

Random Variables

Definition of Random Variable

- A random variable is a function that associates a real number with each element in the sample space.

Example : Two balls are drawn in succession without replacement from an urn containing 4 red balls and 3 black balls. The possible outcomes and the values y of the random variable Y , where Y is the number of red balls, are

Sample Space	Y
RR	2
RB	1
BR	1
BB	0

- If a sample space contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers, it is called a discrete sample space.
- If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a continuous sample space.

A random variable: is called a discrete random variable if its set. of possible outcomes is countable.

Example: Slide 2

The set of ordered pairs $(x, f(x))$ is a probability function, probability mass function, or probability distribution of the discrete random variable X if, for each possible outcome x ,

1. $f(x) \geq 0$,

2. $\sum_x f(x) = 1$,

3. $P(X = x) = f(x)$.

Continuous Random Variable

A Random Variable 'X' which takes all possible values in a given interval is called
1. $f(x) \geq 0$ variable.

2.
$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$P(a < X < b) = \int_a^b f(x) dx$$

A function $f(x)$ which satisfies the above requirements
is called a *probability function*

or *probability distribution* for a continuous random variable,
it is more often called a *probability density function*

DISTRIBUTION FUNCTIONS FOR CONTINUOUS RANDOM VARIABLES

$$F(x) = P(X \leq x) = P(-\infty < X \leq x) = \int_{-\infty}^x f(u) du$$

Properties of c.d.f of a random variable X

$$\text{i)} \quad 0 \leq F(x) \leq 1 \quad -\infty < x < \infty$$

$$\text{ii)} \quad \lim_{x \rightarrow -\infty} F(x) = 0$$

$$\text{iii)} \quad \lim_{x \rightarrow \infty} F(x) = 1$$

$$\text{iv)} \quad P(a \leq x \leq b) = F(b) - F(a)$$

$$\text{v)} \quad F'(x) = \frac{dF(x)}{dx} = f(x) \geq 0 \quad [f(x) \text{ is a p.d.f}]$$

(or)

$$dF(x) = f(x) dx$$

$dF(x)$ is called the probability differential of the random variable X.

By differentiating $F(x)$, we get p.d.f $f(x)$

A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Solution: Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school. Then x can be any of the numbers 0, 1, and 2. Now,

$$f(0) = P(X = 0) = \frac{\binom{3}{0} \binom{5}{2}}{\binom{8}{2}} = \frac{10}{28},$$

$$f(1) = P(X = 1) = \frac{\binom{3}{1} \binom{5}{1}}{\binom{8}{2}} = \frac{15}{28},$$

$$f(2) = P(X = 2) = \frac{\binom{3}{2} \binom{5}{0}}{\binom{8}{2}} = \frac{3}{28}.$$

Thus the probability distribution of X is

x	0	1	2
$f(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

2. If the random variable X takes the values 1,2,3 and 4 such that

$$2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)$$

find the probability distribution and cumulative distribution function of X

A Random variable X has the following probability distribution

x	-2	-1	0	1	2	3
P(x)	0.1	k	0.2	2k	0.3	3k

- i) Find the value of K
- ii) Evaluate $P(X < 2)$ and $P(-2 < X < 2)$
- iii) Cumulative distribution of X
- iv) Mean of X

Practice Problems

1. A r.v X has the following probability function

X	0	1	2	3	4	5	6	7
p(x)	0	K	2K	2K	3K	K²	2K²	7K² + K

- i. Find K
- ii. Evaluate $P(X < 6)$
- iii. If $P(X \leq C) > \frac{1}{2}$ find the minimum value of C .
- iv. Evaluate $P(1.5 < X < 4.5 / X > 2)$
- v. Find $P(X < 2)$, $P(X > 3)$, $P(1 < X < 5)$

1. A continuous random variable 'X' has a p.d.f $f(x) = 3x^2, 0 \leq x \leq 1$.

Find 'a' and 'b' such that

i) $P(x \leq a) = P(x > a)$ and ii) $P(x > b) = 0.05$.

2. Show that the function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases} \text{ is a p.d.f}$$

3.The amount of time ,in hours that a computer functions before breaking down is a continuous random variable with p.d.f is given by

$$f(x) = \begin{cases} \lambda e^{\frac{-x}{100}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

what is the probability that

a)a computer will function between 50 and 150 hrs before breaking down.

b)it will function less than 500 hrs.

4.If a random variable 'X' has the p.d.f

$$f(x) = \begin{cases} \frac{1}{2}(x + 1), & \text{if } -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

- Find mean and variance of x

Summary: Mean & Variance

	Definition	Discrete R.V.s	Continuous R.V.s
Mean: μ	$E(X)$	$\sum_i p(x_i)x_i$	$\int_{-\infty}^{\infty} \mathbf{f}(x)x dx$
Variance: σ^2	$E((X - \mu)^2)$	$\sum_i p(x_i)(x_i - \mu)^2$	$\int_{-\infty}^{\infty} \mathbf{f}(x)(x - \mu)^2 dx$

Practice Problems

1..If the density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ax, & 0 \leq x \leq 1 \\ a, & 1 \leq x \leq 2 \\ 3a - ax, & 2 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

- i. Find the value of a .
- ii. The cumulative distribution functions of X .

1. Verify whether the following is a distribution function

$$\mathbf{F(x)} = \begin{cases} \mathbf{0}, & x < -a \\ \frac{1}{2} \left(\frac{x}{a} + 1 \right) & -a < x < a \\ \mathbf{1} & x > a \end{cases}$$

2. Given the p.d.f of a continuous random variable 'X' as

$$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find cdf for 'X'

3. A random variable 'X' has the density function

$$f(x) = \begin{cases} K \frac{1}{1+x^2} & -\infty < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

a) Find 'K' and the distribution function F(x)

b) Find $P(X > 0)$

1. A random sample 'X' has the probability function

X	0	1	2	3	4	5	6	7	8
p(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

Determine the value of a

If X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} c(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases} \quad \text{what is the value of c?}$$

Given that the p.d.f of a R.V X is $f(x) = Kx, 0 < x < 1$ find K and $P(X > 0.5)$

4.

In a continuous random variable X having the p.d.f

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find $P(0 < X \leq 1)$

For the following c.d.f $F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$

find (i) $P(X > 0.2)$ (ii) $P(0.2 < X \leq 0.5)$

MATHEMATICAL EXPECTATION

Let 'X' be a random variable with p.d.f (or p.m.f).

Then the mathematical expectation of 'X' is denoted by $E(X)$

and is given by $E(X) = \int_{-\infty}^{\infty} xf(x)dx$ (for a continuous random variable)

$$E(X) = \sum_x xP(x) \quad (\text{for a discrete random variable})$$

r^{th} moment (about origin)

Consider a continuous r.v. 'X' with p.d.f (or p.m.f) $f(x)$ then the r^{th} moment (about origin) of the probability distribution is defined as

$$E(X) = \int_{-\infty}^{\infty} x^r f(x) dx .$$

It is denoted by $\mu_r' = \int_{-\infty}^{\infty} x^r f(x) dx$

$$\mu_1' = E(X)$$

$$\mu_2' = E(X^2)$$

$$\text{Mean} = \overline{X} = \mu_1' = E(X)$$

$$\text{Variance} = \mu_2 = \mu_2' - (\mu_1')^2$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

r^{th} moment about mean is denoted by μ_r

$$\mu_r = \int_{-\infty}^{\infty} (x - \bar{X})^r f(x) dx$$

Variance in terms of expectation

$$\text{Variance} = \mu_2 = E[\{X - E(X)\}^2]$$

$$\text{Mean} = \mu_1' = \sum xP(x)$$

$$\text{Variance} = \mu_2 = \sum_x (x - \bar{X})^2 P(x)$$

Addition Theorem (Expectation)

If X and Y are two continuous random variable with pdf (or pmf) then

$$E(X+Y)=E(X)+E(Y)$$

Multiplication theorem of Expectation

If X and Y are independent random variables then

$$E(XY)=E(X).E(Y)$$

Theorem:

If 'X' is a random variable with pdf $f(x)$ and 'a' is a constant ,then

$$(i) E[a G(X)] = a E[G(X)]$$

$$(ii) E[G(X) + a] = E[G(X)] + a$$

where $G(X)$ is a function of 'X' which is also a random variable.

Theorem:

If 'X' is a random variable with pdf $f(x)$ and 'a' and 'b' are constants , then

$$E [aX+b] = a E (X) + b$$

If X is a random variable then

$$\text{Var} (aX + b) = a^2 \text{Var} (X)$$

where ‘ a ’ and ‘ b ’ are constants.

$$\text{Cov} (X, Y) = E (XY) - E (X) .E (Y)$$

where X and Y are random variables.

If X and Y are independent , then

$$E (XY)= E(X) E (Y)$$

Therefore $\text{Cov} (X , Y) = 0$.

$$\mathbf{Cov} (\mathbf{a} \mathbf{X} , \mathbf{b} \mathbf{Y}) = \mathbf{ab} \mathbf{Cov} (\mathbf{X}, \mathbf{Y})$$

$$\mathbf{Cov} (\mathbf{X} + \mathbf{a} , \mathbf{Y} + \mathbf{b}) = \mathbf{Cov} (\mathbf{X}, \mathbf{Y})$$

$$\mathbf{Cov} (\mathbf{aX+b} , \mathbf{cY+d}) = \mathbf{ac} \mathbf{Cov} (\mathbf{X}, \mathbf{Y})$$

$$\mathbf{Var} (\mathbf{X_1+X_2}) = \mathbf{Var(X_1)+Var(X_2)+2Cov(X_1,X_2)}$$

$$\mathbf{Var} (\mathbf{X_1-X_2}) = \mathbf{Var(X_1)+Var(X_2)-2Cov(X_1,X_2)}$$

1. When a die is thrown, 'X' denotes the number that turns up.

Find $E(X)$, $E(X^2)$ and $\text{Var}(X)$.

2. The C.D.F of R.V X is

$$F(x) = 1 - (1 + x) e^{-x}, x > 0.$$

Find the p.d.f of 'X', mean, variance.

3. The density function of a random variable 'X' is given by

$f(x) = k x (2-x), 0 \leq x \leq 2$. Find k, mean, variance and r^{th} moment.

4. If the p.d.f of 'X' is given by

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $E(X^r) = 2/(r+1)(r+2)$.

b) Using this result evaluate $E[(2X+1)^2]$.

5. Let X be a R.V with $E(X) = 10$ and $V(X) = 25$. Find the positive values of a and b such that $Y = aX - b$ has expectation 0 and variance 1.