1. Inequalities of cones of bisection fans

Let F, G be two facet cones of a polytope $P \subset \mathbb{R}^d$ given by $F = \{x \in \mathbb{R}^d : A_1 x \leq 0\}$ and $G = \{x \in \mathbb{R}^d : A_2 x \leq 0\}$, where $A_1 \in \mathbb{R}^{m_1 \times d}$ and $A_2 \in \mathbb{R}^{m_2 \times d}$.

We know that each $\mathrm{bis}_{F,G}(0, a)$ is a polyhedron and that $\mathrm{dist}(x, \cdot)$ restricted to each facet cone is a linear function in x. Let $z_1^\top, z_2^\top$ be the linear functionals corresponding to the distance function restricted to F and G respectively. The H-description of $\mathrm{bis}_{F,G}(0, a)$ can be computed as below:

$$\begin{aligned}
\mathrm{bis}_{F,G}(0,a) &= \mathrm{bis}(0,a) \cap F \cap (G + a) \\
&= \{x \in \mathbb{R}^d : A_1 x \leq 0, A_2(x - a) \leq 0, \mathrm{dist}(x, 0) = \mathrm{dist}(x, a)\} \\
&= \{x \in \mathbb{R}^d : A_1 x \leq 0, A_2 x \leq A_2 a, z_1^\top x = z_2^\top (x - a)\} \\
&= \{x \in \mathbb{R}^d : A_1 x \leq 0, A_2 x \leq A_2 a, (z_1 - z_2)^\top x \leq -z_2^\top a, (z_2 - z_1)^\top x \leq z_2^\top a\} \\
&= \{x \in \mathbb{R}^d : Ax \leq z_a\}
\end{aligned}$$

where $A = \begin{pmatrix} A_1 \\ A_2 \\ z_1^\top - z_2^\top \\ z_2^\top - z_1^\top \end{pmatrix}_{(m_1+m_2+2)\times d}$ and $z_a = \begin{pmatrix} 0 \\ A_2 a \\ -z_2^\top a \\ z_2^\top a \end{pmatrix}_{(m_1+m_2+2)\times 1}$.

We are interested in the set $\mathcal{B}_{F,G} = \{a \in \mathbb{R}^d : \mathrm{bis}_{F,G}(0,a) \neq \emptyset\}$. The fact that it is a convex cone follows from $\mathrm{bis}_{F,G}(0,a)$ being a polyhedron.

Next, we will show that it is also a polyhedral cone and determine its inequalities:

$$\begin{aligned}
\mathcal{B}_{F,G} &= \{a \in \mathbb{R}^d : \mathrm{bis}_{F,G}(0,a) \neq \emptyset\} \\
&= \{a \in \mathbb{R}^d : \exists\, x \in \mathbb{R}^d, Ax \leq z_a\} \\
&= \{a \in \mathbb{R}^d : \forall\, c \in \mathbb{R}^{m_1+m_2+2} \text{ satisfying } cA = 0 \,\text{and}\, c \geq 0, \langle c, z_a \rangle \geq 0\}.
\end{aligned}$$

The last equality holds from the Farkas lemma [1] and allows us to further compute the inequalities of $\mathcal{B}_{F,G}$ explicitly. Let us set $C = \{c \in \mathbb{R}^{m_1+m_2+2} : Xc \geq 0\}$ where $X = \begin{pmatrix} I_{m_1+m_2+2} \\ A^\top \\ -A^\top \end{pmatrix}$

Now write $c \in C$ as $c = (c_1, c_2, c_3, c_4)$ where $c_1 \in \mathbb{R}^{1\times m_1}, c_2 \in \mathbb{R}^{1\times m_2}$ and $c_3, c_4 \in \mathbb{R}$. Then the above condition can be simplified as:

$$\begin{aligned}
\mathcal{B}_{F,G} &= \{a \in \mathbb{R}^d : \forall\, c \in C, \langle c, z_a \rangle \geq 0\} \\
&= \left\{ a \in \mathbb{R}^d : \forall\, (c_1, c_2, c_3, c_4) \in C, (c_1, c_2, c_3, c_4) \begin{pmatrix} 0 \\ A_2 a \\ -z_2^\top a \\ z_2^\top a \end{pmatrix} \geq 0 \right\}
\end{aligned}$$

By the convexity of C, it suffices that the inequality be verified by the rays of C only.

$$\begin{aligned}\mathcal{B}_{F,G} &= \{a \in \mathbb{R}^d : \forall\, c \in \text{rays}(C), \langle c_2, A_2 a\rangle - c_3 z_2^\top a + c_4 z_2^\top a \geq 0, \}\\ &= \{a \in \mathbb{R}^d : \forall\, c \in \text{rays}(C), \langle A_2^\top c - c_3 z_2 + c_4 z_2, a\rangle \geq 0\}\\ &= \{a \in \mathbb{R}^d : Y^C a \geq 0\}\end{aligned}$$

where $Y^C = \left(\text{—}\, A_2^\top c^i - c_3^i z_2 + c_4^i z_2 \,\text{—}\right)_{n\times d}$ and $c^i, i = 1, \ldots, n$ are the rays of C.

This implies that each $\mathcal{A}_{F,G}$ is a polyhedral cone and hence it is of interest to ask if $\mathcal{B} = \{\mathcal{A}_{F,G} : F, G \text{ are facet cones of P}\}$ is a polyhedral fan.

References

[1] G. M. Ziegler, *Lectures on polytopes*, vol. 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.