Comm. Ring A : Prime ideal
Maximal ideal

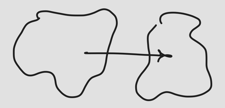
I ideal of $A \Rightarrow \sqrt{I} = \{a \in A \mid \exists n \in \mathbb{N} : a^n \in I\}$

Primary ideal : $\forall x, y \in A : xy \in I \Rightarrow x \in I$ or $y \in \sqrt{I}$

$\Rightarrow q$ primary $\Rightarrow \sqrt{q}$ prime : $q, p = \sqrt{q}$ related / belong to eachother

m maximal , $m^n \subseteq q \neq A \Rightarrow q$ primary , belongs to m

$\operatorname{Spec} A, \Omega A$: $M \subseteq A : V(M) = \{p \in \operatorname{Spec} A \mid M \subseteq p\}$, $D(M) = \operatorname{Spec} A - V(M)$



$f : A \to B$ homo. , $P \in \operatorname{Spec} B$: $\underbrace{f^* : \operatorname{Spec} B \to \operatorname{Spec} A}$

$f^*(P) = f^{-1}(P) := P \cap A$, f^* cont. , P lies over $P \cap A$

$I, \underbrace{p_1, \dots, p_n}$ ideals of $A \Rightarrow \forall i \in [n] : I \not\subseteq p_i \Rightarrow I \not\subseteq \bigcup_{i=1}^{n} p_i$

at most 2 are not prime

I, p_1, p_2 ideals of A, $\underbrace{I \not\subseteq p_1, p_2} \Rightarrow \overset{\text{FSOC}}{I \subseteq \underbrace{p_1 \cup p_2}}$

$\left.\begin{matrix} a \in I - p_1 \\ b \in I - p_2 \end{matrix}\right\} \Rightarrow a, b \in p_1 \cup p_2 \left.\right\} \Rightarrow \left.\begin{matrix} a \in p_2 \\ b \in p_1 \end{matrix}\right\} \Rightarrow a + b \notin p_1 \cup p_2$

$\underbrace{I, \overline{p_1, \dots, p_{n-1}}}, \underbrace{p_n}_{\text{prime}} \Rightarrow I \not\subseteq \bigcup_{i=1}^{n-1} p_i$, $I \subseteq \bigcup_{i=1}^{n} p_i \Rightarrow I p_1 \dots p_{n-1} \not\subseteq p_n$

FSOC

$x \in I p_1 \dots p_{n-1} - p_n$, $S = I - (p_1 \cup \dots p_{n-1}) \neq \emptyset$

$S \subseteq p_n \Rightarrow \forall s \in S : s + x \in S \Rightarrow x \in p_n$

I, J comaximal : $I + J = A \Leftrightarrow \exists i \in I, j \in J : ia + jb = 1$, $a, b \in A$

$I_1, \dots, I_n$ ideals of A : $\forall i, j : I_i + I_j = A \Rightarrow I_1 \dots I_n = I_1 \cap \dots \cap I_n$, $A / \bigcap_{i=1}^{n} I_i = \prod_{i=1}^{n} A / I_i$

J ideal of $A \Rightarrow \underbrace{A \to A/J} \curvearrowright \underbrace{\operatorname{Spec} A \to \operatorname{Spec} A/J}$

$\Rightarrow$ Krull : $\operatorname{Spec} A/J$ has maximal/minimal elements $\Rightarrow \exists$ maximal/minimal ideals cont. J

$1 \in S$ ms. , $0 \notin S \Rightarrow F = \{I$ ideal of $A \mid I \cap S = \emptyset\} \neq \emptyset \Rightarrow \underbrace{\text{Has maximal element } M \Rightarrow M \text{ is prime ideal}}$

$\operatorname{nil}(A) = \sqrt{\{0\}} = \bigcap_{p \in \operatorname{Spec} A} p \Rightarrow \sqrt{I} = \bigcap_{I \subseteq P \in \operatorname{Spec} A} P$

A reduced $\Rightarrow$ Only multiplicative element is $0 \Leftrightarrow \operatorname{nil} A = \{0\} \Leftrightarrow \bigcap_{p \in \operatorname{Spec} A} p = \{0\}$

$A/\operatorname{nil}(A)$ is reduction of A

$\forall \underbrace{M \subseteq A}_{\operatorname{nil} A \subseteq M} : V(M/\operatorname{nil} A) = \{p/\operatorname{nil} A \mid p \in \operatorname{Spec} A, M \subseteq p\}$

m.s. $\underset{0 \notin S}{S}$: $S^{-1}A, A_S = \{\frac{a}{s} \mid a \in A, s \in S\}$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\text{can.}"'] & B \\ S^{-1}A \arrow[ur, dotted, "f_S"'] & \end{tikzcd}$$

$\{ S^{-1}A$ A-algebra, ring

$A \to S^{-1}A \longrightarrow \overline{\operatorname{Spec} A \leftarrow \operatorname{Spec} S^{-1}A}$

$p \cap S \neq \emptyset$, p prime/primary $\Rightarrow p(S^{-1}A)$ prime/primary

I ideal $S^{-1}A \Rightarrow I = (I \cap A) S^{-1}A$, J ideal of A : $J(S^{-1}A) = S^{-1}A \Leftrightarrow J \cap S \neq \emptyset$

S m.s of A, M an A-module : $S^{-1}M = \{\frac{m}{s} \mid m \in M, s \in S\}$

$\{ S^{-1}M$ is a $S^{-1}A$-module , $S^{-1}M \cong S^{-1}A \otimes_A M$

M, N A-modules $\Rightarrow S^{-1}(M \otimes_A N) = S^{-1}M \otimes_{S^{-1}A} S^{-1}N$: $\frac{m \otimes n}{s} \to \frac{m}{s} \otimes \frac{n}{1}$

Finite presentantion : $A^m \to A^n \to M \to 0$ $\Rightarrow \underbrace{S^{-1}(\operatorname{Hom}_A(M, N)) \cong \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)}$

. - - - - $\to M \to 0$

$\to A, B$ $\underbrace{\text{a.a.s}}$ $\Rightarrow \operatorname{Hom}(A, B) = \{f : A \to B \mid f$ homomorphism$\}$

$\varphi : A' \to A$, $\psi : B \to B'$ homomorphism

$\Rightarrow \operatorname{Hom}(\varphi, \psi) : \operatorname{Hom}(A, B) \to \operatorname{Hom}(A', B')$

$\operatorname{Hom}(\varphi, \psi)\, \alpha = \psi \alpha \varphi$, $\forall \alpha \in \operatorname{Hom}(A, B)$

A ring, $p \in \operatorname{Spec} A \Rightarrow A - p$ m.s $\Rightarrow A_p = (A-p)^{-1}A$, $M_p = (A-p)^{-1}M$

$\operatorname{Ann}_A(x) := \{a \in A \mid ax = 0\}$, Ideal quotient : $(I_1 : I_2) = \{a \in A \mid aI_2 \subseteq I_1\}$

$I_1 \subseteq (I_1 : I_2)$, $(I_1 : I_2) I_2 \subseteq I_1$

$\operatorname{Den}(x) := \{a \in A \mid \underbrace{ax \in A}\}$, $x \in A / A - \{0\} \leftarrow A$ integral domain

A, B ring $f : A \to B \Rightarrow A, B, f$ can be viewed as in context of modules

$\overline{f : A \to B}$ ring homo. $S \subseteq A$ m.s. $\Rightarrow f(S) \subseteq B$ ms.

$f(S)^{-1}B = S^{-1}B = S^{-1}A \otimes_A \widetilde{B}$

Special Case : $B = A/I$, can. map $i : A \to A/I$: $\underbrace{i(S)^{-1}(A/I) = S^{-1}A / I(S^{-1}A)}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\text{can.}"'] & B \arrow[dl, "g_{\text{iso.}}"] \\ S^{-1}A & \end{tikzcd}$$

, $\forall b \in B, \exists s \in S$: $\underbrace{f(s) b \in f(A)} \Rightarrow S^{-1}B = S^{-1}A$

A local ring $\Leftrightarrow |\Omega(A)| = 1 \Rightarrow A/m = \kappa(A)$

$p \in \operatorname{Spec} A \Rightarrow (A_p, pA_p, A_p/pA_p)$ is a local ring

$(A, m, \kappa), (B, m', \kappa') \Rightarrow f : A \to B$ ring homo. is local homo if $f(m) \subseteq m'$

$\Rightarrow f' : \kappa \to \kappa'$ is induced

$f : A \to B$ ring homo : $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$, $P \in \operatorname{Spec}(B)$: $f(A - f^{-1}(P)) \subseteq B - P$

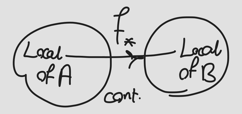
$\Rightarrow \underbrace{\text{Homo} : f_P : A_{f^{-1}(P)} \to B_P}$, $\underbrace{f_P(f^{-1}(P) A_{f^{-1}(P)})} \subseteq PB_P$

$\underbrace{f_P(f^{-1}(P))}_{\subseteq P} \underbrace{f_P(A_{f^{-1}(P)})}_{\subseteq B_P}$

$$\begin{tikzcd} A_{f^{-1}(P)} \arrow[r, "f_P"] \arrow[d, "\text{Localization}"'] & B_P \\ B_{f^{-1}(P)} \arrow[ur, "\text{Localization}"'] & \end{tikzcd}$$

$B_{f^{-1}(P)} = A_{f^{-1}(P)} \otimes_A B$

B as A-module



Jacobson radical : $\operatorname{rad} A = \bigcap_{m \in \Omega A} m$

Semi-Local ring $\Rightarrow |\Omega(A)| < \infty \} \Rightarrow \operatorname{rad} A = \bigcap_{m \in \Omega(A)} m = \prod_{m \in \Omega(A)} m$

$\left.\begin{matrix} x \in \operatorname{rad}(A) \Rightarrow 1 + x \text{ unit} \\ I \text{ ideal}, \forall x \in I : 1 + x \text{ unit} \Rightarrow I \subseteq \operatorname{rad} A \end{matrix}\right\} \Rightarrow \begin{matrix} x \in \operatorname{rad} A \Leftrightarrow \\ \forall y \in A : 1 - xy \text{ unit} \end{matrix}$

Nakayama Lemma : A ring , M finitely generated A-module, I ideal of A , $IM = M$

$\Rightarrow \exists x \in I : (1 + x) M = 0 \Rightarrow \{ I \subseteq \operatorname{rad} A \Rightarrow M = 0$

$\Rightarrow A$ ring , M an A-module , N_1, N_2 submodules of M , I ideal, $M = N_1 + IN$

$\left.\begin{matrix} \text{i) } I \text{ nilpotent} \\ \text{ii) } I \subseteq \operatorname{rad} A, N_2 \text{ finitely generated} \end{matrix}\right\} \Rightarrow M = N_1$

$\alpha : \mathbb{Z} \to A \Rightarrow \operatorname{Ker}(\alpha) = \mathbb{Z}/m\mathbb{Z} \Rightarrow \operatorname{Char}(A) = m$

A $\underbrace{\text{local ring}}$ $\operatorname{char}(A) = 0$ or p^n.

$I + J = A$

$\left.\begin{matrix} \operatorname{char}(A) = ab \\ (a, b) = 1 \end{matrix}\right\} \Rightarrow \begin{matrix} (\alpha(a)) \to I \\ \alpha(b) \to J \end{matrix}$

$\underbrace{1 + \dots + 1}_{ia} = 0$

A ring, M A-module $\Rightarrow a \in A$ is M-regular $\Leftrightarrow m \mapsto am$ injective

Set of M-regulars is m.s. $\to \operatorname{Reg}(M)$

$\Rightarrow \underbrace{\operatorname{Reg}(A)^{-1}A}$ is total quotient ring : $\Phi A \to$ Flatness

Noetherian/Artinian : $\underbrace{\text{ACC}}$, $\underbrace{\text{DCC}}$

A No. $\Leftrightarrow$ Every ideal of A is finitely generated

A No. $\Rightarrow$. - - . . - -

Lasker-Noether theorem :

A ring with $\underbrace{\text{irredundant}}$ $\underbrace{\text{primary decomposition}}$ $\to$ Lasker Ring

$\begin{matrix} \forall i : \bigcap_{j \neq i} q_j \not\subseteq q_i \\ \sqrt{q_i} \neq \sqrt{q_j} \end{matrix}$ $\qquad I = \bigcap_{i=1}^{n} q_i$, q_i primary

$q_1, \dots, q_n$ are p-primary $\Rightarrow \bigcap q_i$ is p-primary

I ideal of A is irreducible : $\forall J_1, J_2$ ideals of A : $J_1 \cap J_2 = I \Rightarrow J_1 = I$ or $J_2 = I$.

Every prime ideal is irreducible : $J_1 \cap J_2 = p \Rightarrow J_1 J_2 \subseteq J_1 \cap J_2 = p \Rightarrow J_1 \subseteq p$ or $J_2 \subseteq p$

No. ring $A \Rightarrow$ Every irreducible ideal is primary :

q irreducible $\Rightarrow a, b \in A$: $ab \in q$, $\underline{a \notin q} \overset{?}{\Rightarrow} b \in \sqrt{q}$

Colon ideal : $\underbrace{(q : (b^n)) = A_n}$, $A_n \uparrow \Rightarrow \underbrace{\exists n_0 \in \mathbb{N} : \forall n \geq n_0 : A_n = A_{n_0}}$

$\underbrace{I := (b^n) + q}$, $\underbrace{J := (a) + q} \Rightarrow \underbrace{I \cap J = q}$, q irreducible

$\Rightarrow q = I$ or $q = J$, $q \neq J \Rightarrow q = I \Rightarrow \underline{b^{n_0} \in q} \Rightarrow b \in \sqrt{q}$

$\underbrace{rb^n + q \in J}$ $Jb \subseteq q \Rightarrow rb^{n+1} + qb = c \in q$

$rb^{n+1} = c - qb \in q \Rightarrow r \in A_{n+1}$

$\Rightarrow \underline{r \in A_n} \Rightarrow \underline{rb^n \in q} \Rightarrow rb^n + q \in q$

$\{$ Radical set of $\{q_1, \dots, q_n\} = \{\sqrt{q_i} \mid i \in [n]\}$

$\{$ $\underline{\operatorname{Ass}(A/I)}$

$\exists$ ideal $I \subseteq A$: I doesn't admit irr. dec. $\mathcal{L}$ = set of all . . . . $\neq \emptyset$

$\Rightarrow \mathcal{L}$ has a maximal element $\Rightarrow \underline{M = I \cap J}$, $\underbrace{M \subsetneq I, M \subsetneq J}_{\cap} \Rightarrow \underline{M \notin \mathcal{L}}$ X

$\bigcap_{i=1}^{n} q_i = I$ finite irr. dec. $\Rightarrow \overset{\text{prime}}{\operatorname{Rad}(\{q_1, \dots, q_n\})} = \underbrace{\{(I : (a)) \mid a \in A/I\}}_{\text{prime}}$

$j \in [n]$, $p_j = \operatorname{Rad} q_j$, $J = \bigcap_{i \neq j} q_i \Rightarrow J \cap q_j = I \subsetneq J$, $\underline{\text{No.} \to \text{Ar.}}$

A No. $\Rightarrow p_j^n \subseteq q_j \Rightarrow J p_j^n \subseteq J \cap p_j^n \subseteq J q_j = I$. n_0 minimum $n \in \mathbb{N}$:

$J p_j^n \subseteq I \Rightarrow J p_j^{n_0 - 1} \not\subseteq I \Rightarrow \underline{\exists a \in J p_j^{n_0 - 1} - I} \Rightarrow a \in J$, $a \notin I$

$\Rightarrow \underline{a \notin q_j}$ $J \cap q_j = I$

$\left.\begin{matrix} \forall b \in A : ab \in I \subseteq q_j \text{ , } q_j \text{ primary} \Rightarrow b \in \sqrt{q_j} \\ a p_j \subseteq J p_j^n \subseteq I \Rightarrow p_j \subseteq (I : (a)) \end{matrix}\right\} \Rightarrow (I : (a)) = p_j$

$\forall a \in A/I$: $\underset{\text{proper}}{p = (I : (a))} \in \operatorname{Spec} A$. $a \neq 0 \Rightarrow a \notin I = \bigcap_{i=1}^{n} q_i \Rightarrow \exists j \in [n] : a \in q_j$

$K = \prod_{a \notin q_i} q_i \Rightarrow aK \subseteq \bigcap_{i=1}^{n} q_i = I \Rightarrow K \subseteq (I : (a)) = p$, p prime , $\prod_{a \notin q_i} q_i \subseteq p$

$\exists k$: $\underline{a \notin q_k}$, $q_k \subseteq p \Rightarrow \sqrt{q_k} \subseteq p$, $\forall x \in p : ax \in I \subseteq q_k$, q_k primary

$\Rightarrow x \in \sqrt{q_k} \Rightarrow p = \sqrt{q_k}$

$\Rightarrow \operatorname{Rad}(q_1, \dots, q_n) = \{(I : (a)) \in \operatorname{Spec} A \mid a \in A/I\} \leftarrow$

A No. , M an A-module $\Rightarrow p$ prime ideal is associated with M : $\exists x \in M : p = \operatorname{Ann}_A(x)$

$\Leftrightarrow M$ contains a submodule isomorphic A/p

$I \subseteq (I : (a)) \Rightarrow A/(I : (a)) \subseteq A/I \Rightarrow (I : (a))$ is an associated prime ideal

A No, q ideal, $p \in \operatorname{Spec} A \Rightarrow q$ is p-primary $\Leftrightarrow \underline{\operatorname{Ass}(A/q) = \{p\}}$

q primary $\Leftrightarrow$ Every zero divisor of A/q is $\underbrace{\text{nilpotent}}$

$\bigcap$ prime ideals $= \underline{\bigcap \text{ minimal prime ideal}}$

A No. zero divisor set of $M = \bigcup_{p \in \operatorname{Ass}(M)} p$

$\{\operatorname{Ann}(x) \mid x \in M, x \neq 0\} = N$, p maximal element of $N \Rightarrow p \in \operatorname{Ass}(M)$

$p = \operatorname{Ann}(x_0)$, $ab \in p$, $b \notin p \Rightarrow bx_0 \neq 0$, $abx_0 = 0$, $\operatorname{Ann}(bx_0) \supseteq \operatorname{Ann}(x_0) = p$

$\Rightarrow \operatorname{Ann}(bx_0) = p \Rightarrow a \in p$

$\operatorname{Ass}(M) \neq \emptyset \Leftrightarrow M = \emptyset$ $\qquad \underline{\sqrt{\{0\}} = \operatorname{nil}}$

$\underbrace{\text{Zero divisors of } A/q} = \bigcup_{p \in \operatorname{Ass}(A/q)} p = \bigcap_{p \in \operatorname{Spec}_{\min}(A/I)} p \overset{\text{Order preserving}}{=} \underbrace{\bigcap_{p \in \operatorname{Ass}_{\min}(A/q)} p}$

$\Rightarrow p$ primary $\Leftrightarrow \bigcap_{p \in \operatorname{Ass}_{\min}(A/q)} p$ consists only of nilpotents

$\operatorname{Ass}(A/q) = \{p\} \Leftrightarrow \operatorname{Ass}_{\min}(A/q) = \{p\}$

$\Rightarrow |\operatorname{Ass}(A/q)| = 1 \Rightarrow \underline{\sqrt{q} = \operatorname{Ass}_{\min}(A/q) = p}$