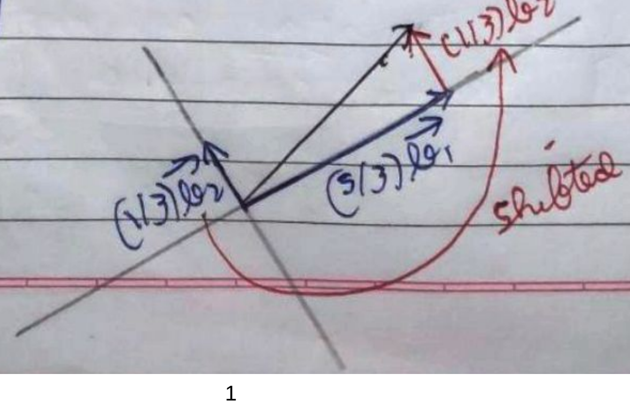


* Now if we use different set of basis vectors say $\vec{b_1}$ and $\vec{b_2}$.

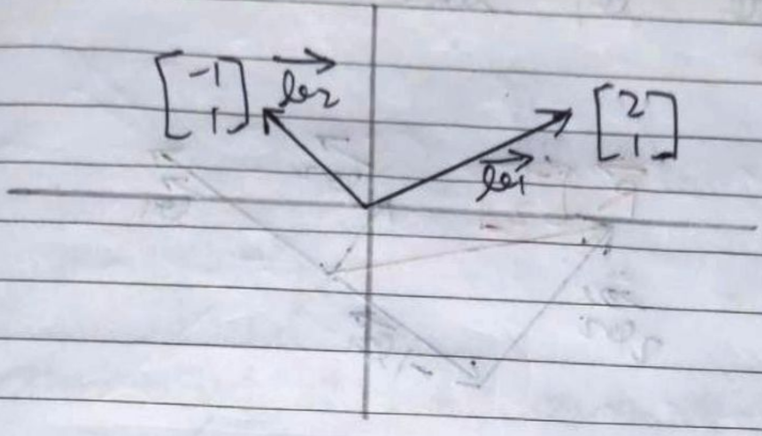


Here the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ ~~at~~, using basis as $\vec{b_1}$ and $\vec{b_2}$ would be described as

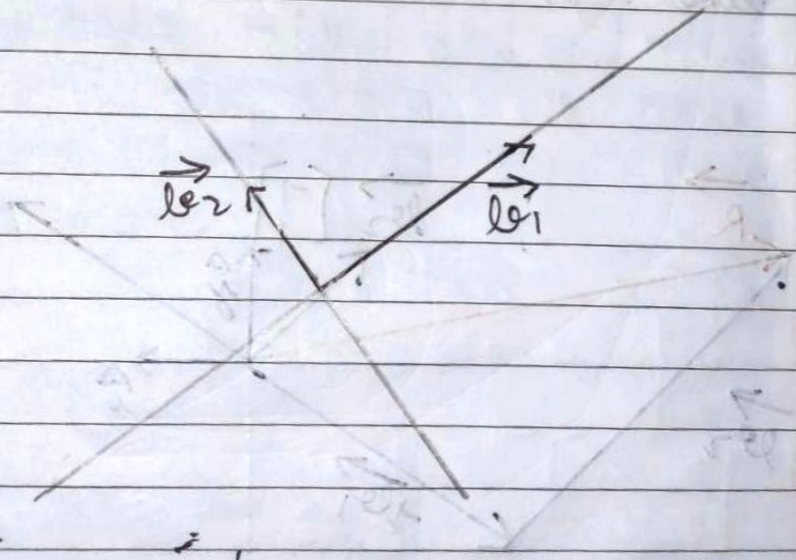
$$\begin{bmatrix} (5/3) \\ (1/3) \end{bmatrix}$$



This difference is because of the difference in basis vectors. $\vec{b_1}$ and $\vec{b_2}$ in a normal grid would appear as



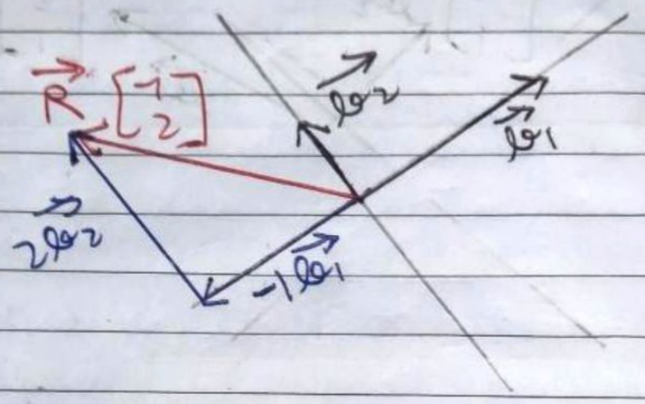
But we look at the grid defined by $\vec{b_1}$ and $\vec{b_2}$ as basis, then the system appears as:



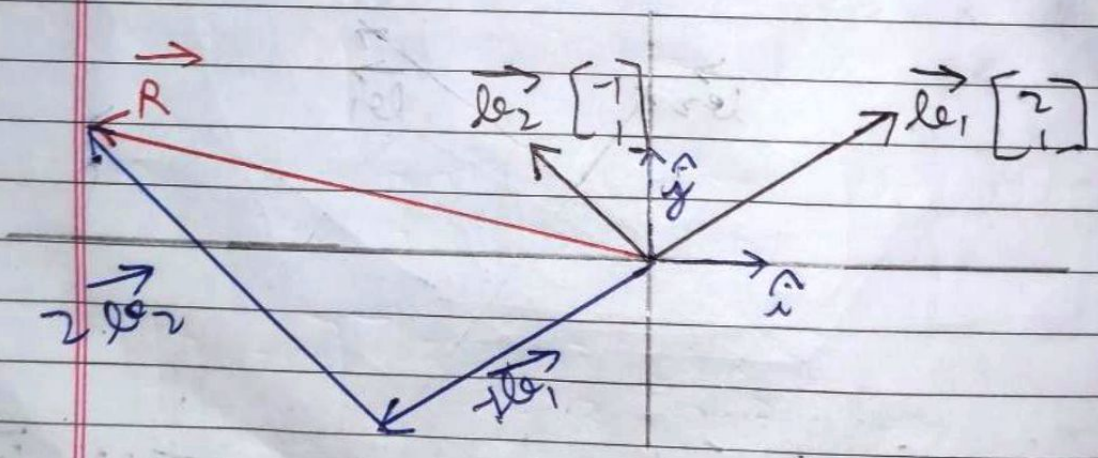
For this grid/system, the vectors $\vec{b_1}$ and $\vec{b_2}$ will be $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ respectively

* How to translate between different co-ordinate systems?

If a vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is defined in a system with $\vec{b_1}$ and $\vec{b_2}$ as basis vectors,



In normal co-ordinate system with $\hat{i}$ and $\hat{j}$ as basis, this would look like:



$\therefore$ In our co-ordinate system, $\vec{b_1}$ is at $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{b_2}$ is at $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

∴ $\vec{R}$, which was at $\begin{bmatrix}-1\\2\end{bmatrix}$ in the co-ordinate system with $\vec{b_1}$ and $\vec{b_2}$ as basis will be at,

$$-1\vec{b_1} + 2\vec{b_2}$$

$$= -1\begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}-1\\1\end{bmatrix}$$

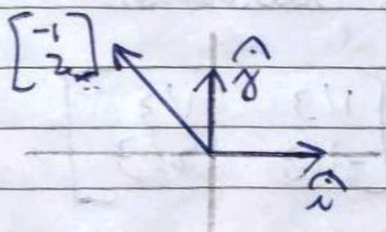
$$= \begin{bmatrix}-4\\1\end{bmatrix}$$

in the normal co-ordinate system.

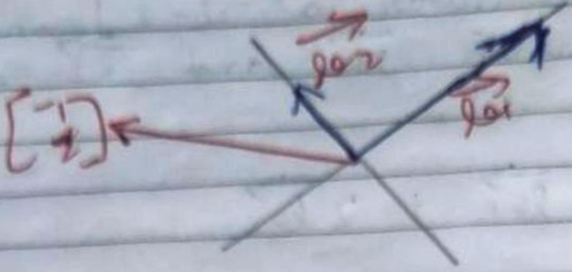
But this process is like matrix vector multiplication

$$\begin{array}{cc} \vec{b_1} & \vec{b_2} \\ \end{array}$$
$$\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = -1\begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-4\\1\end{bmatrix}$$

* In our system,



When we apply the transformation $\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}$ on the vector $\begin{bmatrix}-1\\2\end{bmatrix}$ in our system, it converts our system to the other one.



→ Geometrically, $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ transforms our grid to the inverted one.

→ Numerically, it's transforming vector described in that system to our system.

* But if we ~~reand~~ want to numerically transform a vector described in our system into that system, then we take the inverse of the ~~a~~ transformation matrix $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ ~~an~~ since an inverse of a matrix plays the transformation in reverse direction.

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

→ So to geometrically imagine a vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ described in our system, in that system with $\vec{be_1}$, $\vec{be_2}$ as basis, we:

$$\underbrace{\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}}_{\text{Inverse Change of basis matrix}} \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\text{written in our language}} = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix} \Big\} \text{Same vector in the other system}$$

→ $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ Basis Vectors of that System in our co-ordinates

$$A \begin{bmatrix} x_j \\ y_j \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Vector in that System's co-ordinates

Same Vector in our co-ordinates

$$\begin{bmatrix} x_j \\ y_j \end{bmatrix} = A^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Inverse matrix does the opposite

* When we turn our system 90° anti-clockwise we get our basis vectors like $\uparrow \hat{i}$ $\leftarrow \hat{j}$ but still we maintain the same co-ordinate system/ grid.

Suppose we want do the same thing in co-ordinate system where $\vec{b_1}$ and $\vec{b_2}$ are basis (say Bob's co-ordinate system). What will be the new landing points of $\vec{b_1}$ and $\vec{b_2}$ in Bob's System's language? How to calculate/ translate the matrix?

Ex: Say vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in Bob's system and transformation $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Steps:

① Translate it in our system / language using change of base matrix $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} \overset{\vec{b_1}}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} & \overset{\vec{b_2}}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

This will give same vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in our language.

② Now we apply the transformation to that vector in our language

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$ This gives transformed vector in our system

③ Now we convert this transformed vector back to Bob's system / language by using Inverse change of basis matrix.

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

This will give the transformed vector back in Bob's system.

This process can be done for any vector $\vec{v}$ in Bob's system

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \vec{v}$$

④ This gives us a transformation matrix (TM) in Bob's language which takes in a vector in his language and gives us the transformed vector in his language.

$$TM = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix} \vec{V}$$

→ An expression $A^{-1} \overset{\text{Transformation}}{M} A$:

Shifting between Systems