Two Dimensional Random Variables (X,Y)

Let S be the sample space associated with a random experiment. Let $X = X(s)$ and $Y = Y(s)$ be two functions each of which assigns a real number to each outcome $s \in S$. Then, (X,Y) is called a two dimensional random variable.

Random variable (X,Y)

Discrete $P(x,y)$

Continuous $f(x,y)$

Discrete $P(x,y)$

1. Joint probability mass fun

If (X,Y) is the 2-dim random variable s.t $P(X=x_i, Y=y_j) = P_{ij} = P(x,y)$, then $P(x,y)$ is called the joint probability mass fun, provided the following conditions.

(i) $P(x,y) \geq 0$

(ii) $\sum_y \sum_x P(x,y) = 1$

2. Cumulative distribution fun :

$$P(X \leq x, Y \leq y) = \sum_{-\infty}^{y} \sum_{-\infty}^{x} P(x,y)$$

3. Marginal distribution fun (MDF):

In $P(x,y)$, finding only $P(x)$ or $P(y)$ is MDF

$$P(x) = \sum_y P(x,y), \quad P(y) = \sum_x P(x,y)$$

4. Conditional probability distribution :

$$P(x|y) = \frac{P(x,y)}{P(x)}, \quad P(x) \neq 0$$

$$P(y|x) = \frac{P(x,y)}{P(y)}, \quad P(y) \neq 0$$

5. Condition for independency:

$$P(x,y) = P(x) * P(y)$$

Continuous $f(x,y)$

1. Joint probability density fun

(i) $f(x,y) \geq 0$

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

2.

$$P(X \leq x, Y \leq y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)\, dx\, dy$$

3. In $f(x,y)$, finding only $f(x)$ or $f(y)$

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy, \quad f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

4.

$$f(x|y) = \frac{f(x,y)}{f(y)}, \quad f(y) \neq 0$$

$$f(y|x) = \frac{f(x,y)}{f(x)}, \quad f(x) \neq 0$$

5.

$$f(x,y) = f(x) * f(y)$$

1. Let (X,Y) be two dimensional R.V with joint probability mass fun

$Y \backslash X$	1	2	3
1	$1/12$	$1/6$	0
2	0	$1/9$	$1/5$
3	$1/18$	$1/4$	$2/15$

Evaluate (i) Marginal distribution of X and Y.
(ii) Conditional distribution of X given $Y=2$.
(iii) Conditional distribution of Y given $X=2$.
(iv) $P(X\le 2, Y=3)$ (v) $P(Y\ge 2)$ (vi) $P(X+Y<4)$
(vii) Are X and Y independent?

(i) Marginal distribution of X and Y.

\# $P(X) = \sum_y P(x,y)$ (we need to find $P(x=1), P(x=2), P(x=3)$)

• $P(x=1) = \sum_y P(1,y) = P(1,1)+P(1,2)+P(1,3) = \frac{1}{12}+\frac{1}{0}+\frac{1}{18} = \frac{5}{36}$

• $P(x=2) = \sum_y P(2,y) = P(2,1)+P(2,2)+P(2,3) = \frac{1}{6}+\frac{1}{9}+\frac{1}{4} = \frac{19}{36}$.

• $P(x=3) = \sum_y P(3,y) = 0+\frac{1}{5}+\frac{2}{15} = \frac{1}{3}$

\# $P(y) = \sum_x P(x,y)$ (we need to find $P(y=1), P(y=2), P(y=3)$)

$P(y=1) = \sum_y P(x,1) = P(1,1)+P(2,1)+P(3,1) = \frac{1}{12}+\frac{1}{6}+\frac{1}{0} = \frac{1}{4}$

$P(y=2) = 0+\frac{1}{9}+\frac{1}{5} = \frac{14}{45}$

$P(y=3) = \frac{1}{18}+\frac{1}{4}+\frac{2}{15} = \frac{79}{180}$

(ii) $P(X|Y=2)$ [we need to find $P(x=1|y=2), P(x=2|y=2), P(x=3|y=2)$]

$P(x=1|y=2) = \frac{P(x=1,y=2)}{P(y=2)} = \frac{0}{14/45} = 0$

$P(x=2|y=2) = \frac{P(x=2,y=2)}{P(y=2)} = \frac{1/9}{14/45} = 5/14$

$P(x=3|y=2) = \frac{P(x=3,y=2)}{P(y=2)} = \frac{1/5}{14/45} = 9/14$

(iii) $P(Y|X=2)$

- $P(Y=1 | X=2) = \frac{P(Y=1, X=2)}{P(X=2)} = \frac{1/6}{19/36} = 6/19$
- $P(Y=2 | X=2) = \frac{P(Y=2, X=2)}{P(X=2)} = \frac{1/9}{19/36} = 4/19$
- $P(Y=3 | X=2) = \frac{P(Y=3, X=2)}{P(X=2)} = \frac{1/4}{19/36} = 9/19.$

(iv) $P(X \le 2, Y=3) = P(X=1, Y=3) + P(X=2, Y=3)$

$= \frac{1}{18} + \frac{1}{4} = \frac{11}{36}$

(v) $P(Y \ge 2) = P(Y=2) + P(Y=3)$

$= \frac{14}{45} + \frac{79}{180}$

(vi) $P(X+Y < 4) = P(1,1) + P(1,2) + P(2,1)$

$= \frac{1}{12} + 0 + \frac{1}{6}$

$= \frac{1}{4}$

(vii) ~~Are~~ X and Y independent

$P(x,y) = P(x) * P(y)$

let us take $x=1, y=1$

$P(x=1, y=1) = P(x=1) * P(y=1)$

$\frac{1}{12} \neq \frac{5}{36} * \frac{1}{4}$

$\therefore$ x and y are not independent.

2. The joint probability mass function of (X,Y) is given by

$P(x,y) = k(2x+3y), \quad x=0,1,2; \quad y=1,2,3$

find (i) Marginal distribution of X and Y.

(ii) Conditional distribution of X given $Y=1$.

(iii) probability distribution of $X+Y$.

3. The joint probability density function of 2-D R.V (x, y) is given by

$$f(x,y) = \begin{cases} k(6-x-y), & 0<x<2,\ 2<y<4 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) k (ii) $P(X<1, Y<3)$ (iii) $P(X+Y<3)$ (iv) $P(X<1 \mid Y<3)$

(v) Are X and Y independent?

Solu:

(i) Find k

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_{2}^{4}\int_{0}^{2} k(6-x-y)\,dx\,dy = 1$$

$$\int_{2}^{4} k\left(6x-\frac{x^2}{2}-xy\right)_0^2 dy = 1$$

$$k\int_{2}^{4}(12-2-2y)\,dy = 1$$

$$k\left[10y-\frac{2y^2}{2}\right]_2^4 = 1$$

$$k\left([40-16]-[20-4]\right) = 1$$

$$\boxed{k=\frac{1}{8}}$$

(ii) $P(X<1, Y<3)$

$$= \int_{-\infty}^{3}\int_{-\infty}^{1} f(x,y)\,dx\,dy$$

$$= \int_{2}^{3}\int_{0}^{1} \frac{1}{8}(6-x-y)\,dx\,dy$$

$$= \frac{1}{8}\int_{2}^{3}\left(6x-\frac{x^2}{2}-xy\right)_0^1 dy$$

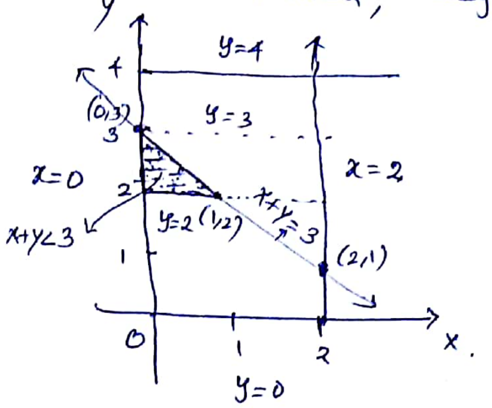
$$= \frac{1}{8}\int_{2}^{3}\left(6-\frac{1}{2}-y\right) dy$$

$$= \frac{3}{8}$$

(iii) $P(x+y<3) = \int_2^3 \int_0^{3-y} \frac{1}{8}(6-x-y)\,dx\,dy$

$= \cancel{15}\ \frac{10}{24}$

$0<x<2, \quad 2<y<4$



limits, For the region of $x+y<3$
$x=0$ to $x=3-y$
$y=2$ to 3

(iv) $P(x<1 \mid y<3) = \dfrac{P(x<1, y<3)}{P(y<3)}$

we have found $P(x<1, y<3) = \frac{3}{8}$

Let us find $P(y<3) = \int_2^3 f(y)\,dy$

Here we need to find $f(y)$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \frac{1}{8}(6-x-y)\,dx$$

$$= \frac{1}{8}\left(6x - \frac{x^2}{2} - xy\right)_0^2$$

$$= \frac{1}{8}\left[12-2-2y\right]$$

$$f(y) = \frac{5-y}{4}$$

$$\Rightarrow P(y<3) = \int_2^3 f(y)\,dy$$

$$= \int_2^3 \left(\frac{5-y}{4}\right) dy = \frac{1}{4}\left[5y - \frac{y^2}{2}\right]_2^3$$

$$= \frac{1}{4}\left[\left(15-\frac{9}{2}\right) - \left(10-\frac{4}{2}\right)\right]$$

$$= \frac{5}{8}$$

$$\therefore P(x<1 \mid y<3) = \frac{P(x<1, y<3)}{P(y<3)} = \frac{3/8}{5/8} = \frac{3}{5}$$

(V) To check X and Y are independent

$$f(x,y) = f(x) * f(y)$$

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{2}^{4} \frac{1}{8}(6-x-y)\,dy$$

$$f(x) = \frac{6-2x}{8}$$

We have found

$$f(y) = \frac{5-y}{4}$$

$$f(x) * f(y) = f(x,y)$$

$$\frac{6-2x}{8} \times \frac{5-y}{4} \neq \frac{1}{8}(6-x-y)$$

$\therefore$ X and Y are dependent.

4. The joint pdf of (X,Y) is given by

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3} & , 0 < x < 1, \; 0 < y < 2 \\ 0 & , \text{otherwise.} \end{cases}$$

Find (i) $P(X > \frac{1}{2})$ (ii) $P(Y < X)$, (iii) $P(Y < \frac{1}{2} \mid X < \frac{1}{2})$.

Covariance and Correlation Coefficient

Convariance :

As the Variance σ_x^2 (or $Var(X) = E(X^2) - [E(X)]^2$) is a measure of the Variation of the Random Variable (R.V) X from its mean Value E(X), the quantity $E[(X-E(X))(Y-E(Y))]$ measures the Simultaneous Variation of two R.V's X and Y from their respective means and hence it is called the Covariance of X, Y and it is denoted as Cov(X,Y).

$$Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$$

Proof :

$E(X) = \bar{x}, \quad E(Y) = \bar{y}$

$$Cov(X,Y) = E[(X-E(X))(Y-E(Y))]$$
$$= E[(X-\bar{x})(Y-\bar{y})]$$
$$= E[XY - X\bar{y} - \bar{x}Y + \bar{x}\bar{y}]$$
$$= E[XY] - E[X\bar{y}] - E[\bar{x}Y] + E[\bar{x}\bar{y}]$$
$$= E[XY] - \bar{y}E[X] - \bar{x}E[Y] + E[\bar{x}\bar{y}]$$
$$= E[XY] - \cancel{\bar{y}\bar{x}} - \bar{x}\bar{y} + \cancel{\bar{x}\bar{y}}$$
$$Cov(x,y) = E[XY] - E[X]\cdot E[Y].$$

Correlation co-efficient :

$$r_{xy} = r(x,y) = \rho(x,y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$$

* Correlation can be -ve but Covariance can't be -ve.

The Correlation Co-efficient is a statistical measure that calculates the strength of the relationship b/w the relative movements of two variables. The values $-1 \le r(x,y) \le 1$. A Correlation -1 shows that perfect negative Correlation, while a correlation 1 shows a perfect positive Correlation. If the Correlation is 0, then there is no relationship b/w variables however other functional relationship may exist.

Raw data:

$$X: \quad x_1 \quad x_2 \quad \cdots \quad x_n$$
$$Y: \quad y_1 \quad y_2 \quad \cdots \quad y_n$$

$$r(x,y) = \frac{E(xy) - E(x)E(y)}{\sigma_x \sigma_y}$$

where $E(X) = \frac{\sum_i x_i}{n}$ $\quad E(y) = \frac{\sum_i y_i}{n}$ $\quad E(xy) = \frac{\sum_i x_i y_i}{n}$

$$\sigma_x = \sqrt{E(x^2) - [E(x)]^2} \quad \text{or} \quad \sigma_x = \sqrt{\frac{\sum (x-\bar{x})^2}{n}}$$

For discrete:

$E(X) = \sum x\, p(x)$ $\qquad E(x^2) = \sum x^2 p(x)$ $\qquad E(xy) = \sum\sum xy\, P(xy)$

$E(y) = \sum y\, p(y)$ $\qquad E(y^2) = \sum y^2 p(y)$

For Continuous:

$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$ $\qquad E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$ $\qquad E(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$

$E(y) = \int_{-\infty}^{\infty} y f(y)\, dy$ $\qquad E(y^2) = \int_{-\infty}^{\infty} y^2 f(y)\, dy$

Note:

(i) $Cov(x,y) = 0$ if x and y are independent.

(ii) $Var(x \pm y) = Var(x) + Var(y) \pm 2Cov(x,y)$

(iii) $Var(x \pm y) = Var(x) + Var(y)$ if x and y are independent.

1. Find the Correlation Coefficient b/w X and Y.

X:	2	4	5	6	8	11
Y:	18	12	10	8	7	5

Solu:

$$r_{xy} = \rho(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{E(xy) - E(x)E(y)}{\sigma_x \sigma_y}$$

- $E(x,y) = \frac{\sum x_i y_i}{n} = \frac{(2\times18)+(4\times12)+(5\times10)+(6\times8)+(8\times7)+(11\times5)}{6} = \frac{293}{6}$

$$E(xy) = 48.83$$

- $E(x) = \frac{\sum x_i}{n} = \frac{2+4+5+6+8+11}{6} = 6.$

- $E(y) = \frac{\sum y_i}{n} = \frac{60}{6} = 10$

$$Cov(x,y) = E(xy) - E(x)E(y)$$

$$Cov(x,y) = -11.17$$

- $\sigma_x = \sqrt{E(x^2) - [E(x)]^2}$

$$E(x^2) = \frac{\sum x_i^2}{n} = \frac{266}{6} = 44.33$$

$$\sigma_x = \sqrt{44.33 - 36} = 2.886.$$

- $\sigma_y = \sqrt{E(y^2) - [E(y)]^2}$

$$E(y^2) = \frac{\sum y_i^2}{n} = 117.67$$

$$\sigma_y = \sqrt{117.67 - 100} = 4.2$$

$$r_{xy} = \rho(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{-11.17}{2.886\times4.2} = -0.9207$$

2. Find $r(x,y)$ for the following

X:	55	56	58	59	60	60	62
Y:	35	38	37	39	44	43	44

3. Let (X,Y) be a 2D-R.VS whose joint pdf is given by

$$f(x,y) = \begin{cases} x+y, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

find $r(X,Y)$.

4. If (X,Y) be a 2D-R.V with joint pmf is given by

$Y \backslash X$	1	2	3
1	$2/21$	$3/21$	$4/21$
2	$3/21$	$4/21$	$5/21$

find $\sigma(X,Y)$.

5. If the joint pdf of 2-D RV is given by

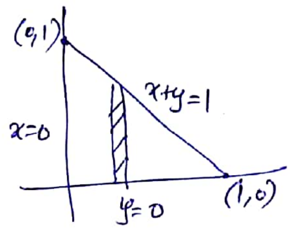
$$f(x,y) = \begin{cases} 3(x+y), & x>0, y>0, x+y\leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find $Cov(X,Y)$, $Var(X)$, $Var(Y)$.

Solu:

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

The marginal density function of X is given by



$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{0}^{1-x} 3(x+y)\,dy$$

$$= 3\left[xy + \frac{y^2}{2}\right]_0^{1-x}$$

$$= 3\left[\left(x(1-x) + \frac{1}{2}(1-x)^2\right) - (0)\right]$$

$$= \frac{3}{2}(1-x^2), \quad 0\leq x \leq 1$$

The marginal density fun of Y

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{0}^{1-y} 3(x+y)\,dx \quad =$$

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$f(y) = \frac{3}{2}(1-y^2), \qquad 0 \le y \le 1.$

- $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_0^1 x\, \frac{3}{2}(1-x^2)\, dx \quad = \frac{3}{2}\int_0^1 (x - x^3)\, dx$$

$$= \frac{3}{2}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{3}{8}$$

- $E(X^2) = \int_{-\infty}^{\infty} X^2 f(X)\, dX$

$$= \int_0^1 x^2\left(\frac{3}{2}(1-x^2)\right) dx \quad = \frac{3}{2}\int_0^1 (x^2 - x^4)\, dx$$

$$= \frac{3}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

- $E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy$

$$= \int_0^1 y\left(\frac{3}{2}(1-y^2)\right) dy \quad = \frac{3}{2}\int_0^1 (y - y^3)\, dy \quad = \frac{3}{8}$$

- $E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\, dy$

$$= \int_0^1 y^2\left(\frac{3}{2}(1-y^2)\right) dy = \frac{3}{2}\int_0^1 (y^2 - y^4)\, dy = \frac{1}{5}$$

$\ast$ $Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{5} - \left(\frac{3}{8}\right)^2 = \frac{19}{320}$

$\ast$ $Var(Y) = \frac{19}{320}$

- $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$

$$= \int_0^1\int_0^{1-y} xy\, 3(x+y)\, dx\, dy \quad = \frac{1}{10}$$

$$= 3\int_0^1$$

$\therefore$ $Cov(X,Y) = E(XY) - E(X)E(Y)$

$$= \frac{1}{10} - \left(\frac{3}{8}\right)\left(\frac{3}{8}\right) = \frac{-13}{320}$$

$Cov(X,Y) = -13/320.$

Characteristic function

$$\phi_x(w) = E[e^{iwx}] = \begin{cases} \sum e^{iwx} p(x), & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{iwx} f(x)\,dx, & \text{if } x \text{ is continuous} \end{cases}$$

Note:

$$* \quad \phi_x(w) = \sum_{r=0}^{\infty} \frac{i^r w^r}{r!} \mu_r'$$

$$* \quad \mu_r' = \frac{1}{i^r} \left\{ \frac{d^r}{dw^r} \phi_x(w) \right\}_{w=0}.$$

mean $= \mu_1'$ variance $= \mu_2' - (\mu_1')^2$.

1. Find the characteristic fun for the R.V X whose pdf is given by $f(x) = \frac{\alpha}{2} e^{-\alpha(|x|)}$, $-\infty < x < \infty$, $\alpha > 0$. and hence find $E(x)$ and $Var(x)$.

Solu:

$$\phi_x(w) = \int_{-\infty}^{\infty} e^{iwx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{iwx} \frac{\alpha}{2} e^{-\alpha|x|}\,dx$$

$$= \frac{\alpha}{2} \left[\int_{-\infty}^{0} e^{iwx} e^{-\alpha(-x)}\,dx + \int_{0}^{\infty} e^{iwx} e^{-\alpha x}\,dx \right]$$

$$= \frac{\alpha}{2} \left[\int_{-\infty}^{0} e^{(\alpha+iw)x}\,dx + \int_{0}^{\infty} e^{-(\alpha-iw)x}\,dx \right].$$

$$= \frac{\alpha}{2} \left[\left(\frac{e^{(\alpha+iw)x}}{(\alpha+iw)} \right)_{-\infty}^{0} + \left(\frac{e^{-(\alpha-iw)x}}{-(\alpha-iw)} \right)_{0}^{\infty} \right]$$

$$= \frac{\alpha}{2} \left[\left(\frac{1}{\alpha+iw} - 0 \right) + \left(0 + \frac{1}{\alpha-iw} \right) \right].$$

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$$= \frac{\alpha}{2}\left[\frac{1}{\alpha+i\omega} + \frac{1}{\alpha-i\omega}\right]$$

$$= \frac{\alpha}{2}\left[\frac{\alpha - \cancel{i\omega} + \alpha + \cancel{i\omega}}{\alpha^2 - \cancel{i\omega\alpha} + \cancel{i\omega\alpha} - i^2\omega^2}\right].$$

$$= \frac{\alpha}{2}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right] \qquad \because i^2 = -1$$

$$\phi_x(\omega) = \frac{\alpha^2}{\alpha^2+\omega^2}$$

Mean = $\mu_1' = \frac{1}{i}\left\{\frac{d}{d\omega}\left(\frac{\alpha^2}{\alpha^2+\omega^2}\right)\right\}_{\omega=0}$

$$= \frac{1}{i}\left\{\frac{(\alpha^2+\omega^2)(0) - \alpha^2(2\omega)}{(\alpha^2+\omega^2)^2}\right\}_{\omega=0}$$

$$\mu_1' = 0.$$

$$\mu_2' = \frac{1}{i^2}\left\{\frac{d^2}{d\omega^2}\left(\frac{\alpha^2}{\alpha^2+\omega^2}\right)\right\}_{\omega=0}.$$

$$= (-1)\left\{\frac{d}{d\omega}\left(\frac{-2\alpha^2\omega}{(\alpha^2+\omega^2)^2}\right)\right\}_{\omega=0}$$

$$= (-1)\left\{\frac{(\alpha^2+\omega^2)^2(-2\alpha^2) + 2\alpha^2\omega\left(2(\alpha^2+\omega^2)\cdot 2\omega\right)}{(\alpha^2+\omega^2)^4}\right\}_{\omega=0}.$$

$$= (-1)\left\{\frac{-2\alpha^4}{\alpha^6}\right\}$$

$$\mu_2' = \frac{2}{\alpha^2}$$

$$Var(X) = \mu_2' - (\mu_1')^2 = \frac{2}{\alpha^2}$$

2. Find the characteristic fun of the R.V where pmf is given by

$$P(X=r) = \frac{e^{-\lambda}\lambda^r}{r!}, \quad r = 0,1,2\ldots \quad \lambda > 0$$