Lecture - 11

* Cross products in the light of linear transformations.

→ Duality: When we find some linear transformation to the number line, we will be able to match it to a vector which is called the dual vector of that transformation, so that performing the linear transformation is same as taking the dot product with that vector.

1. Define a 3D to 1D linear transformation in terms of $\vec{v}$ and $\vec{w}$.

2. Find its dual vector.

3. Show that this dual is $\vec{v} \times \vec{w}$.

$$\vec{u} \times \vec{v} \times \vec{w} = \det\left(\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}\right)$$

where $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

This will give volume of a parallelopiped

Consider $u_1 = x$, $u_2 = y$, $u_3 = z$

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \det\left(\begin{bmatrix} \underbrace{x}_{\text{variable}} & \overset{\vec{v}}{\overbrace{v_1}} & \overset{\vec{w}}{\overbrace{w_1}} \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

Above function is linear $\therefore$ We can think about duality.

$\therefore$ There will be a 1×3 matrix that encodes this transformation (3D to 1D)

Matrix multiplication
$$\begin{bmatrix} ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

But due to duality, we can write the above eq as

Dot product
$$\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

$$\overset{\vec{p}}{\overbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & \overset{\vec{v}}{\overbrace{v_1}} & \overset{\vec{w}}{\overbrace{w_1}} \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

(x, y, z) The dot product of $\vec{p}$ and any other vector gives the same result as plugging in (x, y, z) as the 1^{st} column of a 3×3 matrix whose other 2 columns have co-ordinates of v, w, then computing the determinant.

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$\therefore p_1 \cdot x + p_2 \cdot y + p_3 \cdot z$

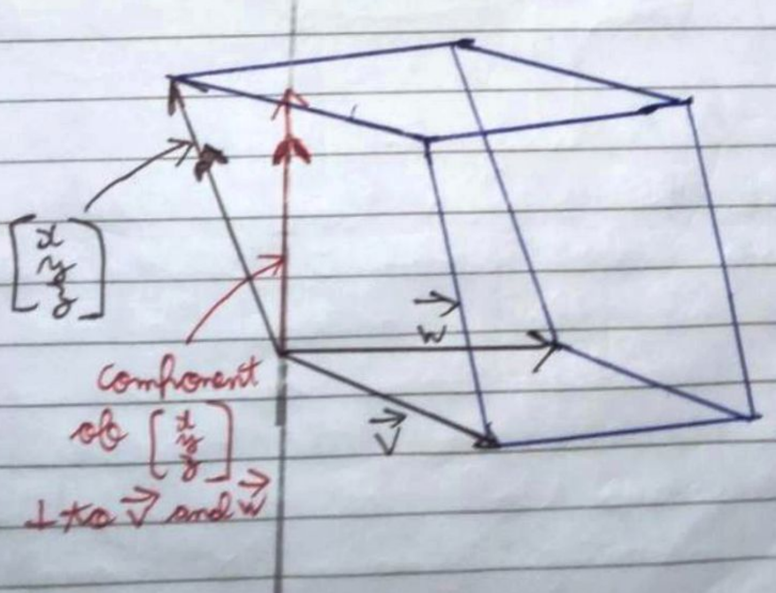
$= x(v_2w_3 - v_3w_2) + y(v_3w_1 - v_1 \cdot w_3) + z(v_1 \cdot w_2 - v_2w_1)$

$\therefore p_1 = v_2w_3 - v_3w_2$
$p_2 = v_3w_1 - v_1w_3$
$p_3 = v_1w_2 - v_2w_1$

$\therefore$ Pulgging in x, y, z is same as plugging in $\hat{i}, \hat{j}, \hat{k}$. $\hat{i}, \hat{j}, \hat{k}$ just indicate that p_1, p_2, p_3 have to be considered as co-ordinate of vectors.

$\rightarrow \because \vec{p} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ = (length of projection) $\times$ (length of $\vec{p}$)

$\therefore$ (Area of parallelogram) $\times$ (Component of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ perpendicular to $\vec{v}$ and $\vec{w}$).



But this is same as taking a dot product between $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and a vector $\perp$ to $\vec{v}$ and $\vec{w}$ with length = Area of parallelogram.

If we choose appropriate direction for that vector, the cases where the dot product is negative will line up with cases where the right hand rule for the orientation for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\vec{v}$ and $\vec{w}$ is -ve.

$\therefore$ Our computation approach and geometric approach verify the same thing.