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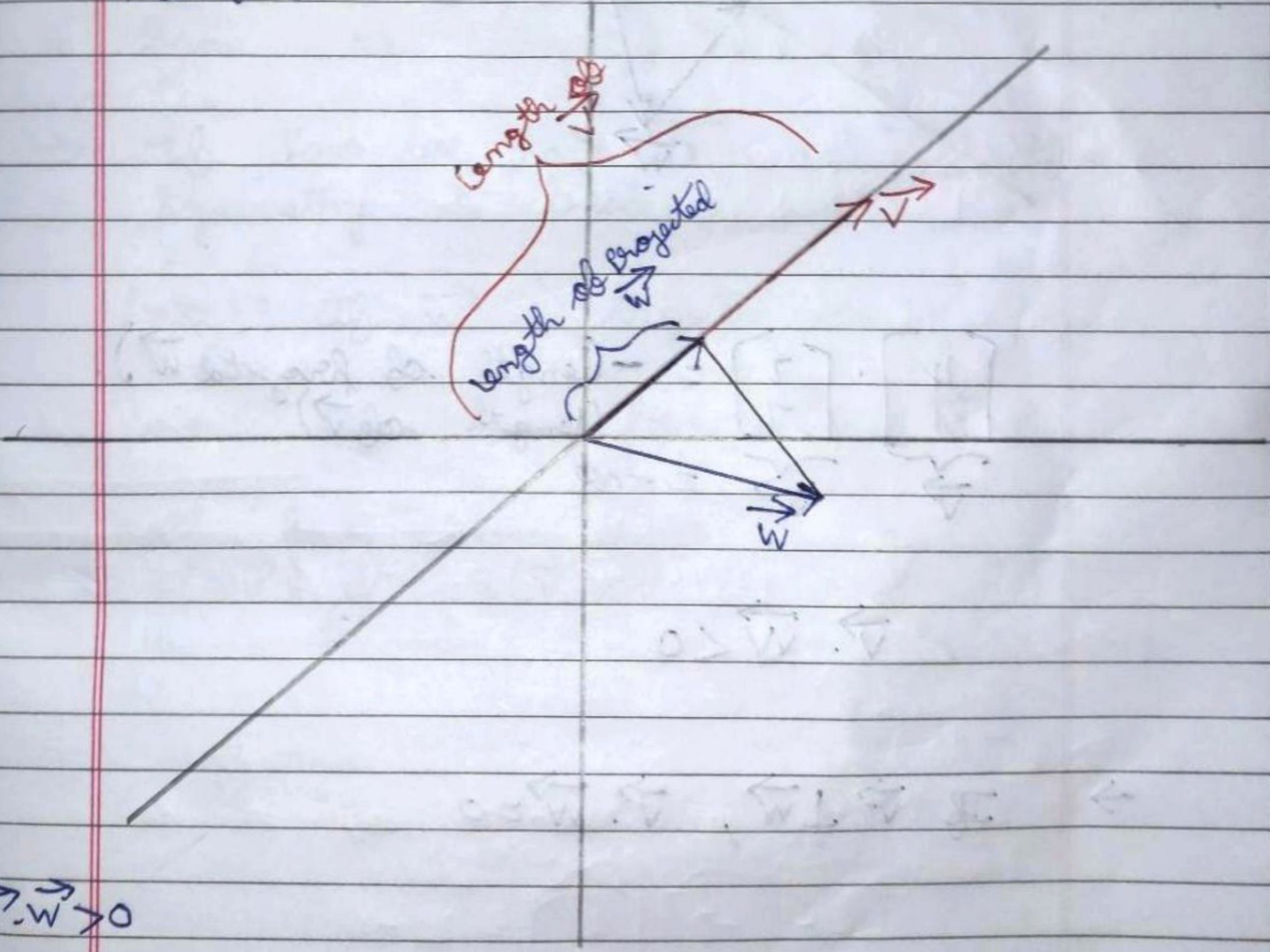
Lecture - 9

* Dot products and duality

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

Geometric Interpretation

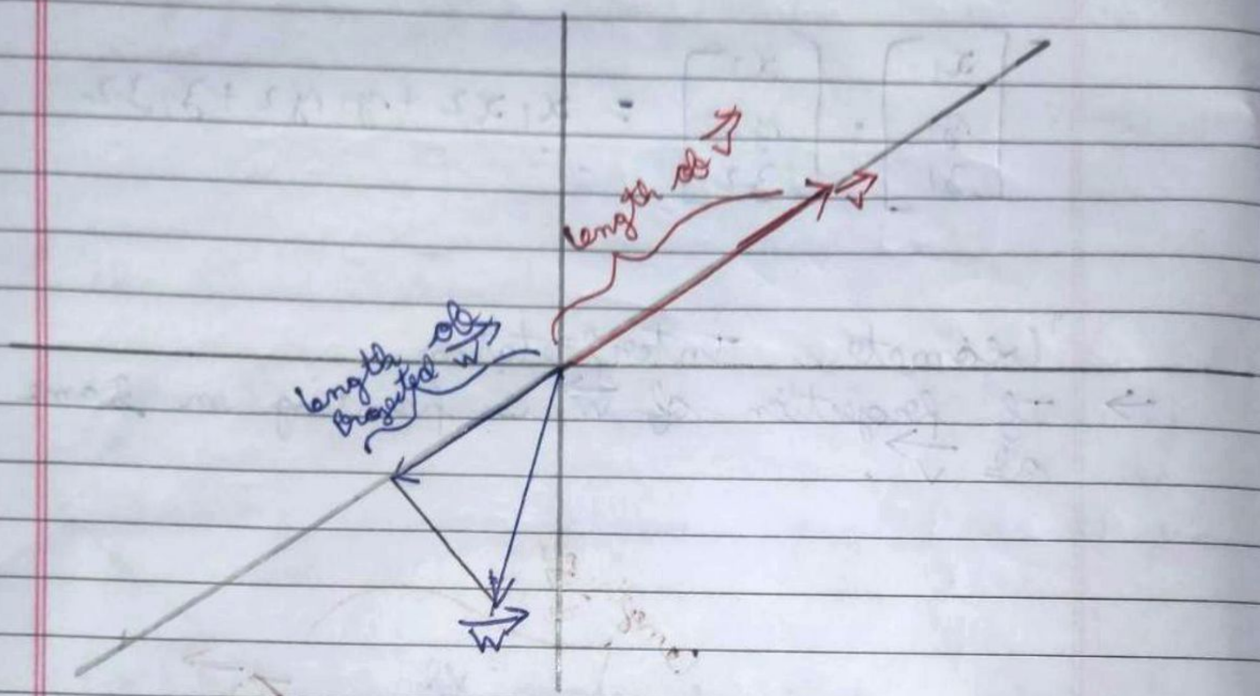
→ If projection of $\vec{w}$ is pointing in same direction as $\vec{v}$.



$\vec{v} \cdot \vec{w} > 0$

$$\underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\vec{v}} \cdot \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\vec{w}} = (\text{Length of projected } \vec{w})(\text{length of } \vec{v}) = +ve$$

→ If projection of $\vec{w}$ is pointing in opposite direction of $\vec{v}$.



$$\underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\vec{v}} \cdot \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\vec{w}} = -(\text{length of projected } \vec{w}).(\text{length of } \vec{v})$$

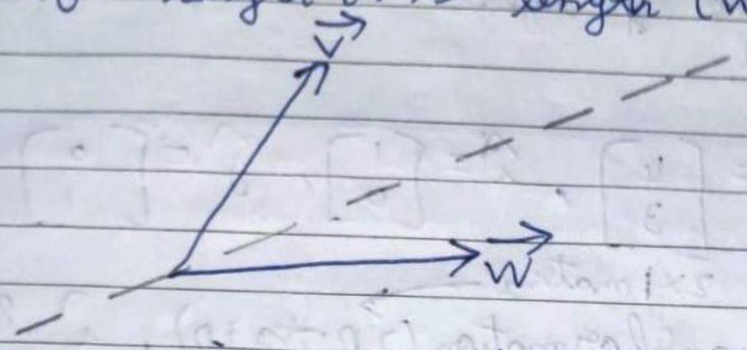
$= -ve$

$$\vec{v} \cdot \vec{w} < 0$$

→ If $\vec{v} \perp \vec{w}$, $\vec{v} \cdot \vec{w} = 0$

* But why is $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$?

→ If length $(\vec{v})$ = length $(\vec{w})$



There is an axis of ~~sys~~ symmetry due to which projecting $\vec{v}$ on $\vec{w}$ would be same as projecting $\vec{w}$ on $\vec{v}$.

→ If Now we have $2\vec{v}$ and $\vec{w}$, then symmetry is broken.

Case 1: If $\vec{w}$ is projected on $2\vec{v}$.

Case 2: If $2\vec{v}$ is projected on $\vec{w}$.

Both have same effect.

$$(2\vec{v}) \cdot \vec{w} = 2(\vec{v} \cdot \vec{w})$$

* Duality

* Why are projections and dot product related ?

Ex: $\vec{v} = \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{2\times 1 \text{ matrix}}, \hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

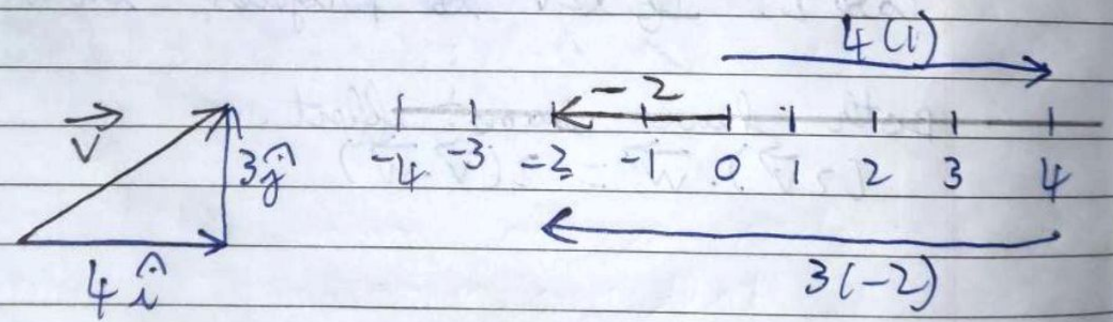
After transformation (2D to 1D), $\hat{i}$ lands on 1, $\hat{j}$ lands on -2.

$\therefore \vec{v} = 4\hat{i} + 3\hat{j}$

After transformation,

$L(\vec{v}) = 4(1) + 3(-2) = -2$

Before | After



$\therefore$ Matrix multiplication of

$$\overbrace{\begin{bmatrix} 1 & -2 \end{bmatrix}}^{\text{transform}} \overbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}^{\text{vector}} = 4 - 6 = -2$$

is same as taking their dot product

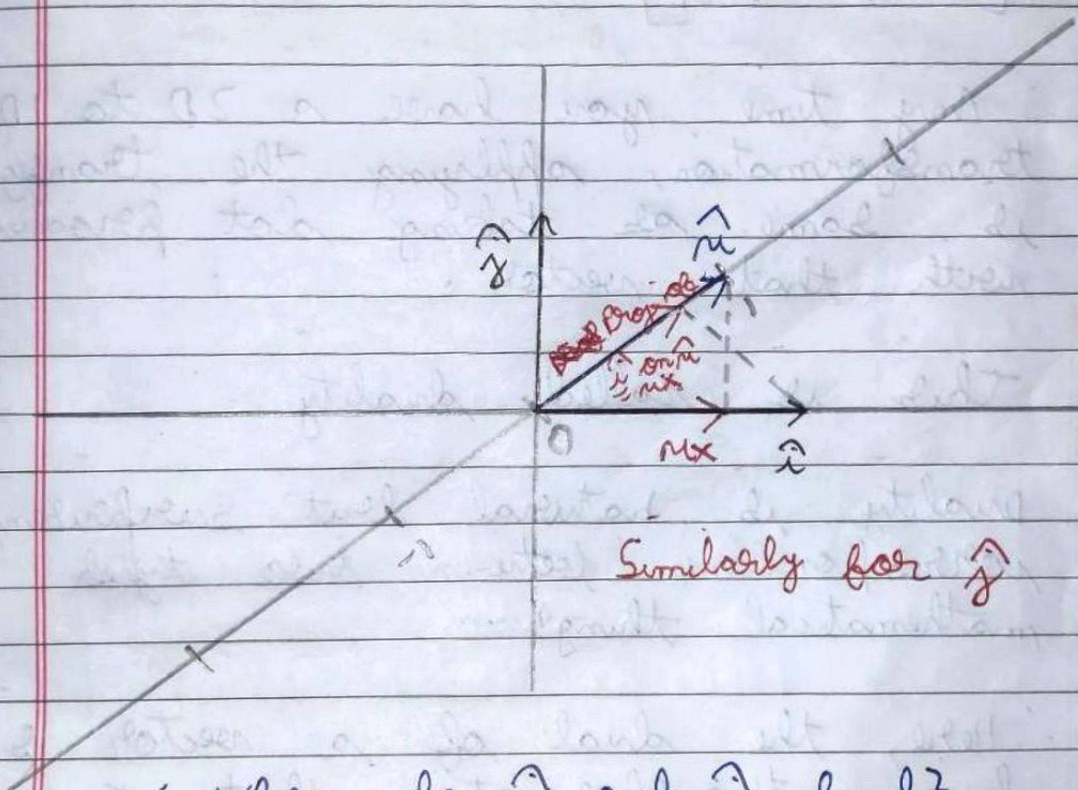
$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 1\cdot 4 + (-2)\cdot 3 = -2$$

1×2 matrices $\xleftrightarrow{\text{Related}}$ 2d vectors

[2 7] $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$

[1 −2] $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

* Imagine 2D unit vector in space overlapping the number line



Similarly for $\hat{j}$

∴ Where do $\hat{i}$ and $\hat{j}$ land?.
Ans: [ux uY]

∴ For an arbitrary vector in 2D space, it's equivalent to taking it's dot product with $\hat{u}$.

$$\underbrace{\begin{bmatrix} ux & uY \end{bmatrix}}_{\text{Transformation}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Vector}} = ux \cdot x + uY \cdot y$$

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = u_x \cdot x + u_y \cdot y$$

For Non unit vector, ex $3\hat{u}$, we first take projections and then multiply it's components by 3.

$[3 u_x \quad 3 u_y]$

∴ Any time you have a 2D to 1D linear transformation, applying the transformation is same as taking dot product with that vector.

This is called duality.

→ Duality is natural but surprising correspondance between two types of mathematical things.

→ ∴ Here, the dual of a vector is the linear transformation that it encodes.

→ The dual of a linear transformation from some space to 1D is a certain vector in that space.