Moduli Spaces of Bundles and Stability Analysis

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Abstract

In a given category, certain group objects can act (via morphisms of the category) on abstract objects. An important moduli problem is then to create a good moduli space for this group action. In the topological category and the smooth category, one can see that the naive quotients lose important data and are far from being faithful. In the category **Var** of algebraic varieties, there are notions of the Kapranov's "Chow quotient" and Mumford's "GIT quotient". In general, the Chow quotient dominates the GIT quotient. However, in the symplectic category, there is the notion of "symplectic reduction" of Marsden-Weinstein and Meyer. Kempf-Ness theorem then states that the GIT quotient is isomorphic to the symplectic reduction. We begin by an analysis of the G-orbits and quotients in each of these categories and proceed by using them to create moduli spaces of bundles and the stability criterion of Hilbert-Mumford. Afterwards, we construct the moduli space of coherent sheaves over a projective variety as a GIT quotient of Grothendieck's Quot scheme and characterize the stable sheaves. Finally, we prove the Kempf-Ness theorem and proceed to explain an infinite-dimensional analogue of the arguments studying the gauge-theoretic moduli spaces. These gauge-theoretic constructions often rise as the infinite dimensional symplectic reductions. Such moduli spaces often admit an algebraic description that appears as GIT quotients. In the end, we state and provide a proof of the Narasimhan-Seshadri theorem, which relates the two notions and can be realized as an infinite-dimensional analogue to the Kempf-Ness theorem.

1 Topological Quotients

Let X be a topological space, and G be a group acting on X. Then we can define the orbits of this action as $X/G := \{G.x \mid x \in X\}$. In other words, defining an equivalence relation $\forall x_1, x_2 \in X : x_1 \overset{G}{\sim} x_2 \iff \exists g \in G : g.x_1 = x_2$, the G-orbits are then identified as $X/\overset{G}{\sim}$. Thus, there is a natural quotient topology of X on X/G. Therefore, X/G is realized as a topological space and is termed the topological quotient. This quotient is generally not well-behaved. One can see for example that if X is compact (resp. connected), then the quotient is also compact (resp. connected). However, the topological quotient does not respect the separation of space. As a concrete example, if $\mathbb{R}$ is the real line with its standard topology (which is Hausdorff), then $\mathbb{Q}$ acts on $\mathbb{R}$ (denoted as $\mathbb{Q} \circlearrowright \mathbb{R}$) by $\forall q \in \mathbb{Q},\, x \in \mathbb{R} : q.x := x + q$. Then the orbit space $\mathbb{R}/\mathbb{Q}$ has trivial topology (thus not Hausdorff).

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Theorem 1. Let G be a compact topological group and X be a Hausdorff space such that $G \circlearrowright X$ and the graph of the action (the map $\Gamma : G \times X \to X \times X$, $(g,\, x) \mapsto (x,\, g.x)$) is a continuous map of topological spaces. Then X/G is a compact Hausdorff topological space.

Proof. First we begin by proving an observation and some well-known propositions of general topology:

Lemma 1. $\mathrm{Im}(\Gamma) \subseteq X \times X$ is a closed subset.

Proof. Let $(x_i,\, y_i) \to (x,\, y)$ be a convergent net in $\mathrm{Im}(\Gamma)$. Therefore, $\forall i \in I : \exists g_i \in G : g_i.x_i = y_i$. Since G is compact, the infinite sequence $\{g_i\}_{i\in I}$ has a convergent subsequence $\{g_{i_j}\}_{j\in J} \to g$. Note also that $\{x_{i_j}\}_{j\in J} \to x$ and $\{y_{i_j}\}_{j\in J} \to y$. Therefore, $y = \lim\{y_{i_j}\} = \lim\{g_{i_j}.x_{i_j}\}$. Since the graph of action is continuous, this is also equal to $(\lim\{g_{i_j}\}).(\lim\{x_{i_j}\}) = g.x$, which implies that $y = g.x \implies \Gamma(g,\, x) = (x,\, g.x) = (x,\, y)$. Therefore, $(x,\, y) \in \mathrm{Im}(\Gamma)$ and $\mathrm{Im}(\Gamma)$ contains its limit points. $\square$

It is clear that the topological quotient of a compact space is compact. Therefore, we only prove the Hausdorffness of X/G. Note that since the graph of action is continuous, the map $x \mapsto g.x$ is a homeomorphism $X \to X$ for all $g \in G$. Therefore, if $U \subseteq X$ is an open subset, $g.U$ is an open set for all $g \in G$, which implies that $G.U$ is also an open set since it is a union of (possibly infinite) open sets. Note that $G.(G.U) = G.U$; that is, $G.U$ is G-invariant. It is well known from general topology that this implies that the quotient map $p : X \to X/G$ is an open map. Take $G.x_1$, $G.x_2$ to be two distinct G-orbits. This means that $G.x_1 \neq G.x_2 \iff (x_1,\, x_2) \notin \mathrm{Im}(\Gamma)$. Note that by lemma 1, $\mathrm{Im}(\Gamma) \subseteq X \times X$ is closed and there exists a neighborhood $(x_1,\, x_2) \in U_1 \times U_2 \subseteq (\mathrm{Im}(\Gamma))^{\complement}$ where $U_1,\, U_2 \subseteq X$ are open sets. Finally, one can see that $p(G.U_1)$, $p(G.U_2)$ are open neighborhoods (in X/G) of $G.x_1$, $G.x_2$ respectively and that $p(G.U_1) \cap (G.U_2) = \varnothing$. This proves that any two G-orbits are separated by disjoint open subsets of X/G which proves that this space is Hausdorff. $\square$

In fact, one can also prove that the topological quotient of a topological manifold of dimension over a compact topological group is again a topological manifold. The smooth analogue of this claim is the quotient manifold theorem:

Proposition 1. Let G be a Lie group acting smoothly, freely, properly on a smooth manifold M. Then the orbit space M/G is a topological manifold of dimension $\dim M - \dim G$ and can be equipped with a unique smooth structure such that the quotient map $p : M \to M/G$ is a smooth submersion. Furthermore, M can be realized as a smooth fibre bundle with base M/G and model fibre G and fibre projection map $p : M \to M/G$.

Definition 1. Let $\mathfrak{X}$ be a groupoid. Then the isotropy group of $\mathfrak{X}$ at $x \in \mathrm{Obj}(\mathfrak{X})$ is the set of isomorphisms of x to itself in $\mathfrak{X}$. Particularly let X be a set and G be a semigroup such that $G \circlearrowright X$. Then this action gives rise to a category structure: Let $\mathfrak{X}$ be the category with $\mathrm{Obj}(\mathfrak{X}) = X$ and $x \to g.x$, $\forall x \in \mathrm{Obj}(\mathfrak{X})$, $g \in G$. Moreover, if G is in fact a group, all morphisms in this category would be isomorphisms. Therefore this category would be a groupoid and therefore the isotropy groups would be isomorphic to $G_x := \{g \in G \mid g.x = x\}$.

Remark 1. Note that an action is then free if and only if all of its isotropy groups are trivial. The definition also extends to arbitrary subset $K \subseteq X$ as $G_K := \{g \in G \mid (g.K) \cap K \neq \varnothing\}$.

Since properness is usually a difficult criteria, the following criteria is often used instead:

Theorem 2. Let G be a Lie group acting continuously on a manifold M. Then the following are equivalent:

1. The action of G on M is proper.
2. For sequences $\{x_i\}$, $\{g_i\}$ in M, G such that $\{x_i\}$, $\{g_i.x_i\}$ converge, $\{g_i\}$ contains a convergent subsequence.
3. For every compact subset $K \subseteq M$, the isotropy group G_K is compact.

Proof. We will attempt to prove (1) ⇒ (2) ⇒ (3) ⇒ (1):

(1) ⇒ (2). Let $\Gamma : G \times X \to X \times X$ be proper. Let the sequences in (2) converge as $\{x_i\} \to p$, $\{g_i.x_i\} \to q$. Note that any manifold is locally compact (since Euclidean spaces are locally compact and manifolds are locally homeomorphic to Euclidean spaces) and therefore there exist compact neighborhoods $p \in U \subseteq M$, $q \in V \subseteq M$. This implies that the points $(x_i,\, g_i.x_i) \in \mathrm{Im}(\Gamma)$ all lie in the compact set $U \times V \subseteq X \times X$ for sufficiently large i. Since Γ is proper, this implies that $\Gamma^{-1}(U \times V)$ is a compact set containing all $(g_i,\, x_i)$ for sufficiently large i. Therefore, there exists a convergent subsequence $(g_{i_j},\, x_{i_j}) \to (g,\, x)$ by compactness of $\Gamma^{-1}(U \times V)$, which proves (2).

(2) ⇒ (3). Let $K \subseteq M$ be a compact subset. Suppose $\{g_i\}_{i \in I}$ is a sequence of points in G_K. By definition of the isotropy group, for any $i \in I$ there exists $x_i \in (g_i.K) \cap K \iff x_i \in K,\, {g_i}^{-1}.x_i \in K$. Therefore by compactness of K, one can see that there exists a subsequence $\{x_{i_j}\}_{j \in J}$ such that $\{x_{i_j}\}$, $\{{g_{i_j}}^{-1}.x_{i_j}\}$ both converge in K. Therefore, by (2), there exists a convergent subsequence $\{g_{i_l}\}_{l \in L}$ such that $\{{g_{i_l}}^{-1}\}$ converges. This implies that $\{g_{i_l}\}$ also converges. This proves (3).

(3) ⇒ (1). Let $L \subseteq M \times M$ be a compact subset. For properness of Γ, one should prove that $\Gamma^{-1}(L) \subseteq G \times M$ is also compact. Let $K = \pi_1(L) \cup \pi_2(L) \subseteq M$. Then one could see that

$$\Gamma^{-1}(L) \subseteq \Gamma^{-1}(K \times K) = \{(g,\, x) \in G \times M \mid x \in K,\, g.x \in K\} \subseteq G_K \times K$$

Note that $L \subseteq M \times M$ is a compact subset of a Hausdorff space and is therefore closed. By continuity of Γ, one can deduce that $\Gamma^{-1}(L)$ is also closed. Moreover, by the last argument, $\Gamma^{-1}(L)$ is a closed subset of the compact set $G_K \times K$ (note that K is compact by compactness of L and that (3) implies that G_K is also compact) and is hence compact and proves (1).

□

But what exactly is the obstruction for Hausdorffness of the topological quotient? A fairly straightforward topological obstruction is given by the topology of the orbits:

Theorem 3. Given a Hausdorff topological space X and a topological group G acting on X, the topological quotient X/G is non-Hausdorff if an orbit is contained in the closure of another orbit.

Proof. First note that open sets of X/G (with the quotient topology) are exactly the images of saturated open sets of X. Now let an orbit $G.x$ be contained in the topological closure of another orbit $G.y$, and that (for the sake of contradiction) $G.x$, $G.y$ can be separated in X/G by disjoint open neighborhoods. This is equivalent to x, y being separated in X by disjoint saturated open neighborhoods. With the equivalence relation at hand, being saturated means being G-invariant. Note that any G-invariant neighborhood of x (resp. y) has to contain all of $G.x$ (resp. $G.y$). Since $G.x \subseteq \overline{G.y}$, any neighborhood of $G.x$ necessarily intersects $G.y$ (and all of its saturated open neighborhoods) and therefore cannot be disjoint. □

As an example, one could visit the previous example of the action $\mathbb{Q} \circlearrowright \mathbb{R}$ where the topological closure of any orbit is the whole real line. To get a more ”geometric” point of view on this obstruction, let us switch argument to the C^∞ category, let a Lie group G act smoothly on a smooth manifold M. Then the orbits inherit a geometric structure.

Definition 2. A smooth manifold equipped with a transitive smooth action of a Lie group G is called a homogeneous G-space or simply a homogeneous space (an action is transitive if it has a single orbit.)

One can then prove that a (set-theoretic) quotient of a Lie group over its closed subgroup can be realized as a homogeneous G-space. This serves as a Lie group analogue of proposition 1.

Theorem 4. (Homogeneous Space Construction Theorem) Let G be a Lie group and $H \leq G$ be a closed subgroup. Then the set-theoretic quotient G/H of left cosets is a topological manifold of dimension $\dim G - \dim H$ and has a unique smooth structure such that $p : G \to G/H$ is a smooth submersion. Furthermore, G acts smoothly, transitively on G/H by left multiplication and turns it into homogeneous G-space.

Proof. Let H act smoothly and freely and properly on G by right multiplication. Then the orbit space of this action is the same as the left coset space G/H. By theorem 2, one can also show that this action is proper:

Lemma 2. The action of H on G is proper.

Proof. Suppose that $\{g_i\}$, $\{h_i\}$ are sequences in G, H respectively, such that $\{g_i\}$, $\{g_i.h_i\}$ converge. Since $h_i = {g_i}^{-1}.(g_i.h_i)$, by continuity of the action, $\{h_i\}$ is also convergent in G. Since $H \leq G$ is a closed subgroup, this limit is moreover contained in H and theorem 2 proves the claim. $\square$

Therefore by proposition 1, G/H is a topological manifold of dimension $\dim M - \dim G$ and admits a unique smooth structure such that the quotient map $p : G \to G/H$ is a smooth submersion. Since product of smooth submersions is a smooth submersion, $\mathrm{Id}_G \times p : G \times G \to G \times G/H$ is a smooth submersion. By the diagram:

$$\begin{tikzcd} G \times G \arrow[r, "\text{mult}"] \arrow[d, "\mathrm{Id}_G \times p"'] & G \arrow[d, "p"] \\ G \times G/H \arrow[r, "(g_1, g_2H) \mapsto (g_1 g_2)H"'] & G/H \end{tikzcd}$$

Since $p \circ \text{mult} : G \times G \to G/H$ is constant on fibres of $\mathrm{Id}_G \times p$ and $\mathrm{Id}_G \times p$ is a surjective smooth submersion of smooth manifolds, it is well-known that one can pass this map down to prove that the action of left multiplication of G on G/H is the unique smooth map that makes the diagram commute. Therefore this action is smooth and well-defined. Finally transitivity is the clear since $\forall g_1H, g_2H \in G/H$, $(g_2 {g_1}^{-1}).(g_1H) = g_2H$, $g_2 {g_1}^{-1} \in G$.[15] $\square$

Before going through the geometric structure of fibres, it is beneeficial to recall the equivariant rank theorem:

Proposition 2. (Equivariant rank theorem): Let M, N be given smooth manifolds with a smooth map $f : M \to N$ and G be a Lie group acting smoothly on M, N such that the action of G on M is transitive. If f is equivariant w.r.t G-actions on M, N, then f is of constant rank. Therefore, if f is injective/surjective/bijective, it would be a smooth immersion/submersion/diffeomorphism.[15]

Theorem 5. Let a Lie group G act smoothly on a smooth manifold M. Then the following hold:[15]

1. $\forall p \in M$, the isotropy group $G_p := \{g \in G \mid g.p = p\} \leq G$ is a closed subgroup.
2. The quotient space G/G_p is a manifold of dimension $\dim G - \dim G_p$ and admits a unique smooth structure making p a smooth submersion. Furthermore, G acts smoothly, transitively on G/G_p by left multiplication.
3. The map $f_p : G \to M$ defined by $\forall g \in G \mapsto g.p \in M$ descends smoothly to a smooth map $F_p : G/G_p \to M$ and is a bijection $G/G_p \longleftrightarrow G.p$. Furthermore, F_p is equivariant w.r.t left actions of G on G/G_p and M.
4. The orbit $G.p$ is a smoothly immersed submanifold of M.

Proof. Noting the assumptions of the theorem, one can go through the prove as follows:

1. Let $\{g_i\}_{i\in I} \to g$ be an arbitrary convergent sequence in G_p. Therefore, by definition, $g_i.p = p$, $\forall i \in I$. Since the action is smooth (and hence continuous) $p = \lim g_i.p = (\lim g_i).p = g.p$ which implies that $g \in G_p$. Therefore G_p contains its limit points and is therefore closed.
2. Immediate corollary of theorem 4 since G_p is a closed subgroup by (1).
3. By the following diagram, note that G, G_p are both smooth manifolds and the quotient map $p : G \to G/G_p$ is a surjective smooth submersion. Furthermore, $f_p : G \to M$ is a smooth map that is constant on fibres of p. Therefore, one can pass f_p smoothly to a unique smooth map F_p that makes the diagram commute.

$$\begin{tikzcd} G \arrow[d, "p"'] \arrow[dr, "f_p"] & \\ G/G_p \arrow[r, dotted, "F_p"'] & M \end{tikzcd}$$

   Bijection of $G/G_p \longleftrightarrow G.p$ under F_p is clear from the commutativity of diagram. For the equivariancy, one has to prove that $F_p(g_1.g_2G_p) = g_1.F_p(g_2G_p)$, $\forall g_1, g_2 \in G$. Note that by commutativity of the diagram, $F_p(g_2G_p) = (F_p \circ p)(g_2) = f_p(g_2) = g_2.p \implies g_1.F_p(g_2G_p) = g_1.(g_2.p) = (g_1g_2).p$. On the other hand, $F_p(g_1.g_2G_p) = F_p((g_1g_2).G_p) = (F_p \circ p)(g_1g_2) = f_p(g_1g_2) = (g_1g_2).p$ that proves equivariancy of F_p.
4. Due to proposition 2 and (3) and since F_p is injective, it is a smooth immersion and its image $G.p$ is a smoothly immersed submanifold. This provides a well-defined notion of the dimension of an orbit. $\square$

Remark 2. Note that theorem 3 implies that one obstruction for Hausdorffness of the topological quotient is that some orbit $G.x$ of the action of G is contained entirely in the closure of another orbit $G.y$. Since orbits of an action are disjoint, this implies that $G.x \subseteq \overline{G.y} \setminus G.y \subseteq \partial(G.y)$. Under suitable conditions over the orbit $G.y$, one can deduce that $\dim G.x \leq \dim \partial(G.y) < \dim G.y$. That is, these orbits are of different dimensions.

Note that under the conditions of theorem 5, $\dim G.p = \dim G/G_p = \dim G - \dim G_p$. This implies that to make all of the orbits of same dimension, $\dim G_p$ should be independent of choice of $p \in M$. This is especially the case when G_p is a discrete subgroup of G for any $p \in M$. In this case, the action is said to be locally free.

Let G be a Lie group acting smoothly, freely, properly on a smooth manifold M. Then due to proposition 1, M is a smooth fibre bundle over M/G with model fibre G. Therefore the local trivialization condition implies that for any $x \in M/G$, there exists an open set $x \in U \subseteq M/G$ such that there exists a homeomorphism $\varphi : p^{-1}(U) \to U \times G$ and

$$\begin{tikzcd}
p^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "p"'] & U \times G \arrow[dl, "\pi_1"] \\
U &
\end{tikzcd}$$

is commutative. Thus the topological space $p^{-1}(x) \subseteq p^{-1}(U)$ is homeomorphic to G for all $x \in U$ (and all $x \in M$). Therefore, unless there are some really specific properties on the group action, the topological quotient would not preserve even the basic topological/geometric properties. Furthermore, theorem 3 suggests that one should "remove" some of the orbits from the manifold to keep it Hausdorff; doing which would further ruin the geometric structure.

2 Through An Algebraic Geometry Analogue

Reminder. Given an algebraically closed field k, the affine n-space over k (denoted by $\mathbb{A}^n$ or $\mathbb{A}^n_k$) is the set of all n-tuples $\mathbb{A}^n = \{(a_1, \cdots, a_n) \mid a_i \in k, \forall i\}$. Given finitely many polynomials $\{f_1, \cdots, f_m\} \subseteq k[X_1, \cdots, X_n]$, one can then define $V(f_1, \cdots, f_m) := \{(a_1, \cdots, a_n) \in \mathbb{A}^n \mid f_i(a_1, \cdots, a_n) = 0, \forall i\}$. It is then immediate that $V(f_1, \cdots, f_m) = V(\langle f_1, \cdots, f_m\rangle)$ where $\langle f_1, \cdots, f_m\rangle \lhd k[X_1, \cdots, X_n]$ is the ideal generated by $f_1, \cdots, f_m$. Furthermore, since $k[X_1, \cdots, X_n]$ is Noetherian due to Hilbert basis theorem, all of its ideals are finitely generated and therefore: $\forall \mathfrak{a} \lhd k[X_1, \cdots, X_n] : \exists f_1, \cdots, f_m \in k[X_1, \cdots, X_n], \mathfrak{a} = \langle f_1, \cdots, f_m\rangle$ and one can then define $V(\mathfrak{a}) = V(f_1, \cdots, f_m)$ which defines V as a map from ideals of $k[X_1, \cdots, X_n]$ to subsets of $\mathbb{A}^n$.
Conversely, one can define for each $X \subseteq \mathbb{A}^n$ the following: $I(X) := \{f \in k[X_1, \cdots, X_n] \mid f(x) = 0, \forall x \in X\}$. It is immediate that $I(X) \lhd k[X_1, \cdots, X_n]$ for all $X \subseteq \mathbb{A}^n$ and that it is a radical ideal. An algebraic set in $\mathbb{A}^n$ is a subset that can be "cut out" by a set of polynomial equations (i.e. subsets that are of form $V(\mathfrak{a})$ for some $\mathfrak{a} \lhd k[X_1, \cdots, X_n]$). The algebraic subsets of $\mathbb{A}^n$ then constitute the closed sets of a topology. This topology on $\mathbb{A}^n$ is then termed the Zariski topology. In other words, an algebraic set is basically a Zariski closed subset of $\mathbb{A}^n$. An algebraic set of $\mathbb{A}^n$ is said to be irreducible if it cannot be realized as union of two proper Zariski closed sets of $\mathbb{A}^n$. An affine variety over k is then defined as an irreducible algebraic subset of $\mathbb{A}^n$.
To any affine algebraic set X, one can then correspond an affine coordinate ring $k[X] := k[X_1, \cdots, X_n]/I(X)$, which is in fact a finitely generated, reduced k-algebra. Note that this definition of the coordinate ring depends on a choice of coordinates $X_1, \cdots, X_n$; which arises from the embedding of X in the affine space. Therefore, we have associated to each affine algebraic set a finitely generated, reduced k-algebra. Conversely, one can associate to any finitely generated, reduced k-algebra R an affine algebraic set $\mathrm{MaxSpec}(R)$ as follows: Since R is finitely generated over k, there exist elements $x_1, \cdots, x_s$ such that the evaluation map $\Phi_{x_1,\cdots,x_s} : k[X_1, \cdots, X_s] \to R$ is surjective. Note that by the first isomorphism theorem this implies that $R \cong k[X_1, \cdots, X_s]/\ker(\Phi_{x_1,\cdots,x_s})$. Furthermore, the kernel $\ker(\Phi_{x_1,\cdots,x_s}) \lhd k[X_1, \cdots, X_s]$ is finitely generated. Thus, there exist $f_1, \cdots, f_t$ in $k[X_1, \cdots, X_s]$ such that $\ker(\Phi_{x_1,\cdots,x_s}) = \langle f_1, \cdots, f_t\rangle$. Then $\mathrm{MaxSpec}(R) := V(\langle f_1, \cdots, f_t\rangle)$. Moreover, MaxSpec does not depend on the choice of generators of R. In fact, one could define $\mathrm{MaxSpec}(R)$ as the set of all maximal ideals of R. This is in fact a subspace of a bigger topological space $\mathrm{Spec}(R)$, defined as the set of all prime ideals, equipped with Zariski topology (where $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{p}\}, \forall \mathfrak{a} \lhd R$ are a basis of closed sets). The space $\mathrm{MaxSpec}(R)$ then inherits this topology as a subspace. In the case where R is finitely generated, the functor $\mathrm{Spec}(R) \mapsto \mathrm{MaxSpec}(R)$ is an equivalence of categories. In the case where k is algebraically closed, maximal ideals of $R \cong k[X_1, \cdots, X_s]/\langle f_1, \cdots, f_t\rangle$ are of the form $\langle X_1 - a_1, \cdots, X_n - a_n\rangle$ where $(a_1, \cdots, a_n) \in V(f_1, \cdots, f_t)$. This yields an isomorphism $\mathrm{MaxSpec}(R) \cong V(f_1, \cdots, f_t)$. An important consequence of this isomorphism is that $k[\mathrm{MaxSpec}(R)] \cong R$ when R is finitely generated and reduced.[14]

Reminder. Given affine varieties $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$, a regular map $\varphi : X \to Y$ is the restriction of a rational map $\varphi(x) = \left(\frac{P_1(x)}{Q_1(X)}, \cdots, \frac{P_m(x)}{Q_m(x)}\right)$, $P_1, Q_1, \cdots, P_m, Q_m \in k[X]$ and that $Q_1(x), \cdots, Q_m(x) \neq 0, \forall x \in X$. Given some open subset $U \subseteq X$, a function $f : U \to \mathbb{A}^1$ is said to be regular at $x_0 \in U$ if it is locally regular at x_0; i.e. there exist affine charts $x \in U \subseteq X$, $f(U) \subseteq V \subseteq \mathbb{A}^1$ such that $f\mid_U : U \to V$ is regular. f is said to be regular if it is regular everywhere. For all $U \subseteq X$, one can then define $\mathcal{O}_X(U)$ as the k-algebra of all regular functions on U. Note that $\mathcal{O}_X(-)$ is then a contravariant functor from the category of open subsets of X to the category of k-algebras. This functor is called the structure sheaf. Furthermore, the k-algebra of regular functions on X is isomorphic to $k[X]$ if X is an affine algebraic variety. This indicates that $k[X]$ is indeed independent of the embedding of X in the affine space, up to isomorphism; i.e. $X \cong Y \iff k[X] \cong k[Y]$.
Given algebraic varieties X, Y, the morphism $\varphi : X \to Y$ is said to be a morphism of varieties if for any open set $U \subseteq Y$ and regular function $f : U \to k$, the pullback $\varphi \circ f : \varphi^{-1}(U) \to k$ is also regular. Two morphisms of varieties are equal everywhere if they agree on a non-empty (Zariski) open set. Therefore, a morphism of varieties $\varphi : X \to Y$ gives rise to a k-algebra homomorphism $\varphi^\sharp : k[Y] \to k[X]$ where $f \mapsto \varphi \circ f$.

Remark 3. Given algebraic sets $V(f_1, \cdots, f_r) = X \subseteq \mathbb{A}^n$, $V(g_1, \cdots, g_s) = Y \subseteq \mathbb{A}^m$, one can consider the product $X \times Y \subseteq \mathbb{A}^{n+m}$. This product can then be realized as an algebraic set as it can be cut out by the polynomials $\{f_i \otimes \mathfrak{u}_m, \mathfrak{u}_n \otimes g_j \mid \forall i, j\}$, where $\mathfrak{u}_n \in k[X_1, \cdots, X_n]$, $\mathfrak{u}_n(x_1, \cdots, x_n) = 1_k$, $\forall x_1, \cdots, x_n \in k$ and

$$\otimes : k[X_1, \cdots, X_n] \times k[X_1, \cdots, X_m] \to k[X_1, \cdots, X_{n+m}]$$

$$(f \otimes g)(x_1, \cdots, x_{n+m}) = f(x_1, \cdots, x_n) \cdot g(x_{n+1}, \cdots, x_{n+m}) \quad , \forall f \in k[X_1, \cdots, X_n], g \in k[X_1, \cdots, X_m]$$

Note that the topology of this algebraic set does not necessarily agree with the product topology of $X \times Y$. If X, Y are also irreducible (hence algebraic varieties), $X \times Y$ is also irreducible (therefore an algebraic variety).

A particularly important family of varieties are separated varieties. Separatedness is an algebraic geometry analogue of topological Hausdorffness property. An algebraic set X is said to be separated if the image of the diagonal morphism Δ_X (which is an algebraic set), is a closed subset in the algebraic set $X \times X$. An abstract algebraic variety is then defined as a separated algebraic variety. From now on, we assume all algebraic varieties to be separated (hence abstract).

Definition 3. An algebraic group is an algebraic variety G with a compatible group structure. This means that the multiplication map $\mathrm{mult} : G \times G \to G$, the inversion map $\iota : G \to G$, and the unit map $\imath : \{e\} \to G$ are morphisms of varieties. A linear algebraic group is an algebraic variety $X \subseteq \mathrm{GL}_n(k)$ (that is; a Zariski closed subset of $\mathrm{GL}_n(k)$) that is also a subgroup of $\mathrm{GL}_n(k)$ under its natural group structure.

Remark 4. Associated to every variety over k is an coordinate ring $k[X] = k[X_1, \cdots, X_n]/I(X)$ which is isomorphic to the k-algebra of regular functions on X (denoted by $\mathcal{O}(X)$). In this sense, the multiplication and inverse maps mult, ι correspond to maps of co-multiplication $\mathrm{mult}^\sharp : k[G] \to k[G] \otimes_k k[G]$, co-inversion $\iota^\sharp : k[G] \to k[G]$ and co-unit $\varepsilon : k[G] \to k$. Therefore, we can realize algebraic groups as finitely generated k-algebras (as coordinate rings of some algebraic variety) equipped with operations of co-multiplication, co-inversion and co-unit. This structure is called a Hopf algebra and the category is thus equivalent to **AlgGrp**.

Note that a group is in fact a set Γ together with a multiplication map $\mathrm{mult} : \Gamma \times \Gamma \to \Gamma$, an inversion map $\iota : \Gamma \to \Gamma$, and unit map $\imath : \{e\} \to \Gamma$, making the following diagrams commute to satisfy the criteria of a group structure:[29]

$$
\begin{tikzcd}
\Gamma\times\Gamma\times\Gamma \arrow[r, "\mathrm{id}_\Gamma\times\mathrm{mult}"] \arrow[d, "\mathrm{mult}\times\mathrm{id}_\Gamma"'] & \Gamma\times\Gamma \arrow[d, "\mathrm{mult}"] \\
\Gamma\times\Gamma \arrow[r, "\mathrm{mult}"'] & \Gamma
\end{tikzcd}
,\quad
\begin{tikzcd}
\Gamma \arrow[r, "(\Delta_e, \mathrm{id}_\Gamma)"] \arrow[d, "(\mathrm{id}_\Gamma, \Delta_e)"'] \arrow[dr, "\mathrm{id}_\Gamma"] & \Gamma\times\Gamma \arrow[d, "\mathrm{mult}"] \\
\Gamma\times\Gamma \arrow[r, "\mathrm{mult}"'] & \Gamma
\end{tikzcd}
,\quad
\begin{tikzcd}
\Gamma \arrow[r, "(\iota, \mathrm{id}_\Gamma)"] \arrow[d, "(\mathrm{id}_\Gamma, \iota)"'] \arrow[dr, "\Delta_e"] & \Gamma\times\Gamma \arrow[d, "\mathrm{mult}"] \\
\Gamma\times\Gamma \arrow[r, "\mathrm{mult}"'] & \Gamma
\end{tikzcd}
$$

corresponding to associativity, existence of unit, and existence of inverse, respectively (where $\Delta_e : \Gamma \to \{e\}$ denotes the constant functor). Before going through the correspondence, the following lemmas are required:

Reminder. An ideal is radical $\iff$ the corresponding quotient ring is reduced (has no proper nilpotent ideal).

Lemma 3. An algebraic set X is an algebraic variety $\iff$ the coordinate ring $k[X]$ is an integral domain.

Proof. Since $k[X] \cong k[X_1, \cdots, X_n]/I(X)$, one can deduce that $k[X]$ is an integral domain if and only if $I(X) \lhd k[X_1, \cdots, X_n]$ is a prime ideal. Therefore, it suffices to show that X is irreducible if and only if $I(X) \lhd k[X_1, \cdots, X_n]$ is a prime ideal. For the sake of contradiction, suppose that X is an irreducible algebraic set but $\exists P, Q \in k[X_1, \cdots, X_n]$ such that $PQ \in I(X)$, $P \notin I(X)$, $Q \notin I(X)$. Also, since X is an algebraic set, let $X = V(f_1, \cdots, f_m)$. It is immediate that $X = V(f_1, \cdots, f_m, PQ) = V(f_1, \cdots, f_m, P) \cup V(f_1, \cdots, f_m, Q)$ which contradicts the irreducibility of X since $V(f_1, \cdots, f_m, P)$, $V(f_1, \cdots, f_m, Q)$ are both proper algebraic sets. Conversely, suppose that $I(X)$ is a prime ideal but $X = V(\mathfrak{a})$ can be realized as union of finitely many proper irreducible algebraic sets $X_i = V(\mathfrak{a}_i)$. By the former discussion, each $I(X_i) \lhd k[X_1, \cdots, X_n]$ is prime. Also, Hilbert Nullstellensatz implies that $I(X_i) = I(V(\mathfrak{a}_i)) = \sqrt{\mathfrak{a}_i}$. Since every prime ideal is radical, this is equal to $\mathfrak{a}_i$. Therefore, $I(X) = I\left(\bigcup X_i\right) = \bigcap I(X_i) = \bigcap \mathfrak{a}_i$. But then $\prod \mathfrak{a}_i \subseteq \bigcap \mathfrak{a}_i = I(X)$. Since $I(X)$ is a prime ideal, $\exists j : \mathfrak{a}_j \subseteq I(X)$. But since $I(X) = \bigcap \mathfrak{a}_i$, one can see that $I(X) \subseteq \mathfrak{a}_j$. Therefore, $I(X) = \mathfrak{a}_j$ which implies that $I(X)$ is irreducible and is thus a contradiction. $\square$

Lemma 4. Given algebraic varieties X, Y, one has $k[X \times Y] \cong k[X] \otimes_k k[Y]$.

Proof. It is well known that if $k[X]$, $k[Y]$ are integral domains and k is an algebraically closed field, then so is $k[X] \otimes_k k[Y]$.[2] Let $k[X] \cong k[X_1, \cdots, X_n]/\langle f_1, \cdots, f_r\rangle$, $k[Y] \cong k[X_1, \cdots, X_m]/\langle g_1, \cdots, g_s\rangle$. One can see $k[X \times Y] \cong k[X_1, \cdots, X_{n+m}]/\langle f_1 \otimes \mathfrak{u}_m, \cdots, f_r \otimes \mathfrak{u}_m, \mathfrak{u}_n \otimes g_1, \cdots, \mathfrak{u}_n \otimes g_s\rangle$ is a finitely generated domain. Define $\otimes : \Big((\langle f_1, \cdots, f_r\rangle + P), (\langle g_1, \cdots, g_s\rangle + Q) \Big) \mapsto (\langle f_1 \otimes \mathfrak{u}_m, \cdots, f_r \otimes \mathfrak{u}_m, \mathfrak{u}_n \otimes g_1, \cdots, \mathfrak{u}_n \otimes g_s\rangle + P \otimes Q)$ which is a canonical map. Then by the universal property of tensor product:

$$
\begin{tikzcd}
\Big(k[X_1, \cdots, X_n]/\langle f_1, \cdots, f_r\rangle\Big) \times \Big(k[X_1, \cdots, X_m]/\langle g_1, \cdots, g_s\rangle\Big) \arrow[r, "f"] \arrow[d, "\otimes"'] & L \\
k[X_1, \cdots, X_{n+m}]/\langle f_1 \otimes \mathfrak{u}_m, \cdots, f_r \otimes \mathfrak{u}_m, \mathfrak{u}_n \otimes g_1, \cdots, \mathfrak{u}_n \otimes g_s\rangle \arrow[ur, dotted] &
\end{tikzcd}
$$

Since any bilinear map f lifts uniquely, $k[X_1, \cdots, X_{n+m}]/\langle f_1 \otimes \mathfrak{u}_m, \cdots, f_r \otimes \mathfrak{u}_m, \mathfrak{u}_n \otimes g_1, \cdots, \mathfrak{u}_n \otimes g_s\rangle$ satisfies the universal property and is isomorphic to $(k[X_1, \cdots, X_n]/\langle f_1, \cdots, f_r\rangle) \otimes_k (k[X_1, \cdots, X_m]/\langle g_1, \cdots, g_s\rangle)$. $\square$

Due to lemma 3, the coordinate ring map is a contravariant functor from the category of algebraic varieties (namely **Var**) to the category of finitely generated, reduced k-algebras that are integral domains (namely **FinaAlgDom**). Similarly, MaxSpec is a contravariant functor from **FinAlgDom** to **Var**. A group functor is a functor from category of k-algebras to the category of groups. Let G be an algebraic variety and $k[G]$ its coordinate ring. Then one can define a set functor G associating to every k-algebra a set $G(R)$ consisting of solutions of polynomials $I(G)$ in R. Then every k-algebra homomorphism $k[G] \to R$ corresponds to a solution of $I(G)$ in R which is identified as an element of $G(R)$. Therefore, we get a natural correspondence between $G(R)$, $\mathrm{Hom}_{\text{k-alg}}(\mathrm{k[G], R})$. Suppose that G is equipped with a compatible group structure. Then $h_G(R) := \mathrm{Hom}_{\text{k-alg}}(k[G], R)$ has an intrinsic group structure and therefore defines a group functor $h_G : \mathbf{FinAlgDom} \to \mathbf{Grp}$ (namely the functor of points of X). Therefore, $X : \mathbf{FinAlgDom} \to \mathbf{Sets}$ is represented by $k[G]$. Note that the map $G \mapsto h_G$ is then a functor $\mathbf{AlgGrp} \to \mathbf{Fun}(\mathbf{FinAlgDom}, \mathbf{Grp})$. Therefore, any morphism of algebraic groups corresponds to a morphism between their functor of points. Furthermore, Yoneda lemma implies that this is in fact a natural correspondence. Now note that:[29]

Lemma 5. Given representable (set) functors F_1, F_2 represented by k-algebras R_1, R_2 respectively, natural transformations $F_1 \to F_2$ correspond to k-algebra homomorphisms $R_2 \to R_1$.

This implies that a map of algebraic varieties $G_1 \to G_2$ corresponds to a map of representable functors $h_{G_1} \to h_{G_2}$ (represented by $k[G_1]$, $k[G_2]$ respectively) which by means of lemma 5, corresponds to a homomorphism of k-algebras $k[G_2] \to k[G_1]$. Now note that as instance, the multiplication map is a map of algebraic varieties $G \times G \to G$. This should in turn correspond to a homomorphism of k-algebras $k[G] \to k[G] \times k[G]$ (co-multiplication) and in turn defines a multiplication on G. The argument is similar for other group operations.

Definition 4. An algebraic action of an algebraic group G on an algebraic variety X (also an algebraic G-action) is a map $\sigma : G \times X \to X$ that is also a morphism of varieties and makes the following diagrams commute:

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mathrm{id}_G \times \sigma"] \arrow[d, "\mathrm{mult} \times \mathrm{id}_X"'] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"'] & X \end{tikzcd} \quad , \quad \begin{tikzcd} \{e\} \times X \arrow[r, "\imath \times \mathrm{id}_X"] \arrow[dr, leftrightarrow, "\cong"'] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

Remark 5. An action $G \circlearrowright X$ corresponds to an action $G \circlearrowright \mathcal{O}(X)$ by $(g.f)(x) := f(g^{-1}.x)$ for all $x \in X$, $g \in G$ and $f \in \mathcal{O}(X)$. The action $G \circlearrowright X$ also gives rise to a coaction $\sigma^\sharp : k[X] \to k[G] \otimes k[X]$ that is a homomorphism of k-algebras since any $f \in \mathcal{O}(X)$ naturally yields a regular map $f(g.x) :\in \mathcal{O}(G \times X) \cong \mathcal{O}(G) \otimes \mathcal{O}(X) \cong k[G] \otimes k[X]$.

Definition 5. A G-variety is defined as an algebraic variety equipped with an algebraic G-action.

Then the algebraic geometry analogue of the quotient construction is to find a "nice" quotient of a G-variety over its algebraic G-action. An immediate simplification comes from the following classical result:[1]

Proposition 3. Every affine algebraic group over k is isomorphic to a linear algebraic group over k.

As long as the field k is algebraically closed, this lets us to use linear algebraic groups and affine algebraic groups interchangeably. Similar to the geometric case, we expect that this quotient is "nice" enough to lie in the category of algebraic varieties. A naive way of defining an algebraic quotient is simply considering the G-orbits of the variety:

Definition 6. Let G be an algebraic group and X be a G-variety equipped with an algebraic G-action σ. The orbit of $x \in X$ is then defined as the image of the morphism $\sigma(-, x)$ and is denoted by $G.x$.

But this quotient does not necessarily admit the structure of an abstract variety. As instance, the orbit space is often not separated. To find a "nice" quotient of X/G (where X is a G-variety and G is a linear algebraic group), one must first identify the orbits of this naive quotient. A morphism $f : X \to Y$ of G-varieties is said to be G-equivariant if $f(g.x) = g.f(x)$, $\forall x \in X$, $g \in G$. A G-equivariant morphism is then said to be G-invariant if the action $G \circlearrowright Y$ is trivial: $f(g.x) = f(x)$, $\forall x \in X$, $g \in G$. Any G-invariant morphism $f : X \to Y$ corresponds to a homomorphism of the coordinate rings $f^\sharp : k[Y] \to k[X]$ that is invariant under the corresponding action of G on $\mathcal{O}(X, Y)$ (morphisms of varieties between X, Y), given by $(g, f) \mapsto (x \mapsto f(g^{-1}.x))$. This means that $f^\sharp(g^{-1}.p) = g.f^\sharp(p) = f^\sharp(p)$, $\forall g \in G$, $p \in k[Y]$. In other words, $\forall p \in f^\sharp(k[Y])$ is invariant under automorphisms $q \mapsto g.q$ of $k[Y]$ for all $g \in G$.

Definition 7. Given a reduced, finitely generated k-algebra R and a group of automorphisms $F \leq \mathrm{Aut}(R)$, one can see that $R^F := \{r \in R \mid f(r) = r, \forall f \in F\}$ is a k-subalgebra of R and is called as the algebra of invariants.

Therefore, by the preceding discussion $f^\sharp(k[Y]) \subseteq k[X]^G$. In other words, $f^\sharp$ is G-invariant. Now note that any such G-invariant morphism $f^\sharp : k[Y] \to k[X]$ is a composition of a map $k[Y] \to k[X]^G$ and the inclusion $k[X]^G \hookrightarrow k[X]$. Therefore any G-invariant morphism $f : X \to Y$ factors through the map $X \to \mathrm{MaxSpec}(k[X]^G)$, provided that $k[X]^G$ is finitely generated. But any G-invariant morphism also factors through the projection map $X \to X/G$. Therefore, we expect that $k[X]^G$ is realized as the orbit space X/G. The first obstruction to this realization is that $k[X]^G$ should be finitely generated. This is widely known as Hilbert's 14th problem and is not true in general (c.f. [21]).

Definition 8. Given an algebraic group G, a linear representation $\rho : G \to \mathrm{GL}(V) \cong \mathrm{GL}_n(k)$ is said to be rational if it is a rational map of algebraic varieties. A linear algebraic group is said to be geometrically reductive if for any rational representation $\rho : G \to \mathrm{GL}(V)$ and any non-zero G-invariant vector $v \in V^G \setminus \{0\}$, there exists a non constant, homogeneous, G-invariant polynomial function $f \in k[V]^G \setminus k$ such that $f(v) \neq 0$.

An important algebraic group is the multiplicative group a $\mathbb{G}_m := \mathrm{Spec}(k[X, Y]/\{xy = 1\}) = \mathrm{Spec}(k[X, X^{-1}])$ where operations of the algebraic group are given by $\mathrm{mult}^\sharp(x) = x \otimes x$, $\iota^\sharp(x) = x^{-1}$. We then any (linear) algebraic group isomorphic to ${\mathbb{G}_m}^n \cong (k^*)^n$ as an algebraic k-torus. Every algebraic group G contains a unique maximal connected solvable normal subgroup, namely the radical of G. An algebraic group is then reductive if its radical is an algebraic k-torus. Finally, we have the celebrated theorem of Nagata (c.f. [5],[20]):

Theorem 6. Let G be a geometrically reductive group acting on a finitely generated k-algebra R. Then R^G is finitely generated. In particular, $k[X]^G$ is finitely generated for any affine variety X.

Remark 6. Since $k[X]$ is an integral domain (following from X being an affine variety respectively), any subring of $k[X]$ (including $k[X]^G$) is an integral domain. Therefore, $\mathrm{MaxSpec}(k[X]^G)$ is an affine variety.

By a theorem of Nagata, every geometrically reductive algebraic group is known to be reductive (c.f. [19]). In characteristic zero, the converse holds by a theorem of Weyl (c.f. [13]) and in positive characteristic, a theorem of Haboush[1]states that all reductive groups are also geometrically reductive (c.f. [10]). On the other hand, by a theorem of Popov (c.f. [22]), G is reductive iff R^G is finitely generated for all reduced, finitely generated k-algebras R equipped with a rational action of G. This shows that the condition of Nagata theorem is not too restrictive.

[1]widely referred to as Mumford conjecture

3 Properties of Algebraic Quotients

To get a notion of quotient with good functorial properties, one should require the quotient to have a universal property (i.e. any other quotient factors through it). This in turn leads to the following definition:

Definition 9. Given an algebraic group G and a G-variety X, a categorical quotient of X is a G-invariant morphism of varieties $\phi : X \to Y$ that satisfies the universal property: Given any G-invariant morphism of varieties $\psi : X \to Z$, there exists a unique morphism (lift) $\widetilde{\psi} : Y \to Z$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "\psi"'] & Y \arrow[d, dotted, "\widetilde{\psi}"] \\ & Z \end{tikzcd}$$

By the universal property, the categorical quotient is unique up to isomorphism and is denoted by $X /\!\!/ G$.

Note that this quotient is not necessarily an orbit space but this may be the case under some "nice" conditions. In particular, since a categorical quotient φ is continuous and constant of fibres of X/G, it is also constant on the closure of these fibres. This shows that for a categorical quotient to be an orbit space, the orbits of $X /\!\!/ G$ should be closed. Thus we define a closed algebraic G-action as an algebraic action of G on the G-variety X such that all of its orbits are closed. Note that similar to the topological case, given an algebraic G-action σ, one can define isotropy groups G_x as fibres of the morphism $\sigma(-, x) : G \to X$ over x. Remind that an algebraic set is connected if it cannot be written as a union of two proper disjoint (Zariski) closed subsets.

Lemma 6. Any algebraic set $X = V(\mathfrak{a})$ admits an irredundant decomposition as a union of finitely many irreducible algebraic sets $X = \bigcup X_i$ (irredundant meaning that $X_i \nsubseteq X_j$, $\forall i \neq j$). Furthermore, this decomposition is unique up to a permutation on the irreducible algebraic sets X_i.

Proof. Remind that $k[X_1, \cdots, X_n]$ is Noetherian as the polynomial ring of a Noetherian ring. Therefore, by primary decomposition theorem it is also a Lasker ring, meaning that any of its ideals can be written as an intersection of finitely many primary ideals. Let $\mathfrak{a} = \mathfrak{q}_0 \cap \cdots \cap \mathfrak{q}_r$ be the primary decomposition of $\mathfrak{a} \lhd k[X_1, \cdots, X_n]$, where each $\mathfrak{q}_i$ is a $\mathfrak{p}_i$-primary ideal. This implies that $X = V(\mathfrak{a}) = \bigcup V(\mathfrak{q}_i)$. Also note that $V(\mathfrak{q}_i) = V(\sqrt{\mathfrak{q}_i}) = V(\mathfrak{p}_i)$. Denote this algebraic set by X_i. To see that it is irreducible, note that since $\mathfrak{p}_i$ is a prime ideal (and therefore radical), by Hilbert Nullstellensatz $I(X_i) = I(V(\mathfrak{p}_i)) = \sqrt{\mathfrak{p}_i} = \mathfrak{p}_i$ is a prime ideal. This implies that $k[X_1, \cdots, X_n]/I(X_i)$ is a domain and proves the claim by lemma 3. One can then easily remove the redundant terms to get an irredundant decomposition of X.
To prove the uniqueness, suppose that $X_1 \cup \cdots \cup X_r = {X_1}' \cup \cdots \cup {X_s}'$ are two irredundant decompositions of X as union of finitely many irreducible algebraic sets. Then ${X_1}' \subseteq X_1 \cup \cdots \cup X_r$. Therefore, ${X_1}' = \bigcup_{i=1}^{r}({X_1}' \cap X_i)$. Note that since ${X_1}'$, X_i's are all Zariski closed (as algebraic sets), ${X_1}'$ is the union of finitely many Zariski closed subsets. By irreducibility of ${X_1}'$, this means that ${X_1}' = {X_1}' \cap X_i$ for some $1 \leq i \leq r$. This implies that ${X_1}' \subseteq X_i$. Similarly, $X_i \subseteq {X_j}'$ for some $1 \leq j \leq s$ and we conclude that ${X_1}' \subseteq X_i \subseteq {X_j}'$. By irredundancy of the decomposition, this implies that $j = 1 \implies X_i = {X_1}'$. WLOG let $i = 1$ which means $X_1 = {X_1}'$. Repeating this procedure we get (WLOG $r \leq s$): $X_i = {X_i}'$, $\forall 1 \leq i \leq r$. If furthermore $r < s$, one concludes that ${X_{r+1}}' \subseteq X_1 \cup \cdots \cup X_r = {X_1}' \cup \cdots \cup {X_r}'$. Similarly, ${X_{r+1}}' = \bigcup_{i=1}^{r}({X_i}' \cap {X_{r+1}}')$ and therefore ${X_{r+1}}' \subseteq {X_l}'$ for some $1 \leq l \leq r$ which contradicts the irredundancy of the decomposition and implies $r = s$. $\square$

Corollary 1. Under the conditions of lemma 6 one has:

1. The irreducible algebraic sets X_i are maximal and are therefore called irreducible components.
2. Any maximal irreducible algebraic subset of X coincides with one of the irreducible components.
3. Any irreducible algebraic subset of X lies in one of the irreducible components of X.

Theorem 7. Let G be an algebraic set with a compatible group structure, as in definition 3 (such a structure is called a pre-algebraic group). Then G is irreducible if and only if it is connected.

Proof. It's immediate (by definition) that any irreducible algebraic set is connected.
Now assume that G is connected. Let $G = G_0 \cup \cdots \cup G_r$ be the irredundant decomposition of G into irreducible components. Note that these irreducible components need not be pairwise disjoint. The irreducible component (maximal irreducible algebraic set) of G containing the identity element of G is (WLOG) denoted by G_0. Given an irreducible component G_i and an arbitrary $g_i \in G_i$, g_iG_0 is an irreducible algebraic subset of G (since the left multiplication morphism $g.- : G \to G$ is a homeomorphism in Zariski topology) and therefore by corollary 1 lies in a unique irreducible component of G, which must be G_i. If $g_iG_0 \subseteq G_i$ does not coincide with G_i, the irreducible algebraic subset ${g_i}^{-1}G_i$ would have G_0 as a proper subset, which contradicts the maximality of G_0. This implies that $g_0G_0 = G_0, \forall g_0 \in G_0$, which means that $G_0 \leq G$ is a subgroup. Furthermore, this proves that G_i's are precisely the left cosets of G_0 in G. Therefore, $\{G_0, \cdots, G_r\} = \{gG_0 \mid g \in G/G_0\}$. Therefore, G_i's are pairwise disjoint and unless $r = 0$, the sets G_0, $G_1 \cup \cdots \cup G_r$ constitute disjoint closed sets covering G, which then contradicts connectedness of G. Therefore $r = 0$, $G = G_0$ which proves the irreducibility of G. □

In fact, lemma 6 extends to the following more general settings:

Theorem 8. Given any Noetherian topological space X and a nonempty closed subset $Y \subseteq X$, there is a unique (up to permutation of components) irredundant decomposition of Y as union of finitely many irreducible closed subsets (called irreducible components). The uniqueness then implies that these components are exactly the maximal irreducible subsets of Y and that any irreducible element lies in one of the components.

Proof. Let $\mathcal{S}$ be the family of all nonempty closed subsets of X that are not representable as union of finitely many irreducible closed subsets. Assume, for the sake of contradiction, that $\mathcal{S} \neq \varnothing$. Since X is Noetherian, $\mathcal{S}$ contains a minimal element. Let $Z \in \mathcal{S}$ be a minimal element. Z is not irreducible, since otherwise it would admit a representation as union (consisting only of Z) of irreducible closed subsets. Therefore, Z can be written as union of two proper closed subsets $Z_1, Z_2 \subsetneq Z$. Since Z is minimal, Z_1, Z_2 both admit representations as union of finitely many irreducible closed subsets and therefore, so does Z. This is then a contradiction with $Z \in \mathcal{S}$ which implies that $\mathcal{S} = \varnothing$. Proof of uniqueness is similar to lemma 6. □

Therefore, a pre-algebraic group decomposes to the irreducible components. In this case, the connected components coincide with the irreducible components. Furthermore, one can see that since G_0 is the maximal connected/irreducible closed subgroup of G, it is in fact a characteristic subgroup. The following theorem is due to Cartier:

Proposition 4. Any pre-algebraic group G over an algebraically closed field k is smooth (c.f. [17]).

Note that if the pre-algebraic group G with irreducible/connected components $G_0, \cdots, G_r$ acts on an algebraic set X, then (since the morphism $\sigma(-, x) : G \to X$ is a continuous topological map) the subsets $G_0.x, \cdots, G_r.x \subseteq G.x$ are all maximal irreducible/connected subsets of $G.x$ and are termed as components of $G.x$.

Definition 10. A subset S of a topological space X is said to be locally closed if it is locally closed at every $s \in S$. This means that there exists a neighborhood $U \in \mathcal{U}_s$ of s in X such that $U \cap S \subseteq S$ is closed in the subspace topology. An irreducible locally closed subset of $\mathbb{P}^n$ is called a quasi-projective variety.

Proposition 5. The following are equivalent for a topological space X and a subset $S \subseteq X$ (c.f. [3]):

1. $S \subseteq X$ is a locally closed subset
2. S is the intersection of an open subset and a closed subset of X
3. S is open in the subspace topology of its closure $\overline{S}$ in X
4. The boundary $\partial S := \overline{S} \setminus S$ is a closed subset of $\overline{S}$ (and therefore a closed subset of X)

The projective space $\mathbb{P}^n = (\mathbb{A}_k^{n+1} \setminus \{0\})/\mathbb{G}_m$ can be covered by the finitely many hyperplanes $U_i := \{(a_0 : \cdots : a_{i-1} : 1 : a_{i+1} : \cdots : a_n) \in \mathbb{P}^n \mid a_0, \cdots, \widehat{a_i}, \cdots, a_n \in k\} = \mathbb{P}^n \setminus V(x_i)$, each of which being isomorphic to $\mathbb{A}^n$. Also note that each $U_i \subseteq \mathbb{P}^n$ is an open subset. Clearly, $X \cap U_i$ is open in the subspace topology of X. Let X be a quasi-projective variety. Then there exists a projective variety $X' \subseteq \mathbb{P}^n$ such that $X \subseteq X'$ is an open subvariety. Since $X' \cap U_i$ is affine, $X \cap U_i \subseteq X' \cap U_i$ is an open subvariety of an affine variety. Therefore, any quasi-projective variety X can be covered by finitely many quasi-affine varieties $X \cap U_i$, each open in the subspace topology of X, (Note that it would have been immediate that $X \cap U_i$ is a quasi-affine variety if $X \cap U_i$ was an open subset in the subspace topology of U_i but this is not always the case)

Definition 11. The family of constructible subsets is defined as the smallest family $\mathcal{F}$ of subsets of X that contains all open subsets and is closed with respect to finite intersections of elements and taking complements.

Proposition 6. The following holds for constructible sets of a topological space (c.f. [11] II Exercise 3.19):

1. A subset is constructible if and only if it is a finite disjoint union of locally closed subsets
2. Inverse image of a constructible subset under a continuous map is constructible
3. If $S \subseteq X$ is a constructible subset, then it contains a subset $S' \subseteq S$ that is relatively open and dense in $\overline{S}$
4. (Chevalley) A morphism of varieties takes constructible sets to constructible sets

Theorem 9. Given an algebraic G-action on an algebraic set X, the orbit $G.x$ is a locally closed subset of X.

Proof. The orbit $G.x$ is the image of the morphism $\sigma(-, x) : G \to X$. Chevalley theorem states that such a a morphism takes constructible subsets of G to constructible subsets of X. Therefore, its image $G.x$ is constructible. By proposition 6, there exists a subset $U \subseteq G.x$ that is dense and open in $\overline{G.x}$. Because G acts transitively on $G.x$ through this morphism, every point of $G.x$ is contained in a translate of U. This shows that $G.x$ is open in $\overline{G.x}$ as union of translates of U, meaning that $G.x$ is locally closed by proposition 5. $\square$

Since $G.x$ is a locally closed subset of X, it is the intersection $V \cap U$ of a closed set $V \subseteq X$ and an open set $U \subseteq X$. Since $U \subseteq X \subseteq \mathbb{A}^n$ is Noetherian (as a subspace of the affine space), the subset $G.x = V \cap U \subseteq U$ is closed in the subspace topology of U and therefore satisfies the criteria of the theorem and can be written as an irredundant union of closed irreducible subsets. If we denote by $\mathrm{cl}_U(S)$ the relative closure of $S \subseteq U$ in U, then the sets $\mathrm{cl}_U(G_i.x)$ are all irreducible closed subsets in the subspace topology of U. This implies that the union $G.x = \mathrm{cl}_U(G.x) = \bigcup \mathrm{cl}_U(G_i.x)$ consists of closed irreducible subsets of U. Note that this union is not necessarily irredundant. It is well-known that for any $S \subseteq U$, $\mathrm{cl}_U(S) = U \cap \overline{S}$ where $\overline{S}$ is the topological closure of S in X. Therefore, the union can be rewritten as $G.x = \bigcup U \cap \overline{G_i.x}$. Since $G_i.x = g_i.(G_0.x)$ and the morphism $\sigma(g_i, -) : X \to X$ is a homeomorphism, it is immediate that $\overline{G_i.x} = \overline{g_i.(G_0.x)} = g_i.\overline{G_0.x}$. Removing the redundant terms of this union, we are left with the union $G.x = \bigcup_{i\in I} g_i.\overline{G_0.x}$ where $I \subseteq G/G_0$. Since this is an irredundant union of closed irreducible subsets of U, these subsets are exactly the irreducible components of $G.x$. Note that a subset of U is irreducible in the original topology of X if and only if it is irreducible in the subspace topology of U. This implies that the subsets $\overline{G_i.x} = g_i.\overline{G_0.x}$ are also the maximal irreducible subsets of $G.x$ in X.

Reminder. Given a subset $X \subseteq \mathbb{A}^n_k$ where k is a commutative ring, one defines the Krull dimension of X as:

$$\dim(X) := \sup\{n \geq 0 \mid \exists Z_0 \subsetneq \cdots \subsetneq Z_n \subseteq X, \forall 0 \leq i \leq n : Z_i \subseteq X \text{ is an irreducible closed subset}\}$$

It is then immediate then that the $\dim(X) = \sup\{\dim(X_i) \mid \forall i : X_i \text{ is an irreducible component of } X\}$ and that one can let Z_n be a irreducible component of X. Note, on the other hand, that Z_0 can be a point of X. The latter is not true for general schemes because unless X has good properties (i.e., being quasi-compact), it is not necessary that they contain any closed points. Note that if $X = \mathrm{Spec}(A)$, then $\dim(X)$ equals the length of a maximal chain of prime ideals in A, which is the original definition of Krull dimension in commutative algebra. Clearly $\dim(X) \leq \dim(\mathbb{A}^n_k) = \dim(\mathrm{Spec}(\mathbb{A}^n_k)) = k[X_1, \cdots, X_n]$, $\forall X \subseteq \mathbb{A}^n_k$. It is a well-known theorem of commutative algebra that $\dim(k[X_1, \cdots, X_n]) = \dim(k) + n$. Therefore, given that k is of finite Krull dimension (as in the case where k is a field where the Krull dimension is zero), any subset of $\mathbb{A}^n_k$ would be of Krull dimension at most n. Given an irreducible closed subset $Y \subseteq X$, one also defines Krull codimension as:

$$\mathrm{codim}(Y, X) := \sup\{n \geq 0 \mid \exists Y = Z_0 \subsetneq \cdots \subsetneq Z_n \subseteq X, \forall 0 \leq i \leq n : Z_i \subseteq X \text{ is an irreducible closed subset}\}$$

As instance, note that if Y is an irreducible component of X then $\mathrm{codim}(Y, X) = 0$ even if X has irreducible components with bigger dimension than Y. Occasionally, $\mathrm{codim}(Y, X)$ is also denoted by $\mathrm{codim}(Y)$.

Definition 12. A morphism of varieties $\varphi : X \to Y$ is dominant if $\overline{\varphi(X)} = Y$, i.e. image of X is dense in Y.

In particular, given an G-variety X, the morphism $\sigma(-, x) : G \to G.x \subseteq X$ is dominant and its fibres are given by $(\sigma(-, x))^{-1}(y) = \{g \in G \mid g.x = y\} \neq \emptyset$. Given some $g_0 \in (\sigma(-, x))^{-1}$, one can see that $g_0 G_x = (\sigma(-, x))^{-1}$.

Proposition 7. Given a dominant morphism of (irreducible) varieties $\varphi : X \to Y$, any irreducible component of the fibre $\varphi^{-1}(x)$ has dimension at least $\dim(X) - \dim(Y)$. Moreover, there exists a nonempty open subset $U \subseteq Y$ such that $\dim(\varphi^{-1}(x)) = \dim(X) - \dim(Y)$, $\forall x \in U$ (c.f. [18] for further discussion).

Theorem 10. Given an algebraic G-action on an affine algebraic variety X:

1. The isotropy group G_x is a closed subvariety of G.
2. The orbit $G.x$ is a smooth subvariety. Any irreducible component of $G.x$ is of dimension $\dim(G)-\dim(G_x)$.
3. The boundary of $G.x$ (defined as $\partial(G.x) = \overline{G.x} - G.x$) is a union of orbits of strictly smaller dimension than $G.x$. Any orbit of minimal dimension is therefore closed and the closure $\overline{G.x}$ contains a closed orbit.

We proceed to prove the statements as follows:

Proof.

1. As G_x is the preimage of the closed subvariety $x \in X$ under the continuous map σ_x, it is a closed subvariety.
2. The image of the constructible set G under the morphism of varieties $\sigma(-, x) : G \to X$ is constructible in X by Chevalley theorem. Therefore, by proposition 6, it contains a subset U relatively open and dense in its closure $\overline{G.x}$. It is well-known that the smooth points in any variety are Zariski dense in that variety (c.f. [11]). Since the smooth points of the affine variety $\overline{G.x}$ are dense in $\overline{G.x}$, the open set $U \subseteq G.x$ contains a smooth point. Since the action of G on the orbit $G.x$ is transitive and each morphism $\sigma(g, -) : G.x \to G.x$ is a homeomorphism, all points of $G.x$ have to be smooth. The rest follows from proposition 7.
3. Remind that since $G.x$ is G-invariant, so is its closure $\overline{G.x}$. Therefore, $\partial(G.x)$ is G-invariant and is therefore a union of G-orbits. Since $G.x$ is locally closed, by proposition 5 its boundary $\partial(G.x)$ (if nonempty) is a proper closed subset of $\overline{G.x}$ and therefore can be written as a union of irreducible components.

Lemma 7. If $X_1, \cdots, X_n$ are the irreducible components of a Noetherian topological space X, the an open subset $U \subseteq X$ is dense in X if and only if $U \cap X_i \neq \varnothing$, $\forall 1 \leq i \leq n$.

Proof. If $U \subseteq X$ is dense in X, $\overline{U} = X$. Since any irreducible component X_i is closed in X, one can see that $\varnothing \neq X_i = X \cap X_i = \overline{U} \cap X_i = \overline{U \cap X_i}$. This implies that $U \cap X_i \neq \varnothing$. Conversely, if $U \cap X_i \neq \varnothing$, $\forall 1 \leq i \leq n$, then $\overline{U} = \overline{U \cap X} = \overline{U \cap (\bigcup X_i)} = \overline{\bigcup (U \cap X_i)} = \bigcup \overline{U \cap X_i}$. Note that $U \cap X_i$ is a nonempty open set in the subspace topology of X_i. It is a basic topological fact that a nonempty open subset of an irreducible set is dense. This implies in particular that $U \cap X_i$ is dense in X_i, meaning that $\overline{U \cap X_i} = X_i$. Therefore, $\overline{U} = \bigcup \overline{U \cap X_i} = \bigcup X_i = X$ so $U \subseteq X$ is dense. □

Since $G.x$ is clearly dense in $\overline{G.x}$, lemma 7 implies that $G.x$ intersects all irreducible components of $\overline{G.x}$. This means that all irreducible components of $\partial(G.x)$ are proper subsets of irreducible components of $\overline{G.x}$. Remind that Krull dimension of a set is the maximum of Krull dimensions of its irreducible components. Thus, since an irreducible component is irreducible and closed, $\dim(\partial(G.x)) < \dim(\overline{G.x})$. By the previous discussion, it's clear that $\dim(\overline{G.x}) = \dim(G.x)$ which proves the claim. The rest is then immediate. □

Therefore, to obtain desirable geometric properties of orbits (such as consistency of dimension and other properties analogue to the topological category), we restrict our study to a certain type of quotients with desirable properties.

4 Good Quotients and Affine Geometric Invariant Theory

Definition 13. Given an algebraic group G acting algebraically on a quasi-projective variety X, a good quotient is a surjective, affine (meaning that preimage of any affine open is affine), G-invariant morphism of quasi-projective varieties $\psi : X \to Y$ such that:

1. The canonical map $\mathcal{O}_Y(U) \to \psi_*(\mathcal{O}_X(U))$ is an isomorphism onto $\psi_*(\mathcal{O}_X(U))^G$ for any open set $U \subseteq Y$.
2. If $V_1, V_2 \subseteq X$ are closed, disjoint and G-invariant, then $\psi(V_1), \psi(V_2) \subseteq Y$ are also closed and disjoint.

We then say that $\psi : X \to Y$ is a good quotient for the action of G on X (or a good G-quotient in short).

As it turns out, good quotients contain useful geometric data about the orbits of the topological quotient. In particular, a good quotient is characterized by the universal property and is therefore proved to be a categorical quotient. We now proceed by proving several geometric properties of the good quotients.

Theorem 11. If $\psi : X \to Y$ is a good quotient for the action of G on X, then it is also a categorical quotient.

Proof. G-invariance is immediate by definition. To prove the universal property, let $\varphi : X \to Z$ be G-invariant:

Lemma 8. Given closed and G-invariant sets $V_1, \cdots, V_n \subseteq X$, one has $\bigcap \psi(V_i) = \psi(\bigcap V_i)$.

Proof. We prove the claim by induction:

Induction base: Note that $\psi(V_1 \cap V_2) \subseteq \psi(V_1) \cap \psi(V_2)$ is clear. To prove the converse, let $x \in \psi(V_1) \cap \psi(V_2)$ and assume for the sake of contradiction that $x \notin \psi(V_1 \cap V_2)$. Then one can also see that $\psi^{-1}(x) \cap V_1 \cap V_2 = (\psi^{-1}(x) \cap V_1) \cap V_2 = \emptyset$. Since $\psi^{-1}(x)$, V_1, V_2 are all closed and G-invariant, $\psi^{-1}(x) \cap V_1$, V_2 are closed and G-invariant. Therefore, by second property of definition 13, $\psi(\psi^{-1}(x) \cap V_1) \cap \psi(V_2) = \emptyset$ which is a contradiction since $x \in \psi(\psi^{-1}(x) \cap V_1) \cap \psi(V_2)$.

Inductive step: Suppose that the claim holds for $n = 2, \cdots, n_0 - 1$ and let $V_1, \cdots, V_{n_0}$ be closed and G-invariant. Then $V_1 \cap \cdots \cap V_{n_0-1}$, V_{n_0} are both closed and G-invariant and therefore by induction base, one deduces $\psi(V_1 \cap \cdots \cap V_{n_0}) = \psi(V_1 \cap \cdots \cap V_{n_0-1}) \cap \psi(V_{n_0}) = \psi(V_1) \cap \cdots \cap \psi(V_{n_0})$. □

1. One could take a finite open quasi-affine cover $\{V_i\}$ of the quasi-projective variety Z. The sets $X \setminus \varphi^{-1}(V_i)$ are closed and G-invariant. By the second part of definition 13, $\psi(X \setminus \varphi^{-1}(V_i)) \subseteq Y$ is closed and $W_i := Y \setminus \psi(X \setminus \varphi^{-1}(V_i)) \subseteq Y$ is open. Then we have a canonical inclusion $\psi^{-1}(W_i) \subseteq \varphi^{-1}(V_i)$. Since $\{V_i\}$ cover Z, $\bigcap(X \setminus \varphi^{-1}(V_i)) = X \setminus \bigcup \varphi^{-1}(V_i) = X \setminus \varphi^{-1}(\bigcup V_i) = X \setminus \varphi^{-1}(Z) = \emptyset$. Therefore, by the former lemma, $\bigcap \psi(X \setminus \varphi^{-1}(V_i)) = \psi\left(\bigcap(X \setminus \varphi^{-1}(V_i))\right) = \psi(\emptyset) = \emptyset$. Thus, $\{W_i\}$ are an open cover of Y.
2. Now, it suffices to construct the lift maps $W_i \to V_i$ so that they agree on the intersections. Then we can glue them to get a map $\widetilde{\psi} : Y \to Z$. Since each V_i is quasi-affine, it is open in some affine variety $\widetilde{V}_i$. Since $\widetilde{V}_i$ is affine, giving a morphism of varieties $W_i \to \widetilde{V}_i$ is equivalent to giving a k-algebra homomorphism $\mathcal{O}_Z(\widetilde{V}_i) \to \mathcal{O}_Y(W_i)$ (c.f. [11] I, Proposition 3.5). Since φ is G-invariant, the image of pullback homomorphism $\mathcal{O}_Z(\widetilde{V}_i) \to \mathcal{O}_X(\varphi^{-1}(\widetilde{V}_i))$ (corresponding to the map $\varphi^{-1}(\widetilde{V}_i) \xrightarrow{\varphi} V_i \hookrightarrow \widetilde{V}_i$) lies in $\mathcal{O}_X(\varphi^{-1}(\widetilde{V}_i))^G$. The canonical inclusion mapping $\psi^{-1}(W_i) \hookrightarrow \varphi^{-1}(V_i) \hookrightarrow \varphi^{-1}(\widetilde{V}_i)$ then corresponds to the homomorphism $\mathcal{O}_X(\varphi^{-1}(\widetilde{V}_i))^G \to \mathcal{O}_X(\psi^{-1}(W_i))^G$. Composing the two homomorphisms, one gets $\mathcal{O}_Z(\widetilde{V}_i) \to \mathcal{O}_X(\varphi^{-1}(\widetilde{V}_i))^G \to \mathcal{O}_X(\psi^{-1}(W_i))^G \cong \mathcal{O}_Y(W_i)$. The last isomorphism is due to definition 13. □

Lemma 9. Let G be a geometrically reductive group G acting algebraically on an affine variety X. Given closed, disjoint and G-invariant sets $V_1, V_2 \subseteq X$, there exists $f \in k[X]^G$ such that $f(V_1) = 0$, $f(V_2) = 1$.

Proof. Note $k[X] = I(\varnothing) = I(V_1 \cap V_2) = \sqrt{I(V_1) + I(V_2)} \implies 1 \in \sqrt{I(V_1) + I(V_2)} \implies 1 \in I(V_1) + I(V_2)$. Consequently, $\exists f_1, f_2 \in k[X] : 1 = f_1 + f_2$, $f_1 \in I(V_1)$, $f_2 \in I(V_2)$, which implies $f_1(V_1) = 0$, $f_1(V_2) = 1$. Since $\langle g.f_1 \mid g \in G \rangle$ is finite dimensional, one can choose a basis $\varphi_1, \cdots, \varphi_n$ and the canonical mapping $\varphi : X \to k^n$, $\varphi(x) = (\varphi_1(x), \cdots, \varphi_n(x))$. Then $\varphi_i(x) = \sum a_i(g).(g, f_1)(x) = \sum a_i(g) f_1(g^{-1}.x)$. Since V_1, V_2 are G-invariant, $\varphi(V_1) = 0$, $\varphi(V_2) = v \neq 0$. Also, $g.\varphi_i = \sum_j a_{ij}(g)\ \varphi_i$. The map $g \mapsto a_{ij}(g)$ then defines a representation $G \to GL_n(k)$. Since $\varphi(V_2) = v$ is a G-invariant point and G is geometrically reductive, there exists some $\psi \in k[X_1, \cdots, X_n]^G$ such that $\psi(v) \neq 0$, $\psi(0) = 0$. Finally, $f = \frac{\psi \circ \varphi}{\psi(v)}$ satisfies the criteria. □

Theorem 12. Let $\psi : X \to Y$ be a good quotient. Then one can prove that:

1. $\overline{G.x_1} \cap \overline{G.x_2} \neq \emptyset$ if and only if $\psi(x_1) = \psi(x_2)$
2. For all $y \in Y$, the fibre $\psi^{-1}(y)$ contains a unique closed orbit
3. If the action is closed (i.e. all orbits are closed), all fibres consist of a single orbit (i.e. geometric quotient).

Proof.

1. Given some $x_3 \in \overline{G.x_1} \cap \overline{G.x_2}$, one can see that $\exists \{g_i\}_{i=1}^{\infty}, \{g'_i\}_{i=1}^{\infty} \in G : \lim g_i.x_1 = x_3 = \lim g'_i.x_2$ which then implies $\psi(x_3) = \psi(\lim g_i.x_1) = \lim(\psi(g_i.x_1)) = \lim(\psi(x_1)) = \psi(x_1)$ and similarly $\psi(x_3) = \psi(x_2)$ which implies $\psi(x_1) = \psi(x_2)$. The converse is immediate by the second property of definition 13.
2. For any $x \in \psi^{-1}(y)$, since ψ is G-invariant, the fibre $\psi^{-1}(y)$ is a G-invariant set. Since singletons are closed in the Zariski topology, $\psi^{-1}(y)$ is a closed subset; which then implies that $\overline{G.x} \subseteq \psi^{-1}(y)$. Therefore, by the third statement of theorem 10, this fibre contains a closed orbit. If this fibre contains two closed orbits (distinct by default), this contradicts the first property and therefore the closed orbit is unique.
3. By the second property, no fibre can contain more than one of the closed orbits. □

Proposition 8. If $\psi : X \to Y$ is a good/geometric quotient, the restriction $\psi\mid_{\psi^{-1}(U)}: \psi^{-1}(U) \to U$ of ψ to an open subset $U \subseteq Y$ is also a good/geometric quotient. Conversely, if $\psi : X \to Y$ is a G-invariant morphism and $\{U_i\}_{i=1}^{n}$ is an open cover of Y such that the restrictions $\psi_{\psi^{-1}(U_i)} : \psi^{-1}(U_i) \to U_i$ are all good quotients, $\psi : X \to Y$ is also a good quotient.

Definition 14. Let G be a reductive group and X be an affine G-variety. Then by remark 6, $k[X]^G$ is finitely generated with no zero divisors. Therefore, one can define the affine GIT quotient $X /\!\!/ G := \mathrm{MaxSpec}(k[X]^G)$. The quotient morphism $X \to X /\!\!/ G$ is induced by inclusion of the coordinate rings $k[X /\!\!/ G] = k[X]^G \hookrightarrow k[X]$.

Lemma 10. Let G be a reductive group G acting algebraically on a finitely generated k-algebra R. If $I \lhd R$ is invariant under G and $f \in (R/I)^G$ then $f^t \in R^G/(I \cap R^G)$ for some positive integer t.

Proof. We can assume $f \neq 0$. There exists $r \in R$ such that $f - r \in I$ and the claim is to find some $r_0 \in R^G$ such that $r_0 - r^t \in I$. Since G acts on R algebraically, the subspace $M = \langle g.r \mid g \in G \rangle \subseteq R$ is finite dimensional. Let $N = M \cap I$. Since $r \neq 0$, it's immediate that $r \notin N$, $g.r - r \in N, \forall g \in G$. Therefore, the elements $g.r - r$ constitute a basis for N, which implies that $\dim M = \dim N + 1$ and that any element $s \in M$ has a unique decomposition as $s = cr + r'$, $c \in k$, $r' \in N$. One can then define the linear G-invariant map $(\iota \in M^*) : M \to k$, $(s = cr + r') \mapsto c$. Extend r to a basis for M such as $\{r_1 = r, r_2, \cdots, r_{\dim M}\}$. Note that $\langle r_2, \cdots, r_{\dim M}\rangle = N$. Identifying the dual space M^* with $k^{\dim M}$, one has $\iota \to (1, 0, \cdots, 0)$. Furthermore, $(1, 0, \cdots, 0)$ is invariant under the induced linear action of G on $k^{\dim M}$. Therefore, since G is geometrically reductive, there exists a non constant, homogeneous, G-invariant polynomial $F \in k[X_1, \cdots, X_{\dim M}]$ such that $F(1, 0, \cdots, 0) \neq 0$ which implies that coefficient of some ${X_1}^t$ is nonzero. We can suppose, without loss of generality that this coefficient is 1. Consider the homomorphism given by $k[X_1, \cdots, X_{\dim M}] \to R$, $X_i \mapsto r_i$. Note that this homomorphism commutes with the action of G and the image r_0 of F under this homomorphism belongs to R^G. Furthermore, $r_0 - r^t \in \langle r_2, \cdots, r_{\dim M}\rangle \lhd I$ which completes the proof. □

Lemma 11. Let G be a reductive group G acting algebraically on a finitely generated k-algebra R. Given $f_1, \cdots, f_s \in R^G$ and $f \in (\sum f_i.R) \cap R^G$, then $f^t \in \sum f_i.R^G$ for some positive integer t.

Proof. We prove the claim by induction on s:

Induction base: For $s = 1$, let $f \in f_1.R \cap R^G$. Therefore, $\exists f' \in R : f = f_1.f'$. Define the ideal $I = \{h \in R \mid f_1 h = 0\} \lhd R$. Since $f_1 \in R^G$, one deduces $f_1(g.f' - f') = 0, \forall g \in G$ and therefore, $f' \overset{I}{\equiv} g.f', \forall g \in G$ which implies that $f' \in (R/I)^G$. Applying lemma 10, we obtain some $f'' \in R^G$ and a positive integer $t \in \mathbb{N}$ such that $f'' - f'^t \in I \cap R^G$. Therefore, $f_1(f'' - f'^t) = 0$ and $f^t = {f_1}^t f'^t = {f_1}^t f'' \in f_1.R^G$

Inductive step: Let $\bar{R} = R/f_1.R$ and $\forall f \in R : \exists \bar{f} \in R : \bar{f} - f \in f_1.R$. Thus, given $f \in (\sum f_i.R) \cap R^G$, the inductive hypothesis implies that $\exists t \in \mathbb{N} : \bar{f}^t \in \sum_{i=2}^s \bar{f}_i \bar{R}^G$. By definition, there exists some $h_1 \in R$ such that $\bar{f} - f = f_1 h_1 \implies f^t = (\bar{f} - f_1 h_1)^t = \bar{f}^t + f_1.(-t h_1 \bar{f}^{t-1} + \cdots + (-1)^t {f_1}^{t-1} {h_1}^t) = \sum f_i.H_i$ where $H_1 := -t h_1 \bar{f}^{t-1} + \cdots + (-1)^t {f_1}^{t-1} {h_1}^t \in R$, $\overline{H_2}, \cdots, \overline{H_s} \in \bar{R}^G$. Now applying lemma 10 to the ideal $I = f_1.R$ and $H_s \in \bar{R}^G = (R/I)^G$, we get a positive integer u and some ${H_s}' \in R^G$ such that ${H_s}' \overset{I}{\equiv} {H_s}^u$. Therefore, $f^{tu} - {f_s}^u {H_s}^u = f^{tu} - {f_s}'({H_s}' + f_1 r_1)$, $r_1 \in R$ which then implies that $f^{tu} - {f_s}^u {H_s}' \in \sum_{i=1}^{s-1} f_i.R$. It is also clear that this is contained in R^G as well. So, by the inductive hypothesis, we get an integer $v \in \mathbb{N}$ such that $(f^{tu} - {f_s}^u {H_s}')^v \in \sum_{i=1}^{s-1} f_i.R^G$ which finally implies $f^{tuv} \in \sum_{i=1}^{s} f_i.R^G$. □

Theorem 13. Given a reductive group G and an affine G-variety X, the affine GIT quotient map $\psi : X \to X /\!\!/ G$ is a good quotient. Furthermore, $X /\!\!/ G$ is in fact an affine variety.

Proof. First note that $X /\!\!/ G$ is an affine variety by remark 6. We now prove that it is a good quotient:

1. Since the GIT morphism is induced by the inclusion $k[X]^G \hookrightarrow k[X]$, it is affine and G-invariant.
2. To prove that ψ is surjective, we have to prove $\forall y \in X /\!\!/ G : \exists x \in X : \psi(x) = y$. Let $\mathfrak{m}_y \lhd k[X /\!\!/ G] = k[X]^G$ be the maximal ideal corresponding to $y \in X /\!\!/ G$. Since $k[X]^G$ is finitely generated by theorem 6, one can assume that $\mathfrak{m}_y = \langle f_1, \cdots, f_s \rangle$. Then one can see that $\psi_*(\mathfrak{m}_y) = \sum f_i.k[X]$.

   Lemma 12. In the above setting, the following holds: $\sum f_i.k[X] \neq k[X]$

   Proof. Assuming the contrary, lemma 11 implies $\forall f \in R^G : \exists t \in \mathbb{N} : f^t \in \sum f_i.R^G = \mathfrak{m}_y \subsetneq R^G$ which leads to a contradiction by choosing $f \in R^G \setminus \mathfrak{m}_y$. □

   Since $\sum f_i.k[X] \neq k[X]$, it is contained in a maximal ideal $\mathfrak{m}_x \lhd k[X]$ corresponding to the closed point $x \in X$. Therefore, since $\psi_*(\mathfrak{m}_y) = \mathfrak{m}_x$, one can deduce that $\psi(x) = y$ which proves surjectivity.
3. The open sets $U_f = \{y \in X /\!\!/ G \mid f(y) \neq 0\}, \forall 0 \neq f \in k[X]^G$ constitute a basis for the Zariski topology on $X /\!\!/ G$. Therefore, it is necessary and sufficient to prove the first property of definition 13 for the basis elements. Note that $\mathcal{O}_{X/\!\!/G}(U_f) = (k[X]^G)_f$ which is in turn equal to $(k[X]_f)^G$ since localization commutes with taking G-invariant. This is then equal to $\mathcal{O}_X(X_f)^G$ and since $X_f = \psi^{-1}(U_f)$, this indicates that the inclusion homomorphism $\mathcal{O}_{X/\!\!/G}(U_f) \to \mathcal{O}_X(\psi^{-1}(U_f))$ is an isomorphism on its image, $\mathcal{O}_X(\psi^{-1}(U_f))$.
4. To prove second property of definition 13, it is necessary and sufficient to prove that given closed, disjoint and G-invariant sets $V_1, V_2 \subseteq X$, the sets $\overline{\psi(V_1)}, \overline{\psi(V_2)}$ are disjoint, which is clear by applying lemma 9. □

Since the action of G is not necessarily closed, this quotient may not be geometric in general. In the case that G is a finite group, each orbit is a finite union of points, which is closed in Zariski topology. Therefore, every good quotient over a finite group is automatically geometric. As we saw in theorem 12, some orbits might contain orbits of smaller dimension in their boundary provided that the action is not closed. Therefore, we essentially "remove" the obstruction we mentioned earlier and hope to have more stable orbits after removing all but the top dimensional closed orbits.

Definition 15. A point $x \in X$ is stable if $\dim G.x = \dim G$ (which is equivalent to $\dim G_x = 0$ due to the orbit-stabilizer theorem) and $G.x \subseteq X$ is closed. Set of stable points is denoted by X^s.

Theorem 14. Given a reductive group G acting on affine G-variety X and a good quotient $\psi : X \to Y$, subsets $Y^s := \psi(X^s) \subseteq Y$ and $X^s = \psi^{-1}(Y^s) \subseteq X$ are open. Furthermore, $\psi\,|_{X^s} : X^s \to Y^s$ is a geometric quotient.

Proof. We prove that X^s is open by taking a point $x_0 \in X^s$ and construct a neighborhood of x_0 inside X^s.

Lemma 13. Given an affine algebraic group X acting algebraically on a G-variety X, the dimension of the stabilizer subgroup is upper semi-continuous, i.e. $\forall n \in \mathbb{N} : \{x \in X \mid \dim G_x \geq n\} \subseteq X$ is closed.

Proof. First, consider the graph $\Gamma : G \times X \to X \times X$ of the action of G on X given by $(g,\, x) \mapsto (x,\, g.x)$ and the fibre product $X \times_{X \times X} (G \times X) = \{x,\, (g,\, x) \mid g.x = x\}$ where the map $\Delta : X \to X \times X$ is given by the diagonal morphism. The map $X \times_{X \times X} (G \times X) \to \mathbb{Z}^{\geq 0}$ sending $(x_0,\, (g_0,\, x_0)) \in X \times_{X \times X} (G \times X)$ to the dimension of the fibre $({\pi_1}^{-1} \circ \pi_1)(x_0,\, (g_0,\, x_0)) = {\pi_1}^{-1}(x_0) = \{(x_0,\, (g,\, x_0)) \mid g \in G_x\}$ is upper semi-continuous being the dimension valuation of a continuous map sending any point to the fibre containing it. Since $\{(x_0,\, (g,\, x_0)) \mid g \in G_x\}$ has essentially the same dimension as G_x, we conclude that $\dim G_x$ is upper semi-continuous. Equivalently, $\{x \in X \mid \dim G_x \geq n\} \subseteq X$ is a closed subset for all $n \in \mathbb{N}$. $\square$

1. Due to this lemma, the subset $X_+ = \{x \in X \mid \dim G_x > 0\} \subseteq X$ is closed. Note that X_+ is also G-invariant. By definition, the orbit $G.x_0$ is closed, disjoint from X_+ and obviously G-invariant. Thus, by lemma 9 one can find $P_{x_0} \in k[X]^G$ such that $f(X_+) = 0$, $f(G.x_0) = 1$. It is immediate that $x \in \operatorname{supp}(P_{x_0})$ and that $\operatorname{supp}(P_{x_0}) \subseteq X$ is open as the inverse image of $k \setminus \{0\}$ under the continuous map P_{x_0}. Finally, we now prove that points of $\operatorname{supp}(P_{x_0}) \subseteq X^s$ have closed orbits and thus lie in X^s: It is clear from the separation that all points of $\operatorname{supp}(P_{x_0})$ have zero-dimensional stabilizers and it is sufficient to prove that points of $\operatorname{supp}(P_{x_0})$ have closed orbits. Suppose that $x \in \operatorname{supp}(P_{x_0})$ has a non-closed orbit; i.e. $G.x \neq \overline{G.x}$. Let $x' \in \overline{G.x} \setminus G.x$. Since P_{x_0} is G-invariant, it is identically nonzero on $G.x'$ and therefore, also nonzero on its closure $\overline{G.x'}$. Thus, $\overline{G.x'} \subseteq \operatorname{supp}(P_{x_0})$ and therefore x' has a zero-dimensional stabilizer or equivalently, $\dim G.x' = \dim G$. But this is a contradiction since by theorem 10, $G.x'$ should be of strictly smaller dimension than $G.x_0$.
2. Given $x_1 \in \operatorname{supp}(P_{x_0})$, $x_2 \in X \setminus \operatorname{supp}(P_{x_0})$, one can see that $\psi(x_1) \neq \psi(x_2)$; since otherwise by the first property of theorem 12, $\overline{G.x_1} \cap \overline{G.x_2} \neq \emptyset$ which is impossible because P_{x_0} is zero on $\overline{G.x_2}$ but identically nonzero on $\overline{G.x_1}$. Therefore, $\operatorname{supp}(P_{x_0}) = \psi^{-1}(\psi(\operatorname{supp}(P_{x_0})))$ which implies $X^s = \psi^{-1}(\psi(X^s)) = \psi^{-1}(Y^s)$ by taking union. Moreover, since $X \setminus \operatorname{supp}(P_{x_0})$ is closed and G-invariant, $\psi(\operatorname{supp}(P_{x_0})) \cap \psi(X \setminus \operatorname{supp}(P_{x_0})) = \emptyset$ and ψ is a good quotient, $\psi(X \setminus \operatorname{supp}(P_{x_0})) = \psi(X) \setminus \psi(\operatorname{supp}(P_{x_0})) \subseteq Y$ is closed; amounting to $\psi(\operatorname{supp}(P_{x_0})) \subseteq Y$ being open. This indicates that $\bigcup_{x_0 \in X^s} \psi(\operatorname{supp}(P_{x_0})) = \psi(\bigcup \operatorname{supp}(P_{x_0})) = \psi(X^s) = Y^s$ is open.
3. Due to proposition 8, $\psi\,|_{X^s} : X^s \to Y^s$ is a good quotient and the orbits of this action are closed by definition of X^s (being the intersection of the open set X^s with the closed orbits of the original action of G on X); which proves that $\psi\,|_{X^s}$ is a geometric quotient by the third property of theorem 12. $\square$

The converse to this claim is not generally true; a counterexample being a G-variety X (with the trivial action $G \circlearrowright X$), with $U = X$ and the geometric quotient $X \xrightarrow{\text{id}} X$. However, it satisfies one of the criteria of the stable locus.

Theorem 15. If $U \subseteq X$ is a G-invariant open subset with $\psi : U \to Z$ a geometric quotient, $G.x \overset{\text{closed}}{\subseteq} U$, $\forall x \in U$.

Proof. Fibres of the geometric quotient are the orbits of the action of G. Therefore, they contain a unique closed orbit by the third statement of theorem 10. Since the whole fibre is an orbit (and orbits of an action are disjoint), the entire fibre is closed. Thus, all orbits of the action are closed in the subspace topology of U. $\square$

5 The Correspondence of Invertible Sheaves and Line Bundles

Reminder. Given a topological space X, a presheaf over X is a contravariant functor $\mathcal{F} : \mathcal{P}(X) \to \textbf{Sets}$. Given any open subset $U \subseteq X$, the elements of $\mathcal{F}(U)$ are called the sections of $\mathcal{F}$ over U. The sections of $\mathcal{F}$ over X are called global sections. A sheaf over X is a presheaf satisfying the following axioms, allowing to recover the sections over an open subset $U \subseteq X$ by their restrictions to an open cover $\{U_i\}_{i\in I}$ ($\forall i \in I : U_i \subseteq U$):

1. **Locality axiom:** If s, t are sections over U. If $s \mid_{U_i} = t \mid_{U_i}$, $\forall i \in I$, then they agree everywhere; i.e. $s = t$.
2. **Gluing axiom:** If $\{s_i \in \mathcal{F}(U_i)\}_{i\in I}$ is an agreeing family of sections (i.e. $s_i \mid_{U_i \cap U_j} = s_j \mid_{U_i \cap U_j}$, $\forall i, j \in I$), then one can glue them together to get a section $s \in \mathcal{F}(U) : s \mid_{U_i} = s_i$, $\forall i \in I$.

One expects sheaves to behave continuously on a net of sets containing $x \in X$ and converging to it. Since the family $\{U_i\}_{i\in I}$ of open subsets of X containing x constitute a direct system of sets, it is reasonable to expect the direct limit to behave nicely. Given a sheaf $\mathcal{F}$ on X, the stalk of $\mathcal{F}$ at $x \in X$ is the direct limit $\mathcal{F}_x := \varinjlim \mathcal{F}(U_i)$. Each element of the stalk therefore corresponds to an equivalence class of sections of $\mathcal{F}$ by the equivalence relation $\forall s, t \in \mathcal{F}(U) : s \sim t \iff \exists x \in U \subseteq X : s \mid_U = t \mid_U$. A ringed space is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$ (namely the structure sheaf) on X. A locally ringed space is a ringed space such that the stalks of the structure sheaf $\mathcal{O}_X$ are local rings; i.e. having a unique maximal ideal. Given a ringed space $(X, \mathcal{O}_X)$, a sheaf of $\mathcal{O}_X$-modules (abbreviated as an $\mathcal{O}_X$-module) is a sheaf $\mathcal{F}$ such that $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module $\forall U \overset{\text{open}}{\subseteq} X$ and restriction morphisms $r_{\mathcal{O}_X}(U \hookrightarrow V) : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$, $r_{\mathcal{F}}(U \hookrightarrow V) : \mathcal{F}(V) \to \mathcal{F}(U)$ are compatible for any open subsets $U \subseteq V$; i.e. $r_{\mathcal{F}}(U \hookrightarrow V)(m \cdot n) = r_{\mathcal{O}_X}(U \hookrightarrow V)(m) \cdot r_{\mathcal{F}}(U \hookrightarrow V)(n)$, $\forall m \in \mathcal{O}_X(V)$, $n \in \mathcal{F}(V)$. Given a morphism of varieties $\varphi : X \to Y$ and a sheaf $\mathcal{G}$ on Y, one can define a presheaf $\varphi^{-1}\mathcal{G}$ given by $\varphi^{-1}{}_{\text{psh}}\,\mathcal{G}(U) = \varinjlim \mathcal{G}(V_i)$ where the limit is taken on the family $\{V_i\}_{i\in I}$ of all open subsets $V_i \subseteq Y$ containing $\varphi(U)$. The reason for taking limits is that $\varphi(U)$ is not necessarily open. The sheafification of this presheaf then gives rise to the inverse image sheaf of $\mathcal{G}$ under φ which is denoted by $\varphi^{-1}\mathcal{G}$.
Given a morphism of varieties $\varphi : X \to Y$ and a sheaf $\mathcal{F}$ on X, one can define the direct image sheaf or the pushforward sheaf given by $\varphi_*\mathcal{F}(V) = \mathcal{F}(\varphi^{-1}(V))$, $\forall V \overset{\text{open}}{\subseteq} Y$. Given a sheaf $\mathcal{F}$ on X and an open subset $U \subseteq X$, one can define the restriction sheaf $\mathcal{F} \mid_U$ given by $\mathcal{F} \mid_U (V) = \mathcal{F}(V)$, $\forall V \overset{\text{open}}{\subseteq} U$. Given sheaves $\mathcal{F}_1$, $\mathcal{F}_2$ on X, one can define a sheaf on X called the sheaf Hom and given by $\text{Hom}(\mathcal{F}_1, \mathcal{F}_2)(U) = \text{Mor}(\mathcal{F}_1 \mid_U, \mathcal{F}_2 \mid_U)$.
Given a ringed space $(X, \mathcal{O}_X)$ and $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{F}'$, one can define a presheaf $\mathcal{F} \otimes^{\text{psh}}_{\mathcal{O}_X} \mathcal{F}'$ that is given by $(\mathcal{F} \otimes^{\text{psh}}_{\mathcal{O}_X} \mathcal{F}')(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}'(U)$, which gives rise to a sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}'$ by sheafification. This sheaf satisfies the universal property of tensor product and commutes with taking stalk: $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}')_p = \mathcal{F}_p \otimes^{\text{psh}}_{\mathcal{O}_{X_p}} \mathcal{F}'_p$.
Given a morphism of (locally) ringed spaces $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and an $\mathcal{O}_Y$-module $\mathcal{G}$, one can prove that $\varphi^{-1}\mathcal{G}$ is an $\varphi^{-1}\mathcal{O}_Y$-module and $\mathcal{O}_X$ is an $\varphi^{-1}Y$-module. Therefore, given a morphism of (locally) ringed spaces $\varphi : X \to Y$ and an $\mathcal{O}_Y$-module $\mathcal{G}$, one can define the pullback sheaf as the $\mathcal{O}_X$-module $\varphi^*\mathcal{G} := \varphi^{-1}\mathcal{G} \otimes_{\varphi^{-1}\mathcal{O}_Y} \mathcal{O}_X$. The pullback sheaf is also characterized by the following adjointness universal property: $\varphi^*\mathcal{G}$ is an $\mathcal{O}_X$-module such that given any $\mathcal{O}_X$-module $\mathcal{F}$, there is a functorial bijection $\text{Hom}_{\mathcal{O}_X}(\varphi^*\mathcal{G}, \mathcal{F}) \longleftrightarrow \text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \varphi_*\mathcal{F})$.
Note that the morphisms of shieves $\varphi \mapsto \varphi^{-1}$, $\varphi \mapsto \varphi_*$ are both functors. Moreover, they are adjoint functors.

Reminder. Given a variety X, a line bundle over X is a variety L equipped with a morphism of varieties $\pi : L \to X$ such that there exists an open covering $\{U_i\}_{i\in I}$ of X satisfying:

1. $\forall i \in I$, there exists an isomorphism $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^1$ such that $\pi = \pi_1 \circ \varphi_i$. This implies that φ_i induces an isomorphism $\varphi_i\,|_{\pi^{-1}(x)}\colon \pi^{-1}(x) \to {\pi_1}^{-1}(x) = \{(x,\,t) \in U_i \times \mathbb{A}^1 \mid \forall t \in \mathbb{A}^1\} \cong \mathbb{A}^1$ and therefore each fibre $\pi^{-1}(x)$ has the linear structure of a one-dimensional vector space
2. $\forall i,\,j \in I$, the transition isomorphism $\zeta_{ij} := \varphi_j \circ {\varphi_i}^{-1} : (U_i \cap U_j) \times \mathbb{A}^1 \to (U_i \times U_j) \times \mathbb{A}^1$ is given by the linear map $(x,\,t) \mapsto (x,\,\alpha_{ij}(x)t)$, $\forall x \in U_i \cap U_j,\, t \in \mathbb{A}^1$ on fibres where $\alpha_{ij} \in \mathcal{O}_X(U_i \cap U_j)^*$. Note that the transition morphisms satisfy a cocycle condition $\zeta_{ij} \circ \zeta_{jk} \circ \zeta_{ki} = \mathrm{id}_{(U_i \cap U_j \cap U_k)\times \mathbb{A}^1}$, $\forall i,\,j,\,k \in I$.

The line bundle L is obtained by gluing the trivial line bundles $\pi_i := \pi \circ {\varphi_i}^{-1} : U_i \times \mathbb{A}^1 \to U_i$ via the linear transition maps α_{ij} (note that $\zeta_{ij} = {\pi_j}^{-1} \circ \pi_i$). Given a line bundle L equipped with the structure morphism $\pi : L \to X$ and a morphism of varieties $\psi : Y \to X$, one can define the pullback bundle as the fibre product

$$\begin{tikzcd} L \times_X Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & L \arrow[d, "\pi"] \\ Y \arrow[r, "\psi"'] & X \end{tikzcd}$$

which is a fibre bundle equipped with the projection map $\pi_2 : L \times_X Y \to Y$ and is obtained by gluing trivial bundles $\eta_i := \psi^{-1} \circ (\pi \circ {\varphi_i}^{-1}) \circ (\psi \otimes \mathrm{id}_{\mathbb{A}^1}) : \psi^{-1}(U_i) \times \mathbb{A}^1 \to \psi^{-1}(U_i)$ via the linear transition maps $(\psi^{-1}(U_i) \cap \psi^{-1}(U_j)) \times \mathbb{A}^1 = \psi^{-1}(U_i \cap U_j) \times \mathbb{A}^1 \to (\psi^{-1}(U_i) \cap \psi^{-1}(U_j)) \times \mathbb{A}^1 = \psi^{-1}(U_i \cap U_j) \times \mathbb{A}^1$ given by

$$\xi_{ij} := {\eta_j}^{-1} \circ \eta_i = (\psi^{-1} \otimes \mathrm{id}_{\mathbb{A}^1}) \circ \varphi_j \circ \pi^{-1} \circ \psi \circ \psi^{-1} \circ \pi \circ {\varphi_i}^{-1} \circ (\psi \otimes \mathrm{id}_{\mathbb{A}^1}) = (\psi^{-1} \otimes \mathrm{id}_{\mathbb{A}^1}) \circ \varphi_j \circ {\varphi_i}^{-1} \circ (\psi \otimes \mathrm{id}_{\mathbb{A}^1})$$

$$= (\psi^{-1} \otimes \mathrm{id}_{\mathbb{A}^1}) \circ \zeta_{ij} \circ (\psi \otimes \mathrm{id}_{\mathbb{A}^1}) \text{ that is of form } (y,\,t) \mapsto (\psi(y),\,t) \mapsto (\psi(y),\,\alpha_{ij}(\psi(y))t) \mapsto (y,\,\alpha_{ij}(\psi(y))t)$$

A morphism between line bundles $\pi : L \to X$ and $\pi' : L' \to Y$ is a morphism of varieties $\varphi : L \to L'$ making

$$\begin{tikzcd} L' \arrow[dr, dotted, "\phi"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "\pi'"'] & & \\ & L \times_X Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & L \arrow[d, "\pi"] \\ & Y \arrow[r, "\psi"'] & X \end{tikzcd}$$

commute (so it keeps the fibres invariant) and the induced map $\varphi\,|_{\pi'^{-1}(y)}\colon \pi'^{-1}(y) \to \pi^{-1}(\psi(x))$, $\forall y \in Y$ between fibres of L', L is linear. The universal property of the fibre product then implies the existence of a unique morphism $\phi : L' \to L \times_X Y$ making the whole diagram commute (i.e. $\pi_1 \circ \phi = \varphi$, $\pi_2 \circ \phi = \pi'$). Given the trivial line bundle $X \times \mathbb{A}^1$ on X, one can see that the only automorphisms on $X \times \mathbb{A}^1$ are in fact of form $(x,\,t) \mapsto (x,\,f(x)t)$, $f \in {\mathcal{O}_X}^*$. Using local trivializations, one can see that the same holds for any line bundle. A section of a line bundle $\pi : L \to X$ is a section of its structure morphism; i.e. a morphism of varieties $s : X \to L$ such that $\pi \circ s = \mathrm{id}_X$. Sections of the trivial line bundle over X are all of the form $(x,\,t) \mapsto (x,\,f(x)t)$, $f \in {\mathcal{O}_X}^*$. Identifying each of the local trivializations $\{U_i\}_{i\in I}$ of the line bundle with the corresponding map $f_i \in \mathcal{O}_X(U_i)^*$, a section of L corresponds to a family of maps $f_i \in \mathcal{O}_X(U_i)^*$ such that $\frac{f_i}{f_j}(x) = \alpha_{ij}(x)$, $\forall x \in U_i \cap U_j$. This indicates that the sections of a line bundle constitute a vector space, denoted by $\Gamma(X,\,L)$.

Reminder. Given the line bundles given by $\pi : L \to X$, $\pi' : L' \to X$ over X, trivialized over the open covers $\{U_i\}_{i\in I}$, $\{{U_i}'\}_{i\in I'}$ respectively, one can readily see that they are both trivialized over the common open covering $\{U_i\}_{i\in I} \cup \{{U_i}'\}_{i\in I'}$ via the transition maps $\{\xi_{ij}\}_{i,\, j\in I\cup I'}$, $\{{\xi_{ij}}'\}_{i,\, j\in I\cup I'}$ respectively. One can then define a new line bundle on X obtained by gluing the trivializations $\{U_i\}_{i\in I} \cup \{{U_i}'\}_{i\in I'}$ via the transition maps $\{{\xi_{ij}}'' := \xi_{ij} \circ {\xi_{ij}}'\}_{i,\, j\in I\cup I'}$ where ${\xi_{ij}}''(x,\, t) = (x,\, \alpha_{ij}(x){\alpha_{ij}}'(x)t)$. This is defined as the tensor product $L \otimes L'$ which is clearly a line bundle over X. It is then clear that the tensor product of line bundles is commutative. Furthermore, each fibre $(L \otimes L')_x$ is given by the tensor product of fibres $L_x \otimes {L'}_x$.
Given a line bundle $\pi : L \to X$, trivialized over the open cover $\{U_i\}_{i\in I}$ of X via the transition maps ξ_{ij}, one can define a line bundle obtained by gluing trivializations $\{U_i\}_{i\in I}$ via transition maps ${\xi^{-1}}_{ij} : U_i \cap U_j \to U_i \cap U_j$ given by ${\xi^{-1}}_{ij}(x,\, t) = (x,\, \alpha_{ij}(x)^{-1}\, t)$. This bundle is called the dual bundle $L^\vee$. It is straightforward to see that $L \otimes L^\vee = L^\vee \otimes L$ is the trivial bundle over X and that $(L^\vee)^\vee \cong L$ for any line bundle L over X.
A local section of $\pi : L \to X$ is a morphism $s : U \to L$ where $U \overset{\text{open}}{\subseteq} X$ such that $\pi \circ s = \mathrm{id}_U$. Thus, the local sections of a line bundle constitute a sheaf on X denoted by $\mathcal{L}(X,\, L)$. One can readily see that the sheaf of local sections of a trivial line bundle is just the structure sheaf $\mathcal{O}_X$ and that $\mathcal{L}$ is a sheaf of $\mathcal{O}_X$-modules. Note that given $s \in \mathcal{L}(X,\, L)(U)$, $t \in \mathcal{L}(X,\, L')(U)$ corresponding to family of maps $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$ respectively, the family of maps $\{f_i g_i\}_{i\in I}$ constitutes a local section $s \otimes t \in \mathcal{L}(X,\, L \otimes L')(U)$ and $\mathcal{L}(X,\, L \otimes L') \cong \mathcal{L}(X,\, L) \otimes \mathcal{L}(X,\, L')$. Therefore, one can see that $\mathcal{L}(X,\, L) \otimes \mathcal{L}(X,\, L^\vee) \cong \mathcal{L}(X,\, L \otimes L^\vee) \cong \mathcal{L}(X,\, \mathcal{O}_X) \cong \mathcal{O}_X$.

Definition 16. Given a ringed space $(X,\, \mathcal{O}_X)$, we say that a sheaf of $\mathcal{O}_X$-modules $\mathcal{F} \in \mathcal{O}_X\mathrm{Mod}$ is invertible if it possesses an inverse with respect to the tensor product of $\mathcal{O}_X$-modules: $\exists \mathcal{F}^\vee \in \mathcal{O}_X\mathrm{Mod} : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \cong \mathcal{O}_X$. Given any sheaf of $\mathcal{O}_X$-modules $\mathcal{F} \in \mathcal{O}_X\mathrm{Mod}$, one can define a dual sheaf $\mathcal{F}^\vee = \mathrm{Hom}(\mathcal{F},\, \mathcal{O}_X)$ which denotes the sheaf given by $\mathrm{Hom}(\mathcal{F},\, \mathcal{O}_X)(U) = \mathrm{Hom}(\mathcal{F}\mid_U,\, \mathcal{O}_X\mid_U)$. One can therefore use the evaluation maps to give a canonical homomorphism $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \to \mathcal{O}_X$. Then $\mathcal{F}$ is invertible $\iff$ this homomorphism is an isomorphism.

Corollary 2. Following the aforementioned argument, the sheaf of local sections $\mathcal{L}(X,\, L)$ is an invertible sheaf.

This maps to the isomorphism class of any line bundle L over X the isomorphism class of an invertible sheaf $\mathcal{L}(X,\, L)$ over X. We now proceed to prove that this is indeed a one-to-one correspondence.

Lemma 14. $\forall M \in R\mathrm{Mod}$ projective over a local ring $(R,\, \mathfrak{m})$ and $x \in M$, M has a free direct summand $M' \ni x$.

Proof. Since M is projective, there exists some R-module N such that $F = M \oplus N$ is free. Choose a basis $\{f_i = m_i + n_i\}_{i\in I}$ such that x has the least number of nonzero coordinates written in the basis among all possible bases of F. Let $x = r_1 f_1 + \cdots + r_t f_t$ where the coefficients f_i, $1 \leq i \leq t$ are nonzero. Since $x \in M$, clearly $n_i = 0$, $\forall 1 \leq i \leq t$. Therefore, $x = r_1 m_1 + \cdots + r_t m_t$. Letting $m_i = a_{i1} f_1 + \cdots + a_{it} f_t + s_i$, $s_i \in \langle \{f_i\}_{i\in I\setminus\{1,\, \cdots,\, t\}}\rangle$ and substituting, we get $x = \sum_{1\leq i,\, j\leq t} r_i a_{ij} f_j + \sum_{1\leq i\leq t} r_i s_i = \sum_{1\leq i\leq t} r_i s_i \in \langle \{f_i\}_{i\in I\setminus\{1,\, \cdots,\, t\}}\rangle$. Since we have $x = \sum_{1\leq i\leq t} r_i f_i$ and $\{f_i\}_{i\in I}$ constitute a basis, we get $\sum_{1\leq i\leq t} r_i s_i = 0$, $\sum_{1\leq i\leq t} r_i a_{ij} = r_j$. Equivalently, $\sum_{1\leq i\leq t,\, i\neq j} r_i a_{ij} = r_j(1 - a_{jj})$, $\forall 1 \leq j \leq t$. If $r_j = \sum_{1\leq i\leq t,\, i\neq j} c_i r_i$, $c_i \in R$, then $x = \sum_{1\leq i\leq t,\, i\neq j} r_i(f_i + c_i f_j)$ which in turn contradicts the minimality property of the basis $\{f_i\}_{i\in I}$. Therefore, all of the elements a_{ij}, $\forall i \neq j$ and $1 - a_{jj}$ are non-units of R, since otherwise one can multiply by the inverse to get to get an expression of some r_i as the linear combination of other r_j's, which is impossible as stated. This indicates that $\langle a_{ij}\rangle \neq R$. Since R is a local ring and $\langle a_{ij}\rangle$ lies in a maximal ideal, one deduces that $a_{ij} \in \mathfrak{m}$, $\forall i \neq j$, $1 - a_{ii} \in \mathfrak{m}$, $\forall i$. Constructing $[A]_{ij} = a_{ij}$, one has $\det(A) \overset{\mathfrak{m}}{\equiv} 1$ which implies that $\det(A) \in R^*$. Therefore, $\{m_1,\, \cdots,\, m_t\} \cup \langle \{f_i\}_{i\in I\setminus\{1,\, \cdots,\, t\}}\rangle$ is also a basis. Finally, $\langle m_1,\, \cdots,\, m_t\rangle$ is a free direct summand of M containing x. $\square$

Lemma 15. **(Kaplansky)** If $(R, \mathfrak{m})$ is a local ring, then any projective R-module is a free module.

Proof. Any projective module is countably generated, so let $P = \langle \{x_i\}_{i=1}^{\infty} \rangle$. By lemma 14, $\langle \{x_i\}_{i=1}^{\infty} \rangle = M_1 \oplus F_1$ where F_1 is free and contains x_1. It is then immediate that M_1 is also projective and that $M_1 = \langle \{x_i{}'\}_{i=2}^{\infty} \rangle$ where $x_i{}'$ is the projection of x_i onto the direct summand M_1. Likewise, since M_1 is projective, by lemma 14, we get a direct decomposition $M_1 = M_2 \oplus F_2$ where F_2 is free and contains $x_2{}'$. Proceeding similarly and using lemma 14 successively, we get to the countable decomposition $M = \bigoplus_{i=1}^{\infty} F_i$ which is a free module. □

One can now prove (by taking a localization) that a projective module over an arbitrary ring is locally free.

Theorem 16. A finitely generated R-module is projective (hence locally free) of rank one $\iff$ it is invertible.

Proof. If an R-module M is projective (hence locally free) of rank one, given any prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$, we get $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}$. Since we have the dual $M^{\vee} = \operatorname{Hom}_R(M, R)$, after localizing we get $M^{\vee}{}_{\mathfrak{p}} \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}})$. Yet, following that M is locally free of rank one, we get $M^{\vee}{}_{\mathfrak{p}} \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}, R_{\mathfrak{p}}) \cong R_{\mathfrak{p}}$. Therefore, the homomorphism $M \otimes M^{\vee} \to R$ is locally just $M_{\mathfrak{p}} \otimes M^{\vee}{}_{\mathfrak{p}} \cong R_{\mathfrak{p}} \otimes R_{\mathfrak{p}} \to R_{\mathfrak{p}}$ which is an isomorphism and proves that M is invertible. For proof of the converse, consult [4] Proposition 19.13. □

Therefore, any invertible module is locally free of rank one. This implies that given an invertible sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, $\mathcal{F}(U)$ is a locally free $\mathcal{O}_X(U)$-module of rank one; i.e. $\forall \mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_X(U)) : \mathcal{F}(U)_{\mathfrak{p}} \cong \mathcal{O}_X(U)_{\mathfrak{p}}$. Given some $x \in X$, one can prove that the ring $\mathcal{O}_{X,\, x}$ (stalk of the structure sheaf at x) is a local ring. Then the canonical isomorphism $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^{\vee} \to \mathcal{O}_X$ induces the map $\mathcal{F}_x \otimes_{\mathcal{O}_{X,\, x}} \mathcal{F}^{\vee}{}_x \to \mathcal{O}_{X,\, x}$ on stalks. We now use the following lemma:

Lemma 16. Let R be a local ring, and L, M be R-modules such that $L \otimes_R M \cong R$. Then $L \cong R$ and $M \cong R$.

Proof. Pick an isomorphism $\alpha : L \otimes_R M \to R$. Since α is surjective, there exists a finite sum $\sigma = \sum_i \ell_i \otimes m_i$ with $\alpha(\sigma) = 1$. In a local ring, at least one summand maps to a unit. So there exist $\ell \in L$, $m \in M$ such that $u := \alpha(\ell \otimes m) \in R^{\times}$. Define an R-linear map $\varphi : L \to R$ by $\varphi(x) := \alpha(x \otimes m)$. Then $\varphi(\ell) = u \in R^{\times}$. Consider the composite $R \xrightarrow{1 \mapsto \ell} L \xrightarrow{\varphi} R$ to be multiplication by u, hence an isomorphism of R-modules. Therefore both arrows split after rescaling by u^{-1}, and we conclude $L \cong R$. Similarly, $M \cong R$. □

Therefore, $\mathcal{F}_x \cong \mathcal{O}_{X,\, x}$. This indeed indicates that $\forall x \in X$, there exist an open neighborhood $U \subseteq X$ such that $\mathcal{F}(U) \cong \mathcal{O}_X(U)$. We term such sheaves of $\mathcal{O}_X$-modules as locally free sheaves of rank one. We say that a sheaf of $\mathcal{O}_X$-modules is free if it is locally isomorphic to $\mathcal{O}_X{}^{\otimes I}$ for some indexing set I, where $|I|$ is the rank of the sheaf.

Definition 17. Given a commutative ring R and an R-module M over R, one can define the tensor algebra of M as $\mathcal{T}(M) = \bigoplus_{n \geq 0} M^{\otimes n}$. Tensor algebra is a noncommutative R-algebra. The symmetric algebra of M is defined as $\mathcal{S}(M) = \mathcal{T}(M)/\langle \{x \otimes y - y \otimes x \mid \forall x,\, y \in M\} \rangle$, where the component of degree n is called the n-th symmetric product of M. It is straightforward to show that the symmetric algebra of M is a commutative R-algebra. The exterior algebra of M is defined as $\bigwedge(M) = \mathcal{T}(M)/\langle \{x \otimes x \mid \forall x \in M\} \rangle$, where the component of degree n is called the n-th exterior power of M. It is clear that the exterior algebra contains all elements of form $x \otimes y + y \otimes x$ and is hence a skew-commutative R-algebra; i.e. $u \wedge v = (-1)^{rs} v \wedge u$, $\forall u \in \bigwedge^r(M)$, $v \in \bigwedge^s(M)$. Given a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, one can then define presheaves $T^{\mathrm{psh}}(\mathcal{F})$, $S^{\mathrm{psh}}(\mathcal{F})$, $\bigwedge^{\mathrm{psh}}(\mathcal{F})$ associating to each $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ the $\mathcal{O}_X$-modules $\mathcal{T}(\mathcal{F}(U))$, $\mathcal{S}(\mathcal{F}(U))$, $\bigwedge(\mathcal{F}(U))$ respectively. One can then define the tensor algebra, symmetric algebra and the exterior algebras of $\mathcal{F}$ as the sheafifications of $T^{\mathrm{psh}}(\mathcal{F})$, $S^{\mathrm{psh}}(\mathcal{F})$, $\bigwedge^{\mathrm{psh}}(\mathcal{F})$ respectively.

Therefore, $\mathcal{T}(\mathcal{F})$, $\mathcal{S}(\mathcal{F})$, $\bigwedge(\mathcal{F})$ are all associative $\mathcal{O}_X$-algebras and each of their homogeneous components is an $\mathcal{O}_X$-module. Alternatively, $\mathcal{T}(\mathcal{F})$, $\mathcal{S}(\mathcal{F})$, $\bigwedge(\mathcal{F})$ are given by the following universal properties. Let M be an R-module:

1. Any module homomorphism $\varphi : M \to \mathcal{A}$ to an associative R-algebra $\mathcal{A}$ can be uniquely extended to an algebra homomorphism$\mathcal{T}(M) \to \mathcal{A}$ making the diagram I commute.
2. Any module homomorphism $\varphi : M \to \mathcal{A}$ to an associative commutative R-algebra $\mathcal{A}$ can be uniquely extended to an algebra homomorphism$\mathcal{T}(M) \to \mathcal{A}$ making the diagram II commute.
3. Any square-zero module homomorphism $\varphi : M \to \mathcal{A}$ (i.e. $\varphi(m) \cdot \varphi(m) = 0,\ \forall m \in M$) to a unital associative R-algebra $\mathcal{A}$ can be uniquely extended to an algebra homomorphism$\mathcal{T}(M) \to \mathcal{A}$ making the diagram III commute.

$$\begin{tikzcd} & \mathcal{A} \\ M \arrow[ur, "\varphi"] \arrow[r, hook, "\text{inclusion}"'] & \mathcal{T}(M) \arrow[u, dotted, "\widetilde{\varphi}"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{A} \\ M \arrow[ur, "\varphi"] \arrow[r, hook, "\text{inclusion}"'] & \mathcal{S}(M) \arrow[u, dotted, "\widetilde{\varphi}"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{A} \\ M \arrow[ur, "\varphi"] \arrow[r, hook, "\text{inclusion}"'] & \bigwedge(M) \arrow[u, dotted, "\widetilde{\varphi}"'] \end{tikzcd}$$

(I) (II) (III)

Definition 18. Given a commutative ring R, one can define (as stated before) a topological space $\operatorname{Spec}(R)$ defined as the set of all prime ideals of R where the sets given by $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$, $\forall a \lhd R$ are a basis of closed sets. One can then attach to $\operatorname{Spec}(R)$ a sheaf of rings and make it a ringed space as follows: Given an open set $D(\mathfrak{a}) = \operatorname{Spec}(R) \setminus V(\mathfrak{a})$, we define $\mathcal{O}_{\operatorname{Spec}(R)}(U)$, $U \overset{\text{open}}{\subseteq} \operatorname{Spec}(R)$ as the set of locally division functions $s : U \to \coprod_{\mathfrak{p} \in U} R_{\mathfrak{p}}$; i.e. $\forall \mathfrak{p} \in U : \exists \mathfrak{p} \in V \overset{\text{open}}{\subseteq} U,\ \exists a, b \in R : \forall \mathfrak{q} \in V,\ b \notin \mathfrak{q} : s(\mathfrak{q}) = \frac{a}{b}$. This set of functions is closed with respect to sums and products and moreover, constitute a commutative ring (with the identity element of product being the map sending every $\mathfrak{p} \in U$ to the identity of each of the localizations $R_{\mathfrak{p}}$.

Let $\mathcal{F}$ be an invertible (hence locally free by the preceding argument) sheaf of $\mathcal{O}_X$-modules on the ringed space $(X, \mathcal{O}_X)$. Let $\mathcal{S}(\mathcal{F})$ denote the symmetric algebra of $\mathcal{F}$ which is in fact an invertible $\mathcal{O}_X$-algebra; mapping to every open $U \overset{\text{open}}{\subseteq} X$ an $\mathcal{O}_X(U)$-algebra. That is, a commutative ring $\mathcal{S}(\mathcal{F})(U)$ equipped with a ring homomorphism denoted by $\iota : \mathcal{O}_X(U) \to \mathcal{S}(\mathcal{F})(U)$. Notice that this equips $\mathcal{S}(\mathcal{F})(U)$ with the structure of an $\mathcal{O}_X(U)$-module by defining the action $r.a := \iota(r).a \in \mathcal{S}(\mathcal{F})(U)$, $\forall r \in \mathcal{O}_X(U)$, $a \in \mathcal{S}(\mathcal{F})(U)$. This induces a map $\iota^* : \operatorname{Spec}(\mathcal{S}(\mathcal{F})(U)) \to \operatorname{Spec}(\mathcal{O}_X(U))$ by pullback of prime ideals. Finally, one can construct the ringed spaces $\operatorname{Spec}(\mathcal{S}(\mathcal{F})(U))$ for any $U \overset{\text{open}}{\subseteq} X$. We can then show that these ringed spaces are conformal in the following sense:

Lemma 17. Given an inclusion of open sets $U_1 \hookrightarrow U_2$, we get a restriction morphism $\mathcal{F}(U_2) \to \mathcal{F}(U_1)$. This then induces a map of the spectra $\operatorname{Spec}(\mathcal{F}(U_1)) \to \operatorname{Spec}(\mathcal{F}(U_2))$. Then the following red diagram is cartesian:

$$\begin{tikzcd} \operatorname{Spec}(S(\mathcal{F})(U_1)) \arrow[r] & \operatorname{Spec}(S(\mathcal{F})(U_2)) \\ S(\mathcal{F})(U_1) \arrow[u] & S(\mathcal{F})(U_2) \arrow[u] \arrow[l] \\ \mathcal{F}(U_1) \arrow[u] & \mathcal{F}(U_2) \arrow[u] \arrow[l] \\ U_1 \arrow[u] \arrow[r, hook, "\text{inclusion}"'] \arrow[uuu, bend left] & U_2 \arrow[u] \arrow[uuu, bend right] \end{tikzcd}$$

Finishing the argument, one can glue the ringed spaces $\mathrm{Spec}(\mathcal{S}(\mathcal{F})(U))$ together to get a ringed space $\mathbf{Spec}(\mathcal{S}(\mathcal{F}))$:

Lemma 18. Let $(Y, \mathcal{O}_Y)$ be a ringed space, and let $\mathcal{A}$ be a sheaf of $\mathcal{O}_Y$-algebras. Then there is a unique ringed space $(X, \mathcal{O}_X)$ and a morphism of locally ringed spaces called the projection morphism $\pi : X \to Y$ such that $\pi^{-1}(V) \cong \mathrm{Spec}(\mathcal{A}(V))$ for any $V \overset{\text{open}}{\subseteq} Y$, and that for any inclusion of open sets $U \hookrightarrow V$, the pullback morphism $\pi^{-1}(U) \to \pi^{-1}(V)$ corresponds to the restriction morphism $\mathcal{A}(V) \to \mathcal{A}(U)$. This ringed space is called the relative spectrum and is denoted by $\mathbf{Spec}_Y(\mathcal{A})$).[2]Relative spectrum is unique up to isomorphism over X.

Proof. Consult §3.3 of [8]. For a proof in the special case of quasi-coherent algebras over schemes, c.f. [25]. $\square$

Back to our initial discussion, define the relative spectrum $\mathbf{Spec}_X(\mathcal{S}(\mathcal{F}))$ and let $\pi : \mathbf{Spec}(\mathcal{S}(\mathcal{F})) \to X$ be the projection morphism. Since $\mathcal{F}$ is locally free, there is an open cover $X = \bigcap_{i \in I} U_i$ such that $\mathcal{F}\mid_{U_i}$ is free. Choose a basis of U_i for all $i \in I$ and let $\psi_i : \pi^{-1}(U_i) \to \mathbb{A}^n_{U_i}$ be the coordinate map identifying $\mathcal{S}(\mathcal{F}(U_i))$ with $\mathcal{O}_X(U_i)[X_i, \forall i \in I]$. Then $\mathbf{Spec}_X(\mathcal{S}(\mathcal{F}))$ constitutes a line bundle over X where the trivializations are given by $\psi_i : \pi^{-1}(U_i) \to \mathbb{A}^n_{U_i}$.

This is called the geometric line bundle associated to $\mathcal{F}$ and is denoted by $V(X, \mathcal{F})$.

Theorem 17. Given an invertible sheaf (hence locally free of rank one) of $\mathcal{O}_X$-modules $\mathcal{F}$ over X, one can construct the geometric line bundle $V(X, \mathcal{F})$ along with its projection map $\pi : V(X, \mathcal{F}) \to X$. One can then derive the sheaf of its local sections $\mathcal{L}(X, V(X, \mathcal{F}))$. Prove that $\mathcal{L}(X, V(X, \mathcal{F})) \cong \mathcal{F}^\vee$.

Proof. Given a section $s \in \mathcal{F}^\vee(U)$ over the open set $U \overset{\text{open}}{\subseteq} X$, one can see s as an element of the sections $\mathrm{Hom}(\mathcal{F}, \mathcal{O}_X)(U) = \mathrm{Hom}(\mathcal{F}(U), \mathcal{O}_X(U))$. According to the universal property of the symmetric algebra, this indeed determines an $\mathcal{O}_X$-algebra homomorphism $\widetilde{s} : \mathcal{S}(\mathcal{F}(U)) \to \mathcal{O}_X(U)$, which in turn defines a map of relative spectra $\mathbf{Spec}(\widetilde{s}) : U \cong \mathbf{Spec}_X(\Gamma(U, \mathcal{O}_X)) = \mathbf{Spec}_X(\mathcal{O}_X(U)) \to \mathbf{Spec}_X(\mathcal{S}(\mathcal{F}(U))) = \mathbf{Spec}_X(\mathcal{S}(\mathcal{F})(U))$. But $\mathbf{Spec}_X(\mathcal{S}(\mathcal{F})(U))$ is isomorphic to $\pi^{-1}(U)$ by definition of the relative spectrum. It is then immediate that $\mathbf{Spec}(\widetilde{s}) \in \mathcal{L}(X, V(X, \mathcal{F}))$. Conversely, given a local section $\sigma : U \to V(X, \mathcal{F})$ where $U \overset{\text{open}}{\subseteq} X$, the definition of local section implies that $\pi \circ \sigma = \mathrm{id}_U$. Therefore, it is clear that $\mathrm{Im}(\sigma) \subseteq \pi^{-1}(U) = \mathbf{Spec}_X(\mathcal{S}(\mathcal{F})(U))$. This induces a map of the global sections $\widetilde{\sigma} : \mathcal{S}(\mathcal{F})(U) \to \mathcal{O}_X(U)$. Composing this map with the canonical inclusion $\mathcal{F}(U) \overset{\iota}{\hookrightarrow} \mathcal{S}(\mathcal{F}(U)) = \mathcal{S}(\mathcal{F})(U)$ gives rise to a map $\sigma \circ \iota : \mathcal{F}(U) \to \mathcal{O}_X(U)$. Finally, $\sigma \circ \iota \in \mathrm{Hom}(\mathcal{F}(U), \mathcal{O}_X(U))$ which is alternatively equal to $\mathrm{Hom}(\mathcal{F}, \mathcal{O}_X)(U) = \mathcal{F}^\vee(U)$.[9] $\square$

Therefore, the map $\mathcal{F} \mapsto V(X, \mathcal{F}^\vee)$ sending any invertible sheaf of $\mathcal{O}_X$-modules to a line bundle on X, and $L \mapsto \Gamma(X, L)$ sending any line bundle on X to an invertible sheaf of $\mathcal{O}_X$-modules are inverse to each other and therefore constitute a one-to-one correspondence between the isomorphism classes of invertible $\mathcal{O}_X$-modules over X and the isomorphism classes of line bundles over X. We therefore use these concepts interchangeably.

Definition 19. The isomorphism classes of line bundles over X constitute an Abelian group with tensor product as the group operation, the dual bundle as the inverse operation, and the isomorphism class of trivial bundle as identity element. This group is called as the Picard group of X and denoted as $\mathrm{Pic}(X)$.

[2]Modification of [11] Ex. 2.5.17.c

Reminder. Given an affine space $\mathbb{A}_k^{n+1}$, one defines the projective space $\mathbb{P}_k^n := (\mathbb{A}_k^{n+1} \setminus \{0\})/\mathbb{G}_m$ where the action of $\mathbb{G}_m$ is canonical ($\mathbb{A}_k^{n+1}$ is then referred to as the affine cone over $\mathbb{P}_k^n$). More generally, any vector space V over k gives rise to a projective space defined by $\mathbb{P}(V) := (V \setminus \{0\})/\mathbb{G}_m$ and V is termed as the affine cone over $\mathbb{P}(V)$. Given homogeneous polynomials $\{f_1, \cdots, f_m\} \subseteq k[X_0, \cdots, X_n]$, the homogeneous ideal $\mathfrak{a} = \langle f_1, \cdots, f_m \rangle \lhd k[X_0, \cdots, X_n]$ is invariant under the canonical action of $\mathbb{G}_m$ and therefore one can define $X := V_P(\mathfrak{a}) := (V(\mathfrak{a}) \setminus \{0\})/\mathbb{G}_m \subseteq \mathbb{P}_k^n$, which is termed as a projective algebraic set. Conversely, given a projective algebraic set X, we define a homogeneous ideal $I_H(X) := \{f \in k[X_0, \cdots, X_n]^+ \mid f(x) = 0, \forall x \in X\}$, where $k[X_0, \cdots, X_n]^+ = \bigoplus_{d \geq 0} k[X_0, \cdots, X_n]_d$ denotes $k[X_0, \cdots, X_n]$ with its natural graded structure. Note that $I_H(X)$ is similarly always a radical ideal, and that the monomials $x_0, \cdots, x_n$ cannot all belong to $I_H(X)$ for some $X \neq \emptyset$. Therefore, the homogeneous ideal $\langle x_0, \cdots, x_n \rangle$ corresponds, in some sense, to the empty set of the projective space. We call a homogeneous ideal $\mathfrak{a} \lhd k[X_0, \cdots, X_n]$ irrelevant if $\langle x_0, \cdots, x_n \rangle = \sqrt{\mathfrak{a}}$. Therefore, I_H, V_P construct a correspondence between nonempty projective algebraic subsets of $\mathbb{P}^n$ and proper, radical and non-irrelevant homogeneous ideals of $k[X_0, \cdots, X_n]$. The projective Nullstellensatz then implies that $I_H(V_P(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. An irreducible projective algebraic set is called a projective algebraic variety.
The homogeneous coordinate ring of a projective variety X is given by $R(X) := k[X_0, \cdots, X_n]/I_H(X)$. Note that since $I_H(X) \lhd k[X_0, \cdots, X_n]$ is a homogeneous ideal, $R(X)$ admits a natural graded algebra structure $R(X) = \bigoplus_{i \geq 0} R_i(X)$. Therefore, $R(X)$ is a reduced, finitely generated, graded k-algebra. Conversely, given a reduced, finitely generated, graded k-algebra $R(X) = \bigoplus_{i \geq 0} R_i(X)$, one can take define the projective spectrum $\mathrm{Proj}(R)$ as the set of all homogeneous prime ideals of R that do not contain the irrelevant ideal given by $R_+(X) = \bigoplus_{i > 0} R_i(X)$, which can then be equipped with Zariski topology. Similar to the affine case, one can also define the maximal projective spectrum $\mathrm{MaxProj}(R)$ as the set of all homogeneous maximal ideals of R that do not contain the irrelevant ideal, and $R(\mathrm{MaxProj}(A)) \cong A$ holds for all reduced, finitely generated k-algebras.
Given projective varieties $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$, a regular map $\varphi : X \to Y$ is the restriction of a rational map given by $\varphi(x) = \left(\frac{P_1(x)}{Q_1(X)}, \cdots, \frac{P_m(x)}{Q_m(x)} \right)$, $\forall 1 \leq i \leq m : P_i, Q_i \in k[X_1, \cdots, X_n]_d$, $d \in \mathbb{N}$ and that $Q_1(x), \cdots, Q_m(x) \neq 0$, $\forall x \in X$. Equivalently, φ is a homogeneous rational function of degree zero (so that it is well-defined on the projective space). Given some open subset $U \subseteq X$, a function $f : U \to \mathbb{A}^1$ is said to be regular at $x_0 \in U$ if it is locally regular at x_0; i.e. there exist affine charts $x \in U \subseteq X$, $f(U) \subseteq V \subseteq \mathbb{A}^1$ such that $f\mid_U : U \to V$ is regular. f is said to be regular if it is regular everywhere.

Remark 7. The projective space $\mathbb{P}^n$ admits two important line bundles. Given the projective space $\mathbb{P}^n$, one can create a line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ over $\mathbb{P}^n$ trivialized over the open cover $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid z_i \neq 0\}$, $\forall 0 \leq i \leq n$ via transition maps $\alpha_{ij} : U_i \cap U_j \to \mathbb{A}^1$ given by $\alpha_{ij}([z_0 : \cdots : z_n]) = z_j {z_i}^{-1}$. This is the hyperplane line bundle. One can also create a line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ over $\mathbb{P}^n$ trivialized over the same open cover via transition maps $\alpha_{ij}([z_0 : \cdots : z_n]) = (z_j {z_i}^{-1})^{-1}$. This is then called the tautological line bundle. It is clear that $\mathcal{O}_{\mathbb{P}^n}(-1)$, $\mathcal{O}_{\mathbb{P}^n}(1)$ are dual to each other. One can then define $\mathcal{O}_{\mathbb{P}^n}(n) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes n}$, $\mathcal{O}_{\mathbb{P}^n}(-n) := \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes n}$. Alternatively, consider the quotient of $\mathbb{A}^{n+1} \setminus \{0\}$ over the canonical action of $\mathbb{G}_m$. This decomposes $\mathbb{A}^{n+1} \setminus \{0\}$ as a space of punctured lines passing through the origin. If k is a field, $\mathbb{G}_m = k \setminus \{0\}$ and therefore by adding points (corresponding to 0) on each fibre, the fibres inherit a linear structure and what we get is $\mathcal{O}_{\mathbb{P}^n}(-1)$. Therefore, $\mathcal{O}_{\mathbb{P}^n}(-1)$ is basically the blow-up of $\mathbb{A}^{n+1}$ in the origin, which is embedded in $\mathbb{P}^n \times \mathbb{A}^{n+1}$. The line bundles $\mathcal{O}_{\mathbb{P}^n}(m)$, $\mathcal{O}_{\mathbb{P}^n}(n)$, $m \neq n$ are easily seen to be non-isomorphic. If k is a UFD, then every line bundle on X is isomorphic to some $\mathcal{O}_{\mathbb{P}^n}(n)$.[24] If $X \hookrightarrow \mathbb{P}^n$ is a projective variety, $\mathcal{O}_X(n) := \mathcal{O}_{\mathbb{P}^n}(n)\mid_X$ are line bundles over X.

Unlike the affine case, the homogeneous coordinate ring of a projective variety actually depends on the embedding of the variety in the projective space; i.e. isomorphic projective varieties do not necessarily have isomorphic homogeneous coordinate rings (e.g. the conic in $\mathbb{P}^2$ cut by $x^2+y^2=z^2$ is isomorphic to $\mathbb{P}^1$ but they have non-isomorphic homogeneous coordinate rings). The proof of coordinate-independentness of the coordinate ring in the affine case also fails to work. In fact, an important theorem is that the only regular maps on a projective variety are the constant maps.[23] Given an embedding $\varphi : X \hookrightarrow \mathbb{P}^n$ of a projective variety, one can get the pullback line bundle $\varphi^*\mathcal{O}_X(1)$.

Reminder. A line bundle/invertible sheaf L over a projective variety X is said to be very ample if there exists some immersion $\varphi : X \hookrightarrow \mathbb{P}^n$ such that $\varphi^*\mathcal{O}_X(1) \cong L$. It is said to be ample if there exists a tensor power $L^{\otimes n}$ of L that is very ample. Ampleness of a line bundle is preserved under pullback by a morphism, which is not the case about very ampleness. Very ampleness is preserved under pullback by a closed immersion.[11]

Given a very ample line bundle L over X, one defines an embedding of X in a projective space $X \hookrightarrow \mathbb{P}(H^0(X, L)^\vee)$ given by $x \mapsto \left[s \mapsto s(x) \in L_x \cong \mathbb{A}^1, \forall s \in H^0(X, L)\right] \in H^0(X, L)^\vee, \forall x \in X$. Therefore, equipping a projective variety with an embedding is equivalent to choosing a very ample line bundle over it.

Reminder. Given a line bundle $\pi : L \to X$, a function $f : L \to k$ of degree d on fibres of L (that is; satisfying $f(\lambda x) = \lambda^d f(x), \forall x \in L_y$) gives a section $s_f \in H^0(X, (L^\vee)^{\otimes d})$ given by $s_f(x)(v^{\otimes d}) = f(v), \forall x \in X, v \in L_x$ (note that a section of $(L^\vee)^{\otimes d}$ maps to each $x \in X$ an element of $((L^\vee)^{\otimes d})_x = ((L^\vee)_x)^{\otimes d} = ({L_x}^\vee)^{\otimes d} = ({L_x}^{\otimes d})^\vee$). Conversely, given a section $s \in H^0(X, (L^\vee)^{\otimes d})$, one can get a function f_x on L given by $f_s(v) = s(\pi(v))(v^{\otimes d})$ that is of degree d on the fibres. Take a homogeneous polynomial P on the projective space $\mathbb{P}(V)$. Given a fibre $\{\lambda v \in V \mid \forall \lambda \in k, v \in V\}$ of the tautological line bundle $\mathcal{O}_{P(V)}(-1)$, it is clear that P defines a function of degree d on this fibre, where $d = \deg P$. Therefore, homogeneous polynomials of degree d on V lift to functions on $\mathcal{O}_{P(V)}(-1)$ that are of degree d on each fibre. Equivalently, any such function corresponds to a section of $(\mathcal{O}_{P(V)}(-1)^\vee)^{\otimes d} = \mathcal{O}_{P(V)}(d)$. Given a projective variety X, this implies that homogeneous polynomials of degree d correspond to sections of $\mathcal{O}_{\mathbb{P}^n}(d) \mid_X = \mathcal{O}_X(d)$. Therefore, corresponding to the canonical grading of homogeneous polynomials over $\mathbb{P}^n$, one can form the direct sum $\bigoplus_{d\geq 0} H^0(X, \mathcal{O}_X(d))$, graded by the canonical isomorphism $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n)$. This is the homogeneous coordinate ring of the naturally polarized projective variety $(X, \mathcal{O}_X(1))$. Remind that a variety is polarized if it is equipped with an ample line bundle. Similar to this construction, one can instead use any ample line bundle L over X. Given an arbitrary ample line bundle L over X, one can construct $R(X, L) = \bigoplus_{d\geq 0} H^0(X, L^{\otimes d})$ for any polarized variety (X, L). In fact, the previous definition of homogeneous coordinate ring coincides with $R(X, \mathcal{O}_X(1))$.

We now try to construct a moduli space of the action of a reductive group on a projective variety. In the affine case, an algebraic action on the affine variety X always lifts to an action of the affine coordinate ring $k[X]$. So we can expect the moduli space to be an affine variety whose coordinate ring consists of G-invariant functions on X; namely $k[X]^G$. In the projective case, we take a polarized projective variety (X, L) along with its homogeneous coordinate ring $R(X, L) = \bigoplus_{d\geq 0} H^0(X, L^{\otimes d})$. Given a projective variety $X \subseteq \mathbb{P}(V)$, there is a canonical action $\mathrm{GL}(V) \circlearrowright \mathbb{P}(V)$.

Definition 20. Given a quasiprojective variety $X \subseteq \mathbb{P}(V)$, an action $G \circlearrowright X$ is said to be linear if there exists a homomorphism $\phi : G \to \mathrm{GL}(V)$ such that $g.x = \phi(g).x, \forall g \in G, x \in X$ where $\phi(g)$ acts on $\mathbb{P}(V)$ canonically.

Remark 8. By definition, a linear action $G \circlearrowright X$ is the restriction of a linear action $G \circlearrowright \mathbb{P}(V)$. A linear action $G \circlearrowright X$ naturally lifts to an action on its affine cone V by $g.v := \phi(g)(v), \forall g \in G, v \in V$. Hence, it also acts on the tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(-1)$. Furthermore, G also acts on the dual vector space V^* by

$g.\alpha \in V^*$, $(g.\alpha)(v) := \alpha(g^{-1}.v) = \alpha(\phi(g^{-1})(v))$, $\forall g \in G$, $\alpha \in V^*$, $v \in V$.

One can see that remark 8 implies that given a linear action $G \circlearrowright X$, this group also acts on the tautological line bundle $\mathcal{O}_{\mathbb{P}(V^*)}(-1) \cong \mathcal{O}_{\mathbb{P}(V)}(1)$ and its restriction $\mathcal{O}_X(1)$, and this action is linear on the fibres (since it comes from a linear map on $\mathbb{P}(V)$) and the structure morphism $\pi : \mathcal{O}_X(1) \to X$ is G-equivariant. If X is equipped with a very ample line bundle (instead of an embedding into a projective space), this amounts to saying that this action lifts to an action on L which is linear on the fibres. Moreover, the structure morphism $\pi : L \to X$ is G-equivariant. Since π is G-equivariant, this action defines a linear map of fibres $g_x := (g.-)\mid_{L_x}: L_x \to L_{g.x}$, $\forall x \in X$, $g \in G$ where L_x indicates the fibre of L over x. Now let $s \in H^0(X, L^{\otimes d})$ be a given global section of $L^{\otimes d}$. Then we can define a new global section $g.s \in H^0(X, L^{\otimes d})$ defined by $(g.s)(x) := (g_{g^{-1}.x})^{\otimes d}(s(g^{-1}.x)) \in {L_x}^{\otimes d} \cong {L^{\otimes d}}_x$. So G also acts canonically on $H^0(X, L^{\otimes d})$ and therefore on $R(X, L) = \bigoplus_{d \geq 0} H^0(X, L^{\otimes d})$ while respecting the graded structure. Thus, to construct a moduli space of the projective G-variety X, we expect the resulting variety to have a coordinate ring consisting of the G-invariant sections of the homogeneous coordinate ring; i.e. $R(X, L)^G = \bigoplus_{d \geq 0} H^0(X, L^{\otimes d})^G$.

Definition 21. Given a polarized projective G-variety (X, L), an action $G \circlearrowright L$ is a lift of $G \circlearrowright X$ only if π is G-equivariant. If a lifted action $G \circlearrowright L$ is also linear on the fibres of L, it said to be a G-linearization of the action $G \circlearrowright X$ with respect to L; i.e. the induced map $(g.-)\mid_{L_x}: L_x \to L_{g.x}$ is a linear automorphism for any $g \in G$, $x \in X$. That is, the induced action of G must commute with the natural $\mathbb{G}_m$-action on the fibres: $g.(a.l) = a.(g.l)$, $\forall g \in G$, $a \in \mathbb{G}_m$, $l \in L$.

Remark 9. Note that to be able to construct a canonical action $G \circlearrowright R(X, L)$, we only need the action $G \circlearrowright X$ to lift to an action $G \circlearrowright L$ that is linear on the fibres and such that π is G-invariant. Therefore, instead of requiring the action $G \circlearrowright X$ to be linear, we only require that L admits a G-linearization of the action $G \circlearrowright X$.

Let $G \circlearrowright X$ be given by $\sigma : G \times X \to X$. We denote by $\pi_2 : G \times X \to X$ the projection on the second component. For any $g \in G$, we define the map $g \times \mathrm{id}_X : X \to G \times X$ given by $x \mapsto (g, x)$. Therefore, one can get the pullback line bundles $(g \times \mathrm{id}_X)^* \circ {\pi_2}^*(L) = (\pi_2 \circ (g \times \mathrm{id}_X))^*(L) = {\mathrm{id}_X}^*(L) = L$ and $(g \times \mathrm{id}_X)^* \circ \sigma^*(L) = (\sigma \circ (g \times \mathrm{id}_X))^*(L) = g^*(L)$, where g^* denotes the pullback by the multiplication morphism given by $x \mapsto g.x$, $\forall x \in X$. This implies that any morphism $\varphi : \sigma^*(L) \to {\pi_2}^*(L)$ of line bundles over $G \times X$ induces morphisms $\varphi_g : g^*(L) \to L$, $\forall g \in G$ of line bundles over X.

Lemma 19. Given a projective G-variety X and a line bundle L over X, there is a bijective correspondence between G-linearizations of L and isomorphisms $\varphi : \sigma^*(L) \to {\pi_2}^*(L) = G \times L$ of line bundles over $G \times X$ such that $\varphi_{g_1 g_2} = \varphi_{g_2} \circ {g_2}^*(\varphi_{g_1})$, $\forall g_1, g_2 \in G$.

Proof. Let $\widetilde{\sigma} : G \times L \to L$ be a G-linearization. Recall that $\sigma^*(L) = (G \times X) \times_X L$ and that the diagram

$$\begin{tikzcd} G \times L \arrow[r, "\widetilde{\sigma}"] \arrow[d, "\mathrm{id}_X \times \pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\sigma"'] & X \end{tikzcd}$$

is commutative. By the universal property of the fibre product, this induces a morphism $\varphi : G \times L \to \sigma^*(L)$. Note that since $\widetilde{\sigma}$ is linear on fibres of L, so is γ. Therefore, the induced morphism $\varphi : G \times L = {\pi_2}^*(L) \to \sigma^*(L)$ is a morphism of line bundles. Since the induced maps $\varphi_g : L \to g^*(L)$, $\forall g \in G$ are all isomorphisms, the morphism $\varphi : {\pi_2}^*(L) \to \sigma^*(L)$ is also an isomorphism. Finally, note that the relation $\widetilde{\sigma}(g_1, \widetilde{\sigma}(g_2, l)) = \widetilde{\sigma}(g_1 g_2, l)$ implies that composing the induced maps ${g_2}^*(\varphi_{g_1}) : {g_2}^*(L) \to {g_2}^*({g_1}^*(L)) = (g_1 g_2)^*(L)$ and $\varphi_{g_2} : L \to {g_2}^*(L)$ is the same as applying $\varphi_{g_1 g_2} : L \to (g_1 g_2)^*(L)$. Therefore, $\varphi_{g_1 g_2} = {g_2}^*(\varphi_{g_1}) \circ \varphi_{g_2}$ and defining the isomorphism $\psi := \varphi^{\text{-}1} : \sigma^*(L) \to {\pi_2}^*(L)$, we get the cocycle condition $\psi_{g_1,\, g_2} = \psi_{g_2} \circ {g_2}^*(\psi_{g_1})$ for all $g \in G$. For proving the converse, note that an isomorphism $\varphi : \sigma^*(L) \to {\pi_2}^*(L)$ extends the fibre product diagram of $\sigma^*(L)$ as follows:

$$
\begin{tikzcd}
\begin{array}{c} {\pi_2}^*(L) \\ = G \times L \end{array} \arrow[dr, "\varphi"] \arrow[drr, dotted, bend left] \arrow[ddr, dotted, bend right] & & \\
& \sigma^*(L) \arrow[r] \arrow[d] & L \arrow[d, "\pi"] \\
& G \times X \arrow[r, "\sigma"'] & X
\end{tikzcd}
$$

This constructs a map $G \times L \to L$ that is a lift of σ. The cocycle condition then similarly proves that associativity of this map, and therefore gives rise to an action $G \times L \circlearrowright L$ that is linear on the fibres. $\square$

Remark 10. A linearization $G \circlearrowright L$ thus corresponds to an isomorphism of line bundles over $G \times X$ given by $\varphi : \sigma^*(L) \to {\pi_2}^*(L)$. This then induces a dual action of $H^0(G, \mathcal{O}_G) \circlearrowright H^0(X, L)$ by composing the following:

$$H^0(X, L) \xrightarrow{\sigma^*} H^0(G \times X, \sigma^*(L)) \xrightarrow{\varphi} H^0(G \times X, {\pi_2}^*(L)) \xrightarrow{\cong} H^0(G, \mathcal{O}_G) \otimes H^0(X, L)$$

where the last isomorphism is an application of the Kunneth formula.

Definition 22. Since the tensor product of G-linearized line bundles and the dual of a G-linearized line bundle both admit canonical G-linearizations, the set of isomorphism classes of all G-linearized line bundles over X form an Abelian group, named the equivariant Picard group of X and denoted by $\mathrm{Pic}_G(X)$. One can then consider the forgetful functor $\mathrm{Pic}_G(X) \to \mathrm{Pic}(X)$ that forgets G-linearization. It is then straightforward to see that a G-equivariant morphism of G-varieties $f : X \to Y$ induces a homomorphism $f^* : \mathrm{Pic}_G(Y) \to \mathrm{Pic}_G(X)$.

Thus, given a G-linearized line bundle L over a G-variety X, the algebraic group G also acts on $H^0(X, L^{\otimes d})$, $\forall d \geq 0$ and therefore on $R(X, L) = \bigoplus_{d \geq 0} H^0(X, L^{\otimes d})$, which corresponds to ring of homogeneous polynomials on X. Similar to the affine case, we expect the moduli space $X /\!\!/ G$ to be a projective variety whose homogeneous coordinate ring consists of G-invariant homogeneous polynomials on X:

Definition 23. Let G be a reductive algebraic group and X be a projective G-variety. Let L be an ample, G-linearized line bundle over X. The inclusion of coordinate rings $\bigoplus_{d \geq 0} H^0(X, L^{\otimes d})^G \hookrightarrow \bigoplus_{d \geq 0} H^0(X, L^{\otimes d})$ induces a rational morphism $X \cong \mathrm{MaxProj} \bigoplus_{d \geq 0} H^0(X, L^{\otimes d}) \to \mathrm{MaxProj} \bigoplus_{d \geq 0} H^0(X, L^{\otimes d})^G$. We define the projective GIT quotient $X /\!\!/_L G$ as $\mathrm{MaxProj} \bigoplus_{d \geq 0} H^0(X, L^{\otimes d})^G$. This rational morphism is undefined on the null cone $N_{R(X,\, L)^G}(X) := \{x \in X \mid f(x) = 0, \forall f \in R(X, L)^G_+\}$, which is a Zariski-closed set. We define the semistable locus X^{ss} as complement of the null cone: $X^{ss} := X \setminus N_{R(X,\, L)^G}(X)$. The quotient morphism of the projective GIT quotient is the regular morphism $X^{ss} \to X /\!\!/_L G = \mathrm{MaxProj} \bigoplus_{d \geq 0} H^0(X, L^{\otimes d})^G$.

For this construction to work, we require that $\bigoplus_{d \geq 0} H^0(X, L^{\otimes d})^G$ is a finitely generated, graded k-algebra with no zero divisors. The graded structure is obvious and it is clear that this ring is reduced. So it suffices to prove:

Theorem 18. If G is geometrically reductive, then the graded ring $\bigoplus_{d\geq 0} H^0(X,\, L^{\otimes d})^G$ is finitely generated.

Proof. First note that the action $G \circlearrowright \bigoplus_{d\geq 0} H^0(X,\, L^{\otimes d})$ is defined on each of the homogeneous components. Therefore, one can see that: $R(X,\, L)^G = \left(\bigoplus_{d\geq 0} H^0(X,\, L^{\otimes d})\right)^G = \bigoplus_{d\geq 0} H^0(X,\, L^{\otimes d})^G$. Note that $R(X,\, L)$ is finitely generated and Noetherian. Therefore, each $H^0(X,\, L^{\otimes d})$ is finitely generated. By theorem 6, one can see that $R(X,\, L)^G$ is also finitely generated. □

Theorem 19. Let G be a reductive algebraic group and X be a projective G-variety. Let L be an ample, G-linearized line bundle over X. Then the projective GIT quotient $\varphi : X^{ss} \to X /\!\!/_L G$ is a good G-quotient.

Proof. For any $f \in R(X,\, L)^G$, note that $k[\varphi(\mathrm{supp}(f))]$ is isomorphic to the degree zero homogeneous piece of the localization $(R(X,\, L)^G)_f$. One can see that $((R(X,\, L)^G)_f)_0 = ((R(X,\, L)_f)^G)_0 = ((R(X,\, L)_f)_0)^G$. Finally, $((R(X,\, L)_f)_0)^G = k[\mathrm{supp}(f)]^G$. Therefore, $\varphi(\mathrm{supp}(f)) \cong \mathrm{MaxSpec}(k[\varphi(\mathrm{supp}(f))]) \cong \mathrm{MaxSpec}(k[\mathrm{supp}(f)]^G)$ which is essentially $\mathrm{supp}(f) /\!\!/ G$. Thus, the map $\varphi\,|_{\mathrm{supp}(f)} : \mathrm{supp}(f) \to \varphi(\mathrm{supp}(f)) \cong \mathrm{supp}(f) /\!\!/ G$ is the affine GIT quotient map and is therefore a good quotient by theorem 13. Since $\bigcup_{f\in R(X,\, L)^G} \mathrm{supp}(f) = X^{ss}$ and $\bigcup_{f\in R(X,\, L)^G} \varphi(\mathrm{supp}(f)) = X /\!\!/_L G$, the claim follows immediately by proposition 8. □

Note that during the proof of theorem 19, we have implicitly used the fact that since L is ample, some power $L^{\otimes r}$ is very ample and embeds X into $\mathbb{P}(H^0(X,\, L^{\otimes r})^\vee)$ by a choice of basis for $H^0(X,\, L^{\otimes r})$. Furthermore, the support $\mathrm{supp}(s) = \{x \in X \mid s(x) \neq 0\}$ of any nonzero section $s \in H^0(X,\, L^{\otimes r})$ is the intersection of the embedded variety with the standard affine chart $\{x \in \mathbb{P}(H^0(X,\, L^{\otimes r})^\vee) \mid s(x) \neq 0\}$, and hence is affine. By the correspondence of (invariant) sections of tensor powers of L with (invariant) elements of the homogeneous coordinate ring, we deduce that for any $f \in R(X,\, L)^G$, the set $\mathrm{supp}(f)$ is affine. One can now prove an analogue of theorem 14 in the projective case:

Definition 24. Given a projective variety X and ample, G-linearized line bundle L over X, a point $x \in X$ is:

1. semistable if the orbit $G.x$ is generated by the invariant sections of some tensor power of L; i.e. there exists some invariant global section $s \in H^0(X,\, L^{\otimes n})^G$ for some $n \geq 1$ such that $s(x) \neq 0$. Equivalently, there exists some $f \in R(X,\, L)^G_+$ such that $f(x) \neq 0$. We denote the semistable locus by X^{ss}.
2. polystable if it is semistable by some invariant section $s \in H^0(X,\, L^{\otimes n})^G$ (equivalently by some invariant function $f \in R(X,\, L)^G_+$), and that the induced action of G on $\mathrm{supp}(s) = \{x \in X \mid s(x) \neq 0\}$ (equivalently $\mathrm{supp}(f) = \{x \in X \mid f(x) \neq 0\}$) is closed. We denote the polystable locus by X^{ps}.
3. stable if it is polystable and $\dim G_x = 0$. We denote the stable locus by X^s.

Note that substituting L with a tensor power does not change the sets X^s, X^{ss}. Therefore, we can always assume the G-linearization L to be very ample. Controlling the dimension of the fibres by the stability condition, we can assure (following the proof in the affine case) that $\varphi(X^s) \subseteq X /\!\!/_L G$ and consequently $X^s = \varphi^{-1}(\varphi(X^s)) \subseteq X^{ss}$ are open and the restriction of the GIT quotient to the stable locus is a geometric quotient. Also, similar to the first statement of theorem 12, one can also prove that $\overline{G.x_1} \cap \overline{G.x_2} \cap X^{ss} \neq \emptyset$ if and only if $\varphi(x_1) = \varphi(x_2)$.

In the most general settings, one could also remove the ampleness condition from L. In fact, for theorem 19 to hold, one only requires that the sets $\mathrm{supp}(f)$ are affine. Therefore, we can lift the ampleness of L and instead require the semistable (and therefore stable) locus to also ensure that the support $\mathrm{supp}(f)$ of the corresponding $f \in R(X,\, L)^G_+$ (that satisfies $f(x) \neq 0$) is affine. In this manner, theorem 19 automatically holds.

Reminder. Given varieties X, Y, a map $f : X \to Y$ is rational if it is regular on an open subset of X (since a variety is irreducible, a nonempty open subset of X is dense in X. Therefore, rational functions are regular almost everywhere). A rational map $f : X \to Y$ is birational if it admits a rational inverse.

Reminder. Given a variety X, one can define the Zariski-Riemann space of X as the inverse limit of the system $\underline{X} := \varprojlim(\pi : Y \to X)$ where the limit is taken over all birational morphisms $\pi : Y \to X$, ordered as follows: $(\pi_1 : Y_1 \to X) \geq (\pi_2 : Y_2 \to X)$ if there exists a birational morphism $\phi : Y_1 \to Y_2$ that lifts π_1 to π_2; i.e. $\pi_1 = \pi_2 \circ \phi$. Note that this is a (locally) ringed space as the projective limit of (locally) ringed spaces (c.f. [7] 4.1.10). Birational varieties are easily shown to have isomorphic (locally ringed spaces) Zariski-Riemann spaces. Moreover, $\underline{X}$, $\underline{Y}$ are isomorphic as locally ringed spaces if and only if X, Y are birational.

The following is the merit of variational GIT. The interested reader is referenced to [6],[26],

Proposition 9. There are finitely many non-isomorphic GIT quotients $X /\!\!/_L G$. All GIT quotients $X /\!\!/_L G$ for different ample G-linearized line bundles L are birational. The family $Y /\!\!/_{L_Y} G$ of all GIT quotients of a variety Y birationally equivalent to X form an inverse system. The inverse limit of this system, $\varprojlim Y /\!\!/_{L_Y} G$ is then isomorphic to the Zariski-Riemann space $\underline{X /\!\!/_L G}$ of any GIT quotient. (c.f. [6])

Reminder. A locally ringed space isomorphic (as locally ringed spaces) to $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for a commutative ring R is said to be an affine scheme. A locally ringed space $(X, \mathcal{O}_X)$, where X, $\mathcal{O}_X$ are the underlying topological space and the structure sheaf of rings, is said to be a scheme if any $x \in X$ admits a neighborhood $U \subseteq X$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. Similarly, a locally ringed space isomorphic (as locally ringed spaces) to $(\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)})$ for a commutative ring R is said to be a projective scheme. A projective scheme is also a scheme. An open subscheme of a projective scheme is said to be a quasi-projective scheme. A scheme is connected (resp. irreducible) if the underlying topological space is connected (resp. irreducible). A scheme is said to be reduced (resp. integral) if for any open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements (resp. is an integral domain). Any scheme is integral if and only if it is both reduced and irreducible. A morphism of schemes is a morphism of locally ringed spaces. Given a scheme Y, a scheme X is said to be a scheme over Y if it is equipped with a morphism of schemes $X \to Y$, named as structure morphism. Given a morphism of schemes $\varphi : X \to Y$, it is said to be locally of finite type if there exists a covering $Y = \bigcup_{i\in I} \mathrm{Spec}(Y_i)$ of Y by open affine schemes such that for all $i \in I$, there exists a covering $\varphi^{-1}(\mathrm{Spec}(Y_i)) = \bigcup_{j\in J_i} \mathrm{Spec}(X_{ij})$ of $\varphi^{-1}(\mathrm{Spec}(Y_i))$ by open affine schemes such that for all $i \in I$, $j \in J_i$, the commutative ring X_{ij} is a finitely generated Y_i-algebra. It is furthermore said to be of finite type if all J_i's are finite. A scheme X over Y is said to be of finite type if its structure morphism is of finite type. It is well-known that the schemes $\mathrm{Spec}(R)$, $\mathrm{Proj}(R)$ are reduced (resp. of finite type over k) if and only if R is reduced (resp. finitely generated over k). An abstract variety is an integral, separated scheme of finite type over an algebraically closed field.

Remark 11. We can extend our theory of GIT quotients to affine and projective schemes. Given an affine G-scheme $\mathrm{Spec}(R)$, we can simply define $X /\!\!/ G := \mathrm{Spec}(R^G)$, which is an affine G-scheme. Similarly, given a projective G-scheme $\mathrm{Proj}(R)$ linearized by an ample line bundle L, one can define $X /\!\!/_L G := \mathrm{Proj} \bigoplus_{d\geq 0} H^0(X, L^{\otimes d})^G$. Note that $X /\!\!/ G$ (resp. $X /\!\!/_L G$) is an abstract variety if X itself is an abstract variety. One can similarly prove (by taking affine charts) that the resulting quotient is still a good quotient and the restriction to the stable locus would result in a geometric quotient. The case is similar for quasi-projective schemes.

7 Hilbert Mumford Numerical Criterion for Stability

Consider a geometrically reductive group G acting linearly on a projective variety $X \subseteq \mathbb{P}(V)$ by a group homomorphism $\phi : G \to \mathrm{GL}(V)$. Given a one-parameter subgroup of G (i.e. a group homomorphism $\lambda : \mathbb{G}_m \to G$), one can restrict the action $G \circlearrowright \mathbb{P}(V)$ to an action $\mathbb{G}_m \circlearrowright \mathbb{P}(V)$. Since $\mathbb{G}_m$ is represented by the Abelian algebra $k[X, X^{-1}]$, the action $\mathbb{G}_m \circlearrowright \mathbb{P}(V)$ is diagonalizable; i.e. can be written as sum of one-dimensional representations. Therefore, the linear transformations $\phi(\lambda(t)) \in \mathrm{GL}(V)$ admit a common set of eigenvectors $\{v_1, \cdots, v_{\dim V}\} \subseteq \mathbb{V}$ such that $\phi(\lambda(t))(v_i) = t^{r_i} v_i, \forall t \in \mathbb{G}_m$ where $r_i \in \mathbb{Z}$. Given some $x \in \mathbb{P}(V)$, choose a nonzero lift $\widetilde{x} = \sum x^i e_i$ over x and define $\mu(x, \lambda) = -\min\{r_i \mid x_i \neq 0\}$. One can see that this value does not depend on the choice of a lift $\widetilde{x}$. This is called the Hilbert-Mumford weight of x at λ.

8 Vanishing Theorems and Mumford-Castelnuovo Regularity

Definition 25. Let $(X, \mathcal{O}_X)$ be a ringed topological space and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. $\mathcal{F}$ is said to be:

1. generated by its global sections at $x \in X$ if the homomorphism $\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_{X,x} \to \mathcal{F}_x$ is surjective.
2. generated by its global sections if it is generated by its global sections at every $x \in X$.
3. finitely generated if for every $x \in X$, there exists an open neighborhood U of x and a surjective homomorphism of sheaves of $\mathcal{O}_X$-modules ${\mathcal{O}_X}^n|_U \to \mathcal{F}|_U$ for positive integer $n \geq 1$.

Remark 12. $\mathcal{F}$ is generated by its global sections if and only if there exist a surjective homomorphism of sheaves of $\mathcal{O}_X$-modules ${\mathcal{O}_X}^{\oplus I} \to \mathcal{F}$ for some indexing set I (for proof , see [16] §5.1.1).

Definition 26. Let $(X, \mathcal{O}_X)$ be a ringed topological space and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. $\mathcal{F}$ is said to be:

1. quasi-coherent if for every $x \in X$, there exists some neighborhood U of x and indexing sets I, J (possibly infinite) such that the sequence ${\mathcal{O}_X}^{\oplus J}|_U \cong {\mathcal{O}_U}^{\oplus J} \to {\mathcal{O}_X}^{\oplus I}|_U \cong {\mathcal{O}_U}^{\oplus I} \to \mathcal{F}|_U \to 0$ is exact.
2. coherent if it is finitely generated and that for every open subset $U \subseteq X$ and every homomorphism $\varphi : {\mathcal{O}_X}^n|_U \to \mathcal{F}|_U$ for some positive integer $n \geq 1$, the kernel of this homomorphism is finitely generated.

Remark 13. Suppose $(X, \mathcal{O}_X)$ is a ringed topological space and $\mathcal{F}$ is a locally free sheaf of $\mathcal{O}_X$-modules. Note that $\mathcal{O}_X$ is clearly a quasi-coherent sheaf. If $\mathcal{O}_X$ is furthermore coherent, then $\mathcal{O}_X|_U$ is also coherent for every open subset $U \subseteq X$. Since $\mathcal{F}$ is locally free, $\mathcal{F}|_U \cong {\mathcal{O}_X}^{\oplus I}|_U \cong (\mathcal{O}_X|_U)^{\oplus I}$ and is therefore coherent. It is also well known that the structure sheaf of quasi-projective varieties (as well as all locally Noetherian schemes) is coherent. Therefore, we discuss in the case of coherent sheaves and note that the results also hold for locally free sheaves, as long as the underlying space is a quasi-projective variety.

Proposition 10. Let $(X, \mathcal{O}_X(1) = L)$ be a polarized projective variety where L is an ample line bundle. Then for any coherent sheaf $\mathcal{F}$, there exists an integer n_0 such that for all $n \geq n_0$, the twisted sheaf $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes n} = \mathcal{F} \otimes \mathcal{O}_X(n)$ is generated by (a finite number of) its global sections. (c.f. [28] Theorem 16.2.2)

This implies that the canonical evaluation map $\mathrm{ev} : H^0(X, \mathcal{F}(n)) \otimes \mathcal{O}_X \to \mathcal{F}(n)$ is surjective. This gives the following short exact sequence of sheaves of $\mathcal{O}_X$-modules:

$$0 \longrightarrow \ker(\mathrm{ev}) \hookrightarrow H^0(X, \mathcal{F}(n)) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} \mathcal{F}(n) \longrightarrow 0$$

We can then fix an isomorphism $H^0(X, \mathcal{F}(n)) \cong \mathbb{C}^{\dim H^0(X, \mathcal{F}(n))}$. The short exact sequence then becomes:

$$0 \longrightarrow \ker(\mathrm{ev}) \hookrightarrow \mathbb{C}^{\dim H^0(X, \mathcal{F}(n))} \otimes \mathcal{O}_X \cong \mathcal{O}_X{}^{\oplus \dim H^0(X, \mathcal{F}(n))} \longrightarrow \mathcal{F}(n) \longrightarrow 0$$

Untwisting this sequence, we get the following exact sequence:

$$0 \longrightarrow \ker(\mathrm{ev}) \otimes \mathcal{O}_X(-n) \hookrightarrow \mathcal{O}_X{}^{\oplus \dim H^0(X, \mathcal{F}(n))} \otimes \mathcal{O}_X(-n) \cong \mathcal{O}_X(-n)^{\oplus \dim H^0(X, \mathcal{F}(n))} \longrightarrow \mathcal{F}(n) \otimes \mathcal{O}_X(-n) \cong \mathcal{F} \longrightarrow 0$$

Therefore, we can express $\mathcal{F}$ as a quotient of $\mathcal{O}_X(-n)^{\oplus \dim H^0(X, \mathcal{F}(n))}$ (up to a choice of isomorphism for $H^0(X, \mathcal{F}(n))$). Now we proceed to find a family of sheaves where n is uniformly bounded.

Theorem 20. (Serre's Vanishing Theorem) Given a polarized projective variety $(X, \mathcal{O}_X(1) = L)$ and a coherent sheaf $\mathcal{F}$ over X, there exists some integer n_0 such that for any $n \geq n_0$, $H^i(X, \mathcal{F}(n)) = 0$, $\forall i > 0$.

Proof. Since $\pi : X \hookrightarrow \mathbb{P}(H^0(X, L)^\vee)$ is a closed embedding, the natural map $H^i(\mathbb{P}(H^0(X, L)^\vee), \pi_*\mathcal{F}) \to H^i(X, \mathcal{F})$ is an isomorphism. Therefore, we only need to prove the claim for $X = \mathbb{P}^n$. According to proposition 10, there exists some integer m such that $\mathcal{F}(m)$ is generated by its global sections. By remark 12, we get a short exact sequence $0 \longrightarrow \ker(\varphi) \hookrightarrow \mathcal{O}^{\oplus I} \xrightarrow{\varphi} \mathcal{F}(m) \longrightarrow 0$. Untwisting, we get the short exact sequence $0 \longrightarrow \ker(\varphi) \otimes \mathcal{O}(-m) \hookrightarrow \mathcal{O}^{\oplus I} \otimes \mathcal{O}(-m) \cong \mathcal{O}(-m)^{\oplus I} \longrightarrow \mathcal{F}(m) \otimes \mathcal{O}(-m) \cong \mathcal{F} \longrightarrow 0$. Note that $\ker(\varphi) \otimes \mathcal{O}(-m)$ is also a coherent sheaf. Twisting by $\mathcal{O}(r)$ we get that $0 \longrightarrow \ker(\varphi) \otimes \mathcal{O}(r-m) \hookrightarrow \mathcal{O}(r-m)^{\oplus I} \xrightarrow{\varphi} \mathcal{F}(r) \longrightarrow 0$ is a short exact sequence of coherent sheaves. Note that $H^n(\mathbb{P}^n, \mathcal{O}(r-m)^{\oplus I}) = H^n(\mathbb{P}^n, \mathcal{O}(r-m))^{\oplus I} = 0$ for sufficiently large r (precisely whenever $r - m \geq -n$). Furthermore, we also have the long exact sequence:

$$0 \to H^0(\mathbb{P}^n, \ker(\varphi) \otimes \mathcal{O}(r-m)) \to H^0(\mathbb{P}^n, \mathcal{O}(r-m)^{\oplus I}) \to H^0(\mathbb{P}^n, \mathcal{F}(r)) \to H^1(\mathbb{P}^n, \ker(\varphi) \otimes \mathcal{O}(r-m)) \to \cdots$$

$$\to H^n(\mathbb{P}^n, \ker(\varphi) \otimes \mathcal{O}(r-m)) \to H^n(\mathbb{P}^n, \mathcal{O}(r-m)^{\oplus I}) = 0 \to H^n(\mathbb{P}^n, \mathcal{F}(r)) \to 0$$

Therefore, $H^n(\mathbb{P}^n, \mathcal{F}(r)) = H^n(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{O}(r)) = 0$. Since $\mathcal{F}$ can be taken to be any coherent sheaf, taking it to be $\ker(\varphi)$ one deduces that $H^n(\mathbb{P}^n, \ker(\varphi) \otimes \mathcal{O}(r))$ also vanishes. Since $H^i(\mathbb{P}^n, \mathcal{O}(r-m)) = 0$, $\forall 0 < i < n$, one can simply induct downwards to prove the claim. Note that since $\mathbb{P}^n$ is covered by $n+1$ affine open subsets, $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$, $\forall i > n$ and therefore all nonzero cohomologies of $\mathcal{F}(r)$ vanish. $\square$

Theorem 21. Given an exact sequence $\mathcal{F}_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{m-1}} \mathcal{F}_m$ of coherent sheaves over $\mathbb{P}^n$, there exists some integer k_0 such that for any $t \geq t_0$, the sequence of global sections $\mathcal{F}_1(t)(\mathbb{P}^n) \to \cdots \to \mathcal{F}_m(t)(\mathbb{P}^n)$ is exact.

Proof. The sequence breaks down to short exact sequences $0 \to \ker(\varphi_i) \hookrightarrow \mathcal{F}_i \twoheadrightarrow \mathrm{Im}(\varphi_i) = \ker(\varphi_{i+1}) \to 0$. By theorem 20, one can choose k_i such that $H^1(\mathbb{P}^n, \ker(\varphi_{i+1})(t)) = 0$ for all $t \geq t_i$. Let $t_0 = \max t_i$. Twisting by $\mathcal{O}(t)$, we get the short exact sequence $0 \to \ker(\varphi_i)(t) \to \mathcal{F}_i(t) \to \ker(\varphi_{i+1})(t) \to 0$. Taking cohomology, we get: $0 \to H^0(\mathbb{P}^n, \ker(\varphi_i)(t)) \to H^0(\mathbb{P}^n, \mathcal{F}_i(t)) \to H^0(\mathbb{P}^n, \ker(\varphi_{i+1})(t)) \to H^1(\mathbb{P}^n, \ker(\varphi_i)(t)) \to \cdots$ where $H^1(\mathbb{P}^n, \ker(\varphi_{i+1})(t))$ vanishes for $t \geq t_0$. In this case, we get a short exact sequence of the global sections given by $0 \to H^0(\mathbb{P}^n, \ker(\varphi_i)(t)) \xrightarrow{\alpha_i} H^0(\mathbb{P}^n, \mathcal{F}_i(t)) \xrightarrow{\beta_i} H^0(\mathbb{P}^n, \ker(\varphi_{i+1})(t)) \to 0$. Note that all α_i are injective and all β_i are surjective. Finally, one can check that it's possible to glue the short exact sequences to get the long exact sequence $H^0(\mathbb{P}^n, \mathcal{F}_1(t)) \to \cdots \to H^0(\mathbb{P}^n, \mathcal{F}_m(t))$ whenever $t \geq t_0$. $\square$

Definition 27. A coherent sheaf is said to be m-regular if $H^i(X, \mathcal{F}(m-i)) = 0, \forall i > 0$. We then define the Mumford-Castelnuovo regularity of $\mathcal{F}$ as $\operatorname{reg}(\mathcal{F}) = \inf\{m \in \mathbb{Z} \mid \mathcal{F} \text{ is m-regular}\}$.

Remark 14. Note that the set $\{m \in \mathbb{Z} \mid \mathcal{F} \text{ is m-regular}\}$ is non-empty by theorem 20 and therefore such infimum exists. Furthermore, $\operatorname{reg}(\mathcal{F}) = -\infty$ if and only if $\mathcal{F}$ is zero-dimensional.

Definition 28. Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules on X. A point $x \in X$ is said to be associated to $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_{X,x}$ is an associated prime of the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$. We denote the set of associated points of $\mathcal{F}$ by $\operatorname{Ass}_X(\mathcal{F})$.

Proposition 11. Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules on X. Let $\operatorname{Spec}(A) \subseteq X$ be an affine open of X. Note that $\mathcal{F}(\operatorname{Spec}(A))$ is by definition a $\mathcal{O}_X(\operatorname{Spec}(A)) = A$-module. Let $x \in \operatorname{Spec}(A)$ and let $\mathfrak{p} \lhd A$ be the corresponding prime ideal. Then $x \in \operatorname{Ass}_X(\mathcal{F})$ if $\mathfrak{p} \in \operatorname{Ass}_A(\mathcal{F}(\operatorname{Spec}(A))$. If $\mathfrak{p}$ is finitely generated, the converse also holds. This happens specifically when X is locally Noetherian.

Remark 15. Remind that given a Noetherian ring A, a finite A-module has a finite set of associated primes. Thus, the associated primes of a quasi-coherent sheaf of $\mathcal{O}_X$-modules on a locally Noetherian scheme is finite.

Theorem 22. Let $(X, \mathcal{O}_X)$ be a Noetherian scheme and $\mathcal{F}$ be a coherent sheaf on X. Let $\ell : \mathcal{O}_X \to \mathcal{O}_X(d)$ be a global section of $\mathcal{O}_X(d)$. This defines a morphism of sheaves $\ell_{\mathcal{F}} : \mathcal{F} \otimes \mathcal{O}_X \cong \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_X(d) = \mathcal{F}(d)$. Then $\ell_{\mathcal{F}}$ is an injective morphism of sheaves if and only if ℓ does not vanish on any points of $\operatorname{Ass}_X(\mathcal{F})$.

Proof. Select an affine open $U = \operatorname{Spec}(R)$. Then $\mathcal{F}\mid_U$ is isomorphic to the sheaf $\widetilde{M}$ corresponding to the finitely generated (due to the coherence of $\mathcal{F}$) R-module $M = \mathcal{F}(\operatorname{Spec}(R))$, by the functorial correspondence of quasi-coherent sheaves on $\operatorname{Spec}(R)$ and R-modules. Note that on an affine scheme, every invertible sheaf is free of rank one and therefore isomorphic to $\mathcal{O}_U$. Also, $\ell\mid_U : \mathcal{O}_X\mid_U = \mathcal{O}_U \to \mathcal{O}_X\mid_U(d) = \mathcal{O}_U(d) \cong \mathcal{O}_U$ corresponds (by the functorial correspondence) to an R-module homomorphism $f : R \to R$. This induces another module homomorphism $f' : M \cong M \otimes_R R \xrightarrow{\mathrm{id}_M \otimes f} M \otimes_R R \cong M$, corresponding to the morphism of sheaves $\ell_{\mathcal{F}}\mid_U : \mathcal{F}\mid_U \to \mathcal{F}\mid_U(d) \cong \mathcal{F}\mid_U$. Clearly, a module homomorphism $f : R \to R$ is given by multiplication by some element $r \in R$. Note that $\ell_{\mathcal{F}}$ is injective if and only if all $\ell_{\mathcal{F}}\mid_U$ are injective (since a morphism of sheaves is injective if and only if it is injective on the stalks). Furthermore, $\ell_{\mathcal{F}}\mid_U$ is injective if and only if the module homomorphism f' given by multiplication by r is injective, which is equivalent to r not being a zero divisor of M. On the other hand, any $x \in U = \operatorname{Spec}(R)$ is associated to $\mathcal{F}$ if and only if the corresponding prime ideal $\mathfrak{p} \lhd R$ is associated to the R-module M (since X is a Noetherian scheme). Finally, note that the set of zero divisors of an R-module M is the union of the associated primes of M, if R is a Noetherian ring. □

Lemma 20. Let $(X, \mathcal{O}_X(1) = L)$ be a polarized projective variety and $\mathcal{F}$ be an m-regular sheaf on X. Let $H \xhookrightarrow{i} X$ be a hyperplane that contains no associated point of $\mathcal{F}$. Then $\mathcal{F}\mid_H = i^*\mathcal{F} = \mathcal{F} \otimes \mathcal{O}_H$ is also m-regular.

Proof. Let $s \in H^0(X, \mathcal{O}_X(1))$ denote the global section of the hyperplane bundle corresponding to H. Note that H is the zero locus of this global section. By theorem 22, since s does not vanish on any associated points of $\mathcal{F}$, the morphism of sheaves $s : \mathcal{O}_X \to \mathcal{O}_X(1)$ is injective and we get an exact sequence of coherent sheaves:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\ s\ } \mathcal{O}_X(1) \longrightarrow \mathcal{O}_X(1) \mid_H \longrightarrow 0$$

Twisting by $\mathcal{F}(m-i-1)$, we get the following exact sequence of coherent sheaves:

$$0 \longrightarrow \mathcal{F}(m-i-1) \xrightarrow{\ s\ } \mathcal{F}(m-i) \longrightarrow \mathcal{F}(m-i) \mid_H \longrightarrow 0$$

Taking cohomology, we get the long exact sequence of coherent sheaves:

$$\cdots \to H^i(X, \mathcal{F}(m-i-1)) \to H^i(X, \mathcal{F}(m-i)) \to H^i(X, \mathcal{F}(m-i) \mid_H) \to H^{i+1}(X, \mathcal{F}(m-i-1)) \to \cdots$$

Since $\mathcal{F}$ is m-regular, we know that $H^i(X, \mathcal{F}(m-i)) = H^{i+1}(X, \mathcal{F}(m-i-1)) = 0$. This implies that, $H^i(X, \mathcal{F}(m-i) \mid_H) = H^i(X, \mathcal{F} \mid_H (m-i)) = 0$. This completes the proof since it is true for all $i > 0$. $\square$

Theorem 23. (Castelnuovo Theorem) Given a m-regular sheaf $\mathcal{F}$ over a projective variety $(X,\, \mathcal{O}_X(1) = L)$:

1. The canonical morphism $H^0(X,\, \mathcal{O}_X(1)) \otimes H^0(X,\, \mathcal{F}(n)) \to H^0(X,\, \mathcal{F}(n+1))$ is surjective for $n \geq m$.
2. The sheaf $\mathcal{F}$ is $(m+1)$-regular. Equivalently, $H^i(X,\, \mathcal{F}(j)) = 0,\ \forall i + j \geq m,\ i > 0$.
3. The sheaf $\mathcal{F}(n)$ is generated by its global sections for all $n \geq m$.

Proof. We begin by proving 1,2. Note that as in the proof of theorem 20, we only need to prove these statements for $X = \mathbb{P}^n$. We begin by induction on n. The base case clearly holds for $n = 0$. Since the underlying field k is algebraically closed, it is automatically infinite (one can even lift the condition of k being algebraically closed. Specifically, by base change to the algebraic closure, one can still assume without loss of generality that the underlying field is algebraically closed and therefore infinite). Hence, one can choose a hyperplane H not containing any of the associated points of $\mathcal{F}$. By lemma 20, $\mathcal{F}\mid_H$ is also an m-regular sheaf. Since $\dim(H) = n-1$, the induction hypothesis indicates that these statements hold for $\mathcal{F}\mid_H$.

2. We proceed by induction on $i + j$. The base of induction is clear for $i + j = m$ as a consequence of the m-regularity. Assume the claim holds for all $i,\, j$ such that $i+j \leq m+k$. Let $i',\, j'$ be such that $i'+j' = m+k+1$. Using the long exact sequence in lemma 20, we get:

$$\cdots \to H^{i'}(\mathbb{P}^n,\, \mathcal{F}(j'-1)) \to H^{i'}(\mathbb{P}^n,\, \mathcal{F}(j')) \to H^{i'}(\mathbb{P}^n,\, \mathcal{F}\mid_H (j'-1)) \to H^{i'+1}(\mathbb{P}^n,\, \mathcal{F}(j'-1)) \to \cdots$$

Since $\mathcal{F},\, \mathcal{F}\mid_H$ are both m-regular, we get $H^{i'}(\mathbb{P}^n,\, \mathcal{F}(j'-1)) = H^{i'}(\mathbb{P}^n,\, \mathcal{F}\mid_H (j'-1)) = 0$ by the induction hypothesis. Therefore, $H^{i'}(\mathbb{P}^n,\, \mathcal{F}(j')) = 0$ which completes the proof of the inductive step.

1. Using the long exact sequence in lemma 20, we get the following commutative diagram:

$$\begin{tikzcd}
 & H^0(\mathbb{P}^n,\, \mathcal{F}(n)) \otimes H^0(\mathbb{P}^n,\, \mathcal{O}_{\mathbb{P}^n}(1)) \arrow[r, "\delta_n \otimes \rho_H"] \arrow[d, "\sigma_{n,1}"] & H^0(H,\, \mathcal{F}\mid_H (n)) \otimes H^0(H,\, \mathcal{O}_H(1)) \arrow[d, "\sigma_{n,2}"] & \\
H^0(\mathbb{P}^n,\, \mathcal{F}(n)) \arrow[r, "\gamma_n"'] & H^0(\mathbb{P}^n,\, \mathcal{F}(n+1)) \arrow[r, "\delta_{n+1}"'] & \begin{array}{c} H^0(\mathbb{P}^n,\, \mathcal{F}\mid_H (n+1)) \\ \cong H^0(H,\, \mathcal{F}\mid_H (n+1)) \end{array} \arrow[r] & H^1(\mathbb{P}^n,\, \mathcal{F}(n))
\end{tikzcd}$$

where ρ_H is the projection morphism on H. By the induction hypothesis, we know that $\sigma_{n,2}$ is surjective. Furthermore, by the result of part 2, we know that $H^1(\mathbb{P}^n,\, \mathcal{F}(n)) = 0$. This implies that δ_{n+1} is also surjective. The same argument proves that the morphism $\delta_n : H^0(\mathbb{P}^n,\, \mathcal{F}(n)) \to H^0(H,\, \mathcal{F}\mid_H (n))$, and therefore $\delta_n \otimes \rho_H$, is also surjective. This implies that $\sigma_{n,2} \circ (\delta_n \otimes \rho_H) = \delta_{n+1} \circ \sigma_{n,1}$ is a surjective morphism of modules. Therefore, $H^0(\mathbb{P}^n,\, \mathcal{F}(n+1)) = \mathrm{Im}(\sigma_{n,1}) + \ker(\delta_{n+1})$. Since the bottom row is exact, $\mathrm{Im}(\gamma_n) = \ker(\delta_{n+1})$ and therefore $H^0(\mathbb{P}^n,\, \mathcal{F}(n+1)) = \mathrm{Im}(\sigma_{n,1}) + \mathrm{Im}(\gamma_n)$. Finally, noting that $\mathrm{Im}(\gamma_n) \subseteq \mathrm{Im}(\sigma_{n,1})$, the claim is proved.

3. Applying part 1 repeatedly, we get a surjection $H^0(\mathbb{P}^n,\, \mathcal{O}_X(1))^{\otimes k} \otimes H^0(\mathbb{P}^n,\, \mathcal{F}(n)) \to H^0(\mathbb{P}^n,\, \mathcal{F}(n+k))$. This then factors through the surjective morphism $H^0(\mathbb{P}^n,\, \mathcal{O}_X(k)) \otimes H^0(\mathbb{P}^n,\, \mathcal{F}(n)) \to H^0(\mathbb{P}^n,\, \mathcal{F}(n+k))$ by the map $H^0(\mathbb{P}^n,\, \mathcal{O}_X(1))^{\otimes k} \to H^0(\mathbb{P}^n,\, \mathcal{O}_X(k))$. To prove that $\mathcal{F}(m)$ is generated by its global sections, we have to prove that the canonical morphism $H^0(\mathbb{P}^n,\, \mathcal{F}(m)) \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\varphi} \mathcal{F}(m)$ is surjective. Immediately, we have the exact sequence of the morphism φ given by: $0 \to \ker(\varphi) \to H^0(\mathbb{P}^n,\, \mathcal{F}(m)) \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\varphi} \mathcal{F}(m) \to \mathrm{coker}(\varphi) \to 0$. By theorem 21, there exists t_0 such that for every $t \geq t_0$, the following sequence of global sections is exact:

$$0 \to H^0(\mathbb{P}^n,\, \ker(\varphi)(t)) \to H^0(\mathbb{P}^n,\, \mathcal{F}(m)) \otimes H^0(\mathbb{P}^n,\, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(\mathbb{P}^n,\, \mathcal{F}(m+t)) \to H^0(\mathbb{P}^n,\, \mathrm{coker}(\varphi)(t)) \to 0$$

Since the middle arrow is surjective, $H^0(\mathbb{P}^n,\, \mathrm{coker}(\varphi)(t)) = 0,\ \forall t \geq t_0 \implies \mathrm{coker}(\varphi) = 0$ and φ is surjective. $\square$

Therefore, one can consider the family of m-regular coherent sheaves of fixed Hilbert polynomial P. Note that by theorem 23, the m-twist of an m-regular sheaf is generated by its global sections. Hirzebruch-Riemann-Roch theorem implies that $\chi(\mathcal{F}(m)) = \sum(-1)^i \dim H^i(X, \mathcal{F}(m))$. By m-regularity of $\mathcal{F}$, the nonzero cohomologies vanish and $\chi(\mathcal{F}(m)) = \dim H^0(X, \mathcal{F}(m))$. Thus, such sheaves are quotients of $\mathcal{O}_X(-m)^{\oplus\chi(\mathcal{F}(m))} = {\mathcal{O}_X}^{\oplus\chi(\mathcal{F}(m))} \otimes \mathcal{O}_X(-m) = {\mathcal{O}_X}^{\oplus P(m)} \otimes \mathcal{O}_X(-m)$, up to a choice of isomorphism $H^0(X, \mathcal{F}(m)) \cong H^0(X, \mathcal{O}_X)^{P(m)} = k^{P(m)}$; i.e. an element of $\mathrm{SL}_{P(m)}(k)$. Note that $\mathcal{F}$ is a quotient of ${\mathcal{O}_X}^{\oplus P(m)} \otimes \mathcal{O}_X(-m)$ if and only if $\mathcal{F} \otimes \mathcal{O}_X(m)$ is a quotient of ${\mathcal{O}_X}^{\oplus P(m)}$.

9 The Quot Scheme

Quot scheme is the fine moduli space of all quotients of a coherent sheaf $\mathcal{F}$ over a projective scheme X. The naive moduli problem is given by $\{q : \mathcal{F} \twoheadrightarrow \mathcal{E}$, where $\mathcal{E} \mid \mathcal{E} \in \mathrm{Coh}_X\}/(q : \mathcal{F} \to \mathcal{E}) \sim (q' : \mathcal{F} \to \mathcal{E}') \iff \ker(q) = \ker(q')$. This is equivalent to the existence of a sheaf isomorphism $\varphi : \mathcal{E} \to \mathcal{E}'$ such that the following diagram is commutative:

$$\begin{tikzcd} & \mathcal{E}' \\ \mathcal{F} \arrow[ur, "q'"] \arrow[r, "q"'] & \mathcal{E} \arrow[u, dotted, "\varphi"'] \end{tikzcd}$$

For constructing this moduli space, we give a

Theorem 24. Given a nonnegative integer N and a polarized projective variety X embedded in $\mathbb{P}^n$, there is a polynomial $F_{N,n}(X_0, \cdots, X_n) \in \mathbb{Z}[X_0, \cdots, X_n]$ such that given any coherent sheaf $\mathcal{F}$ over X that is a quotient of ${\mathcal{O}_X}^{\oplus N}$ with Hilbert polynomial written as $\chi(\mathcal{F})(m) = \sum_{i=0}^{n} a_i \binom{m}{i}$, the sheaf $\mathcal{F}$ is $F_{N,n}(a_0, \cdots, a_n)$-regular.

Proof. As in the proof of theorem 20, theorem 23, we can assume that the base field is infinite and that $X = \mathbb{P}^n$. We prove the claim by induction on n. The base case $n = 0$ is clear. For $n \geq 1$, one can choose a hyperplane H not containing any of the associated points of $\mathcal{F}$. Restricting to H (i.e. tensoring with $\mathcal{O}_H$), one sees that ${\mathcal{O}_{\mathbb{P}^n}}^{\oplus N} \otimes \mathcal{O}_H = {\mathcal{O}_H}^{\oplus N} \to \mathcal{F}_H$ is a surjective morphism and therefore $\mathcal{F}_H$ is a quotient of ${\mathcal{O}_H}^{\oplus N}$. Taking the short exact sequence of lemma 20, we get $\mathcal{F}(r-1) \to \mathcal{F}(r) \to \mathcal{F}_H(r) \to 0$. Taking Euler characteristics, we get $\chi(\mathcal{F}(r-1)) + \chi(\mathcal{F}_H(r)) = \chi(\mathcal{F}(r))$. Therefore, writing the Hilbert polynomial of $\mathcal{F}_H(r)$ in terms of the binomial coefficients, we get $b_i = p_i(a_0, \cdots, a_n)$, $p_i \in \mathbb{Z}[X_0, \cdots, X_n]$. Note that p_i is independent of k, $\mathcal{F}$. By applying the induction hypothesis on $\mathcal{F}_H$, we get a polynomial $F_{N,n-1} \in \mathbb{Z}[X_0, \cdots, X_{n-1}]$ such that $\mathcal{F}_H$ is $F_{N,n-1}(b_0, \cdots, b_{n-1})$-regular. Substituting p_i's, we find out that there is a polynomial $P \in \mathbb{Z}[X_0, \cdots, X_n]$, independent of k, $\mathcal{F}$ such that $\mathcal{F}_H$ is $P(a_0, \cdots, a_n)$-regular. Taking cohomology we get an exact sequence:

$$0 \to H^0(\mathbb{P}^n, \mathcal{F}(r-1)) \to H^0(\mathbb{P}^n, \mathcal{F}(r)) \to H^0(\mathbb{P}^n, \mathcal{F}_H(r)) \to H^1(\mathbb{P}^n, \mathcal{F}(r-1)) \to H^1(\mathbb{P}^n, \mathcal{F}(r)) \to H^1(\mathbb{P}^n, \mathcal{F}_H(r))$$

But $H^1(\mathbb{P}^n, \mathcal{F}_H(r)) = 0, \forall r \geq P(a_0, \cdots, a_n)$. Furthermore, the morphisms $H^i(\mathbb{P}^n, \mathcal{F}(r-1)) \to H^i(\mathbb{P}^n, \mathcal{F}(r))$ are isomorphisms for all $i \geq 2$. By theorem 20, $H^i(\mathbb{P}^n, \mathcal{F}(r')) = 0$ for sufficiently large values of r'. This implies that $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0, i \geq 2, r \geq P(a_0, \cdots, a_n) - 1$. Note that the morphism $H^1(\mathbb{P}^n, \mathcal{F}(r-1)) \to H^1(\mathbb{P}^n, \mathcal{F}(r))$ is a surjection. Therefore, the sequence $\dim H^1(\mathbb{P}^n, \mathcal{F}(r))$ is decreasing for $r \geq P(a_0, \cdots, a_n) - 1$. Suppose that the sequence is not strictly decreasing at some point. Then $\dim H^1(\mathbb{P}^n, \mathcal{F}(r_0)) = \dim H^1(\mathbb{P}^n, \mathcal{F}(r_0 - 1))$ for some $r_0 \geq P(a_0, \cdots, a_n)$. This can only happen if the morphism $H^0(\mathbb{P}^n, \mathcal{F}(r_0)) \to H^0(\mathbb{P}^n, \mathcal{F}_H(r_0))$ is a surjection. Looking at the commutative diagram in theorem 23, this means that δ_{r_0} is surjective. Therefore, $\delta_{r_0} \otimes \rho_H$ is also surjective. Since $r_0 \geq P(a_0, \cdots, a_n)$, we also know that $\mathcal{F}_H$ is an r_0-regular sheaf. Therefore, by part 1 of the theorem 23, this implies that $\sigma_{r_0,2}$ is surjective. This amounts to $\sigma_{r_0,2} \circ (\delta_{r_0} \otimes \rho_H) = \delta_{r_0+1} \circ \sigma_{r_0,1}$ being surjective; which gives the surjectivity of δ_{r_0+1}. Thus, δ_r will be surjective for all $r \geq r_0$. Taking a look at the long exact sequence above, this means that the morphism $H^1(\mathbb{P}^n, \mathcal{F}(r-1)) \to H^1(\mathbb{P}^n, \mathcal{F}(r))$ is an injection. As this morphism is also a surjection, we get an isomorphism. As a result, $\dim H^1(\mathbb{P}^n, \mathcal{F}(r-1)) = \dim H^1(\mathbb{P}^n, \mathcal{F}(r))$ for all $r \geq r_0$. By theorem 20, this implies that $\dim H^1(\mathbb{P}^n, \mathcal{F}(r_0)) = 0$. Therefore, this sequence is strictly decreasing until it hits zero. This implies that $H^1(\mathbb{P}^n, \mathcal{F}(r)) = 0, \forall r \geq c + \dim H^1(\mathbb{P}^n, \mathcal{F}(c))$ where $c = P(a_0, \cdots, a_n)$. We finally need to bound $\dim H^1(\mathbb{P}^n, \mathcal{F}(P(a_0, \cdots, a_n)))$. Since higher cohomologies vanish at $P(a_0, \cdots, a_n)$, we get $\dim H^1(\mathbb{P}^n, \mathcal{F}(c)) = \dim H^0(\mathbb{P}^n, \mathcal{F}(c)) - \chi(\mathcal{F}(c))$. Since $\mathcal{F}$ is a quotient of ${\mathcal{O}_{\mathbb{P}^n}}^{\oplus N}$, it is well-known that $\dim H^0(\mathbb{P}^n, \mathcal{F}(c)) \leq N\binom{n+c}{n}$. Wrapping everything up, we get $\dim H^1(\mathbb{P}^n, \mathcal{F}(c)) \leq N\binom{n+c}{n} - \sum a_i \binom{c}{i}$. Plugging $c = P(a_0, \cdots, a_n)$ back in, we get $R(a_0, \cdots, a_n) = N\binom{n+P(a_0, \cdots, a_n)}{n} - \sum a_i \binom{P(a_0, \cdots, a_n)}{i} \in \mathbb{Z}[X_0, \cdots, X_n]$. Finally, we get $\dim H^1(\mathbb{P}^n, \mathcal{F}(r)) = 0, \forall r \geq P(a_0, \cdots, a_n) + R(a_0, \cdots, a_n)$ and $F_{N,n} := P + R$. $\square$

10 The Kempf-Ness Theorem

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