Prof Gilbert Strang

Lect 2

Elimination with matrices (Gauss elimination)

All softwares solve by elimination.

* Forward substitution / elimination

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$
$$4y + z = 2$$

$$Ax = b$$

1st pivot

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

$R_2 - 3R_1$

2nd pivot

$$(2,1) \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

Now we've eliminated x, 2 eqs in y and z.

Note: Matlab first solves 'A' by using elimination completely then moves on to 'b' to apply the same transformations.

$R_3 - 2R_2$

$$(3,2) \rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix}}_{U \text{ matrix} = \text{Upper triangular}}$$

3rd pivot

U matrix = Upper triangular

Note: Any pivot can't be '0'.

If a pivot is zero, try exchanging the rows to get a non-zero pivot.

→ We call it a temporary failure if pivot is 0 but we can do a row exchange.

→ We call it a permanent failure if pivot is 0 and there is nothing below it to exchange.

* Now matlab does the same to matrix 'b'. Although, we directly perform all operations on Augment matrix.

$$b = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}$$

$R_2 - 3R_1$

$$= \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

$R_3 - 2R_1$

$$= \underbrace{\begin{bmatrix} 2 \\ 6 \\ -10 \end{bmatrix}}_{c}$$

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* Back Substitution

$$x + 2y + z = 2$$
$$2y - 2z = 6$$
$$5z = -10$$

$\therefore z = -2$
$\therefore y = 1$
$\therefore x = 2$

* We know that,

$$\begin{array}{ccc} C_1 & C_2 & C_3 \end{array}$$
$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Matrix × Column = Column

$= 3C_1 + 4C_2 + 5C_2$

Now,

$$\underset{1\times 3}{\begin{bmatrix} 1 & 2 & 7 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}}$$

Row × Matrix = Row

$= 1\times$ row1 $+ 2\times$ row 2 $+ 7\times$ row3

This is a linear combination of rows.

How does the row transformations we just performed happen?

~~Ans~~ ~~Permutation matrix~~

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

Step 1 1^{st} operation we performed was $R_2 - 3R_1$.

Perform $R_2 - 3R_1$ on $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

$\therefore E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is our elementary/elimination matrix

It eliminates element at a_{21} position

Step 2 $R_3 - 2R_2$

Perform $R_3 - 2R_2$ on $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_{32}$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

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* All these steps we have discussed can be compressed into one small equation.

$$E_{32}(E_{21}A) = U \longrightarrow ①$$

we perform multiplication from left to right

* If we start with matrix A and want to end up having matrix U, then which is the single matrix that can make it happen?

Ans: $(E_{32}\, E_{21})$

Eq ① can be written as:

$$(E_{32}\, E_{21})\, A = U \quad \text{(Associative Law)}$$

* Which is the matrix which is used to exchange rows?

Ans Permutation matrix (P)

Exchange rows 1 and 2.

i.e. Perform $R_1 \leftrightarrow R_2$ on $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\therefore \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

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* What if we needed to exchange columns using permutation matrix :

Note: For row operations, we put permutation and other required matrices on the left of 'A'.

For column operations, we put permutation and other required matrices on the right of 'A'.

~~Perform~~

To Exchange $C_1 \Leftrightarrow C_2$,

Perform $C_1 \Leftrightarrow C_2$ on $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

* Inverses

Coming back to $(E_{32} E_{21}) A = U$

If $E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

What matrix will reverse our 1st row operation $R_2 - 3R_1$?

Ans It has to be the matrix which does $R_2 + 3R_1$.

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i.e. Perform R_2+3R_1 on $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$