Lecture - 7

* Inverse matrices, column space, rank and Null space.

* Linear system of equations

$2x + 5y + 3z = -3$
$4x + 0y + 8z = 0$
$1x + 3y + 0z = 2$

$$\underbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}_{\vec{v}}$$

This means that the vector $\vec{x}$ after transformation by A lands on $\vec{v}$

→ Case 1: $\det(A) \neq 0$ (i.e. Area is not squished to a single line or point)

In this case, we get 1 vector which will land on $\vec{v}$. To find that vector, we play the transformation in reverse. We start from $\vec{v}$ and will end up on $\vec{x}$. This is called inverse of a matrix. ($A^{-1}$).

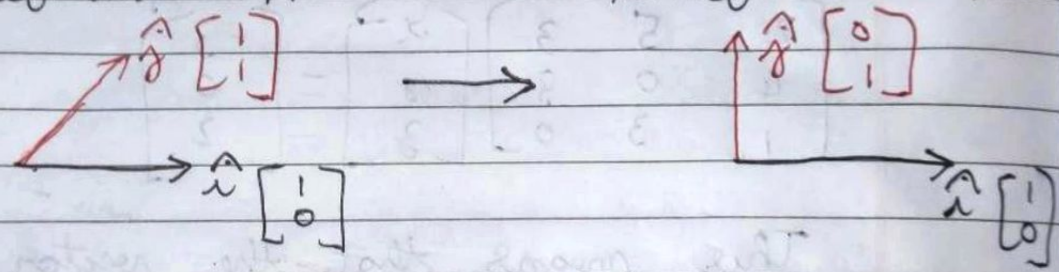
$A^{-1} =$

→ If $A$ was a clockwise rotation by $90°$, $A^{-1}$ will be a counter-clockwise rotation by $90°$.

If $A$ was a rightward shear that pushes $\hat{j}$ 1 unit to the right, $A^{-1}$ would be a leftward shear that pushes $\hat{j}$ 1 unit to the left.

After applying transformation A

After applying transformation $A^{-1}$



$\therefore$ If we first apply $A$, then $A^{-1}$ i.e.

$$A^{-1}A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we get back to where we started from. This is represented by identity matrix (no transformation).

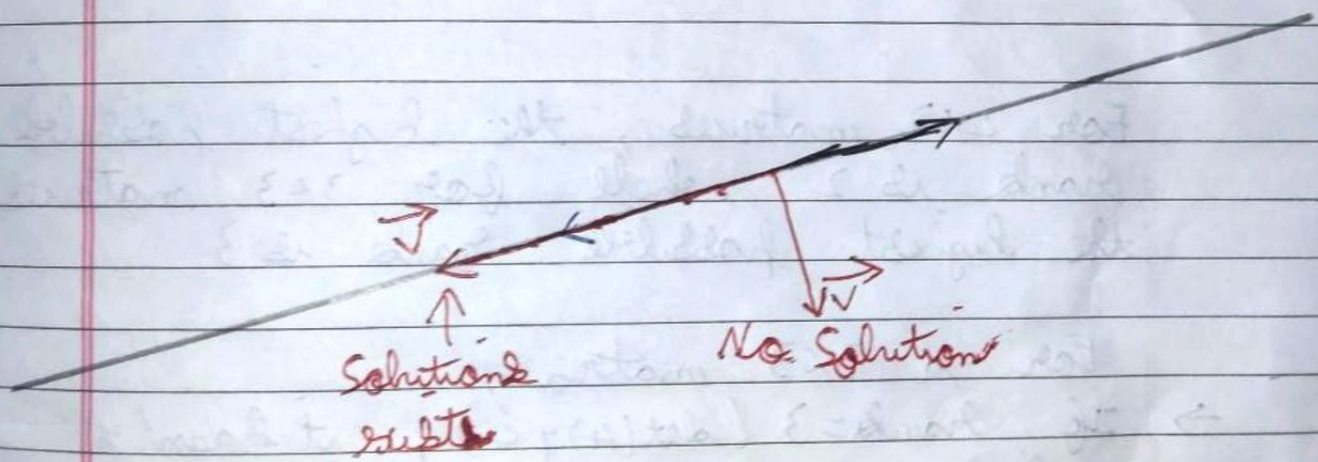
$$\therefore A\vec{x} = \vec{v}$$
$$\therefore A^{-1}A\vec{x} = A^{-1}\vec{v}$$
$$\therefore \vec{x} = A^{-1}\vec{v}$$

This means that we follow $\vec{v}$ to $\vec{x}$ i.e. play the transformation in reverse.

→ Case 2: If $\det(A) = 0$

If space is squished to a line, then we cannot use inverse transformation to make it space again as before.

→ Solution can exist even when $\det(A) = 0$. But for this, we need $\vec{v}$ on the line on which entire space is squished after transformation by A.




→ In 3-D, it is harder for solution to exist when it squishes space onto a line ($\det(A) = 0$) as compared to when it squishes space onto a plane ($\det(A) = 0$)

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* Rank

→ When output of a transformation is a line (i.e. it's 1-D), we say that the transformation has a rank of one.

→ When all ~~xx~~ vectors land on some 2-D plane, we say the transformation has a rank of two.

"Rank" ⟷ No of dimensions in the output

For 2×2 matrices, the highest possible rank is 2 while for 3×3 matrices the highest possible rank is 3.

For a 3×3 matrix,

→ If rank = 3 ($det(A) \neq 0$), it hasn't collapsed.

→ If rank = 2 ($det(A) = 0$), it's collapsed onto a plane.

→ If rank = 1 ($det(A) = 0$), it's collapsed onto a line.

* Column space,

It's the span of the columns of your matrix. i.e. it's the span of $\hat{i}$ new and $\hat{j}$ new.
It gives the set of all possible values whether it may be a plane, a line, etc.

$\therefore$ Rank is the number of dimensions in column space.

vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ will always be included in column space ($\because$ origin must be fixed in place)

→ For a full rank transformation, only $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ lands on $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

→ For matrices that aren't full rank, i.e. which squish to a smaller dimension, you can have ~~bit~~ bunch of vectors that land on zero.

## * Null Space / Kernel

→ If a 2D transformation squishes space onto a line, there's a separate line in different direction full of vectors that got squished onto the origin.

→ If a 3D transformation squishes space onto a plane, there's also a full line of vectors that land on the origin.

→ If a 3D transformation squishes space onto a line, then there's a plane full of vectors that land on the origin.

This set of vectors that ~~lies~~ lands on the origin is called the null space or Kernel of your matrix. It's a space of all vectors that become null, i.e. that they land on the zero vector.

In linear system of eqs, $A\vec{x} = \vec{v}$, when $\vec{v} = \begin{bmatrix}0\\0\end{bmatrix}$, the null space gives you all possible solutions to the eq.