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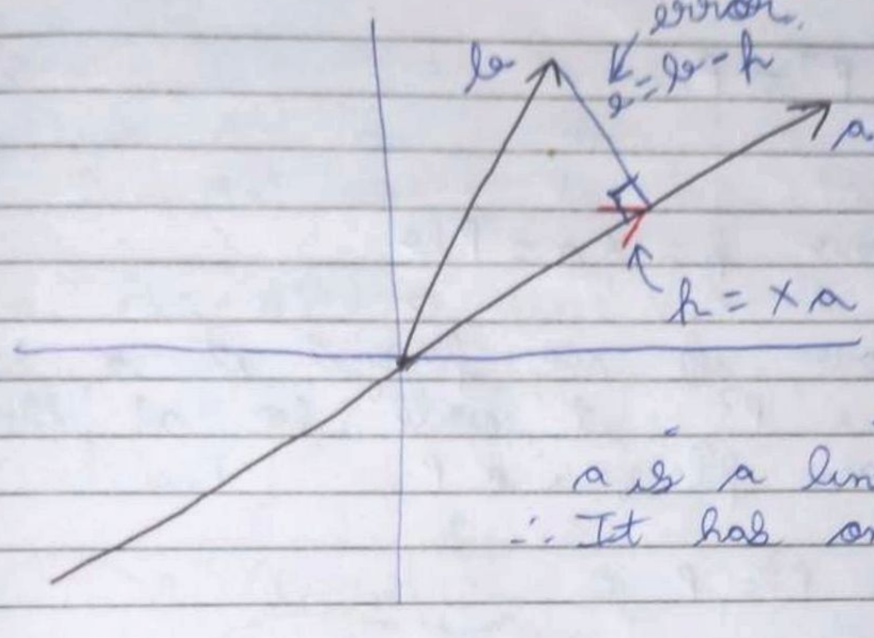
Lecture 15

Projections onto subspaces



a is a line,
$\therefore$ It has only 1 column

a is $\perp$ to e.

$\therefore a^T \cdot e = 0$
$\therefore a^T \cdot (b - xa) = 0$
$\therefore x\, a^T a = a^T b$
$\therefore \boxed{x = \dfrac{a^T b}{a^T a}}$, $\boxed{p = ax}$

$\therefore \boxed{p = a \dfrac{a^T b}{a^T a}}$

proj $p = P b$

where P is the projection matrix,
$P = \dfrac{a a^T}{a^T a}$ [$a a^T$ is a matrix whereas $a^T a$ will just be a number]

This is because if we multiply 'b' by a number, p will also increase by that factor but this won't happen when 'a' is increased.

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* $C(P)$ = line through 'a'
* rank $(P) = 1$
* $P^T = P$

Properties of Project matrix

Now, $p = xa = Pb$

Now, if we project it a second time i.e. P^2, it will be at the exact same place i.e. P.

* $\therefore P^2 = P$

* Why projection?

Because $Ax = b$ may have no solution (i.e. b is not in column space of A)

$\therefore$ Solve $A\hat{x} = p$ instead where p is projection of b onto column space.

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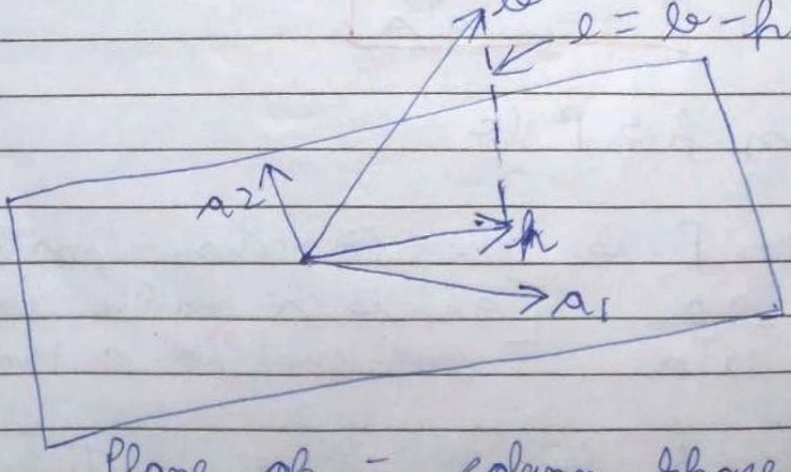

Plane of a_1, a_2 = column space of A = $\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$

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$p = \hat{x}_1 a_1 + \hat{x}_2 a_2$

$\therefore p = A\hat{x}$

Find $\hat{x}$

Key: $b - A\hat{x}$ is $\perp$ to plane

$\therefore b - A\hat{x}$ is $\perp$ to a_1 and a_2.

$\therefore a_1^T (b - A\hat{x}) = 0, \quad a_2^T (b - A\hat{x}) = 0$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore A^T \underbrace{(b - A\hat{x})}_{} \overset{e}{} = 0$$

$\therefore e$ is in nullspace of A^T

$\therefore e$ is $\perp C(A)$ $[\because N(A^T) \perp C(A)]$

$$\boxed{A^T A\hat{x} = A^T b}$$

$$\therefore \boxed{\hat{x} = (A^T A)^{-1} A^T b}$$

$$\boxed{p = A\hat{x} = A(A^T A)^{-1} A^T b}$$

$\cdot$ It's same as 1D case $p = \dfrac{a a^T b}{a^T a}$

But $a^T a$ was a number there. $\therefore$ It was in DR.

Here $A^T A$ is a matrix $\therefore$ We write $(A^T A)^{-1}$.

Here $P = A(A^T A)^{-1} A^T$

$\therefore P = A(A^TA)^{-1}A^T$

→ But $P = AA^{-1}(A^T)^{-1}A^T = I$

We cannot write this because A is a rectangular matrix. $\therefore A^{-1}$ does not exist.

But $(A^TA)^{-1}$ is always a square matrix. $\therefore$ It's inverse exists. Therefore we can write $P = A(A^TA)^{-1}A^T$ but we can't write $P = AA^{-1}(A^T)^{-1}A^T = I$

→ If A was a square invertible matrix, P would be equal to I. $\therefore$ b would already be in the column space of A.

For $P = A(A^TA)^{-1}A^T$,

$$P^T = A\left((A^TA)^{-1}\right)^T A^T = A(A^TA)^{-1}A^T = P$$

$\therefore P^T = P$

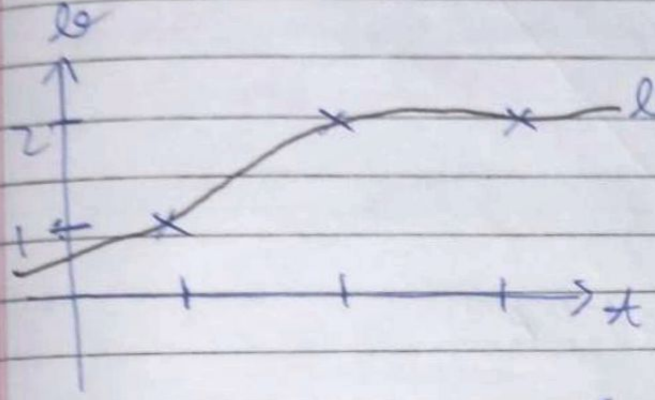
$$P^2 = A(A^TA)^{-1}\underbrace{\left[A^TA(A^TA)^{-1}\right]}_{I}A^T$$

$\therefore P^2 = A(A^TA)^{-1}A^T = P$

$\therefore P^2 = P$

* Least squares,
Fitting by a line

Fit (1,1), (2,2), (3,2) on the best line possible.



Three eqs of line for 3 pts

$C + D = 1$
$C + 2D = 2$
$C + 3D = 2$

These eqs won't be solvable otherwise it would mean we could pass all the pts through a single line.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$A \quad x = b$$

We can't solve $Ax = b$ but we can solve $A^T A \hat{x} = A^T b$

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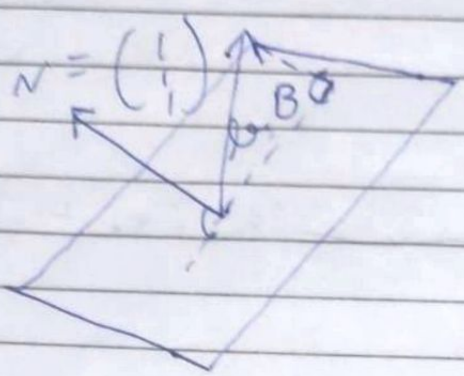
Q. Find the orthogonal projection matrix onto the plane:
$x + y - z = 0$



$P = A(A^TA)^{-1}A^T$

$A = \begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix}$

Any 2 columns which span the subspace

$a_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^TA = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$(A^TA)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$P = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Projection matrix P takes the normal vector to the plane down to zero.

$\therefore PN = 0$

Another approach,

$$I\, b = \underbrace{P\, b}_{\text{Projectn onto plane}} + \underbrace{P_{\wedge}\, b}_{\text{Projectn onto Normal / orthogonal complement}}$$

$$I = P + P_{\wedge}$$

$$\therefore P = I - P_{\wedge}$$

$$\therefore P_{\wedge} = N(N^{T}N)^{-1}N^{T}$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \frac{1}{3} (1 \;\; 1 \;\; -1)$$

$$P_{\wedge} = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\therefore P = I - P_{\wedge}$$

$$\therefore P = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$