Suggested Problems for Topology HW (Separation Axioms, compactness) *

Problem 1. Prove that a continuous bijection from a compact space X into a Hausdorff space Y is always a homeomorphism.

Problem 2. A topological space Y is said to have the **unique extension property** if for any topological space X and every pair of continuous maps $f, g : X \to Y$ agreeing on a dense subset of X, f, g agree on all of X. Prove that Y has the unique extension property if and only if it is Hausdorff.

Problem 3. Prove the following claims:

1. Let E^1 (Euclidean line) denote $\mathbb{R}$ equipped with the standard topology. Let X be the set of irrationals in $\mathbb{R}$ and let $f : X \to E^1$ be a map that is always positive. For each $n \in \mathbb{Z}^{>0}$ let $H_n := \{x \mid f(x) \geq \frac{1}{n}\}$. Prove that there exists an m and an open interval U, such that $V \cap H_m \neq \emptyset$ for every open interval $V \subset U$.
2. Let ${E^1}_U$ denote $\mathbb{R}$ equipped with the upper limit topology. Show that in ${E^1}_U \times {E^1}_U$ the disjoint closed sets $A := \{(r, -r) \mid r \in \mathbb{Q}\}$ and $B := \{(r, -r) \mid r \notin \mathbb{Q}\}$ cannot be separated.

Problem 4. Prove the following claims on regularity of a topological space:

1. A T_1 space X is regular iff $\forall x \in X$ and every neighbourhood $x \in V \subseteq X$, there exists a neighbourhood U that $\overline{U} \subseteq V$.
2. Let X be regular. Prove that a compact set A and a disjoint closed set F can be separated by disjoint open sets.

Problem 5. A topological space is regular if and only if any closed set is the intersection of its closed neighborhoods.

Problem 6. Let $(A, >)$ be a linearly ordered set. Then A can be equipped with the **open interval topology** which is generated by the open base including $(a, b) := \{x \mid a < x, x < b\}$, $(-\infty, b) := \{x \mid x < b\}$, $(a, \infty) := \{x \mid a < x\}$. A linearly ordered topological space (LOTS) is a linear ordered set endowed with the open interval topology. Prove that every LOTS is normal.

Problem 7. Given a set X and a family of topologies $\{\tau_\alpha\}_{\alpha \in \mathcal{A}}$ on X (where $\mathcal{A}$ is the indexing set), define $\bigvee_{\alpha \in \mathcal{A}} \tau_\alpha$ as the topology generated by the subbasis $\bigcup_{\alpha \in \mathcal{A}} \tau_\alpha$. Given any family $\mathcal{F} := \{(Y_\alpha, \tau_\alpha), f_\alpha \mid \alpha \in \mathcal{A}\}$ of topological spaces (sets Y_α, topologies τ_α on them) and maps $f_\alpha : X \to Y_\alpha$, define **projective limit topology of X generated by** $\mathcal{F}$ as $\bigvee_{\alpha \in \mathcal{A}} {f_\alpha}^{-1}(\tau_\alpha)$. Prove that if X is completely regular, the topology of X is equal to the projective limit topology generated by the family of all continuous maps $f : X \to [0, 1]$.

Problem 8. In the following exercise, we'll define the topology generated by a set of morphisms:

1. Let $X, \{Y_\alpha\}_{\alpha \in \mathcal{A}}$ be topological spaces where $\mathcal{A}$ is the indexing set. Let $f_\alpha : X \to Y_\alpha$ be given maps for all $\alpha \in \mathcal{A}$. Tikhonov defined the **diagonal product** of $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ as the map $F : X \to Y := \prod_{\alpha \in \mathcal{A}} Y_\alpha$ where $F(x) := \{f_\alpha(x)\}_{\alpha \in \mathcal{A}}$ for all $x \in X$. Prove that the diagonal product of continuous mappings is continuous.
2. For every set X and every $\mathcal{F} \subseteq \mathbb{R}^X$ (a set of functions $f : X \to \mathbb{R}$), let $\mathcal{T}_\mathcal{F}$ be the coarsest topology in which all $f \in \mathcal{F}_0$ are continuous. One would define $\overline{\mathcal{F}}$, the **local completion** of $\mathcal{F}$ to be the set of all functions $f : X \to \mathbb{R}$ such that for all $x \in X$, there exists open neighbourhood $x \in U \in \mathcal{T}$ and a finite set of functions $f_1, \cdots, f_k \in \mathcal{F}$, along with a continuous map $F : \mathbb{R}^k \to \mathbb{R}$ such that $F(f_1(y), \cdots, f_k(y)) = f(y), \forall y \in U$. ($f$ is said to be locally generated by $\mathcal{F}$)
   (a) Prove that $\mathcal{T}_\mathcal{F} = \mathcal{T}_{\overline{\mathcal{F}}}$.
   (b) Prove that a subspace of a (completely) regular space is (completely) regular.
   (c) Prove that a product of a (completely) regular spaces is (completely) regular. Deduce $\mathbb{R}^\mathcal{F}$ is (completely) regular.
   (d) Assuming that $\mathcal{T}_{\overline{\mathcal{F}}}$ is T_0, prove that $\mathcal{T}_{\overline{\mathcal{F}}}$ is (completely) regular.

Problem 9. The **density of a space** X is defined as $d(X) := \inf\{|A| : A \subseteq X \text{ is a dense subset}\}$ and **weight of** X as $\omega(X) := \inf\{|A| : A \text{ is a base of } X\}$. Prove that (for avoiding set theoretic complexions, one could assume that X is finite):

1. For every topological space X, $d(X) \leq \omega(X)$.
2. For every T_0 topological space we have $|X| \leq 2^{\omega(X)}$.
3. For every regular space we have $\omega(X) \leq 2^{d(X)}$.
4. For every compact space we have $\omega(X) \leq |X|$.

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