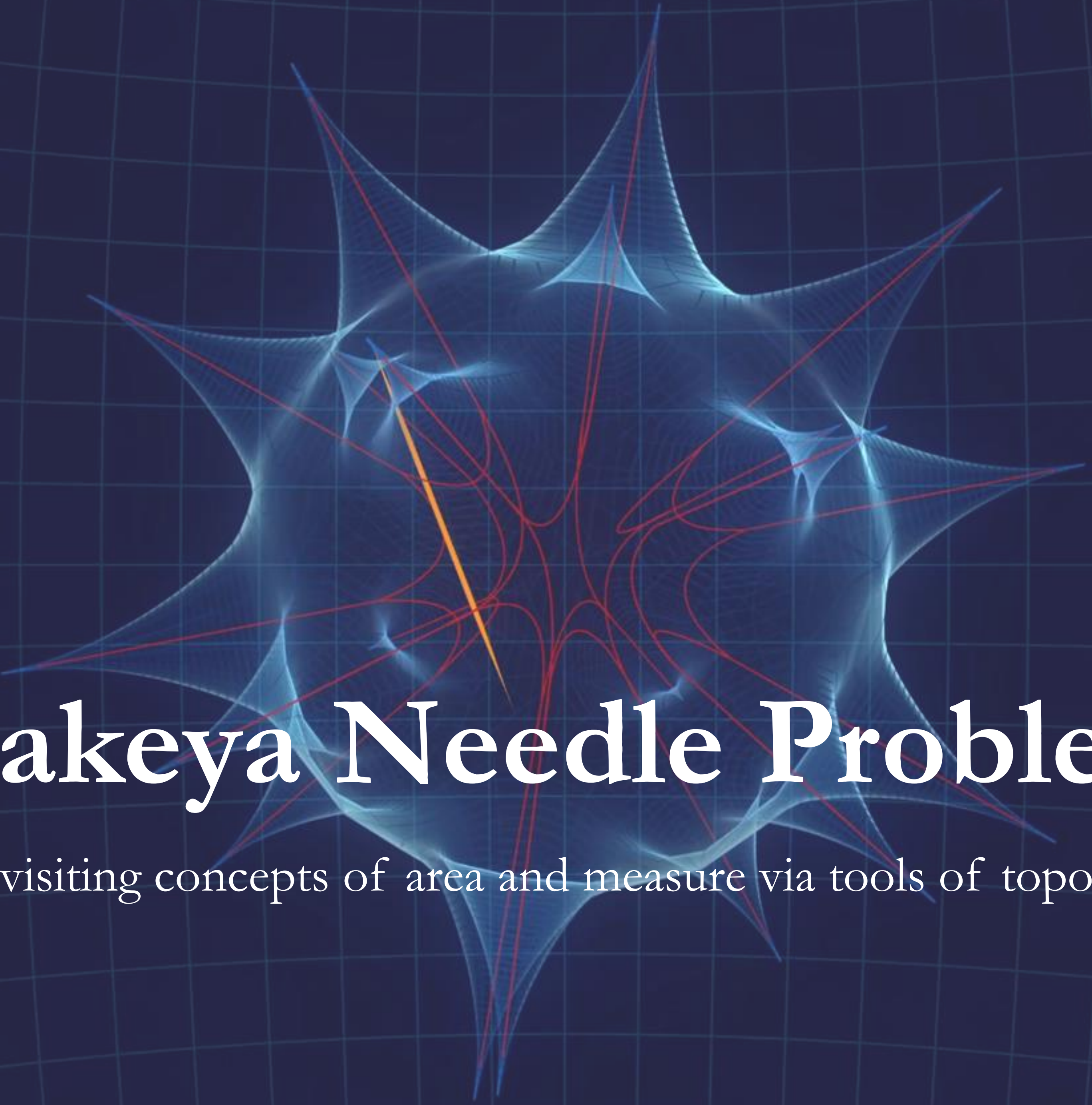


Kakeya Needle Problem

Revisiting concepts of area and measure via tools of topology

What do we expect of an measure functional on $\mathbb{R}^n$?

- Monotonicity of measure function: $\forall A \subseteq B\ :\ \mu(A) \leq \mu(B)$
- Building blocks of measure : Unit cubes, relativization of measure
- Positive-definiteness of the measure functional: $\forall A \subseteq \mathbb{R}^n : \mu(A) \geq 0$
- Finite additivity: For any two disjoint subsets $A, B \subseteq \mathbb{R}^n, A \cap B = \emptyset$, one has $\mu(A \cup B) = \mu(A) + \mu(B)$. This generalizes by induction to finitely many pairwise disjoint subsets of $\mathbb{R}^n$
- Continuity of μ: Convergent sequences of subsets$\{A_i\}_{i\in\mathbb{N}} \to A$ of $\mathbb{R}^n$ correspond to convergent sequence of measures $\mu(A_i) \to \mu(A)$. A useful technique in measure theory is to convert convergent sequences to monotone sets: Given sequence $\{A_i\}_{i\in\mathbb{N}} \to A$, one can define $\overline{A_i} \coloneqq \bigcup_{j\geq i} A_i$, $\underline{A_i} = \bigcap_{j\geq i} A_i$. Therefore one can see $\overline{A}_1 \supseteq \overline{A_2} \supseteq \cdots$ and $\underline{A_1} \subseteq \underline{A_2} \subseteq \cdots$ and $\lim \underline{A_i} = \lim \overline{A_i} = \lim A_i$. Continuity hence translates to that $\{A_i\} \uparrow A$ and $\{A_i\} \downarrow A$ both imply $\mu(A_i) \to \mu(A)$. This in extremely important for assigning measures to subsets
- The two latter conditions imply that for a countable set of pairwise disjoint sets $\{A_i\}$, $\mu(\cup A_i) = \sum \mu(A_i)$

Can such a functional assign a measure to every set?

Let X be a set and $\sim$ be a relation on X. Then $\sim$ is said to be an equivalence relation if it satisfies:

- Reflexivity: $\forall x \in X : x \sim x$
- Symmetry: $\forall x, y \in X : x \sim y \iff y \sim x$
- Transitivity: $\forall x, y, z \in X : x \sim y, y \sim z \Rightarrow x \sim z$

Given an equivalence relation $\sim$ on a set X, one can define equivalence classes $\forall x \in X : [x] \coloneqq \{y \in X \mid y \sim x\}$. A well-known result is that X is then partitioned into the equivalence classes. Set of equivalence classes is then denoted by $X/\sim$ and is termed as the quotient of X by the equivalence relation $\sim$.

Any $X \subseteq \mathbb{R}^n$ induces a relation on $\mathbb{R}^n$ as follows: $\forall x, y \in \mathbb{R}^n : x \sim y \Leftrightarrow x - y \in X$. This relation is especially an equivalence relation if it is closed under vector addition and inverses (negative of vector). One can therefore consider the quotient $\mathbb{R}/\mathbb{Q}$. For each $[x] \in \mathbb{R}/\mathbb{Q}$, choose a representative of $[x]$ in $[0,1]$ (this depends on axiom of choice). A set containing such a choice of representatives is then called a Vitali set.

Any Vitali set is uncountable. For a Vitali set X and $x, y \in X$ it is immediate that $x - y \notin \mathbb{Q}$.

Let X be a Vitali set. The rational translations $X_q = \{x + q \mid x \in X\}$, where $q \in \mathbb{Q}$ is a rational number, are pairwise disjoint and cover all $\mathbb{R}$. In particular, $[0,1] \subseteq \bigcup_{\mathbb{Q}\cap[-1,1]} X_q \subseteq [-1,2]$ and $\mathbb{Q} \cap [-1,1]$ is in fact a countable subset of $\mathbb{R}$. This implies that due to countable additivity and monotonicity, we would expect that $\mu([0,1]) \leq \sum_{\mathbb{Q}\cap[-1,1]} \mu(X_q) \leq \mu([-1,2])$. It is also natural that we expect the measure functional of $\mathbb{R}$ to be translation-invariant. Therefore one would expect that $\mu(X_q) = \mu(X), \forall q \in \mathbb{Q}$. But this would then also imply that $1 \leq \mu([0,1]) = |\mathbb{Q} \cap [-1,1]| \cdot \mu(X) \leq \mu([-1,2]) = 3$. A direct consequence is that $\mu(X)$ cannot take any real value which means that Vitali sets are in fact not measurable!

Note that the existence of such (non-measurable) Vitali sets heavily depends on the ZF axiom of choice. In fact a shocking consequence is that existence of non-measurable sets is equivalent to the axiom of choice.

Therefore, the “natural” conditions on the measure functional cause an obstruction to the measurability of certain sets. We hereby denote the measurable sets as $\mathcal{F}$.

What do we expect of $\mathcal{F}$?

- The total space (namely X) is measurable: Deletion of redundant points
- $\mathcal{F}$ is closed under complementation: $\forall S \subseteq X : S \in \mathcal{F} \Leftrightarrow S^C \in \mathcal{F}$
- $\mathcal{F}$ is closed under countable union (due to the conditions on the measure functional)

For a given set X, such a family of subsets of X is called a σ-algebra on X.

- Definition of open sets in $\mathbb{R}^n$
- All open sets are measurable
- $\mathcal{F}$ contains the smallest σ-algebra containing all of open sets: Borel σ-algebra
- Not all measurable sets are Borel sets

What is a measure?

Given a set X and a σ-algebra on X denoted by $\mathcal{F}$ (considered as the family of measurable sets), a map $\mu : \mathcal{F} \to \mathbb{R}$ is said to be a measure if it satisfies the following:

- $\mu(\emptyset) = 0$
- Positive-definiteness: $\forall F \in \mathcal{F} : \mu(F) \geq 0$
- σ-additivity: For any countable family $\{F_i\}$ of pairwise disjoint measurable sets: $\mu(\cup F_i) = \sum \mu(F_i)$

The triple $(X, \mathcal{F}, \mu)$ is then said to be a measure space. A subset $A \subseteq X$ is called a null set if there exists a superset of A such that $A \subseteq B \in \mathcal{F}, \mu(B) = 0$. A measure space is complete if every null set is measurable in it.

Examples:

- Counting measure
- Point mass measure
- Lebesgue measure: Unique complete translation-invariant measure (up to scalar multiple and extension)

Properties of measure

- A point is a set of measure zero in $\mathbb{R}^n, n \geq 2$
- Consequently, a set of countably many points is of measure zero in $\mathbb{R}^n, n \geq 2$
- The converse does not hold. Cantor set is in fact an uncountable set of measure zero
- Monotonicity: $\forall A, B \in \mathcal{F}, A \subseteq B : \mu(A) \leq \mu(B)$
- Upper/Lower continuity: $\{A_i\} \uparrow A$ and $\{A_i\} \downarrow A$ both imply $\mu(A_i) \to \mu(A)$

Kakeya Needle problem in 2D

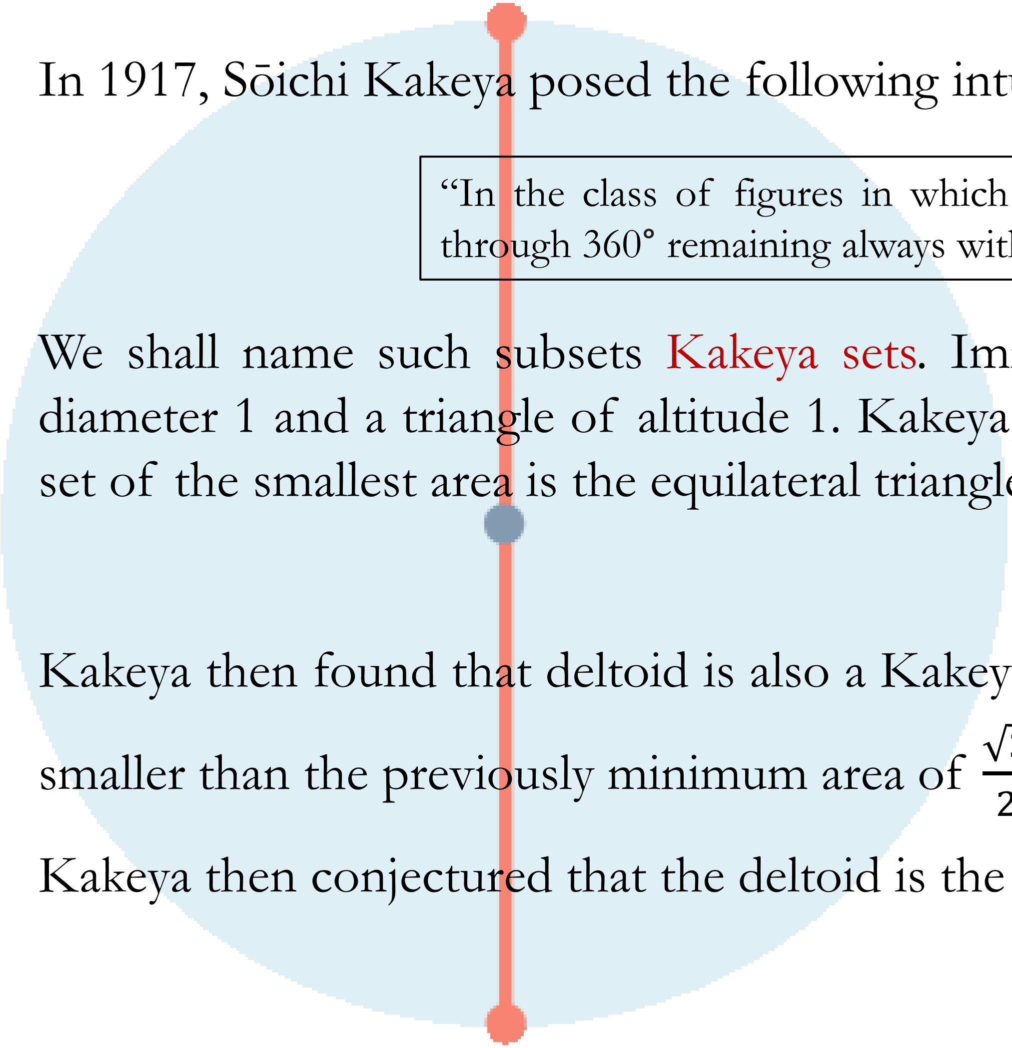
In 1917, Sōichi Kakeya posed the following intuitive problem:

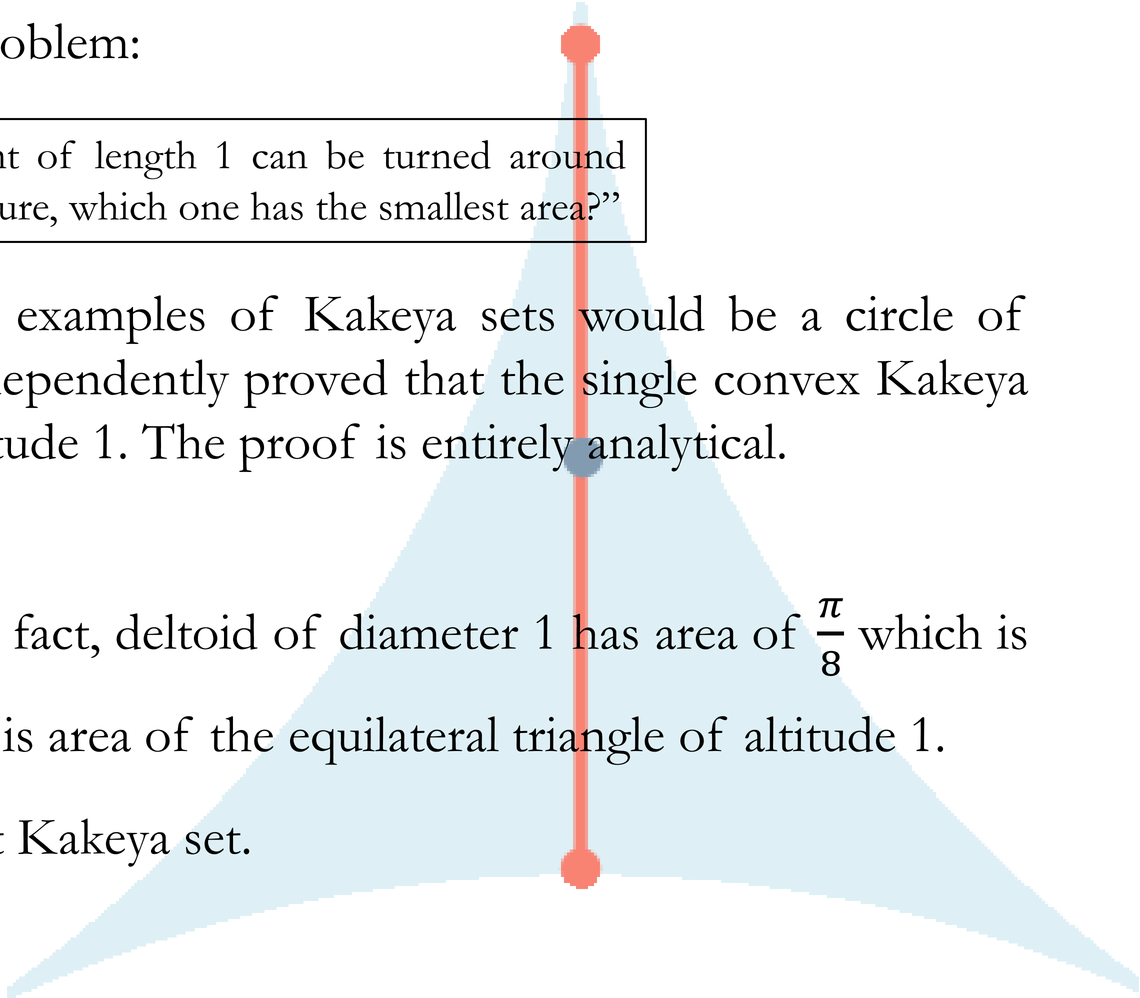
> "In the class of figures in which a segment of length 1 can be turned around through 360° remaining always within the figure, which one has the smallest area?"

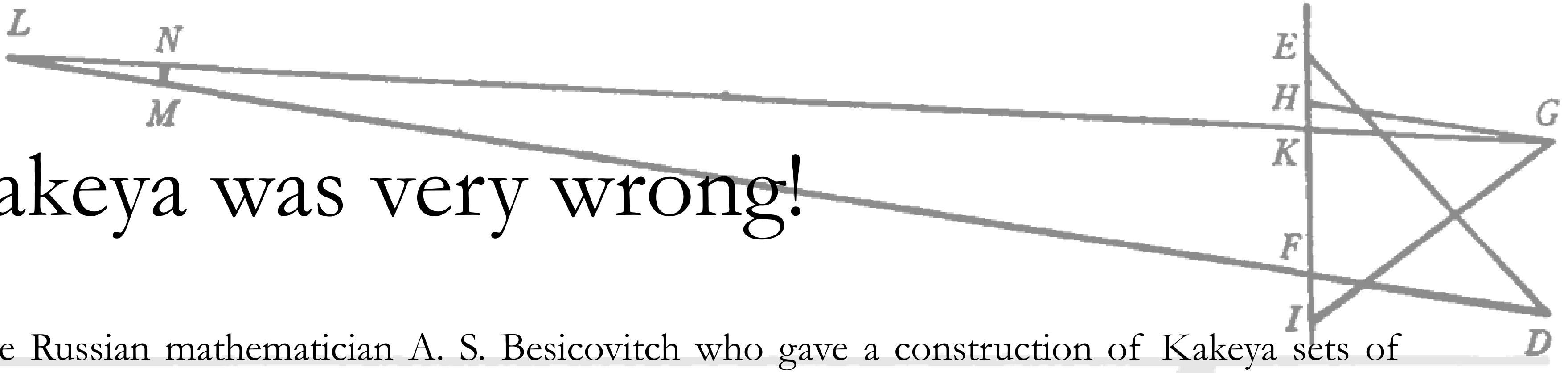
We shall name such subsets Kakeya sets. Immediate examples of Kakeya sets would be a circle of diameter 1 and a triangle of altitude 1. Kakeya, Pal independently proved that the single convex Kakeya set of the smallest area is the equilateral triangle of altitude 1. The proof is entirely analytical.

Kakeya then found that deltoid is also a Kakeya set. In fact, deltoid of diameter 1 has area of $\frac{\pi}{8}$ which is smaller than the previously minimum area of $\frac{\sqrt{3}}{2}$ which is area of the equilateral triangle of altitude 1.

Kakeya then conjectured that the deltoid is the smallest Kakeya set.





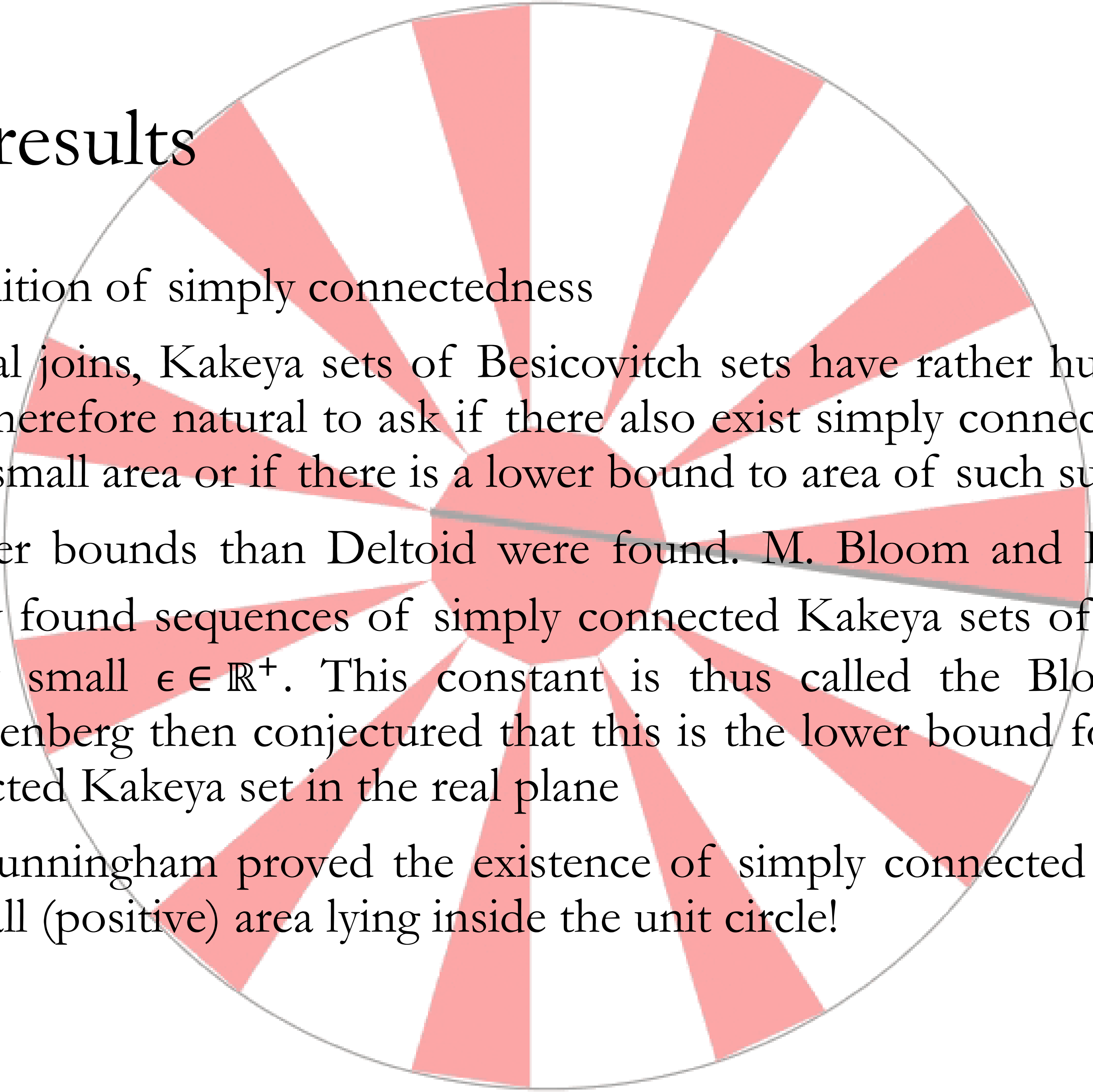


Apparently, Kakeya was very wrong!

In 1918, the problem reached the Russian mathematician A. S. Besicovitch who gave a construction of Kakeya sets of arbitrarily small area. The construction was then refined several times by J. Pal and I. J. Schoenberg into what we now call the Perron tree. The construction goes on in the following steps:

- Taking a square of side length 2 and subdividing it into 4n “elementary triangles” with equal bases
- Enumerate the elementary triangles in counter-clockwise order. This will be the order of “sweeping”
- Parallel transporting the elementary triangles would still let the needle “sweep” through each of them. Therefore, sweeping through all of them in the mentioned order would eventually cause a full rotation
- How to transport between consecutive elementary triangles? Introducing Pal “joins” + Area penalty analysis
- How to transport at corners? Introducing parallel sweeps + Area penalty analysis
- Reduction of problem to giving a parallel transportation scheme of elementary triangles on each side of the square such that the total area is eventually very small
- For any $p \geq 2$, creating a p-level hierarchy of isosceles right triangles of altitudes $\frac{1}{p}, \ldots, \frac{p}{p} = 1$
- Bisection & expansion to the upper level of hierarchy. Increase of area during bisecting & expanding an elementary triangle (due to its “end pieces”) would be at most twice the area of its “top end” + Area analysis

Relevant results



- Intuitive definition of simply connectedness
- Due to the Pal joins, Kakeya sets of Besicovitch sets have rather huge fundamental groups. It is therefore natural to ask if there also exist simply connected Kakeya sets of arbitrarily small area or if there is a lower bound to area of such subsets?
- In 1965, better bounds than Deltoid were found. M. Bloom and I. J. Schoenberg independently found sequences of simply connected Kakeya sets of area $\frac{5-2\sqrt{2}}{24}\pi+\epsilon$ for arbitrarily small $\epsilon \in \mathbb{R}^+$. This constant is thus called the Bloom-Schoenberg number. Schoenberg then conjectured that this is the lower bound for measure of a simply connected Kakeya set in the real plane
- In 1971, F. Cunningham proved the existence of simply connected Kakeya sets of arbitrarily small (positive) area lying inside the unit circle!

Can there exist Kakeya sets of measure zero?

- Suppose $X \subseteq \mathbb{R}^2$ is a Kakeya set. Therefore there exists a continuous map of needle rotation in X:
  $$\tau : [0,1] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \to X \quad , \tau(t,s) = \{x(t) + s \cdot \cos \omega(t) \, , y(t) + s \cdot \sin \omega(t)\} \; , x, y : [0,1] \to X \, , \omega : [0,1] \to [0,2\pi) \in C^0$$
- A theorem of analysis implies that any continuous function f on [0,1] (which means that this function satisfies $\forall t \in [0,1] \, , \epsilon \in \mathbb{R}^+ : \exists \delta \in \mathbb{R}^+ : \forall t' \in [0,1], \; |t' - t| < \delta \Rightarrow |f(t') - f(t)| < \epsilon$) is uniformly continuous as well: $\forall \epsilon \in \mathbb{R}^+ : \exists \delta \in \mathbb{R}^+ : \forall t, t' \in [0,1], \; |t' - t| < \delta \Rightarrow |f(t') - f(t)| < \epsilon$ (this is due to compactness of $[0,1]$).
- Fix $\epsilon \in \mathbb{R}^+$. Therefore $\exists \delta \in \mathbb{R}^+ : \forall t, t' \in [0,1], |t' - t| < \delta \Rightarrow |x(t') - x(t)|, |y(t') - y(t)|, \; |\omega(t') - \omega(t)| < \epsilon$.
- Map ω is clearly non-constant. Therefore, there exist $t_0, t_1 \in [0,1] \, , |t_0 - t_1| < \delta \, , \omega(t_0) \neq \omega(t_1)$.
- WLOG $x(t_0) = y(t_0) = \omega(t_0) = 0$. For sufficiently small ϵ, the line $\tau(t, -)$ intersects the vertical line $x = a$ in some point $(a, y_t(a))$ for all $a \in \left[-\frac{1}{2} + \gamma, \frac{1}{2} - \gamma\right] \, , t \in [t_0, t_1]$. The value of γ can be arbitrarily small by choosing sufficiently small value of ϵ. Note that the point $(a, y_t(a))$ lies in X.
- The map $y_t(a)$ is then continuous in $t \in [t_0, t_1]$. Therefore, the set $\{(a, y_t(a)) \mid t \in [t_0, t_1]\}$ is a segment on the vertical line $x = a$ and this segment lies entirely in X. Denote this segment by $\mathcal{L}_a$.
- Finally $\mathcal{L}_a$ depends continuously on a. Therefore the set $\bigcup_{a \in \left[-\frac{1}{2} + \gamma, \frac{1}{2} - \gamma\right]} \mathcal{L}_a \subseteq X$ has nonempty interior.

- Besicovitch notes in his article that a "twin" of this problem is as follows:

> "In the class of figures containing a segment of length 1 in every direction, which one has the smallest area?"

Such subsets of Euclidean space are called Besicovitch sets. The Perron tree construction indeed proves the existence of Besicovitch sets of arbitrarily small positive area (i.e. Lebesgue measure). Before knowing about the Kakeya problem, Besicovitch proved the existence of Besicovitch sets of zero Lebesgue measure! This denotes a fundamental difference between Besicovitch sets and Kakeya sets.

Since Kakeya problem could not possibly generalize to higher-dimensional Euclidean spaces, Besicovitch suggested the following conjecture that is essentially the same as Kakeya problem:

> "Must a Besicovitch set in $\mathbb{R}^n$ necessarily be n-dimensional in the sense of Hausdorff dimension?"

This is known as Kakeya conjecture. Kakeya conjecture is clear in $\mathbb{R}^1, \mathbb{R}^2$. In 1995, Wolff showed that the dimension of a Besicovitch set in $\mathbb{R}^n$ is at least $\frac{n}{2} + 1$. In 2002, Katz, Tao improved Wolff's previous bound to $(2 - \sqrt{2})(n - 4) + 3$. Finally in late February 2025, Wang, Zahl published a proof for 3D Kakeya conjecture which settled one of the most important problems in geometric measure theory.