Prof Gilbert Strang

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Lecture 11

Matrix Spaces; Rank 1; Small world Graphs

→ Basis for M = all 3×3's

Dimension of basis = 9

Basis:

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix},$$

$$- - - - , \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

→ Dim of all 3×3 symmetric matrices (S) = 6

Why?

$$\begin{bmatrix}a&\alpha&\beta\\\alpha&b&\gamma\\\beta&\gamma&c\end{bmatrix}$$ ← Symmetric matrix

We only have 6 unknowns $a, b, c, \alpha, \beta, \gamma$. ∴ No of Basis vectors = 6

→ Dim of all 3×3 upper triangular matrices(U) = 6

why ?

$$\begin{bmatrix} a & \alpha & \beta \\ 0 & b & \gamma \\ 0 & 0 & c \end{bmatrix} \leftarrow \text{UT matrix}$$

Again = no of unknowns = 6. ∴ No of basis vectors = 6

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots$$

→ $S \cap U$ = Diagonal 3×3 matrices (D)
Dim (D) = 3

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & \cancel{\alpha} \\ 0 & 0 & c \end{bmatrix}$$

→ $S \cup U$, also can be represented as $S+U$ = any element of S + any element of U. = all 3×3's

$\therefore \dim(S \cup U) = \dim(S+U) = 9.$

→ ∴ we observe that,
$\dim(S) = \dim(U) = 6$
$\dim(S \cap U) = 3$
$\dim(S+U) = 9 = \dim(S \cup U)$

$\therefore \dim(S) + \dim(U) = \dim(S \cup U) + \dim(S \cap U)$

* Rank 1 matrices

$$\underset{2\times3}{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} \qquad r = 1$$

∴ Only 1st row and 1st column are independent.

$\therefore \dim C(A) = \dim C(A^T) = \text{Rank}(r)$

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \underset{2\times1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \underset{1\times3}{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}}$$

$\therefore A = uv^T$
Where u and v both are column matrices

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→ If we have a 5×17 matrix of rank 4, we will need 4 rank 1 matrices.

→ let M be all 5×17 matrices with rank 4. Is M a subspace ?

Ans: No.

∵ Rank = No of non-zero rows = 4 here
∴ We won't get the 0 matrix.
∴ It's not a subspace.

→ In R^4, $V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

S = All v in R^4 with $v_1 + v_2 + v_3 + v_4 = 0$

Is it a subspace ?

Ans Yes. ∵ $u + v = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$K u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

∴ It satisfies both axioms. ∴ It's a subspace.

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But S can be a nullspace to a matrix A such that,

Pivot Var

$A = \begin{bmatrix} 1 & \underbrace{1 \quad 1 \quad 1}_{\text{Free Var}} \end{bmatrix}_{1\times4} \quad \therefore n = 4$

rank (r) = 1.

$\therefore \dim N(A) = \dim (S) = n - r = 4 - 1 = 3$

Basis for S (N(A)),

$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$C(A) = R^1$

$N(A^T) = \{0\} = R^0$

Dim (Row Space of A $C(A^T)$) = 1

* Graphs

Graph = { Nodes, Edges }

Q. Show that the set of 2×3 matrices whose nullspace contains $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is a vector subspace, and find a basis for it.
What about the set of those whose column space contains $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

Soln :-

$$\underset{2\times 3}{A} \underset{3\times 1}{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}, \quad B \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(A+B) \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = A \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + B \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad c \text{ a scalar}$$

$$(cA) \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = c \left(A \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ It's a subspace.

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$\therefore \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is in the nullspace of A,

the dot product of each row of A with $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ must be 0.

(Remember explanation for why row space and null space are orthogonal) $\therefore$

Each row of A must be

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0 \qquad (\therefore 2a+b+c=0) \quad (\therefore c = -2a-b)$$

$\therefore$ Each row of A must be

$$\begin{bmatrix} a & b & -2a-b \end{bmatrix}$$
$$= \begin{bmatrix} a & 0 & -2a \end{bmatrix} + \begin{bmatrix} 0 & b & -b \end{bmatrix}$$
$$= a\begin{bmatrix} 1 & 0 & -2 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$$

$\therefore$ Each row of A must be a linear combination of $\begin{bmatrix} 1 & 0 & -2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$

$$\therefore \text{Basis} = \left\{ \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \right\}$$

Consider a 2×3 matrix,

$$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A does not contain $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in it's column space.

∴ Set of 2×3 matrix whose column space has $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ will not contain the 0 matrix (null matrix).

∴ It's not a subspace.