Prof Gilbert Strang

Lect 4

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Factorization into $A = LU$

→ $(AB)^{-1} = B^{-1}A^{-1}$

→ $(AB)(B^{-1}A^{-1}) = I$

→ $AA^{-1} = I$

$(AA^{-1})^T = (I)^T$

$\therefore (A^{-1})^T A^T = I$

$(A^{-1})^T = (A^T)^{-1}$

*

$$\begin{array}{ccc} E_{21} & A & U \\ \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = & \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \end{array}$$

If $A = LU$,

Then, $L = E_{21}^{-1}$

$$\begin{array}{ccc} A & = L & U \\ \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} & = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \end{array}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{D \leftarrow \text{Pivots}} \underbrace{\begin{bmatrix} 1 & 1/2 \\ 0 & 1/3 \end{bmatrix}}_{U}$$

Take 2 common

Take 3 common

L = Lower triangular matrix.
U = Upper triangular matrix

* consider a 3×3 matrix A.

$E_{32} E_{31} E_{21} A = U$ (No row exchanges)

$\therefore A = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} U$

$\therefore A = L U$

Q. Why is using $A = LU$ better than $E_{32} E_{31} E_{21} A = U$

let $E_{31} = I$ (i.e. It's not eliminating because A_{31} was already 0)

$$\therefore \underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix} = E \left(\text{left of } A\right)$$

i.e. $EA = U$

$$\underset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L \left(\text{left of } U\right)$$

i.e. $A = LU$

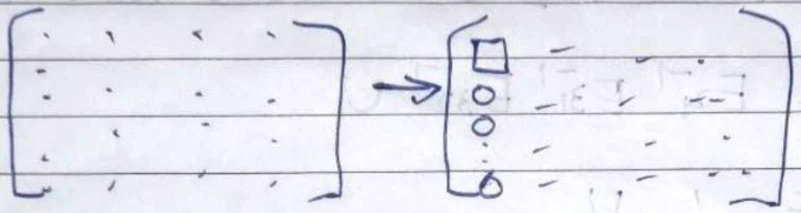
* Finding L is easy. Just keep the record of the multipliers from both the matrices (Here 2, 5)

$A = LU$

1. If no row exchanges, the multipliers go directly into L.

What is the cost of $E_{32} E_{31} E_{21}$?
i.e. Performing elimination on $AX = B$

Suppose A is $n \times n$ matrix where $n = 100$
∴ How many operations on it ? How many E matrices would we have to multiply ?



∴ we do row operations on $n-1$ rows for bringing 0 in the 1st column and also there are n elements in each row, we require $n(n-1) \simeq n^2$ operations i.e. 100^2 here.

Then for the 2nd column, to bring zeros, we will require approx 99^2 operations i.e. $(n-1)^2$ operations.

∴ Total number of operations

$$= n^2 + (n-1)^2 + (n-2)^2 + \dots 3^2 + 2^2 + 1^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$\simeq \frac{1}{3} n^3 \text{ operations}$$

You can think of this as

$$\int_0^n x^2 \, dx = \left[\frac{x^3}{3}\right]_0^n = \frac{n^3}{3}$$

This is the cost of elimination on LHS i.e. on A.

The cost of these row operations on RHS i.e. on B is n^2. (elimination)
It's lesser than A because B is only a column vector $[n+(n-1)+(n-2)+\dots = n^2]$

* Now what if row exchanges are allowed now ?

We would do it if 0 shows up in first position

Permutations : (3x3 matrices)

All 3x3 permutation matrices are : 6 P's

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ doesn't do anything, $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

If we multiply 2 permutation matrices or take inverse of a permutation matrix or take transpose of a permutation matrix, we will get a matrix from the group of these 6 matrices only.

→ $P^{-1} = P^T$

→ For 4x4 matrices, there are 24 permutation matrices

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Q. Find the LU- decomposition of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{pmatrix}$$

when it exists.
For which real numbers a and b does it exist?

Soln :-

$$\begin{pmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{pmatrix}$$

$R_2 - aR_1$

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ b & b & a \end{pmatrix}$$

$R_3 - bR_1$

$$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & b & a-b \end{pmatrix}$$

* Here to eliminate 'b' which is at a_{32} position we have to assume our pivot 'a' at a_{22} position is non-zero. LU decompostn can't be performed using elimination if pivot is 0.

$\therefore$ $\boxed{\text{Assume } a \neq 0}$

$R_3 - \frac{b}{a} R_2$

$$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -b/a & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{pmatrix}$$

$$\therefore \boxed{U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{pmatrix}}$$

$\therefore E_{32} E_{31} E_{21} A = U$

$$\therefore A = \underbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}_{L} U$$

$$\therefore L = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b/a & 1 \end{pmatrix}$$

$$\therefore \boxed{L = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & b/a & 1 \end{pmatrix}}$$

Condition: It exists when $a \neq 0$

Note: In U, $(a-b)$ i.e. a_{33} here can be equal to 0 because we wouldn't have to do a row exchange to get U even if $a-b=0$.
Singular matrices can undergo LU decomposition.