Nielsen-Thurston Classification of Automorphisms of Compact, Orientable Surfaces

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Hyperbolic Sets and Anosov Maps

Definition (Hyperbolic Sets)

Let M be a compact Riemannian manifold and $f : M \to M$ be a diffeomorphism. Given a compact f-invariant subset $\Lambda \subseteq M$, we say that it is a hyperbolic set for f if there exist $c > 0, \lambda \in (0,1)$, along with an invariant decomposition of the tangent bundle given by $T_\Lambda M = E^s_\Lambda \oplus E^u_\Lambda$ such that:

- $||Df^n(x)(v)|| \leq c\lambda^n ||v||, \forall n \geq 0, v \in E^s_x, x \in \Lambda$
- $||Df^{-n}(x)(v)|| \leq c\lambda^n ||v||, \forall n \geq 0, v \in E^u_x, x \in \Lambda$
- $Df(x)(E^s_x) = E^s_{f(x)}, Df(x)(E^u_x) = E^u_{f(x)}, \forall x \in \Lambda$

Definition (Anosov Diffeomorphism)

A C^1 diffeomorphism $f : M \to M$ of a compact manifold M is called an Anosov diffeomorphism if M is a hyperbolic set for f.

Classification of Toral Automorphisms: An Introduction

Correspondence of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{Aut}(\mathbb{T}^2)$

- The torus $\mathbb{T}^2$ can be regarded as the quotient $\mathbb{R}^2/\mathbb{Z}^2$ equipped with a fixed orientation. Note that a map $\mathbb{T}^2 \to \mathbb{T}^2$ lifts to a map $\mathbb{R}^2 \to \mathbb{T}^2$.
- Since $\mathbb{R}^2$ is a covering space of $\mathbb{T}^2$, the lifting criterion implies that any map $f : \mathbb{R}^2 \to \mathbb{T}^2$ lifts to a map $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$.
- This implies that $\widetilde{f}(x+v) - \widetilde{f}(x) \in \mathbb{Z}^2$ for all $x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, v \in \mathbb{Z}^2$. Since the map $\widetilde{f}(x+v) - \widetilde{f}(x) \in \mathbb{Z}^2$ in continuously ranging on a discrete set, it is locally constant on x. Since $\mathbb{R}^2$ is connected, this map is essentially constant on x. We therefore define $F(v) := \widetilde{f}(x+v) - \widetilde{f}(x)$.
- Therefore, any homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ gives rise to some group automorphism $F : \mathbb{Z}^2 \to \mathbb{Z}^2$, which correspond to elements of $\mathrm{GL}_2(\mathbb{Z})$.
- Such group automorphism preserves orientations if and only if $F \in \mathrm{SL}_2(\mathbb{Z})$.
- Conversely, given $F \in \mathrm{SL}_2(\mathbb{Z})$, one can recover $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$.
- One can therefore classify $f \in \mathrm{Aut}(\mathbb{T}^2)$ by the corresponding $F \in \mathrm{SL}_2(\mathbb{Z})$, and $F \in \mathrm{SL}_2(\mathbb{Z})$ by its eigenvalues $\mathrm{Spec}(F) = \{\lambda, \lambda^{-1}\}$.

Classification of $\mathrm{Aut}(\mathbb{T}^2)$ by Classifying $\mathrm{SL}_2(\mathbb{Z})$

- $\lambda, \lambda^{-1} \in \mathbb{C} \setminus \mathbb{R} \implies \mathrm{tr}(F) = 0, 1, -1$. In this case, Cayley-Hamilton implies that $F^{12} = \mathrm{id}_{\mathbb{R}^2}$. Therefore, $f^{12} = \mathrm{id}_{\mathbb{R}^2/\mathbb{Z}^2}$ and f is periodic.
- $\lambda = \lambda^{-1} = \pm 1 \implies \mathrm{tr}(F) = \pm 2$. In this case (and only in this case), F has an integer eigenvalue. Therefore, it has an integer eigenvector $v \in \mathbb{R}^2$. The image of v under the quotient map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is then an invariant (under f), essential (not null-homotopic), simple closed curve. In this case we say that f is reducible.
- $\lambda, \lambda^{-1} \in \mathbb{R}, \lambda \neq \lambda^{-1} \implies |\mathrm{tr}(F)| > 2$. Suppose that $|\lambda| > 1 > |\lambda^{-1}|$ and v_1, v_2 are the corresponding real eigenvalues. One can immediately see that F (and therefore f) is of infinite order and leaves no simple closed curve invariant. One can also see that F (and therefore f) stretches tangent vectors by a factor of λ in the direction of x_1 and shrinks tangent vectors by a factor of λ in the direction of x_2. Therefore, f is Anosov.

Towards A Generalization of Toral Classification

Basics of Hyperbolic Geometry

- Poincare open disk model of $\mathbb{R}^2$: One point compactification $\mathbb{R}^2 \cup \{\infty\}$.
- Geodesics in Poincare disk (corresponding to Euclidean lines) are circles through infinity that meet the boundary of Poincare disk orthogonally.
- Inversion in $\mathbb{R}^2$ with respect to a line or circle gives rise to an involution of Poincare disk. Inversions carry circles of $\mathbb{R}^2 \cup \{\infty\}$ to circles in $\mathbb{R}^2 \cup \{\infty\}$.
- If $C \cap \mathbb{H}^2$ is a geodesic between two points, inversion with respect to C is called the reflection in $C \cap \mathbb{H}^2$. An isometry is a product of reflections.
- Isometries preserve the metric $\frac{ds}{1-r^2}$ in the Poincare disk. In fact, the geodesics with respect to this metric are exactly the geodesics of Poincare disk. We call $\frac{2ds}{1-r^2}$ the hyperbolic metric of $\mathbb{H}^2$.
- Let $f : \mathbb{H}^2 \to \mathbb{H}^2$ be an orientation-preserving isometry. By Brouwer's fixed point theorem, f has a fixed point. Then either $f \equiv \mathrm{id}_{\mathbb{H}^2}$, or f has a fixed point inside $\mathbb{H}^2$ and none on the boundary (elliptic), or it has no fixed points inside $\mathbb{H}^2$ and one or two fixed points on the boundary (parabolic and hyperbolic respectively).

Hyperbolic Structures

- Hyperbolic structures on surfaces are defined by hyperbolic atlas of charts.
- In a complete hyperbolic surface, all geodesics extend indefinitely.
- Any complete, connected, simply-connected, hyperbolic surface is isometric to $\mathbb{H}^2$. Thus, $\mathbb{H}^2$ is the universal cover of compact hyperbolic surfaces.
- Given a closed hyperbolic surface M, a geodesic in M is the image of a completely extended geodesic in the universal cover $\mathbb{H}^2$. A geodesic in M is simple if it has no transverse self-intersections.
- A non-empty closed subset $L \subseteq M$ which is a disjoint union of simple geodesics in M is called a geodesic lamination of M. Each of the simple geodesics are called the leaves of the lamination.
- The set of all geodesic laminations on a closed orientable hyperbolic surface constitute a metric space via the Hausdorff distance.

Geometry of Geodesic Laminations

- A component of $M \setminus L$ is called a principal region of L.
- Given a principal region U, a leaf $l \subseteq L$ is a boundary leaf if for all $x \in l$ there exists some $\epsilon > 0$ such that $\mathcal{U}_\epsilon(x) \cap U$ contains at least one component of $\mathcal{U}_\epsilon(x) \setminus \mathcal{C}$ where $\mathcal{C}$ is the segment of l of length 2ϵ centered at x.
- Any lamination has finitely many principal regions and each principal region has finitely many boundary leaves

A closed 1-submanifold is a disjoint union of simple closed curves. An essential 1-submanifold is a closed 1-submanifold where every component is essential and no two components are homotopic. An automorphism $f : M \to M$ of a closed orientable surface is said to be periodic if f^n is homotopic to identity for some $n > 0$. By a theorem of Fenchel and Nielsen, this implies that f is in fact isotopic to an automorphism g such that $g^n \equiv \text{id}$. An automorphism $f : M \to M$ is reducible if it isotopic to some automorphism that keeps invariant some closed, essential 1-submanifold of M.

Observation of Non-periodic Irreducible Automorphisms

Penner's Construction

By a theorem of M.Dehn, the Dehn twists generate the mapping class group of isotopy classes of orientation-preserving homeomorphisms of closed, oriented surfaces. W.B.R. Lickorish showed that in fact, $3g-1$ specific Dehn twists are enough to generate the mapping class group. This number was later improved by S.P. Humphries to $2g+1, g>1$, which he showed was minimal.

Theorem

Let M be a closed orientable surface. Choose two families $\mathcal{A}=\{\alpha_i\}, \mathcal{B}=\{\beta_j\}$ of simple closed curves where any element $M \setminus (A \cup B)$ is homeomorphic to a disk. Then any product of positive Dehn twists about α_i's and negative Dehn twists about β_j's using all of the curves at least once gives rise to a non-periodic irreducible automorphism of M.

Nielsen-Thurston Classification Theorem

By a theorem of Cheeger and Gromov, any orientation-preserving homeomorphism of closed oriented hyperbolic surfaces $f : M_1 \to M_2$ induces a homeomorphism of unit tangent bundles $\hat{f} : \mathrm{UT}(M_1) \to \mathrm{UT}(M_2)$. Let $f : M \to M$ be a non-periodic irreducible automorphism of a closed orientable hyperbolic surface. Any lift of $f^n, n > 0$ to the universal covering space $\mathbb{H}^2$ then has finitely many fixed points on $\partial\mathbb{H}^2$. Furthermore, M admits a unique geodesic lamination L^s, invariant under $\hat{f}$, such that the preimage $\widetilde{L^s}$ contains the geodesics joining contracting fixed points of $f^n, \forall n > 0$. This is known as the stable lamination. The stable lamination of f^{-1} is known as the unstable lamination.

Theorem

One can finally prove that f is isotopic to a homeomorphism $f' : M \to M$ such that $f'(L^s) = L^s, f'(L^u) = L^u$ (this is not necessarily a diffeomorphism).

Pseudo-Anosov Mappings

Definition

A singular foliation $\mathcal{M}$ on a surface M is the decomposition of M as a disjoint union of leaves. Any point $x \in M$ is singular if there exists some neighborhood U of x in M such that the local chart $\varphi : \mathcal{M} \cap U \to W_k$ takes the intersection to the singularity with k separatrices. Singular foliations $\mathcal{M}_1, \mathcal{M}_2$ are transverse if they have the same singular set and are transverse at any other points.

Theorem

Let $f : M \to M$ be a non-periodic, irreducible automorphism of a closed, oriented, hyperbolic surface. Then f is isotopic to an automorphism $f' : M \to M$ such that there exist a pair of transverse singular foliations $\mathcal{M}^s, \mathcal{M}^u$ for which $f'(\mathcal{M}^s) = \mathcal{M}^s, f'(\mathcal{M}^u) = \mathcal{M}^u$.

In fact, this construction guarantees an important dynamical condition on the foliations, as known by pseudo-Anosovness (explained further).

Definition

A transverse measure to a singular foliation $\mathcal{M}$ defines on each arc α transverse to $\mathcal{M}$ a non-negative Borel measure $\mu\big|_{\alpha}$ satisfying the following:

- If $\beta \subseteq \alpha$ is a sub-arc. then $\mu\big|_{\beta} \equiv \mu\big|_{\alpha}\big|_{\beta}$.
- If α_0, α_1 are two arcs transverse to $\mathcal{M}$ and homotopic by a homotopy $F : I \times I \to M$, such that $F(a_0 \times I)$ is entirely contained in a leaf of $\mathcal{M}$ for all $a_0 \in I$, then $\mu\big|_{\alpha_0} \equiv \mu\big|_{\alpha_1}$.

Definition

An automorphism $f : M \to M$ of a closed orientable surface is pseudo-Anosov if there are transverse singular foliations $\mathcal{M}^s, \mathcal{M}^u$ equipped with transverse measures μ^s, μ^u such that there exists some $\lambda > 1$ that:

$$f(\mathcal{M}^s, \mu^s) = (\mathcal{M}^s, \lambda\mu^s) \quad , \quad f(\mathcal{M}^u, \mu^u) = (\mathcal{M}^u, \lambda^{-1}\mu^u)$$