Prof Gilbert Strang

lect 3

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Multiplication and Inverse matrices

①

$$\begin{array}{c} \\ \text{row 3} \\ \\ \end{array}\begin{bmatrix} \\ a_{31} & a_{32} & \cdots \\ \\ \end{bmatrix}_{m\times n} \begin{array}{c}\text{column 4} \\ \begin{bmatrix} b_{14} \\ b_{24} \\ \vdots \\ \vdots \end{bmatrix}_{n\times p}\end{array} = \begin{bmatrix} c_{34} & \\ & \cdot \\ & \end{bmatrix}$$

$C = AB$

$c_{34} = (\text{row 3 of } A) \cdot (\text{column 4 of } B)$

$= a_{31} b_{14} + a_{32} b_{24} + \cdots$

$= \sum_{k=1}^{n} a_{3k} b_{k4}$

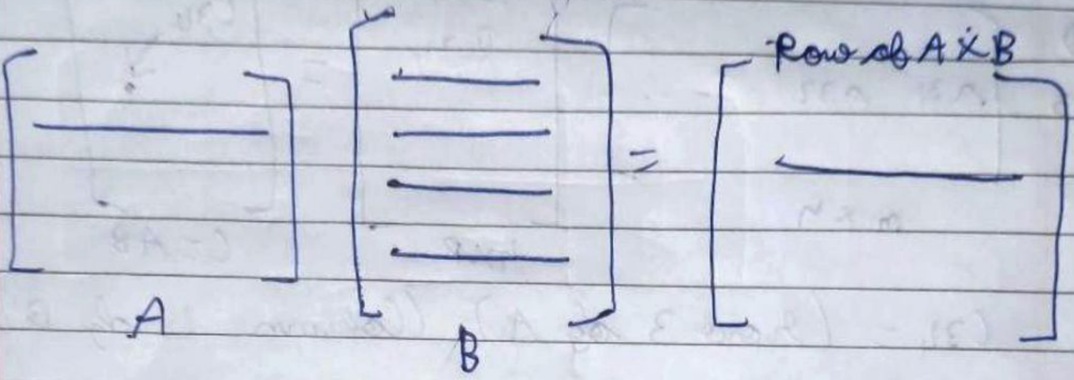
Another method to multiply matrices is by looking at columns.

②

$$\underset{m\times n}{\underset{A}{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}} \ \underset{n\times p}{\underset{B}{\begin{bmatrix} \text{col1} & \cdots & \\ & & \\ & & \end{bmatrix}}} = \underset{C_{m\times p}}{\begin{bmatrix} A\,\text{col1} & \cdots & \\ & & \\ & & \end{bmatrix}}$$

Columns of C are combinations of columns of A. It makes sense because length of columns of A is 'm', and length of columns of C is 'm' too. Every column of C is some combination of columns of A and the numbers in column of B tells us that combination

③ We can multiply them by looking at rows too.



Rows of C are combinations of rows of B.

④ Another way is,

Column of A × row of B
$m \times 1$ $\quad$ $1 \times p$

$$\overset{A}{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & 6 \end{bmatrix}} = \overset{C}{\begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}}$$

As discussed above, we can see that the columns of the resultant matrix(C) are multiples/combinations of columns of A.

The rows of matrix C are multiples/combinations of rows of B.

∴ The 4^{th} way to multiply matrices is

$$AB = \text{Sum of}\left[\text{Cols of } A \times \text{Rows of } B\right]$$

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix} C$$

The row space for matrix C (i.e. all the combinations of the rows) is just a line for this matrix. The row space is the line through vector [1 6]. All rows lie on that line.

The column space is also a line. All the columns lie on the line through the ~~s~~ vector ~~[2 1 3]~~ $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

(5) We can also multiply matrices by dividing them in blocks.

$$\underbrace{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]}_{A} \underbrace{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]}_{B}$$

Suppose A is 20×20. $\therefore A_1$ is a 5×5 matrix.

$$\therefore AB = \left[\begin{array}{c|c} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ \hline A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{array}\right]$$

* Inverses (Square matrices)

If A^{-1} exists, then

$$AA^{-1} = I = A^{-1}A$$

and A is an invertible, non-singular ($\det(A) \neq 0$) matrix.

If a matrix is singular ($\det = 0$), then it's inverse does not exist.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

Why can't A^{-1} exist?

Ans Because if A^{-1} existed, then it would have had to satisfy the property,

$$AA^{-1} = I$$

But the columns of the resultant of two matrices must be the combination of the columns of the 1st matrix. But here for A, the column span of A is a single line i.e. both the columns lie on a single line.

Now the columns of I are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which clearly are not combination of columns of A. ∴ They do not lie on the line of column span of A.

* The actual reason why the inverse of A does not exist is because we can find a non-zero vector X with $AX = 0$

$$AX = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ $x + 3y = 0 \rightarrow ①$
$2x + 6y = 0$ i.e. $x + 3y = 0 \rightarrow ②$

If $x = 3$, $y = -1$, eqs ①, ② are satisfied.

$$\therefore AX = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Why is this statement true?

Ans Let's consider A^{-1} exists.
We have proved that there exists a non-zero vector X such that $AX = 0$

$$AX = 0$$

Pre-multiplying by A^{-1} on both sides.

$$(A^{-1}A)X = A^{-1}0$$
$$\therefore IX = 0$$
$$\therefore X = 0$$

But X can't be a null matrix (0). We already proved that $X = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Hence our assumption that A^{-1} exists is wrong.
$\therefore$ A^{-1} can't exist.

Conclusion : For singular matrices, some combination of their columns gives a zero column.

* Now consider an invertible matrix A.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

$$\therefore \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

$A \times$ Column j of A^{-1} = Column j of I.

Gauss - Jordan (Solve 2 eqs at once)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Augmented matrix $\rightarrow$ $$\left\{ \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{A} \middle| \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \right.$$

$R_2 - 2R_1$

$$\sim \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$R_1 - 3R_2$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$I \qquad A^{-1}$ ← This has to be A^{-1}.

Start with $A \mid I$ and end up with $I \mid A^{-1}$.

→ How do we look at this?

$E \;[A \quad I]$

E matrix to perform some row operations

Whatever E is, it converted A by performing some operations into I.

$\therefore EA = I$

But if $EA = I$

$\therefore E$ has to be A^{-1}.

$\therefore E[A \quad I] = [EA \quad EI]$

But $E = A^{-1}$

$\therefore [A^{-1}A \quad A^{-1}I]$

$\therefore [I \quad A^{-1}]$

$\therefore$ We started from $[A \quad I]$ and ended up on $[I \quad A^{-1}]$