Linear Algebra in Orbit-finite Dimension

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Abstract

This thesis investigates systems of linear equations and inequalities that are orbit-finite, i.e. finite up to permutations of variables. Such systems naturally arise in the study of models of computation with infinite data, such as register automata and data Petri nets. The work is organised around four main contributions. First, we construct an orbit-finite basis for the vector space of orbit-finite functionals over an orbit-finite set. Second, we establish that the solvability of orbit-finite systems of linear equations is decidable. Third, we prove that while determining integer solutions for orbit-finite systems of linear inequalities is undecidable, the problem becomes decidable when restricted to rational solutions; this result is further extended to orbit-finite linear programming. Finally, we demonstrate that weak duality for finite linear programs generalises to the orbit-finite setting, and although strong duality does not hold in the general class, it can be recovered for the subclasses of row-finite and column-finite orbit-finite linear programs.

Arka[1]: Add poly-par, say it is an interesting connection. Add keywords

Contents

Chapter 1

Introduction

Contents

1.1 Orbit-finite systems

Solvability of finite systems of equations and inequalities is one of the oldest and one of the most fundamental algorithmic questions. In this thesis we study systems of linear equations and inequalities which are orbit-finite, i.e. possibly infinite but finite up to (certain) symmetries.

Throughout the thesis, we use the symbol $\mathbb{A}$ to denote a fixed countably infinite set. Elements of $\mathbb{A}$ are called *atoms*. We start with an example of an orbit-finite system of linear equations. For any $n \in \mathbb{N}$, let $\mathbb{A}^{(n)}$ denote the set of non-repeating tuples of n atoms

$$\mathbb{A}^{(n)} \stackrel{\text{def}}{=} \left\{\alpha_1 \ldots \alpha_n \in \mathbb{A}^n \ : \ \alpha_i \neq \alpha_j \text{ for all } i \neq j\right\} \ .$$

For succinctness of notation, we omit the commas and brackets, and write $\alpha_1 \ldots \alpha_n$ to represent an n-tuple instead of $(\alpha_1, \ldots, \alpha_n)$. Consider an infinite set of variables $\left\{\mathbf{x}(\alpha\beta)\ :\ \alpha\beta \in \mathbb{A}^{(2)}\right\}$ [1], and a system of equations indexed by elements of $\mathbb{A}$ (i.e. contains one equation for every $\alpha \in \mathbb{A}$):

$$\sum_{\beta \neq \alpha} \mathbf{x}(\alpha\beta) = 1 \qquad (\alpha \in \mathbb{A})\ .$$

Note that this is an infinite system of infinite equations (i.e., infinitely many variables appear in every equation). Every permutation π of $\mathbb{A}$ induces the permutation $\mathbf{x}(\alpha\beta) \mapsto \mathbf{x}(\pi(\alpha)\pi(\beta))$ of the variables. This in turn takes the set of variables $\{\mathbf{x}(\alpha\beta)\ :\ \beta \neq \alpha\}$, for any fixed $\alpha \in \mathbb{A}$, to the set of variables

$$\{\mathbf{x}(\pi(\alpha)\pi(\beta))\ :\ \pi(\beta) \neq \pi(\alpha)\} = \{\mathbf{x}(\pi(\alpha)\beta)\ :\ \beta \neq \pi(\alpha)\}$$

and hence induces the permutation

$$\sum_{\beta \neq \alpha} \mathbf{x}(\alpha\beta) = 1 \qquad \mapsto \qquad \sum_{\beta \neq \pi(\alpha)} \mathbf{x}(\pi(\alpha)\beta) = 1$$

on the set of equations. Notice that, although the variables and equations get permuted, the set of variables and the system of equations remain the same. Moreover, there are finitely many variables and equations up to these permutations (namely just one). Stated in the language of group theory, the numbers of orbits of variables and equations in the system are finite. Hence, we may call this system *orbit-finite* (in this particular example the system is a single orbit).

For any function $f : \mathbb{A} \to \mathbb{A}$ such that $f(\alpha) \neq \alpha$ for all $\alpha \in \mathbb{A}$, the vector $\mathbf{x}_f : \mathbb{A}^{(2)} \to \mathbb{R}$ defined as

$$\mathbf{x}_f(\alpha\beta) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{, if } \beta = f(\alpha) \\ 0 & \text{, otherwise} \end{cases}$$

is a solution of the system of equations. We return to this example in Chapters 2 and 4.

1.2 Why study orbit-finite systems?

We outline some motivations to investigate orbit-finite systems: we recall applications to automata, and advocate infinite systems as approximations of large finite ones.

[1] Note that we slightly deviate from the usual convention of writing the index in the subscript, and write $\mathbf{x}(\alpha\beta)$ instead of $\mathbf{x}_{\alpha\beta}$.

1.2.1 Applications in automata theory

A system of linear equations or inequalities, or a set in general, is *definable* if and only if it is orbit-finite ([5, Theorem 4.10]) [2]. The consequence of this in automata theory is that the systems of equations and inequalities that we encounter while studying models of computations over infinite alphabets such as data Petri nets ([19, Definition 2.1]) and register automata ([5, Definition 1.2]), are orbit-finite. We give an example of this phenomenon in the case of data Petri nets.

Example 1.1. Consider the data Petri net in Figure 1.1, which has a single place (drawn as a circle) and two transitions (drawn as rectangles), and whose tokens carry data values from the set $B = \{\star, \#\} \cup \mathbb{A}^{(2)}$. For any $\alpha\beta\gamma \in \mathbb{A}^{(3)}$, the

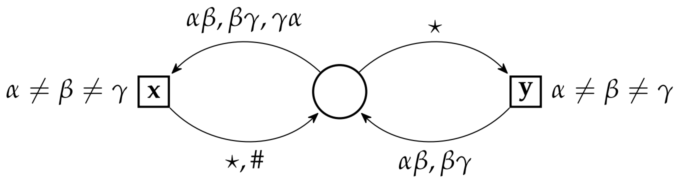

Figure 1.1

transition **x** can input three tokens, respectively carrying the values $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$, and output two tokens, one carrying the value $\star$ and the other carrying the value #. Similarly, for any $\alpha\beta\gamma \in \mathbb{A}^{(3)}$, the transition **y** can input one token carrying the value $\star$, and output two tokens, one of them carrying the value $\alpha\beta$ and the other carrying the value $\beta\gamma$.

Say we want to check whether the configuration $\mathbf{t} = 2 \cdot \#$ containing two tokens, each carrying the value #, is reachable from the configuration $\mathbf{s} = \star$ containing one token carrying the value $\star$

$$\star \longrightarrow^* 2 \cdot \# \,.$$

The configuration **t** is not reachable from **s** since from **s** we can only take the transition **y**, which does not lead to **t** but the Petri nets reaches a deadlock, i.e. no further transitions can be taken.

Now say we want to check the reachability in the relaxed semantics when the configurations of the Petri net are allowed to have a *negative* number of tokens. In this case, **t** is reachable from **s** if and only if the following system of (state) equations has a non-negative integer solution which assigns 0 to all but finitely many variables (variables $\mathbf{x}(\alpha\beta\gamma)$ and $\mathbf{y}(\alpha\beta\gamma)$, respectively, corresponds

[2]Here, by *definable*, we mean definable using a set builder expression [5, Definition 4.1].

to the number of firings of the corresponding transitions)

$$\sum_{\alpha\beta\gamma\in\mathbb{A}^{(3)}} \mathbf{x}(\alpha\beta\gamma) - \sum_{\alpha\beta\gamma\in\mathbb{A}^{(3)}} \mathbf{y}(\alpha\beta\gamma) = -1$$

$$\sum_{\alpha\beta\gamma\in\mathbb{A}^{(3)}} \mathbf{x}(\alpha\beta\gamma) = 2$$

$$\sum_{\gamma\neq\alpha,\beta} (\mathbf{y}(\alpha\beta\gamma) + \mathbf{y}(\gamma\alpha\beta) - \mathbf{x}(\alpha\beta\gamma) - \mathbf{x}(\beta\gamma\alpha) - \mathbf{x}(\gamma\alpha\beta)) = 0 \quad (\alpha\beta \in \mathbb{A}^{(2)}) \,.$$

There are three orbits of equations: two singleton orbits and one infinite orbit indexed by $\mathbb{A}^{(2)}$. The first equation ensures that the total number of consumed tokens carrying the value $\star$ is one more than the total number produced token with the same value. The second equation ensures that the total number of produced tokens carrying the value # is 2 (note that these tokens can not be consumed). The remaining set of equations, which form a single orbit, ensure that for every $\alpha\beta \in \mathbb{A}^{(2)}$, the number of consumed tokens carrying the value $\alpha\beta$ is equal to the number of produced tokens with the same value.

For any $\alpha\beta\gamma \in \mathbb{A}^{(3)}$ the vector which assigns 1 to the variables $\mathbf{y}(\alpha\beta\gamma)$, $\mathbf{y}(\beta\gamma\alpha)$, $\mathbf{y}(\gamma\alpha\beta)$, $\mathbf{x}(\alpha\beta\gamma)$, $\mathbf{x}(\beta\gamma\alpha)$, and 0 to the rest, is a solution to this system. From this solution we can construct the following run of the Petri net from $\mathbf{s}$ to $\mathbf{t}$. Configurations of the Petri net are written as finite formal sums of elements from $\mathbb{A}^{(3)} \cup \{\star, \#\}$ with integer coefficients.

$$\begin{tikzcd}
\mathbf{s} = \star \arrow[r, "\mathbf{y}(\alpha\beta\gamma)"] & (\alpha\beta + \beta\gamma) \arrow[r, "\mathbf{y}(\beta\gamma\alpha)"] & (\alpha\beta + 2\cdot\beta\gamma + \gamma\alpha - \star) \arrow[d, "\mathbf{x}(\alpha\beta\gamma)"] \\
\mathbf{t} = 2\cdot\# & (\alpha\beta + \beta\gamma + \gamma\alpha + \# - \star) \arrow[l, "\mathbf{x}(\beta\gamma\alpha)"'] & \beta\gamma + \# \arrow[l, "\mathbf{y}(\gamma\alpha\beta)"]
\end{tikzcd}$$

◀

1.2.2 Approximations of large but highly symmetric systems

Inspired by the slogan "The infinite is a good approximation of the very large finite" [3], we study orbit-finite systems as approximations of systems which are large but also highly symmetric, i.e., with small number of orbits. Consider an orbit-finite system of constraints (i.e. equations or inequalities) $\mathcal{U}$. Let $\mathcal{U}_n$ be the system that we get from $\mathcal{U}$ assuming $\mathbb{A}$ has n elements, instead of infinitely many. Although the number of constraints in $\mathcal{U}_n$ increases with n, for every sufficiently large n the system $\mathcal{U}_n$ has the same number of orbits of constraints

[3] This was the title of a lecture given by László Lovász at the Jagiellonian University, Krakow in June 2024.

as $\mathcal{U}$.

Example 1.2. For illustration, consider the system $\mathcal{U}$ consisting of the inequalities

$$\begin{aligned} \sum_{\alpha \in \mathbb{A}} \mathbf{v}(\alpha) &\leqslant 2 \\ \sum_{\beta \in \mathbb{A} \setminus \{\alpha\}} \mathbf{e}(\beta\alpha) &\leqslant \mathbf{v}(\alpha) + 3 \qquad (\alpha \in \mathbb{A}) \\ \sum_{\alpha\beta \in \mathbb{A}^{(2)}} \mathbf{e}(\alpha\beta) &\geqslant 16 \,. \end{aligned}$$

Intuitively, unknowns $\mathbf{v}(\alpha)$ and $\mathbf{e}(\alpha\beta)$ respectively correspond to vertices α and edges $\alpha\beta$ of an infinite directed clique. A solution of this system is an assignment of weights to vertices and edges of the infinite clique such that:

1. the total weight of vertices is at most 2,
2. for every vertex, the total weight of edges incoming to the vertex is at most the weight of the vertex plus 3, and
3. the total weight of edges is at least 16.

This system has three orbits of inequalities. If we assume $\mathbb{A}$ to have $n \geqslant 2$ elements instead of being infinite, we get a system $\mathcal{U}_n$ which has $n+2$ inequalities. But, irrespective of n, the system $\mathcal{U}_n$ has the same number of orbits of inequalities as $\mathcal{U}$, namely 3. ◀

One might wonder, whether the solvability of $\mathcal{U}$ coincides with the solvability of $\mathcal{U}_n$ for large enough n. That is, does there exists an $N \in \mathbb{N}$ such that solvability (existence of a solution) of $\mathcal{U}_n$ coincides with solvability of $\mathcal{U}$ for $n \geqslant N$? This is true for the system in the above example, since it is solvable and $\mathcal{U}_n$ is solvable for $n \geqslant 4$, but not for $n < 4$ (we leave it to the reader to verify these claims). In Chapter 7, which is the last chapter of this thesis, we show that this is not true for orbit-finite systems in general, but is true for some interesting subclasses.

1.3 Contributions of the thesis

This thesis is about lifting classical results in linear algebra from finite dimension to orbit-finite dimension. The main contributions of this thesis are summarised in the following paragraphs.

Orbit-finite bases: One of the fundamental results in linear algebra is that every vector space has a basis, and that any two bases of a vector space are isomorphic as sets (i.e. they have the same cardinality). In Chapter 3 we investigate to what extent these hold in the orbit-finite setting. In [6] the authors have also shown that there exists vector spaces with orbit-finite spanning sets but no orbit-finite basis. We complement their result by giving an example of a vector space having two non-isomorphic orbit-finite bases (§ 3.4).

Another important result in [6] is to show that the vector space of definable functions from an orbit-finite set to a field has an orbit-finite spanning set. As our first main result in Chapter 3 we strengthen their result to show such a vector space even has an orbit-finite basis (Theorem 3.1). This fundamental observation is used in later chapters where we study solvability of orbit-finite systems of equations and inequalities.

Linear Equations: Solvability of a finite systems of linear equations over integers is in PTIME ([34]). In Chapter 4 we show that solvability of orbit-finite systems of equations is decidable in EXPTIME, and in PTIME for fixed atom-dimension [4] (Theorem 4.6). Our algorithm is general enough to work within any commutative ring under mild effectivity assumptions.

Linear Programming: In Chapter 5 we discuss solvability of orbit-finite systems of linear inequalities. We show that existence of integer solutions is undecidable (Theorem 5.4), while existence of real solutions is decidable in EXPTIME, and in PTIME for fixed atom-dimension (Theorem 5.3). In short, in the orbit-finite setting, linear programming is decidable while integer linear programming is not. This is in contrast with the classical setting, where linear programming is in PTIME and integer linear programming is NP-complete. In Chapter 6 we extend our results to optimisation of an orbit-finite linear function with respect to an orbit-finite linear system of linear inequalities as constraints. Contrary to the finite dimension case, existence of an optimal solution is not guaranteed in our setting (Example 6.4); however the optimum can be computed in EXPTIME, and PTIME for fixed atom-dimension (Theorem 6.7).

As an intermediate step for deciding solvability of orbit-finite systems of linear inequalities, we introduce *polynomially parametrised* systems of linear inequalities. These are finite systems of linear inequalities where the coefficients are univariate polynomials over an integer *parameter*. For example, the following system is parametrised over n:

$$\begin{aligned} n \cdot (n-1) \cdot x + y &\leqslant 4 \cdot n^2 - 15 \\ x - n \cdot y &\geqslant 1\,. \end{aligned} \tag{1.1}$$

In § 5.5 we provide a reduction from solvability of orbit-finite systems to the

[4] To be defined later (Definition 2.17). For instance, the set $\mathbb{A}^{(n)}$ has atom dimension 4.

almost-all-solvability of polynomially parametrised systems (does there exist one and the same solution for almost all values of the parameter?). For instance $x = 1$ and $y = 0$ is a solution of (1.1) for all $n \geqslant 3$. We also give a polynomial-time decision procedure for the almost-all-solvability (Theorem 5.16).

Duality in Linear Programming: In Chapter 7 we investigate to which extent duality in finite linear programming extends to orbit-finite linear programming. We show that weak duality holds in the orbit-finite setting (Theorem 7.1) but strong duality doesn't (§ 7.3). However, strong duality can be recovered in two interesting subclasses of orbit-finite linear programming (Theorem 7.12). In this chapter we also discuss to what extent orbit-finite systems approximate large but highly symmetric finite systems.

1.3.1 Source materials

Arka[2]: TODO: say about new proofs

Chapters 3 and 4 are based on the paper

> Arka Ghosh, Piotr Hofman, and Sławomir Lasota: **Solvability of orbit-finite systems of linear equations**. LICS 2022.

Chapters 5 and 6 are based on the paper

> Arka Ghosh, Piotr Hofman, and Sławomir Lasota. **Orbit-finite linear programming**. LICS 2023 (*distinguished paper*) (*full version published in the Journal of the ACM*).

Chapter 7 consists of unpublished work done in collaboration with Joanna Fijalkow, Piotr Hofman, Sławomir Lasota and Szymon Toruńczyk.

1.4 Document conventions

Text formatting

Following the usual convention we *italicise* a piece of text to emphasise it. An italicised phrases or a symbol is *coloured in red* to denote that the surrounding text defines it. Instances of this can be found in page 7. These are the only instances in this chapter where this convention is used.

Sometimes, we fix the meaning of a symbol for an entire section or chapter. In such cases, the relevant sentences are coloured in blue, making them easier to locate and track. An instance of this can be found in page 25.

End-of-proof/Q.E.D. symbols

We use two symbols to denote end of proofs. For proofs of lemmas and theorems we use $\blacksquare$. For proofs of claims appearing inside proofs of lemmas and theorems, we use $\square$. Finally, we use $\blacktriangleleft$ to denote the ends of examples.

Some specific notations used in this document

We write $X \subseteq_{\mathrm{FIN}} Y$ to denote that the set X is a finite subset of the set Y.

As we mentioned in the beginning of this chapter, for atoms $\alpha_1, \ldots, \alpha_n$ we write $\alpha_1 \ldots \alpha_n$ to denote the tuple $(\alpha_1, \ldots, \alpha_n)$.

The symbol $\stackrel{\text{def}}{=}$ is used when an equality defines its left hand side.

Index

This thesis contains a significant number of definitions. To make it easier to keep track of them we have an added an index at the end of thesis to indicate the page numbers where each important term is defined.

Chapter 2

Preliminaries

Contents

2.1 Sets with atoms

Our definitions rely on basic notions and results of the theory of *sets with atoms* [5], also known as *nominal sets* [30].

Informally speaking, a set with atoms is a set that can have atoms, or other sets with atoms, as elements. Formally, we define the universe of sets with atoms by a suitably adapted *cumulative hierarchy* of sets, by transfinite induction: the only set of rank 0 is the empty set; and for a cardinal κ, a set of rank κ may contain, as elements, sets of rank smaller than κ as well as atoms. In particular, non-empty subsets of $\mathbb{A}$ have rank 1. The power set $\mathcal{P}(\mathbb{A})$ of $\mathbb{A}$ has rank 2.

The group $\mathrm{Aut}(\mathbb{A})$ of all permutations of $\mathbb{A}$, called in this thesis *automorphisms*, acts on sets with atoms hereditarily. Formally, by another transfinite induction, for $\pi \in \mathrm{Aut}(\mathbb{A})$ we define $\pi(X) \stackrel{\text{def}}{=} \{\pi(x) \ : \ x \in X\}$.

Example 2.1. Pick three distinct atoms α,β and γ. Let π be the automorphism such that $\pi(\alpha) = \gamma$, $\pi(\gamma) = \alpha$ and $\pi(\delta) = \delta$ for all $\delta \neq \alpha, \gamma$. Then $\pi(\{\mathbb{A} \setminus \{\alpha\}, \beta\}) = \{\mathbb{A} \setminus \{\gamma\}, \beta\}$. ◀

Via standard set-theoretic encodings of pairs or finite sequences we obtain, in particular, the point-wise action on pairs $\pi(x, y) = (\pi(x), \pi(y))$, and likewise on finite sequences. Relations and functions from X to Y are considered as subsets of $X \times Y$.

We restrict to sets with atoms that only depend on finitely many atoms, in the following sense. For $S \subseteq \mathbb{A}$, let

$$\mathrm{Aut}_S(\mathbb{A}) \stackrel{\text{def}}{=} \{\pi \in \mathrm{Aut}(\mathbb{A}) \ : \ \pi(\alpha) = \alpha \text{ for every } \alpha \in S\}$$

be the set of automorphisms that *fix* S. Automorphisms in $\mathrm{Aut}_S(\mathbb{A})$ are called *S-automorphisms*. A finite set $S \subseteq_{\text{FIN}} \mathbb{A}$ *supports* a set x if for all $\pi \in \mathrm{Aut}_S(\mathbb{A})$ we have $\pi(x) = x$. In this case we also say that x is *S-supported*.

Example 2.2. Any finite set of atoms is supported by itself. Any co-finite set of atoms is supported by its complement. For any $S \subseteq_{\text{FIN}} \mathbb{A}$, the set $\{T \subseteq \mathbb{A} \ : \ T \supseteq S\}$ is supported by S. The set $\mathbb{A}^*$ of finite sequences of atoms is supported by the empty set. ◀

A Function $f : B \to C$ is identified with its graph, i.e., the set $\{(b, f(b)) \ : \ b \in B\}$. For any $\pi \in \mathrm{Aut}(\mathbb{A})$ we have $\pi(b, f(b)) = (\pi(b), \pi(f(b))$. This implies

Lemma 2.3. *For any function $f : B \to C$, $\pi \in \mathrm{Aut}(\mathbb{A})$ and $b \in B$,*

$$\pi(f)(\pi(b)) = \pi(f(b)) \, .$$

Lemma 2.4. *A function $f : B \to C$ is supported by some $S \subseteq_{\text{FIN}} \mathbb{A}$ if and only if for all $\pi \in \mathrm{Aut}_S(A)$ and $b \in B$ we have $\pi(f(b)) = f(\pi(b))$.*

Lemma 2.5. *The domain of a function supported by $S \subseteq_{\text{FIN}} \mathbb{A}$ is also supported by S.*

We leave the proofs of the above three lemmas as exercises for the reader.

An S-supported set is also S'-supported, as long as $S \subseteq S'$. A set x is *finitely supported* if it has some finite support; in this case x has *the least support*, denoted $\mathrm{support}(x)$, called *the support* of x (cf. [5, Sect. 6]). Sets supported by the empty set $\emptyset$ we call *equivariant*. Finitely supported sets are preserved under finite union, intersection and cartesian products. Equivariant sets are preserved under arbitrary union, intersection and cartesian products.

Example 2.6. Given $\alpha, \beta \in \mathbb{A}$, the support of the set $\mathbb{A} \setminus \{\alpha, \beta\}$ is $\{\alpha, \beta\}$. The set $\mathbb{A}^2$ and the projection function $(\alpha, \beta) \mapsto \alpha : \mathbb{A}^2 \to \mathbb{A}$ are both equivariant; and the support of a tuple $\alpha_1 \ldots \alpha_n \in \mathbb{A}^n$, encoded as a set in a standard way, is the set of atoms $\{\alpha_1, \ldots, \alpha_n\}$ appearing in it. ◀

Lemma 2.7. support $(-)$ *is an equivariant function on the collection of finitely supported sets. That is, for any finitely supported x and $\pi \in \mathrm{Aut}(\mathbb{A})$,*

$$\mathrm{support}\,(\pi(x)) = \pi(\mathrm{support}\,(x))\ .$$

Proof. We start by proving the following claim.

Claim 2.7.1. *If x is supported by $S \subseteq_{\mathrm{FIN}} \mathbb{A}$ then $\pi(x)$ is supported by $\pi(S)$.*

Proof. Pick arbitrary $\sigma \in \mathrm{Aut}_{\pi(S)}(\mathbb{A})$. Then $\pi^{-1} \circ \sigma \circ \pi \in \mathrm{Aut}_S(\mathbb{A})$. Hence

$$(\pi^{-1} \circ \sigma \circ \pi)(x) = x\ .$$

Applying π to both sides we get

$$\sigma(\pi(x)) = \pi(x)\ .$$

As σ is chosen arbitrarily, we deduce that $\pi(S)$ supportes $\pi(x)$. □

Using the above claim we conclude

$$\mathrm{support}\,(\pi(x)) \subseteq \pi(\mathrm{support}\,(x))\ . \tag{2.1}$$

By the same claim, replacing x with $\pi(x)$ and π with π^{-1}, we get

$$\mathrm{support}\,(x) \subseteq \pi^{-1}(\mathrm{support}\,(\pi(x)))\ .$$

which implies

$$\pi(\mathrm{support}\,(x)) \subseteq \mathrm{support}\,(\pi(x))\ . \tag{2.2}$$

By (2.1) and (2.2) we get support $(\pi(x)) = \pi(\mathrm{support}\,(x))$. ■

A set is called *atom-less* if no atoms appear inside it, or its elements, or the elements of its elements, and so on. Clearly, atom-less sets are equivariant. The most common examples of atom-less sets we will see in this thesis are numbers and finite sets.

A set is called *hereditarily finitely supported* if it's finitely supported, its elements are finitely supported, element of elements are finitely supported, and so on. For example, for any $S \subseteq_{\text{FIN}} \mathbb{A}$, both the sets

$$\{T \subseteq_{\text{FIN}} \mathbb{A} \,:\, T \supseteq S\} \quad \text{and} \quad \{T \subseteq \mathbb{A} \,:\, T \supseteq S\}$$

are finitely supported but only the first one is hereditarily finitely supported. From now on, we will only consider the finitely supported sets which are also hereditarily finitely supported.

2.1.1 Orbit-finite sets

For $S \subseteq_{\text{FIN}} \mathbb{A}$, two atoms or sets with atoms x, y are *in the same S-orbit* if $\pi(x) = y$ for some $\pi \in \text{Aut}_S(\mathbb{A})$. This equivalence relation splits all sets with atoms into equivalence classes, which we call *S-orbits*; $\emptyset$-orbits we call equivariant orbits. By the very definition, every S-orbit O is S-supported: support $(O) \subseteq S$ and, even if the inclusion is strict (which may happen only for singleton orbits), O is also a $(\text{support}\,(O))$-orbit. When the set S is irrelevant, we simply speak of an *orbit*, meaning an S-orbit for some $S \subseteq_{\text{FIN}} \mathbb{A}$. Every S-supported set is a union of (necessarily disjoint) S-orbits; the *set is orbit-finite* if this union is finite. As the next lemma says, orbit-finiteness is stable under orbit-refinement and taking finitely supported subsets:

Lemma 2.8 ([5, Theorem 3.16])**.** *For any $S \subseteq T \subseteq_{\text{FIN}} \mathbb{A}$, a finite union of S-orbits is also a finite union of T-orbits (although the number of orbits may increase).*

Notice that orbit-finite sets are finitely supported by definition.

Example 2.9. Some examples of orbit-finite sets are:

1. $\mathbb{A}$ (1 orbit);
2. $\mathbb{A} \setminus \{\alpha\}$ for some $\alpha \in \mathbb{A}$ (1 orbit);
3. $\mathbb{A}^2$ (2 orbits: diagonal $\{\alpha\alpha \,:\, \alpha \in \mathbb{A}\}$ and non-diagonal $\mathbb{A}^{(2)} = \{\alpha\beta \,:\, \alpha \neq \beta \in \mathbb{A}\}$);
4. $\mathbb{A}^3$ (5 orbits, corresponding to the equality types of triples);
5. non-repeating n-tuples of atoms $\mathbb{A}^{(n)} = \{\alpha_1 \ldots \alpha_n \in \mathbb{A}^n \,:\, \alpha_i \neq \alpha_j \text{ for all } i \neq j\}$ (1 orbit);
6. n-sets of atoms $\binom{\mathbb{A}}{n} \stackrel{\text{def}}{=} \{X \subseteq \mathbb{A} \,:\, |X| = n\}$ (1 orbit).
7. Any finite set X containing only finitely supported sets or atoms is an orbit-finite set supported by $S = \cup_{x \in X} \text{support}\,(x)$. The set of its S-orbits is $\{\{x\} \,:\, x \in X\}$.

Lemma 2.23-vii, stated later in this section implies that elements of X need to be finitely supported for X to be finitely supported, and hence also for being orbit-finite.

The set $\mathcal{P}_{fin}(\mathbb{A})$ of all finite subsets of atoms is orbit-infinite, since for every $n \in \mathbb{N}$, the set $\{S \subseteq \mathbb{A} \ : \ |S| = n\} \subseteq \mathcal{P}_{\text{fin}}(\mathbb{A})$ is an equivariant orbit. ◀

Example 2.10. For $n, m \in \mathbb{N}$, equivariant orbits inside $\mathbb{A}^{(n)} \times \mathbb{A}^{(m)}$ are exactly of the form

$$\left\{(\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_m) \in \mathbb{A}^{(n)} \times \mathbb{A}^{(m)} \ : \ \alpha_i = \beta_j \iff \iota(i) = j\right\}$$

where $\iota : \{1, \ldots, n\} \to \{1, \ldots, m\}$ is a partial injection. We leave it to the reader to verify the details. ◀

Notation 2.11. For an atom or set with atoms x and $S \subseteq_{\text{FIN}} \mathbb{A}$, by $\text{orbit}_S(x)$ we denote the S-orbit containing x. That is,

$$\text{orbit}_S(x) \stackrel{\text{def}}{=} \{\pi(x) \ : \ \pi \in \text{Aut}_S(\mathbb{A})\} \ .$$

For example,

$$\text{orbit}_S(\alpha\beta) = (\mathbb{A} \setminus S)^{(2)}$$

for any $\alpha, \beta \in \mathbb{A} \setminus S$. When $S = \emptyset$ we write $\text{orbit}(x)$ instead of $\text{orbit}_\emptyset(x)$. For an S-supported set X, by $\text{Orbits}_S(X)$ we denote the set of S-orbits inside X. That is,

$$\text{Orbits}_S(X) \stackrel{\text{def}}{=} \{\text{orbit}_S(x) \ : \ x \in X\} \ .$$

When $S = \emptyset$ we write $\text{Orbits}(X)$ instead of $\text{Orbits}_\emptyset(X)$.

As an immediate corollary of Lemma 2.8 we get:

Lemma 2.12. *Orbit-finite sets are closed under finite unions and intersections.*

Proof. Pick two orbit-finite sets X and Y. Let

$$S \stackrel{\text{def}}{=} \text{support}\,(X) \cup \text{support}\,(Y) \ .$$

Lemma 2.8 implies $\text{Orbits}_S(X)$ and $\text{Orbits}_S(Y)$ are both finite. The sets $X \cup Y$ and $X \cap Y$ are supported by S. Moreover,

$$\text{Orbits}_S(X \cup Y) = \text{Orbits}_S(X) \cup \text{Orbits}_S(Y) \supseteq \text{Orbits}_S(X \cap Y) \, .$$

Hence $X \cup Y$ and $X \cap Y$ are also orbit-finite. ■

Recall that a function $f : B \to C$ is identified with its graph

$$\{(b, f(b)) \; : \; b \in B\} \; .$$

If f is finitely supported, we call it orbit-finite if the set $\{(b, f(b)) \; : \; b \in B\}$ is orbit-finite.

Lemma 2.13. *A finitely supported function $f : B \to C$ is orbit-finite if and only if B is orbit-finite.*

Proof. Let S be the support of f. Then S also supports B (Lemma 2.5).

Claim 2.13.1. *For $b, b' \in B$, the pairs $(b, f(b))$ and $(b', f(b'))$ are in the same S-orbit if and only if b and b' are in the same S-orbit.*

The above claim implies that for every subset U of B, U is an S-orbit if and only if the subset $\{(b, f(b)) \; : \; b \in U\}$ of $\{(b, f(b)) \; : \; b \in B\}$ is an S-orbit. Since the function $b \mapsto (b, f(b))$ is a bijection, this implies B and $\{(b, f(b)) \; : \; b \in B\}$ have the same number of S-orbits. This proves the lemma. ■

A set is called *hereditarily orbit-finite* if it is orbit-finite, its elements are orbit-finite, the elements of its elements are orbit-finite, and so on. For example, the set $\left\{(\mathbb{A} \setminus T)^2 \; : \; T \subseteq (\mathbb{A} \setminus S), \; |T| = 3\right\}$ is hereditarily orbit-finite. The set $\{T \subseteq \mathbb{A} \; : \; \text{both } T \text{ and } \mathbb{A} \setminus T \text{ are infinite}\}$ is an equivariant orbit but is not hereditarily orbit-finite. From now on, we will only consider those orbit-finite sets which are also hereditarily orbit-finite.

Remark 2.14. The reason behind the above assumption is that hereditarily orbit-finite sets are finitely representable ([5, Thereom 4.10]) and all finitely-supported transformations between these are effectively computable (for a detailed presentation we refer to [10] or [5, Sect. 4, 8, 9]). They also enjoy better closure properties that general orbit-finite sets. For example, orbit-finite sets are not closed under taking products (Example 2.15) but hereditarily orbit-finite sets are (Lemma 2.16).

Example 2.15. Let X be the orbit $\{T \subseteq \mathbb{A} \; : \; \text{both } T \text{ and } \mathbb{A} \setminus T \text{ are infinite}\}$. For every $n \in \mathbb{N}$ the set

$$\{(T_1, T_2) \; : \; |T_1 \cap T_2| = n\}$$

is an equivariant orbit inside $X \times X$. Hence $X \times X$ cannot be orbit-finite. ◀

Lemma 2.16. *Hereditarily orbit-finite sets are closed under taking finite products.*

Proof. Let X and Y be two hereditarily orbit-finite sets. We show $X \times Y$ is also hereditarily orbit-finite. It is clear that elements of $X \times Y$ are orbit-finite. Hence, we only need to show that $X \times Y$ is orbit-finite. Let

$$S \stackrel{\text{def}}{=} \text{support}\,(X) \cup \text{support}\,(Y)\ .$$

Then S supports $X \times Y$. We show that $\text{Orbits}_S(X \times Y)$ is finite. Firstly,

$$\text{Orbits}_S(X \times Y) = \bigcup_{B \in \text{Orbits}_S(X)} \left(\bigcup_{C \in \text{Orbits}_S(Y)} \text{Orbits}_S(B \times C) \right) ,$$

and hence it suffices to show that $\text{Orbits}_S(B \times C)$ is finite for every $B \in \text{Orbits}_S(X)$ and $C \in \text{Orbits}_S(Y)$.

Pick such B and C. Pick $b \in B$. Let $S_b \stackrel{\text{def}}{=} S \cup \text{support}\,(b)$. For every $D \in \text{Orbits}_{S_b}(C)$ pick $c_D \in D$. By Lemma 2.8 the set $\text{Orbits}_{S_b}(C)$ is finite. To finish the proof we show

$$B \times C = \bigcup_{D \in \text{Orbits}_S(C)} \text{orbit}_S((b, c_D))\ .$$

Pick $(b', c') \in B \times C$. There exists $\pi \in \text{Aut}_S(\mathbb{A})$ such that $\pi(b) = b'$. Let $D' \stackrel{\text{def}}{=} \text{orbit}_{S_b}(\pi^{-1}(c'))$. There exists $\sigma \in \text{Aut}_{S_b}(\mathbb{A})$ such that $\sigma(c_D) = \pi^{-1}(c')$. Since $S \subseteq S_b$, we have $\sigma \in \text{Aut}_S(\mathbb{A})$. Hence $\pi \circ \sigma \in \text{Aut}_S(\mathbb{A})$. This implies,

$$(\pi \circ \sigma)(b, c_D) = \pi(b, \pi^{-1}(c')) = (b', c')\ .$$

Since $(b', c') \in B \times C$ was arbitrarily chosen, this finishes the proof. ■

Definition 2.17. Let $S \subseteq_{\text{FIN}} \mathbb{A}$. We define the *S-atom dimension of an S-orbit* O, written S-dim(O), as the size of support (x) for some (every) element $x \in O$, but not counting elements of S:

$$S\text{-dim}(O) \stackrel{\text{def}}{=} |\text{support}\,(x) \setminus S|.$$

The choice of x is irrelevant due to Lemma 2.7.

S-atom dimension of an orbit-finite set supported by S is the maximum of atom-dimension of its S-orbits. When S is clear from the context we omit S and speak of *atom dimension*.

Notice that the assumption of herditarily finitely supportedness guarantees atom-dimension of orbit-finite sets are finite.

2.1.2 Canonical orbits

For this subsection, fix S to be an arbitrary finite subset of $\mathbb{A}$. For a positive integer $k > 0$, denote by $\mathbf{S}_k$ the symmetric group on $\{1, \ldots, k\}$. Given a subgroup G of $\mathbf{S}_k$, we denote by $(\mathbb{A} \setminus S)^{(k)}/G$ the set of non-repeating k-tuples of atoms from $(\mathbb{A} \setminus S)$ modulo coordinate permutations from the group G. More formally, we define an equivalence in $(\mathbb{A} \setminus S)^{(k)}$, where a tuple $\overline{\alpha} = \alpha_1 \ldots \alpha_k \in (\mathbb{A} \setminus S)^{(k)}$ is equivalent to every tuple $\overline{\alpha} \circ \sigma \stackrel{\text{def}}{=} \alpha_{\sigma(1)} \ldots \alpha_{\sigma(k)}$, for $\sigma \in G$. The equivalence classes are thus finite.

Notation 2.18. For $\overline{\alpha} = \alpha_1 \ldots \alpha_k \in (\mathbb{A} \setminus S)^{(k)}$, let $\overline{\alpha}/G$ denotes its equivalence class. That is,

$$\overline{\alpha}/G \stackrel{\text{def}}{=} \left\{\alpha_{\sigma(1)} \ldots \alpha_{\sigma(k)} \ : \ \sigma \in G\right\}$$

This definition is easily extended to subsets. That is, for $X \subseteq (\mathbb{A} \setminus S)^{(k)}$,

$$X/G \stackrel{\text{def}}{=} \{\overline{\alpha}/G \ : \ \overline{\alpha} \in X\} \ .$$

Example 2.19. Let $k = 3$ and let G be the subgroup of $\mathbf{S}_3$ generated by the cyclic shift σ to the right: $\sigma \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right)$. Then for any $\alpha\beta\gamma \in \mathbb{A}^{(3)}$

$$\alpha\beta\gamma/G = \{\alpha\beta\gamma, \beta\gamma\alpha, \gamma\alpha\beta\} \ ,$$

and

$$\mathbb{A}^{(3)}/G = \left\{\{\alpha\beta\gamma, \beta\gamma\alpha, \gamma\alpha\beta\} \ : \ \alpha\beta\gamma \in \mathbb{A}^{(3)}\right\} \ .$$

◀

Definition 2.20. Orbits of the form $(\mathbb{A} \setminus S)^{(k)}/G$ are called *canonical S-orbits*, and a finite union of canonical S-orbits is called a *canonical S-orbit-finite set*. Canonical S-orbit-finite sets of the form

$$(\mathbb{A} \setminus S)^{(k_1)} \uplus \cdots \uplus (\mathbb{A} \setminus S)^{(k_n)}$$

are called *straight*.

The following lemma justifies calling orbits of the form $(\mathbb{A} \setminus S)^{(k)}/G$ to be canonical.

Lemma 2.21. *Every S-orbit is in S-supported bijection with $(\mathbb{A} \setminus S)^{(k)}/G$, where k is the S-atom dimension of the S-orbit and some subgroup G of $\mathbf{S}_k$.*

The proof of this lemma is almost identical to the proof of Theorem 6.3 in [5], so we skip it.

Remark 2.22. We defined the cumulative hierarchy of sets with atoms starting from the infinite set $\mathbb{A}$, seen as a structure $(\mathbb{A}, =)$ with no additional relations other than equality (hence called *equality atoms*). We can also define the cumulative hierarchy starting from an arbitrary fixed logical structure $\mathbb{X}$, in which case $\mathrm{Aut}(\mathbb{X})$ denotes the set of all structure preserving automorphisms of $\mathbb{X}$, and acts hereditarily on sets in this cumulative hierarchy. For a finite subset S of $\mathbb{X}$

$$\mathrm{Aut}_S(\mathbb{X}) \stackrel{\text{def}}{=} \{\pi \in \mathrm{Aut}(X) \ : \ \pi(\alpha) = \alpha \text{ for all } \alpha \in S\} \ ,$$

and an S-orbit is a set of the form $\{\pi(x) \ : \ \pi \in \mathrm{Aut}_S(\mathbb{X})\}$ for some set or atom x. A set will be called orbit-finite if it has finitely many S-orbits for some $S \subseteq_{\text{FIN}} \mathbb{X}$.

Some common examples of atoms other than equality that appear in the literature are:

1. $(\mathbb{D}, <)$, a countable dense linear order without endpoints, also called *ordered atoms*. $\mathrm{Aut}(\mathbb{D})$ is the set of all order preserving bijections. This structure is isomorphic to the set of rational with the usual ordering.

2. $(\mathbb{Z}, <)$, integers with the natural order. $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}$ is the set of all translations $n \mapsto (n + k)$ for $k \in \mathbb{Z}$.

3. An infinite dimensional vector space over a finite field. When the finite field is $\mathbb{F}_2$ it is called *bit-vector atoms*. Automorphisms are vector space isomorphisms.

4. The *Rado graph* [1], also called *graph atoms*. Automorphisms are graph isomorphisms.

In this thesis we focus on orbit-finite systems of equations and inequalities with equality atoms.

2.1.3 Collection of lemmas

We finish this section with a collection of lemmas are used in later parts of the thesis. Proofs of most of these lemmas are left to the reader as exercise.

Lemma 2.23. *For any arbitrary $S \subseteq_{\text{FIN}} \mathbb{A}$, S-supported sets X and Y, and S-supported function $f : X \to Y$:*

(i) Every $\pi \in \mathrm{Aut}_S(\mathbb{A})$ induces a permutation on X such that for every S-orbit O included in X we have $\pi(O) = O$.

(ii) For every $\pi, \sigma \in \mathrm{Aut}(\mathbb{A})$, if $\pi(\alpha) = \sigma(\alpha)$ for all $\alpha \in S$, then $\pi(X) = \sigma(X)$.

[1] `https://en.wikipedia.org/wiki/Rado_graph`

(iii) *For every $T_1 \supseteq T_2 \supseteq S$, T_1-orbit $X_1 \subseteq X$ and T_2-orbit $X_2 \subseteq X$, either $X_1 \cap X_2 = \emptyset$ or $X_1 \subseteq X_2$.*

(iv) *For every orbit finite subset $Z \subseteq X$ the set $f(Z)$ is also orbit finite.*

(v) *For every $x \in X$, $f(x)$ is supported by $S \cup \text{support}(x)$. Moreover, if f is injective then $\text{support}(f(x)) \cup S = \text{support}(x) \cup S$.*

(vi) *The set X contains an element x such that $\text{support}(x) \subseteq (S \cup T)$ if and only if the size of T is more than the S-atom-dimension of X.*

(vii) *If X is finite then each of its elements are supported by S.*

Notation 2.24. For any $T \subseteq_{\text{FIN}} \mathbb{A}$ we use $\text{Aut}(T)$ to denote the set of automorphisms π of $\mathbb{A}$ such that $\pi(\alpha) = \alpha$ for all $\alpha \in \mathbb{A} \setminus T$. For any $S \subseteq_{\text{FIN}} \mathbb{A} \setminus T$ we define $\text{Aut}_{S \cup \{T\}}(\mathbb{A})$ to be the set of automorphisms $\pi \in \text{Aut}_S(\mathbb{A})$ such that $\pi(T) = T$. Notice that

$$\text{Aut}(T) \subseteq \text{Aut}_{S \cup \{T\}}(\mathbb{A}) \subseteq \text{Aut}(\mathbb{A}) .$$

Lemma 2.25. *For any $S \subseteq_{\text{FIN}} \mathbb{A}$, $T \subseteq_{\text{FIN}} \mathbb{A} \setminus S$ and S-orbit O:*

(i) *For every element $x' \in O$ supported by $(S \cup T)$*

$$\begin{aligned} &\{x \in O \ : \ \text{support}(x) \subseteq (S \cup T)\} \\ = &\left\{\pi(x') \ : \ \pi \in \text{Aut}(T)\right\} \\ = &\left\{\pi(x') \ : \ \pi \in \text{Aut}_{S \cup \{T\}}(\mathbb{A})\right\} . \end{aligned}$$

(ii) *The number of elements of O supported by $(S \cup T)$ is at most $|T|!$.*

(iii) *O admits an S-supported surjection from $(\mathbb{A} \setminus S)^{(d)}$ for some $d \in \mathbb{N}$ such that inverse image of every $p \in O$ is finite.*

(iv) *O is either a singleton set or is infinite.*

(v) *Let O' be another S-orbit of S-atom dimension d'. The number of S-orbits in $O \times O'$ is at most $(d + d')^{d+d'}$.*

Proof. The first two items are left for the reader to verify. We provide proofs for the remaining.

(iii): This follows from Lemma 2.21.

(iv): Consider some element $x \in O$. If $\text{support}(x) \subseteq S$ then for every S-automorphism $\pi \in \text{Aut}_S(\mathbb{A})$ we have $\pi(x) = x$, and hence $O = \{x\}$. Otherwise, choose any $\alpha \in \text{support}(x) \setminus S$. For each $\beta \in \mathbb{A} \setminus (\text{support}(x) \cup S)$ there exists an S-automorphism π_β that maps α to β and preserves support $(x) \setminus \{\alpha\}$. By Lemma 2.7, support $(\pi_\beta(x)) \neq$ support $(\pi_\gamma(x))$ for $\beta \neq \gamma$, which implies $\pi_\beta(x) \neq \pi_\gamma(x)$ for $\beta \neq \gamma$. Therefore O is infinite.

(v): Using Lemma 2.25-iii, let $f : (\mathbb{A} \setminus S)^{(d)} \to O$ and $f' : (\mathbb{A} \setminus S)^{(d')} \to O'$ be two S-supported surjections. Then

$$f \times f' : ((\mathbb{A} \setminus S)^{(d)} \times (\mathbb{A} \setminus S)^{(d')}) \to (O \times O')$$

defined as

$$(f \times f')(\overline{\alpha}, \overline{\alpha'}) = (f(\overline{\alpha}), f'(\overline{\alpha}'))$$

is also an S-supported surjection. This implies that for any S-orbit $V \subseteq (O \times O')$ the inverse image $(f \times f')^{-1}(V)$ of V under $(f \times f')$ is supported by S and hence is a union of S-orbits. As a consequence, the number of S-orbits in $(O \times O')$ is at most the number of S-orbits in $((\mathbb{A} \setminus S)^{(d)} \times (\mathbb{A} \setminus S)^{(d')})$. To finish the proof we show that the latter is at most $(d + d')^{d+d'}$.

Pick arbitrary $R \subseteq_{\text{FIN}} (\mathbb{A} \setminus S)$ of size $d + d'$. Using Lemma 2.23-vi, every S-orbit of $((\mathbb{A} \setminus S)^{(d)} \times (\mathbb{A} \setminus S)^{(d')})$ contains at least one element supported by R. Such an element has to come from the set $(R^{(d)} \times R^{(d')})$. Hence the number of such elements cannot be larger than $(d + d')^{d+d'}$. As a consequence the number of S-orbits in $((\mathbb{A} \setminus S)^{(d)} \times (\mathbb{A} \setminus S)^{(d')})$ is at most $(d + d')^{d+d'}$. ■

2.2 Finitely supported vector spaces

For the remainder of this chapter fix R to be an arbitrary commutative ring with identity. For any orbit-finite set B, the set of all functions from B to R is closed under point-wise addition and multiplication by scalars, and hence forms a vector space. [2] We denote this space by $B \to R$. An element of this space is called an *R vector indexed by B* or simply a *vector* when B and R are clear from the context. We use lowercase boldface letters $\mathbf{a}, \mathbf{b}, \ldots$ to denote vectors. Let $S \subseteq_{\text{FIN}} \mathbb{A}$ to be the support of B. The vector space $B \to R$ is supported by S, i.e., it is supported by S as a set and the addition and scalar multiplication operations commute with S-automorphisms.

Lemma 2.26. *For $r \in R$, $\mathbf{x}, \mathbf{y} \in B \to R$ and $\pi \in \mathrm{Aut}_S(\mathbb{A})$ we have $\pi(r \cdot \mathbf{x}) = r \cdot \pi(\mathbf{x})$ and $\pi(\mathbf{x} + \mathbf{y}) = \pi(\mathbf{x}) + \pi(\mathbf{y})$.* [3]

We skip the proof since it is straightforward. An *orbit-finite vector* is a vector

[2] Technically it is a module since R is not a field. However, with a slight abuse of nomenclature, we will call it a vector space.

[3] The equality holds even if π does not support S. But then $\pi(\mathbf{x})$ and $\pi(\mathbf{y})$ may not be elements of $B \to \mathbb{R}$.

which is orbit-finite as a function. The set of orbit-finite vectors in $B \to R$ forms a subspace since:

Lemma 2.27. *Orbit-finite vectors in $B \to R$ are closed under sum and multiplication by scalars.*

Proof. For any two orbit-finite vectors $\mathbf{a}, \mathbf{b} \in B \to R$ and $r \in R$ the vectors $\mathbf{a} + \mathbf{b}$ and $r \cdot \mathbf{v}$ are respectively supported by $(\text{support}(\mathbf{a}) \cup \text{support}(\mathbf{b}))$ and $\text{support}(\mathbf{a})$. Since the set B is orbit-finite, lemma 2.13 implies $\mathbf{a} + \mathbf{b}$ is also orbit-finite. ∎

The space of all orbit-finite vectors in $B \to R$ is denoted by $\text{Lin}_R(B)$. For a subset $X \subseteq B$, let $\mathbf{1}_X : B \to R$ denote the characteristic function on X. If X is orbit-finite, $\mathbf{1}_X \in \text{Lin}_R(B)$. When $X = \{b\}$, we write $\mathbf{1}_b$ instead of $\mathbf{1}_{\{b\}}$.

Notation 2.28. Consider $S \subseteq_{\text{FIN}} \mathbb{A}$ and a S-supported vector $\mathbf{v} \in \text{Lin}_R(B)$. Then $\mathbf{v}$ is constant, restricted to every S-orbit $O \subseteq B$. This allows us to write $\mathbf{v}(O) \in R$ in place of $\mathbf{v}(x)$ for $x \in O$ and represent $\mathbf{v}$ as the linear combination

$$\mathbf{v} = \sum_{O \in \text{Orbits}_S(B)} \mathbf{v}(O) \cdot \mathbf{1}_O \,, \tag{2.3}$$

of characteristic vectors $\mathbf{1}_O$ of $O \in \text{Orbits}_S(B)$.

The *domain of a vector* $\mathbf{v} \in B \to R$, denoted as $\text{dom}(\mathbf{v})$ is defined as

$$\text{dom}(\mathbf{v}) \stackrel{\text{def}}{=} \{b \in B \,:\, \mathbf{v}(b) \neq 0\} \,.$$

A *finite vector* is a vector whose domain is finite. It is easy to see that finite vectors in $B \to R$ form a vector subspace of $B \to R$. We denote this space as $\text{FinLin}_R(B)$.

Notation 2.29. $\text{FinLin}_R(B)$ can also be thought of as the vector space generated by B. Keeping this is mind we equate B with the set of vector $\{\mathbf{1}_b \,:\, b \in B\}$. Any finite vector $\mathbf{v} \in \text{FinLin}(B)$ can be written as

$$\mathbf{v} = \sum_{b \in \text{dom}(\mathbf{v})} \mathbf{v}(b) \cdot b \,.$$

As observed above, the set of vectors $\{\mathbf{1}_b \,:\, b \in B\}$ forms a basis of the space $\text{FinLin}_R(B)$. A vector $\mathbf{1}_b$, for $b \in B$, is supported by $\text{support}(b) \cup S$ and hence is an orbit-finite vector. This implies $\text{FinLin}_R(B)$ is a subspace of $\text{Lin}_R(B)$.

2.2.1 Finitely supported subspaces

A subspace of $\mathrm{FinLin}(B)$ or $\mathrm{Lin}(B)$ is called finite supported if it is finitely supported as a set.

Example 2.30. For any orbit-finite set B, the set $V_0(B)$ of all $\mathbf{v} \in \mathrm{FinLin}_R(B)$ such that

$$\sum_{b \in B} \mathbf{v}(b) = 0$$

is a subspace of $\mathrm{FinLin}(B)$ supported by $S = \mathrm{support}\,(B)$. ◀

Example 2.31. The set of all vectors $\mathbf{v} \in \mathrm{Lin}(\mathbb{A})$ such that $\mathbf{v}(\alpha) = \mathbf{v}(\beta)$ for all $\alpha, \beta \in \mathbb{A}$ is an equivariant subspace of $\mathrm{Lin}(\mathbb{A})$. ◀

Example 2.32. The set of all vectors $\mathbf{v} \in \mathrm{FinLin}\left(\mathbb{A}^{(2)}\right)$ such that $\mathbf{v}(\alpha\beta) + \mathbf{v}(\beta\alpha) = 0$ for all $\alpha\beta \in \mathbb{A}^{(2)}$ is an equivariant subspace of $\mathrm{FinLin}\left(\mathbb{A}^{(2)}\right)$. ◀

Since we do not investigate finitely supported vector spaces in this thesis, we mention some important results of [6, 36] regarding them.

Definition 2.33. For $S \subseteq_{\mathrm{FIN}} \mathbb{A}$ and a vector space V with $S = \mathrm{support}\,(V)$, define the *length* of V as the length n of longest strictly increasing chain

$$V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n \subsetneq V$$

of S-supported subspaces of V.

Example 2.34. Recall Example 2.30. The length of $\mathrm{FinLin}(\mathbb{A})$ is at least 2 since $\{\mathbf{0}\} \subsetneq V_0(\mathbb{A}) \subsetneq \mathrm{FinLin}(\mathbb{A})$. ◀

Definition 2.35. For vector spaces V an W, their *product* $V \times W$ is the vector space with elements $(\mathbf{v}, \mathbf{w})$ for $\mathbf{v} \in V$ and $\mathbf{w} \in W$, with addition and scalar multiplication defined pointwise.

Lemma 2.36. *Any S-supported vector space V of finite length is spanned by an orbit-finite set.*

Proof. Construct a sequence of vectors $\mathbf{0} = \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$ in V such that for $n \geqslant 1$, $\mathbf{v}_n$ is not inside the vector space V_{n-1} spanned by $\cup_{k<n}\mathrm{orbit}_S(\mathbf{v}_k)$. Then $V_1 \subsetneq V_2 \subsetneq$ is a strictly increasing chain of S-supported vector subspaces of V. Which must be finite since the S-length of V is finite. Say the sequence stops at $\mathbf{v}_N$ for some $N \in \mathbb{N}$. Then $\cup_{k \leqslant N}\mathrm{orbit}_S(\mathbf{v}_k)$ must span V. ∎

Lemma 2.37 ([6, Lemma 4.3])**.** *For $S \subseteq_{\mathrm{FIN}} \mathbb{A}$, S-supported vector spaces W and V:*

1. *The S-length of $V \times W$ is the sum of S-lengths of V and W.*

2. *If $V \subsetneq W$, then the S-length of V is smaller than the S-length of W.*

The authors of [6] also gave an upper bound on lengths of vector spaces of the form $\mathrm{FinLin}\left((\mathbb{A} \setminus S)^{(n)}\right)$ for $n \in \mathbb{N}$ and $S \subseteq_{\mathrm{FIN}} \mathbb{A}$ ([6, Corollary 4.9]). This was improved in [36] where a matching lower and upper bound is given.

Theorem 2.38 ([36, Corollary 4.4])**.** *For $n \in \mathbb{N}$ and $S \subseteq_{\mathrm{FIN}} \mathbb{A}$, the length of $\mathrm{FinLin}\left((\mathbb{A} \setminus S)^{(n)}\right)$ is equal to a_n where $(a_k)_{k \in \mathbb{N}}$ is the OEIS sequence A005425(k).*

Combining Theorem 2.38, lemma 2.37, and item Lemma 2.25-iii we can conclude:

Corollary 2.39. *For any $S \subseteq_{\mathrm{FIN}} \mathbb{A}$ and orbit-finite set B supported by S, the length of $\mathrm{FinLin}(B)$ is finite.*

Proof. Using Lemma 2.25-iii, there is an S-supported surjection $f : B' \to B$ from a straight orbit-finite set B' supported by S (Definition 2.20) to B such that $f^{-1}(b)$ is finite for every $b \in B$. Recall Notation 2.29. The function f induces the linear map $\mathrm{FinLin}(f)$ from $\mathrm{FinLin}(B)$ to $\mathrm{FinLin}(B')$ defined as:

$$\mathrm{FinLin}(f)(\mathbf{v}) \stackrel{\mathrm{def}}{=} \sum_{b \in B} \mathbf{v}(b) \cdot \left(\sum_{b' \in f^{-1}(b)} b' \right) .$$

We leave it to the reader to the verify the following:

1. $\mathrm{FinLin}(f)(\mathbf{v})$ is a finite vector for every $\mathbf{v} \in \mathrm{FinLin}(B)$,
2. $\mathrm{FinLin}(f)$ is supported by S, and
3. $\mathrm{FinLin}(f)$ is injective. [4]

Injectivity of $\mathrm{FinLin}(f)$ implies that every strictly increasing chain

$$V_0 \subsetneq \cdots \subsetneq V_k \subsetneq \mathrm{FinLin}(B)$$

of S-supported vector spaces $V_0, \ldots V_k$ of $\mathrm{FinLin}(B)$ gets mapped to a strictly increasing chain

$$\mathrm{FinLin}(f)(V_0) \subsetneq \cdots \subsetneq \mathrm{FinLin}(f)(V_k) \subsetneq \mathrm{FinLin}(f)(\mathrm{FinLin}(B)) \subseteq \mathrm{FinLin}\left(B'\right)$$

of S-supported vector subspaces of $\mathrm{FinLin}(B')$. Hence if $\mathrm{FinLin}(B')$ has finite length, so does $\mathrm{FinLin}(B)$. We show $\mathrm{FinLin}(B')$ has finite length.

The set B' can be written as a disjoint finite union of sets of the form $(\mathbb{A} \setminus S)^{(k)}$ for $k \in \mathbb{N}$. Hence $\mathrm{FinLin}(B')$ can be written as a product of vector spaces of the

[4]This can be done in two steps. First, show that $\mathrm{FinLin}(f)$ is injective if and only if $\mathrm{FinLin}(f)^{-1}(\mathbf{0}) = \{\mathbf{0}\}$. Then show $\mathrm{FinLin}(f)^{-1}(\mathbf{0}) = \{\mathbf{0}\}$

form $\mathrm{FinLin}\Big((\mathbb{A}\setminus S)^{(k)}\Big)$ for $k \in \mathbb{N}$. Using Lemma 2.37 and Theorem 2.38 we conclude $\mathrm{FinLin}(B')$ also has finite length. ■

2.3 Orbit-finite matrices

For orbit-finite sets B and C, an *orbit-finite $B\times C$-matrix* is an orbit-finite function from $B\times C$ to R. When B and C are irrelevant, we speak of an *orbit-finite matrix*, meaning orbit-finite $(B\times C)$-matrix for some orbit-finite sets B and C. For the remainder of this section arbitrarily fix orbit-finite sets B, C and D. The set of orbit-finite $B\times C$-matrices forms a vector space under point-wise addition and scalar multiplication. We denote this space by $\mathrm{Lin}_R(B\times C)$. We use boldface capital letters $\mathbf{A}, \mathbf{B}, \ldots$ to denote orbit-finite matrices. Following usual conventions, matrices with one row are called *row vectors*, matrices with one column are called *column vectors*, and vectors are identified with column vectors. Let $\mathbf{A} \in \mathrm{Lin}_R(B\times C)$. For $b' \in B$, let $\mathbf{A}(b', -)$ denote the row indexed by b', i.e. the row vector defined as

$$\mathbf{A}(b', -)(c) \stackrel{\text{def}}{=} \mathbf{A}(b', c) \text{ for } c \in C.$$

Similarly for $c \in C$, let $\mathbf{A}(-, c)$ denote the column indexed by c, i.e. the (column) vector defined as

$$\mathbf{A}(-, c')(b) \stackrel{\text{def}}{=} \mathbf{A}(b, c') \text{ for } b \in B.$$

The *transpose* $\mathbf{A}^\mathsf{T}$ of the matrix $\mathbf{A}$ is defined to be the $C\times B$-matrix $\mathbf{A}^\mathsf{T}(c, b) \stackrel{\text{def}}{=} \mathbf{A}(b, c)$ for $(b, c) \in B\times C$. Clearly, orbit-finite matrices are closed under taking transposes. The transpose of a row(column) vector is a column(row) vector.

Definition 2.40. Following the convention that functions are identified with their graphs, If $\mathbf{A} \in \mathrm{Lin}_R(B\times C)$ is supported by S, then its *S-atom-dimension* is the atom-dimension of the orbit-finite finite set of pairs $\{((b, c), \mathbf{A}(b, c)) \ : \ b \in B,\ c \in C\}$. When S is clear from the context, we speak of simple *atom-dimension*.

Remark 2.41. The S-atom-dimension of an S-supported matrix $\mathbf{A} \in \mathrm{Lin}(B\times C)$ can be bounded by the sum of S-atom-dimension of

Lemma 2.5 says

Example 2.42. Define the $\mathbb{A}\times\mathbb{A}^{(2)}$-matrix $\mathbf{A}$ as

$$\mathbf{A}(\alpha, \alpha\beta) \stackrel{\text{def}}{=} 1 \quad \text{and} \quad \mathbf{A}(\alpha, \beta\gamma) \stackrel{\text{def}}{=} 0$$

for $\alpha\beta \in \mathbb{A}^{(2)}$ and $\gamma \in \mathbb{A}\setminus\{\beta\}$. The matrix $\mathbf{A}$ is equivariant. For $\alpha\beta \in \mathbb{A}^{(2)}$ we have $\mathbf{A}(-,\alpha\beta) = \mathbf{1}_\alpha$ and $\mathbf{A}(\alpha,-) = \mathbf{1}^\top_{X_\alpha}$ where

$$X_\alpha \stackrel{\text{def}}{=} \{\alpha\gamma \,:\, \gamma \in \mathbb{A} \setminus \{\alpha\}\} \ .$$

◀

For $\mathbf{x}, \mathbf{y} : C \to R$ *the inner product* $\mathbf{x}^\top \cdot \mathbf{y}$ *is well-defined* if there are only finitely many $c \in C$ for which both $\mathbf{x}(c)$ and $\mathbf{y}(c)$ are non-zero (equivalently, the intersection $\mathrm{dom}(\mathbf{x}) \cap \mathrm{dom}(\mathbf{y})$ is finite). In particular $\mathbf{x}^\top \cdot \mathbf{y}$ is always well-defined if either $\mathbf{x}$ or $\mathbf{y}$ is finite.

The inner product of orbit-finite vectors extend to product of orbit-finite matrices in the obvious way. The *product* $\mathbf{A} \cdot \mathbf{B}$ *of two orbit-finite matrices* $\mathbf{A} \in \mathrm{Lin}_R(B \times C)$ and $\mathbf{B} \in \mathrm{Lin}_R(C \times D)$ is well-defined when the $\mathbf{A}(b,-) \cdot \mathbf{B}(-,d)$ is well-defined for every $(b,d) \in B \times D$.

Remark 2.43. For the special case when $R = \mathbb{R}$ well-definedness of inner product coincides with unconditional convergence. Since orbit-finite vectors contain only finitely many different numbers (Notation 2.28), the sum

$$\sum_{c \in C} \mathbf{x}(c) \cdot \mathbf{y}(c)$$

is *unconditionally convergent* (i.e., convergent to the same value irrespectively of the order in which the elements $c \in C$ are enumerated) exactly when $\mathrm{dom}(\mathbf{x}) \cap \mathrm{dom}(\mathbf{y})$ is finite. [5]

Example 2.44. Pick arbitrary atoms $\alpha \neq \beta \in \mathbb{A}$. Define two sets $X, Y \subseteq \mathbb{A}^{(2)}$ as

$$X \stackrel{\text{def}}{=} \{\alpha\gamma \,:\, \gamma \neq \alpha\} \quad \text{and} \quad Y \stackrel{\text{def}}{=} \{\gamma\beta \,:\, \gamma \neq \beta\} \ .$$

Then $\mathbf{1}^\top_X \cdot \mathbf{1}_Y = 1$ and $\mathbf{1}^\top_{\mathbb{A}^{(2)}} \cdot \mathbf{1}_Y$ is not well-defined. ◀

Notation 2.45. When $R = \mathbb{R}$, we write $\mathrm{Lin}(B)$ and $\mathrm{FinLin}(B)$ instead of $\mathrm{Lin}_\mathbb{R}(B)$ and $\mathrm{FinLin}_\mathbb{R}(B)$.

Remark 2.46. When B is finite, the inner product over $\mathrm{Lin}(B)$ is always defined and coincides with the usual definition of inner product of finite dimensional vectors.

2.3.1 Collection of lemmas

We end this section with a collection of lemmas which are used in later parts of the thesis. Proofs of some of these lemmas are left to the reader as an exercise.

[5] We are grateful to Szymon Toruńczyk for this remark.

Lemma 2.47. *For any vector $\mathbf{v} \in \mathrm{FinLin}_R(B)$, $\mathbf{v}$ is supported by*

$$\left(\bigcup_{b \in \mathrm{dom}(\mathbf{v})} \mathrm{support}\,(b)\right) \cup \mathrm{support}\,(B) \ .$$

Proof. Let

$$T \stackrel{\text{def}}{=} \left(\bigcup_{b \in D} \mathrm{support}\,(b)\right) \cup \mathrm{support}\,(B) \ .$$

The set T is finite due to our inherent assumption of hereditarily finitely-supported-ness. Pick $\pi \in \mathrm{Aut}_T(\mathbb{A})$ and $b \in B$. Recall Notation 2.29. We have

$$\begin{aligned}
\pi(\mathbf{v}) &= \pi\left(\sum_{b \in D} \mathbf{v}(b) \cdot b\right) && \text{(by definition of } D\text{)} \\
&= \sum_{b \in D} \mathbf{v}(b) \cdot \pi(b) && \text{(Lemma 2.26)} \\
&= \sum_{b \in D} \mathbf{v}(b) \cdot b && \text{(by definition of } T\text{)} \\
&= \mathbf{v} \ .
\end{aligned}$$

This finishes the proof. ∎

Lemma 2.48. *For any vectors $\mathbf{x}, \mathbf{y} \in \mathrm{Lin}_R(C)$, and $S \subseteq_{\mathrm{FIN}} \mathbb{A}$ supporting $\mathbf{x}$ and $\mathbf{y}$, if $\mathbf{x}^{\mathsf{T}} \cdot \mathbf{y}$ is well-defined then*

$$\mathbf{x}^{\mathsf{T}} \cdot \mathbf{y} = \sum_{c \in C_S} \mathbf{x}(c) \cdot \mathbf{y}(c) \ ,$$

where C_S is the set of elements in C which are supported by S

$$C_S = \{c \in C \ : \ \mathrm{support}\,(c) \subseteq S\} \ .$$

Proof. Assume $\mathbf{x}^{\mathsf{T}} \cdot \mathbf{y}$ is well defined. We show for all $c \in C$ if $\mathbf{x}(c) \cdot \mathbf{y}(c) \neq 0$ then $\mathrm{support}\,(c) \subseteq S$. Assume otherwise. Let $c \in C$ be such that $\mathbf{x}(c) \cdot \mathbf{y}(c) \neq 0$ and $\mathrm{support}\,(c) \nsubseteq S$. Pick $\alpha \in \mathrm{support}\,(c) \setminus S$. Following a similar line of argument as in the proof of Lemma 2.25-iv we can show that $\mathrm{orbit}_S(c)$ is infinite. Since both $\mathbf{x}$ and $\mathbf{y}$ are supported by S, for every $c' \in \mathrm{orbit}_S(c)$ we have $\mathbf{x}(c') \cdot \mathbf{y}(c') \neq 0$. This contradicts well-defined-ness of $\mathbf{x}^{\mathsf{T}} \cdot \mathbf{y}$. ∎

Lemma 2.49. *For any matrix $\mathbf{A} \in \mathrm{Lin}(B \times C)$, $S \subseteq_{\mathrm{FIN}} \mathbb{A}$ supporting B and C and $\mathbf{A}$, $\pi \in \mathrm{Aut}_S(\mathbb{A})$, $b \in B$ and $c \in C$:*

(i) $\pi(\mathbf{A}(b, -)) = \mathbf{A}(\pi(b), -)$ *and* $\pi(\mathbf{A}(-, c)) = \mathbf{A}(-, \pi(c))$.

(ii) $\mathrm{support}\,(\mathbf{A}(b,-)) \subseteq \mathrm{support}\,(\mathbf{A}) \cup \mathrm{support}\,(b)$ *and* $\mathrm{support}\,(\mathbf{A}(-,c)) \subseteq \mathrm{support}\,(\mathbf{A}) \cup \mathrm{support}\,(c)$.

Lemma 2.50. *For any matrices* $\mathbf{A} \in \mathrm{Lin}(B \times C)$ *and* $\mathbf{B} \in \mathrm{Lin}(C \times D)$, *and* $S \subseteq_{\mathrm{FIN}} \mathbb{A}$ *supporting* B, C, D, $\mathbf{A}$ *and* $\mathbf{B}$, *if* $\mathbf{A} \cdot \mathbf{B}$ *is well-defined then:*

(i) for any $\pi \in \mathrm{Aut}(\mathbb{A})$, $\pi(\mathbf{A} \cdot \mathbf{B})$ *is also well-defined and* $\pi(\mathbf{A} \cdot \mathbf{B}) = \pi(\mathbf{A}) \cdot \pi(\mathbf{B})$.

(ii) for any $(b,d) \in B \times D$,

$$(\mathbf{A} \cdot \mathbf{B})(b,d) = \sum_{\mathrm{support}(c) \subseteq S'} \mathbf{A}(b,c) \cdot \mathbf{B}(c,d)\,,$$

where $S' = S \cup \mathrm{support}\,(b) \cup \mathrm{support}\,(d)$.

Proof. The first item is left to the reader to verify. The second item follows from the first one and Lemma 2.48.

We have $(\mathbf{A} \cdot \mathbf{B})(b,d) = \mathbf{A}(b,-) \cdot \mathbf{B}(-,d)$. The first item and Lemma 2.48 together imply

$$\mathbf{A}(b,-) \cdot \mathbf{B}(-,d) = \sum_{\mathrm{support}(c) \subseteq S'} \mathbf{A}(b,c) \cdot \mathbf{B}(c,d)\,,$$

as required. ∎

2.4 Orbit-finite systems of linear constraints

In this section we formally define orbit-finite systems of linear constraints (i.e., equations or inequalities) and give their matrix formulations. We focus on systems of equations since inequalities can be defined analogously.

For this section fix an orbit-finite set C and let $S \stackrel{\mathrm{def}}{=} \mathrm{support}\,(C)$. A *linear equation* over C is an expression of the form $\mathbf{a}^\top \cdot \mathbf{x} = t$ where $\mathbf{a} \in \mathrm{Lin}_R(C)$, $t \in R$ and $\mathbf{x}$ is a vector of variables indexed by C. Sometimes we also write equations as

$$\sum_{c \in C} \mathbf{a}(c) \cdot \mathbf{x}(c) = t$$

where $\mathbf{x}(c)$ denotes the variable indexed by c and $\mathbf{a}(c)$ its coefficient. A vector $\mathbf{y} : C \to R$ is called a *solution* of the equation $\mathbf{a}^\top \cdot \mathbf{x} = t$, if the product $\mathbf{a}^\top \cdot \mathbf{y}$ is well-defined and equal to t. A permutation $\pi \in \mathrm{Aut}_S(\mathbb{A})$ induces a permutation $\mathbf{x}(c) \mapsto \mathbf{x}(\pi(c))$ of the variables (Lemma 2.23-i). This in turn induces the permutation

$$\sum_{c \in C} \mathbf{a}(c) \cdot \mathbf{x}(c) = t \qquad \mapsto \qquad \sum_{c \in C} \mathbf{a}(c) \cdot \mathbf{x}(\pi(c)) = t$$

of linear equations over C. Rewriting the RHS using Lemma 2.3 we get

$$\begin{aligned}\sum_{c\in C}\mathbf{a}(c)\cdot\mathbf{x}(\pi(c)) &= \sum_{c\in C}\mathbf{a}(\pi^{-1}(c))\cdot\mathbf{x}(c)\\ &= \sum_{c\in C}\pi(\mathbf{a})(c)\cdot\mathbf{x}(c)\\ &= \pi(\mathbf{a})^{\mathsf{T}}\cdot\mathbf{x}\,.\end{aligned}$$

Hence, the permutation of equations induced by π can also be written as

$$\mathbf{a}^{\mathsf{T}}\cdot\mathbf{x} = t \qquad \mapsto \qquad \pi(\mathbf{a})^{\mathsf{T}}\cdot\mathbf{x} = t\,.$$

The equation $\mathbf{a}^{\mathsf{T}}\cdot\mathbf{x} = t$ *is supported by* $T \subseteq_{\mathrm{FIN}} \mathbb{A}$ if the vector $\mathbf{a}$ is supported by T.

Example 2.51. For example consider $C = \mathbb{A}^{(2)}$. For any atom $\alpha \in \mathbb{A}$ consider the equation

$$\mathbf{1}_{\alpha\_}^{\mathsf{T}}\cdot\mathbf{x} = 1\,,$$

where $\alpha\_ \stackrel{\text{def}}{=} \left\{\alpha\beta \in \mathbb{A}^{(2)} \;:\; \beta \neq \alpha\right\}$. This equation is supported by $\{\alpha\}$. It can also be written as

$$\sum_{\beta\neq\alpha}\mathbf{x}(\alpha\beta) = 1\,. \tag{2.4}$$

For any atom $\beta \neq \alpha$, the vector $\mathbf{1}_{\alpha\beta} \in \mathrm{FinLin}\left(\mathbb{A}^{(2)}\right)$ is an example solution of this equation. ◀

Let $S \subseteq_{\mathrm{FIN}} \mathbb{A}$ be an arbitrary finite subset of atoms. The *S-orbit of an equation* $\mathbf{a}^{\mathsf{T}}\cdot\mathbf{x} = t$ consists of all equations of the form $\pi(\mathbf{a})^{\mathsf{T}}\cdot\mathbf{x} = t$ for $\pi \in \mathrm{Aut}_S(\mathbb{A})$. In this chapter we are interested in *orbit-finite systems of linear equations* i.e., systems of linear equations which have finitely many orbits. A solution of such a system is a vector $\mathbf{y}$ which satisfies all the equations in the system.

Example 2.52. The equivariant orbit of the equation presented in Example 2.51 contains for every atom α the equation (2.4). We write:

$$\sum_{\beta\neq\alpha}\mathbf{x}(\alpha\beta) = 1 \qquad (\alpha \in \mathbb{A})\,.$$

For any function $f : \mathbb{A} \to \mathbb{A}$ such that $f(\alpha) \neq \alpha$ for all $\alpha \in \mathbb{A}$, the vector $\mathbf{x}_f : C \to \mathbb{R}$ defined as

$$\mathbf{x}_f(\alpha\beta) \stackrel{\text{def}}{=} \begin{cases}1 & \text{if } \beta = f(\alpha)\\ 0 & \text{otherwise}\end{cases}$$

is a solution of this orbit of equations. Notice that the vector $\mathbf{x}_f$ may be orbit-finite or not depending on the choice of f. For a positive example, pick arbitrary $\alpha, \beta \in \mathbb{A}$, and define f as

$$f(\gamma) \stackrel{\text{def}}{=} \begin{cases} \beta & \text{if } \gamma = \alpha \text{, and} \\ \alpha & \text{otherwise.} \end{cases}$$

Then $\mathbf{x}_f$ is supported by $\{\alpha, \beta\}$ and is orbit-finite. For a negative example, consider any enumeration $\alpha_1, \alpha_2, \ldots$ of atoms and the function f defined as

$$f(\alpha_n) \stackrel{\text{def}}{=} \alpha_{n+1} \quad (n \in \mathbb{N})\,.$$

The vector $\mathbf{x}_f$ is not finitely supported, and hence not orbit-finite. ◀

2.4.1 Solvability

An orbit-finite system of equations or inequalities is called *solvable* if it is has an orbit-finite solution. So the *solvability problem* asks whether a given orbit-finite system has an orbit-finite solution. It has the following variants:

Finitary solvability : existence of a finite solution.

Unrestrcited solvability : existence of a solution which need not be orbit-finite.

Support preserving solvability : existence of a solution which is supported by the support of the system.

In this thesis we focus on decidability of solvability and finitary solvability. Decidability of unrestricted solvability we leave as an open question. Support preserving solvability is decidable since:

Lemma 2.53. *Support preserving solvability reduces to solvability of finite systems of equations/inequalities.*

Proof. We do the proof for a single orbit of equations

$$\pi(\mathbf{a})^\top \cdot \mathbf{x} = t \qquad (\pi \in \mathrm{Aut}_S(\mathbb{A})) \tag{2.5}$$

for some orbit-finite set C, orbit-finite vector $\mathbf{a} \in \mathrm{Lin}(C)$ and $S \subseteq_{\text{FIN}} \mathbb{A}$. We leave it to reader to extend the proof to the general case where the system consists of a finite number of orbits of equations or inequalities. Let $C_1, \ldots, C_n$ be the S-orbits inside C. For any S-supported vector $\mathbf{y} \in \mathrm{Lin}(C)$ and $\pi \in \mathrm{Aut}_S(\mathbb{A})$, we have

$$\begin{aligned}
&\pi(\mathbf{a})^{\top} \cdot \mathbf{y} \\
&= \pi(\mathbf{a})^{\top} \cdot \pi(\mathbf{y}) && (\mathbf{y} \text{ is supported by } S) \\
&= \pi(\mathbf{a}^{\top} \cdot \mathbf{y}) && (\text{Lemma 2.25-i}) \\
&= \mathbf{a}^{\top} \cdot \mathbf{y} && (\mathbf{a}^{\top} \cdot \mathbf{y} \text{ is atom-less (page 17)}).
\end{aligned}$$

Hence the system (2.5) has an S-supported solution if and only if the equation $\mathbf{a}^{\top} \cdot \mathbf{x} = t$ has an S-supported solution. Let $\mathbf{a}_i$ be the restriction of $\mathbf{a}$ to C_i. Let $I \subseteq \{1, \ldots, n\}$ be the set of i's such that $\mathbf{a}_i$ is finite. We show that the equation $\mathbf{a}^{\top} \cdot \mathbf{x} = t$ has an S-supported solution if and only if the equation

$$\sum_{i \in I} \left(\sum_{c \in C_i} \mathbf{a}_i(c) \right) \cdot x_i = t \,, \tag{2.6}$$

with variables $\{x_i \,:\, i \in I\}$, has a solution.

If (2.6) has a solution $(y_i)_{i \in I}$ then

$$\sum_{i \in I} y_i \cdot \mathbf{1}_{C_i}$$

is a solution of $\mathbf{a}^{\top} \cdot \mathbf{x} = t$.

Conversely, any S-supported solution $\mathbf{y}'$ of $\mathbf{a}^{\top} \cdot \mathbf{x} = t$ can be written as

$$\mathbf{y}' = \sum_{i=1}^{n} y'_i \cdot \mathbf{1}_{C_i} \,.$$

We show $(y'_i)_{i \in I}$ is a solution of (2.6). The product $\mathbf{a}^{\top} \cdot \mathbf{y}'$ can be written as

$$\sum_{i=1}^{n} y'_i \cdot \mathbf{a}_i^{\top} \cdot \mathbf{1}_{C_i} \,.$$

This product is well-defined only if $y'_i = 0$ for every $i \notin I$. This implies

$$\sum_{i \in I} \left(\sum_{c \in C_i} \mathbf{a}_i(c) \right) \cdot y'_i = \sum_{i \in I} \mathbf{a}_i^{\top} \cdot \mathbf{1}_{C_i} \cdot y'_i = \mathbf{a}^{\top} \cdot \mathbf{y}' = t.$$

■

We present some examples to show that the solvability questions defined above can give different answers for the same system of equations. Moreover, the answer can also change with the underlying ring (for example, for a system of equations with integer coefficients, the underlying ring can be both integers or rationals). For these examples assume $R = \mathbb{Q}$.

Example 2.54 ([5, Example 5.30])**.** Consider the system of equations which contains, for every $\{\alpha, \beta\} \in \binom{\mathbb{A}}{2}$ (cf. Example 2.9), the equation

$$\mathbf{x}(\alpha\beta) + \mathbf{x}(\beta\alpha) = 1 \,.$$

We show that this system

1. has an equivariant rational solution,
2. does not have any orbit-finite integer solution,
3. does not have any finite solution, and
4. has an orbit-infinite integer solution.

The constant vector $\frac{1}{2} \cdot \mathbf{1}_{\mathbb{A}^{(2)}} : \alpha\beta \mapsto \frac{1}{2}$ is an orbit-finite rational solution of this system. It has no finite solution since every variable appears only in finitely many equations and RHS of infinitely many equations is non-zero. Furthermore, the system has no orbit-finite integer solution either, as any such solution $\mathbf{y}$ would necessarily satisfy, for every distinct atoms $\alpha, \beta \in \mathbb{A} \setminus \mathrm{support}\,(\mathbf{y})$, the equality $\mathbf{y}(\alpha\beta) = \mathbf{y}(\beta\alpha)$, which is in contradiction with $\mathbf{y}(\alpha\beta) + \mathbf{y}(\beta\alpha) = 1$. But this system has orbit-infinite integer solutions. For example, for any total order $<$ on $\mathbb{A}$ the vector $\mathbf{y}_< : \mathbb{A}^{(2)} \to \mathbb{Z}$, defined as

$$\mathbf{y}_<(\alpha\beta) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \alpha < \beta\,, \\ 0 & \text{otherwise,} \end{cases}$$

is a solution. ◀

Example 2.55. We recall the following system which was presented in Example 2.52.

$$\sum_{\beta \neq \alpha} \mathbf{x}(\alpha\beta) = 1 \qquad (\alpha \in \mathbb{A})\,.$$

We argue that this syste

1. has an orbit-finite integer solution but,
2. does not have any finite or equivariant solution.

For $\alpha \in \mathbb{A}$ define

$$\alpha_{\_} \stackrel{\text{def}}{=} \left\{ \alpha\beta \in \mathbb{A}^{(2)} \;:\; \beta \neq \alpha \right\} \quad \text{and} \quad {}_{\_}\alpha \stackrel{\text{def}}{=} \left\{ \beta\alpha \in \mathbb{A}^{(2)} \;:\; \beta \neq \alpha \right\} \,.$$

Take any two fixed atoms $\gamma, \eta \in \mathbb{A}$. The vector

$$\mathbf{y} \stackrel{\text{def}}{=} \mathbf{1}_{\text{-}\gamma} + \mathbf{1}_{\gamma\eta}.$$

is an orbit-finite integer solution of this system. Indeed, for $\alpha \neq \gamma$ we have $\mathbf{1}_{\alpha\_}^{\mathsf{T}} \cdot \mathbf{y} = \mathbf{1}_{\alpha\gamma} \cdot \mathbf{1}_{\alpha\gamma} = 1$ as required. Furthermore, we have $\mathbf{1}_{\gamma\_}^{\mathsf{T}} \cdot \mathbf{y} = \mathbf{1}_{\gamma\eta} \cdot \mathbf{1}_{\gamma\eta} = 1$ as required. The system has no finite solution for the same reason as in the previous example. It also does not have any equivariant solution, because the only equivariant vectors over $\mathbb{A}^{(2)}$ are constant ones $q \cdot \mathbf{1}$, and the inner product $\mathbf{1}_{\alpha\_} \cdot \mathbf{1}$ is ill-defined for every $\alpha \in \mathbb{A}$ as long as $q \neq 0$). ◀

Example 2.56. Consider the following system of equations which contains for every $\alpha \in \mathbb{A}$, the equation

$$\sum_{\beta \neq \alpha} \mathbf{x}(\{\alpha, \beta\}) = 1\ .$$

We argue that the system

1. has no orbit-finite rational solution (despite the apparent similarity to the system in Example 2.52), but
2. has orbit-infinite integer solutions.

First, towards a contradiction, suppose the system has a solution $\mathbf{y}$, supported by some $T \subseteq_{\text{FIN}} \mathbb{A}$. Thus, it is constant on every T-orbit in $\binom{\mathbb{A}}{2}$. An infinite T-orbit in $\binom{\mathbb{A}}{2}$ is either the set $\binom{\mathbb{A} \setminus T}{2}$ of all 2-sets disjoint from T or, for some fixed $\alpha \in T$, the set of all 2-sets with one element α and the other element not in T:

$$\{\{\alpha, \gamma\}\ :\ \gamma \in \mathbb{A} \setminus T\}$$

Recall the definitions of subsets $\alpha\_$ and $\_\alpha$ of $\mathbb{A}^{(2)}$ from Example 2.55. Each infinite T-orbit in $\binom{\mathbb{A}}{2}$ intersects infinitely with $\alpha\_$ for some $\alpha \in \mathbb{A}$. In consequence, $\mathbf{y}$ is necessarily 0 when restricted to any infinite T-orbit in $\binom{\mathbb{A}}{2}$ as otherwise $\mathbf{1}_{\alpha\_}^{\mathsf{T}} \cdot \mathbf{y}$ would be ill-defined for some $\alpha \in \mathbb{A}$. Therefore $\mathbf{y}$ is forcedly finite, and the argument of the previous examples applies.

Now we construct an orbit-infinite integer solution of this system. Take any enumeration $\mathbb{A} = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ of atoms. The vector $\mathbf{y} : \binom{\mathbb{A}}{2} \to R$ that maps each set $\{\alpha_{2n}, \alpha_{2n+1}\}$ to 1, for $n = 0, 1, \ldots$, and all other sets to 0, satisfies all equations. Note that $\mathbf{y}$ is not finitely supported (and hence is not orbit-finite), i.e., there is no finite $T \subseteq \mathbb{A}$ such that $\pi(\mathbf{y}) = \mathbf{y}$ for all $\pi \in \mathrm{Aut}_T(\mathbb{A})$. ◀

2.4.2 Matrix formulations

Just like finite systems of linear equations can be represented using finite dimensional matrices and vectors, orbit-finite systems of equations can be represented using orbit-finite matrices and vectors.

Example 2.57. The system of equations presented in Examples 2.51 and 2.55 can be represented as $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ where $\mathbf{A}$ is the matrix defined in Example 2.42 and $\mathbf{b} = \mathbf{1}_{\mathbb{A}}$. We recall the definition of $\mathbf{A}$:

$$\mathbf{A}(\alpha, \alpha\beta) = 1 \quad \text{and} \quad \mathbf{A}(\alpha, \beta\gamma) = 0$$

for $\alpha\beta\gamma \in \mathbb{A}^{(3)}$. Notice that $\mathbf{A}$ and $\mathbf{b}$ are both equivariant. Following the convention that vectors are considered implicitly as column vectors, and using Notation 2.29 we can also define $\mathbf{A}$ specifying its columns:

$$\mathbf{A}(-, \alpha\beta) = \alpha$$

for $\alpha\beta \in \mathbb{A}^{(2)}$. ◀

Example 2.58. The system in Example 2.54 can be represented as $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ where $\mathbf{A} \in \operatorname{Lin}\left(\binom{\mathbb{A}}{2} \times \mathbb{A}^{(2)}\right)$ and $\mathbf{b} \in \operatorname{Lin}\left(\binom{\mathbb{A}}{2}\right)$ are defined as

$$\mathbf{A}(\{\alpha, \beta\}, \gamma\delta) = \begin{cases} 1 & \text{if } \{\alpha, \beta\} = \{\gamma, \delta\} \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{b}(\{\alpha, \beta\}) = 1 \,.$$

Using the conventions that row vectors are considered as transposes of column vectors, $\mathbf{A}$ can be written more concisely by specifying its rows

$$\mathbf{A}(\{\alpha, \beta\}, -) = (\alpha\beta + \beta\alpha)^{\mathsf{T}}$$

for $\{\alpha, \beta\} \in \binom{\mathbb{A}}{2}$. ◀

Formally, for $\mathbf{A} \in \operatorname{Lin}_R(B \times C)$ and $\mathbf{b} \in \operatorname{Lin}_R(B)$, the matrix formulation $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ represents the system consisting of the equations

$$\mathbf{A}(b, -) \cdot \mathbf{x} = \mathbf{b}(b) \qquad (b \in B) \,. \tag{2.7}$$

We say it is a $(B \times C)$-system to denote that $\mathbf{A} \in \operatorname{Lin}_R(B \times C)$ and $\mathbf{b} \in \operatorname{Lin}_R(B)$. We claim that the system is orbit-finite. Indeed, let S be a finite set supporting both $\mathbf{A}$ and $\mathbf{b}$. For any $b \in B$ and $\pi \in \operatorname{Aut}_S(\mathbb{A})$ the element $\pi(b) \in B$ indexes the equation

$$e \quad : \quad \mathbf{A}(\pi(b), -) \cdot \mathbf{x} = \mathbf{b}(\pi(b)) \,.$$

Using Lemma 2.49-i we get $\mathbf{A}(\pi(b), -) = \pi(\mathbf{A}(b, -))$, and since $\mathbf{b}$ is supported by S we have $\mathbf{b}(\pi(b)) = \mathbf{b}(b)$. Hence, the above equation is equal to

$$\pi(e) \quad : \quad \pi(\mathbf{A}(b, -)) \cdot \mathbf{x} = \mathbf{b}(b) \; .$$

This implies that every S-orbit in B indexes an S-orbit of equations. Since B is considered to be orbit-finite, the system of equations is also orbit-finite.

Conversely, every orbit-finite system of linear equations is representable in matrix form ((2.7)). Indeed, pick an orbit-finite system of linear equations

$$\begin{gathered} \mathrm{orbit}_S(\mathbf{a}_1^\top \cdot \mathbf{x} = t_1) \\ \vdots \\ \mathrm{orbit}_S(\mathbf{a}_n^\top \cdot \mathbf{x} = t_n) \end{gathered} \tag{2.8}$$

where the variables are indexed by an orbit-finite set C. Let S be the support of this system. Let $B_i = \mathrm{orbit}_S(\mathbf{a}_i) \times \{t_i\}$, and $B = B_1 \cup \cdots \cup B_n$. Define $\mathbf{A} : (B \times C) \to R$ and $\mathbf{b} : B \to R$ as

$$\mathbf{A}((\mathbf{a}, t), c) = \mathbf{a}(c) \qquad \text{and} \qquad \mathbf{b}(\mathbf{a}, t) = t$$

Clearly, $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ represents the system (2.8). We just need to show that $\mathbf{A}$ and $\mathbf{b}$ are orbit-finite. The set C is orbit-finite by assumption. Each $B_i \subseteq B$ is an S-orbit and hence B is also orbit-finite. Using lemma 2.13 we just need to prove that both $\mathbf{A}$ and $\mathbf{b}$ are finitely supported. For any $\pi \in \mathrm{Aut}_S(\mathbb{A})$

$$\begin{aligned} \mathbf{A}(\pi(\mathbf{a}, t), \pi(c)) &= \mathbf{A}((\pi(\mathbf{a}), t), \pi(c)) \\ &= \pi(\mathbf{a})(\pi(c)) \\ &= \mathbf{a}(c) && \text{(Lemma 2.3)} \end{aligned}$$

and

$$\mathbf{b}(\pi(\mathbf{a}, t)) = \mathbf{b}(\pi(\mathbf{a}), t) = t \; .$$

Using Lemma 2.4 we conclude that they are supported by S.

2.5 State of the art

This thesis belongs to a wider research program that aims at lifting different aspects of theory of computation from finite to orbit-finite sets [7, 4, 10, 9, 8, 24, 12, 23, 25, 11, 22]. In this section we summarise the recent advancements in this area regarding solvability of equations and theory of vector spaces. Most of the prior work concentrated on equality atoms only, and therefore we mention the choice of atoms only if this is not the case.

2.5.1 Orbit-finite systems of equations and inequalities

Solvability of orbit-finite systems have already been studied in several works, most often for equality atoms. Prior to this thesis, only partial results were known, applying to very restricted special cases. In [20] the authors show that orbit-finite systems of linear equations where rows are indexed by $\mathbb{A} \times F$ for some finite set F and the columns are finite, existence of a finite integer or rational solution is decidable in PTIME, and decidability of existence of a finite non-negative integer solution NP-complete. These results are extended in [21] for the case when the rows are indexed by $\binom{\mathbb{A}}{n} \times F$ for some fixed natural number n and finite set F, and columns are finite as before. In this case, for a fixed n, existence of a finite integer solution is decidable in PTIME, but existence of a finite non-negative integer solution is decidable in EXPTIME.

In [23] the authors prove decidability of solvability of orbit-finite system of linear equations over finite fields, where every equation contains appearances of finitely many variables. Additionally they do not require the solutions to be orbit-finite.

Article [18] extends the results of [20] to ordered atoms and show that existence of a finite integer/rational solution is decidable in PTIME, but it becomes equivalent to VAS reachability (and hence is Ackermann-complete [28, 15, 27]) if we additionally restrict solutions to be non-negative.

2.5.2 Orbit-finitely generated vector spaces

Orbit-finitely generated vector spaces were investigated prior to this thesis in the context of solvability problems, but also in the context of algorithmic verification of data enriched models of computation (for example register automata, an orbit-finite counterpart of NFA [8]). Two first papers which study orbit-finite dimensional vector spaces explicitly was [6], where the authors prove that orbit-finitely generated vector spaces have finite length (the length of an equivariant vector space is the length of the longest strictly increasing chain of equivariant subspaces inside it), and [21], investigating vector spaces of the form $\mathrm{FinLin}\left(\binom{\mathbb{A}}{n}\right)$ in the context of solvability of systems of equations. The finite length property is used in [6] to extend the classical algorithms for deciding equivalence and minimisation of weighted automata to weighted register automata. These results have recently been improved in the master's thesis of Jingjie Yang [36], which provides an explicit formula for the lengths of equivariant vector spaces. He has also proven that orbit-finitely generated vector spaces with graph atoms and bit vectors atoms have finite length [37].

In [6] the authors have also proven that for equality and ordered atoms, spaces of the form $\mathrm{Lin}(B)$ for orbit-finite sets B have orbit-finite spanning sets. This result has been significantly extended in [31], which shows that these spaces have orbit-finite bases for *ω-categorical and ω-stable structures*, and also for ordered atoms.

In [13] the author has shown finite generation of equivariant polynomial ideals with equality atoms as variables. This result implies that for equality atoms, finitely supported subspaces of $\mathrm{FinLin}(B)$ for orbit-finite sets B have orbit-finite spanning sets. The latter result is a weaker version of the finite length property proved in [6].

In [26] the authors study equivariant cones inside $\mathrm{FinLin}(\mathbb{A} \times F)$ where F is an arbitrary finite set, and extend accordingly theorems of Carathéodory and Minkowski-Weyl.

Chapter 3

Bases and Dimension

Contents

3.1 Introduction

One of the fundamental results in linear algebra is that any vector space has a basis, and any two bases of a vector space are isomorphic (as sets). In this chapter we want to examine, to which extent the same statements hold in the orbit-finite setting. For this chapter we fix R to be an commutative ring with identity, denoted as 1. For any orbit-finite set B we are interested in two vector spaces defined using B,

1. $\mathrm{FinLin}_R(B)$, the vector space of finite vectors in $B \to \mathbb{R}$, and

2. $\mathrm{Lin}_R(B)$, the vector space of orbit-finite vectors in $B \to \mathbb{R}$.

The set $\{\mathbf{1}_b : b \in B\}$ is, by the very definition, an orbit-finite basis of the vector space $\mathrm{FinLin}_R(B)$. We prove that this set can be extended to an orbit-finite basis of the larger space $\mathrm{Lin}_R(B)$.

Theorem 3.1 (Orbit-finite basis theorem)**.** *For any orbit-finite set B, the vector space* $\mathrm{Lin}_R(B)$ *has an orbit-finite basis.*

§ 3.2 is devoted to the proof of this theorem.

Contrary to the classical setting, not all orbit-finitely spanned vector spaces have an orbit-finite basis. A counterexample was given in [6], which we recall in § 3.3. Finally, in § 3.4 we show that dimension of an orbit-finitely generated vector space is not always well-defined, i.e., there exists a vector space with two orbit-finite bases which are not related by an orbit-finite bijection. This is also in contrary to the classical setting where any two bases of a vector space are isomorphic.

3.2 Orbit-finite basis theorem

Throughout this section we fix B to be an orbit-finite set and let $S \subseteq_{\text{FIN}} \mathbb{A}$ denote its support. We start with an example proving the theorem for a special case.

Example 3.2. Let $B = \mathbb{A}^{(2)}$. For $\gamma \in \mathbb{A}$, let $\gamma_{\_} \stackrel{\text{def}}{=} \{\gamma\alpha \;:\; \alpha \in \mathbb{A} \setminus \{\gamma\}\} \subseteq B$; and symmetrically let ${}_{\_}\gamma \stackrel{\text{def}}{=} \{\alpha\gamma \;:\; \alpha \in \mathbb{A} \setminus \{\gamma\}\} \subseteq B$. One obtains a basis $\textsc{Basis}(B)$ of $\mathrm{Lin}(B)$ by extending $\{\mathbf{1}_{\alpha\beta} \;:\; \alpha\beta \in B\}$ with the constant vector $\mathbf{1}_B$ that maps every pair $\alpha\beta \in B$ to 1, and also, for every $\gamma \in \mathbb{A}$, with the characteristic vector $\mathbf{1}_{\gamma_{\_}}$ that maps all pairs in $\gamma_{\_}$ to 1 and all others to 0, and the characteristic vector $\mathbf{1}_{{}_{\_}\gamma}$ that maps all pairs in ${}_{\_}\gamma$ to 1 and all others to 0:

$$\textsc{Basis}(B) = \{\mathbf{1}_{\alpha\beta} \;:\; \alpha\beta \in B\} \cup \{\mathbf{1}_{\alpha_{\_}} \;:\; \alpha \in \mathbb{A}\} \cup \{\mathbf{1}_{{}_{\_}\beta} \;:\; \beta \in \mathbb{A}\} \cup \{\mathbf{1}_B\} \;.$$

Towards seeing that this is indeed a basis, pick any vector $\mathbf{v}$ in $\mathrm{Lin}\left(\mathbb{A}^{(2)}\right)$. Let $T \stackrel{\text{def}}{=} \mathrm{support}\,(\mathbf{v})$. Let

$$O_{\bullet\bullet} \stackrel{\text{def}}{=} (\mathbb{A} \setminus T)^{(2)} \;.$$

For $\alpha \in T$, let

$$O_{\alpha\bullet} \stackrel{\text{def}}{=} \{\alpha\} \times (\mathbb{A} \setminus T) \quad \text{and} \quad O_{\bullet\alpha} \stackrel{\text{def}}{=} (\mathbb{A} \setminus T) \times \{\alpha\} \;.$$

Note that all these are T-orbits. Moreover, note that $O_{\alpha\bullet} \neq \alpha_{\_}$ and $O_{\bullet\alpha} \neq {}_{\_}\alpha$ except for the case when $T = \{\alpha\}$.

We observe the following equality which is implied by the inclusion exclusion rule:

$$\mathbf{v} \quad = \quad \mathbf{v}(O_{\bullet\bullet}) \cdot \mathbf{1}_B \; + \; \sum_{\alpha \in T} \; (\mathbf{v}(O_{\alpha\bullet}) - \mathbf{v}(O_{\bullet\bullet})) \cdot \mathbf{1}_{\alpha_{\_}} \; +$$

$$\sum_{\beta \in T} \quad (\mathbf{v}(O_{\bullet\beta}) - \mathbf{v}(O_{\bullet\bullet})) \cdot \mathbf{1}_{\_\beta} \;+$$
$$\sum_{\alpha\beta \in T^{(2)}} (\mathbf{v}(\alpha\beta) - \mathbf{v}(O_{\alpha\bullet}) - \mathbf{v}(O_{\bullet\beta}) + \mathbf{v}(O_{\bullet\bullet})) \cdot \mathbf{1}_{\alpha\beta}.$$

This yields a representation of $\mathbf{v}$ in the base $\textsc{Basis}(B)$, hence $\textsc{Basis}(B)$ spans $\mathrm{Lin}(B)$. Moreover, this representation is unique, hence $\textsc{Basis}(B)$ is a linearly independent set of vectors. ◀

3.2.1 Reduction to single-orbit B

We claim that we can assume, WLOG, that B is a single orbit. Indeed, let $S \stackrel{\text{def}}{=} \text{support}\,(B)$ and let $B = B_1 \uplus \cdots \uplus B_n$ be the partition into S-orbits. Then $\mathrm{Lin}_R(B)$ is isomorphic to the Cartesian product $\mathrm{Lin}_R(B_1) \times \cdots \times \mathrm{Lin}_R(B_n)$. Denote by $\iota_i : \mathrm{Lin}_R(B_i) \to \mathrm{Lin}_R(B)$ the natural embedding that extends a vector $\mathbf{v} : \mathrm{Lin}_R(B_i)$ by 0 for all $b \in B \setminus B_i$:

$$\iota_i(\mathbf{v})(b) \stackrel{\text{def}}{=} \begin{cases} \mathbf{v}(b) & \text{if } b \in B_i \\ 0 & \text{otherwise.} \end{cases}$$

Supposing we have orbit-finite bases $\textsc{Basis}(B_1), \ldots, \textsc{Basis}(B_n)$ of the vector spaces $\mathrm{Lin}_R(B_1), \ldots, \mathrm{Lin}_R(B_n)$, respectively, we get the basis $\textsc{Basis}(B)$ of $\mathrm{Lin}_R(B)$ as the union of embeddings of $\textsc{Basis}(B_1), \ldots, \textsc{Basis}(B_n)$:

$$\textsc{Basis}(B) \stackrel{\text{def}}{=} \iota_1(\textsc{Basis}(B_1)) \,\cup\, \ldots \,\cup\, \iota_n(\textsc{Basis}(B_n))\,. \tag{3.1}$$

From now on we assume that B is a single orbit.

3.2.2 Tight orbits

Definition 3.3. An orbit O is called *tight* if $\text{support}\,(O) \subseteq \text{support}\,(x)$ for every $x \in O$.

In particular, every singleton is tight, and so is also every equivariant orbit.

Example 3.4. Recall Example 3.2. In case of $B = \mathbb{A}^{(2)}$, the tight orbits $O \subseteq B$ are the following ones:

$$B \qquad \alpha\_ \qquad \_\beta \qquad \{\alpha\beta\}$$

where α, β range over atoms and $\alpha \neq \beta$. The set B is an equivariant orbit, $\alpha\_$ is an $\{\alpha\}$-orbit, $\_\beta$ is a $\{\beta\}$-orbit, and $\{\alpha, \beta\}$ is an $\{\alpha, \beta\}$-orbit. Contrarily, for two fixed and distinct $\alpha, \beta \in \mathbb{A}$, the $\{\alpha, \beta\}$-orbit

$$\neq\alpha\beta \stackrel{\text{def}}{=} \{\gamma\beta \ : \ \gamma \notin \{\alpha, \beta\}\},$$

is not tight, since for any atom $\gamma \notin \{\alpha, \beta\}$ we have $\gamma\beta \in (\neq\alpha\beta)$, however

$$\operatorname{support}(\gamma\beta) = \{\gamma, \beta\} \not\supseteq \{\alpha, \beta\} = \operatorname{support}(\neq\alpha\beta) \ .$$

◀

Using the following lemma, WLOG we can assume that B is tight, i.e., $S \subseteq \operatorname{support}(b)$ for every $b \in B$.

Lemma 3.5. *Let $T \subseteq_{\text{FIN}} \mathbb{A}$. Every T-orbit O is in a T-supported bijection with a tight T-orbit.*

Proof. Let $O' \stackrel{\text{def}}{=} O \times \{S\}$. Then O' and O are related by a S-supported bijection, and O' is tight. ■

From now on we assume that B is a tight S-orbit.

Lemma 3.6. *The set $\{O \ : \ O \subseteq B,\ O$ is a tight orbit$\}$ is orbit-finite.*

Proof. By the very definition, for any tight T-orbit $O \subseteq B$, the size of T is at most the size of the support of elements of B. Let $d \stackrel{\text{def}}{=} |\operatorname{support}(b)|$ for some (every) $b \in B$ (d is well-defined since B is assumed to be a single orbit). Then $|T| \leqslant d$.

Define a function that maps a subset $T \subseteq \mathbb{A}$ of size $\leqslant d$, and an element $b \in B$ to the T-orbit of b:

$$T, b \mapsto \operatorname{orbit}_T(b) \ .$$

This function is supported by S, its domain is orbit-finite and hence its range is also orbit-finite (Lemma 2.23-iv). All tight orbits $O \subseteq B$ belong to the range, an hence there are only orbit-finitely many of them. ■

3.2.3 Definition of the basis

We define $\textsc{Basis}(B)$ as the set of characteristic vectors of all tight orbits $O \subseteq B$:

$$\textsc{Basis}(B) \stackrel{\text{def}}{=} \{\mathbf{1}_O \ : \ O \subseteq B \text{ a tight orbit}\}\,.$$

Once B is fixed, the set $\textsc{Basis}(B)$ is orbit-finite due to Lemma 3.6. Since every singleton is a tight orbit, $\mathbf{1}_b \in \textsc{Basis}(B)$ for every $b \in B$, i.e., $\textsc{Basis}(B)$ extends the basis $\{\mathbf{1}_b \ : \ b \in B\}$ of $\operatorname{FinLin}_R(B)$.

Remark 3.7. The definition of $\textsc{Basis}(B)$ is essentially independent of the ring R, which is why we omit R from the notation $\textsc{Basis}(B)$.

Example 3.8. Continuing Example 3.2, where $B = \mathbb{A}^{(2)}$, the basis vectors are the following ones:

$$\mathbf{1} \qquad \mathbf{1}_{\alpha\_} \qquad \mathbf{1}_{\_\beta} \qquad \mathbf{1}_{\alpha\beta},$$

for any $\alpha\beta \in \mathbb{A}^{(2)}$. ◀

We argue that $\textsc{Basis}(B)$ is indeed a basis, that is it spans the whole space $\mathrm{Lin}_R(B)$ and is linearly independent.

3.2.4 Spanning

Fix an arbitrary finite subset $T \subseteq_{\text{FIN}} \mathbb{A}$ such that $S \subseteq T$. We distinguish the set of all tight T'-orbits for $S \subseteq T' \subseteq T$:

$$\mathrm{TO}(S,T) \stackrel{\text{def}}{=} \{O \subseteq B \ : \ O \text{ is a tight orbit}, S \subseteq \text{support}\,(O) \subseteq T\}\,.$$

The set $\mathrm{TO}(S,T)$ is finite since, due to Lemma 2.8, B includes only finitely many T'-orbits for every fixed $T' \subseteq_{\text{FIN}} \mathbb{A}$. Define

$$\textsc{Basis}(B)_T \ \stackrel{\text{def}}{=} \ \{\mathbf{1}_O \ : \ O \in \mathrm{TO}(S,T)\} \ \subseteq \ \textsc{Basis}(B).$$

Note that the set $\textsc{Basis}(B)_T$ is finite, as $\mathrm{TO}(S,T)$ is so. The following lemma implies that $\textsc{Basis}(B)$ spans $\mathrm{Lin}_R(B)$.

Lemma 3.9 (Spanning)**.** *Each T-supported vector $\mathbf{v} \in \mathrm{Lin}_R(B)$ is a linear combination of vectors from $\textsc{Basis}(B)_T$.*

Proof. Recalling Notation 2.28 we define the *T-orbit-domain* of $\mathbf{v}$ as follows:

$$T\text{-orbit-dom}(\mathbf{v}) \ \stackrel{\text{def}}{=} \ \{O \ : \ O \subseteq B \text{ an } T\text{-orbit}, \ \mathbf{v}(O) \neq 0\} \ .$$

For two T-supported vectors $\mathbf{w}, \mathbf{w}' \in \mathrm{Lin}_R(B)$, we write $\mathbf{w} \prec \mathbf{w}'$ if the set T-orbit-dom$(\mathbf{w})$ is obtained from T-orbit-dom$(\mathbf{w}')$ by removing one T-orbit and replacing it by arbitrarily many T-orbits of strictly smaller T-atom dimension.

Example 3.10. For illustration, continuing Example 3.2,

$$T\text{-dim}(O_{\bullet\bullet}) > T\text{-dim}(O_{\bullet\alpha}) = T\text{-dim}(O_{\alpha\bullet}) > T\text{-dim}\big(O_{\alpha\beta}\big) \text{ for } \alpha,\beta \in T\,.$$

Hence, $(\mathbf{1}_{O_{\alpha\bullet}} + \mathbf{1}_{O_{\bullet\beta}} + 2\cdot\mathbf{1}_{\alpha\beta}) \prec (\mathbf{1}_{O_{\bullet\bullet}} + 2\cdot\mathbf{1}_{\alpha\beta})$. ◀

We define a representation of $\mathbf{v}$ in basis $\textsc{Basis}(B)$ by structural induction with respect to the transitive closure of $\prec$. Concerning the induction base, if T-orbit-dom$(\mathbf{v})$ is empty then $\mathbf{v} = 0$ and the lemma holds vacuously. Otherwise, suppose the claim holds for all strictly smaller vectors $\mathbf{w} \prec \mathbf{v}$. Take a T-orbit $O \in T$-orbit-dom$(\mathbf{v})$ of maximal T-atom dimension. Let

$$T' \stackrel{\text{def}}{=} \text{support}\,(x) \cap T \tag{3.2}$$

for some (every) $x \in O$. Note that $S \subseteq T'$ as $S \subseteq \text{support}\,(x)$ (since B is tight) and $S \subseteq T$. We define the T'-orbit O' as T'-closure of O:

$$O' \stackrel{\text{def}}{=} \{\pi(x) \ : \ x \in O,\ \pi \in \text{Aut}_{T'}(\mathbb{A})\}$$

(note that when $O = \{x\}$ is a singleton then $T' = \text{support}\,(x)$ and $O' = O$). The set O' is supported by T' and T' is included in the support of every element of O', therefore the orbit O' is tight, and hence $\mathbf{1}_{O'} \in \textsc{Basis}(B)_T$. As $T' \subseteq T$, every T-orbit in B is either included in O' or disjoint from it, and hence O' is a finite union of T-orbits (Lemma 2.23-iii). We claim that O has the largest T-atom dimension among all T-orbits included in O':

Claim 3.10.1. *For every T-orbit M included in O' but different than O, we have* T-dim$(M) < T$-dim(O).

Consider the vector

$$\mathbf{w} \quad \stackrel{\text{def}}{=} \quad \mathbf{v} \ - \ \mathbf{v}(O) \cdot \mathbf{1}_{O'}. \tag{3.3}$$

Note that $\mathbf{w}$ is supported by T as both $\mathbf{v}$ and $\mathbf{1}_{O'}$ are so, and $\mathbf{w}(O) = 0$. By Claim 3.10.1 we infer $\mathbf{w} \prec \mathbf{v}$, and therefore by the induction assumption $\mathbf{w}$ is a linear combination of vectors from $\textsc{Basis}(B)_T$. By (3.3) we deduce the same for $\mathbf{v}$. This completes the proof of Lemma 3.9 modulo the proof of Claim 3.10.1.

Before proving Claim 3.10.1 we give an example instance of it.

Example 3.11. Continuing Examples 3.2 and 3.10, pick two atoms $\alpha, \beta \in \mathbb{A}$. Let

$$\mathbf{v} \stackrel{\text{def}}{=} 2 \cdot \mathbf{1}_{O_{\alpha\bullet}} + \mathbf{1}_{O_{\bullet\beta}} - \mathbf{1}_{\beta\alpha}\ .$$

Then $T = \{\alpha, \beta\}$ and

$$T\text{-orbit-dom}(\mathbf{v}) = \big\{O_{\alpha\bullet}, O_{\bullet\beta}, \{\beta\alpha\}\big\}\ .$$

The orbits $O_{\alpha\bullet}$ and $O_{\bullet\beta}$ are of maximal T-atom dimension in T-orbit-dom$(\mathbf{v})$. Take $O = O_{\alpha\bullet}$ and $x = \alpha\gamma \in O_{\alpha\bullet}$. Then $T' = \{\alpha\}$ and $O' = \alpha_{\_}$. We have

$$\text{Orbits}_T(O') = \{O, \{\beta\alpha\}\}\ .$$

Clearly O has the largest atom dimension among all T-orbits inside O'. ◀

Proof of Claim 3.10.1. Recall that $T' \subseteq T \cap \text{support}(x')$ for every $x' \in O'$. Consider the subset $N \subseteq O'$ containing those elements $x' \in O'$ for which $T' = \text{support}(x') \cap T$. The set N is supported by T and hence both N and $O' \setminus N$ are finite union of T-orbits in O'. For all $x' \in O' \setminus N$ we have $T' \subsetneq \text{support}(x') \cap T$, which implies that each T-orbit $M \subseteq O' \setminus N$ has strictly smaller T-atom dimension than O. To prove the claim it is enough to prove that $N = O$.

By the definition of T' (3.2) we have $O \subseteq N$. We prove $N \subseteq O$, by showing that every element $y \in N$ is related by an T-automorphism to some element of $z \in O$. Indeed, consider any $z \in O$ and $y = \pi'(z) \in N$, where $\pi' \in \text{Aut}_{T'}(\mathbb{A})$. We have

$$T' = \text{support}(z) \cap T = \text{support}\left(\pi'(z)\right) \cap T$$

and hence there is some $\pi \in \text{Aut}_T(\mathbb{A})$, possibly different than π', that coincides with π' on support (z). Using Lemma 2.23-ii we conclude $y = \pi(z)$, as required. The two inclusions imply $N = O$. □

Now that we have proven Claim 3.10.1, the proof of Lemma 3.9 is also complete. ■

Example 3.12. Continuing Example 3.11,

$$\mathbf{w} = \mathbf{v} - \mathbf{v}(O) \cdot \mathbf{1}_O = \mathbf{v} - 2 \cdot \mathbf{1}_{\alpha_-} = \mathbf{1}_{O_{\bullet\beta}} - 2 \cdot \mathbf{1}_{\alpha\beta} - \mathbf{1}_{\beta\alpha} .$$

We have

$$\begin{aligned} T\text{-orbit-dom}(\mathbf{v}) &= \left\{O_{\alpha\bullet}, O_{\bullet\beta}, \{\beta\alpha\}\right\} \\ T\text{-orbit-dom}(\mathbf{w}) &= \left\{O_{\bullet\beta}, \{\beta\alpha\}, \{\alpha\beta\}\right\} \end{aligned}$$

Clearly $\mathbf{w} \prec \mathbf{v}$. ◀

3.2.5 Linear independence

We rely on the following property of tight orbits (this property is not true for arbitrary orbits):

Lemma 3.13. *If orbits $O, O_1, \ldots, O_n$ are tight and $O \subseteq O_1 \cup \ldots \cup O_n$ then $O \subseteq O_i$ for some $i = 1, \ldots, n$.*

Proof. If O is a singleton then the claim holds vacuously. Relying on Lemma 2.25-iv we may thus assume that O is infinite.

Suppose $O \subseteq O_1 \cup \ldots \cup O_n$ for a tight orbit O and arbitrary tight orbits $O_1, \ldots, O_n$. Let $K \stackrel{\text{def}}{=} \text{support}(O)$. Take any $x \in O$ and let $K' \stackrel{\text{def}}{=} \text{support}(x) \setminus$

K. Consider elements $\pi(x) \in O$ for all $\pi \in \mathrm{Aut}_K(\mathbb{A})$, thus ranging over all elements of the orbit O. At least one of the orbits $O_1, \ldots, O_m$, say the orbit O_1, necessarily contains $\pi(x)$ and $\pi'(x)$, for some two K-atoms automorphisms π, π', such that the sets $\pi(K')$ and $\pi'(K')$ are disjoint. Let $K_1 \stackrel{\text{def}}{=} \mathrm{support}\,(O_1)$. By tightness of O_1 (and relying on Lemma 2.7) we get $K_1 \subseteq \mathrm{support}\,(\pi(x)) \subseteq K \cup \pi(K')$ and $K_1 \subseteq \mathrm{support}\,(\pi'(x)) \subseteq K \cup \pi'(K')$, and hence $K_1 \subseteq K$. This implies

$$O = \mathrm{orbit}_K(\pi(x)) \subseteq \mathrm{orbit}_{K_1}(\pi(x)) = O_1 .$$

□

We now argue that the set $\textsc{Basis}(B)$ is linearly independent. Towards contradiction, suppose that the zero vector is obtainable as a linear combination of basis vectors

$$q_1 \cdot \mathbf{1}_{O_1} \; + \; \ldots \; + \; q_n \cdot \mathbf{1}_{O_n} \quad = \quad \mathbf{0}, \tag{3.4}$$

for some tight pairwise-different orbits $O_1, \ldots, O_n \subseteq B$ and $q_1, \ldots, q_n \in \mathbb{R} \setminus \{0\}$. Take any inclusion-maximal orbit among $O_1, \ldots, O_n$, say O_1. We distinguish two cases.

Case 1 If $O_1 \subseteq O_2 \cup \ldots \cup O_n$ then using Lemma 3.13 we arrive at a contradiction with the inclusion-maximality of O_1.

Case 2 Otherwise $O_1 \not\subseteq O_2 \cup \ldots \cup O_n$. Taking any $x \in O_1 \setminus (O_2 \cup \ldots \cup O_n)$ we arrive at a contradiction, as the value of the left-hand side of (3.4) on x is non-zero:

$$\left(q_1 \cdot \mathbf{1}_{O_1} \; + \; \ldots \; + \; q_n \cdot \mathbf{1}_{O_n}\right)(x) \;=\; q_1 \;\neq\; 0,$$

while the value of the right-hand side is $\mathbf{0}(x) = 0$. ■

Remark 3.14. Theorem 3.1 is effective. Indeed, the transformation from B to $\textsc{Basis}(B)$ is equivariant, and the set $\textsc{Basis}(B)$ as well as the transformation from $\mathbf{v} \in [R]B$ to its basis representation in $\mathrm{FinLin}_R(\textsc{Basis}(B))$ are supported by $\mathrm{support}\,(B)$, and therefore is computable (Remark 2.14).

Remark 3.15. The number of S-orbits inside $\textsc{Basis}(B)$ is linear in the number of S-orbits inside B and exponential in the S-atom-dimension of B (recall that S was fixed to be the support of B in the beginning of this section). The linearity in numnber of S-orbits of B follows from (3.1). We quickly show that when B is an S-orbit of atom-dimension k, there are at most 2^k many S-orbits inside $\textsc{Basis}(B)$. Indeed, using Lemmas 3.5 and 2.21, B is isomorphic to a set of the form $\{S\} \times (\mathbb{A} \setminus S)^{(k)}/G$ for some $k \in \mathbb{N}$ and subgroup G of $\mathbf{S}_k$, by an

S-supported bijection. Vectors in $\textsc{Basis}(B)$ are of the form $\mathbf{1}_O$ where $O \subseteq B$ is a tight orbit. Tight orbits inside $\{S\} \times (\mathbb{A} \setminus S)^{(k)}/G$ are of the form

$$\mathrm{orbit}_{S \cup T}((S, \alpha_1 \ldots \alpha_k/G))$$

for $\alpha_1, \ldots, \alpha_k \in \mathbb{A} \setminus S$ and $T \subseteq \{\alpha_1, \ldots, \alpha_k\}$. Tight orbits $\mathrm{orbit}_{S \cup T}((S, \alpha_1 \ldots \alpha_k/G))$ and $\mathrm{orbit}_{S \cup T'}((S, \alpha'_1 \ldots \alpha'_k/G))$ are in the same S-orbit when

$$\{i \in \{1, \ldots, k\} \;:\; \alpha_i \in T\} = \left\{i' \in \{1, \ldots, k\} \;:\; \alpha'_{i'} \in T'\right\} \;.$$

This means the number of S-orbits of tight orbits inside $\{S\} \times (\mathbb{A} \setminus S)^{(k)}/G$ is at most 2^k.

Remark 3.16. Theorem 3.1 is an improvement upon [6, Theorem 6.7] which says that $\mathrm{Lin}_R(B)$ has an orbit-finite spanning set for orbit-finite sets B. In fact, the basis $\textsc{Basis}(B)$ of $\mathrm{Lin}_R(B)$ that we construct in this section is a subset of the spanning set of $\mathrm{Lin}_R(B)$ constructed in the proof of [6, Theorem 6.7].

3.3 Non-existence of bases

Contrary to the classical setting, not all orbit-finitely spanned vector spaces have an orbit-finite basis. An example was provided in [6] without a complete proof. We repeat the example and supply the proof as well.

Let $V \subseteq \mathrm{FinLin}\left(\mathbb{A}^{(2)}\right)$ be the subspace spanned by the set of vectors

$$X \stackrel{\text{def}}{=} \left\{\alpha\beta - \beta\alpha \;:\; \alpha\beta \in \mathbb{A}^{(2)}\right\} \;.$$

We show that V does not have an orbit-finite basis. Towards a contradiction, suppose $B \subseteq V$ is an orbit-finite basis of V supported by some $S \subseteq_{\mathrm{FIN}} \mathbb{A}$. For $T \subseteq_{\mathrm{FIN}} \mathbb{A} \setminus S$, let

$$V_{S \cup T} \stackrel{\text{def}}{=} V \cap \left\{\mathbf{v} \in V \;:\; \mathrm{dom}(\mathbf{v}) \subseteq (S \cap T)^{(2)}\right\} \quad \text{and} \quad B_{S \cup T} \stackrel{\text{def}}{=} B \cap V_{S \cup T} \;.$$

Lemma 3.17. *For any $T \subseteq_{\mathrm{FIN}} \mathbb{A} \setminus S$, $B_{S \cup T}$ is a basis of $V_{S \cup T}$.*

Proof. Pick arbitrary $T \subseteq_{\mathrm{FIN}} \mathbb{A} \setminus S$. Since $B_{S \cup T} \subseteq B$ and B is linearly independent, $B_{S \cup T}$ is also linearly independent. So we just have to show that $B_{S \cup T}$ spans $V_{S \cup T}$.

Towards a contradiction, suppose that the unique basis representation of some vector $\mathbf{v} \in V_{S \cup T}$ uses some base vector not belonging to $B_{S \cup T}$. That is, for some $\mathbf{v} \in V_{S\cup T}$, $r_1, \ldots, r_n \neq 0$ and $\mathbf{u}_1, \ldots, \mathbf{u}_n \in B$,

$$\mathbf{v} = r_1 \cdot \mathbf{u}_1 + \cdots + r_n \cdot \mathbf{u}_n$$

and $\mathbf{u}_1 \notin B_{S\cup T}$. Pick some atom $\alpha \in \operatorname{support}(\mathbf{u}_1) \setminus (S \cup T)$ and

$$\beta \in \mathbb{A} \setminus (S \cup T \cup \operatorname{support}(\mathbf{u}_2) \cup \cdots \cup \operatorname{support}(\mathbf{u}_n)) \ .$$

Let $\pi \in \operatorname{Aut}(\mathbb{A})$ be the automorphism of atoms which swaps α and β and fixes all other atoms. Then $\pi \in \operatorname{Aut}_{S\cup T}(\mathbb{A})$. Hence $\pi(\mathbf{v}) = \mathbf{v}$ and $\pi(\mathbf{u}_i) \in B$ for all $i \in \{2, \ldots, n\}$. Moreover:

Claim 3.17.1. $\mathbf{u}_1 \neq \pi(\mathbf{u}_i)$ *for all* $i = 2, \ldots, n$.

Proof. For every i we have $\beta \notin \operatorname{support}(\mathbf{u}_i)$ and hence $\alpha \notin \operatorname{support}(\mathbf{u}_i)$. As $\alpha \in \operatorname{support}(\mathbf{u}_1)$, we deduce $\mathbf{u}_1 \neq \pi(\mathbf{u}_i)$. □

Since $\mathbf{v} \in V_{S\cup T}$, $\pi(\mathbf{v}) = \mathbf{v}$. Hence,

$$\sum_{i=1}^{n} r_i \cdot \mathbf{u}_i - \sum_{i=1}^{n} r_i \cdot \pi(\mathbf{u}_i) = \mathbf{v} - \pi(\mathbf{v}) = 0\,.$$

Since $r_1 \neq 0$ and $\mathbf{u}_1 \neq \mathbf{u}_2, \ldots, \mathbf{u}_n, \pi(\mathbf{u}_1), \ldots, \pi(\mathbf{u}_n)$, this contradicts linear independence of B. ■

Lemma 3.18. *For any* $T \subseteq_{\mathrm{FIN}} \mathbb{A} \setminus S$*, The dimension of* $V_{S\cup T}$ *is* $\binom{|S\cup T|}{2}$.

Proof. Pick arbitrary $T \subseteq_{\mathrm{FIN}} \mathbb{A} \setminus S$. Let $n \stackrel{\text{def}}{=} |S \cup T|$, and let $\alpha_1, \ldots, \alpha_n$ be an enumeration of the elements of $S \cup T$. Then

$$\left\{\alpha_i\alpha_j - \alpha_j\alpha_i \ : \ i \leqslant j\right\}$$

is a basis of $V_{S\cup T}$. Hence dimension of $V_{S\cup T}$ is $\binom{n}{2}$. ■

As an immediate consequence of the above two lemmas we get that for any $T \subseteq_{\mathrm{FIN}} \mathbb{A} \setminus S$ the set $B_{S\cup T}$ has $\binom{|S\cup T|}{2}$ elements. Let $m \stackrel{\text{def}}{=} |S|$. Pick $\alpha\beta \in (\mathbb{A} \setminus S)^{(2)}$. Then we have

$$|B_S| = \binom{m}{2},$$

$$|B_{S\cup\{\alpha\}}| = |B_{S\cup\{\beta\}}| = \binom{m+1}{2}, \text{and}$$

$$|B_{S\cup\{\alpha,\beta\}}| = \binom{m+2}{2}.$$

Moreover, as $B_{S\cup\{\alpha\}} \cap B_{S\cup\{\beta\}} = B_S$, by the exclusion-inclusion principle we have:

$$|B_{S\cup\{\alpha\}} \cup B_{S\cup\{\beta\}}| = |B_{S\cup\{\alpha\}}| + |B_{S\cup\{\beta\}}| - |B_S| \,.$$

In consequence,

$$|B_{S\cup\{\alpha,\beta\}} \setminus (B_{S\cup\{\alpha\}} \cup B_{S\cup\{\beta\}})| = \binom{m+2}{2} - 2\cdot\binom{m+1}{2} + \binom{m}{2} = 1$$

We show that this leads to a contradiction. Let $\mathbf{v}$ be the unique element in $B_{S\cup\{\alpha,\beta\}} \setminus (B_{S\cup\{\alpha\}} \cup B_{S\cup\{\beta\}})$. It can be written as

$$\mathbf{v} = r\cdot(\alpha\beta - \beta\alpha) + \mathbf{u}$$

for some $r \neq 0$ and $\mathbf{u} \in V_{S\cup\{\alpha\}} + V_{S\cup\{\beta\}}$. Let π be the automorphism which swaps α and β and fixes all other atoms. Since $\alpha, \beta \notin S$ we have $\pi \in \mathrm{Aut}_S(\mathbb{A})$. The set B is supported by S. Hence

$$\pi(\mathbf{v}) = r\cdot(\beta\alpha - \alpha\beta) + \pi(\mathbf{u}) \in B \,.$$

Clearly $\pi(\mathbf{v}) \neq \mathbf{v}$ and $\pi(\mathbf{v}) \in B_{S\cup\{\alpha,\beta\}} \setminus (B_{S\cup\{\alpha\}} \cup B_{S\cup\{\beta\}})$, which contradicts $B_{\{S\cup\alpha,\beta\}} \setminus (B_{S\cup\{\alpha\}} \cup B_{S\cup\{\beta\}})$ being a singleton. ■

3.4 Non-isomorphic bases

In this section we give an example of a vector space with two orbit-finite bases which are non-isomorphic as orbit-finite sets (i.e. there is no orbit-finite bijection from one to the other).

Let H, K_{12} and K_{23} be the subgroups of $\mathbf{S}_3$ (the symmetric group of order 3) generated respectively by the permutations

$$(1\mapsto 2, 2\mapsto 3, 3\mapsto 1)\,,\quad (1\mapsto 2, 2\mapsto 1)\,,\quad (2\mapsto 3, 3\mapsto 2)\,.$$

Recall Notation 2.18. For any $\alpha\beta\gamma \in \mathbb{A}^{(3)}$

$$\begin{array}{ccccc} \alpha\beta\gamma/H & = & \beta\gamma\alpha/H & = & \gamma\alpha\beta/H \\ \alpha\beta\gamma/K_{12} & = & \beta\alpha\gamma/K_{12} & \neq & \alpha\gamma\beta/K_{12} \\ \alpha\beta\gamma/K_{23} & = & \alpha\gamma\beta/K_{23} & \neq & \beta\alpha\gamma/K_{23} \end{array} .$$

Define equivariant orbit-finite sets B and C as

$$B \stackrel{\text{def}}{=} \left(\mathbb{A}^{(3)}/H\right) \uplus \left(\mathbb{A}^{(3)}/K_{12}\right) \uplus \left(\mathbb{A}^{(3)}/K_{23}\right)$$
$$C \stackrel{\text{def}}{=} \binom{\mathbb{A}}{3} \uplus \binom{\mathbb{A}}{3} \uplus \mathbb{A}^{(3)} .$$

To distinguish the copies of $\binom{\mathbb{A}}{3}$ inside C we write an arbitrary element of the first copy as $\{\alpha, \beta, \gamma\}$ and an element of the second copy as $[\alpha, \beta, \gamma]$. An arbitrary element of $\mathbb{A}^{(3)} \subseteq C$ is written as $\alpha\beta\gamma$.

Lemma 3.19. *There exists an equivariant linear isomorphism from* $\mathrm{FinLin}(B)$ *to* $\mathrm{FinLin}(C)$.

Lemma 3.20. *There is no orbit-finite bijection from B to C.*

In conseqeunce of the two lemmas, the space $\mathrm{FinLin}(B)$ (and $\mathrm{FinLin}(C)$) has two bases not related by an orbit-finite bijection.

Proof of Lemma 3.19. We define equivariant linear functions

$$f : \mathrm{FinLin}(B) \to \mathrm{FinLin}(C) \quad \text{and} \quad g : \mathrm{FinLin}(C) \to \mathrm{FinLin}(B)$$

such that $f \circ g$ and $g \circ f$ are both identities. Note that it is enough to define the functions on the generators. Recall Notation 2.29. Define $f : \mathrm{FinLin}(B) \to \mathrm{FinLin}(C)$ as

$$f(\alpha\beta\gamma/H) \stackrel{\text{def}}{=} \alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta$$
$$f(\alpha\beta\gamma/K_{12}) \stackrel{\text{def}}{=} [\alpha, \beta, \gamma] + \alpha\beta\gamma + \beta\alpha\gamma$$
$$f(\alpha\beta\gamma/K_{23}) \stackrel{\text{def}}{=} \{\alpha, \beta, \gamma\} + \alpha\beta\gamma + \alpha\gamma\beta$$

for $\alpha\beta\gamma \in \mathbb{A}^{(3)}$. Define $g' : \mathrm{FinLin}(C) \to \mathrm{FinLin}(B)$ as

$$g'(\{\alpha, \beta, \gamma\}) \stackrel{\text{def}}{=} \alpha\beta\gamma/K_{23} + \beta\gamma\alpha/K_{23} + \gamma\alpha\beta/K_{23} - (\alpha\beta\gamma/H + \beta\alpha\gamma/H)$$
$$g'([\alpha, \beta, \gamma]) \stackrel{\text{def}}{=} \alpha\beta\gamma/K_{12} + \beta\gamma\alpha/K_{12} + \gamma\alpha\beta/K_{12} - (\alpha\beta\gamma/H + \beta\alpha\gamma/H)$$
$$g'(\alpha\beta\gamma) \stackrel{\text{def}}{=} \alpha\beta\gamma/K_{23} - \beta\gamma\alpha/K_{23} + \alpha\beta\gamma/K_{12} - \alpha\gamma\beta/K_{12} + \alpha\beta\gamma/H$$

Define $g : \mathrm{FinLin}(C) \to \mathrm{FinLin}(B)$ as

$$g(\mathbf{x}) \stackrel{\text{def}}{=} \frac{g'(\mathbf{x})}{3} \text{ for all } \mathbf{x} \in \mathrm{FinLin}(C) .$$

We leave it to the reader to check that f and g are well-defined and $f \circ g$ and $g \circ f$ are both identities. Notice that it is enough to check it on the generators. ∎

Proof of Lemma 3.20. We prove it by contradiction. Say there exists orbit-finite functions $p : B \to C$ and $q : C \to B$ such that both $p \circ q$ and $q \circ p$ are identities. Let S be the union of supports of p and q

$$S \stackrel{\text{def}}{=} \text{support}(p) \cup \text{support}(q) \ .$$

Then S supports both p and q. Pick $\alpha\beta\gamma \in (\mathbb{A} \setminus S)^{(3)}$. Since q is injective, applying Lemma 2.23-v we get $\text{support}(q(\alpha\beta\gamma)) \cup S = \{\alpha, \beta, \gamma\} \cup S$. Hence

$$q(\alpha\beta\gamma) \in \{\alpha\beta\gamma/H, \alpha\beta\gamma/K_{12}, \alpha\beta\gamma/K_{23}\} \ .$$

We show $q(\alpha\beta\gamma)$ cannot be equal to $\alpha\beta\gamma/H$, and leave it to the reader to check that it cannot be equal to $\alpha\beta\gamma/K_{12}$ or $\alpha\beta\gamma/K_{23}$ either. Let π be the automorphism defined as

$$\begin{aligned}
\pi(\alpha) &\stackrel{\text{def}}{=} \beta \\
\pi(\beta) &\stackrel{\text{def}}{=} \gamma \\
\pi(\gamma) &\stackrel{\text{def}}{=} \alpha \\
\pi(\delta) &\stackrel{\text{def}}{=} \delta \text{ for all } \delta \neq \alpha, \beta, \gamma \ .
\end{aligned}$$

Since p is inverse of q, it must hold

$$\alpha\beta\gamma = p(\alpha\beta\gamma/H) \ . \tag{3.5}$$

By definition of H,

$$\alpha\beta\gamma/H = \pi(\alpha\beta\gamma/H) \ . \tag{3.6}$$

We have chosen α, β, γ to be outside S, so $\pi \in \text{Aut}_S(\mathbb{A})$. Since p is supported by S, using Lemma 2.4

$$p(\pi(\alpha\beta\gamma/H)) = \pi(p(\alpha\beta\gamma/H)) \ . \tag{3.7}$$

Combining (3.5), (3.6) and (3.7) we get

$$\alpha\beta\gamma \stackrel{(3.5)}{=} p(\alpha\beta\gamma/H) \stackrel{(3.6)}{=} p(\pi(\alpha\beta\gamma/H)) \stackrel{(3.7)}{=} \pi(p(\alpha\beta\gamma/H)) \stackrel{(3.5)}{=} \pi(\alpha\beta\gamma) \ ,$$

which is a contradiction. ■

Remark 3.21. The reason why such there exists an vector space with two orbit-finite bases which are not isomorphic as orbit-finite sets is because the uniqueness of bases fails in the category of G-sets; there exists a finite group

G and non-isomorphic G-sets X and Y whose associated representations are isomorphic [1]. In fact, if we assume $\mathbb{A}$ to be a set of cardinality three instead of being countable and suitably adapt Lemmas 3.19 and 3.20, our example shows this for $G = \mathbf{S}_3$.

[1] `https://math.stackexchange.com/questions/4103510/injectivity-of-the-functor-from-group-actions-to-permutation-representations`

Chapter 4

Linear Equations

Contents

4.1 Introduction

In this chapter we are interested in solvability of orbit-finite systems of equations. For this chapter fix R to be an arbitrary commutative ring with multiplicative unit 1, which satisfies the following effectivity assumption:

Assumption 4.1. *Given a finite system of linear equations with coefficients in R, one can decide whether it has a solution and compute a solution if there is one.*

Remark 4.2. The above assumption in particular implies that the addition and multiplication operations of R are computable. For $a, b \in R$, $(a+b)$ is the unique solution to z for the system if equations

$$x = a \qquad y = b \qquad z = x + y\,,$$

likewise, $a \cdot b$ is the unique solution to v for the system of equations

$$u = b \qquad v = a \cdot x\,.$$

The effectivity assumption is satisfied by commonly used rings such as $\mathbb{Q}$, $\mathbb{Z}$, the field of algebraic numbers, ring of integers modulo m for $m \in \mathbb{Z}_+$. We assume that the coefficients of vectors and matrices in this chapter to be from R.

We recall from § 2.4.2 that, for orbit-finite sets B and C, an orbit-finite $(B \times C)$-system of equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ is given by a matrix $\mathbf{A} \in \mathrm{Lin}_R(B \times C)$ and $\mathbf{b} \in \mathrm{Lin}_R(B)$. We are interested in the decidability of the following problems.

EQ(R)
Input: An orbit-finite system of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$.
Question: Does it have an **orbit-finite** solution?

FIN-EQ(R)
Input: An orbit-finite system of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$.
Question: Does it have a **finite** solution?

We show EQ(R) reduces to FIN-EQ(R) and FIN-EQ(R) is decidable:

Theorem 4.3. EQ(R) *reduces to* FIN-EQ(R).

Theorem 4.4. FIN-EQ(R) *is decidable.*

And as a consequence we get:

Theorem 4.5. EQ(R) *is decidable.*

For $R = \mathbb{Z}, \mathbb{Q}$ we get a more precise complexity.

Theorem 4.6. EQ($\mathbb{Z}$) *and* EQ($\mathbb{Q}$) *are both in* EXPTIME, *and in* PTIME *for fixed atom-dimension of the input system.*

Atom dimension of an orbit-finite system of equations is defined in the following subsection.

4.1.1 Representation of input

There are several possible ways of representing an orbit-finite $B \times C$-system of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ as input to our algorithms. One possibility is to rely on the equivalence between (hereditary) orbit-finite sets and *definable* sets [5, Sect. 4]. We choose another standard possibility, as specified in items (1)–(3) below. First, the representation includes:

(1) a common support $S \subseteq_{\mathrm{FIN}} \mathbb{A}$ of $\mathbf{A}$ and $\mathbf{b}$.

Second, knowing that B and C are disjoint unions of S-orbits, and relying on Lemma 2.21, the representation also includes:

(2) a list of all S-orbits included in B and C, each one represented by some tuple $\overline{\alpha} \in (\mathbb{A} \setminus S)^{(n)}$ and subgroup G of $\mathbf{S}_n$; and a list of S-orbits included in $B \times C$, each one represented by one of its element.

Finally, relying on Notation 2.28, we assume that the representation contains:

(3) a list of values $\mathbf{t}(O), \mathbf{s}(O), \mathbf{A}(O) \in R$, respectively, for all S-orbits O included in B, C, and $B \times C$. Integers are represented in binary.

The *atom-dimension* of such a representation is the maximum of atom-dimensions of B, C, $\mathbf{A}$ and $\mathbf{b}$ (see Definitions 2.17 and 2.40).

Organisation of the chapter

In § 4.2 we prove Theorem 4.3, i.e. we show that solvability reduces to finitary solvability. Then in § 4.3 we introduce the *order equivariant finitary solvability problem* and give a reduction of finitary solvability to order equivariant finitary solvability (§ 4.4). Finally in § 4.5 we show the latter problem is decidable, thus proving Theorem 4.4. We end with a rough estimation of complexity in § 4.6 to justify Theorem 4.6.

4.2 Solvability reduces to finitary solvability

In this section we prove Theorem 4.3, i.e. we show that $\mathrm{E{\scriptsize Q}}(R)$ reduces to $\mathrm{F{\scriptsize IN}}$-$\mathrm{E{\scriptsize Q}}(R)$. For this section, fix orbit-finite sets B and C, and an orbit-finite $(B \times C)$-system of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$. WLOG, we assume $\mathrm{support}(\mathbf{A}) = \mathrm{support}(\mathbf{b}) = \mathrm{support}(B) = \mathrm{support}(C) = S$, for some $S \subseteq_{\mathrm{FIN}} \mathbb{A}$, and that B and C are finite unions of tight S-orbits. [1]

[1] This can be achieved by taking $S = \mathrm{support}(\mathbf{A}) \cup \mathrm{support}(\mathbf{b})$ and replacing B and C with $\{(S, b) \,:\, b \in B\}$ and $\{(S, c) \,:\, c \in C\}$, respectively.

4.2.1 Spans

For a subset $P \subseteq \mathrm{Lin}_R(B)$, we define $\textsc{Fin-Span}(P) \subseteq \mathrm{Lin}_R(B)$ as the set of all linear combinations of vectors from P, forming a subspace of $\mathrm{Lin}_R(B)$:

$$\textsc{Fin-Span}(P) \stackrel{\text{def}}{=} \left\{ \sum_{\mathbf{v} \in P'} q_{\mathbf{v}} \cdot \mathbf{v} \; : \; q_{\mathbf{v}} \in R,\ P' \subseteq_{\text{FIN}} P \right\} .$$

Given a vector $\mathbf{v} \in \mathrm{Lin}_R(C)$ the product $\mathbf{A} \cdot \mathbf{v}$ can be also seen as the orbit-finite linear combination

$$\sum_{U \in \mathrm{Orbits}_T(C)} \mathbf{v}(U) \cdot \sum_{c \in U} \mathbf{A}(-, c) \qquad \text{(recall Notation 2.28)}$$

of column vectors $\mathbf{A}(-, c)$, where $T \stackrel{\text{def}}{=} S \cup \mathrm{support}\,(\mathbf{v})$. This sum is well-defined as a vector in $\mathrm{Lin}_R(B)$ and is equal to $\mathbf{A} \cdot \mathbf{v}$ when the latter is well-defined (recall well-definedness of products of matrices from § 2.3). This allows us to define the span of $\mathbf{A}$ by seeing it as the orbit-finite set of vectors $\{\mathbf{A}(-, c) \; : \; c \in C\}$:

$$\textsc{Span}(\mathbf{A}) \stackrel{\text{def}}{=} \{\mathbf{A} \cdot \mathbf{v} \; : \; \mathbf{v} \in \mathrm{Lin}_R(C),\ \mathbf{A} \cdot \mathbf{v} \text{ well-defined}\} . \tag{4.1}$$

Deciding existence of an orbit-finite solution of the system $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ amounts to deciding if $\mathbf{b} \in \textsc{Span}(\mathbf{A})$.

For a finite vector $\mathbf{u} \in \mathrm{FinLin}(C)$ the product $\mathbf{A} \cdot \mathbf{u}$ is always well-defined, and we define:

$$\begin{aligned} \textsc{Fin-Span}(\mathbf{A}) &\stackrel{\text{def}}{=} \{\mathbf{A} \cdot \mathbf{u} \; : \; \mathbf{u} \in \mathrm{FinLin}(C)\} \\ &= \textsc{Fin-Span}(\{\mathbf{A}(-, c) \; : \; c \in C\}), \end{aligned}$$

We prove Theorem 4.3 by effectively constructing a matrix $\widetilde{\mathbf{A}}$ with the same row-indexing set B as $\mathbf{A}$, such that $\textsc{Span}(\mathbf{A}) = \textsc{Fin-Span}(\widetilde{\mathbf{A}})$.

4.2.2 Well-definedness and exactness

Let $\mathbf{x} \in \mathrm{Lin}_R(C)$ a vector. We start by a characterisation of vectors $\mathbf{x} \in \mathrm{Lin}_R(C)$ for which the product $\mathbf{y} = \mathbf{A} \cdot \mathbf{x}$ is well-defined. Recall that $\mathbf{y}(b)$ is well-defined if and only if there are only finitely many $c \in C$ such that $\mathbf{A}(b, c) \neq 0$ and $\mathbf{x}(c) \neq 0$. Let

$$T \stackrel{\text{def}}{=} \mathrm{support}\,(\mathbf{A}) \cup \mathrm{support}\,(\mathbf{x}) = \mathrm{support}\,((\mathbf{A}, \mathbf{x})) .$$

We say that the pair $(\mathbf{A}, \mathbf{x})$ is *exact* if for every $b \in B$ and $c \in C$ such that $\mathbf{A}(b,c) \neq 0$ and $\mathbf{x}(c) \neq 0$, it holds

$$\text{support}\,(c) \subseteq \text{support}\,(b) \cup T. \tag{4.2}$$

Lemma 4.7. $\mathbf{A} \cdot \mathbf{x}$ *is well-defined if and only if* $(\mathbf{A}, \mathbf{x})$ *is exact.*

Proof. For $b \in B$, define $T_b \stackrel{\text{def}}{=} \text{support}\,(b) \cup T$.

For the if direction, suppose $(\mathbf{A}, \mathbf{x})$ is exact, and consider an arbitrary fixed $b \in B$. By (4.2) the support of every c satisfying $\mathbf{A}(b,c) \neq 0$ and $\mathbf{x}(c) \neq 0$ is included in T_b. By Lemma 2.25-i, every orbit $V \subseteq C$ contains at most $|T_b|!$ elements $c \in V$ such that $\text{support}\,(c) \subseteq T_b$. Since C has finitely many orbits, there are only finitely many $c \in C$ satisfying $\mathbf{A}(b,c) \neq 0$ and $\mathbf{x}(c) \neq 0$. The product $\mathbf{A} \cdot \mathbf{x}$ is thus well-defined, as required.

For the opposite direction, suppose $(\mathbf{A}, \mathbf{x})$ is not exact, i.e., for some $b \in B$ and $c \in C$ we have:

$$\mathbf{A}(b,c) \neq 0\,, \quad \mathbf{x}(c) \neq 0\,, \quad \text{and} \quad \text{support}\,(c) \not\subseteq T_b\,.$$

According to the latter condition, some atom $\alpha \in \mathbb{A}$ satisfies

$$\alpha \in \text{support}\,(c) \quad \text{and} \quad \alpha \notin T_b\,.$$

Pick an infinite set of atoms $\{\beta_1, \beta_2, \ldots\} \subseteq (\mathbb{A} \setminus T_b)$. For $n \in \mathbb{N}$, define $\pi_n \in \text{Aut}(\mathbb{A})$ as $\pi_n(\alpha) \stackrel{\text{def}}{=} \beta_n$, $\pi_n(\beta_n) \stackrel{\text{def}}{=} \alpha$ and $\pi_n(\gamma) \stackrel{\text{def}}{=} \gamma$ for $\gamma \in \mathbb{A} \setminus \{\alpha, \beta_n\}$. Then $\pi_n \in \text{Aut}_{T_b}(\mathbb{A})$ which implies it preserves $\mathbf{A}$, $\mathbf{x}$ and b. Applying Lemma 2.3 we get that for all $n \in \mathbb{N}$ we have

$$\mathbf{A}(b, \pi_n(c)) = \pi_n(\mathbf{A})(\pi_n(b), \pi_n(c)) = \mathbf{A}(b,c) \neq 0, \text{ and}$$

$$\mathbf{x}(\pi_n(c)) = \pi_n(\mathbf{x})(c) = \mathbf{x}(c) \neq 0\,.$$

Applying Lemma 2.7 we get that $\beta_n \in \text{support}\,(\pi_m(c))$ if and only if $n = m$. This implies the set $\{\pi_n(c) \,:\, n \in \mathbb{N}\}$ is an infinite set, and in consequence the product $\mathbf{A} \cdot \mathbf{x}$ is not well-defined on b. This completes the proof. ∎

4.2.3 Proof of Theorem 4.3

We are going to construct effectively a matrix $\widetilde{\mathbf{A}}$ with the same row-indexing set B as $\mathbf{A}$, which satisfies $\text{SPAN}(\mathbf{A}) = \text{FIN-SPAN}(\widetilde{\mathbf{A}})$. We claim that it is enough to consider the special case when C is a single S-orbit. Indeed, otherwise we can split the matrix $\mathbf{A}$ into m matrices

$$\mathbf{A} = \left[\mathbf{A}_1 | \dots | \mathbf{A}_m\right]$$

each corresponding to one S-orbit $C_i \subseteq C$. Assuming matrices $\widetilde{\mathbf{A}_i}$ such that $\textsc{Span}(\mathbf{A}_i) = \textsc{Fin-Span}(\widetilde{\mathbf{A}_i})$ for $i = 1, \dots, m$, we construct a matrix $\widetilde{\mathbf{A}}$ as

$$\widetilde{\mathbf{A}} \stackrel{\text{def}}{=} \left[\widetilde{\mathbf{A}_1} | \dots | \widetilde{\mathbf{A}_m}\right] .$$

We have

$$\begin{aligned} \textsc{Span}(\mathbf{A}) &= \textsc{Span}(\mathbf{A}_1) + \cdots + \textsc{Span}(\mathbf{A}_m) \\ &= \textsc{Fin-Span}(\widetilde{\mathbf{A}_1}) + \cdots + \textsc{Fin-Span}(\widetilde{\mathbf{A}_m}) \\ &= \textsc{Fin-Span}(\widetilde{\mathbf{A}}) \end{aligned}$$

We thus proceed under the assumption that C is a single S-orbit. WLOG we also assume C to be a tight S-orbit (Lemma 3.5).

Lemma 4.8. *For any $\mathbf{v} \in \mathrm{Lin}_R(C)$, if $\mathbf{A} \cdot \mathbf{v}$ is well-defined and $\mathbf{u} \in \textsc{Basis}(C)$ appears in the basis representation of $\mathbf{v}$, then $\mathbf{A} \cdot \mathbf{u}$ is well-defined too.*

Proof. Let $\mathbf{v} \in \mathrm{Lin}_R(C)$, and let $T \stackrel{\text{def}}{=} \mathrm{support}\,(\mathbf{v}) \cup S$. We follow the definition of the basis representation of T-supported vector $\mathbf{v}$ by structural induction with respect to the transitive closure of $\prec$, as in the proof of Lemma 3.9. If T-orbit-dom$(\mathbf{v})$ is empty then $\mathbf{v}$ is the zero vector and the claim holds vacuously. Otherwise, suppose the lemma holds for all strictly smaller T-supported vectors $\mathbf{w}$. As in the proof of Lemma 3.9, take a T-orbit $O \in T$-orbit-dom$(\mathbf{v})$ of maximal S-dimension. Let

$$T' \stackrel{\text{def}}{=} \mathrm{support}\,(c) \cap T \tag{4.3}$$

for some (every) $c \in O$. Since $S \subseteq T$ and $S \subseteq \mathrm{support}\,(c)$ (as every S-orbit in C is tight), we deduce

$$S \subseteq T' . \tag{4.4}$$

We define the T'-orbit O' as T'-closure of O:

$$O' \stackrel{\text{def}}{=} \{\pi(c) \ : \ c \in O,\ \pi \in \mathrm{Aut}_{T'}(\mathbb{A})\} .$$

By definition, T' is included in the support of every element of O', therefore the orbit O' is tight, and hence $\mathbf{1}_{O'} \in \textsc{Basis}(C)$. According to the proof of Lemma 3.9, the vector $\mathbf{1}_{O'}$ appears in the basis representation of $\mathbf{v}$, together with the vectors appearing in the basis representation of the vector

$$\mathbf{w} \quad \stackrel{\text{def}}{=} \quad \mathbf{v} \; - \; \mathbf{v}(O) \cdot \mathbf{1}_{O'}. \tag{4.5}$$

Note that $\mathbf{w}$ is supported by T as both $\mathbf{v}$ and $\mathbf{1}_{O'}$ are so. Moreover, $\mathbf{w}(O) = 0$. By Claim 3.10.1 we infer that $\mathbf{w} \prec \mathbf{v}$ and therefore, relying on the induction assumption, it is sufficient to show that $\mathbf{A} \cdot \mathbf{1}_{O'}$ is well-defined.

According to the assumption and Lemma 4.7 we know that $(\mathbf{A}, \mathbf{v})$ is exact. Using Lemma 4.7 again, it is sufficient to show that $(\mathbf{A}, \mathbf{1}_{O'})$ is exact too.

Choose an arbitrary element $c \in O'$ and $b \in B$ such that $\mathbf{A}(b, c) \neq 0$, and an arbitrary $\pi \in \mathrm{Aut}_{T'}(\mathbb{A})$ such that $\pi(c) \in O$. $\mathbf{A}$ is S-supported so it is also T'-supported (Equation (4.4)). Hence using Lemma 2.4 we get $\mathbf{A}(\pi(b), \pi(c)) \neq 0$. As $(\mathbf{A}, \mathbf{v})$ is exact and $\mathbf{v}(O) \neq 0$, we have:

$$\text{support}\,(\pi(c)) \subseteq \text{support}\,(\pi(b)) \cup T. \tag{4.6}$$

By definition of T' (Equation (4.3)), as $\pi(c) \in O$, we have $T' = \text{support}\,(\pi(c)) \cap T$, and thus the inclusion (4.6) can be strenghtened to

$$\text{support}\,(\pi(c)) \subseteq \text{support}\,(\pi(b)) \cup T'.$$

Applying π^{-1} and using Lemma 2.7 to both sides yields

$$\text{support}\,(c) \subseteq \text{support}\,(b) \cup T'\,.$$

As b and c were chosen arbitrarily, we conclude that $(\mathbf{A}, \mathbf{1}_{O'})$ is exact, as required. ■

As the indexing set of $\widetilde{\mathbf{A}}$ we take those basis vectors $\mathbf{v} \in \textsc{Basis}(C)$ for which $\mathbf{A} \cdot \mathbf{v}$ is well-defined:

$$\widetilde{C} \stackrel{\text{def}}{=} \{\mathbf{v} \in \textsc{Basis}(C) \;:\; \mathbf{A} \cdot \mathbf{v} \text{ is well-defined}\}\,.$$

The new indexing set $\widetilde{C}$ is orbit-finite as $\textsc{Basis}(C)$ is so, and is S-supported since both $\mathbf{A}$ and $\textsc{Basis}(C)$ are S-supported. We define the new matrix $\widetilde{\mathbf{A}}$: $\mathrm{Lin}_R\left(B \times \widetilde{C}\right)$ as follows

$$\widetilde{\mathbf{A}}(-, \mathbf{v}) \;\stackrel{\text{def}}{=}\; \mathbf{A} \cdot \mathbf{v}.$$

Note the injection $c \mapsto \mathbf{1}_c$ of C into $\widetilde{C}$, as $\mathbf{A} \cdot \mathbf{1}_c = \mathbf{A}(-, c)$ is always well-defined. Therefore $\widetilde{\mathbf{A}}$ extends $\mathbf{A}$, as $\widetilde{\mathbf{A}}(-, \mathbf{1}_c) = \mathbf{A} \cdot \mathbf{1}_c = \mathbf{A}(-, c)$. It is now sufficient to prove:

Lemma 4.9. $\textsc{Span}(\mathbf{A}) = \textsc{Fin-Span}(\widetilde{\mathbf{A}})$.

Proof. WLOG we assume that $\mathbf{A}$ contains non-zero column vectors only (otherwise, since C is a single orbit, *all* column vectors in $\mathbf{A}$ are zero vectors and the claim holds vacuously). In one direction, consider any vector $\mathbf{v} \in \text{FIN-SPAN}(\widetilde{\mathbf{A}})$, i.e.,

$$\mathbf{v} \;=\; q_1 \cdot (\mathbf{A} \cdot \mathbf{v}_1) + \ldots + q_n \cdot (\mathbf{A} \cdot \mathbf{v}_n)$$

for $q_1, \ldots, q_n \in \mathbb{Q}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \widetilde{C}$, which immediately yields the required membership in $\text{SPAN}(\mathbf{A})$:

$$\mathbf{v} \;=\; \mathbf{A} \cdot (q_1 \cdot \mathbf{v}_1 + \ldots + q_n \cdot \mathbf{v}_n) \;\in\; \text{SPAN}(\mathbf{A}).$$

In the opposite direction, let $\mathbf{v} = \mathbf{A} \cdot \mathbf{x}$ be well-defined for some $\mathbf{x} \in \text{Lin}_R(C)$. We are going to prove that $\mathbf{v} \in \text{FIN-SPAN}(\widetilde{\mathbf{A}})$. Consider the representation of $\mathbf{x}$ in the basis $\text{BASIS}(C)$

$$\mathbf{x} \;=\; q_1 \cdot \mathbf{v}_1 + \ldots + q_\ell \cdot \mathbf{v}_\ell.$$

Due to Lemma 4.8 we know that $\mathbf{A} \cdot \mathbf{v}_i$ is well-defined and hence $\mathbf{v}_i \in \widetilde{C}$ for all $i = 1, \ldots, \ell$. Therefore

$$\begin{aligned} \mathbf{v} \;=\; & \mathbf{A} \cdot (q_1 \cdot \mathbf{v}_1 + \ldots + q_\ell \cdot \mathbf{v}_\ell) \;=\; \\ & q_1 \cdot (\mathbf{A} \cdot \mathbf{v}_1) + \ldots + q_\ell \cdot (\mathbf{A} \cdot \mathbf{v}_\ell) \;=\; \\ & q_1 \cdot \widetilde{\mathbf{A}}(-, \mathbf{v}_1) + \ldots + q_\ell \cdot \widetilde{\mathbf{A}}(-, \mathbf{v}_\ell) \;\in\; \text{FIN-SPAN}(\widetilde{\mathbf{A}}), \end{aligned}$$

as required. ∎

Since $C \mapsto \text{BASIS}(C)$ is an effective construction (Remark 3.14), the transformation from $\mathbf{A}$ to $\widetilde{\mathbf{A}}$ is effective as well. This completes the proof of Theorem 4.3. ∎

Remark 4.10. Assuming $\mathbf{A}$ is either integer or rational matrix, the blow-up in the reduction $\mathbf{A} \mapsto \widetilde{\mathbf{A}}$ is exponential in the atom dimension of $\mathbf{A}$, and polynomial in the number of orbits of B and C.

The matrix $\widetilde{\mathbf{A}}$ is indexed by $B \times \widetilde{C}$, and $\widetilde{C} \subseteq \text{BASIS}(C)$. Let S be the support of $\mathbf{A}$ and d be its atom-dimension (§ 4.1.1). Then S also supports the system $\widetilde{\mathbf{A}}$. Remark 3.15 says that the number of S-orbits of $\text{BASIS}(C)$ is linear in the number of S-orbits of C and exponential in d. Elements of $\text{BASIS}(C)$ are of the form $\mathbf{1}_O$ where O is a tight $S \cup T$-orbit for some $T \subseteq \mathbb{A} \setminus S$. By definition of tight orbits, $O = \text{orbit}_{S \cup T}(c)$ for some $c \in O \subseteq C$ and $T \subseteq \text{support}\,(c)$. Which means

$$\text{support}\,(O) \setminus S = T \subseteq \text{support}\,(c) \setminus S = S\text{-dim}(O)\ .$$

This implies the atom-dimension of $\text{BASIS}(C)$, and hence also of $\widetilde{C}$, is at most d.

Since the atom-dimension of B and $\widetilde{C}$ is both less than d. Using Lemma 2.25-v we get that the number of S-orbits of $B \times \widetilde{C}$ is linear in the number of S-orbits of $B \times C$ and exponential in d. To finish the proof, we show that the entries of $\widetilde{\mathbf{A}}$ only blow up by a multiplicative factor which is exponential in d. The entries of $\widetilde{\mathbf{A}}$ are of the form $\mathbf{A}(b, -) \cdot \mathbf{1}_O$ for some $b \in B$ and tight orbit O in C such that $\mathbf{A} \cdot \mathbf{1}O$ is well-defined. Let $T = (\text{support}\,(O) \cup \text{support}\,(b)) \setminus S$. Then $|T| \leqslant 2d$. Using Lemma 2.49-ii and Lemma 2.48 we get

$$\mathbf{A}(b, -) \cdot \mathbf{1}_O = \sum_{c \in O_{S \cup T}} \mathbf{A}(b, c)\,,$$

where $O_{S \cup T} = \{c \in O \,:\, \text{support}\,(c) \subseteq S \cup T\}$. Lemma 2.25-ii implies that $O_{S \cup T}$ contains at most $|T|! \leqslant (2d)!$ many elements. This implies the size of the entry $\mathbf{A}(b, -) \cdot \mathbf{1}_O$ of $\widetilde{\mathbf{A}}$ is at most $(2d)! \cdot \max\{|\mathbf{A}(b, c)| \,:\, b \in B,\ c \in C\}$.

4.3 Order equivariant finitary solvability

Let $<$ be a dense linear order on $\mathbb{A}$ without endpoints. For concreteness, one can equate $\mathbb{A}$ with the set of rationals with the usual ordering. A permutation $\pi \in \text{Aut}_<(\mathbb{A})$ is called an *$<$-automorphism* if it is order preserving:

$$\pi(\alpha) < \pi(\beta) \text{ for all } \alpha < \beta\,.$$

The set of all $<$-automorphisms in $\text{Aut}(\mathbb{A})$ forms a subgroup, and is denoted as $\text{Aut}_<(\mathbb{A})$:

$$\text{Aut}_<(\mathbb{A}) \stackrel{\text{def}}{=} \{\pi \in \text{Aut}(\mathbb{A}) \,:\, \pi \text{ is a } <\text{-automorphism}\}\,.$$

For x an atom or a set, the *$<$-orbit* of x is defined as:

$$\text{orbit}_<(x) \stackrel{\text{def}}{=} \{\pi(x) \,:\, \pi \in \text{Aut}_<(\mathbb{A})\}\,.$$

A set is *order equivariant* if it is invariant under $<$-automorphisms. Since $\text{Aut}_<(\mathbb{A}) \subseteq \text{Aut}(\mathbb{A})$, every equivariant set is automatically order equivariant. Every order equivariant set splits into $<$-orbits. The set of $<$-orbits inside a order equivariant set B is denoted as

$$\text{Orbits}_<(B) \stackrel{\text{def}}{=} \{O \subseteq B \,:\, O \text{ is a } <\text{-orbit}\}\,.$$

B is called *$<$-orbit-finite* if $\text{Orbits}_<(B)$ is finite.

Remark 4.11. Order equivariant sets, $<$-orbits and $<$-orbit-finite sets are respectively equivariant sets, equivariant orbits and equivariant orbit-finite sets with ordered atoms (see Remark 2.22).

Example 4.12. For any $n \in \mathbb{N}$, the set $\binom{\mathbb{A}}{n}$ is a $<$-orbit, and the set $\mathbb{A}^{(n)}$ is order equivariant. For any permutation g in $\mathbf{S}_n$ the subset $\binom{\mathbb{A}}{n}_g$ of $\mathbb{A}^{(n)}$ defined as

$$\binom{\mathbb{A}}{n}_g \stackrel{\text{def}}{=} \left\{\alpha_{g(1)} \dots \alpha_{g(n)} \ : \ \alpha_1 < \dots < \alpha_n\right\}$$

is a $<$-orbit. It is isomorphic to $\binom{\mathbb{A}}{n}$ by the order equivariant bijection

$$\alpha_1 \dots \alpha_n \mapsto \{\alpha_1, \dots, \alpha_n\} : \binom{\mathbb{A}}{n}_g \to \binom{\mathbb{A}}{n} .$$

For any $n \in \mathbb{N}$, the set $\mathbb{A}^{(n)}$ is the union of the $n!$ disjoint $<$-orbits $\binom{\mathbb{A}}{n}_g$, where g ranges over elements of $\mathbf{S}_n$.

Note that, for $n \geqslant 2$ none of the sets $\binom{\mathbb{A}}{n}_g$ are equivariant or even finitely supported (under the action of $\mathrm{Aut}(\mathbb{A})$). We leave it to the reader to verify this claim. ◀

The following lemma generalises Example 4.12 from $\mathbb{A}^{(n)}$ to arbitrary equivariant orbit-finite sets.

Lemma 4.13. *Every equivariant orbit-finite set is also $<$-orbit-finite.*

Proof. We show that every equivariant orbit splits into finitely many $<$-orbits.

Consider an orbit $X = \mathrm{orbit}(x)$. Using our assumption of hereditary finite support (page 17) let

$$\mathrm{support}\,(x) = \{\alpha_1 < \dots < \alpha_n\} \ .$$

For $g \in \mathbf{S}_n$ define $\sigma_g \in \mathrm{Aut}(\mathbb{A})$ as

$$\begin{aligned} \sigma_g(\alpha_i) &\stackrel{\text{def}}{=} \alpha_{g(i)} && \text{for } i \in \{1, \dots, n\} \\ \sigma_g(\alpha) &\stackrel{\text{def}}{=} \alpha && \text{for } \alpha \notin \{\alpha_1, \dots, \alpha_n\}. \end{aligned}$$

We show that

$$\mathrm{orbit}(x) = \bigcup_{g \in \mathbf{S}_n} \mathrm{orbit}_<(\sigma_g(x)) \ .$$

Pick $\pi \in \mathrm{Aut}(\mathbb{A})$. We need to find $g \in \mathbf{S}_n$ and $\pi' \in \mathrm{Aut}_<(\mathbb{A})$ such that $\pi'(\sigma_g(x)) = \pi(x)$. Let $\{\beta_1 < \dots < \beta_n\} \stackrel{\text{def}}{=} \pi(\{\alpha_1, \dots, \alpha_n\})$. We define g to be the permutation of $\{1, \dots, n\}$ satisfying $\pi(\alpha_i) = \beta_{g(i)}$, for $i \in \{1, \dots, n\}$. Let π' be any order preserving permutation such that $\pi'(\alpha_i) = \beta_i$. Clearly

$(\pi' \circ \sigma_g)(\alpha_i) = \beta_{g(i)}$ for all $i \in \{1, \ldots, n\}$. Applying Lemma 2.23-ii now we get

$$\pi'(\sigma_g(x)) = \pi(x)$$

which finishes the proof. ■

Definition 4.14. A $<$-orbit-finite set is called *canonical* if it is a finite disjoint union of orbits of the form $\binom{\mathbb{A}}{k}$ for $k \in \mathbb{N}$. The *atom-dimension* of such a set is the maximum k such that it has an orbit of the form $\binom{\mathbb{A}}{k}$.

Remark 4.15. Note that a canonical orbit-finite set is also equivariant in the usual sense (i.e. w.r.t. the action of $\mathrm{Aut}(\mathbb{A})$), and its atom-dimension, when considered as an equivariant set (again w.r.t. the action of $\mathrm{Aut}(\mathbb{A})$), is the same as its atom-dimension when considered as a canonical $<$-orbit-finite set.

Let B be a canonical $<$-orbit-finite set. Relying on § 4.2.1 we define the *order equivariant finitary solvability problem*, which is a variant of FIN-EQ(R):

FIN-EQ$_<(R)$

Input:	An order equivariant vector $\mathbf{v} \in \mathrm{FinLin}_R(B)$ and an $<$-orbit-finite set $P \subseteq \mathrm{FinLin}_R(B)$, where B is a canonical $<$-orbit-finite set.
Question:	Is $\mathbf{v} \in$ FIN-SPAN(P)?

Theorem 4.16. FIN-EQ(R) *reduces to* FIN-EQ$_<(R)$.

Theorem 4.17. FIN-EQ$_<(R)$ *is decidable.*

These two theorems together imply Theorem 4.4. Theorems 4.16 and 4.17 are proven in § 4.4 and § 4.5, respectively .

4.4 Finitary solvability to order equivariant finitary solvability

This section is devoted to proving that FIN-EQ(R) reduced to FIN-EQ$_<(R)$ (Theorem 4.4). Fix an orbit-finite $(B \times C)$-system of equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ supported by some $S \subseteq \mathbb{A}$. We want to decide if it has a finite solution. We start with two simplifying assumptions in § 4.4.1, and then reduce FIN-EQ(R) to FIN-EQ$_<(R)$ in § 4.4.2.

4.4.1 Simplifying assumptions

Column-finiteness

A $(B' \times C')$-system of equations $\mathbf{A}' \cdot \mathbf{x}' = \mathbf{b}'$ is called *column-finite* if the columns of the augmented matrix $(\mathbf{A}'|\mathbf{b}')$[2] are finite, i.e.

$$\{\mathbf{A}'(-, c') \; : \; c' \in C'\} \cup \{\mathbf{b}'\} \subseteq \mathrm{FinLin}_R(B') \; .$$

Lemma 4.18. *WLOG we can assume that the system $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ is column-finite.*

Proof. We define a column-finite system system which has a finite solution if and only if the system $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ also has a finite solution. For this proof, let

$$f : \mathrm{Lin}(B) \to \mathrm{FinLin}(\textsc{Basis}(B))$$

be the linear isomorphism which given an orbit-finite vector outputs its basis decomposition, i.e. for all $\mathbf{v} \in \mathrm{Lin}(B)$

$$\mathbf{v} = \sum_{\mathbf{b} \in \textsc{Basis}(B)} (f(\mathbf{v}))(\mathbf{b}) \cdot \mathbf{b} \; .$$

Let $\mathbf{A}' \in \mathrm{Lin}(\textsc{Basis}(B) \times C)$ be the matrix whose columns are defined as

$$\mathbf{A}'(-, c) \stackrel{\text{def}}{=} f(\mathbf{A}(-, c))$$

for $c \in C$. We leave it to the reader to verify that the system $\mathbf{A}' \cdot \mathbf{x} = \mathbf{b}'$ has the same finite solutions as the system $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$. Note that we can use the same vector of variables $\mathbf{x}$ for both of the systems, since columns of both $\mathbf{A}$ and $\mathbf{A}'$ are indexed by C. ■

Equivariance and Straightness

Recall the definition of straight orbit-finite sets (Definition 2.20).

Lemma 4.19. *WLOG we can assume that B and C are equivariant straight orbit-finite sets and, $\mathbf{A}$ and $\mathbf{b}$ are equivariant.*

Proof. The proof will be done in two steps. In the first step we show that WLOG B and C can be assumed to be straight orbit-finite sets. Then in the second step we show that furthermore WLOG we can assume B,C, $\mathbf{A}$ and $\mathbf{b}$ to be equivariant.

[2] `https://en.wikipedia.org/wiki/Augmented_matrix`

Step 1: Using Lemma 2.25-iii we know there exists straight orbit-finite sets B' and C' and S-supported surjective functions $f : B' \to B$ and $g : C' \to C$. Define $\mathbf{A}' \in \mathrm{Lin}(B' \times C')$ and $\mathbf{b}' \in \mathrm{Lin}(B')$ as

$$\mathbf{A}'(b', c') \stackrel{\text{def}}{=} \mathbf{A}(f(b'), g(c')) \quad \text{and} \quad \mathbf{b}'(b') \stackrel{\text{def}}{=} \mathbf{b}(f(b')) .$$

It is easy to check that both $\mathbf{A}'$ and $\mathbf{b}'$ are supported by S. We need to show that the system $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has a finite solution if and only if the system $\mathbf{A}' \cdot \mathbf{x}' = \mathbf{b}'$ has a finite solution.

Recall Notation 2.29. In one direction, suppose $\mathbf{A}' \cdot \mathbf{y}' = \mathbf{b}'$ for some

$$\mathbf{y}' = \sum_{i=1}^{n} r_i \cdot c_i' \ \in \ \mathrm{FinLin}_R(C') \ ,$$

where $r_i \in R$ and $c_i' \in C'$. We claim that $\mathbf{A} \cdot \mathbf{y} = \mathbf{b}$, where

$$\mathbf{y} = \sum_{i=1}^{n} r_i \cdot g(c_i') \ .$$

Indeed, for any $b' \in B'$ we have

$$\begin{aligned}
(\mathbf{A} \cdot \mathbf{y})(f(b')) &= \sum_{i=1}^{n} \mathbf{A}(f(b'), g(c_i')) \cdot \mathbf{y}(g(c_i')) \\
&= \sum_{i=1}^{n} \mathbf{A}(f(b'), g(c_i')) \cdot r_i \\
&= \sum_{i=1}^{n} \mathbf{A}'(b', c_i') \cdot \mathbf{y}'(c_i') \\
&= (\mathbf{A}' \cdot \mathbf{y}')(b') \\
&= \mathbf{b}'(b') \\
&= \mathbf{b}(f(b))
\end{aligned}$$

Since f is surjective, this says $\mathbf{A} \cdot \mathbf{y} = \mathbf{b}$.

For the opposite direction, suppose $\mathbf{A} \cdot \mathbf{z} = \mathbf{b}$ for some

$$\mathbf{z} = \sum_{j=1}^{m} s_j \cdot d_j \in \mathrm{FinLin}(C)$$

Since g is surjective, for every j there exists a $d_j' \in C'$ such that $g(d_j') = d_j$. Define $\mathbf{z}' \in \mathrm{FinLin}(C')$ as

$$\mathbf{z}' = \sum_{j=1}^{m} s_j \cdot d_j' \ .$$

We show $\mathbf{A}' \cdot \mathbf{z}' = \mathbf{b}'$. For any $b' \in B'$ we have

$$\begin{aligned}(\mathbf{A}' \cdot \mathbf{z}')(b') &= \sum_{j=1}^{m} \mathbf{A}'(b', d'_j) \cdot \mathbf{z}'(d'_j) \\ &= \sum_{j=1}^{m} \mathbf{A}(f(b'), g(d'_j)) \cdot s_j \\ &= \sum_{j=1}^{m} \mathbf{A}(f(b'), d_j) \cdot \mathbf{z}(d_j) \\ &= (\mathbf{A} \cdot \mathbf{z})(f(b')) \\ &= \mathbf{b}(f(b')) \\ &= \mathbf{b}'(b') \ .\end{aligned}$$

This finishes the first step of the proof.

Step 2: Redefine the set of atoms to be $(\mathbb{A} \setminus S)$ instead of $\mathbb{A}$. We have

$$\mathrm{Aut}_S(\mathbb{A}) = \mathrm{Aut}(\mathbb{A} \setminus S) \ . \tag{4.7}$$

Both B and C are finite unions of sets of the form $(\mathbb{A} \setminus S)^{(n)}$ for $n \in \mathbb{N}$. Hence both of them are equivariant sets with atoms $(\mathbb{A} \setminus S)$. The matrix $\mathbf{A}$ and the vector $\mathbf{b}$ are S-supported as sets with atoms $\mathbb{A}$. Hence using (4.7) we can say they are equivariant sets with atoms $\mathbb{A} \setminus S$. ∎

We illustrate the steps by the following examples.

Example 4.20. We give an example of application of the first step. Recall Notation 2.18. Let $B = \binom{\mathbb{A}}{2}$ and $C = \mathbb{A}^{(3)}/H$ where H is generated by the cyclic permutation

$$1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1 \ .$$

Recall Notation 2.29. Define the matrix $\mathbf{A} \in \mathrm{Lin}(B \times C)$ with columns

$$\mathbf{A}(-, \alpha\beta\gamma/H) = \{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\}$$

for $\alpha\beta\gamma \in \mathbb{A}^{(3)}$. We leave it to the reader to verify that $\mathbf{A}$ is well-defined and equivariant. Fix $\alpha_1, \alpha_2 \in \mathbb{A}$, and let

$$\mathbf{b} = \{\alpha_1, \alpha_2\} \ .$$

Then the system $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ is supported by $S = \{\alpha_1, \alpha_2\}$. As a result of applying Step 1 to this system we get $\mathbb{A}^{(2)} \times \mathbb{A}^{(3)}$-system $\mathbf{A}' \cdot \mathbf{x}' = \mathbf{b}'$, where

$$\mathbf{A}'(-, \alpha\beta\gamma) = \alpha\beta + \beta\alpha + \beta\gamma + \gamma\beta + \gamma\alpha + \alpha\gamma$$

for $\alpha\beta\gamma \in \mathbb{A}^{(3)}$, and

$$\mathbf{b}' = \alpha_1\alpha_2 + \alpha_2\alpha_1 .$$

◀

Example 4.21. We give an example of application of the second step. Let $B = \mathbb{A}$ and $C = \mathbb{A}^{(2)}$. Define $\mathbf{A} \in \mathrm{Lin}(B \times C)$ and $\mathbf{b} \in \mathrm{FinLin}(B)$ as

$$\mathbf{A}(-, \alpha\beta) = 2 \cdot \alpha + \beta$$

for $\alpha\beta \in \mathbb{A}^{(2)}$ and

$$\mathbf{b} = 2 \cdot \alpha .$$

The system of equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ is supported by $\{\alpha\}$. Let $\mathbb{A}_\alpha = \mathbb{A} \setminus \{\alpha\}$. We have

$$\mathrm{Orbits}_{\{\alpha\}}(B) = \{\mathbb{A}_\alpha, \{\alpha\}\}$$

and

$$\mathrm{Orbits}_{\{\alpha\}}(C) = \left\{ (\mathbb{A}_\alpha \times \{\alpha\}, \{\alpha\} \times \mathbb{A}_\alpha, \mathbb{A}_\alpha^{(2)} \right\} .$$

For applying Step 2 we define our set of atoms to be $\mathbb{A}_\alpha$ and replace α with an equivariant symbol a to emphasise that it is not an element of $\mathbb{A}_\alpha$. Then we get the $(\{a\} \cup \mathbb{A}_\alpha) \times (\mathbb{A}_\alpha \uplus \mathbb{A}_\alpha \uplus (\mathbb{A}_\alpha^{(2)})$-system

$$\begin{array}{cc} & \begin{array}{ccc} \mathbb{A}_\alpha & \mathbb{A}_\alpha & \mathbb{A}_\alpha^{(2)} \end{array} \\ \begin{array}{c} \{a\} \\ \mathbb{A}_\alpha \end{array} & \left[\begin{array}{c|c|c} 2 \cdot \mathbf{1}_{\mathbb{A}_\alpha}^{\mathsf{T}} & \mathbf{1}_{\mathbb{A}_\alpha}^{\mathsf{T}} & \mathbf{0}^{\mathsf{T}} \\ \hline \mathrm{Id}_{\mathbb{A}_\alpha} & 2 \cdot \mathrm{Id}_{\mathbb{A}_\alpha} & \mathbf{A}_\alpha \end{array}\right] \end{array} \cdot \mathbf{x} = \left[\begin{array}{c} 2 \\ \hline \mathbf{0} \end{array}\right] \tag{4.8}$$

Where $\mathbf{A}_\alpha$ is simply the restriction of $\mathbf{A}$ to $\mathbb{A}_\alpha^{(2)} \times \mathbb{A}_\alpha$. Notice that this system is equivariant. ◀

Remark 4.22. Both Step 1 and 2 preserve column-finiteness of the system. For Step 1 it follows from the fact that Lemma 2.25-iii gives us f and g such that for every $b \in B$ and $g \in C$ both $f^{-1}(b)$ and $g^{-1}(c)$ are finite. Step 2 only changes the set of atoms and does not modify the matrix.

Remark 4.23. Step 2 in the above proof does not increase the size or atom-dimension of a system. We argue that Step 1 does not increase the atom-dimension of a system and can only increase the size of a system by a multiplicative factor of $2^{O(d \log(d))}$ where d is the atom-dimension of the system. Let n and m, respectively, be the number of orbits of B and C. Lemma 2.25-iii guarantees that the atoms dimension of B' and C' is at most d, and the number of orbits of B' and C' is, respectively, n and m. The number of orbits of $B\times C$ is at least nm. Lemma 2.25-v implies that the number of orbits of $B'\times C'$ is at most $(2d)^{2d} \cdot nm = 2^{O(d\log(d))} \cdot nm$, and hence at most $2^{O(dlog(d))}$ times the number of orbits in $B\times C$.

4.4.2 The reduction

Due to § 4.4.1, the system $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ is assumed to be equivariant, straight and column-finite. The system $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has a finite solution if and only if the vector $\mathbf{b} \in \mathrm{FinLin}(B)$ is in the equivariant subspace spanned by the equivariant orbit-finite set of vectors

$$P = \{\mathbf{A}(-,c) \; : \; c \in C\} \subseteq \mathrm{FinLin}(B) \; .$$

Since $\mathbf{b}$ and P are equivariant, they are also order equivariant. Moreover, using Lemma 4.13 the set of vectors P is also $<$-orbit-finite. In fact the proof of Lemma 4.13 is effective, i.e. it gives us a method to compute representatives of $<$-orbits of P from representatives of equivariant orbits of P (as illustrated by Example 4.24 below). The only thing left to do is to make B a canonical $<$-orbit-finite set. This follows from the assumption that B is equivariant orbit-finite and straight, and the fact that for any $n \in \mathbb{N}$, the set $\mathbb{A}^{(n)}$ splits into $<$-orbits $\binom{\mathbb{A}}{n}_g$ for $g \in \mathbf{S}_n$ (Example 4.12), each of which is isomorphic to $\binom{\mathbb{A}}{n}$.

Example 4.24. Consider the system (4.8) in Example 4.21. Writing $\mathbb{A}$ instead of $\mathbb{A}_\alpha$, we get $B = \{a\} \uplus \mathbb{A}$, $\mathbf{b} = 2 \cdot a$ and

$$P = \mathrm{orbit}(2 \cdot a + \beta) \cup \mathrm{orbit}(a + 2 \cdot \beta) \cup \mathrm{orbit}(2 \cdot \beta + \gamma)$$

for some (every) $\beta, \gamma \in \mathbb{A}$. B is already a canonical $<$-equivariant set. WLOG assume $\beta < \gamma$. Applying the construction in Lemma 4.13 we split P into four $<$-orbits

$$\begin{aligned} P \quad = \quad & \mathrm{orbit}_<(2 \cdot a + \beta) \;\cup\; \mathrm{orbit}_<(a + 2 \cdot \gamma) \cup \\ & \mathrm{orbit}_<(2 \cdot \beta + \gamma) \;\cup\; \mathrm{orbit}_<(\beta + 2 \cdot \gamma) \; . \end{aligned}$$

◀

4.5 Deciding order equivariant finitary solvability

For the remainder of this section, fix a canonical $<$-orbit-finite set B.

Definition 4.25. Define the *$<$-orbit summation function*

$$\gamma_< : \mathrm{FinLin}_R(B) \to R^{\mathrm{Orbits}_<(B)}$$

as follows:

$$\gamma_<(\mathbf{v}) : O \in \mathrm{Orbits}_<(B) \mapsto \sum_{b \in O} \mathbf{v}(b)\ .$$

The following lemma follows immediately from the definition:

Lemma 4.26. *The $<$-orbit summation function $\gamma_<$ is order equivariant.*

Corollary 4.27. *For every vector $\mathbf{x}$, we have $\gamma_<(\mathrm{orbit}_<(\mathbf{x})) = \{\gamma_<(\mathbf{x})\}$.*

Proof. Since $\gamma_<$ is order equivariant (Lemma 4.26), for every vector $\mathbf{y} \in \mathrm{orbit}_<(\mathbf{x})$ we have $\gamma_<(\mathbf{x}) = \gamma_<(\mathbf{y})$. Hence

$$\gamma_<(\mathrm{orbit}_<(\mathbf{x})) = \{\gamma_<(\mathbf{x})\}\ .$$

■

Example 4.28. Consider $R = \mathbb{Z}$ and $B = \{\star\} \uplus \binom{\mathbb{A}}{2} \uplus \binom{\mathbb{A}}{2}$. Then

$$\mathrm{FinLin}_{\mathbb{Z}}(B) \cong \mathbb{Z} \times \mathrm{FinLin}_{\mathbb{Z}}\binom{\mathbb{A}}{2} \times \mathrm{FinLin}_{\mathbb{Z}}\binom{\mathbb{A}}{2}$$

and we may write an element of $\mathrm{FinLin}_{\mathbb{Z}}(B)$ as a tuple $(r, \mathbf{p}, \mathbf{q})$ where $r \in \mathbb{Z}$ and $\mathbf{p}, \mathbf{q} \in \mathrm{FinLin}_{\mathbb{Z}}\binom{\mathbb{A}}{2}$. For $\alpha < \beta < \gamma$ define $\mathbf{s}_{\alpha\beta}, \mathbf{t}_{\alpha\beta\gamma}$ as

$$\mathbf{s}_{\alpha\beta} \overset{\mathrm{def}}{=} (1, \{\alpha, \beta\}, \{\alpha, \beta\}) \qquad \mathbf{t}_{\alpha\beta\gamma} \overset{\mathrm{def}}{=} (0, \{\alpha, \beta\} + \{\beta, \gamma\}, 2 \cdot \{\alpha, \gamma\})\ .$$

Let $\mathbf{v} \overset{\mathrm{def}}{=} (1, \mathbf{0}, \mathbf{0})$ and

$$P \overset{\mathrm{def}}{=} \{\mathbf{s}_{\alpha\beta} \ :\ \alpha < \beta \in \mathbb{A}\} \cup \{\mathbf{t}_{\alpha\beta\gamma} \ :\ \alpha < \beta < \gamma \in \mathbb{A}\}\ .$$

For $\alpha < \beta < \gamma \in \mathbb{A}$ we have

$$\begin{aligned} \gamma_<(\mathbf{v}) &= (1, 0, 0) \\ \gamma_<(\mathbf{s}_{\alpha\beta}) &= (1, 1, 1) \\ \gamma_<(\mathbf{t}_{\alpha\beta\gamma}) &= (0, 2, 2)\ . \end{aligned}$$

◀

Theorem 4.29. *For every order equivariant vector $\mathbf{b} \in \mathrm{FinLin}_R(B)$ and order equiv-*

ariant set of vectors $P \subseteq \mathrm{FinLin}_R(B)$

$$\mathbf{b} \in \textsc{Fin-Span}(P) \iff \gamma_<(\mathbf{b}) \in \textsc{Fin-Span}(\gamma_<(P)) .$$

Note that we do not assume P to be orbit-finite in the statement of theorem. However, in all the applications of the theorem in this chapter, P is orbit-finite.

The above theorem implies decidability of $\textsc{Fin-Eq}_<(R)$. Let $\mathbf{b}$ be an arbitrary order equivariant vector in $\mathrm{FinLin}(B)$, and $P = \mathrm{orbit}_<(\mathbf{p}_1) \cup \cdots \cup \mathrm{orbit}_<(\mathbf{p}_n)$ be an arbitrary orbit-finite set of vectors inside $\mathrm{FinLin}(B)$. Applying Corollary 4.27 we get

$$\textsc{Fin-Span}(\{\gamma_<(P)\}) = \textsc{Fin-Span}(\{\gamma_<(\mathbf{p}_1), \ldots, \gamma_<(\mathbf{p}_n)\}) .$$

Now Theorem 4.29 implies that, $\mathbf{b} \in \textsc{Fin-Span}(P)$ if and only if

$$\gamma_<(\mathbf{b}) \in \textsc{Fin-Span}(\{\gamma_<(\mathbf{p}_1), \ldots, \gamma_<(\mathbf{p}_n)\}) .$$

The latter is an instance of solvability of a finite system of linear equations over R, which is decidable by our effectivity assumption on R (Assumption 4.1).

Example 4.30. Continuing Example 4.28, by Theorem 4.29 we get $\mathbf{v} \notin \textsc{Fin-Span}(P)$. However, if we take $R = \mathbb{Q}$ instead of $\mathbb{Z}$, then $\mathbf{v} \in \textsc{Fin-Span}(P)$. ◀

The remainder of this section is devoted to proving Theorem 4.29.

4.5.1 Proof of Theorem 4.29

Let n be the atom-dimension of B. Then B can be written as

$$B = B' \uplus \widetilde{B}$$

where

$$B' = \binom{\mathbb{A}}{n_1} \uplus \cdots \uplus \binom{\mathbb{A}}{n_m}$$

for some $m \in \mathbb{N}$ and $0 \leqslant n_1 \leqslant \cdots \leqslant n_m < n$, and

$$\widetilde{B} = \underbrace{\binom{\mathbb{A}}{n} \uplus \cdots \uplus \binom{\mathbb{A}}{n}}_{\ell\text{-times}}$$

for some $\ell \in \mathbb{Z}_+$. Hence $\mathrm{FinLin}_R(B)$ can be written as

$$\mathrm{FinLin}_R(B) = \mathrm{FinLin}_R(B') \times \mathrm{FinLin}_R\left(\widetilde{B}\right) .$$

Noting this, we can write every vector $\mathbf{v} \in \mathrm{FinLin}(B)$ as

$$\mathbf{v} = (\mathbf{v}', \widetilde{\mathbf{v}})\,,$$

where

$$\mathbf{v}' \in \mathrm{FinLin}_R(B') \cong \mathrm{FinLin}_R\binom{\mathbb{A}}{n_1} \times \ldots \times \mathrm{FinLin}_R\binom{\mathbb{A}}{n_m}$$
$$\widetilde{\mathbf{v}} \in \mathrm{FinLin}_R\left(\widetilde{B}\right) \cong \left(\mathrm{FinLin}_R\binom{\mathbb{A}}{n}\right)^{\ell}.$$

For $n \in \mathbb{N}$, define a mapping

$$f_n : \mathrm{FinLin}_R\binom{\mathbb{A}}{n} \to \mathrm{FinLin}_R\binom{\mathbb{A}}{n-1}^{n}$$

as the unique linear extension of the map

$$\{\alpha_1 < \cdots < \alpha_n\} \mapsto (\{\alpha_2, \ldots, \alpha_n\}, \{\alpha_1, \alpha_3, \ldots, \alpha_n\}, \ldots, \{\alpha_1, \ldots, \alpha_{n-1}\})\,.$$

Equivalently,

$$f_n(\mathbf{v}) = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

where

$$\mathbf{v}_i(\{\beta_1, \ldots, \beta_{n-1}\}) = \sum_{\beta_{i-1} < \beta < \beta_i} \mathbf{v}(\{\beta_1 < \cdots < \beta_{n-1}\} \cup \{\beta\})$$

assuming $\beta_0 = -\infty$ and $\beta_n = +\infty$. The function f_n is readily extended to spaces of the form $\left(\mathrm{FinLin}_R\binom{\mathbb{A}}{n}\right)^{\ell}$ for $\ell \in \mathbb{Z}_+$ as

$$f_n(\mathbf{v}_1, \ldots, \mathbf{v}_\ell) = (f_n(\mathbf{v}_1), \ldots, f_n(\mathbf{v}_\ell))$$

and to spaces of the form $\mathrm{FinLin}_R\binom{\mathbb{A}}{k_1} \times \cdots \times \mathrm{FinLin}_R\binom{\mathbb{A}}{k_m} \times \left(\mathrm{FinLin}_R\binom{\mathbb{A}}{n}\right)^{\ell}$ for $0 \leqslant k_1 \leqslant k_2 \leqslant k_m < n$ and $\ell \in \mathbb{Z}_+$ as

$$f_n(\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{v}) = (\mathbf{v}_1, \ldots, \mathbf{v}_m, f_n(\mathbf{v}))$$

Remark 4.31. The definition of the function f_n is borrowed from [6, page 14].

Now we state the main lemma for proving Theorem 4.29.

Lemma 4.32. *Let C be a canonical $<$-orbit-finite set of atom-dimension $n \geqslant 1$ (Definition 4.14). For every order equivariant vector $\mathbf{v} \in \mathrm{FinLin}(C)$ and order equivariant* *set of vectors U in* $\mathrm{FinLin}(C)$*,*

$$\mathbf{v} \in \text{Fin-Span}(U) \quad \textit{if and only if} \quad f_n(\mathbf{v}) \in \text{Fin-Span}(f_n(U)) \ .$$

Note that U is not assumed to be $<$-orbit-finite in the statement of the lemma. Before proving the lemma, let's prove Theorem 4.29 using it.

Proof of Theorem 4.29 using Lemma 4.32

If $\mathbf{b} \in \text{Fin-Span}(P)$, then $\gamma_<(\mathbf{b}) \in \text{Fin-Span}(\gamma_<(P))$ by linearity of $\gamma_<$. We need to prove the other direction.

Assume $\gamma_<(\mathbf{b}) \in \text{Fin-Span}(\gamma_<(P))$. Let n be the atom-dimension of B. If $n = 0$, them $\gamma_< = \mathrm{Id}$ on $\mathrm{FinLin}_R(B)$ and $\mathbf{b} \in \text{Fin-Span}(P)$. Otherwise, applying Lemma 4.32 n-times we get

Corollary 4.33. $\mathbf{b} \in \text{Fin-Span}(P)$ *if and only if* $f_*(\mathbf{b}) = \text{Fin-Span}(f_*(P))$*, where* $f_* \stackrel{\text{def}}{=} f_1 \circ \ldots \circ f_n$*.*

Each of the functions f_i is equivariant and linear, and hence so is f_*. Let V be the vector space which is the co-domain of f_*. Then

$$V = \left(\mathrm{FinLin}_R\binom{\mathbb{A}}{0}\right)^p \cong R^p$$

for some $p \in \mathbb{Z}_+$, and hence is atomless. By Corollary 4.33, to prove that $\mathbf{b}$ is in $\text{Fin-Span}(P)$ it is enough to prove that $f_*(\mathbf{b}) \in \text{Fin-Span}(f_*(P))$, which we do now.

Since f_* is equivariant and its co-domain V is atomless, for every $<$-orbit O of B and $b, b' \in O$, f_* maps $\mathbf{1}_b$ and $\mathbf{1}_{b'}$ to the same vector,

$$f_*(\mathbf{1}_b) = f_*(\mathbf{1}_{b'}) \ .$$

Denote this vector as $f_*(O)$. Define the map $g : R^{\mathrm{Orbits}_<(B)} \to R^p$ as

$$g(\mathbf{x}) = \sum_{O \in \mathrm{Orbits}_<(B)} \mathbf{x}(O) \cdot f_*(O) \ .$$

Clearly g is linear.

Claim 4.33.1. $g \circ \gamma_< = f_*$

Proof. We prove it for vectors $\mathbf{1}_b$ for $b \in B$. Since $\{\mathbf{1}_b \ : \ b \in B\}$ forms a basis of the vector space $\mathrm{FinLin}(B)$, the result extends to all elements of $\mathrm{FinLin}(B)$ by linearity.

For $b \in B$ the vector $\gamma_<(\mathbf{1}_b) \in R^{\mathrm{Orbits}_<(B)}$ maps $\mathrm{orbit}_<(b)$ to 1 and every other $<$-orbit of B to 0. Hence $g(\gamma_<(\mathbf{1}_b)) = f_*(\mathrm{orbit}_<(b)) = f_*(\mathbf{1}_b)$. □

Using the above claim and our assumption that $\gamma_<(\mathbf{b}) \in \textsc{Fin-Span}(\gamma_<(P))$ we conclude that

$$\begin{aligned} f_*(\mathbf{b}) &= g(\gamma_<(\mathbf{b})) \\ &\in g(\textsc{Fin-Span}(\gamma_<(P))) \\ &= \textsc{Fin-Span}((g \circ \gamma_<)(P)) \\ &= \textsc{Fin-Span}(f_*(P)) \; . \end{aligned}$$

This finishes the proof. ■

Now we prove Lemma 4.32. The proof is almost identical to the proof of [6, Claim 4.7].

4.5.2 Proof of Lemma 4.32

We start with the easy direction. Since f_n is linear it is clear that if $\mathbf{v}$ is in $\textsc{Fin-Span}(U)$ then

$$f_n(\mathbf{v}) \in f_n(\textsc{Fin-Span}(U)) = \textsc{Fin-Span}(f_n(U)) \; .$$

From this point on we prove the other direction. Suppose

$$f_n(\mathbf{v}) \in \textsc{Fin-Span}(f_n(U)) \; .$$

Write $\mathrm{FinLin}_R(C)$ as

$$\mathrm{FinLin}_R(C) = \mathrm{FinLin}_R(C') \times \mathrm{FinLin}_R\left(\widetilde{C}\right)$$

where

$$\mathrm{FinLin}_R(C') \cong \mathrm{FinLin}_R\binom{\mathbb{A}}{k_1} \times \cdots \times \mathrm{FinLin}_R\binom{\mathbb{A}}{k_m}$$

and

$$\mathrm{FinLin}_R\left(\widetilde{C}\right) \cong \left(\mathrm{FinLin}_R\binom{\mathbb{A}}{n}\right)^{\ell} \tag{4.9}$$

for some $0 \leqslant k_1 \leqslant \cdots \leqslant k_m < n$ and $\ell \in \mathbb{Z}_+$. Recall that we can identify every vector $\mathbf{x} \in \mathrm{FinLin}(C)$ with the pair $(\mathbf{x}', \widetilde{\mathbf{x}})$ where $\mathbf{x}' \in \mathrm{FinLin}(C')$ and $\widetilde{\mathbf{x}} \in \mathrm{FinLin}\left(\widetilde{C}\right)$. Since $\mathbf{v}$ is an order equivariant vector and $n \geqslant 1$ we have $\widetilde{\mathbf{v}} = 0$. Hence, to finish the proof we have to argue that $(\mathbf{v}', \mathbf{0}) \in \textsc{Fin-Span}(U)$. For a vector

$$\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_\ell) \in \mathrm{FinLin}\left(\widetilde{C}\right) = \left(\mathrm{FinLin}_R\binom{\mathbb{A}}{n}\right)^{\ell} ,$$

define $\widetilde{\mathrm{dom}}(\mathbf{y})$ as

$$\widetilde{\mathrm{dom}}(\mathbf{y}) \stackrel{\text{def}}{=} \bigcup_{i=1}^{\ell} \mathrm{dom}(\mathbf{y}_i) \ .$$

Clearly, for any finite vector $\mathbf{y} \in \mathrm{FinLin}\left(\widetilde{C}\right)$, $\widetilde{\mathrm{dom}}(\mathbf{y})$ is finite and

$$\widetilde{\mathrm{dom}}(\mathbf{y}) \subseteq \binom{\mathrm{support}\,(\mathbf{y})}{n} \ .$$

Example 4.34. Consider $C = \mathbb{A} \cup \binom{\mathbb{A}}{2}$ and $\mathbf{v} = \alpha + \{\beta,\gamma\} - \{\beta,\delta\} \in \mathrm{FinLin}_{\mathbb{Z}}(C)$ for some $\alpha < \beta < \gamma < \delta \in \mathbb{A}$ Then

$$\begin{aligned} \widetilde{C} &= \binom{\mathbb{A}}{2} \ , \\ \mathrm{support}\,(\mathbf{v}) &= \{\alpha,\beta,\gamma,\delta\} \ , \\ \widetilde{\mathbf{v}} &= \{\beta,\gamma\} - \{\beta,\delta\} \ , \\ \mathrm{dom}(\mathbf{v}) &= \{\alpha, \{\beta,\gamma\}, \{\beta,\delta\}\} \ , \\ \widetilde{\mathrm{dom}}(\widetilde{\mathbf{v}}) &= \{\{\beta,\gamma\}, \{\beta,\delta\}\} \ . \end{aligned}$$

◀

Due to our assumption that

$$f_n(\mathbf{v}) \in \textsc{Fin-Span}(f_n(U)) = f_n(\textsc{Fin-Span}(f_n(U))) \ ,$$

there exists a vector $\mathbf{w} \in \textsc{Fin-Span}(U)$ such that $f_n(\mathbf{w}) = f_n(\mathbf{v})$. Applying the definition of f_n we get

$$(\mathbf{w}', f_n(\widetilde{\mathbf{w}})) = (\mathbf{v}', f_n(\widetilde{\mathbf{v}})) = (\mathbf{v}', \mathbf{0}) \ .$$

Let $S = \mathrm{support}\,(\mathbf{w})$. Then we know that

$$\widetilde{\mathrm{dom}}(\widetilde{\mathbf{w}}) \subseteq \binom{S}{n} \ .$$

Let $<^S_{\ell\mathrm{ex}}$ denote the lexicographic order on $\binom{S}{n}$. That is, for two n-element subsets $\{\alpha_1 < \cdots < \alpha_n\}, \{\beta_1 < \cdots < \beta_n\}$ of S, we define

$$\{\alpha_1 < \cdots < \alpha_n\} <^S_{\ell\mathrm{ex}} \{\beta_1 < \cdots < \beta_n\}$$

if there exists $i \in \{1,\ldots,n\}$ such that $\alpha_j = \beta_j$ for all $j > i$ and $\alpha_i < \beta_i$. For every vector $\widetilde{\mathbf{x}} \in \mathrm{FinLin}\left(\widetilde{C}\right)$ such that $\widetilde{\mathrm{dom}}(\widetilde{\mathbf{x}})$ is a non-empty subset $\binom{S}{n}$ we write

$\max(\widetilde{\mathrm{dom}}(\widetilde{\mathbf{x}}))$ to denote the maximum element in $\widetilde{\mathrm{dom}}(\widetilde{\mathbf{x}})$ with respect to $<^S_{\ell\mathrm{ex}}$. Since $\binom{S}{n}$ is a finite set:

Claim 4.32.1. *$<^S_{\ell\mathrm{ex}}$ is a well-order.*

Let X be the set of all vectors $\mathbf{x} \in \textsc{Fin-Span}(U)$ such that $\mathbf{x}' = \mathbf{v}'$, $f_n(\widetilde{\mathbf{x}}) = \mathbf{0}$ and $\widetilde{\mathrm{dom}}(\widetilde{\mathbf{x}}) \subseteq \binom{S}{n}$:

$$X \stackrel{\text{def}}{=} \left\{ \mathbf{x} \in \textsc{Fin-Span}(U) \;:\; \mathbf{x}' = \mathbf{v}', f_n(\widetilde{\mathbf{x}}) = \mathbf{0} \text{ and } \widetilde{\mathrm{dom}}(\widetilde{\mathbf{x}}) \subseteq \binom{S}{n} \right\} .$$

The vector $\mathbf{w}$ is an element of X, hence X is non-empty. If there exists $\mathbf{x} \in X$ such that $\widetilde{\mathbf{x}} = \mathbf{0}$ then we get

$$\mathbf{x} = (\mathbf{x}', \mathbf{0}) = (\mathbf{v}', \mathbf{0}) = \mathbf{v} .$$

We show that this is the only possible case.

Suppose otherwise, i.e. suppose that for every $\mathbf{x} \in X$ we have $\widetilde{\mathbf{x}} \neq \mathbf{0}$. Which means that for every $\mathbf{x} \in X$ the set $\widetilde{\mathrm{dom}}(\widetilde{\mathbf{x}})$ is non-empty and $\max(\widetilde{\mathrm{dom}}(\widetilde{\mathbf{x}}))$ is well-defined. Since X is non-empty and $<^S_{\ell\mathrm{ex}}$ is a well order (Claim 4.32.1), there exists $\mathbf{u} \in X$ which is minimal in the following sense: for all $\mathbf{x} \in X$

$$\max(\widetilde{\mathrm{dom}}(\widetilde{\mathbf{u}})) \leqslant^S_{\ell\mathrm{ex}} \max(\widetilde{\mathrm{dom}}(\widetilde{\mathbf{x}})) .$$

By definition of X we have $\mathbf{u}' = \mathbf{v}'$. Let

$$\{\beta_1 < \cdots < \beta_n\} = \max(\widetilde{\mathrm{dom}}(\widetilde{\mathbf{u}})) . \tag{4.10}$$

Pick $\alpha_1, \ldots, \alpha_n \in \mathbb{A} \setminus S$ such that

$$\alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_n < \beta_n .$$

For every $I \subseteq \{1, \ldots, n\}$ pick $\pi_I \in \mathrm{Aut}_<(\mathbb{A})$ such that

$$\pi_I(\beta_i) = \begin{cases} \alpha_i & \text{if } i \in I \\ \beta_i & \text{otherwise,} \end{cases}$$

and $\pi_I(\gamma) = \gamma$ for all $\gamma \in S \setminus \{\beta_1 < \cdots < \beta_n\}$. Define a vector

$$\mathbf{x} \in \textsc{Fin-Span}(\mathrm{orbit}_<(\{\mathbf{u}\})) \subseteq \textsc{Fin-Span}(U)$$

as

$$\mathbf{x} \stackrel{\text{def}}{=} \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \cdot \pi_I(\mathbf{u}) .$$

Using (4.9), we write $\widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{x}}$ as

$$\widetilde{\mathbf{u}} = (\mathbf{u}_1, \ldots, \mathbf{u}_\ell) \qquad \widetilde{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_\ell)$$

for $\mathbf{u}_i, \mathbf{x}_i \in \mathrm{FinLin}\binom{\mathbb{A}}{n}$. Recall Notation 2.29.

Claim 4.32.2. *For every $i \in \{1, \ldots, \ell\}$ we have:*

$$\mathbf{x}_i = \mathbf{u}_i(\{\beta_1 < \cdots < \beta_n\}) \cdot \left(\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \cdot (\{\alpha'_i \,:\, i \in I\} \cup \{\beta_i \,:\, i \notin I\}) \right) \tag{4.11}$$

Claim 4.32.3. *There exists $\alpha'_1, \ldots, \alpha'_n \in S$ such that*

$$\alpha'_1 < \beta_1 < \alpha'_2 < \cdots < \alpha'_n < \beta_n .$$

We finish the proof of Lemma 4.32 using the above claim. The claims themselves are proven at the end of this section.

Pick $\pi \in \mathrm{Aut}_<(\mathbb{A})$ such that $\pi(\alpha_i) = \alpha'_i$ and $\pi(\beta_i) = \beta_i$ for all $i \in \{1, \ldots, n\}$. Define $\mathbf{y} \stackrel{\text{def}}{=} \mathbf{u} - \pi(\mathbf{x}) \in \textsc{Fin-Span}(U)$. The following claim contradicts the minimality of $\mathbf{u}$ and finishes the proof of Lemma 4.32.

Claim 4.32.4.
1. $\mathbf{y}' = \mathbf{v}'$,
2. $f_n(\widetilde{\mathbf{y}}) = \mathbf{0}$,
3. $\widetilde{\mathrm{dom}}(\widetilde{\mathbf{y}}) \subseteq \binom{S}{n}$, *and*
4. $\max(\widetilde{\mathrm{dom}}(\widetilde{\mathbf{y}})) <^S_{\ell\mathrm{ex}} \{\beta_1 < \cdots < \beta_n\} = \max(\widetilde{\mathrm{dom}}(\widetilde{\mathbf{u}}))$.

Proof. **Item 1:** First we show $\mathbf{y}' = \mathbf{v}'$. We use multiple times the fact that both the functions $\mathbf{z} \mapsto \mathbf{z}'$ and $\mathbf{z} \mapsto \widetilde{\mathbf{z}}$ are linear and equivariant. This fact will be used multiple times in this proof. We already have $\mathbf{u}' = \mathbf{v}'$. Hence,

$$\mathbf{y}' = (\mathbf{u} - \pi(\mathbf{x}))' = \mathbf{u}' - \pi(\mathbf{x})' = \mathbf{v}' - \pi(\mathbf{x})' = \mathbf{v}' - \pi(\mathbf{x}') .$$

Hence, to prove $\mathbf{y}' = \mathbf{v}'$ it is enough to show that $\mathbf{x}' = \mathbf{0}$. The vector $\mathbf{v}$ is equivariant by assumption, hence so is the vector $\mathbf{u}' = \mathbf{v}'$. Pick arbitrary $I \subseteq \{1, \ldots, n\}$. We have $(\pi_I(\mathbf{u}))' = \pi_I(\mathbf{u}')$. We also have $\pi_I(\mathbf{u}') = \mathbf{u}'$ since $\mathbf{u}'$ is order equivariant. Now applying the definition of $\mathbf{x}$ we get

$$\mathbf{x}' = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \cdot (\pi_I(\mathbf{u}))' = \left(\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \right) \cdot \mathbf{u}' = \mathbf{0} .$$

Since $n \geqslant 1$, we have $\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} = (1-1)^n = 0$. Hence $\mathbf{x}' = \mathbf{0}$ As a consequence $\mathbf{y}' = \mathbf{v}'$.

Item 2: Now we show $f_n(\widetilde{\mathbf{y}}) = \mathbf{0}$. We already know $f_n(\widetilde{\mathbf{u}}) = \mathbf{0}$. Hence,

$$f_n(\widetilde{\mathbf{y}}) = f_n(\widetilde{\mathbf{u}}) - f_n(\pi(\widetilde{\mathbf{x}})) = -\pi(f_n(\widetilde{\mathbf{x}}))$$

So we just have to show $f_n(\widetilde{\mathbf{x}}) = \mathbf{0}$. Applying the definition of $\mathbf{x}$ and using the fact that $f_n(\widetilde{\mathbf{u}}) = \mathbf{0}$ we get

$$f_n(\widetilde{\mathbf{x}}) = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \pi_I(f_n(\widetilde{\mathbf{u}})) = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \pi_I(\mathbf{0}) = \mathbf{0}\,.$$

Item 3: Now we show $\widetilde{\operatorname{dom}}(\widetilde{\mathbf{y}}) \subseteq \binom{S}{n}$. Since $\widetilde{\operatorname{dom}}(\widetilde{\mathbf{u}}) \subseteq S$, we just need to show $\widetilde{\operatorname{dom}}(\widetilde{\pi(\mathbf{x})}) = \widetilde{\operatorname{dom}}(\pi(\widetilde{\mathbf{x}})) \subseteq \binom{S}{n}$. Claim 4.32.2 implies $\widetilde{\operatorname{dom}}(\mathbf{x}) \subseteq \binom{S'}{n}$ where $S' = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$. Since $\pi(\alpha_i) = \alpha_i'$ and $\pi(\beta_i) = \beta_i$ for all i, we have

$$\widetilde{\operatorname{dom}}(\pi(\widetilde{\mathbf{x}})) \subseteq \binom{S''}{n} \subseteq \binom{S}{n}$$

where $S'' = \left\{\alpha_1', \ldots, \alpha_n', \beta_1, \ldots, \beta_n\right\}$.

Item 4: Finally, we show $\max(\widetilde{\operatorname{dom}}(\widetilde{\mathbf{y}})) <^S_{\ell\mathrm{ex}} \max(\widetilde{\operatorname{dom}}(\widetilde{\mathbf{u}}))$. Using (4.9), we write $\widetilde{\mathbf{y}}$ as

$$\widetilde{\mathbf{y}} = (\mathbf{y}_1, \ldots, \mathbf{y}_\ell)$$

for $\mathbf{y}_i \in \operatorname{FinLin}\binom{\mathbb{A}}{n}$. Clearly $\mathbf{y}_i = \mathbf{u}_i - \mathbf{x}_i$ for all i. Recall that $\{\beta_1 < \cdots < \beta_n\} = \max(\widetilde{\operatorname{dom}}(\widetilde{\mathbf{u}}))$ and

$$\alpha_1' < \beta_1 < \cdots < \alpha_n' < \beta_n\,.$$

Using Claim 4.32.2 we get $\mathbf{x}_i(\{\beta_1 < \cdots < \beta_n\}) = \mathbf{u}_i(\{\beta_1 < \cdots < \beta_n\})$. Since

$$\{\beta_1 < \cdots < \beta_n\} \in \widetilde{\operatorname{dom}}(\widetilde{\mathbf{x}})$$

there exists $i \in \{1, \ldots, n\}$ such that $\mathbf{u}_i(\{\beta_1 < \cdots < \beta_n\}) \neq 0$. Using Claim 4.32.2 again we deduce

$$\widetilde{\operatorname{dom}}(\widetilde{\mathbf{x}}) = \left\{\{\gamma_1 < \cdots < \gamma_n\} \;:\; \gamma_i \in \left\{\alpha_i', \beta_i\right\} \text{ for } i \in \{1, \ldots, n\}\right\}\,.$$

Hence

$$\max(\widetilde{\operatorname{dom}}(\widetilde{\mathbf{x}})) = \{\beta_1 < \cdots < \beta_n\}\,.$$

Moreover, for all i

$$\mathbf{y}_i(\{\beta_1 < \cdots < \beta_n\}) = \mathbf{u}_i(\{\beta_1 < \cdots < \beta_n\}) - \mathbf{x}_i(\{\beta_1 < \cdots < \beta_n\}) = 0\,.$$

Which implies $\{\beta_1 < \cdots < \beta_n\} \notin \mathrm{dom}(\widetilde{\mathbf{y}})$. Hence,

$$\widetilde{\mathrm{dom}}(\widetilde{\mathbf{y}}) \subseteq (\widetilde{\mathrm{dom}}(\widetilde{\mathbf{u}}) \cup \widetilde{\mathrm{dom}}(\widetilde{\mathbf{x}})) \setminus \{\{\beta_1 < \cdots < \beta_n\}\} \ .$$

As a result

$$\max(\widetilde{\mathrm{dom}}(\widetilde{\mathbf{y}})) <^S_{\ell\mathrm{ex}} \{\beta_1 < \cdots < \beta_n\} \ .$$

This finishes the proof of Claim 4.32.4. □

The proof of Lemma 4.32 is thus completed, modulo the proofs of Claims 4.32.2 and 4.32.3, which we do now.

Missing proofs

Proof of Claim 4.32.2. Pick i arbitrarily. Firstly,

$$\begin{aligned} &\sum_{I \subseteq \{1,\ldots,n\}} (-1)^I \cdot \pi_I(\{\beta_1 < \cdots < \beta_n\}) \\ = &\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \cdot (\{\alpha_i \ : \ i \in I\} \cup \{\beta_i \ : \ i \notin I\}) \ . \end{aligned} \tag{4.12}$$

The above equation implies (4.11) if we can prove that for every $\gamma_1 < \cdots < \gamma_n \in S$ such that $\{\gamma_1 < \cdots < \gamma_n\} \neq \{\beta_1 < \cdots < \beta_n\}$ we have

$$\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \cdot \pi_I(\{\gamma_1 < \cdots < \gamma_n\}) = 0 \,, \tag{4.13}$$

which we do now. Pick such atoms $\gamma_1 < \cdots < \gamma_n \in S$. Let j be such that $\gamma_j \neq \beta_j$. For every $J \subseteq \{1, \ldots, n\} \setminus \{j\}$ we have

$$\pi_J(\{\gamma_1 < \cdots < \gamma_n\}) = \pi_{J \cup \{j\}}(\{\gamma_1 < \cdots < \gamma_n\}) \ .$$

Using this we conclude

$$\begin{aligned} &\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \cdot \pi_I(\{\gamma_1 < \cdots < \gamma_n\}) \\ = &\sum_{J \subseteq \{1,\ldots,n\} \setminus \{j\}} \Big((-1)^{|J|} \cdot \pi_J(\{\gamma_1 < \cdots < \gamma_n\}) + \\ &\qquad (-1)^{|J|+1} \cdot \pi_{J \cup \{j\}}(\{\gamma_1 < \cdots < \gamma_n\}) \Big) \\ = &\sum_{J \subseteq \{1,\ldots,n\} \setminus \{j\}} (-1)^{|J|} \cdot \big(\pi_J(\{\gamma_1 < \cdots < \gamma_n\}) - \pi_J(\{\gamma_1 < \cdots < \gamma_n\}) \big) \end{aligned}$$

$$= 0\,.$$

□

Proof of Claim 4.32.3. We know $f_n(\widetilde{\mathbf{u}}) = \mathbf{0}$. Using (4.9) write $\widetilde{\mathbf{u}}$ as $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_\ell)$. Since $\{\beta_1 < \cdots < \beta_n\} \in \widetilde{\mathrm{dom}}(\widetilde{\mathbf{u}})$ there exists i such that

$$\mathbf{u}_i(\{\beta_1 < \cdots < \beta_n\}) \neq 0\,.$$

Pick $j \in \{1, \dots, n\}$. Let $S_j = \{\beta_1 < \cdots < \beta_n\} \setminus \{\beta_j\}$. We know

$$f_n(\mathbf{u}_i)(j)(S_j) = 0\,.$$

Using the definition of f_n, this implies

$$\sum_{\beta_{j-1} < \beta < \beta_{j+1}} \mathbf{u}_i(S_j \cup \{\beta\}) = 0$$

where $\beta_0 = -\infty$ and $\beta_{n+1} = +\infty$. We have $\mathbf{u}_i(\{\beta_1 < \cdots < \beta_n\}) \neq 0$ which implies there exists β such that $\beta_{j-1} < \beta < \beta_{j+1}$, $\mathbf{u}_i(S_j \cup \{\beta\}) \neq 0$ and $\beta \neq \beta_j$. Hence $\beta \in S$. To finish the proof we show $\beta < \beta_j$. Since $\mathbf{u}_i(S_j \cup \{\beta_j\}) \neq 0$ we have $S_j \cup \{\beta_j\} \in \widetilde{\mathrm{dom}}(\widetilde{\mathbf{u}})$. By definition, $\{\beta_1 < \cdots < \beta_n\}$ is the largest element in $\widetilde{\mathrm{dom}}(\widetilde{\mathbf{u}})$. Hence $S_j \cup \{\beta\} <^S_{\ell\mathrm{ex}} \{\beta_1 < \cdots < \beta_n\}$. This is possible only if $\beta < \beta_j$. Applying the above argument for all $j \in \{1, \dots, n\}$ we get the claim. □

With all the claims being proven, the proof of Lemma 4.32 is now finished. ■

Remark 4.35. FIN-EQ$_<(R)$ is a special case of FIN-EQ(R) in the setting of ordered atoms (see Remark 2.22) where the system is assumed to be equivariant and column-finite. In fact, the general problem can be reduced to this specific version, but we leave that for future work.

Remark 4.36. Reducing FIN-EQ(R) to FIN-EQ$_<(R)$ was necessary for solving the former problem, namely Theorem 4.29 is not true in the setting of equality atoms, with order equivariance replaced by equivariance: there exists an equivariant orbit-finite set B, equivariant vector $\mathbf{b} \in \mathrm{FinLin}_{\mathbb{Z}}(B)$ and an orbit-finite set of vectors P such that $\mathbf{b} \notin$ FIN-SPAN$_{\mathbb{Z}}(P)$ but $\gamma(\mathbf{b}) \in$ FIN-SPAN$_{\mathbb{Z}}(\gamma(P))$, for the orbit-summation function

$$\gamma : \mathrm{FinLin}_R(B) \to R^{\mathrm{Orbits}(B)}$$

defined as

$$\gamma(\mathbf{x}) : O \in \text{Orbits}(B) \mapsto \sum_{b \in O} \mathbf{x}(b) \ .$$

Here is a counterexample. Let $B = \mathbb{A}^{(3)} \uplus \{\star\}$ where $\star$ is an equivariant element, i.e. $\pi(\star) = \star$ for all $\pi \in \text{Aut}(\mathbb{A})$. Recall Notation 2.29. For $\alpha\beta\gamma \in \mathbb{A}^{(3)}$ define $\mathbf{x}_{\alpha\beta\gamma} = \alpha\beta\gamma - \beta\alpha\gamma + \star \in \text{FinLin}_{\mathbb{Z}}(B)$. Let $\mathbf{b} = \star \in \text{FinLin}_{\mathbb{Z}}(B)$ and

$$P = \left\{ \mathbf{x}_{\alpha\beta\gamma} \ : \ \alpha\beta\gamma \in \mathbb{A}^{(3)} \right\} \ .$$

For $\alpha\beta\gamma \in \mathbb{A}^{(3)}$, the orbit-summation function yields

$$\gamma(\mathbf{b}) = \gamma(\mathbf{x}_{\alpha\beta\gamma}) = (0, 1) \ ,$$

where the first and second co-ordinate, respectively, correspond to the equivariant orbits $\mathbb{A}^{(3)}$ and $\{\star\}$. Thus $\gamma(\mathbf{b}) \in \text{FIN-SPAN}_{\mathbb{Z}}(\gamma(P))$. We argue that $\mathbf{b} \notin \text{FIN-SPAN}_{\mathbb{Z}}(P)$. Towards a contradiction, suppose there exists a finite set S and integers $r_{\alpha\beta\gamma}$ for $\alpha\beta\gamma \in S^{(3)}$ such that

$$\mathbf{b} = \sum_{\alpha\beta\gamma \in S^{(3)}} r_{\alpha\beta\gamma} \cdot \mathbf{x}_{\alpha\beta\gamma} \ . \tag{4.14}$$

Putting $\mathbf{x} = \alpha\beta\gamma - \beta\alpha\gamma + \star$ we get

$$\mathbf{b} \ = \ \left(\sum_{\alpha\beta\gamma \in S^{(3)}} r_{\alpha\beta\gamma} \right) \cdot \star \ + \ \sum_{\gamma \in S} \ \sum_{\alpha\beta \in S \setminus \{\gamma\}^{(2)}} (r_{\alpha\beta\gamma} - r_{\beta\alpha\gamma}) \cdot \alpha\beta\gamma \ , \tag{4.15}$$

which implies

$$\sum_{\alpha\beta\gamma \in S^{(3)}} r_{\alpha\beta\gamma} = 1 \text{ , and} \tag{4.16}$$

$$r_{\alpha\beta\gamma} - r_{\beta\alpha\gamma} = 0 \text{ for all } \alpha\beta\gamma \in S^{(3)}. \tag{4.17}$$

Let $<$ be a total order on S. Using (4.17) we rearrange the LHS of (4.16) to get

$$\begin{aligned} \sum_{\alpha\beta\gamma \in S^{(3)}} r_{\alpha\beta\gamma} &= \sum_{\gamma \in S} \ \sum_{\{\alpha<\beta\} \in \binom{S\setminus\{\gamma\}}{2}} r_{\alpha\beta\gamma} + r_{\beta\alpha\gamma} \\ &= \sum_{\gamma \in S} \ \sum_{\{\alpha<\beta\} \in \binom{S\setminus\{\gamma\}}{2}} 2 \cdot r_{\alpha\beta\gamma} \ . \end{aligned}$$

But this means the LHS of (4.16) is even, which contradicts the equations itself.

4.6 Complexity

For proving Theorem 4.6, we do a rough estimation of complexity of solving an orbit-finite $(B \times C)$-system of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ with respect to the number of orbits in B and C, and the atom-dimension of the system $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$. The blow-up of reduction of Theorem 4.3 and Theorem 4.16 is exponential in the atom dimension of the system, but polynomial in the number of orbits in B and C. Likewise is the number and size of finite systems of equations that are produced in the procedure of Theorem 4.17. Summing up, the combined algorithm for $\text{EQ}(R)$ produces exponentially many finite systems of exponential size (polynomially many finite systems of polynomial size, when atom dimension of input is fixed), and answers positively exactly when all these systems are solvable. For $R = \mathbb{Q}$ or $\mathbb{Z}$, finite systems are solvable in PTIME. Therefore, the problems $\text{EQ}(\mathbb{Q})$ and $\text{EQ}(\mathbb{Z})$ are decidable in EXPTIME, and likewise are $\text{FIN-EQ}(\mathbb{Q})$ and $\text{FIN-EQ}(\mathbb{Z})$. When atom dimension of input is fixed, all these problems are decidable in PTIME.

Chapter 5

Linear Inequalites

Contents

5.1 Introduction

In this chapter we focus on solvability of orbit-finite systems of linear inequalities. In this and the following chapters, we will only consider vectors and matrices with coefficients from $\mathbb{R}$ (or possibly its subsets like $\mathbb{Q}$ or $\mathbb{Z}$), instead of an arbitrary commutative ring. For orbit-finite sets B and C, an *orbit-finite $(B \times C)$-system of linear inequalities* $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ is given by a matrix $\mathbf{A} \in \mathrm{Lin}(B \times C)$ and a vector $\mathbf{b} \in \mathrm{Lin}(B)$. When B and C are irrelevant, we speak of *orbit-finite systems of linear inequalities*. For $\mathbb{F} \subseteq \mathbb{R}$, an *$\mathbb{F}$-solution* of the system $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ is a vector $\mathbf{x}' : C \to \mathbb{F}$ such that the product $\mathbf{A} \cdot \mathbf{x}'$ is well-defined and $\mathbf{A} \cdot \mathbf{x}' \leqslant \mathbf{b}$.

Notation 5.1. We write $\mathbf{A} \cdot \mathbf{x} \geqslant \mathbf{b}$ to denote the system $(-\mathbf{A}) \cdot \mathbf{x} \leqslant (-\mathbf{b})$.

We are interested in the following decision problems regarding solvability of orbit-finite systems of linear inequalities.

INEQ($\mathbb{F}$)
Input: An orbit-finite system of linear inequalites $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$.
Question: Does it have an **orbit-finite** $\mathbb{F}$-solution?

FIN-INEQ($\mathbb{F}$)
Input: An orbit-finite system of linear inequalities $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$.
Question: Does it have a **finite** $\mathbb{F}$-solution?

We are also interested in the following decision problems regarding existence of non-negative solutions of orbit-finite systems of linear equations, which are closely related to the above problems.

NONNEG-EQ($\mathbb{F}$)
Input: An orbit-finite system of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$.
Question: Does it have an **orbit-finite non-negative** $\mathbb{F}$-solution?

FIN-NONNEG-EQ($\mathbb{F}$)
Input: An orbit-finite system of linear equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$.
Question: Does it have a **finite non-negative** $\mathbb{F}$-solution?

The main results in this chapter are the following:

Theorem 5.2. *For $\mathbb{F} = \mathbb{R}, \mathbb{Z}$ the problems* FIN-INEQ($\mathbb{F}$), NONNEG-EQ($\mathbb{F}$) *and* INEQ($\mathbb{F}$) *are inter-reducible in* EXPTIME *and in* PTIME *in fixed atom-dimension.*

Theorem 5.3. FIN-INEQ($\mathbb{R}$), INEQ($\mathbb{R}$) *and* NONNEG-EQ($\mathbb{R}$) *are decidable in* EXPTIME, *and in* PTIME *for fixed atom-dimension.*

Theorem 5.4. FIN-INEQ($\mathbb{Z}$), INEQ($\mathbb{Z}$) *and* NONNEG-EQ($\mathbb{Z}$) *are undecidable.*

Theorem 5.5. FIN-NONNEG-EQ($\mathbb{R}$) *is decidable in* EXPTIME *and in* PTIME *for fixed atom-dimension.*

Theorem 5.6. FIN-NONNEG-EQ($\mathbb{Z}$) *is decidable in* 2-EXPTIME.

Example 5.7. For illustration, consider the set $\mathbb{A}\times\mathbb{A}$-system of inequalities

$$\begin{bmatrix} 0 & 1 & 1 & \\ 1 & 0 & 1 & \cdots \\ 1 & 1 & 0 & \\ & \vdots & & \ddots \end{bmatrix} \cdot \mathbf{x} \quad \geqslant \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} \tag{5.1}$$

Equivalently written as,

$$\sum_{\alpha \in \mathbb{A} \setminus \{\beta\}} \mathbf{x}(\alpha) \geqslant 1 \qquad (\beta \in \mathbb{A}). \tag{5.2}$$

This system is solvable. For example, for a finite set of atoms $S = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{A}$ where $n \geqslant 2$, the finite vector $\mathbf{1}_S$ is a solution a solution of this system, so is the finite vector $\mathbf{x}_n : \mathbb{A} \to \mathbb{R}$ defined as:

$$\mathbf{x}_n(\alpha) \stackrel{\text{def}}{=} \frac{1}{n-1} \text{ if } \alpha \in S, \qquad \mathbf{x}_n(\alpha) \stackrel{\text{def}}{=} 0 \text{ if } \alpha \notin S\,.$$

◀

Example 5.8. Consider the following system of inequalities with variables indexed by $\mathbb{A} \uplus \mathbb{A}^{(2)}$. Intuitively, unknowns correspond to vertices α and edges $\alpha\beta$ of an infinite directed clique. The system contains an inequality

$$\sum_{\alpha \in \mathbb{A}} \mathbf{x}(\alpha) \geqslant 1 \tag{5.3}$$

enforcing the sum of values assigned to all vertices to be at least 1, and the inequalities

$$\sum_{\beta \in \mathbb{A} \setminus \{\alpha\}} \mathbf{x}(\alpha\beta) - \mathbf{x}(\alpha) - \sum_{\beta \in \mathbb{A} \setminus \{\alpha\}} \mathbf{x}(\beta\alpha) \geqslant 0 \qquad (\alpha \in \mathbb{A})\,, \tag{5.4}$$

enforcing that for each vertex $\alpha \in \mathbb{A}$, the sum of values assigned to all outgoing edges to be larger or equal to the sum of values assigned to all incoming edges, plus the value assigned to the vertex α. In matrix form (0 entries of the matrix are omitted):

$$\begin{array}{c} \\ \mathbb{A} \\ \\ \{*\} \end{array} \begin{array}{c} \begin{array}{cc} \quad\mathbb{A}\quad & \quad\mathbb{A}^{(2)} \end{array} \\ \left[\begin{array}{ccc|c} -1 & & & \\ & -1 & & \mathbf{A} \\ & & \ddots & \\ \hline 1 & 1 & \cdots & \end{array}\right] \end{array} \cdot \mathbf{x} \geqslant \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 1 \end{array}\right] \tag{5.5}$$

where $\mathbf{A}$ is the oriented incidence matrix, namely for every distinct atoms $\alpha, \beta \in \mathbb{A}$,

$$\mathbf{A}(\alpha, \alpha\beta) = 1 \qquad \mathbf{A}(\alpha, \beta\alpha) = -1,$$

and all other entries of $\mathbf{A}$ are 0. Solutions of the system correspond to directed graphs, whose vertices and edges are labelled in accordance with constraints

(5.3) and (5.4).

This system has no finite solution. For illustration, we show how this follows from our results in Example 5.25 inside § 5.4. It can also be proven independently. Consider an arbitrary finite solution $\mathbf{y} \in \mathrm{FinLin}\left(\mathbb{A}^{(2)}\right)$ of the above system of inequalities. There exists a finite set of atoms $T \subseteq_{\mathrm{FIN}} \mathbb{A}$ such that $\mathbf{y}(\alpha)$ or $\mathbf{x}(\alpha\beta)$ are non-zero only if $\alpha, \beta \in T$. But then

$$\begin{aligned}
0 &\leqslant \sum_{\alpha \in T} \left(\sum_{\beta \in \mathbb{A} \setminus \{\alpha\}} \mathbf{y}(\alpha\beta) - \mathbf{y}(\alpha) - \sum_{\beta \in \mathbb{A} \setminus \{\alpha\}} \mathbf{y}(\beta\alpha) \right) \\
&= \sum_{\alpha\beta \in T^{(2)}} \mathbf{y}(\alpha\beta) - \sum_{\alpha \in T} \mathbf{y}(\alpha) - \sum_{\alpha\beta \in T^{(2)}} \mathbf{y}(\alpha\beta) \\
&= -\sum_{\alpha \in T} \mathbf{y}(\alpha) \\
&\leqslant -1
\end{aligned}$$

The system (5.3)-(5.4) has no orbit-finite solutions either, since using similar arguments as Example 2.55, any orbit-finite solution of this system has to be finite.

However, this system has orbit-infinite solutions. For example, consider any enumeration $\alpha_0, \alpha_1, \ldots$ of atoms. The vector

$$\alpha_0 + \sum_{n \in \mathbb{N}} \alpha_n \alpha_{n+1}$$

is a solution of this system, as evident from the following diagram.

$$\begin{tikzcd}
\overset{1}{\bullet}_{\alpha_0} \arrow[r, "1"] & \bullet_{\alpha_1} \arrow[r, "1"] & \bullet_{\alpha_2} \arrow[r, "1"] & \cdots
\end{tikzcd}$$

◀

We prove Theorems 5.2, 5.5 and 5.6, postponing the Theorems 5.3 and 5.4 to the following sections.

Proof of Theorem 5.2. Consider arbitrary $\mathbf{A} \in \mathrm{Lin}(B \times C)$ and $\mathbf{b} \in \mathrm{Lin}(C)$.

($\textbf{NONNEG-EQ}(\mathbb{F}) \to \textbf{INEQ}(\mathbb{F})$)**:** The system of equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has a non-negative orbit-finite $\mathbb{F}$-solution if and only if the system

$$\left[\begin{array}{c} \mathbf{A} \\ \hline -\mathbf{A} \\ \hline -\mathrm{Id} \end{array}\right] \cdot \mathbf{x} \leqslant \left[\begin{array}{c} \mathbf{b} \\ \hline -\mathbf{b} \\ \hline \mathbf{0} \end{array}\right]$$

has an orbit-finite $\mathbb{F}$-solution. This shows NONNEG-EQ($\mathbb{F}$) is reducible to INEQ($\mathbb{F}$).

(INEQ($\mathbb{F}$) $\rightarrow$ NONNEG-EQ($\mathbb{F}$)): The system of inequalities $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ has an orbit-finite $\mathbb{F}$-solution if and only if the system

$$\left[\begin{array}{c|c|c} \mathbf{A} & -\mathbf{A} & \mathrm{Id} \end{array}\right] \cdot \mathbf{y} = \mathbf{b}$$

has a non-negative orbit-finite $\mathbb{F}$-solution. This shows INEQ($\mathbb{F}$) is reducible to NONNEG-EQ($\mathbb{F}$).

(FIN-INEQ($\mathbb{F}$) $\rightarrow$ INEQ($\mathbb{F}$)): The system of inequalities $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ has a finite $\mathbb{F}$-solution if and only if the system

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{0} \\ \hline \mathbf{1}^{\mathsf{T}} & 1 \end{array}\right] \cdot \mathbf{y} \leqslant \left[\begin{array}{c} \mathbf{b} \\ \hline 0 \end{array}\right]$$

has an orbit-finite $\mathbb{F}$-solution, since any orbit-finite solution $\mathbf{y}$ of this system has to be finite for $(\mathbf{1}^{\mathsf{T}}|1) \cdot \mathbf{y}$ to be well-defined. This shows FIN-INEQ($\mathbb{F}$) is reducible to INEQ($\mathbb{F}$).

(INEQ($\mathbb{F}$) $\rightarrow$ FIN-INEQ($\mathbb{F}$)): Lemma 4.9 implies the system $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ has an orbit-finite solution if and only if the system $\widetilde{\mathbf{A}} \cdot \mathbf{y} \leqslant \mathbf{b}$ has a finite solution ($\widetilde{\mathbf{A}}$ is defined in § 4.2.3). Hence INEQ($\mathbb{F}$) is reducible to FIN-INEQ($\mathbb{F}$).

The first three reductions can be easily argued to be computable in EXPTIME and in PTIME for fixed atom-dimension. The complexity of the reduction (INEQ($\mathbb{F}$) $\rightarrow$ FIN-INEQ($\mathbb{F}$)) follows from Remark 4.10. ■

Proof of Theorem 5.5. Decidability of FIN-NONNEG-EQ($\mathbb{R}$) follows by a direct reduction of FIN-NONNEG-EQ($\mathbb{R}$) to NONNEG-EQ($\mathbb{R}$) (similar to the reduction of FIN-INEQ(F) to INEQ($\mathbb{F}$)) together with Theorem 5.3. Since the reduction is PTIME and NONNEG-EQ($\mathbb{R}$) is decidable in EXPTIME and PTIME for fixed atom-dimension, we get FIN-NONNEG-EQ($\mathbb{R}$) is also decidable in EXPTIME and PTIME for fixed atom-dimension. ■

Proof of Theorem 5.6. For an orbit-finite set of vectors X and $K \subseteq \mathbb{Z}$ we use FIN-SPAN$_K(X)$ to denote the set of vectors of the form

$$\sum_{\mathbf{x} \in Y} r_{\mathbf{x}} \cdot \mathbf{x}$$

where $Y \subseteq_{\text{FIN}} X$ and $r_{\mathbf{x}} \in K$.

Let $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ be an instance of FIN-NONNEG-EQ($\mathbb{Z}$), where $\mathbf{A} \in \mathrm{Lin}_{\mathbb{Z}}(B \times C)$ and $\mathbf{b} \in \mathrm{Lin}_{\mathbb{Z}}(B)$. Consider the set of column vectors

$$P \stackrel{\text{def}}{=} \{\mathbf{A}(-, c) \ : \ c \in C\} \subseteq \mathrm{Lin}(B)$$

of $\mathbf{A}$. The system of equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has a finite non-negative integer solution if and only if

$$\mathbf{b} \in \text{FIN-SPAN}_{\mathbb{N}}(P)\,. \tag{5.6}$$

We rely on Theorem 3.1 which says that $\mathrm{Lin}(B)$ has an orbit-finite basis. Let $\mathrm{BASIS}(B) \subseteq \mathrm{Lin}(B)$ be such a basis. This implies that there exists a linear isomorphism $\varphi : \mathrm{Lin}(B) \to \mathrm{FinLin}(\mathrm{BASIS}(B))$. In consequence, (5.6) is equivalent to

$$\varphi(\mathbf{b}) \in \text{FIN-SPAN}_{\mathbb{N}}(\varphi(P)). \tag{5.7}$$

By [21, Remark 11.16] we can compute a finite set of vectors $\{\mathbf{b}'_1, \ldots, \mathbf{b}'_k\} \subseteq \mathrm{FinLin}(\mathrm{BASIS}(B))$ and an orbit-finite subset $P' \subseteq \varphi(P)$ such that (5.7) holds if and only if

$$\mathbf{b}'_i \in \text{FIN-SPAN}_{\mathbb{Z}}(P') \tag{5.8}$$

for some $i \in \{1, \ldots, k\}$. This is an instance of EQ($\mathbb{Z}$) which we have proven to be decidable (Theorem 4.6). Since the computation of $\mathbf{b}'_1, \ldots, \mathbf{b}'_k$ can be done in NEXPTIME and EQ(Z) is decidable EXPTIME, FIN-NONNEG-EQ($\mathbb{Z}$) is decidable in 2-EXPTIME. ■

Remark 5.9 (Strict inequalities)**.** We consider system of *non-strict* inequalities, for the sake of presentation. The decision procedures of Theorem 5.3 work equally well if both *non-strict and strict* inequalities are allowed. Reductions between FIN-INEQ($\mathbb{F}$) and INEQ($\mathbb{F}$) work as well, but not the reductions between NONNEG-EQ($\mathbb{F}$) to INEQ($\mathbb{F}$) as we can not simulate equalities and non-strict inequalities with strict inequalities.

Organisation of the chapter

The remainder of this chapter is organised as follows, In § 5.2 we prove undecidability of FIN-EQ($\mathbb{Z}$) (Theorem 5.4). Then, in § 5.3 and we define *polynomially parametrised* systems of inequalities. In § 5.4 we introduce the concept of setwise support, which is then used in § 5.5 to show that FIN-EQ($\mathbb{R}$) is reducible to solvability of polynomially parametrised systems of inequalities. In § 5.6

we show solvability of the latter systems is in PTIME, proving decidability of FIN-INEQ($\mathbb{R}$) (Theorem 5.3).

5.2 Undecidability of integer solvability

We prove Theorem 5.4 by showing undecidability of FIN-INEQ($\mathbb{Z}$). We proceed by reduction from the reachability problem of counter machines. We conveniently define a *d-counter machine* M as a finite set of instructions I, where each instruction is a function

$$i : \{1, \ldots, d\} \to \mathbb{Z} \cup \{\text{ZERO}\}$$

that specifies, for each counter $k \in \{1, \ldots, d\}$, either the additive update of k (if $i(k) \in \mathbb{Z}$) or the zero-test of k (if $i(k) = \text{ZERO}$). Configurations of M are nonnegative vectors $c \in \mathbb{N}^d$, and each instruction induces steps between configurations: $c \xrightarrow{i} c'$ if $c'(k) = c(k) + i(k)$ whenever $i(k) \in \mathbb{Z}$ and $c(k) + i(k) \in \mathbb{N}$, and $c'(k) = c(k) = 0$ whenever $i(k) = \text{ZERO}$. A run of M is defined as a finite sequence of steps

$$c_0 \xrightarrow{i_1} c_1 \xrightarrow{i_2} \ldots \xrightarrow{i_n} c_n. \tag{5.9}$$

The reachability problem asks, given a machine M and two its configurations, a source c_0 and a target c_f, if M admits a run from c_0 to c_f. The problem is undecidable, as counter machines can easily simulate classical Minsky machines.[1]

For $k \in \{1, \ldots, d\}$ we denote by $\text{ZERO}(k) \stackrel{\text{def}}{=} \{i \in I \;:\; i(k) = \text{ZERO}\}$ the set of instructions that zero-test counter k, and $\text{UPD}(k) \stackrel{\text{def}}{=} \{i \in I \;:\; i(k) \in \mathbb{Z}\}$ the set of instructions that update counter k.

Given a d-counter machine M and two configurations c_0, c_f, we construct an orbit-finite system of inequalities $\mathcal{S}$ such that M admits a run from c_0 to c_f if and only if $\mathcal{S}$ has a finite nonnegative integer solution. Nonegativeness is *enforced* by adding inequalities $x \geqslant 0$ for all unknowns x. We describe the construction of $\mathcal{S}$ gradually, on the way giving intuitive explanations and sketching the proof of the if direction.

The system $\mathcal{S}$ has unknowns $\mathbf{e}_{\alpha\beta}$ indexed by pairs of distinct atoms $\alpha\beta \in \mathbb{A}^{(2)}$, and contains the following inequalities:

$$\mathbf{e}_{\alpha\beta} \leqslant 1 \qquad\qquad (\alpha\beta \in \mathbb{A}^{(2)}). \tag{5.10}$$

[1] A d-counter machine resembles a vector addition system with zero tests. A Minsky machine with n states and k counters can be simulated by an $(n+k)$-counter machine, by encoding control states into additional counters.

Therefore, in every solution the unknowns $\mathbf{e}_{\alpha\beta}$ define a directed graph G, where atoms are vertices, $\mathbf{e}_{\alpha\beta} = 1$ encodes an edge from α to β and $\mathbf{e}_{\alpha\beta} = 0$ encodes a non-edge. In case of a finite solution, the graph G is finite (when atoms with no adjacent edges are dropped). Let us fix two distinct atoms $\iota, \zeta \in \mathbb{A}$. The system $\mathcal{S}$ contains the following further equations and inequalities:

$$\sum_{\beta \neq \alpha} \mathbf{e}_{\beta\alpha} \;=\; \sum_{\beta \neq \alpha} \mathbf{e}_{\alpha\beta} \leqslant 1 \qquad\qquad (\alpha \in \mathbb{A} \setminus \{\iota, \zeta\}) \tag{5.11}$$

enforcing that in-degree of every vertex, except for ι and ζ, is the same as its out-degree, and equal 0 or 1, and also

$$\sum_{\beta \neq \iota} \mathbf{e}_{\beta\iota} = 0 \qquad \sum_{\beta \neq \iota} \mathbf{e}_{\iota\beta} = 1 \qquad \sum_{\beta \neq \zeta} \mathbf{e}_{\beta\zeta} = 1 \qquad \sum_{\beta \neq \zeta} \mathbf{e}_{\zeta\beta} = 0 \tag{5.12}$$

enforcing that in-degree of ι and out-degree of ζ are 0, while out-degree of ι and in-degree of ζ are 1. Thus atoms split into three categories: inner nodes (with in- and out-degree equal 1), end nodes (ι and ζ) and non-nodes (with in- and out-degree equal 0). Therefore, the graph G defined by a finite solution consists of a directed path from ι to ζ plus a number of vertex disjoint directed cycles. The path will be used below to encode a run of M: each edge, intuitively speaking, will be assigned a configuration of M, while each inner node will be assigned an instruction of M.

The system $\mathcal{S}$ has also unknowns $\mathbf{t}_{i\alpha}$ indexed by instructions $i \in I$ of M and atoms $\alpha \in \mathbb{A}$, and the following equations:

$$\sum_{i \in I} \mathbf{t}_{i\alpha} = \sum_{\beta \neq \alpha} \mathbf{e}_{\alpha\beta} \qquad\qquad (\alpha \in \mathbb{A} \setminus \{\iota, \zeta\}). \tag{5.13}$$

Therefore in every finite solution, for each inner node α of the above-defined graph G, there is exactly one instruction $i \in I$ such that $\mathbf{t}_{i\alpha}$ equals 1 (intuitively, this instruction i is *assigned* to node α), and $\mathbf{t}_{i\alpha}$ equals 0 for all other instructions. (This applies to *all* inner nodes of G, both those on the path as well as those on cycles.) For non-nodes α, all $\mathbf{t}_{i\alpha}$ are necessarily equal 0. Note that the values of unknowns $\mathbf{t}_{i\iota}$ and $\mathbf{t}_{i\zeta}$ are unrestricted, as they are irrelevant.

Finally, the system $\mathcal{S}$ contains unknowns $\mathbf{c}_{\alpha\beta\gamma k}$ indexed by $\alpha\beta\gamma \in \mathbb{A}^{(3)}$ and $k \in \{1, \ldots, d\}$. The following inequalities:

$$\mathbf{c}_{\alpha\beta\gamma k} \leqslant \mathbf{e}_{\alpha\beta} \qquad\qquad (\alpha\beta\gamma \in \mathbb{A}^{(3)}, k \in \{1, \ldots, d\}) \tag{5.14}$$

enforce that, whatever atom γ is, the value of unknown $\mathbf{c}_{\alpha\beta\gamma k}$ may be 0 or 1 when $\alpha\beta$ is an edge (i.e., when $\mathbf{e}_{\alpha\beta} = 1$), but $\mathbf{c}_{\alpha\beta\gamma k}$ is forcedly 0 when $\alpha\beta$ is a non-edge (i.e., when $\mathbf{e}_{\alpha\beta} = 0$). The underlying intuition is that for each

$k \in \{1, \ldots, d\}$, we represent the kth coordinate of the configuration *assigned* to the edge $\alpha\beta$ by the (necessarily finite) sum

$$\sum_{\gamma \notin \{\alpha,\beta\}} \mathbf{c}_{\alpha\beta\gamma k}. \tag{5.15}$$

(In particular, configurations assigned to non-edges are necessarily zero on all coordinates.) In agreement with this intuition, we add to $\mathcal{S}$ the requirement that the configuration assigned to the edge outgoing from ι is the source c_0, and the configuration assigned to the edge incomming to ζ is the target c_f:

$$\sum_{\beta,\gamma \neq \iota} \mathbf{c}_{\iota\beta\gamma k} = c_0(k) \qquad\qquad \sum_{\beta,\gamma \neq \zeta} \mathbf{c}_{\beta\zeta\gamma k} = c_f(k) \qquad\qquad (k \in \{1, \ldots, d\}).$$

Furthermore, in order to enforce correctness of encoding of a run of M, we add to $\mathcal{S}$ equations that relate, intuitively speaking, two consecutive configurations. Recall that, due to (5.11)–(5.12) and (5.14), for every $\alpha \in \mathbb{A} \setminus \{\iota, \zeta\}$, unknowns $\mathbf{c}_{\beta\alpha\gamma k}$ may be positive for at most one $\beta \in \mathbb{A}$; likewise unknowns $\mathbf{c}_{\alpha\beta\gamma k}$. For $\alpha \in \mathbb{A} \setminus \{\iota, \zeta\}$ and $k \in \{1, \ldots, d\}$), we add to $\mathcal{S}$ the following equations:

$$\sum_{\beta,\gamma \neq \alpha} \mathbf{c}_{\beta\alpha\gamma k} \;+ \sum_{i \in \mathrm{UPD}(k)} i(k) \cdot \mathbf{t}_{i\alpha} \;=\; \sum_{\beta,\gamma \neq \alpha} \mathbf{c}_{\alpha\beta\gamma k} \tag{5.16}$$

These equalities say that for every inner node or non-node α (i.e., every atom except for the end nodes ι and ζ), on every coordinate k, the configuration incomming to α differs from the configuration outgoing from α exactly by the sum

$$\sum_{i \in \mathrm{UPD}(k)} i(k) \cdot \mathbf{t}_{i\alpha}$$

ranging over those instructions i of M that update counter k. Remembering that for each α there is at most one instruction i satisfying $\mathbf{t}_{i\alpha} \neq 0$, we get that the configurations differ on coordinate k by exactly $i(k)$ (if i updates counter k) or the configuration are equal on coordinate k (if i zero-tests counter k, or there is no instruction i such that $\mathbf{t}_{i\alpha} \neq 0$).

In order to deal with zero tests, we add to $\mathcal{S}$ not just the inequalities (5.14), but the following strengthening thereof:

$$\mathbf{c}_{\alpha\beta\gamma k} + \sum_{i \in \mathrm{ZERO}(k)} \mathbf{t}_{i\alpha} \leqslant \mathbf{e}_{\alpha\beta} \qquad (\alpha\beta\gamma \in \mathbb{A}^{(3)}, k \in \{1, \ldots, d\}). \tag{5.17}$$

In consequence, for every edge $\alpha\beta$, if the instruction i assigned to α updates counter k, or no instruction is executed at α, (5.17) does not restrict further the kth coordinate of the configuration assigned to $\alpha\beta$. But if the instruction i

assigned to α zero-tests counter k, the sum

$$\sum_{i \in \text{ZERO}(k)} \mathbf{t}_{i\alpha}$$

equals 1 and therefore the kth coordinate of the configuration assigned to $\alpha\beta$, encoded by (5.15), is necessarily 0 (the same applies also to the configuration incomming to α, due to inequalities (5.16) below). The above considerations apply to *all* edges of G, both those on the path as well as those on cycles. As a further consequence, for a non-edge $\alpha\beta$, the configuration assigned at $\alpha\beta$, encoded by (5.15), is necessarily the zero configuration.

The construction of $\mathcal{S}$ is thus completed, and it remains to argue towards its correctness:

Lemma 5.10. *M admits a run from c_0 to c_f if and only if $\mathcal{S}$ has a finite nonnegative integer solution.*

Proof. For the 'if' direction, given a finite nonnegative integer solution of $\mathcal{S}$, we consider the graph G determined by values of unknowns $\mathbf{e}_{\alpha\beta}$, as discussed in the course of construction, consisting of inner nodes and two end nodes, and having the form of a finite directed path plus (possibly) a number of directed cycles. By the construction of $\mathcal{S}$, each edge of G has assigned a configuration of M, and each inner node has assigned an instruction of M, so that the configuration on the edge outgoing from an inner node is exactly the result of executing its instruction on the configuration assigned to the incomming edge. (As above, this applies to *all* inner nodes and edges of G, both those on the path as well as those on cycles.) Ignoring the cycles of G, we conclude that the sequence of configurations and instructions along the path of G is a run of M from c_0 to c_f.

For the 'only if' direction, given a run of M from c_0 to c_n as in (5.9), one constructs a solution of $\mathcal{S}$ in the form of a sole path involving end nodes $\alpha_0 = \iota, \alpha_{n+1} = \zeta$, n inner nodes $\alpha_1, \ldots, \alpha_n$, and $n+1$ edges $\alpha_j\alpha_{j+1}$. Thus unknowns $\mathbf{e}_{\alpha_j,\alpha_{j+1}}$ are equal 1. The values of unknowns $\mathbf{t}_{i\alpha_j}$ are determined by instructions i_j used in the run, and the values of the unknowns $\mathbf{c}_{\alpha_j\alpha_{j+1}k}$, for sufficiently many fresh atoms γ, are determined by configurations c_j. All other unknowns are equal 0. ∎

Remark 5.11. The proof does not adapt to FIN-NONNEG-EQ($\mathbb{Z}$). Indeed, the standard way of transforming inequalities into equations involves adding an infinite set of additional unknowns, that might be all non-zero.

Remark 5.12. The atom-dimension of the system defined in this section is 3. We do not know whether existence of finite integer solutions is decidable for systems with atom diemension $\leqslant 2$. We leave it is an open question.

5.3 Polynomially-parametrised inequalities

We now introduce a core problem that will serve as a target of reductions in the proof of Theorem 5.3 in § 5.5. Consider a finite inequality $\mathcal{E}$ of the form:

$$p_1(n) \cdot x_1 + \ldots + p_k(n) \cdot x_k \leqslant q(n), \tag{5.18}$$

where $p_1, \ldots, p_n$ and q are univariate polynomials with integer coefficients, and $x_1, \ldots, x_n$ are unknowns. The special unknown n plays a role of a nonnegative integer parameter, and that is why we call such an inequality a *polynomially parametrised inequality*. For every fixed value $n \in \mathbb{N}$, by evaluating all polynomials in n we get an ordinary inequality $\mathcal{E}(n)$ with integer coefficients. Also, if n does not appear in $\mathcal{E}$, i.e., all polynomials are constants, $\mathcal{E}$ is an ordinary inequality. A *polynomially parametrised system of inequalities* is a system of linear inequalities parametrised over the same parameter.

Example 5.13. For instance, the system (5.1) in Example 5.7 is transformed to the following two inequalities with one unknown x, which are polynomially (actually, linearly) parametrised in a parameter n (the details are exposed in Example 5.37):

$$\begin{bmatrix} n-1 \\ n \end{bmatrix} \cdot x \geqslant \begin{bmatrix} n \\ n \end{bmatrix} \tag{5.19}$$

This system is solvable for every $n > 1$, an example solution being $x = \frac{n}{n-1}$. ◀

In the sequel we study solvability of finite systems P of such inequalities (5.18). Again, by evaluating all polynomials in n we get an ordinary system $P(n)$. We use the matrix form $P(n) \equiv \mathbf{A}(n) \cdot \mathbf{x} \leqslant \mathbf{b}(n)$ when convenient. A fundamental problem is to check if for some value $n \in \mathbb{N}$, the system $P(n)$ has a real solution:

POLY-INEQ

Input: A finite system of polynomially-parametrised inequalities P.

Question: Does $P(n)$ have a real solution for some $n \in \mathbb{N}$?

Theorem 5.14. POLY-INEQ *is decidable.*

Proof. Consider a fixed system P of polynomially-parametrised inequalities over unknowns $x_1, \ldots, x_k$ and n where n is the only non-linear variable. Let $\sigma_P(n, x_1, \ldots, x_k)$ be the conjunction of inequalities in P, each of the form (5.18). It is thus a quantifier-free real arithmetic formula which says that a tuple

$\vec{x} = (x_1, \ldots, x_k)$ is a solution of $P(n)$. The existential real arithmetic formula $\psi(n) \overset{\text{def}}{=} \exists \vec{x} : \sigma_P(n, x_1, \ldots, x_k)$, with one free variable n, says that $P(n)$ has a real solution. Using quantifier elimination of real arithmetic ([3, Theorem 10.1]) we can compute a quantifier free formula $P'(n)$ equivalent to $\exists \vec{x} : \sigma_P(n, x_1, \ldots, x_k)$. $P'(n)$ is thus a boolean combination of polynomial inequalities. The satisfiability set of each inequality is a finite union of points and intervals such that, the points, and the endpoints of the intervals are the zeros of the underlying polynomial. Hence the satisfiability set of $P'(n)$ is also a finite union of points and intervals, such that the points, and the endpoints of the intervals are the zeros of one or more polynomials appearing in $P'(n)$. A real root of a polynomial $\sum_{k=0}^{d} a_k x^k$ can not be larger than $\frac{\sum_{k=0}^{d-1} |a_k|}{|a_d|} + 1$. [2] Therefore, there exists a computable natural number N such that all real roots of the polynomials appearing in $P'(n)$ are smaller than N. Hence $P'(N)$ is true if and only if $P'(n)$ is true for all $n \geqslant N$. Which means to check whether $P'(n)$ has a non-negative integer solution it is enough if any of the sentences $P(0), P(1), \ldots, P(N)$ is true. ∎

Remark 5.15. Polynomially parametrised systems are also studied in [2], where the authors show that given a polynomially parametrised system $P(r)$, one can decide in NP whether there exists $r \in \mathbb{Q}$ such that $P(r)$ has a rational solution ([2, Corollary 3.3]).

In the sequel we will not use the decision procedure of Theorem 5.14, but rather the algorithm of Theorem 5.16 stated below, since our later applications only use *monotonic* instances of POLY-INEQ, which can be solved in PTIME due to this theorem.

Monotonic systems

A system P is *monotonic* if there is some $n_0 \in \mathbb{N}$ such that every solution of $P(n)$, for an integer $n \geqslant n_0$, is also a solution of $P(n+1)$. Note that monotonicity trivially holds (with any value of n_0) if n does not appear in P, i.e., when P is an ordinary (non-parametrised) system. Instead of POLY-INEQ we use in the sequel the following core problem, where we do not assume monotonicity but seek for a solution of $P(n)$ for *almost all* (all sufficiently large) values of the parameter $n \in \mathbb{N}$:

ALL-POLY-INEQ

Input: A finite system P of polynomially-parametrised inequalities.

Question: Is there $n_0 \in \mathbb{N}$ and a vector $(x_1, \ldots, x_k)$ which is a solution of $P(n)$ for every integer $n \geqslant n_0$?

[2] We leave it to the reader to verify the details.

From now on, a vector $(x_1, \dots, x_k)$ which is a solution of an inequality $\mathcal{E}(n)$ (resp. a system $P(n)$) for almost all $n \in \mathbb{N}$ we call an *almost-all-solution* of $\mathcal{E}$ (resp. P). For monotonic systems, the problems POLY-INEQ and ALL-POLY-INEQ coincide.

Theorem 5.16. ALL-POLY-INEQ *is in* PTIME.

This theorem will be proven in § 5.6.

5.4 Finitely setwise-supported sets

For $T \subseteq_{\text{FIN}} \mathbb{A}$ define

$$\mathrm{Aut}_{\{T\}}(\mathbb{A}) \stackrel{\text{def}}{=} \{\pi \in \mathrm{Aut}(\mathbb{A}) \ : \ \pi(T) = T\} \ .$$

This is a special case of Notation 2.24 with $S = \emptyset$.

Definition 5.17. A *$\{T\}$-orbit* is a set of the form $\left\{\pi(x) \ : \ \pi \in \mathrm{Aut}_{\{T\}}(\mathbb{A})\right\}$ where x is an atom or set with atoms.

Remark 5.18. We have $\mathrm{Aut}_T(\mathbb{A}) \subseteq \mathrm{Aut}_{\{T\}}(\mathbb{A}) \subseteq \mathrm{Aut}(\mathbb{A})$. Hence every equivariant orbit splits into finitely many $\{T\}$-orbits, each of which splits in turn into finitely many T-orbits.

Definition 5.19. A set x is called *$\{T\}$-supported* if for all $\pi \in \mathrm{Aut}_{\{T\}}(\mathbb{A})$ we have $\pi(x) = x$.

Note that each $\{T\}$-supported set is T-supported, but the opposite implication is not true. Finally notice that a $\{T\}$-supported set is not necessarily $\{T'\}$-supported for $T \subseteq T'$. For any $\emptyset \subsetneq T \subsetneq T' \subseteq_{\text{FIN}} \mathbb{A}$ the set T itself is an example of a set which is $\{T\}$-supported but not $\{T'\}$-supported.

Example 5.20. Let $B = \mathbb{A}^{(2)}$. Pick two atoms $\alpha, \beta \in \mathbb{A}$ and let $T = \{\alpha, \beta\}$. The function $\mathbf{v} : B \to \mathbb{R}$ defined, for $\eta, \gamma \in \mathbb{A} \setminus \{\alpha, \beta\}$, by

$$\begin{aligned}
\mathbf{v}(\alpha\eta) &\stackrel{\text{def}}{=} \mathbf{v}(\eta\alpha) \stackrel{\text{def}}{=} -1 & \qquad \mathbf{v}(\alpha\beta) &\stackrel{\text{def}}{=} \mathbf{v}(\beta\alpha) \stackrel{\text{def}}{=} 3 \\
\mathbf{v}(\beta\eta) &\stackrel{\text{def}}{=} \mathbf{v}(\eta\beta) \stackrel{\text{def}}{=} -2 & \qquad \mathbf{v}(\eta\gamma) &\stackrel{\text{def}}{=} 0 \,.
\end{aligned}$$

is an integer vector over B, supported by T. It is not $\{T\}$-supported. Indeed,

$$\pi(\mathbf{v})(\alpha, \gamma) = \mathbf{v}(\beta, \gamma) \neq \mathbf{v}(\alpha, \gamma)$$

for any $\gamma \notin T$ and $\pi \in \mathrm{Aut}_{\{T\}}(\mathbb{A})$ that swaps α and β but preserves all other atoms. However, the vector $\mathbf{v} + \pi(\mathbf{v})$ is $\{T\}$-supported. ◀

Clearly, with the size of T increasing towards infinity, the number of T-orbits included in one equivariant orbit may increase towards infinity as well. The crucial property of $\{T\}$-supported sets is that they do not suffer from this unbounded growth: the number of $\{T\}$-orbits included in a fixed equivariant orbit is bounded, no matter how large T is. We will need this property for $\{T\}$-orbits $U \subseteq \mathbb{A}^{(n)}$, $n \in \mathbb{N}$, and it follows immediately by the next lemma.

Lemma 5.21. *Let $T \subseteq_{\text{FIN}} \mathbb{A}$ of size $|T| \geq n$. Each $\{T\}$-orbit $U \subseteq \mathbb{A}^{(n)}$ is of the form*

$$U = \left\{\alpha_1 \ldots \alpha_n \in \mathbb{A}^{(n)} \ : \ \alpha_i \in T \text{ for } i \in I \text{ and } \alpha_i \in \mathbb{A} \setminus T \text{ for } i \notin I\right\} \tag{5.20}$$

for some fixed $I \subseteq \{1, \ldots, n\}$.

Proof. Consider any tuple $\alpha_1 \ldots \alpha_n \in \mathbb{A}^{(n)}$. Let

$$I \stackrel{\text{def}}{=} \{i \in \{1, \ldots, n\} \ : \ \alpha_i \in T\}$$

denote the positions in $\alpha_1 \ldots \alpha_n$ filled by atoms from T. By applying all $\{T\}$-automorphisms to $\alpha_1 \ldots \alpha_n$, we obtain all tuples, where positions from I are arbitrarily filled by elements of T, and positions outside of I are arbitrarily filled by elements of $\mathbb{A} \setminus T$. ∎

Definition 5.22. We define the *orbit summation function*

$$\gamma : \text{FinLin}(C) \mapsto \mathbb{R}^{\text{Orbits}(C)}$$

which takes a vector $\mathbf{x} \in \text{FinLin}(C)$ and outputs the vector

$$\gamma(\mathbf{x}) \stackrel{\text{def}}{=} \left(O \in \text{Orbits}(C) \mapsto \sum_{c \in O} \mathbf{x}(c)\right).$$

Remark 5.23. The $<$-orbit summation function (Definition 4.25), which was used for solving order equivariant finitary solvability in Chapter 4, is a slight variation of the above definition.

Lemma 5.24. *If an equivariant system of inequalities $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ has a finite T-supported solution $\mathbf{x}$ then it also has a finite $\{T\}$-supported one $\mathbf{y}$ such that $\gamma(\mathbf{x}) = \gamma(\mathbf{y})$.*

Proof. Let $\mathbf{x} \in \text{FinLin}(C)$ be a solution of the system, namely $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$. Let $T = \text{support}\,(\mathbf{x})$ and $n = |T|$. We recall the definition

$$\text{Aut}(T) = \text{Aut}_{\mathbb{A} \setminus \{T\}}(T)$$

given in Notation 2.24, i.e. $\text{Aut}(T)$ is the set of permutations of $\text{Aut}(\mathbb{A})$ which only permutes elements inside T and fixes everything else. As $\mathbf{A}$ and $\mathbf{b}$ are both equivariant, for every $\rho \in \mathrm{Aut}(T)$ the vector $\rho(\mathbf{x})$ is also a solution: $\mathbf{A} \cdot \rho(\mathbf{x}) \leqslant \mathbf{b}$. Knowing that the size of $\mathrm{Aut}(T)$ is $n!$, we have

$$\mathbf{A} \cdot \left(\sum_{\rho \in \mathrm{Aut}(T)} \rho(\mathbf{x}) \right) \leqslant \; n! \cdot \mathbf{b},$$

and hence the vector $\mathbf{y}$ defined by averaging

$$\mathbf{y} \overset{\text{def}}{=} \frac{1}{n!} \cdot \sum_{\rho \in \mathrm{Aut}(T)} \rho(\mathbf{x}) \tag{5.21}$$

is also a solution of the system, namely $\mathbf{A} \cdot \mathbf{y} \leqslant \mathbf{b}$. We notice that for finite $\mathbf{x}$, the vector $\mathbf{y}$ is also finite. By the very definition, the averaging (5.21) preserves the orbit-sum: $\gamma(\mathbf{x}) = \gamma(\mathbf{y})$. Furthermore, we claim that the vector $\mathbf{y}$ is $\{T\}$-supported. To prove this, we fix an arbitrary $\pi \in \mathrm{Aut}_{\{T\}}(\mathbb{A})$, aiming at showing that $\pi(\mathbf{y}) = \mathbf{y}$. It factors through $\pi = \sigma \circ \rho$ for some $\rho \in \mathrm{Aut}(T)$ and $\sigma \in \mathrm{Aut}_T(\mathbb{A})$. Indeed, ρ acts as π on T but is identity elsewhere, while σ acts as π outside of T but is identity on T. A crucial but simple observation is that, by the very construction of $\mathbf{y}$, we have

$$\rho(\mathbf{y}) = \mathbf{y}. \tag{5.22}$$

Indeed, as $\mathbf{y}$ is defined by averaging over all $\rho' \in \mathrm{Aut}(T)$,

$$\rho\Big(\sum_{\rho' \in \mathrm{Aut}(T)} \rho'(\mathbf{x}) \Big) \; = \sum_{\rho' \in \mathrm{Aut}(T)} \rho \circ \rho'(\mathbf{x}) \; = \sum_{\rho' \in \mathrm{Aut}(T)} \rho'(\mathbf{x})$$

which implies $\rho(\mathbf{y}) = \mathbf{y}$. Moreover, as action of automorphisms commutes with support (Lemma 2.7), we have

$$\mathrm{support}\left(\rho'(\mathbf{x})\right) = \rho'(\mathrm{support}\,(\mathbf{x}))$$

for every $\rho' \in \mathrm{Aut}(T)$, and therefore

$$\mathrm{support}\left(\rho'(\mathbf{x})\right) = \mathrm{support}\,(\mathbf{x})$$

for every $\rho' \in \mathrm{Aut}(T)$. Therefore T supports the right-hand side of (5.21), which means that $\mathrm{support}\,(\mathbf{y}) \subseteq T$ and implies

$$\sigma(\mathbf{y}) = \mathbf{y}. \tag{5.23}$$

Combining (5.22) and (5.23) we obtain $\pi(\mathbf{y}) = \mathbf{y}$, as required. ∎

Example 5.25. Recall the system of inequalities from Example 5.8. Its finitary solutions correspond to finite directed graphs, whose vertices and edges are labelled by real numbers satisfying constraints (5.3) and (5.4). According to Lemma 5.24, if such a finite directed graph existed, there would also exist a finite directed clique, where labels of all vertices are pairwise equal, and labels of all edges are pairwise equal as well, which satisfying constraints (5.3) and (5.4). In particular, all edges incoming to a vertex would carry the same value as all outgoing edges. This requirement is clearly contradictory with constraints (5.3) and (5.4), and hence the system has no finitary solutions. ◀

In the next section we rely on the fact that existence of a $\{T\}$-supported solution implies existence of such a $\{T'\}$-supported solution for any $T' \supseteq_{\text{FIN}} T$. The fact follows immediately from Lemma 5.24, as every $\{T\}$-supported vector is trivially T'-supported, for every superset T' of T:

Corollary 5.26. *If an equivariant system of inequalities $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ has a finite $\{T\}$-supported solution $\mathbf{y}$ for some $T \subseteq_{\text{FIN}} \mathbb{A}$, then for every superset $T' \supseteq_{\text{FIN}} T$, the system has a finite $\{T'\}$-supported solution $\mathbf{y}'$ such that $\gamma(\mathbf{y}) = \gamma(\mathbf{y}')$.*

Remark 5.27. Our technique discussed in Example 5.8, and developed formally in this section, seems to be reminiscent of (but independent from) the techniques in the recent work [1].

5.5 Orbit-finite to polynomially parametrised

In this section we prove Theorem 5.3 by a reduction of FIN-INEQ($\mathbb{R}$) to ALL-POLY-INEQ introduced in § 5.3. We start with some simpliflying assumptions in § 5.5.1 and an intuitive explanation of the reduction in § 5.5.2. In § 5.5.3 we describe the reduction and prove its correctness. Finally in § 5.5.4 we briefly discuss the complexity of this reduction.

5.5.1 Simplifying assumptions

Consider an orbit-finite system of inequalities $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ given by an integer matrix $\mathbf{A} \in \text{Lin}(B \times C)$ and integer vector $\mathbf{b} \in \text{Lin}(B)$. The proof of Lemma 4.19 can be suitably adapted to prove that:

Lemma 5.28. *WLOG we can assume that B and C are disjoint unions of equivariant orbits $\mathbb{A}^{(k)}$, $k \in \mathbb{N}$:*

$$B = \mathbb{A}^{(n_1)} \uplus \ldots \uplus \mathbb{A}^{(n_s)} \qquad\qquad C = \mathbb{A}^{(m_1)} \uplus \ldots \uplus \mathbb{A}^{(m_r)} \tag{5.24}$$

(see the figure below), and that $\mathbf{A}$ and $\mathbf{b}$ are equivariant.

$$
\mathbf{A} \;=\; \begin{array}{c} \\ \mathbb{A}^{(n_1)} \\ \mathbb{A}^{(n_2)} \\ \ldots \\ \mathbb{A}^{(n_s)} \end{array}
\begin{array}{c} \begin{array}{cccc} \mathbb{A}^{(m_1)} & \mathbb{A}^{(m_2)} & \ldots & \mathbb{A}^{(m_r)} \end{array} \\
\left[\begin{array}{c|c|c|c} \; & \; & \; & \; \\ \hline & & & \\ \hline & & & \\ \hline & & & \end{array}\right] \end{array}
\qquad
\mathbf{b} \;=\; \left[\begin{array}{c} \; \\ \hline \; \\ \hline \; \\ \hline \; \end{array}\right]
$$

As mentioned in Remark 4.23, the size blow-up is exponential in atom dimension, but polynomial when atom dimension is fixed.

5.5.2 Idea of the reduction

Suppose only finite T-supported solutions are sought, for a fixed $T \subseteq_{\text{FIN}} \mathbb{A}$. FIN-INEQ($\mathbb{R}$) then reduces to a finite system of inequalities $\mathbf{A}' \cdot \mathbf{x}' \leqslant \mathbf{b}'$ obtained from $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ as follows:

(1) Keep only columns indexed by T-tuples (= elements of finite T-orbits) $c \in C$, discarding all other columns.

(2) Pick arbitrary representatives of all T-orbits included in B, and keep only rows of $\mathbf{A}$ and entries of $\mathbf{b}$ indexed by the representatives, discarding all others.

The system $\mathbf{A}' \cdot \mathbf{x}' \leqslant \mathbf{b}'$ is solvable if and only if the original one $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ has a finite T-supported solution. Indeed, discarding unknowns as in (1) is justified as a finite T-supported solution of $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ assigns 0 to each non-T-tuple (Lemma 2.47). Discarding inequalities as in (2) is also justified. Indeed, each inequality in the original system is obtained by applying some atom T-automorphism to an inequality in $\mathbf{A}' \cdot \mathbf{x}' \leqslant \mathbf{b}'$, while atom T-automorphisms preserve T-supported solutions of $\mathbf{A}' \cdot \mathbf{x}' \leqslant \mathbf{b}'$, which implies that every T-supported solutions of $\mathbf{A}' \cdot \mathbf{x}' \leqslant \mathbf{b}'$ is also a solution of all inequalities in the original system. The above reduction yields no algorithm yet, as we do not know a priori any bound on size of T, and the size of $\mathbf{A}' \cdot \mathbf{x}' \leqslant \mathbf{b}'$ depends on the number of T-orbits and hence grows unboundedly when T grows. We overcome this difficulty by using $\{T\}$-orbits instead of T-orbits, and relying on Lemmas 5.21 and 5.24. The latter one guarantees that the number of $\{T\}$-orbits is constant - independent of T. Once we additionally merge (sum up) all columns indexed by elements of the same $\{T\}$-orbit, we get $\mathbf{A}'$ of size independent of T. This still does not yield an algorithm, as entries of $\mathbf{A}'$ change when T grows. We however crucially discover that the growth of the entries of $\mathbf{A}'$ is *polynomial* in $n = |T|$, for sufficiently large n. Therefore, $\mathbf{A}'$ is a matrix of polynomials in one unknown n, and solvability of $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ is equivalent to solvability of

$\mathbf{A}' \cdot \mathbf{x}' \leqslant \mathbf{b}'$ for some value $n \in \mathbb{N}$. As argued in § 5.3, the latter solvability is decidable.

5.5.3 The reduction

Fix an equivariant system $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$. We construct a finite system P_2 of polynomially-parametrised inequalities such that $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ has a finite solution if and only if $P_2(n)$ has a solution for almost all $n \in \mathbb{N}$.

Let us denote by $d \stackrel{\text{def}}{=} \max\{n_1, \ldots, n_s, m_1, \ldots, m_r\}$ the maximal atom dimension of orbits included in B and C. Let $T \subseteq_{\text{FIN}} \mathbb{A}$ be an arbitrary finite subset of atoms. Both B and C split into $\{T\}$-orbits, refining (5.24):

$$B = B_1 \uplus \ldots \uplus B_N \qquad C = C_1 \uplus \ldots \uplus C_{M'}. \tag{5.25}$$

Let $C_1, \ldots, C_M$ be the *finite* $\{T\}$-orbits among $C_1, \ldots, C_{M'}$ (in principle, M and N may depend on T). Importantly, by Lemma 5.21, M and N do not depend on T as long as $|T| \geqslant d$. In fact M is same as the number of equivariant orbits included in C, as by Lemma 5.21 we deduce:

Lemma 5.29. *Assuming $|T| \geqslant \ell$, the equivariant orbit $\mathbb{A}^{(\ell)}$ includes exactly one finite $\{T\}$-orbit, namely $T^{(\ell)}$.*

Our reduction proceeds in two steps. In the first step we derive a finite polynomially-parametrised system P_1 which is yet not necessarily monotonic. In the second step, we transform it further to a monotonic system P_2.

Step 1 (finite polynomially-parametrised system) Our construction is parametric in T. Let $b_1 \in B_1, \ldots, b_N \in B_N$ be arbitrarily chosen representatives of $\{T\}$-orbits included in B. Given $\mathbf{A}$ and $\mathbf{b}$, we define an $N \times M$ matrix $\mathbf{A}_1(T)$ and a vector $\mathbf{b}_1(T) \in \mathbb{Z}^N$ as follows:

(1) Pick columns of $\mathbf{A}(T)$ indexed by elements of all finite $\{T\}$-orbits included in C, and discard other columns; this yields a matrix $\mathbf{A}'(T)$ with finitely many columns (number thereof depending on T).

(2) Merge (sum up) columns of $\mathbf{A}'(T)$ indexed by elements of the same $\{T\}$-orbit C_i; this yields a matrix $\mathbf{A}''(T)$ with M columns (M independent of T).

(3) Pick N rows of $\mathbf{A}''$, indexed by $b_1, \ldots, b_N$, and discard other rows; this yields an $N \times M$ matrix $\mathbf{A}_1(T)$.

(4) Likewise pick the corresponding entries of $\mathbf{b}$ and discard others, thus yielding a finite vector $\mathbf{b}_1(T) \in \mathbb{Z}^N$.

For $b \in B$ and $C_j \subseteq C$, $j \in \{1, \ldots, M\}$, we write $\mathbf{A}^{\Sigma}(b, C_j)$ for the finite sum ranging over elements of C_j:

$$\mathbf{A}^{\Sigma}(b, C_j) \overset{\text{def}}{=} \sum_{c \in C_j} \mathbf{A}(b, c),$$

which allows us to formally define the $B \times M$ matrix $\mathbf{A}''(T)$, the $N \times M$ matrix $\mathbf{A}_1(T)$ and the vector $\mathbf{b}_1(T) \in \mathbb{Z}^N$:

$$\mathbf{A}''(T)(b, j) \overset{\text{def}}{=} \mathbf{A}^{\Sigma}(b, C_j) \tag{5.26}$$

$$\mathbf{A}_1(T)(i, j) \overset{\text{def}}{=} \mathbf{A}''(T)(b_i, j) = \mathbf{A}^{\Sigma}(b_i, C_j) \tag{5.27}$$

$$\mathbf{b}_1(T)(i) \overset{\text{def}}{=} \mathbf{b}(b_i). \tag{5.28}$$

Example 5.30. We apply the above construction to the system (5.1) presented in Example 5.7. Fix a non-empty $T \subseteq_{\text{FIN}} \mathbb{A}$. The set $\mathbb{A}$ includes just one finite $\{T\}$-orbit, namely T. Therefore the matrix $\mathbf{A}'(T)$ has $|T|$ columns, $\mathbf{A}''(T)$ has just one column, and the system $(\mathbf{A}_1(T), \mathbf{b}_1(T))$ has just one unknown. Furthermore, the set $\mathbb{A}$ includes two $\{T\}$-orbits, the finite one T plus the infinite one $\mathbb{A} \setminus T$, and therefore the system $(\mathbf{A}_1(T), \mathbf{b}_1(T))$ has two inequalities. Pick arbitrary representatives of the $\{T\}$-orbits, $b_1 \in T$ and $b_2 \in (\mathbb{A} \setminus T)$. We have

$$\mathbf{A}_1(T)(1,1) \;=\; \sum_{c \in T} \mathbf{A}(b_1, c) = |T| - 1$$

$$\mathbf{A}_1(T)(2,1) \;=\; \sum_{c \in T} \mathbf{A}(b_2, c) = |T|.$$

Replacing $|T|$ with n yields the system $(\mathbf{A}_1(T), \mathbf{b}_1(T))$:

$$\begin{bmatrix} n-1 \\ n \end{bmatrix} \cdot x \;\geqslant\; \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5.29}$$

which happens to be monotonic. In general, the system obtained so far needs not be monotonic, but we will ensure monotonicity in the subsequent steps. ◀

The choice of representatives b_i is irrelevant, and hence $\mathbf{A}_1(T)$ and $\mathbf{b}_1(T)$ are well defined, since rows of $\mathbf{A}''$ indexed by any two elements of B belonging the same $\{T\}$-orbit are equal, and likewise the corresponding entries of $\mathbf{b}$:

Lemma 5.31. *If $b, b' \in B$ are in the same $\{T\}$-orbit, then $\mathbf{b}(b) = \mathbf{b}(b')$ and $\mathbf{A}^{\Sigma}(b, C_j) = \mathbf{A}^{\Sigma}(b', C_j)$ for every $j \in \{1, \ldots, M\}$.*

Proof. Let $\pi \in \mathrm{Aut}_{\{T\}}(\mathbb{A})$ be such that $\pi(b) = b'$. As $\mathbf{b}$ is equivariant, it is necessarily constant on the whole equivariant orbit to which b and b' belong, and hence $\mathbf{b}(b') = \mathbf{b}(b)$. For the second point fix $j \in \{1, \ldots, M\}$. As $\mathbf{A}$ is equivariant, it is constant over the orbit included in $B \times C$ to which (b, c) belongs, for every $c \in C$, and hence $\mathbf{A}(b, c) = \mathbf{A}(\pi(b), \pi(c))$. This implies

$$\sum_{c \in C_j} \mathbf{A}(b, c) = \sum_{c \in C_j} \mathbf{A}(\pi(b), \pi(c)) = \sum_{c \in C_j} \mathbf{A}(b', \pi(c)).$$

Since $\pi \in \mathrm{Aut}_{\{T\}}(\mathbb{A})$, when restricted to the $\{T\}$-orbit C_j it is a bijection $C_j \to C_j$, and hence the two sums below differ only by the order of summation and are thus equal:

$$\sum_{c \in C_j} \mathbf{A}(b', \pi(c)) = \sum_{c \in C_j} \mathbf{A}(b', c).$$

The two above equalities imply the claim, namely

$$\sum_{c \in C_j} \mathbf{A}(b, c) = \sum_{c \in C_j} \mathbf{A}(b', c) .$$

∎

Notation 5.32. Let $T \subseteq_{\mathrm{FIN}} \mathbb{A}$. Due to Lemma 5.29, the set of finite $\{T\}$-orbits $\{C_1, \ldots, C_M\}$ included in C is in bijection with the set $\mathrm{Orbits}(C) = \{U_1, \ldots, U_M\}$ of equivariant orbits included in C (Lemma 5.29). WLOG assume $C_j \subseteq U_j$ for $j = 1 \ldots M$. Take any finite $\{T\}$-supported vector $\mathbf{x} \in \mathrm{FinLin}(C)$. It is non-zero only inside finite $\{T\}$-orbits C_j, which implies

$$\sum_{c \in C_j} \mathbf{x}(c) = \sum_{c \in U_j} \mathbf{x}(c) = \gamma(\mathbf{x})(U_j)$$

for $j = 1 \ldots M$. Furthermore, $\mathbf{x}$ is constant inside each C_j. We denote this constant by $\dot{\mathbf{x}}(j)$. Finally, we note the (obvious) relation between $\dot{\mathbf{x}}$ and $\gamma(\mathbf{x})$:

$$\gamma(\mathbf{x})(C_j) \;=\; |C_j| \cdot \dot{\mathbf{x}}(j). \tag{5.30}$$

The following lemma, being a cornerstone of correctness of the whole reduction, is now not difficult to prove:

Lemma 5.33. *Let $|T| \geqslant d$ and $\mathbf{x} : \mathrm{FinLin}(C)$ a finite $\{T\}$-supported vector. The following conditions are equivalent:*

- $\mathbf{x}$ *is solution of* $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$*;*

- *$\dot{\mathbf{x}}$ is a solution of $P_1(T) = (\mathbf{A}_1(T), \mathbf{b}_1(T))$.*

Proof. Take any $\{T\}$-supported vector $\mathbf{x}$: FinLin(C), and let $\mathbf{x}'$ be the restriction of $\mathbf{x}$ to $U = C_1 \uplus \ldots \uplus C_M$. We argue that the following four conditions are equivalent, which implies the claim:

1. $\mathbf{x}$ is solution of $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$;
2. $\mathbf{x}'$ is solution of $(\mathbf{A}'(T), \mathbf{b})$;
3. $\dot{\mathbf{x}}$ is solution of $(\mathbf{A}''(T), \mathbf{b})$;
4. $\dot{\mathbf{x}}$ is a solution of $(\mathbf{A}_1(T), \mathbf{b}_1(T))$.

First, as $\mathbf{x}$ is finite, we have $\mathbf{x}(c) = 0$ for all $c \notin U$, and hence $\mathbf{A}'(T) \cdot \mathbf{x}' = \mathbf{A} \cdot \mathbf{x}$. This implies equivalence of (1) and (2). Second, as $\mathbf{A}''$ is obtained from $\mathbf{A}'$ by summing columns over a $\{T\}$-orbit where the vector $\mathbf{x}$, being $\{T\}$-supported, is constant, we have $\mathbf{A}''(T) \cdot \dot{\mathbf{x}} = \mathbf{A}'(T) \cdot \mathbf{x}'$. This implies equivalence of (2) and (3). Finally, (3) implies (4) as $(\mathbf{A}_1(T), \mathbf{b}_1(T)$ is obtained from $(\mathbf{A}''(T), \mathbf{b})$ by removing inequalities. For the reverse implication, we recall that Lemma 5.31 shows that $\mathbf{A}''(b, j) = \mathbf{A}''(b_i, j)$ and $\mathbf{b}(b) = \mathbf{b}(b_i)$ for every $i \in \{1, \ldots, N\}$ and $b \in B_i$, and therefore $\mathbf{A}''(T)$ contains the same inequalities as $\mathbf{A}_1(T)$. In consequence, (4) implies (3). ∎

The function $T \mapsto P_1(T)$ is equivariant, i.e., invariant under action automorphisms. In consequence, the entries of $\mathbf{A}_1(T)$ and $\mathbf{b}_1(T)$ do not depend on the set T itself, but only on its size $|T|$. Indeed, if $|T| = |T'|$ then $\pi(T) = T'$ for some automorphism π, and hence $\pi(P_1(T)) = P_1(T')$. Since the system $P_1(T)$ is atomless we have also $\pi(P_1(T)) = P_1(T)$, which implies $P_1(T) = P_1(T')$. We can thus meaningfully write $P_1(|T|) = (\mathbf{A}(|T|), \mathbf{b}(|T|))$, i.e. $P_1(n) = (\mathbf{A}_1(n), \mathbf{b}_1(n))$ for $n \in \mathbb{N}$ (cf. Example 5.30).

We argue that the dependence on $|T|$ is polynomial, as long as $|T| \geqslant 2d$:

Lemma 5.34. *There are univariate polynomials $p_{ij}(n) \in \mathbb{Z}[n]$ such that for $n \geqslant 2d$ we have $\mathbf{A}_1(n)(i, j) = p_{ij}(n)$.*

Proof. Let $n = |T|$. Fix a $\{T\}$-orbits $B_i \subseteq B$ and a finite $\{T\}$-orbit $C_j \subseteq C$. Each of them is included in a unique equivariant orbit, say:

$$B_i \subseteq B' = \mathbb{A}^{(p)} \qquad C_j \subseteq C' = \mathbb{A}^{(\ell)}$$

(cf. the partitions (5.24)). Recall Lemma 5.21: B_i is determined by the subset $I \subseteq \{1, \ldots, p\}$ of positions where atoms of T appear in tuples belonging to B_i. Let $m = |I|$. On the other hand $C_j = T^{(\ell)}$ (cf. Lemma 5.29). Note that $m = |T \cap \text{support}(b_i)|$.

We are going to demonstrate that the value $\mathbf{A}_1(n)(i,j) = \mathbf{A}^{\Sigma}(b_i, C_j)$ is polynomially depending on $n = |T|$. We will use the polynomials $n^{(w)}$ of degree w, for $w \leqslant d$, defined by

$$n^{(w)} \stackrel{\text{def}}{=} n \cdot (n-1) \cdot \ldots \cdot (n-w+1). \tag{5.31}$$

In the special case of $w = 0$, we put $n^{(w)} = 1$. The value $n^{(w)}$ can be interpreted as follows:

Claim 5.34.1. *For $n \geqslant w$, $n^{(w)}$ is equal to the number of arrangements of w items chosen from n objects into a sequence.*

Denote by $\mathcal{D}$ the set of equivariant orbits $U \subseteq B' \times C'$. For $U \in \mathcal{D}$, we put $U(b_i, C_j) \stackrel{\text{def}}{=} \{c \in C_j \;:\; (b_i, c) \in U\}$. As $\mathbf{A}$ is equivariant, the value $\mathbf{A}(b_i, c)$ depends only on the orbit to which (b_i, c) belongs. We write $\mathbf{A}(U)$, for $U \in \mathcal{D}$, and get:

Claim 5.34.2. $\mathbf{A}^{\Sigma}(b_i, C_j) = \sum_{U \in \mathcal{D}} \mathbf{A}(U) \cdot |U(b_i, C_j)|$.

Recall Example 2.10. Orbits inside $B' \times C'$ are in one-to-one correspondence with partial injections $\iota : \{1, \ldots, p\} \to \{1, \ldots, \ell\}$. We write U_ι for the orbit corresponding to ι. Let $\mathrm{dom}(\iota) = \{x \;:\; \iota(x) \text{ is defined}\}$ denote the domain of ι.

Claim 5.34.3. $U_\iota(b_i, C_j) \neq \emptyset$ *if and only if* $\mathrm{dom}(\iota) \subseteq I$.

Indeed, recall again Example 2.10 which yields

$$U_\iota(b_i, C_j) = \left\{c \in C_j \;:\; \forall x\, \forall y\; b_i(x) = c(y) \iff \iota(x) = y\right\}.$$

If $\mathrm{dom}(\iota) \subseteq I$, the set $U_\iota(b_i, C_j)$ contains tuples $c \in C_j$ with fixed values on positions $J = \{\iota(x) \;:\; x \in \mathrm{dom}(\iota)\}$, namely

$$b_i(x) = c(\iota(x)), \tag{5.32}$$

and arbitrary other atoms from T elsewhere, and therefore is nonempty. If there is $x \in \mathrm{dom}(\iota) \setminus I$ then $b_i(x) \notin T$ and therefore no $c \in C_j$ satisfies (5.32). Claim 5.34.3 is thus proved.

Claim 5.34.4. *Let $k = |\mathrm{dom}(\iota)|$ be the number of pairs related by ι. If $U_\iota \neq \emptyset$ then $|U_\iota(b_i, C_j)| = (n-m)^{(\ell-k)}$.*

According to (5.32), tuples $c \in U_\iota(b_i)$ have fixed values on k positions in J. The remaining $\ell - k$ positions in tuples $c \in U_\iota(b_i, C_j)$ are filled arbitrarily using $n - m$ atoms from $T \setminus \text{support}\,(b_i)$. Due to the assumption that $n \geqslant 2d$, we have

$n - m \geqslant d \geqslant \ell - k$, and therefore using Claim 5.34.1 (for $w = \ell - k$) we deduce $|U_\iota(b_i, C_j)| = (n-m)^{(\ell-k)}$. thus proving Claim 5.34.4.

Once $b_i \in B_i$ and $U \in \mathcal{D}$ are fixed, the values k, ℓ and m are fixed too, and the formula of Claim 5.34.4 is an univariate polynomial of degree $\ell - k$. The formula of Claim 5.34.2 yields the required polynomial[3] $\mathbf{A}(n)(i,j) = p_{ij}(n)$ and hence the proof of Lemma 5.34 is completed. ∎

Relying on Lemma 5.34 we get a polynomially-parametrised system $P_1(n) \stackrel{\text{def}}{=} (\mathbf{A}_1(n), \mathbf{b}_1(n))$.

Step 2 (monotonicity) The system P_1 constructed so far, does *not* have to be monotonic in general. As an immediate corollary of Lemma 5.33 and Corollary 5.26, we only know that If $P_1(n)$ has a solution for $n \geqslant d$, then $P_1(n+1)$ has a (potentially different) solution. We slightly modify the system P_1 in order to achieve monotonicity.

Before formally defining the new system $P_2(n) = (\mathbf{A}_2(n), \mathbf{b}_2(n))$, we point to our objective: we want to replace the orbit-value vector $\dot{\mathbf{x}}$ in Lemma 5.33 by the orbit-sum vector $\gamma(\mathbf{x})$, as in Lemma 5.36 below. Noting that $\text{Orbits}(C) = M$ we consider $\gamma(\mathbf{x})$ to be an M-dimensional vector where the j-th coefficient is equal to $\gamma(\mathbf{x})(C_j)$. In other terms, we want the solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ of $P_1(n)$ and $P_2(n)$, respectively, differ on position j by the multiplicative factor of $|C_j|$, namely

$$\mathbf{y}_2(j) \;=\; |C_j| \cdot \mathbf{y}_1(j) \tag{5.33}$$

for $j = 1, \ldots, M$ (cf. (5.30)). The size $|C_j|$ of the $\{T\}$-orbit C_j, where $|T| = n$, is equal to

$$|C_j| \;=\; n^{(e_j)}, \tag{5.34}$$

where $C_j \subseteq \mathbb{A}^{(e_j)}$, i.e., e_j is the atom dimension of the equivariant orbit including C_j, assuming $|T| \geqslant e_j$. These considerations lead to the following formal definition of P_2:

$$\mathbf{A}_2(n)(i,j) \;\stackrel{\text{def}}{=}\; \mathbf{A}_1(n)(i,j) \cdot \frac{n^{(d)}}{n^{(e_j)}} \qquad\qquad \mathbf{b}_2(i) \;\stackrel{\text{def}}{=}\; \mathbf{b}_1(i) \cdot n^{(d)} \tag{5.35}$$

where $\mathbf{A}_1(n)(i,j) = p_{ij}(n)$ We rely on the following fact:

Claim 5.34.5. $n^{(w)} \cdot (n-w)^{(u)} \;=\; n^{(w+u)}$.

[3]This confirms, in particular, that $\mathbf{A}(T)(i,j)$ is independent from the actual set T, and only depend on its size $n = |T|$.

By the claim, all coefficients in (5.35) are polynomials, namely: $\mathbf{A}_2(n)(i,j) = p_{ij}(n) \cdot (n-e_j)^{(d-e_j)}$, since $e_j \leqslant d$. It remains to conclude that the systems $P_1(n)$ and $P_2(n)$ have the same solutions modulo (5.33):

Lemma 5.35. *Let $n \geqslant d$ and let $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^M$ satisfy $\mathbf{y}_2(j) = n^{(e_j)} \cdot \mathbf{y}_1(j)$ for $j = 1, \ldots, M$. Then $\mathbf{y}_1$ is a solution of $P_1(n)$ if and only if $\mathbf{y}_2$ is a solution of $P_2(n)$.*

Combination of equations (5.30) and (5.34), and Lemmas 5.33 to 5.35 yields:

Lemma 5.36. *Let $|T| = n \geqslant 2d$ and $\mathbf{x} \in \mathrm{FinLin}(C)$ a finite $\{T\}$-supported vector. The following conditions are equivalent:*

- *$\mathbf{x}$ is solution of $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$;*
- *$\gamma(\mathbf{x})$ is a solution of $P_2(n)$.*

Example 5.37. The system (5.19) in Example 5.13 is obtained from (5.29) in Example 5.30 by applying the equation (5.35). Indeed, $M = d = e_1 = 1$ and hence $\mathbf{A}$ stays unchanged, while the right-hand side vector $\mathbf{b}$ gets multiplied by $n^{(1)} = n$. ◀

Lemma 5.38 (Monotonicity)**.** *Let $n \geqslant d$. Every solution of $P_2(n)$ is also a solution of $P_2(n+1)$.*

Proof. Suppose $\mathbf{y}$ is a solution of $P_2(n)$, for $n \geqslant d$. Let $T \subseteq_{\text{FIN}} \mathbb{A}$ be any subset of atoms of size $|T| = n$. Let $\mathbf{x}$ be the finite $\{T\}$-supported vector uniquely determined by

$$\gamma(\mathbf{x}) \;=\; \mathbf{y}. \tag{5.36}$$

By Lemma 5.36, $\mathbf{x}$ is a solution of $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$. We apply Corollary 5.26 to obtain another finite solution $\mathbf{x}'$ of $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$, $\{T'\}$-supported by some T' of size $|T|' = n+1$, and having the same orbit-summation mapping:

$$\gamma(\mathbf{x}) \;=\; \gamma(\mathbf{x}'). \tag{5.37}$$

Equalities (5.36) and (5.37) imply $\gamma(\mathbf{x}') = \mathbf{y}$. By Lemma 5.36 again, $\gamma(\mathbf{x}') = \mathbf{y}$ is a solution of $P_2(n+1)$, as required. ■

Combining Lemmas 5.24, 5.34, 5.36 and 5.38 we derive correctness of the reduction (the constraint $n \geqslant 2d$ is inherited from the assumption in Lemma 5.34):

Corollary 5.39. *The following conditions are equivalent:*

- *$\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ has a finite solution,*

- *$P_2(n)$ has a solution for some integer $n \geqslant 2d$,*
- *$P_2(n)$ has a solution for almost all $n \in \mathbb{N}$.*

To finish the reduction of FIN-INEQ($\mathbb{R}$) to ALL-POLY-INEQ it remains to argue that P_2 is computable from $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$. Computability of P_2 follows immediately from computability of P_1, which we focus now on:

Lemma 5.40. *The system P_1 is computable from $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$.*

Proof. It is enough to range over representations of $\{T\}$-orbits B_i and C_j of B and C, respectively (such representations are given by Lemma 5.21), and for each pair of such orbits proceed with computations outlined in the proof of Lemma 5.34, applied to an arbitrarily chosen representative $b_i \in B_i$. ∎

By Corollary 5.39 and Lemma 5.40, FIN-INEQ($\mathbb{R}$) reduces to ALL-POLY-INEQ.

5.5.4 Complexity

Concerning computational complexity, the number of $\{T\}$-orbits included in an equivariant orbit $\mathbb{A}^{(\ell)}$ is exponential in ℓ (cf. Lemma 5.21). That is why the size of P_2 may be exponential in atom dimension d of $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$. On the other hand, the size of P_2 is only polynomial (actually, linear) in the number of orbits included in $B \times C$. In consequence, for fixed atom dimension we get a polynomial-time reduction and hence, relying on Theorem 5.16, the decision procedure for FIN-INEQ($\mathbb{R}$) in PTIME. Without fixing atom dimension, we get an exponential-time reduction and hence the decision procedure is in EXPTIME.

The same complexity bounds apply to the algorithm for the optimisation problem presented in Chapter 6.

5.6 Almost always solvability in PTIME

In this section we prove Theorem 5.16. Consider a polynomially-parametrised inequality $\mathcal{E}$ of the form:

$$p_1(n) \cdot x_1 + \ldots + p_k(n) \cdot x_k \;\geqslant\; q(n). \tag{5.38}$$

Let d be the maximal degree of polynomials $p_1, \ldots, p_k, q$ appearing in $\mathcal{E}$. We call d the *degree* of $\mathcal{E}$, and denote it also as $\deg(\mathcal{E})$. Let $a_1, \ldots, a_k, b$ be the (integer) coefficients of the monomial n^d in $p_1, \ldots, p_k, q$, respectively. Therefore

$$\begin{aligned} p_1(n) &= a_1 \cdot n^d + p'_1(n) \\ &\vdots \\ p_k(n) &= a_k \cdot n^d + p'_k(n) \\ q(n) &= b \cdot n^d + q'(n) \end{aligned} \tag{5.39}$$

for some polynomials $p'_1, \ldots, p'_k, q$ of degree strictly smaller than d. The ordinary inequality with integer coefficients

$$a_1 \cdot x_1 + \ldots + a_k \cdot x_k \geqslant b, \tag{5.40}$$

we call the *head inequality* of $\mathcal{E}$, and denote by $\text{HEAD}(\mathcal{E})$. Furthermore, the polynomially-parametrised inequality

$$p'_1(n) \cdot x_1 + \ldots + p'_k(n) \cdot x_k \;\geqslant\; q'(n), \tag{5.41}$$

obtained by removing all appearances of the monomial n^d, we call the *tail inequality* of $\mathcal{E}$, and denote it by $\text{TAIL}(\mathcal{E})$. We also consider below the strict strengthening of the head inequality (5.40), denoted as $\text{HEAD}_{>}(\mathcal{E})$, and the equality, denoted as $\text{HEAD}_{=}(\mathcal{E})$:

$$a_1 \cdot x_1 + \ldots + a_k \cdot x_k > b \quad , \qquad a_1 \cdot x_1 + \ldots + a_k \cdot x_k = b. \tag{5.42}$$

As $\mathcal{E}$ is equal to the sum of its head $\text{HEAD}(\mathcal{E})$ multiplied by n^d, and its tail $\text{TAIL}(\mathcal{E})$, we immediately deduce:

Claim 5.40.1. *For every $n \in \mathbb{N}$, every solution of $\text{HEAD}_{=}(\mathcal{E})$ is either a solution of both $\mathcal{E}(n)$ and $\text{TAIL}(\mathcal{E})(n)$, or of none of them.*

We now provide under- and over-approximations of the solution set of $\mathcal{E}$ (in Claims 5.40.2 and 5.40.3).

Claim 5.40.2. *Every almost-all-solution of $\mathcal{E}$ is also a solution of $\text{HEAD}(\mathcal{E})$.*

Proof. Consider an inequality $\mathcal{E}$ (5.38) and any its almost-all-solution $\mathbf{x} = (x_1, \ldots, x_k)$. Let $d = \deg(\mathcal{E})$. We thus have

$$\frac{p_1(n)}{n^d} \cdot x_1 + \ldots + \frac{p_k(n)}{n^d} \cdot x_k \;\geqslant\; \frac{q(n)}{n^d}$$

for all sufficiently large $n \in \mathbb{N}$. Using the decomposition (5.39), we rewrite the above inequality to

$$\left(a_1 + \frac{p'_1(n)}{n^d}\right) \cdot x_1 + \ldots + \left(a_k + \frac{p'_k(n)}{n^d}\right) \cdot x_k \;\geqslant\; b + \frac{q'(n)}{n^d}.$$

As the degrees of all polynomials $p'_1, \ldots, p'_k, q'$ are smaller than d, all the fractions tend to 0 when n tends to ∞, and we may deduce

$$a_1 \cdot x_1 + \ldots + a_k \cdot x_k \;\geqslant\; b,$$

i.e., $\mathbf{x}$ is a solution of $\mathrm{HEAD}(\mathcal{E})$, as required. □

Claim 5.40.3. *Every solution of* $\mathrm{HEAD}_{>}(\mathcal{E})$ *is also an almost-all-solution of* $\mathcal{E}$.

Proof. Let $d = \deg(\mathcal{E})$. Consider any vector $\mathbf{x} = (x_1, \ldots, x_k)$ satisfying the strict inequality $\mathrm{HEAD}_{>}(\mathcal{E})$ (in (5.42) on the left). Therefore for any polynomials $p'_1, \ldots, p'_k, q'$ of degree strictly smaller than d, the inequality

$$\left(a_1 + \frac{p'_1(n)}{n^d}\right) \cdot x_1 + \ldots + \left(a_k + \frac{p'_k(n)}{n^d}\right) \cdot x_k > b + \frac{q'(n)}{n^d}$$

is satisfied for all sufficiently large $n \in \mathbb{N}$. Applying the above inequality to polynomials appearing in (5.39), we obtain:

$$\frac{p_1(n)}{n^d} \cdot x_1 + \ldots + \frac{p_k(n)}{n^d} \cdot x_k \;>\; \frac{q(n)}{n^d}$$

for all sufficiently large $n \in \mathbb{N}$. We multiply both sides by n^d in order to derive that $\mathbf{x}$ is a solution of $\mathcal{E}(n)$ for all sufficiently large $n \in \mathbb{N}$, as required. □

Consider an instance P of POLY-INEQ, i.e., a finite system of polynomially-parametrised inequalities of the form (5.38). Let

$$\mathrm{HEAD}(P) \stackrel{\text{def}}{=} \{\mathrm{HEAD}(\mathcal{E}) \;:\; \mathcal{E} \in P\}$$

be the system of head inequalities (note that degrees of different inequalities in P may differ), and let $\mathrm{HEAD}_{>}(P) \stackrel{\text{def}}{=} \{\mathrm{HEAD}_{>}(\mathcal{E}) \;:\; \mathcal{E} \in P\}$. Using Claims 5.40.2 and 5.40.3 we derive:

Claim 5.40.4. *Every almost-all-solution of P is also a solution of* $\mathrm{HEAD}(P)$.

Claim 5.40.5. *Every solution of* $\mathrm{HEAD}_{>}(P)$ *is also an almost-all-solution of P.*

For time estimation, as the size measure $|\mathcal{E}|$ of an inequality $\mathcal{E}$ we take the total number of monomials appearing in $\mathcal{E}$. In particular, $|\mathcal{E}| > |\mathrm{TAIL}(\mathcal{E})|$. The size of a system P is the sum of sizes of all its inequalities. For two systems P',

P'' of inequalities, we denote their union by $P' \cup P''$ (clearly, union of systems corresponds to conjunction of constraints). We write $P \cup \mathcal{E}$ instead of $P \cup \{\mathcal{E}\}$. By $P \setminus \mathcal{E}$ we denote the system obtained from P by removing an inequality $\mathcal{E}$.

The algorithm. A decision procedure for ALL-POLY-INEQ iteratively transforms an instance of the form $P \cup \Gamma$, where P is a system of polynomially-parametrised inequalities, and Γ is a system of ordinary (non-parametrised) equalities over the same unknowns (each equality can be represented as two opposite inequalities). Initially, Γ is empty. We define a transformation step that given such an instance $P \cup \Gamma$, either confirms its solvability (existence of an almost-all-solution), or confirms its non-solvability (non-existence of an almost-all-solution), or outputs an instance $P' \cup \Gamma'$ which has the same almost-all-solutions as $P \cup \Gamma$, and such that $|P'| < |P|$. ALL-POLY-INEQ is solved by iterating the transformation step until it confirms either solvability or non-solvability. Termination after a polynomial number of iterations is guaranteed, as $|P|$, while being nonnegative, strictly decreases in each iteration. The transformation step invokes a PTIME procedure for ordinary linear programming (as detailed in (5.43) and (5.44) below). Here is a pseudo-code of the algorithm:

Algorithm 1 (ALL-POLY-INEQ)

Input: A monotonic polynomially-parametrised system P.
$\Gamma \leftarrow \emptyset$
repeat
    if $\textsc{head}(P) \cup \Gamma$ (5.43) is non-solvable **then** ▷ see Claim 5.40.4
        report non-solvability of P
    else
        if $\textsc{head}_>(\mathcal{E}) \cup \textsc{head}(P \setminus \mathcal{E}) \cup \Gamma$ (5.44) is solvable for all $\mathcal{E} \in P$ **then** ▷ see Claims 5.40.5 and 5.40.6
            report solvability of P
        else
            choose any $\mathcal{E} \in P$ **such that** $\textsc{head}_>(\mathcal{E}) \cup \textsc{head}(P \setminus \mathcal{E}) \cup \Gamma$ (5.44) is non-solvable ▷ see Claim 5.40.7
            $P \leftarrow (P \setminus \mathcal{E}) \cup \textsc{tail}(\mathcal{E})$
            $\Gamma \leftarrow \Gamma \cup \textsc{head}_=(\mathcal{E})$
until solvability or non-solvability of P is reported

Transformation step. The step, defined by the body of the **repeat** loop, proceeds as follows. If the ordinary system

$$\textsc{head}(P) \cup \Gamma \tag{5.43}$$

is non-solvable, non-solvability of $P \cup \Gamma$ is reported. This is correct due to Claim 5.40.4. Otherwise, knowing that (5.43) is solvable, the algorithm checks, for every $\mathcal{E} \in P$, whether the strengthened system

$$\text{HEAD}_{>}(\mathcal{E}) \;\cup\; \text{HEAD}(P \setminus \mathcal{E}) \;\cup\; \Gamma, \tag{5.44}$$

obtained from (5.43) by replacing the inequality $\text{HEAD}(\mathcal{E})$ with $\text{HEAD}_{>}(\mathcal{E})$, is also solvable. If this is the case for every $\mathcal{E} \in P$, solvability of $P \cup \Gamma$ is reported. This is correct due to Claim 5.40.5 combined with the following one:

Claim 5.40.6. *Solvability of* (5.44) *for every inequality $\mathcal{E}$ in P, implies solvability of*

$$\text{HEAD}_{>}(P) \;\cup\; \Gamma. \tag{5.45}$$

Proof. Suppose that for every inequality $\mathcal{E}$ in P, the system (5.44) has a solution, $\mathbf{x}_{\mathcal{E}}$. The average of all these solutions $\frac{1}{m} \cdot \sum_{\mathcal{E} \in P} \mathbf{x}_{\mathcal{E}}$ is then a solution of $\text{HEAD}_{>}(P) \;\cup\; \Gamma$, where m is the number of inequalities in P. □

Otherwise, we know that some inequality $\mathcal{E}$ in P is *degenerate*, namely (5.44) is non-solvable. In other words, the equality $\text{HEAD}_{=}(\mathcal{E})$ is implied by (5.43). The algorithm chooses a degenerate inequality $\mathcal{E} \in P$ and creates a new instance $P' \cup \Gamma'$, where

$$P' \;=\; (P \setminus \mathcal{E}) \;\cup\; \text{TAIL}(\mathcal{E}) \qquad\qquad \Gamma' \;=\; \Gamma \;\cup\; \text{HEAD}_{=}(\mathcal{E}).$$

In words, P' is obtained from P by replacing $\mathcal{E}$ with $\text{TAIL}(\mathcal{E})$, and Γ' is obtained from Γ by adding $\text{HEAD}_{=}(\mathcal{E})$. As $|\text{TAIL}(\mathcal{E})| < |\mathcal{E}|$, we have $|P'| < |P|$, as required. This completes description of the transformation step.

Correctness. By Claim 5.40.1 we derive:

Claim 5.40.7. *Systems $P \cup \Gamma$ and $P' \cup \Gamma'$ have the same almost-all-solutions.*

Proof. In one direction, consider an almost-all-solution $\mathbf{x}$ of $P' \cup \Gamma'$. It is trivially a solution of Γ. Furthermore, being a solution of $\text{HEAD}_{=}(\mathcal{E})$ and of $\text{TAIL}(\mathcal{E})(n)$ for almost all $n \in \mathbb{N}$, by Claim 5.40.1 it is a solution of $\mathcal{E}(n)$ for almost all n, and hence an almost-all-solution of P.

Conversely, consider an almost-all-solution $\mathbf{x}$ of $P \cup \Gamma$. By Claim 5.40.4, it is a solution of $\text{HEAD}(P) \cup \Gamma$ and hence, as $\mathcal{E}$ is degenerate, also a solution of $\text{HEAD}_{=}(\mathcal{E})$. Therefore $\mathbf{x}$ is a solution of Γ'. Furthermore, being a solution of $\text{HEAD}_{=}(\mathcal{E})$ and of $\mathcal{E}(n)$ for all sufficiently large $n \in \mathbb{N}$, by Claim 5.40.1 it is also a solution of $\text{TAIL}(\mathcal{E})(n)$ for all sufficiently large $n \in \mathbb{N}$, and hence an almost-all-solution of P'. □

Complexity. We note that the main loop of the algorithm always terminates, at latest when $P = \emptyset$, as in this case the system (5.44) is vacuously solvable for all $\mathcal{E} \in P$. Solvability of (5.43) in line 4 is checked by one solvability test of an ordinary system of inequalities. Solvability of (5.44) in line 7 is also checkable in polynomial time due to the following claim applied to $Q = \textsc{head}(P) \cup \Gamma$:

Claim 5.40.8. *Given an ordinary system Q of linear inequalities and $\mathcal{E} \in Q$, one can check, in* PTIME, *solvability of $\mathcal{E}_> \cup (Q \setminus \mathcal{E})$, where $\mathcal{E}_>$ is the strict strengthening of $\mathcal{E}$.*

Proof. We invoke ordinary linear programming twice (in PTIME, see e.g. [29, Section 8.7]). Let $\mathcal{E}$ be of the form $a_1 \cdot x_1 + \ldots + a_k \cdot x_k \geqslant b$. If Q is non-solvable, the algorithm reports non-solvability of $\mathcal{E}_> \cup (Q \setminus \mathcal{E})$. Otherwise, the algorithm computes the supremum $M \in \mathbb{Q} \cup \{\infty\}$ of the objective function

$$S(x_1, \ldots, x_k) = a_1 \cdot x_1 + \ldots + a_k \cdot x_k,$$

constraint by $Q \setminus \mathcal{E}$, by invoking ordinary linear programming. By solvability of Q we know that $M \geqslant b$. If $M > b$, the algorithm reports solvability, otherwise it reports non-solvability. □

Number of iterations of transformation step is polynomial (as $|P|$ decreases in each iteration) and hence so is the number of inequalities in Γ. In consequence, the number of invocations of ordinary linear programming is polynomial in each transformation step, and hence polynomial in total, and each its instance of ordinary linear programming is also polynomial. Summing up, our decision procedure for ALL-POLY-INEQ works in PTIME.

The proof of Theorem 5.16 is thus completed. ■

Example 5.41. Recall the system (5.19) presented in Example 5.13. It contains two polynomially-parametrised inequalities. They have the same head inequality $x \geqslant 1$, which is trivially solvable, and hence the algorithm reports solvability after the first iteration. Both the tail inequalities, $-y \geqslant 1$ and $0 \geqslant 1$, are ordinary (non-parametrised).

The following instance P_0 admits three iterations of the main loop of the algorithm:

$$\begin{array}{rcl} n^2 \cdot x \;-\; n^2 \cdot y \;+\; n \cdot z & \geqslant & 0 \\ -n \cdot x \;+\; (n+3) \cdot y \qquad & \geqslant & 0 \end{array}$$

The head inequalities of these two inequalities are $x - y \geqslant 0$ and $-x + y \geqslant 0$, respectively. Therefore the system $\textsc{head}(P_0)$ is equivalent to $x = y$ and hence solvable, while $\textsc{head}_>(P_0)$ is not, and both inequalities in P_0 are degenerate. Supposing the first one is chosen by the algorithm, after the first iteration we get the following systems P_1 (left) and Γ_1 (right):

$$\begin{aligned} n \cdot z &\geqslant 0 \\ -n \cdot x + (n+3) \cdot y &\geqslant 0 \end{aligned} \qquad\qquad x - y = 0$$

In the second iteration, the system $\textsc{head}(P_1) \cup \Gamma_1$ (left) is solvable but the system $\textsc{head}_>(P_1) \cup \Gamma_1$ (right) is not:

$$\begin{aligned} z &\geqslant 0 \\ -x + y &\geqslant 0 \\ x - y &= 0 \end{aligned} \qquad\qquad \begin{aligned} z &> 0 \\ -x + y &> 0 \\ x - y &= 0 \end{aligned}$$

The algorithm picks up the second inequality in P_1, the only degenerate one, and sets P_2 (left) and Γ_2 (right):

$$\begin{aligned} n \cdot z &\geqslant 0 \\ 3 \cdot y &\geqslant 0 \end{aligned} \qquad\qquad \begin{aligned} x - y &= 0 \\ -x + y &= 0 \end{aligned}$$

In the last third iteration, the system $\textsc{head}_>(P_2) \cup \Gamma_2$ (obtained by replacing the inequality $n \cdot z \geqslant 0$ by $z > 0$) is solvable, and hence solvability of P_0 is reported. ◀

Chapter 6

Linear Programming

Contents

6.1 Introduction

Let B and C be two arbitrary orbit-finite sets. Let $\mathbf{A}$ be an arbitrary matrix in $\mathrm{Lin}(B \times C)$, $\mathbf{b}$ be an arbitrary vector in $\mathrm{Lin}(B)$ and $\mathbf{c}$ be an arbitrary vector in $\mathrm{Lin}(C)$. Using $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ we can define an optimisation problem

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \in \mathrm{Lin}(C) \end{array} \tag{6.1}$$

which asks to maximise $\mathbf{c}^{\mathsf{T}} \cdot \mathbf{x}$ for vectors $\mathbf{x} \in \mathrm{Lin}(C)$, for which the products $\mathbf{c}^{\mathsf{T}} \cdot \mathbf{x}$ and $\mathbf{A} \cdot \mathbf{x}$ are well-defined and $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$. A problem such as this is an instance of an *orbit-finite linear maximisation problem*. The orbit-finite partial linear functional $\mathbf{x} \mapsto \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x}$ is called its *objective function*. A vector $\mathbf{x} \in \mathrm{Lin}(C)$ is called a *solution* of this problem if the objective function is well-defined at $\mathbf{x}$ and it satisfies the orbit-finite system of inequalities $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$. The *value of a solution* $\mathbf{x}$ is the number $(\mathbf{c}^{\mathsf{T}} \cdot \mathbf{x})$. The *supremum* of the maximisation problem is the supremum of values of its solutions.

The *support of the linear maximisation problem* is

$$\mathrm{support}(\mathbf{A}) \cup \mathrm{support}(\mathbf{b}) \cup \mathrm{support}(\mathbf{c}) \ .$$

The *atom-dimension of the maximisation problem* is the maximum of atom-dimension of $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$.

Symmetrically, a problem such as the following is an instance of an *orbit-finite linear minimisation problem*

$$\begin{array}{ll} \text{minimise} & \mathbf{c}^\mathsf{T} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \geqslant \mathbf{b} \\ & \mathbf{x} \in \operatorname{Lin}(C) \end{array} \tag{6.2}$$

which asks to compute the infimum of values $\mathbf{c}^\mathsf{T} \cdot \mathbf{x}$ for vectors $\mathbf{x} \in \operatorname{Lin}(C)$, for which the products $\mathbf{c}^\mathsf{T} \cdot \mathbf{x}$ and $\mathbf{A} \cdot \mathbf{x}$ are well-defined and $\mathbf{A} \cdot \mathbf{x} \geqslant \mathbf{b}$. The objective function, solutions, infimum, support, and atom-dimension of an orbit-finite minimisation problem are defined accordingly. Instances of orbit-finite maximisation or minimisation problems are known as *orbit-finite linear programs*. The *optimum* of such a problem is its supremum or infimum depending on whether it is a maximisation or minimisation problem. A solution $\mathbf{x}$ is called *optimal* if its value is equal to the optimum of the linear program. Sometimes we restrict the solution of the linear program to be finite instead of orbit-finite.

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^\mathsf{T} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \in \operatorname{FinLin}(C) \end{array} \qquad \begin{array}{ll} \text{minimise} & \mathbf{c}^\mathsf{T} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \geqslant \mathbf{b} \\ & \mathbf{x} \in \operatorname{FinLin}(C) \end{array}$$

Orbit-finite linear programs of this form are said to be of the *finitary variant*.

Remark 6.1. An orbit-finite linear minimisation problem

$$\begin{array}{ll} \text{minimise} & \mathbf{c}^\mathsf{T} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \geqslant \mathbf{b} \\ & \mathbf{x} \in \operatorname{Lin}(C) \end{array} \tag{6.3}$$

is equivalent to the orbit-finite linear maximisation problem

$$\begin{array}{ll} \text{maximise} & (-\mathbf{c})^\mathsf{T} \cdot \mathbf{x} \\ \text{subject to} & (-\mathbf{A}) \cdot \mathbf{x} \leqslant (-\mathbf{b}) \\ & \mathbf{x} \in \operatorname{Lin}(C) \end{array} \tag{6.4}$$

in the sense that, $\mathbf{x}$ is a solution of (6.3) with value $\mathbf{c}^\mathsf{T} \cdot \mathbf{x} = r$ (say), if and only if $\mathbf{x}$ is a solution of (6.4) with value $-r = (-\mathbf{c})^\mathsf{T} \cdot \mathbf{x}$. This in particular implies that if the optimum of (6.3) is R, then the optimum of (6.4) is $-R$. And if (6.3) has an optimal solution, then (6.4) also has an optimal solution.

Remark 6.2. Sometimes we add an explicit non-negativity constraint to our linear programs which leads to linear programs of the form:

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \\ & \mathbf{x} \in \mathrm{Lin}(C) \end{array}$$

Linear programs of the above form are as general as the linear programs of the form (6.1), since the linear program (6.1) is equivalent to the linear program

$$\begin{array}{ll} \text{maximise} & \left[\begin{array}{c|c} \mathbf{c}^{\mathsf{T}} & -\mathbf{c}^{\mathsf{T}} \end{array}\right] \cdot \mathbf{x}' \\ \text{subject to} & \left[\begin{array}{c|c} \mathbf{A} & -\mathbf{A} \end{array}\right] \cdot \mathbf{x}' \leqslant \mathbf{b} \\ & \mathbf{x}' \geqslant \mathbf{0} \\ & \mathbf{x}' \in \mathrm{Lin}(C \uplus C) \quad . \end{array}$$

Remark 6.3. An instance

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \\ & \mathbf{x} \in \mathrm{FinLin}(C) \end{array}$$

of an orbit-finite maximisation problem of the finitary variant can be considered as the instance

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x} \\ \text{subject to} & \left[\begin{array}{c} \mathbf{A} \\ \hline -\mathbf{1}^{\mathsf{T}} \end{array}\right] \cdot \mathbf{x} \leqslant \left[\begin{array}{c} \mathbf{b} \\ \hline 0 \end{array}\right] \\ & \mathbf{x} \geqslant \mathbf{0} \\ & \mathbf{x} \in \mathrm{Lin}(C) \end{array}$$

of an orbit-finite maximisation problem, because the added constraint

$$(-\mathbf{1}^{\mathsf{T}}) \cdot \mathbf{x} \leqslant \mathbf{0}$$

enforces the solution to be finite and is satisfied for any non-negative finite vector.

Orbit-finite linear programming faces phenomena not present in the classical setting. For instance, in classical linear programming whenever the optimum is finite, there exists an optimal solution (i.e. a solution with value equal to the optimum). This does not always happen in orbit-finite linear programming, as the following example illustrates.

Example 6.4. Suppose that we aim at minimising the objective function

$$2 \cdot \sum_{\alpha \in \mathbb{A}} \mathbf{x}(\alpha). \tag{6.5}$$

with respect to the constraints (5.2). The function is equivariant, and its value is always greater than 2. Indeed, for every solution $\mathbf{x} : \mathbb{A} \to \mathbb{R}$ there is necessarily some $\beta \in \mathbb{A}$ such that $\mathbf{x}(\beta) > 0$, and hence

$$2 \cdot \sum_{\alpha \in \mathbb{A}} \mathbf{x}(\alpha) \; > \; \left(2 \cdot \sum_{\alpha \in \mathbb{A} \setminus \{\beta\}} \mathbf{x}(\alpha) \right) \geqslant 2. \tag{6.6}$$

What is the minimal value of the objective function? For solutions $\mathbf{x}_n$ defined in Example 5.7, the value $2 \cdot \sum_{\alpha \in \mathbb{A}} \mathbf{x}_n(\alpha) = \frac{2n}{n-1}$ may be arbitrarily close to 2 but, according to (6.6), never achieves 2. ◀

We are interested in the following computational problems.

LIN-PROG
Input: An orbit-finite linear program.
Output: Optimum of the linear program.

FINLIN-PROG
Input: An orbit-finite linear program of the finitary variant.
Output: Optimum of the linear program.

The main results in this chapter are the following.

Theorem 6.5. LIN-PROG *is reducible to* FLIN-PROG *in* EXPTIME *and in* PTIME *for fixed atom-dimension.*

Theorem 6.6. FINLIN-PROG *is computable in* EXPTIME *and in* PTIME *for fixed atom-dimension.*

As a corollary we get:

Theorem 6.7. LIN-PROG *is computable in* EXPTIME *and in* PTIME *for fixed atom-dimension.*

Proof of Theorem 6.5. We do the proof only for maximising variants. Consider an orbit-finite linear program

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \in \mathrm{Lin}(C) \end{array} \tag{6.7}$$

Let $\mathbf{M}$ be the augmented matrix

$$\mathbf{M} \stackrel{\text{def}}{=} \left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{c} \end{array}\right]$$

Then $\widetilde{\mathbf{M}}$ (defined in § 4.2.3) can also be written as an augmented matrix

$$\widetilde{\mathbf{M}} = \left[\begin{array}{c} \mathbf{A}' \\ \hline \mathbf{c}' \end{array}\right]$$

Lemma 4.9 says that

$$\mathrm{S{\small PAN}}(\mathbf{M}) = \mathrm{F{\small IN}\text{-}S{\small PAN}}(\widetilde{\mathbf{M}}). \tag{6.8}$$

Using $\mathbf{A}'$ and $\mathbf{c}'$ we define the orbit-finite linear program

$$\begin{array}{ll} \text{maximise} & \mathbf{c}'^{\mathsf{T}} \cdot \mathbf{x}' \\ \text{subject to} & \mathbf{A}' \cdot \mathbf{x}' \leqslant \mathbf{b} \\ & \mathbf{x}' \in \mathrm{FinLin}(C') \end{array} \tag{6.9}$$

By the equality (6.8), for every $r \in \mathbb{R}$ we have the following: there exists an orbit-finite vector $\mathbf{x}$ such that $\mathbf{A} \cdot \mathbf{x} \geqslant \mathbf{b}$ and $\mathbf{c} \cdot \mathbf{x} = r$ if and only if there exists a finite vector $\mathbf{x}'$ such that $\mathbf{A}' \cdot \mathbf{x}' \geqslant \mathbf{b}$ and $\mathbf{c}' \cdot \mathbf{x}' = r$. In consequence, the systems (6.7) and (6.9) have the same supremum. Theorem 6.5 is thus proved. ∎

Remark 6.8. Since the existence of integer solutions of orbit-finite systems of inequalities is undecidable we do not consider orbit-finite integer linear programming. However, existence of finite non-negative integer solutions of orbit-finite systems of inequalities is still decidable, hence solvability of linear programs of the form

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} = \mathbf{b} \\ & \mathbf{x} \in \mathrm{Lin}_{\mathbb{N}}(C) \end{array}$$

Is a valid question, which we leave as open. Here, $\mathrm{Lin}_{\mathbb{N}}(C)$ denotes the set of orbit-finite functions from C to $\mathbb{N}$.

Organisation of the chapter

The remainder of this chapter is organised as follows, In § 6.2 and we define *polynomially parametrised linear programs* and show they are solvable in PTIME. Then in § 6.3 we show FINLIN-PROG is reducible to solvability of polynomially parametrised systems of linear programming, proving linear programming is also computable (Theorems 6.6 and 6.7).

6.2 Polynomially-parametrised linear programs

A *polynomially parametrised maximisation problem*

$$\begin{array}{ll} \text{maximise} & S(x_1, \ldots, x_k) \\ \text{subject to} & P(n, x_1, \ldots, x_n) \end{array}$$

asks to maximise an *ordinary* (non-parametrised) objective function S given by a linear map

$$S(x_1, \ldots, x_k) = a_1 \cdot x_1 + \ldots + a_k \cdot x_k.$$

with respect to a polynomially parametrised system of constraints $P(n, x_1, \ldots, x_n)$ with n being the only non-linear variable. As in § 5.3, by an *almost-all-solution* of a system P we mean in this section a solution of $P(n)$ for almost all $n \in \mathbb{N}$. The *supremum* of such an instance, denoted as $\sup(P, S)$ is defined to be the supremum of the objective function over almost-all-solutions,

$$\sup(P, S) \stackrel{\text{def}}{=} \sup \{S(\mathbf{x}) \ : \ \mathbf{x} \text{ is an almost-all-solution of } P\},$$

Referring to the standard terminology, we can say that the system is *infeasible* if the supremum is $-\infty$, and it is *unbounded* if it is ∞.

Symmetrically we define *polynomially parametrised minimisation problems* and their *infimums* (denoted as $\inf(P, S)$). Instances of polynomially parametrised maximisation or minimisation problems are called *polynomially parametrised*

!linear programs. The *optimum* of such a problem is its supremum or infimum depending on whether it is a maximising or minimising instance. Interestingly, the optimum cannot be irrational (see Corollary 6.10 below). In this section we study the following problem:

ALL-POLY-OPT
Input: A polynomially parametrised optimisation problem.
Output: Its optimum.

The problem generalises ordinary (non-parametrised) linear programming, and can be solved similarly to POLY-INEQ (of which it is a strengthening):

Theorem 6.9. ALL-POLY-OPT *is in* PTIME.

Proof. Let (P_0, S) be an instance. The algorithm is essentially the same as Algorithm 1 for POLY-INEQ in the proof of Theorem 5.16, and proceeds by iterating the transformation step until either unsolvability or solvability is reported. Recall that solution set is preserved by the transformation step (Claim 5.40.7). If unsolvability is reported, the algorithm returns $-\infty$. If solvability is reported — let $P \cup \Gamma$ be the system examined in the last iteration — the decision procedure computes and returns $\sup(\text{HEAD}(P) \cup \Gamma, S)$, the supremum of S constrained by the ordinary system of inequalities $\text{HEAD}(P) \cup \Gamma$, by invoking any PTIME procedure for ordinary linear programming. Correctness follows by the two claims formulated below. First, since solution set is preserved by the transformation step, we have:

Claim 6.9.1. $\sup(P_0, S) = \sup(P \cup \Gamma, S)$.

Second, the supremum does not change if the polynomially parametrised constraints P are replaced by the over-approximation $\text{HEAD}(P)$:

Claim 6.9.2. $\sup(P \cup \Gamma, S) = \sup(\text{HEAD}(P) \cup \Gamma, S)$.

For the claim it is enough to prove the inequality

$$\sup(\text{HEAD}_{>}(P) \cup \Gamma, S) \geqslant \sup(\text{HEAD}(P) \cup \Gamma, S)$$

as, according to Claims 5.40.4 and 5.40.5, we have $\sup(\text{HEAD}_{>}(P) \cup \Gamma, S) \leqslant \sup(P \cup \Gamma, S) \leqslant \sup(\text{HEAD}(P) \cup \Gamma, S)$. Take any solution $\mathbf{y}$ of $\text{HEAD}(P) \cup \Gamma$, and any solution $\mathbf{x}$ of $\text{HEAD}_{>}(P) \cup \Gamma$ (we rely here on solvability of the latter system). For every $k \in \mathbb{N}$, the vector $\mathbf{x}_k = \frac{\mathbf{x}+k\mathbf{y}}{k+1}$ is a solution of $\text{HEAD}_{>}(P) \cup \Gamma$, and $S(\mathbf{x}_k)$ tends to $S(\mathbf{y})$ when k tends to ∞. Hence $\sup(\text{HEAD}_{>}(P) \cup \Gamma, S) \geqslant \sup(\text{HEAD}(P) \cup \Gamma, S)$, as required. ∎

By Claims 6.9.1 and 6.9.2 in the proof of Theorem 6.9, the optimum of a polynomially parametrised instance, if not $-\infty$ nor ∞, is equal to the optimum of an ordinary linear program and hence is rational:

Corollary 6.10. *The optimum of a polynomially parametrised instance belongs to* $\mathbb{Q} \cup \{-\infty, +\infty\}$.

Remark 6.11. The next example illustrates that like orbit-finite linear programs, polynomially parametrised linear programs also may not admit optimal solutions even when its optimum is finite. However, once the optimum $s \in \mathbb{Q}$ is computed, one can easily check if S achieves its supremum, by adding to the system an equation $S(x_1, \ldots, x_k) = s$ and checking if the system is still solvable.

Example 6.12. Consider the polynomially parametrised linear program

$$\begin{array}{ll} \text{minimise} & 2\text{x} \\[1em] \text{subject to} & \begin{bmatrix} n-1 \\ n \end{bmatrix} \cdot x \geqslant \begin{bmatrix} n \\ n \end{bmatrix} \end{array}$$

The value of any solution is always greater than 2. Indeed, for every solution $y \in \mathbb{R}$ there exits $n \in \mathbb{N}$ such that $2 \cdot y \geqslant \frac{2 \cdot n}{n-1} > 2$. The infimum of this program is 2. Since $y_n = \frac{n}{n-1}$ is an almost-all-solution for any $n \geqslant 2$ with $S(y_n) = \frac{2 \cdot n}{n-1}$, which may be arbitrarily close to 2. ◀

6.3 Orbit-finite to polynomially parametrised

In this section we prove Theorem 6.6 by a reduction of FINLIN-PROG to ALL-POLY-OPT. We only sketch the reduction as it amounts to a slight adaptation of the reduction of § 5.5.3. Remark 6.1 says it is enough to start with a maximisation variant. Consider a orbit-finite maximisation problem

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \\ & \mathbf{x} \in \mathrm{FinLin}(C) \end{array} \tag{6.10}$$

Lemmas 4.19 and 5.28 can easily be extended to linear programming as well.

Lemma 6.13. WLOG *we may assume that* $\mathbf{A}$, $\mathbf{b}$ *and* $\mathbf{c}$ *to be equivariant and* B *and* C *to be straight.*

Proof. In the proof of Lemma 4.19 take $\mathbf{c}' = \mathbf{c} \circ g$ in step 1. ∎

We compute an instance

$$\begin{array}{ll} \text{maximise} & S'(x_1, \ldots, x_k) \\ \text{subject to} & P_2(n, x_1, \ldots, x_k) \end{array}$$

of a polynomially parametrised linear program, where the finite system $P_2(n) = (\mathbf{A}_2(n), \mathbf{t}_2(n))$ of polynomially-parametrised inequalities is exactly as in § 5.5.3, and the objective function is

$$S'(x_1, \ldots, x_k) = a_1 \cdot x_1 + \ldots + a_M \cdot x_M, \tag{6.11}$$

where $a_j = \mathbf{c}(C_j)$ for $j = 1 \ldots M$. More concisely, the vector $\mathbf{a} = (a_1, \ldots, a_M)$ is defined as $\mathbf{a} = \dot{\mathbf{c}}$ (recall Notation 5.32). We apply Lemmas 5.24 and 5.36 to obtain:

Lemma 6.14. *Supremum of* (6.10) *is equal to* $\sup(P_2, S')$.

Proof. Let $\mathbf{x} \in \mathrm{Lin}(C)$. By equivariance of $\mathbf{c}$ and the definition of S' we have the equality $\mathbf{c}^\top \cdot \mathbf{x} = S'(\gamma(\mathbf{x}))$, that is, the value of the objective function $\mathbf{c}^\top \cdot \mathbf{x}$ depends only on the image of $\mathbf{x}$ under the orbit-sum function $\gamma(\mathbf{x}) : \mathrm{Orbits}(C) \to \mathbb{R}$. As Lemmas 5.24 and 5.36 preserve orbit-sum, we deduce that for every $T \subseteq_{\mathrm{FIN}} \mathbb{A}$ of size $|T| = n \geqslant 2d$, the values $\mathbf{c}^\top \cdot \mathbf{x}'$ for finite T-supported solutions $\mathbf{x}'$ of $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ are the same as the values of S' on solutions of $P_2(n)$. By Lemma 5.38, the solutions of $P_2(n)$ for some $n \geqslant 2d$ are exactly the same as the almost-all-solutions of P_2. In consequence, the two suprema are equal. ∎

We illustrate the reduction with the following two examples.

Example 6.15. The system in Example 6.4 is transformed to the system in Example 6.12. ◀

Example 6.16. Consider the modification of the system in Example 5.8:

$$\sum_{\alpha \in \mathbb{A}} \alpha \ \geqslant\ 1 \tag{6.12}$$

$$\sum_{\beta \in \mathbb{A}} \alpha\beta \ -\ \alpha \ -\ 2 \cdot \sum_{\beta \in \mathbb{A}} \beta\alpha \ \geqslant\ 0 \qquad (\alpha \in \mathbb{A}). \tag{6.13}$$

It enforces, for each vertex $\alpha \in \mathbb{A}$, the sum of values assigned to all outgoing edges to be larger than *double* the sum of values assigned to all ingoing edges, plus the value assigned to the vertex α. The indexing sets $B = \mathbb{A} \cup \{*\}$ and $C = \mathbb{A} \cup \mathbb{A}^{(2)}$ and the shape of the matrix (5.5) are the same. We identify the singletons $\{*\} = \mathbb{A}^{(0)}$. We consider maximisation of triple the sum of values assigned to edges: $\mathbf{s}^\top \cdot \mathbf{x}$, where $\mathbf{s} = 3 \cdot \mathbf{1}_{\mathbb{A}^{(2)}}$. According to Lemma 5.29, the set C includes exactly 2 finite $\{T\}$-orbits, namely $T \subseteq \mathbb{A}$ and $T^{(2)} \subseteq \mathbb{A}^{(2)}$, and therefore the system computed by the reduction has 2 unknowns, x_1 and x_2. By Lemma 5.21, for any nonempty $T \subseteq_{\text{FIN}} \mathbb{A}$, the set B includes three $\{T\}$-orbits, namely T, $\mathbb{A} \setminus T$ and $\{*\}$, and therefore the system P_1 computed in the first step has 3 inequalities:

$$\begin{aligned} -x_1 \;-\; (n-1) \cdot x_2 \;&\geqslant\; 0 && (T) \\ 0 \;&\geqslant\; 0 && (\mathbb{A} \setminus T) \\ n \cdot x_1 \;&\geqslant\; 1 && (\{*\}) \end{aligned} \tag{6.14}$$

For instance, the coefficient $-(n-1)$ in the first inequality arises as:

$$\begin{aligned} &\mathbf{A}(U_{\text{out}}) \cdot |U_{\text{out}}(\alpha, T^{(2)})| \;+\; \mathbf{A}(U_{\text{in}}) \cdot |U_{\text{in}}(\alpha, T^{(2)})| \\ &= 1 \cdot (n-1) - 2 \cdot (n-1) \;=\; -(n-1) \end{aligned}$$

(cf. Claim 5.34.2), for some arbitrary $\alpha \in T$ and the following two orbits included in $\mathbb{A} \times \mathbb{A}^{(2)}$:

$$U_{\text{out}} = \{(\alpha, \alpha\beta) \,:\, \beta \neq \alpha\}\,, \;\; U_{\text{in}} = \{(\alpha, \beta\alpha) \,:\, \beta \neq \alpha\}\,.$$

Likewise, the coefficient n in the last inequality arises as $\mathbf{A}(U) \cdot |O(*, T)| = 1 \cdot n = n$, for the orbit $U = \{*\} \times \mathbb{A}$. According to (5.35), the system P_2 is obtained from (6.14) by multiplying all occurrences of x_1 by $(n-1)^{(1)} = n-1$, and by multiplying all right-hand sides by $n^{(2)} = n(n-1)$ [1] (the trivial second inequality is omitted):

$$\begin{aligned} -(n-1) \cdot x_1 \;-\; (n-1) \cdot x_2 \;&\geqslant\; 0 \\ n(n-1) \cdot x_1 \;&\geqslant\; n(n-1) \end{aligned} \tag{6.15}$$

Finally, the objective function produced by the reduction, as in (6.11), is $S'(x_1, x_2) = \mathbf{s}(\mathbb{A}^{(2)}) \cdot x_2 \;=\; 3 \cdot x_2$. It achieves -3 as its supremum, as the system (6.15) is equivalent to the ordinary system (its head):

$$x_1 \geqslant 1 \qquad\qquad x_2 \leqslant -x_1.$$

For every $n \geqslant 2$, the optimal solution $x_1 = 1$, $x_2 = -1$ corresponds, via the constructions of § 5.5.3, to a clique of n vertices where each vertex is assigned $\frac{1}{n}$, and each edge is assigned $-\frac{1}{n(n-1)}$. ◀

[1] Recall in Chapter 5 we defined $n^{(w)} = n \cdot (n-1) \cdot \ldots \cdot (n-w+1)$ for $w \leqslant n \in \mathbb{N}$.

Chapter 7

Duality in Linear Programming

Contents

7.1 Introduction

For orbit-finite sets B and C, and an orbit-finite $(B \times C)$-linear program

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \\ & \mathbf{x} \in \mathrm{Lin}(C) \end{array} \tag{7.1}$$

called the *primal*, its *dual* is defined to be the following $(C \times B)$-orbit-finite linear program

$$\begin{array}{ll} \text{minimise} & \mathbf{b}^{\mathsf{T}} \cdot \mathbf{y} \\ \text{subject to} & \mathbf{A}^{\mathsf{T}} \cdot \mathbf{y} \geqslant \mathbf{c} \\ & \mathbf{y} \geqslant \mathbf{0} \\ & \mathbf{y} \in \mathrm{Lin}(B) \end{array} \tag{7.2}$$

Every such pair (7.1)-(7.2) of linear programs is known as a *primal-dual pair*. Duals of finite linear programs are clearly finite themselves. The *weak duality theorem* states that the optimum of every finite linear program is dominated by the optimum of its dual. The *strong duality theorem* states that for every primal-dual pair of finite linear programs, if the optimum of one of the linear programs is finite, then the optimum of the other linear program is also finite and the two optima are equal. In this chapter we investigate whether these duality theorems can be extended to orbit-finite linear programs. We show that, weak duality holds for orbit-finite linear programs and strong duality does not. However, both weak and strong duality hold between two natural subclasses of orbit-finite linear programs, called *column-finite* and *row-finite* linear programs, respectively.

Organisation of the chapter

What about 7.4? I think something is wrong with this overview.

In § 7.2 we prove weak duality does not hold for orbit-finite linear programs. In § 7.3 we show strong duality does not hold for orbit-finite linear programs by constructing a counterexample. In § 7.5 we define column-finite and row-finite linear programs and state and prove duality between them (Theorem 7.12). This proof uses several facts which are demonstrated in §§ 7.7 to 7.10. Finally, in § 7.11 we answer the question raised in § 1.2.2: whether orbit-finite linear programs approximate large linear programs. In particular we show that orbit-finite linear programs in general do not, however both column-finite and row-finite linear programs do.

7.2 Weak duality

Theorem 7.1 (Weak Orbit-finite Duality)**.** *The optimum of any orbit-finite linear program is dominated by the optimum of its dual.*

Proof of Theorem 7.1

Let B and C be two arbitrary orbit-finite sets. Consider a primal-dual pair of orbit-finite linear programs

$$\begin{array}{llll}
\text{maximise} & \mathbf{c}^\top \cdot \mathbf{x} & \text{minimise} & \mathbf{b}^\top \cdot \mathbf{y} \\
\text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} & \text{subject to} & \mathbf{A}^\top \cdot \mathbf{y} \geqslant \mathbf{c} \\
& \mathbf{x} \geqslant \mathbf{0} & & \mathbf{y} \geqslant \mathbf{0} \\
& \mathbf{x} \in \mathrm{Lin}(C) & & \mathbf{y} \in \mathrm{Lin}(B)
\end{array}$$

where $\mathbf{A} \in \mathrm{Lin}(B \times C)$, $\mathbf{b} \in \mathrm{Lin}(B)$ and $\mathbf{c} \in \mathrm{Lin}(C)$. To prove that the optimum of the primal is dominated by the optimum of the dual, it is sufficient to show that for any pair of vectors $(\mathbf{x}, \mathbf{y}) \in \mathrm{Lin}(C) \times \mathrm{Lin}(B)$ such that:

1. $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{y} \geqslant \mathbf{0}$,
2. the products $\mathbf{c}^\top \cdot \mathbf{x}$, $\mathbf{A} \cdot \mathbf{x}$, $\mathbf{b}^\top \cdot \mathbf{y}$ and $\mathbf{A}^\top \cdot \mathbf{y}$ are all well defined, and
3. $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$ and $\mathbf{A}^\top \cdot \mathbf{y} \geqslant \mathbf{c}$,

we have $\mathbf{c}^\top \cdot \mathbf{x} \leqslant \mathbf{b}^\top \cdot \mathbf{y}$. Fix such a pair of vectors $(\mathbf{x}, \mathbf{y}) \in \mathrm{Lin}(C) \times \mathrm{Lin}(B)$. If both the products $\mathbf{y}^\top \cdot (\mathbf{A} \cdot \mathbf{x})$ and $(\mathbf{y}^\top \cdot \mathbf{A}) \cdot \mathbf{x}$ are well-defined then they are equal. In this case, we can replicate the proof of weak duality for finite linear programs ([35, page 435]) and conclude

$$\mathbf{c}^\top \cdot \mathbf{x} \leqslant (\mathbf{y}^\top \cdot \mathbf{A}) \cdot \mathbf{x} = \mathbf{y}^\top \cdot (\mathbf{A} \cdot \mathbf{x}) \leqslant \mathbf{y}^\top \cdot \mathbf{b} \,. \tag{7.3}$$

Unfortunately, the products $\mathbf{y}^\top \cdot (\mathbf{A} \cdot \mathbf{x})$ and $(\mathbf{y}^\top \cdot \mathbf{A}) \cdot \mathbf{x}$ may not be well-defined even in very simple cases (for example, consider $B = C = \mathbb{A}$, $\mathbf{A} = \mathrm{Id}_{\mathbb{A}}$, $\mathbf{b} = \mathbf{c} = \mathbf{0}$ and $\mathbf{x} = \mathbf{y} = \mathbf{1}_{\mathbb{A}}$). So we need to be more careful.

The matrix $\mathbf{A}$ and the vectors $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x}$ and $\mathbf{y}$ are all finitely supported. Let $S \subseteq \mathbb{A}$ be the union of their supports

$$S = \mathrm{support}\,(\mathbf{A}) \cup \mathrm{support}\,(\mathbf{b}) \cup \mathrm{support}\,(\mathbf{c}) \cup \mathrm{support}\,(\mathbf{x}) \cup \mathrm{support}\,(\mathbf{y}) \ .$$

Then, S supports all of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x}$ and $\mathbf{y}$. Let $B_S \subseteq B$ and $C_S \subseteq C$ be the subsets of elements of respectively B and C which are supported by S:

$$B_S = \{b \in B \ : \ \mathrm{support}\,(b) \subseteq S\} \qquad C_S = \{c \in C \ : \ \mathrm{support}\,(c) \subseteq S\} \ .$$

Lemma 2.25-ii implies that B_S and C_S are both finite. Let $\mathbf{A}_S \in \mathrm{Lin}(B_S \times C_S)$, $\mathbf{b}_S \in \mathrm{Lin}(B_S)$, $\mathbf{c}_S \in \mathrm{Lin}(C_S)$, $\mathbf{x}_S \in \mathrm{Lin}(C_S)$ and $\mathbf{y}_S \in \mathrm{Lin}(B_S)$ be the respective restrictions of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x}$ and $\mathbf{y}$. Lemma 2.48 implies

$$\mathbf{c}^\top \cdot \mathbf{x} = \mathbf{c}_S^\top \cdot \mathbf{x}_S \quad \text{and} \quad \mathbf{b}^\top \cdot \mathbf{y} = \mathbf{b}_S^\top \cdot \mathbf{y}_S \,. \tag{7.4}$$

Hence, it is enough to show $\mathbf{c}_S^\top \cdot \mathbf{x}_S \leqslant \mathbf{b}_S^\top \cdot \mathbf{y}_S$. This will follow from the next two claims.

Claim 7.1.1. $\mathbf{A}_S \cdot \mathbf{x}_S \leqslant \mathbf{b}_s$.

Claim 7.1.2. $\mathbf{A}_S^\top \cdot \mathbf{y}_S \geqslant \mathbf{c}_s$.

Before giving the proofs we show how these claims imply $\mathbf{c}_S^\top \cdot \mathbf{x}_S \leqslant \mathbf{b}_S^\top \cdot \mathbf{y}_S$. The vectors $\mathbf{x}$ and $\mathbf{y}$ are both non-negative. Hence, their restrictions $\mathbf{x}_S$ and $\mathbf{y}_S$ are also non-negative. The matrix $\mathbf{A}_S$ and the vectors $\mathbf{b}_S$, $\mathbf{c}_S$, $\mathbf{x}_S$ and $\mathbf{y}_S$ are all finite dimensional. Hence, the products $\mathbf{c}_S^\top \cdot \mathbf{x}_S$, $(\mathbf{y}_S^\top \cdot \mathbf{A}_S) \cdot \mathbf{x}_S$, $\mathbf{y}_S^\top \cdot (\mathbf{A}_S \cdot \mathbf{x}_S)$ and $\mathbf{y}_S^\top \cdot \mathbf{b}_S$ are all well-defined. Using non-negativity of $\mathbf{x}_S$ and $\mathbf{y}_S$, and Claims 7.1.1 and 7.1.2 we conclude

$$\mathbf{c}_S^\top \cdot \mathbf{x}_S \leqslant (\mathbf{y}_S^\top \cdot \mathbf{A}_S) \cdot \mathbf{x}_S = \mathbf{y}_S^\top \cdot (\mathbf{A}_S \cdot \mathbf{x}_S) \leqslant \mathbf{y}_S^\top \cdot \mathbf{b}_S \,.$$

so we only prove the first claim. I think this version is better

Now we prove the claims. The proof of the two claims are similar, so we concentrate on the first claim only.

Proof of Claim 7.1.1. Pick arbitrary $b \in B_S$. We show

$$(\mathbf{A}_S \cdot \mathbf{x}_S)(b) \leqslant \mathbf{b}_S(b) \,. \tag{7.5}$$

Since $\mathbf{b}_S$ is a restriction of $\mathbf{b}$,

$$\mathbf{b}_S(b) = \mathbf{b}(b) \,. \tag{7.6}$$

The vector $\mathbf{x}$ satisfies $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$. Hence,

$$(\mathbf{A} \cdot \mathbf{x})(b) \leqslant \mathbf{b}(b) \,. \tag{7.7}$$

By definition of matrix multplication

$$(\mathbf{A} \cdot \mathbf{x})(b) = \mathbf{A}(b, -) \cdot \mathbf{x} \,. \tag{7.8}$$

The set S supports both b and $\mathbf{A}$. Using Lemma 2.25-ii we can write

$$\mathbf{A}(b, -) \cdot \mathbf{x} = \sum_{c \in C_S} \mathbf{A}(b, c) \cdot \mathbf{x}(c) = \mathbf{A}_S(b, -) \cdot \mathbf{x}_S \,, \tag{7.9}$$

and by definition of $\mathbf{A}_S$ and $\mathbf{x}_S$ we can rewrite the latter term as

$$\mathbf{A}_S(b, -) \cdot \mathbf{x}_S = (\mathbf{A}_S \cdot \mathbf{x}_S)(b) \,. \tag{7.10}$$

We get (7.5) as a consequence of (7.6)-(7.10) ■

7.3 Counterexample to strong duality

In this section we show strong duality does not hold for orbit-finite linear programs by giving a counterexample. We construct a primal-dual pair of orbit-finite linear programs such that the optimum of the primal system is finite but the optimum of the dual system is not. Let $B = C = \mathbb{A} \uplus \{\star\}$. Consider the $B \times C$ linear program

$$\text{maximise} \quad \begin{array}{c} \begin{array}{ccc} \quad\;\; \mathbb{A} & \quad\quad & \star \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & \cdots & 0 \end{array}\right] \cdot \mathbf{x} \end{array}$$

$$\text{subject to} \quad \begin{array}{c} \mathbb{A} \\ \\ \\ \star \end{array} \left[\begin{array}{ccc|c} 1 & & & 1 \\ & 1 & & 1 \\ & & \ddots & \vdots \\ \hline 0 & 0 & \dots & -1 \end{array}\right] \cdot \mathbf{x} \leqslant \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ \hline 1 \end{array}\right] \tag{7.11}$$

$$\mathbf{x} \geqslant \mathbf{0}$$
$$\mathbf{x} \in \mathrm{Lin}(C)$$

Written explicitly, this is the linear program

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \\ & \mathbf{x} \in \mathrm{Lin}(C) \end{array}$$

where $\mathbf{A} \in \mathrm{Lin}(B \times C)$, $\mathbf{b} \in \mathrm{Lin}(B)$ and $\mathbf{c} \in \mathrm{Lin}(C)$ are defined as

$$\begin{array}{ll} \mathbf{A}(\alpha, \alpha) = 1, \text{ for } \alpha \in \mathbb{A} & \\ \mathbf{A}(\alpha, \beta) = 0, \text{ for } \alpha \neq \beta \in \mathbb{A} & \mathbf{b} = \mathbf{1}_{\{\star\}} \in \mathrm{Lin}(B) \\ \mathbf{A}(\star, \alpha) = 1, \text{ for } \alpha \in \mathbb{A} & \\ \mathbf{A}(\alpha, \star) = 0, \text{ for } \alpha \in \mathbb{A} & \mathbf{c} = \mathbf{1}_{\mathbb{A}} \in \mathrm{Lin}(C) \\ \mathbf{A}(\star, \star) = -1 & \end{array}$$

It is clear from the definitions that **A**, **b** and **c** are all equivariant. The dual linear program of (7.11) is the $(C \times B)$-linear program

$$\begin{array}{ll}
\text{minimise} & \begin{array}{c} \begin{array}{cccc} & \mathbb{A} & & \star \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & \cdots & 1 \end{array}\right] \end{array} \cdot \mathbf{y} \\[2em]
\text{subject to} & \begin{array}{c} \mathbb{A} \\ \\ \\ \star \end{array} \left[\begin{array}{ccc|c} 1 & & & 0 \\ & 1 & & 0 \\ & & \ddots & \vdots \\ \hline 1 & 1 & \ldots & -1 \end{array}\right] \cdot \mathbf{y} \geqslant \left[\begin{array}{c} 1 \\ 1 \\ \vdots \\ \hline 0 \end{array}\right] \\[2em]
 & \mathbf{y} \geqslant \mathbf{0} \\
 & \mathbf{y} \in \mathrm{Lin}(B)
\end{array} \tag{7.12}$$

We argue that the pair (7.11)-(7.12) does not satisfy strong duality (**??**):

Lemma 7.2. *The optimum of the linear program* (7.11) *is* 0*.*

Lemma 7.3. *The optimum of the linear program* (7.12) *is* $+\infty$*, i.e. it is infeasible.*

Proof of Lemma 7.2. The vector $\mathbf{0} \in \mathrm{Lin}(C)$ satisfies the system of constraints in (7.11) since

$$\mathbf{A} \cdot \mathbf{0} = \mathbf{0} \leqslant \mathbf{b} .$$

Moreover, $\mathbf{c}^{\mathsf{T}} \cdot \mathbf{0} = 0$. Hence, the optimum of (7.11) is at least 0. We claim that $\mathbf{0}$ is the only solution of the system of constraints in (7.11). Consider arbitrary $\mathbf{x} \in \mathrm{Lin}(C)$ which satisfies

$$\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \quad \text{and} \quad \mathbf{x} \geqslant \mathbf{0} .$$

Then, for any $\alpha \in \mathbb{A}$

$$\mathbf{x}(\alpha) + \mathbf{x}(\star) \leqslant 0 .$$

Since $\mathbf{x} \geqslant \mathbf{0}$, this can only be true if $\mathbf{x}(\alpha) = 0$ for all $\alpha \in \mathbb{A}$ and $\mathbf{x}(\star) = 0$, equivalently, when $\mathbf{x} = \mathbf{0}$. ∎

Proof of Lemma 7.3. We argue that the system of constraints $\mathbf{A}^{\mathsf{T}} \cdot \mathbf{y} \geqslant \mathbf{c}$ is not satisfiable. Towards arriving at a contradiction, consider an arbitrary vector $\mathbf{y} \in \mathrm{Lin}(B)$ such that $\mathbf{A}^{\mathsf{T}} \cdot \mathbf{y}$ is well-defined and $\mathbf{A}^{\mathsf{T}} \cdot \mathbf{y} \geqslant \mathbf{c}$. From the definition of $\mathbf{A}$ we get $(\mathbf{A}^{\mathsf{T}} \cdot \mathbf{y})(\alpha) = \mathbf{y}(\alpha)$ for $\alpha \in \mathbb{A}$, and

$$(\mathbf{A}^{\mathsf{T}} \cdot \mathbf{y})(\star) = \left(\sum_{\alpha \in \mathbb{A}} \mathbf{y}(\alpha) \right) - \mathbf{y}(\star) .$$

The condition $\mathbf{A}^\top \cdot \mathbf{y} \geqslant \mathbf{c}$ implies $\mathbf{y}(\alpha) \geqslant 1$ for every $\alpha \in \mathbb{A}$. But then, the sum $\sum_{\alpha \in \mathbb{A}} \mathbf{y}(\alpha)$ is not well-defined. This implies that $\mathbf{A}^\top \cdot \mathbf{y}$ is not well defined, and we arrive at a contradiction. ∎

Remark 7.4. The above counterexample also works even if we relax the constraints by dropping the conditions $\mathbf{x} \in \mathrm{Lin}(C)$ and $\mathbf{y} \in \mathrm{Lin}(B)$ in the primal and the dual system, respectively.

I would add: i.e. $\mathbf{x}$ and $\mathbf{y}$ are not finitly supported.

Remark 7.5. For a primal-dual pair of linear program, their *duality gap* is the difference between the optimums of the primal and the dual. The duality gap between the primal-dual pairs of linear programs defined in this section is infinite. We do not know whether there exists a primal-dual pair of orbit-finite linear programs where the duality gap is finite and non-zero.

Remark 7.6. Weak-duality does not hold for infinite linear programs in general. For a counterexample, look at [32, Section 3].

7.4 Duality for orbit-infinite linear programs

In this section, exceptionally, we do not assume the linear programs and their solutions to be orbit-finite.

Definition 7.7. A matrix is called *column-finite* if its column vectors are finite (i.e. non-zero only on finitely many co-ordinates). Symmetrically it is called *row-finite* if its row vectors are finite.

Definition 7.8. A (not necessarily orbit-finite) linear program

$$\begin{array}{rl} \text{maximise/minimise} & \mathbf{c}^\top \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0}. \end{array}$$

is called *column-finite* if the matrix $\mathbf{A}$ is column-finite and $\mathbf{b}$ is a finite vector. Symmetrically, it is called *row-finite* if the matrix $\mathbf{A}$ is row-finite and $\mathbf{c}$ is a finite vector.

One can try to remedy the fact that weak duality does not hold for infinite linear programs (Remark 7.6) by adding extra assumptions. As we saw in § 5.4, orbit-finiteness is such an assumption. Another possibility is to assume that the primal linear program is column-finite and restrict the solutions to be finite (i.e. non-zero only on finitely many variables) as well. [1] This means that the primal

[1] Note that column-finiteness of the system does not enforce finiteness of the solutions.

is of the form

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \\ & \mathbf{x} \text{ is finite.} \end{array} \tag{7.13}$$

where columns $\mathbf{A}$ is column-finite, $\mathbf{b}$ is a finite vector (as mentioned before, we do not assume $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ to be orbit-finite). Then the dual becomes row finite

I think it is fine, and i dont know how to do it better, but as b is a finite vector it is orbit finite.

$$\begin{array}{ll} \text{minimise} & \mathbf{b}^{\mathsf{T}} \cdot \mathbf{y} \\ \text{subject to} & \mathbf{A}^{\mathsf{T}} \cdot \mathbf{y} \geqslant \mathbf{c} \\ & \mathbf{y} \geqslant \mathbf{0} \\ & \mathbf{y} : B \to \mathbb{R} \end{array} \tag{7.14}$$

Multiplication with column-finite matrices is always well-defined, and the product of two column-finite matrices is also column-finite.

Lemma 7.9. *For matrices $\mathbf{X} : B \times C \to \mathbb{R}$ and $\mathbf{Y} : (C \times D) \to \mathbb{R}$, if $\mathbf{Y}$ is column-finite then the product $\mathbf{X} \cdot \mathbf{Y}$ is well-defined, and if $\mathbf{X}$ is also column-finite then $\mathbf{X} \cdot \mathbf{Y}$ is also column-finite.*

Proof. First we prove $\mathbf{X} \cdot \mathbf{Y}$ is well-defined when $\mathbf{Y}$ is column-finite. Pick arbitrary $(b, d) \in (B \times D)$. Then $(\mathbf{X} \cdot \mathbf{Y})(b, d) = \mathbf{A}(b, -) \cdot \mathbf{B}(-, d)$. Since $\mathbf{Y}$ is column-finite, $\mathbf{Y}(-, d)$ is a finite vector. Hence $\mathbf{X}(b, -) \cdot \mathbf{Y}(-, d)$ is well-defined.

Now we show $\mathbf{X} \cdot \mathbf{Y}$ is column-finite assuming both $\mathbf{X}$ and $\mathbf{Y}$ are column-finite. Pick arbitrary $d \in D$. We show $(\mathbf{X} \cdot \mathbf{Y})(-, d)$ is finite. Let $C' \subseteq C$ be the (necessarily finite) subset of elements of $c \in C$ such that $\mathbf{Y}(c, d) \neq 0$.

$$C' = \{c \in C \ : \ \mathbf{Y}(c, d) \neq 0\}$$

For every $c \in C'$ let B_c be the subset of elements of $b \in B$ such that $\mathbf{X}(b, c) \neq 0$. Since $\mathbf{X}$ is column-finite, B_c is finite for every $c \in C'$. Let $B' = \cup_{c \in C'} B_c$. Since C' is finite and B_c is finite for every $c \in B'$, the set B' is also finite. We claim $(\mathbf{X} \cdot \mathbf{Y})(b, d) \neq 0$ only if $b \in B'$. Pick arbitrary $b \in (B \setminus B')$. Since $\mathbf{Y}(c, d) \neq 0$ only if $c \in C'$, we have

$$(\mathbf{X} \cdot \mathbf{Y})(b, d) = \sum_{c \in C} \mathbf{X}(b, c) \cdot \mathbf{Y}(c, d) = \sum_{c \in C'} \mathbf{X}(b, c) \cdot \mathbf{Y}(c, d) \, .$$

And for $c \in C'$ we have $\mathbf{X}(b, c) \neq 0$ only if $b \in B'$. Hence,

$$\sum_{c \in C'} \mathbf{X}(b, c) \cdot \mathbf{Y}(c, d) = 0 \, .$$

Combining the above equations we get

$$(\mathbf{X} \cdot \mathbf{Y})(b, d) = 0 .$$

Thus $(\mathbf{X} \cdot \mathbf{Y})(b, d) \neq 0$ only if $b \in B'$, which implies finiteness of $(\mathbf{X} \cdot \mathbf{Y})(-, d)$. ■

Remark 7.10. We emphasize that Lemma 7.9 does not assume the matrices to be orbit-finite. For a detailed discussion on column-finite matrices outside the orbit-finite setting see [14]. We acknowledge Lorenzo Clemente for pointing out that Lemma 7.9 holds without the orbit-finiteness assumption, and also for making us aware of the reference.

As a consequence of the above lemma, the proof of weak-duality for finite linear programs also extends to this setting: for every solutions $\mathbf{x}$ of (7.13) and $\mathbf{y}$ of (7.14), we have

$$\mathbf{c}^{\mathsf{T}} \cdot \mathbf{x} \leqslant (\mathbf{y}^{\mathsf{T}} \cdot \mathbf{A}) \cdot \mathbf{x} = \mathbf{y}^{\mathsf{T}} \cdot (\mathbf{A} \cdot \mathbf{x}) \leqslant \mathbf{y}^{\mathsf{T}} \cdot \mathbf{b} ,$$

since the terms and subterms appearing in this sequence of equations and inequalities are all well-defined. This implies that the optimum of the primal system (7.13) cannot be bigger that the optimum of the dual system (7.14).

However, like orbit-finiteness, restricting the primal system to be column-finite and its solutions to be finite does not guarantee strong duality, as described by the following example.

Example 7.11. Let $<$ be a dense linear order on $\mathbb{A}$. Let $\star$ be an element not in $\mathbb{A}$. Let $B = \{\star\} \uplus \mathbb{A}$ and $C = \binom{\mathbb{A}}{2}$. Consider the $(B \times C)$-linear program

$$\begin{array}{lll}
\text{maximise} & \sum_{\{\alpha,\beta\} \in C} \mathbf{x}(\{\alpha, \beta\}) & \\
\text{subject to} & \sum_{\{\alpha,\beta\} \in C} \mathbf{x}(\{\alpha, \beta\}) \leqslant 1 & \\
 & \sum_{\beta > \alpha} \mathbf{x}(\{\alpha, \beta\}) - \sum_{\beta < \alpha} \mathbf{x}(\{\alpha, \beta\}) \leqslant 0 & (\alpha \in \mathbb{A}) \\
 & \mathbf{x} \geqslant \mathbf{0} & \\
 & \mathbf{x} \text{ is finite.} &
\end{array} \tag{7.15}$$

Considering this to be the primal system, the dual system becomes

$$\begin{array}{lll}
\text{minimise} & \mathbf{y}(\star) & \\
\text{subject to} & \mathbf{y}(\star) + \mathbf{y}(\alpha) - \mathbf{y}(\beta) \geqslant 1 & (\{\alpha < \beta\} \in C) \\
 & \mathbf{y} \geqslant \mathbf{0} & \\
 & \mathbf{y} : B \to \mathbb{R} . &
\end{array} \tag{7.16}$$

The dual is row-finite, therefore, the primal is column-finite. The following two claims says that the pair (7.15)-(7.16) violates strong duality.

Claim 7.11.1. *The optimum of the the primal system is* 0*.*

Claim 7.11.2. *The optimum of the dual system is* 1*.*

We now prove the claims.

Proof of Claim 7.11.1. The vector $\mathbf{0}$ is a solution to the primal system, hence the optimum is at least 0. The optimum is equal to 0 since it is the only solution to the primal system. To see this, pick any non-negative finite vector $\mathbf{v} : C \to \mathbb{R}$. Let S be the union of all sets $\{\alpha, \beta\}$ such that $\mathbf{v}(\{\alpha, \beta\})$ is non-zero. Since $\mathbf{v}$ is finite and not equal to $\mathbf{0}$ the set S is finite and non-empty. Let α_0 be the smallest element in S. Then

$$\sum_{\beta > \alpha_0} \mathbf{v}(\{\alpha, \beta\}) - \sum_{\beta < \alpha_0} \mathbf{v}(\{\alpha, \beta\}) = \sum_{\beta > \alpha_0} \mathbf{v}(\{\alpha, \beta\}) > 0$$

which means $\mathbf{v}$ cannot be a solution of the primal system. □

Proof of Claim 7.11.2. The vector $\mathbf{1}_\star : B \to \mathbb{R}$ is a solution to the dual system. Hence the optimum is at most 1. It is equal to 1 since for any solution $\mathbf{z} : B \to \mathbb{R}$ of the dual system, we have $\mathbf{z}(\star) \geqslant 1$. To see this, pick any solution $\mathbf{z} : B \to \mathbb{R}$ of the dual system. Pick $\alpha < \beta \in \mathbb{A}$. Since we assumed $\mathbb{A}$ is dense linear order for any n there exists n atoms $\alpha < \alpha_1 < \cdots < \alpha_n < \beta$ between α and β. Putting $\alpha_0 = \alpha$ and $\alpha_{n+1} = \beta$ we get

$$\mathbf{z}(\alpha) - \mathbf{z}(\beta) = \sum_{i=0}^{n} \mathbf{z}(\alpha_i) - \mathbf{z}(\alpha_{i+1}) \geqslant n \cdot (1 - \mathbf{z}(\star)) .$$

This cannot be true unless $\mathbf{z}(\star) \geqslant 1$. □

◀

Interestingly, assuming both orbit-finiteness and column-finiteness gives us strong duality (Theorem 7.12). §§ 7.5 to 7.10 are devoted to proving this theorem.

7.5 Column-finite and row-finite linear programs

In this and the following sections, we study orbit-finite linear programs that are either column-finite or row-finite. That is, they are of the form

$$\begin{array}{llcll}
\text{maximise} & \mathbf{c}^\top \cdot \mathbf{x} & \qquad\qquad & \text{minimise} & \mathbf{b}^\top \cdot \mathbf{y} \\
\text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} & & \text{subject to} & \mathbf{A}^\top \cdot \mathbf{y} \geqslant \mathbf{c} \\
& \mathbf{x} \geqslant \mathbf{0} & & & \mathbf{y} \geqslant \mathbf{0} \\
& \mathbf{x} \in \mathrm{Lin}(C) & & & \mathbf{y} \in \mathrm{Lin}(B)
\end{array}$$

where B and C are orbit-finite sets, the matrix $\mathbf{A} \in \mathrm{Lin}(B \times C)$ is column-finite, and the vector $\mathbf{b} \in \mathrm{FinLin}(B)$. Note that we wrote the column-finite linear program in primal form and the row-finite linear program in the dual form, we follow this convention for the rest of the chapter.

These linear programs are better behaved than orbit-finite linear programs. Firstly, they satisfy strong duality.

Theorem 7.12 (Strong Duality)**.** *For any primal-dual pair of orbit-finite linear programs, if the primal or the dual is column-finite and the optimum of any of linear programs is finite, then it is equal to the optimum of the other.*

Secondly, column-finite and row-finite linear programs are more robust compared to orbit-finite linear programs in general, and they admit optimal solutions with small support when their optimum is finite.

Definition 7.13. For a vector $\mathbf{x} : C \to \mathbb{R}$, by $\|\mathbf{x}\|_1$ we denote its *ℓ^1-norm*, i.e.

$$\|\mathbf{x}\|_1 \stackrel{\text{def}}{=} \sum_{c \in C} |\mathbf{x}(c)| \,,$$

where $|\mathbf{x}(c)|$ denotes the absolute value of $\mathbf{x}(c)$. We write $\|\mathbf{x}\|_1 < \infty$ to denote that the sum is finite, i.e. converges to a real number. Otherwise we write $\|\mathbf{x}\|_1 = \infty$.

Vectors $\mathbf{x} : C \to \mathbb{R}$ such that $\|\mathbf{x}\|_1 < \infty$ forms a vector space (we leave it to the reader to verify the details), which we denote by $\ell^1(C)$.

Remark 7.14. For every $\mathbf{v} \in \mathrm{Lin}(C)$, the set $\{\mathbf{v}(c) \ : \ c \in C\}$ of its coefficients is finite, and hence $\|\mathbf{v}\|_1 < \infty$ if and only if $\mathbf{v} \in \mathrm{FinLin}(C)$.

Example 7.15. For any enumeration $\alpha_1, \alpha_2, \ldots$ of the atoms, the vector $\mathbf{y} : \mathbb{A}^{(2)} \to \mathbb{R}$ defined as

$$\mathbf{y}(\alpha\beta) = \begin{cases} \frac{1}{2^n} & \text{if } \alpha = \alpha_n \text{ and } \beta = \alpha_{n+1} \text{ for some } n \in \mathbb{N}, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

is a vector in $\ell^1(\mathbb{A}^{(2)})$ which is not in $\mathrm{FinLin}\left(\mathbb{A}^{(2)}\right)$. ◀

Theorem 7.16. *Consider an S-supported column-finite linear program of atom dimension d. For any $T \subseteq_{\mathrm{FIN}} (\mathbb{A} \setminus S)$ of size at least d:*

1. *The optimum of the linear program does not change if we restrict the solutions to be finite and supported by $S \cup T$, or allow them to be orbit-infinite but of finite ℓ^1-norm.*

2. *If the optimum is finite then it has an optimal solution which is finite and supported by $S \cup T$.*

Theorem 7.17. *For any S-supported row-finite linear program:*

1. *Its optimum does not change if we restrict the solutions to be supported by S, or allow them to be orbit-infinite.*

2. *If its optimum is finite then it has an optimal solution supported by S.*

Example 7.18. We give an example of a column-finite system with a non-trivial orbit-infinite solution with bounded ℓ^1-norm.

Pick $\alpha \in \mathbb{A}$, Consider the $(\mathbb{A} \times \mathbb{A}^{(2)})$-system

$$\begin{array}{ll} \text{maximise} & \mathbf{1}_{\mathbb{A}}^{\mathsf{T}} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{1}_{\alpha} \\ & \mathbf{x} \geqslant \mathbf{0} \\ & \mathbf{x} \in \mathrm{Lin}\left(\mathbb{A}^{(2)}\right) \end{array} \tag{7.17}$$

where columns of the matrices are defined (using Notation 2.29) as

$$\mathbf{A}(-, \alpha\beta) = \alpha - \frac{1}{2} \cdot \beta\,.$$

The vector $\mathbf{y}$ defined in Example 7.15 is a solution of the above system assuming the enumeration $\alpha_1, \alpha_2, \ldots$ defining $\mathbf{y}$ starts from $\alpha_1 = \alpha$. For any two atoms $\beta, \gamma \neq \alpha$, the vector $\mathbf{z} = \alpha\beta + \alpha\gamma + \frac{1}{2} \cdot (\beta\gamma + \gamma\beta)$ is a finite solution of (**??**) with the same value as $\mathbf{z}$

$$\mathbf{1}_{\mathbb{A}}^{\mathsf{T}} \cdot \mathbf{y} = \mathbf{1}_{\mathbb{A}}^{\mathsf{T}} \cdot \mathbf{z} = 2\,.$$

◀

7.6 Proof of strong duality

In this section we prove Theorems 7.12, 7.16 and 7.17. We start by defining six functions and state one *main lemma* for each of them. The proof of above

this is incorrect. You give 4 functions and some lemmas (somtimes more than one sometimes it is not clear if there is any as in lemma 7.24 the function γ_S does not appear. Next there is theorems follow almost immediately from these lemmas. The proofs of the lemmas appear in later sections.

For the remainder of the chapter, fix an arbitrary $S \subseteq_{\mathrm{FIN}} \mathbb{A}$, orbit-finite sets B and C supported by S and an arbitrary column-finite maximisation problem

$$\begin{aligned} \text{maximise} \quad & \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x} \\ \text{subject to} \quad & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \\ & \mathbf{x} \in \mathrm{Lin}(C) \end{aligned} \tag{7.18}$$

such that $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ are supported by S. Call this linear program $\mathcal{U}$. Fix $d \in \mathbb{N}$ to be the atom-dimension of $\mathcal{U}$. Fix an arbitrary $T \subseteq (\mathbb{A} \setminus S)$ of size at least d.

Definition 7.19. Define the *orbit restriction function* $\zeta : \mathrm{Lin}(C) \to \mathrm{FinLin}(C)$ as

$$(\zeta(\mathbf{x}))(c) = \begin{cases} \mathbf{x}(c), & \text{if support}\,(c) \subseteq (S \cup \text{support}\,(\mathbf{x})), \\ 0, & \text{otherwise.} \end{cases}$$

The co-domain of ζ is indeed $\mathrm{FinLin}(C)$ since Lemma 2.25-ii implies that for every vector $\mathbf{x} \in \mathrm{Lin}(C)$, the vector $\zeta(\mathbf{x})$ is finite.

Lemma 7.20. *For any orbit-finite solution $\mathbf{x}$ of $\mathcal{U}$, $\zeta(\mathbf{x})$ is a finite solution of $\mathcal{U}$ and $\mathbf{c}^{\mathsf{T}} \cdot \zeta(\mathbf{x}) = \mathbf{c}^{\mathsf{T}} \cdot \mathbf{x}$.*

Arka[3]: TODO prove the above lemma

Definition 7.21. For an orbit-finite set D supported by S, define the *orbit summation function*

$$\gamma_S : \ell_1(D) \to \mathbb{R}^{\mathrm{Orbits}_S(D)}$$

as

$$\gamma_S(\mathbf{v}) : (K \in \mathrm{Orbits}_S(D)) \mapsto \sum_{b \in K} \mathbf{v}(b)$$

Remark 7.22. Theorem 3.55 in [33] implies that $\gamma_S(\mathbf{x})$ is well-defined for vectors $\mathbf{x}$ in $\ell^1(C)$: for any enumeration $c_1, c_2, \dots$ of elements of O, the sequence of *partial sums*

$$\sum_{i=1}^{n} \mathbf{x}(c_i)$$

converges to a real number, which is independent of the enumeration.

Remark 7.23. The function γ (Definition 5.22) which was useful in solving systems of inequalities is a special case of γ_S with $S = \emptyset$. A similar definition (Definition 4.25) was also used in solving linear equations.

Lemma 7.24. *For any* $\mathbf{x} \in \ell^1(C)$ *and* $\mathbf{y} \in \mathrm{Lin}(C)$ *the vector*

$$c \mapsto \mathbf{x}(c) \cdot \mathbf{y}(c) \ : \ C \to \mathbb{R}$$

is also in $\ell^1(C)$.

Is it the lemma for definition 7.21? γ_S does not appear here.

Proof. Let $R = \max(\{|\mathbf{y}(c)| \ : \ c \in C\})$. Since $\mathbf{y} \in \mathrm{Lin}(C)$, R is finite. Then

$M \to R$ in the equation

$$\sum_{c \in C} |\mathbf{x}(c) \cdot \mathbf{y}(c)| \leqslant M \cdot \sum_{c \in C} |\mathbf{x}(c)| < \infty\,.$$

■

The orbit summation function γ_S is used to convert column-finite and row-finite linear programs into finite linear programs. To do this, we extend it to matrices.

Definition 7.25. For orbit-finite sets D and E and column-finite matrix $\mathbf{B}$ in $\mathrm{Lin}(D \times E)$, all of them supported by S, define $\Gamma_S(\mathbf{B})$ to be the $\mathrm{Orbits}_S(D) \times \mathrm{Orbits}_S(E)$-matrix with columns

$$(\Gamma_S(\mathbf{B}))(-, K) \stackrel{\text{def}}{=} \gamma_S(\mathbf{B}(-, e)), \text{ for some } e \in K$$

and $K \in orbits(E)$

In § 7.7 we prove Γ_S is well defined for S-supported matrices (Lemma 7.40), and it commutes with matrix multiplication (Lemma 7.42). Using Γ_S we get from $\mathcal{U}$ the linear program $\Gamma_S(\mathcal{U})$:

$$\begin{array}{ll} \text{maximise} & \Gamma_S(\mathbf{c}^\mathsf{T}) \cdot \mathbf{x} \\ \text{subject to} & \Gamma_S(\mathbf{A}) \cdot \mathbf{x} \leqslant \Gamma_S(\mathbf{b}) \\ & \mathbf{x} \geqslant \mathbf{0}\,. \end{array} \tag{7.19}$$

Note that $\Gamma_S(\mathbf{b})$ is well-defined since $\mathbf{b}$ is supported by S. Since $\mathbf{b}$ is a column-vector (i.e. a matrix with only one column)

$$\Gamma_S(\mathbf{b}) = \gamma_S(\mathbf{b})\,.$$

Lemma 7.26. *For any solution* $\mathbf{x} \in \ell^1(C)$ *of* $\mathcal{U}$, $\gamma_S(\mathbf{x})$ *is a solution of* $\Gamma_S(\mathcal{U})$ *and* $\Gamma_S(\mathbf{c}^\mathsf{T}) \cdot \gamma_S(\mathbf{x}) = \mathbf{c}^\mathsf{T} \cdot \mathbf{x}$.

Example 7.27. Let $S = \emptyset$. Let $\star$ be an equivariant element, i.e. $\pi(\star) = \star$ for every $\pi \in \mathrm{Aut}(\mathbb{A})$. Let $B = \{\star\} \uplus \mathbb{A}$ and $C = \mathbb{A}^2$. The set B has two

This example is very complex and it didin't help me, maybe it will be different for other people. I don't know what to do with it, probably leave it as it is.

equivariant orbits, namely $\{\star\}$ and $\mathbb{A}$, and C also has two equivariant orbits, namely $\mathbb{A}^{(2)} = \{\alpha\beta \in C \;:\; \alpha \neq \beta\}$ and $I = \{\alpha\alpha \;:\; \alpha \in \mathbb{A}\}$. For every $(\alpha, \beta) \in C$ define $\mathbf{v}_{\alpha\beta} \in \mathrm{FinLin}(B)$ as

$$\mathbf{v}_{\alpha\beta} \stackrel{\text{def}}{=} \begin{cases} \alpha + \beta + \star & \text{if } \alpha \neq \beta \\ \star - \alpha & \text{otherwise.} \end{cases}$$

For every $(\alpha, \beta) \in C$ we have $\gamma_S(\mathbf{v}_{\alpha\beta}) \in \mathbb{R}^{\mathrm{Orbits}_S(B)} = \mathbb{R}^2$ and

$$\gamma_S(\mathbf{v}_{\alpha\beta}) = \begin{cases} (2,1) & \text{if } \alpha \neq \beta \\ (1,-1) & \text{otherwise.} \end{cases}$$

(assuming that the first and the second coordinate, respectively, correspond to the orbits $\mathbb{A}$ and $\{\star\}$). Define $\mathbf{A}$ to be the $(B \times C)$ matrix with columns $\mathbf{A}(-, \alpha\beta) = \mathbf{v}_{\alpha\beta}$ for $c \in C$. Then $\mathbf{A}$ is a column-finite matrix. We have

$$\Gamma_S(\mathbf{A}) \quad = \quad \begin{array}{cc} \begin{array}{cc} \mathbb{A}^{(2)} & I \end{array} & \\ \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} & \begin{array}{c} \mathbb{A} \\ \{\star\} \end{array} \end{array}$$

Define $\mathbf{b} \in \mathrm{FinLin}(B)$ and $\mathbf{c} \in \mathrm{Lin}(C)$ as

$$\mathbf{b} = \star \quad \text{and} \quad \mathbf{c} = 2 \cdot \mathbf{1}_{\mathbb{A}^{(2)}} + \mathbf{1}_I \,.$$

Using $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ we form $\mathcal{U}$ to be the column-finite linear program

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^\top \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \\ & \mathbf{x} \in \mathrm{Lin}(C) \end{array} \tag{7.20}$$

The vector $\mathbf{b}$ is an equivariant finite vector. Hence $\Gamma_S(\mathbf{b}) = \gamma_S(\mathbf{b}) = (0,1)$. Since $\mathbf{c}$ is not a finite vector $\gamma_S(\mathbf{c})$ is not well-defined. However, following usual convention we consider $\mathbf{c}^\top$ to be a row vector (i.e. matrix with only one row), and hence it is automatically column-finite. Moreover $\mathbf{c}^\top$ is equivariant. Hence $\Gamma_S(\mathbf{c}^\top)$ is well-defined and is equal to

$$\Gamma_S(\mathbf{c}^\top) \quad = \quad \begin{array}{c} \begin{array}{cc} \mathbb{A}^{(2)} & I \end{array} \\ \begin{bmatrix} 2 & 1 \end{bmatrix} \end{array} .$$

Using Γ_S we get the finite linear program

$$\Gamma_S(\mathcal{U}): \quad \begin{array}{ll} \text{maximise} & 2 \cdot x_1 + x_2 \\ \text{subject to} & 2 \cdot x_1 - x_2 \leqslant 0 \\ & x_1 + x_2 \leqslant 1 \end{array}$$

◀

If definition of d and T are in one line then if you want to recall them then inline them instead of making references.

Recall the definition of d and T fixed in the beginning of this section. Also recall Notation 2.29.

Definition 7.28. For an S-supported orbit-finite set D of atom dimension at most d, define the *semi-orbit distribution function*

$$\delta_T^S : \mathbb{R}^{\mathrm{Orbits}_S(D)} \to \mathrm{FinLin}(D)$$

as

$$\delta_T^S(\mathbf{x}) \stackrel{\text{def}}{=} \sum_{K \in \mathrm{Orbits}_S(D)} \left(\frac{\mathbf{x}(K)}{|K_{S \cup T}|} \cdot \sum_{b \in K_{S \cup T}} b \right)$$

I would use $\sum_{b \in K_{S \cup T}} 1_b$ instead of $\sum_{b \in K_{S \cup T}} b$

where for $K \in \mathrm{Orbits}_S(D)$

$$K_{S \cup T} = \{ b \in K \;:\; \mathrm{support}\,(b) \subseteq S \cup T \}$$

Lemma 7.29. *For any solution $\mathbf{x}$ of $\Gamma_S(\mathcal{U})$, $\delta_T^S(\mathbf{x})$ is an $(S \cup T)$-supported finite solution of $\mathcal{U}$ and $\mathbf{c}^\mathsf{T} \cdot \delta_T^S(\mathbf{x}) = \Gamma_S(\mathbf{c}^\mathsf{T}) \cdot \mathbf{x}$.*

Definition 7.30. Two orbit-finite linear programs are called *equivalent* if:

1. they have the same optimum, and
2. in case one of them has an optimal solution then so does the other.

An immediate corollary of Lemmas 7.26 and 7.29 is:

Corollary 7.31. *The linear programs $\mathcal{U}$ and $\Gamma_S(\mathcal{U})$ are equivalent.*

Now we are ready to prove Theorem 7.16 using Lemmas 7.26 and 7.29. The proof of these lemmas will appear in respectively § 7.7 and § 7.8.

Proof of Theorem 7.16

The optimum of $\mathcal{U}$ can only decrease if we restrict to finite solutions supported by $S \cup T$. Lemmas 7.20, 7.26 and 7.29 together imply that for any solution $\mathbf{x}$ of $\mathcal{U}$, $(\delta_T^S \circ \gamma_S)(\zeta(\mathbf{x}))$ is a finite $(S \cup T)$-supported solution of $\mathcal{U}$ with

$$\mathbf{c}^\mathsf{T} \cdot (\delta_T^S \circ \gamma_S)(\zeta(\mathbf{x})) = \mathbf{c}^\mathsf{T} \cdot \mathbf{x}$$

Hence the optimum of $\mathcal{U}$ does not change if we restrict to finite $(S \cup T)$-supported solutions.

Now assume the optimal of $\mathcal{U}$ is finite (say $r \in \mathbb{R}$). Corollary 7.31 implies the optimum of $\Gamma_S(\mathcal{U})$ is also r. Finite linear programs admit optimal solutions when their optimums are finite ([29, Theorem 2.6]). Let $\mathbf{z}$ be an optimal solution of $\Gamma_S(\mathcal{U})$. Then $\delta_T^S(\mathbf{z})$ is a $(S \cup T)$-supported finite solution of $\mathcal{U}$ with

$$\mathbf{c}^{\mathsf{T}} \cdot \delta_T^S(\mathbf{z}) = r$$

Since r is the optimum of $\mathcal{U}$, $\delta_T^S(\mathbf{z})$ is also an optimal solution. ■

Lemmas 7.26 and 7.29, and the proof of Theorem 7.16 using them can be summarised by the following diagram. In this diagram, r represents an arbitrary real number.

$$\begin{tikzcd}
\text{orbit-finite solutions of } \mathcal{U} \text{ with value } r \arrow[rr, "\zeta"] & & \text{finite solutions of } \mathcal{U} \text{ with value } r \\
(S \cup T)\text{-supported finite solutions of } \mathcal{U} \text{ with value } r \arrow[u, phantom, "\rotatebox{90}{$\subseteq$}"] & & \text{possibly orbit-infinite solutions of } \mathcal{U} \text{ of bounded } \ell^1\text{-norm and with value } r \arrow[u, phantom, "\rotatebox{90}{$\supseteq$}"] \arrow[dl, "\gamma_S"] \\
& \text{solutions of } \Gamma_S(\mathcal{U}) \text{ with value } r \arrow[ul, "\delta_T^S"] &
\end{tikzcd}$$

The dual of the column-finite linear program $\mathcal{U}$ is the row-finite linear program $\mathcal{U}^{\mathsf{T}}$:

$$\begin{array}{ll} \text{minimise} & \mathbf{b}^{\mathsf{T}} \cdot \mathbf{y} \\ \text{subject to} & \mathbf{A}^{\mathsf{T}} \cdot \mathbf{y} \geqslant \mathbf{c} \\ & \mathbf{y} \geqslant \mathbf{0} \\ & \mathbf{y} \in \mathrm{Lin}(B) \end{array} \tag{7.21}$$

The dual of the finite linear program $\Gamma_S(\mathcal{U})$ is the linear program $\Gamma_S(\mathcal{U})^{\mathsf{T}}$:

$$\begin{array}{ll} \text{minimise} & \Gamma_S(\mathbf{b})^{\mathsf{T}} \cdot \mathbf{y} \\ \text{subject to} & \Gamma_S(\mathbf{A})^{\mathsf{T}} \cdot \mathbf{y} \leqslant \Gamma_S(\mathbf{c}^{\mathsf{T}})^{\mathsf{T}} \\ & \mathbf{y} \geqslant \mathbf{0} \end{array} \tag{7.22}$$

Definition 7.32. For an S-supported orbit-finite set D, define the *orbit distribution function*

$$\delta_S : \mathbb{R}^{\mathrm{Orbits}_S(D)} \to \mathrm{Lin}(D)$$

as

$$\delta_S(\mathbf{x}) : c \mapsto \mathbf{x}(\mathrm{orbit}_S(c)) \ .$$

Lemma 7.33. *If* $\mathbf{y}$ *is a solution of* $\Gamma_S(\mathcal{U})^\intercal$ *then,* $\delta_S(\mathbf{y})$ *is a solution of* $\mathcal{U}^\intercal$ *and*

$$\mathbf{b}^\intercal \cdot \delta_S(\mathbf{y}) = \Gamma_S(\mathbf{b})^\intercal \cdot \mathbf{y}$$

Recall the definition of d and T fixed in the beginning of this section.

Definition 7.34. For an S-supported orbit-finite set D of atom-dimension at least d, define the *semi-orbit summation function*

$$\gamma_T^S : (D \to \mathbb{R}) \to (\mathbb{R}^{\mathrm{Orbits}_S(D)})$$

as

$$\gamma_T^S(\mathbf{v}) : (K \in \mathrm{Orbits}_S(D)) \mapsto \left(\frac{1}{|K_{S \cup T}|} \cdot \sum_{b \in K_{S \cup T}} \mathbf{v}(b) \right)$$

where for $K \in \mathrm{Orbits}_S(D)$

$$K_{S \cup T} = \{ b \in X \ : \ b \text{ is supported by } S \cup T \} \ .$$

Lemma 7.35. *If* $\mathbf{y}$ *is a solution of* $\mathcal{U}^\intercal$ *then,* $\gamma_T^S(\mathbf{y})$ *is a solution of* $\Gamma_S(\mathcal{U})^\intercal$ *with*

$$\mathbf{b}^\intercal \cdot \mathbf{y} = \Gamma_S(\mathbf{b})^\intercal \cdot \gamma_T^S(\mathbf{y})$$

As an immediate corollary of Lemmas 7.33 and 7.35 we get:

Corollary 7.36. *The linear programs* $\mathcal{U}^\intercal$ *and* $\Gamma_S(\mathcal{U})^\intercal$ *are equivalent.*

Now we are ready to prove Theorem 7.17 using Lemmas 7.33 and 7.35. The proof of these lemmas will appear in respectively in § 7.9 and § 7.10.

Proof of Theorem 7.17

The optimum of $\mathcal{U}^\intercal$ can only decrease if we allow the solutions to be orbit-infinite and can only increase if we restrict the solutions to be supported by S. Lemmas 7.33 and 7.35 together imply that for any solution $\mathbf{y}$ of $\mathcal{U}^\intercal$ (be it orbit-infinite or orbit-finite), $(\delta_S \circ \gamma_T^S)(\mathbf{y})$ is an S-supported solution of $\mathcal{U}^\intercal$ with

$$\mathbf{b}^\intercal \cdot (\delta_S \circ \gamma_T^S)(\mathbf{y}) = \mathbf{c}^\intercal \cdot \mathbf{y}$$

I think it should be $\mathbf{b}^\top \cdot (\delta_S \circ \gamma^S_T)(\mathbf{y}) = \mathbf{c}^\top \cdot \mathbf{y}$

Hence the optimum of $\mathcal{U}^\top$ does not change if either we allow the solutions to be orbit-infinite or we restrict the solutions to be S-supported.

Now assume the optimum of $\mathcal{U}^\top$ is finite (say $r \in \mathbb{R}$). Corollary 7.36 implies the optimum of $\Gamma_S(\mathcal{U})^\top$ is also r. Finite linear programs admit optimal solutions when their optimums are finite. Let $\mathbf{z}$ be an optimal solution of $\Gamma_S(\mathcal{U})$. Then $\delta_S(\mathbf{z})$ is an S-supported finite solution of $\mathcal{U}^\top$ with

$$\mathbf{b}^\top \cdot \delta_S(\mathbf{z}) = r$$

Since r is the optimum of $\mathcal{U}^\top$, $\delta_S(\mathbf{z})$ is also an optimal solution. ■

Lemmas 7.33 and 7.35, and the proof of Theorem 7.17 using them can be summarised by the following diagram. In this diagram, r represents an arbitrary real number.

$$\begin{tikzcd}
\begin{array}{l}S\text{-supported solutions}\\ \text{of } \mathcal{U}^\top \text{ with value } r\end{array} & \subseteq & \begin{array}{l}\text{(possibly) orbit-infinite}\\ \text{solutions of } \mathcal{U}^\top \text{ with value } r\end{array} \arrow[dl, "\gamma^S_T"] \\
& \begin{array}{l}\text{solutions of } \Gamma_S(\mathcal{U})^\top\\ \text{with value } r\end{array} \arrow[ul, "\delta_S"] &
\end{tikzcd}$$

Finally, we are ready to prove Theorem 7.12.

Proof of Theorem 7.12

Consider the primal-dual pair $\mathcal{U} - \mathcal{U}^\top$. Corollary 7.31 says that $\mathcal{U}$ and $\Gamma_S(\mathcal{U})$ have the same optimum. Similarly, Corollary 7.36 says that $\mathcal{U}^\top$ and $\Gamma_S(\mathcal{U})^\top$ have the same optimum. The pair $\Gamma_S(\mathcal{U}) - \Gamma_S(\mathcal{U})^\top$ is a primal-dual pair of finite linear programs. Hence if the optimum of any of the linear programs $\mathcal{U}$ or $\mathcal{U}^\top$ is finite then using the classical duality theorem we can conclude that the it is equal to the optimum of the other. ■

Example 7.37. Continuing Example 7.27, fix some $\alpha_1\beta_1\tau_1 \in \mathbb{A}^{(3)}$. Define $\mathbf{x} \in \mathrm{FinLin}(B)$ as

$$\mathbf{x} = \frac{1}{9} \cdot (2 \cdot \alpha_1\beta_1 + \beta_1\tau_1 + 2 \cdot \alpha_1\alpha_1 + 3 \cdot \beta_1\beta_1 + \tau_1\tau_1) .$$

Then $\mathbf{x}$ is a non-negative finite solution with value $\frac{4}{3}$, since

$$\mathbf{A} \cdot \mathbf{x} = \star = \mathbf{b}, \quad \text{and} \quad \mathbf{c}^\top \cdot \mathbf{x} = \frac{4}{3} .$$

We have $\gamma_S(\mathbf{x}) = \left(\frac{1}{3}, \frac{2}{3}\right)$. Since

$$\Gamma_S(\mathbf{A}) \cdot \gamma_S(\mathbf{x}) = (1,0) = \Gamma_S(\mathbf{b}), \quad \text{and} \quad \Gamma_S(\mathbf{c}^\mathsf{T}) \cdot \gamma_S(\mathbf{x}) = \frac{4}{3}$$

the vector $\gamma_S(\mathbf{x})$ is a non-negative solution of $\Gamma_S(\mathcal{U})$ with value $\frac{4}{3}$. Coincidentally it is also an optimal solution, and in this case the unique one. The atom dimension of $\mathcal{U}$ is 2. Let $T = \{\alpha_1, \beta_1\}$. Then

$$\delta_T^S(\gamma_S(\mathbf{x})) = \frac{1}{6} \cdot (\alpha_1\beta_1 + \beta_1\alpha_1) + \frac{1}{3} \cdot (\alpha_1\alpha_1 + \beta_1\beta_1)$$

The vector $\delta_T^S(\gamma_S(\mathbf{x}))$ is non-negative. Furthermore, $\mathbf{A} \cdot \delta_T^S(\gamma_S(\mathbf{x})) = \star = \mathbf{b}$, and $\mathbf{c}^\mathsf{T} \cdot \delta_T^S(\gamma_S(\mathbf{x})) = \frac{4}{3}$. Hence $\delta_T^S(\gamma_S(\mathbf{x}))$ is a solution of $\mathcal{U}$ and in this case an optimal one. Notice that $\delta_T^S(\gamma_S(\mathbf{x}))$ uses fewer atoms that $\mathbf{x}$.

Pick an infinite subset $W \subseteq \mathbb{A}$. Define $\mathbf{y}_W = 3 \cdot \star + \mathbf{1}_W + 2 \cdot \mathbf{1}_{(\mathbb{A} \setminus W)}$. The vector $\mathbf{y}_W$ is an orbit-infinite solution of $\mathcal{U}^\mathsf{T}$ with value 3. The result $\gamma_T^S(\mathbf{y}_W)$ of applying γ_T^S to $\mathbf{y}_W$ depends on the intersection $\{\alpha, \beta\} \cap W$. We focus on the case where $\{\alpha, \beta\} \cap W = \{\alpha\}$, the remaining cases can be dealt with similarly. In this case, $\gamma_T^S(\mathbf{y}_W) = (\frac{3}{2}, 3)$ (assuming that the first and the second co-ordinate, respectively, corresponds to the orbits $\mathbb{A}$ and $\{\star\}$). Dualising $\Gamma_S(\mathcal{U})$ we get

$$\Gamma_S(\mathcal{U})^\mathsf{T} : \quad \begin{array}{ll} \text{minimise} & y_2 \\ \text{subject to} & 2 \cdot y_1 + y_2 \geqslant 2 \\ & -y_1 + y_2 \geqslant 1 \end{array}$$

The vector $\gamma_T^S(\mathbf{y}_W)$ is a solution with of $\Gamma_S(\mathcal{U})^\mathsf{T}$ with value 3. Applying δ_S to $\gamma_T^S(\mathbf{y}_W)$ we get

$$\delta_S(\gamma_T^S(\mathbf{y}_W)) = 3 \cdot \star + \frac{3}{2} \cdot \mathbf{1}_A$$

which is an equivariant solution of $\mathcal{U}^\mathsf{T}$ with value 3. Note that although $\mathbf{y}_W$ is orbit-infinite, but $\delta_S(\gamma_T^S(\mathbf{y}_W))$ is equivariant irrespective of W. ◀

It now remains to demonstrate Lemmas 7.26, 7.29, 7.33 and 7.35. Which we do in the following §§ 7.7 to 7.10.

7.7 The orbit summation function

add some short intro explaining the goal of this section

Lemma 7.38. *For any orbit-finite set D supported by S, every element in $\mathbb{R}^{\mathrm{Orbits}_S(D)}$ is supported by S.*

Proof. Pick arbitrary $\mathbf{v} \in \mathbb{R}^{\mathrm{Orbits}_S(D)}$, $\pi \in \mathrm{Aut}_S(\mathbb{A})$ and $K \in \mathrm{orbit}_S(D)$. Using

Lemma 2.23-i we get $\pi^{-1}(K) = K$. Applying Lemma 2.3 we conclude

$$\pi(\mathbf{v})(K) = \mathbf{v}(\pi^{-1}(K)) = \mathbf{v}(K) .$$

■

Recall the definition of γ_S (Definition 7.21).

Lemma 7.39. *The function γ_S is an S-supported monotonic linear map.*

Proof. Linearity and monotonicity of γ_S follows easily from its definition. We focus on proving that it is supported by S.

Let D be an arbitrary orbit-finite set supported by S. Pick arbitrary $\mathbf{v} \in \ell^1(D)$ and $\pi \in \mathrm{Aut}_S(\mathbb{A})$. For any $K \in \mathrm{Orbits}_S(D)$

$$\begin{aligned}
\gamma_S(\pi(\mathbf{v}))(K) &= \sum_{b \in K} \pi(\mathbf{v})(b) \\
&= \sum_{b \in K} \mathbf{v}(\pi^{-1}(b)) \\
&= \sum_{b \in K} \mathbf{v}(b) && \text{(Lemma 2.23-i)} \\
&= \gamma_S(\mathbf{v})(K) && .
\end{aligned}$$

Hence $\gamma_S(\pi(\mathbf{v})) = \gamma_S(\mathbf{v})$. Now using Lemma 7.38 we get $\gamma_S(\mathbf{v}) = \pi(\gamma_S(\mathbf{v}))$. Hence $\gamma_S(\pi(\mathbf{v})) = \pi(\gamma_S(\mathbf{v}))$, which finishes the proof. ■

Lemma 7.40. *The function Γ_S is well-defined for S-supported column-finite matrices.*

Proof. Pick orbit-finite sets D and E and column-finite matrix $\mathbf{B} \in \mathrm{Lin}(D \times E)$, all of them supported by S. We show for any two elements c and c' in the same S-orbit of E.

$$\gamma_S(\mathbf{B}(-, c)) = \gamma_S(\mathbf{B}(-, c') .$$

Pick such $c, c' \in E$ arbitrarily. Since they are in the same S-orbit, there exists $\pi \in \mathrm{Aut}_S(\mathbb{A})$ such that $\pi(c) = c'$. Using Lemma 7.38 we get

$$\gamma_S(\mathbf{B}(-, c)) = \pi(\gamma_S(\mathbf{B}(-, c))) .$$

Since γ_S is supported by S (Lemma 7.39), from Lemma 2.4 it follows that

$$\pi(\gamma_S(\mathbf{B}(-, c))) = \gamma_S(\pi(\mathbf{B}(-, c))) .$$

The matrix $\mathbf{B}$ is assumed to be supported by S. Using Lemma 2.49-i we conclude

$$\gamma_S(\pi(\mathbf{B}(-,c))) = \gamma_S(\mathbf{B}(-,\pi(c))) = \gamma_S(\mathbf{B}(-,c')) \ .$$

As a consequence of the above equalities we get

$$\gamma_S(\mathbf{B}(-,c)) = \gamma_S(\mathbf{B}(-,c')) \ .$$

■

Lemma 7.41. *For any vector* $\mathbf{x} \in \ell^1(C)$ *we have* $\gamma_S(\mathbf{A} \cdot \mathbf{x}) = \Gamma_S(\mathbf{A}) \cdot \gamma_S(\mathbf{x})$ *and* $\mathbf{c}^\top \cdot \mathbf{x} = \Gamma_S(\mathbf{c}^\top) \cdot \gamma_S(\mathbf{x})$.

Proof. Pick $\mathbf{x} \in \ell^1(C)$. We prove $\gamma_S(\mathbf{A} \cdot \mathbf{x}) = \Gamma_S(\mathbf{A}) \cdot \gamma_S(\mathbf{x})$. The proof of $\mathbf{c}^\top \cdot \mathbf{x} = \Gamma_S(\mathbf{c}^\top) \cdot \gamma_S(\mathbf{x})$ is similar.

Pick arbitrary $K \in \mathrm{Orbits}_S(B)$. We show

$$\gamma_S(\mathbf{A} \cdot \mathbf{x})(K) = (\Gamma_S(\mathbf{A}) \cdot \gamma_S(\mathbf{x}))(K) \ .$$

The following sequence of equations finish the proof

$$\begin{aligned}
\gamma_S(\mathbf{A} \cdot \mathbf{x})(K) &= \sum_{b \in K} (\mathbf{A} \cdot \mathbf{x})(b) && \text{(Definition 7.21)} \\
&= \sum_{b \in K} \sum_{c \in C} \mathbf{A}(b,c) \cdot \mathbf{x}(c) \\
&= \sum_{b \in K} \sum_{M \in \mathrm{Orbits}_S(C)} \sum_{c \in M} \mathbf{A}(b,c) \cdot \mathbf{x}(c) \\
&= \sum_{M \in \mathrm{Orbits}_S(C)} \sum_{c \in M} \mathbf{x}(c) \cdot \left(\sum_{b \in K} \mathbf{A}(b,c) \right) && \text{(rearrangement)} \\
&= \sum_{M \in \mathrm{Orbits}_S(C)} \sum_{c \in M} \mathbf{x}(c) \cdot \Gamma_S(\mathbf{A})(K,M) && \text{(Definition 7.25)} \\
&= \sum_{M \in \mathrm{Orbits}_S(C)} \Gamma_S(\mathbf{A})(K,M) \cdot \sum_{c \in M} \mathbf{x}(c) && \text{(rearrangement)} \\
&= \sum_{M \in \mathrm{Orbits}_S(C)} \Gamma_S(\mathbf{A})(K,M) \cdot \gamma_S(\mathbf{x})(M) && \text{(Definition 7.21)} \\
&= (\Gamma_S(\mathbf{A}) \cdot \gamma_S(\mathbf{x}))(K) \ .
\end{aligned}$$

Note that we can rearrange the sums freely due to Lemma 7.24 and Remark 7.22.

■

The following lemma is not used in our current context. But we believe it is a fundamental result and therefore worth stating.

Lemma 7.42. *The function Γ_S commutes with matrix multiplication.*

Proof. Pick arbitrary orbit-finite sets D, E and F, and column-finite matrices $\mathbf{B} \in \mathrm{Lin}(D \times E)$ and $\mathbf{C} \in \mathrm{Lin}(E \times F)$, all supported by S. We prove that $\Gamma_S(\mathbf{B} \cdot \mathbf{C}) = \Gamma_S(\mathbf{B}) \cdot \Gamma_S(\mathbf{C})$.

Let L be an arbitrary S-orbit of F. We show

$$\gamma_S(\mathbf{B} \cdot \mathbf{C})(-, L) = (\Gamma_S(\mathbf{B}) \cdot \Gamma_S(\mathbf{C}))(-, L)$$

Let d be an arbitrary element of L.

I think the next equation should start from γ_S not from Γ_S.

$$\begin{aligned}
&(\Gamma_S(\mathbf{B} \cdot \mathbf{C}))(-, L) \\
&= \gamma_S((\mathbf{B} \cdot \mathbf{C})(-, d)) && \text{(Definition 7.25)} \\
&= \gamma_S(\mathbf{B} \cdot \mathbf{C}(-, d)) \\
&= \Gamma_S(\mathbf{B}) \cdot \gamma_S(\mathbf{C}(-, d)) && \text{(Lemma 7.41)} \\
&= \Gamma_S(\mathbf{B}) \cdot (\Gamma_S(\mathbf{C}))(-, L) && \text{(Definition 7.25)} \\
&= (\Gamma_S(\mathbf{B}) \cdot \Gamma_S(\mathbf{C}))(-, L) \,.
\end{aligned}$$

∎

Proof of Lemma 7.26

Consider an arbitrary solution $\mathbf{x} \in \ell^1(C)$ of $\mathcal{U}$. By monotonicity of γ_S (Lemma 7.39), $\gamma_S(\mathbf{x})$ is non-negative. Applying Lemma 7.41 we get

$$\Gamma_S(\mathbf{A}) \cdot \gamma_S(\mathbf{x}) = \gamma_S(\mathbf{A} \cdot \mathbf{x}) \,. \tag{7.23}$$

Since $\mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b}$, again using monotonicity of γ_S (Lemma 7.39) we get

$$\gamma_S(\mathbf{A} \cdot \mathbf{x}) \leqslant \gamma_S(\mathbf{b}) \,. \tag{7.24}$$

Combining equation (7.23) with inequality (7.24) we get

$$\Gamma_S(\mathbf{A}) \cdot \gamma_S(\mathbf{x}) \leqslant \gamma_S(\mathbf{b}) = \Gamma_S(\mathbf{b}) \,.$$

Hence, $\gamma_S(\mathbf{x})$ is a solution of $\Gamma_S(\mathcal{U})$. Finally another use of Lemma 7.41 gives us $\Gamma_S(\mathbf{c}^\mathsf{T}) \cdot \gamma_S(\mathbf{x}) = \mathbf{c}^\mathsf{T} \cdot \mathbf{x}$. ∎

7.8 The semi-orbit distribution function

add an overview

Recall the definition of δ_T^S (Definition 7.28). Immediately from the definition we get:

Lemma 7.43. *The function δ_T^S is a monotonic linear map.*

Recall the definition of $\mathrm{Aut}_{S\cup\{T\}}(\mathbb{A})$ given in Notation 2.24.

Definition 7.44. A set x is said to be *supported by $S \cup \{T\}$* if $\pi(x) = x$ for any $\pi \in \mathrm{Aut}_{S\cup\{T\}}(\mathbb{A})$.

Remark 7.45. Definition 5.19, which was useful in solving orbit-finite linear programs is a special case of the above definition.

Lemma 7.46. *If a set x is supported by S then it is also supported by $(S \cup \{T\})$.*

Proof. Follows from the fact that $\mathrm{Aut}_{S\cup T}(\mathbb{A}) \cup \mathrm{Aut}(T) \subseteq \mathrm{Aut}_S(\mathbb{A})$. ■

Lemma 7.47. *For any orbit-finite set D supported by S and $\mathbf{x} \in \mathbb{R}^{\mathrm{Orbits}_S(D)}$ the vector $\delta_T^S(\mathbf{x})$ is finite and is supported by $S \cup \{T\}$.*

$K_{S\cup T} = \{b \in K \ : \ \text{support}\,(b) \subseteq (S \cup T)\}$

Proof. First we show $\delta_T^S(\mathbf{x})$ is supported by $S \cup T$. For $K \in \mathrm{Orbits}_S(D)$ define

$$K_{S\cup T} = \{b \in D \ : \ \text{support}\,(b) \subseteq (S \cup T)\} \ .$$

B or D in the statement of the Lemma is D

The vector $(\delta_T^S(\mathbf{x}))(b) \neq 0$ only if $b \in K_{S\cup T}$ for some $K \in \mathrm{Orbits}_S(B)$. Lemma 2.25-ii implies that for every $K \in \mathrm{Orbits}_S(B)$, the set $K_{S\cup T}$ is finite. Hence $\delta_T^S(\mathbf{x})$ is also a finite vector. By definition, every element of $K_{S\cup T}$ is supported by $S \cup T$. Applying Lemma 2.47 we conclude $\delta_T^S(\mathbf{x})$ is supported by $S \cup T$.

Pick arbitrary $\tau \in \mathrm{Aut}_{S\cup\{T\}}(\mathbb{A})$. We prove $\tau(\delta_T^S(\mathbf{x})) = \delta_T^S(\mathbf{x})$. There exists $\sigma \in \mathrm{Aut}_{S\cup T}(\mathbb{A})$ and $\pi \in \mathrm{Aut}(T)$ (recall Notation 2.24) such that $\tau = \pi \circ \sigma$. We have proved that $\delta_T^S(\mathbf{x})$ is supported by $S \cup T$. Hence $\sigma(\delta_T^S(\mathbf{x})) = \delta_T^S(\mathbf{x})$. To finish the proof we show $\pi(\delta_T^S(\mathbf{x})) = \delta_T^S(\mathbf{x})$. Recall Notation 2.29. Since addition of vectors commutes with the action of automorphims (Lemma 2.26), we get

$$\pi(\delta_T^S(\mathbf{x})) = \left(\sum_{K \in \mathrm{Orbits}_S(B)} \frac{\mathbf{x}(K)}{|K_{S\cup T}|} \cdot \sum_{b \in K_{S\cup T}} \pi(b) \right) \tag{7.25}$$

Lemma 2.25-i implies that $\pi(K_{S\cup T}) = K_{S\cup T}$. Then π induces a permutation of $K_{S\cup T}$ (with π^{-1} inducing the inverse permutation). Hence

$$\sum_{b \in K_{S\cup T}} \pi(b) = \sum_{b \in K_{S\cup T}} b \tag{7.26}$$

Using equations (7.25) and (7.26) we conclude

$$\pi(\delta_T^S(\mathbf{x})) = \left(\sum_{K \in \mathrm{Orbits}_S(B)} \frac{\mathbf{x}(K)}{|K_{S\cup T}|} \cdot \sum_{b \in K_{S\cup T}} b \right) = \delta_T^S(\mathbf{x})$$

■

Lemma 7.48. $\gamma_S \circ \delta^S_T = \mathrm{Id}$.

Proof. The proof is an easy application of the definitions of γ_S and δ^S_T and is left to the reader. ■

Lemma 7.49. *For any S-supported orbit-finite set D and* $(S \cup \{T\})$*-supported finite vector* $\mathbf{x} \in \mathrm{FinLin}(D)$

$$(\delta^S_T \circ \Gamma_S)(\mathbf{x}) = \mathbf{x}\,.$$

Proof of Lemma 7.49. Pick arbitrary $b \in D$. We show

$$((\delta^S_T \circ \gamma_S)(\mathbf{x}))(b) = \mathbf{x}(b)$$

We split the proof into two cases.

(Case 1: b is not supported by $S \cup T$) The vector $\mathbf{x}$ is a finite vector supported by $S \cup T$. Lemma 7.47 implies $(\delta^S_T \circ \gamma_S)(\mathbf{x})$ is also a finite vector supported by $S \cup T$. Applying Lemma 2.47

$$((\delta^S_T \circ \gamma_S)(\mathbf{x}))(b) = 0 = \mathbf{x}(b)$$

(Case 2: b is supported by $S \cup T$) For $K \in \mathrm{orbit}_S(D)$ define $K_{S \cup T}$ as done in Definition 7.28. Expanding the expression $(\delta^S_T \circ \gamma_S)(\mathbf{x})$ using the definition of γ_S and δ^S_T (Definitions 7.21 and 7.28)

$$(\delta^S_T \circ \gamma_S)(\mathbf{x}) = \sum_{K \in \mathrm{Orbits}_S(B)} \frac{(\gamma_S(\mathbf{x}))(K)}{|K_{S \cup T}|} \cdot \left(\sum_{b' \in K_{S \cup T}} b' \right)$$

Let $L = \mathrm{orbit}_S(b)$. Since b is supported by $S \cup T$, we have $b \in L_{S \cup T}$. Hence

$$((\delta^S_T \circ \gamma_S)(\mathbf{x}))(b) = \frac{(\gamma_S(\mathbf{x}))(L)}{|L_{S \cup T}|}$$

To finish the proof we need to show

$$\mathbf{x}(b) = \frac{(\gamma_S(\mathbf{x}))(L)}{|L_{S \cup T}|}$$

Claim 7.49.1. *For any* $b' \in L_{S \cup T}$ *we have* $\mathbf{x}(b) = \mathbf{x}(b')$.

Proof. Pick arbitrary $b' \in L_{S \cup T}$. Using Lemma 2.25-i we can find $\pi \in \mathrm{Aut}(T)$

such that $\pi(b) = b'$. Since $\mathbf{x}$ is assumed to be supported by $(S \cup \{T\})$, we have

$$\pi^{-1}(\mathbf{x}) = \mathbf{x}$$

Using Lemma 2.3 we get

$$\mathbf{x}(b') = \mathbf{x}(\pi(b)) = (\pi^{-1}(\mathbf{x}))(b) = \mathbf{x}(b)$$

This finishes the proof of the claim. $\square$

Since $\mathbf{x}$ is supported by $S \cup T$, if $\mathbf{x}(b') \neq 0$ for some $b' \in L$ then $b' \in L_{S\cup T}$. This implies

$$\sum_{b' \in L_{S\cup T}} \mathbf{x}(b') = \sum_{b' \in L} \mathbf{x}(b') = (\gamma_S(\mathbf{x}))(L)$$

Now the Claim 7.49.1 implies

$$\mathbf{x}(b) = \frac{(\gamma_S(\mathbf{x}))(L)}{|L_{S\cup T}|}$$

which finishes the proof for this case and also of the lemma. $\blacksquare$

Lemma 7.50. *For any vector* $\mathbf{x} \in \mathbb{R}^{\mathrm{Orbits}_S(C)}$

$$\mathbf{A} \cdot \delta^S_T(\mathbf{x}) = \delta^S_T(\Gamma_S(\mathbf{A}) \cdot \mathbf{x}) \quad \textit{and} \quad \mathbf{c}^\mathsf{T} \cdot \delta^S_T(\mathbf{x}) = \Gamma_S(\mathbf{c}^\mathsf{T}) \cdot \mathbf{x}\,.$$

Proof. We prove $\mathbf{A} \cdot \delta^S_T(\mathbf{x}) = \delta^S_T(\Gamma_S(\mathbf{A}) \cdot \mathbf{x})$. The proof of $\mathbf{c}^\mathsf{T} \cdot \delta^S_T(\mathbf{x}) = \Gamma_S(\mathbf{c}^\mathsf{T}) \cdot \mathbf{x}$ is similar.

Claim 7.50.1. *The vector* $\mathbf{A} \cdot \delta^S_T(\mathbf{x})$ *is supported by* $S \cup \{T\}$.

Proof. Pick arbitrary $\tau \in \mathrm{Aut}_{S\cup\{T\}}(\mathbb{A})$. We show $\tau(\mathbf{A} \cdot \delta^S_T(\mathbf{x})) = \mathbf{A} \cdot \delta^S_T(\mathbf{x})$. There exists $\sigma \in \mathrm{Aut}_{S\cup T}(\mathbb{A})$ and $\pi \in \mathrm{Aut}(T)$ such that

$$\tau = \pi \circ \sigma\,.$$

First we show $\sigma(\mathbf{A} \cdot \delta^S_T(\mathbf{x})) = \mathbf{A} \cdot \delta^S_T(\mathbf{x})$. The matrix $\mathbf{A}$ is supported by S and hence also by $S \cup T$. Lemma 7.47 implies that the vector $\delta^S_T(\mathbf{x})$ is supported by $S \cup T$. The matrix $\mathbf{A}$ is supported by S and hence also by $S \cup T$. Now using the fact that automorphisms commute with matrix multiplication (Lemma 2.25-i) we get

This sentence is repeated line above, remove one of them

$$\sigma(\mathbf{A} \cdot \delta^S_T(\mathbf{x})) = \sigma(\mathbf{A}) \cdot \sigma(\delta^S_T(\mathbf{x})) = \mathbf{A} \cdot \delta^S_T(\mathbf{x})\,.$$

Now we show that

$$\pi(\mathbf{A} \cdot \delta^S_T(\mathbf{x})) = \mathbf{A} \cdot \delta^S_T(\mathbf{x})$$

Since $\mathrm{Aut}(T) \subseteq \mathrm{Aut}_S(\mathbb{A})$ and $\mathbf{A}$ is supported by S we have $\pi(\mathbf{A}) = \mathbf{A}$. The vector $\delta_T^S(\mathbf{x})$ is supported by $S \cup \{T\}$ (Lemma 7.47). Hence

$$\pi(\delta_T^S(\mathbf{x})) = \delta_T^S(\mathbf{x})\ .$$

Applying Item Lemma 2.25-i again we get

$$\pi(\mathbf{A} \cdot \delta_T^S(\mathbf{x})) = \pi(\mathbf{A}) \cdot \pi(\delta_T^S(\mathbf{x})) = \mathbf{A} \cdot \delta_T^S(\mathbf{x})\ .$$

□

Since $\delta_T^S \circ \gamma_S$ acts as identity on $S \cup \{T\}$-supported vectors (Lemma 7.49), the above claim gives us

$$\mathbf{A} \cdot \delta_T^S(\mathbf{x}) = (\delta_T^S \circ \gamma_S)(\mathbf{A} \cdot \delta_T^S(\mathbf{x})) \tag{7.27}$$

Applying Lemma 7.41 we get

$$(\delta_T^S \circ \gamma_S)(\mathbf{A} \cdot \delta_T^S(\mathbf{x})) = \delta_T^S(\Gamma_S(\mathbf{A}) \cdot (\gamma_S \circ \delta_T^S)(\mathbf{x})) \tag{7.28}$$

Since $\gamma_S \circ \delta_T^S = \mathrm{Id}$ (Lemma 7.48), we have $(\gamma_S \circ \delta_T^S)(\mathbf{x}) = \mathbf{x}$. Hence

$$\delta_T^S(\Gamma_S(\mathbf{A}) \cdot (\gamma_S \circ \delta_T^S)(\mathbf{x})) = \delta_T^S(\Gamma_S(\mathbf{A}) \cdot \mathbf{x}) \tag{7.29}$$

Combining equations (7.27), (7.28) and (7.29) we get

$$\mathbf{A} \cdot \delta_T^S(\mathbf{x}) = \delta_T^S(\Gamma_S(\mathbf{A}) \cdot \mathbf{x})\ ,$$

which finishes the proof of lemma. ■

Proof of Lemma 7.29

Pick a solution $\mathbf{x}$ of $\Gamma_S(\mathcal{U})$. We have to show $\delta_T^S(\mathbf{x})$ is a $(S \cup T)$-supported solution of $\mathcal{U}$ such that

$$\mathbf{c}^\mathsf{T} \cdot \delta_T^S(\mathbf{x}) = \Gamma_S(\mathbf{c}^\mathsf{T}) \cdot \mathbf{x}\ .$$

Lemma 7.50 gives us $\mathbf{c}^\mathsf{T} \cdot \delta_T^S(\mathbf{x}) = \Gamma_S(\mathbf{c}^\mathsf{T}) \cdot \mathbf{x}$. Lemma 7.47 implies $\delta_T^S(\mathbf{x})$ is a finite vector supported by $S \cup T$. It remains to show $\delta_T^S(\mathbf{x})$ is a solution of $\mathcal{U}$.

Because $\mathbf{x}$ is a solution of $\Gamma_S(\mathcal{U})$ it is non-negative and

$$\Gamma_S(\mathbf{A}) \cdot \mathbf{x} \leqslant \Gamma_S(\mathbf{b})\ .$$

Using Lemma 7.50 we conclude $\delta_T^S(\mathbf{x})$ is non-negative as well. Now Lemmas 7.43 and 7.50 gives us

$$\mathbf{A} \cdot \delta_T^S(\mathbf{x}) = \delta_T^S(\Gamma_S(\mathbf{A}) \cdot \mathbf{x}) \leqslant \delta_T^S(\Gamma_S(\mathbf{b}))\,. \tag{7.30}$$

The vector $\mathbf{b}$ is supported by S and hence also by $S \cup \{T\}$. Using Lemma 7.49 we conclude $\delta_T^S(\Gamma_S(\mathbf{b})) = \mathbf{b}$. Along with (7.30), this implies $\mathbf{A} \cdot \delta_T^S(\mathbf{x}) \leqslant \mathbf{b}$ and finishes the proof. ■

7.9 The orbit distribution function

Immediately from the definition of the orbit distribution function δ_S we get:

Lemma 7.51. *The function δ_S is a monotonic linear function.*

Lemma 7.52. *For any vector $\mathbf{y} \in \mathbb{R}^{\mathrm{Orbits}_S(B)}$*

$$\mathbf{A}^{\mathsf{T}} \cdot \delta_S(\mathbf{y}) = \delta_S(\Gamma_S(\mathbf{A})^{\mathsf{T}} \cdot \mathbf{y}) \quad \textit{and} \quad \mathbf{b}^{\mathsf{T}} \cdot \delta_S(\mathbf{y}) = \Gamma_S(\mathbf{b})^{\mathsf{T}} \cdot \mathbf{y}\,.$$

Proof. We prove the first equality, the proof of the second is similar.

Pick arbitrary $c \in C$. Let $L = \mathrm{orbit}_S(c)$. We have the following sequence of equations proving $(\mathbf{A}^{\mathsf{T}} \cdot \delta_S(\mathbf{y}))(c) = (\delta_S(\Gamma_S(\mathbf{A}) \cdot \mathbf{y}))(c)$.

$(b) \rightarrow (c)$ in second and third lines of the equation

$$\begin{aligned}
& (\mathbf{A}^{\mathsf{T}} \cdot (\delta_S(\mathbf{y})))(c) \\
= \; & \sum_{b \in b} \mathbf{A}^{\mathsf{T}}(c,b) \cdot (\delta_S(\mathbf{y}))(b) \\
= \; & \sum_{b \in b} \mathbf{A}(b,c) \cdot (\delta_S(\mathbf{y}))(b) \\
= \; & \sum_{K \in \mathrm{Orbits}_S(B)} \sum_{b \in K} \mathbf{A}(b,c) \cdot \mathbf{y}(K) && \text{(Definition 7.32)} \\
= \; & \sum_{K \in \mathrm{Orbits}_S(B)} \mathbf{y}(K) \cdot \sum_{b \in K} \mathbf{A}(b,c) && \text{(rearrangement)} \\
= \; & \sum_{K \in \mathrm{Orbits}_S(B)} \mathbf{y}(K) \cdot \Gamma_S(\mathbf{A})(K,L) && \text{(Definition 7.25)} \\
= \; & \sum_{K \in \mathrm{Orbits}_S(B)} \Gamma_S(\mathbf{A})^{\mathsf{T}}(L,K) \cdot \mathbf{y}(K) \\
= \; & (\Gamma_S(\mathbf{A}) \cdot \mathbf{y})(L) \\
= \; & (\delta_S(\Gamma_S(\mathbf{A}) \cdot \mathbf{y}))(c)\,.
\end{aligned}$$

■

Proof of Lemma 7.33

This lemma follows from Lemmas 7.51 and 7.52 in the same way that Lemma 7.26 follows from Lemmas 7.39 and 7.41. ■

7.10 The semi-orbit summation function

Lemma 7.53. *The function γ_T^S is a $(S \cup \{T\})$-supported monotonic linear map.*

Proof. Left to the reader. ■

Lemma 7.54. *For any S-supported vector* $\mathbf{y}$ *we have* $(\delta_S \circ \gamma_T^S)(\mathbf{y}) = \mathbf{y}$.

Proof. Follows from the definitions of δ_S and γ_T^S and left to the reader. ■

Lemma 7.55. *For any orbit-finite set D and vector $\mathbf{y} \in \mathrm{Lin}(D)$, both supported by S*

$$(\gamma_T^S(\mathbf{y}))^\mathsf{T} = \Gamma_S(\mathbf{y}^\mathsf{T})\ .$$

Proof. Using the same notation as Definition 7.34 let

$$K_{S \cup T} = \{b \in K \ : \ \mathrm{support}\,(b) \subseteq S \cup T\}$$

We have

$$\begin{aligned}
& (\gamma_T^S(\mathbf{y}))(K) \\
= \ & \frac{1}{|K_{S\cup T}|} \cdot \left(\sum_{b \in K_{S\cup T}} \mathbf{y}(b) \right) \\
= \ & \frac{1}{|K_{S\cup T}|} \cdot \left(\sum_{b \in K_{S\cup T}} \mathbf{y}(K) \right) \quad \text{(recall Notation 2.28)} \\
= \ & \mathbf{y}(K) \\
= \ & (\Gamma_S(\mathbf{y}^\mathsf{T}))^\mathsf{T}(K)
\end{aligned}$$

This finishes the proof of the lemma. ■

Lemma 7.56. *For any S-supported orbit-finite set D, vector $\mathbf{y} : D \to \mathbb{R}, K \in$* $\mathrm{Orbits}_S(D)$ *and $(S \cup T)$-supported element $b \in K$,*

$$(\gamma_T^S(\mathbf{y}))(K) = \left(\frac{1}{|\mathrm{Aut}(T)|} \cdot \sum_{\pi \in \mathrm{Aut}(T)} \pi(\mathbf{y}) \right)(b)\ .$$

Proof. Define $K_{S\cup T}$ as done inside the definition of γ_T^S (Definition 7.34). We have

$$(\gamma_T^S(\mathbf{y}))(K) = \frac{1}{|K_{S\cup T}|} \cdot \sum_{b\in K_{S\cup T}} \mathbf{y}(b)$$

Let

$$\mathbf{z} = \frac{1}{|\mathrm{Aut}(T)|} \cdot \sum_{\pi\in \mathrm{Aut}(T)} \pi(\mathbf{y})$$

Then

$$\begin{aligned}
\mathbf{z}(b) = & \left(\frac{1}{|\mathrm{Aut}(T)|} \cdot \sum_{\pi\in \mathrm{Aut}(T)} \pi(\mathbf{y})\right)(b) \\
= & \frac{1}{|\mathrm{Aut}(T)|} \cdot \sum_{\pi\in \mathrm{Aut}(T)} (\pi(\mathbf{y}))(b) \\
= & \frac{1}{|\mathrm{Aut}(T)|} \cdot \sum_{\pi\in \mathrm{Aut}(T)} \mathbf{y}(\pi^{-1}(b))
\end{aligned}$$

Using Lemma 2.25-i we get

$$\begin{aligned}
& \frac{1}{|\mathrm{Aut}(T)|} \cdot \sum_{\pi\in \mathrm{Aut}(T)} \mathbf{y}(\pi^{-1}(b)) \\
= & \frac{1}{|\mathrm{Aut}(T)|} \cdot \left(\sum_{b'\in K_{S\cup T}} \left|\left\{\pi \in \mathrm{Aut}(T) \ : \ \pi^{-1}(b) = b'\right\}\right| \cdot \mathbf{y}(b')\right) \\
= & \frac{1}{|\mathrm{Aut}(T)|} \cdot \left(\sum_{b'\in K_{S\cup T}} \left|\left\{\pi \in \mathrm{Aut}(T) \ : \ \pi(b') = b\right\}\right| \cdot \mathbf{y}(b')\right)
\end{aligned}$$

Hence

$$\mathbf{z}(b) = \frac{1}{|\mathrm{Aut}(T)|} \cdot \left(\sum_{b'\in K_{S\cup T}} \left|\left\{\pi \in \mathrm{Aut}(T) \ : \ \pi(b') = b\right\}\right| \cdot \mathbf{y}(b')\right)$$

To finish the proof we show that for every $b' \in K_{S\cup T}$

$$\left|\left\{\pi \in \mathrm{Aut}(T) \ : \ \pi(b') = b\right\}\right| = \frac{\mathrm{Aut}(T)}{|K_{S\cup T}|}$$

For $b' \in K_{S\cup T}$ let $W_{b'} = \{\pi \in \mathrm{Aut}(T) \ : \ \pi(b') = b\}$. Then $\mathrm{Aut}(T)$ is the disjoint union of the sets $W_{b'}$ for b'. Hence

$$|\mathrm{Aut}(T)| = \sum_{b'\in K_{S\cup T}} |W_{b'}| \tag{7.31}$$

Pick any $b' \in K_{S\cup T} = \mathrm{Aut}(T) \cdot \{b\}$. Let $\sigma \in \mathrm{Aut}(T)$ be such that $\sigma(b') = b$. Then $\pi \mapsto (\sigma \circ \pi)$ is a bijection from $W_{b'}$ to W_b with $\pi \mapsto (\sigma^{-1} \circ \pi)$ being its inverse. Since b' is an arbitrary element of $K_{S\cup T}$, this implies for every $b' \in K_{S\cup T}$ we have $W_{b'} = W_b$. Which together with equation (7.31) imply that for every $b' \in K_{S\cup T}$

$$|W_{b'}| = \frac{|\mathrm{Aut}(T)|}{|K_{S\cup T}|}$$

This finishes the proof of the lemma. ∎

Lemma 7.57. *For any vector* $\mathbf{y} : B \to \mathbb{R}$

$$\Gamma_S(\mathbf{A})^\top \cdot \gamma^S_T(\mathbf{y}) = \gamma^S_T(\mathbf{A}^\top \cdot \mathbf{y}) \quad \textit{and} \quad \Gamma_S(\mathbf{b})^\top \cdot \gamma^S_T(\mathbf{y}) = \mathbf{b}^\top \cdot \mathbf{y}\,.$$

Proof of Lemma 7.57. We prove $\Gamma_S(\mathbf{A})^\top \cdot \gamma^S_T(\mathbf{y}) = \gamma^S_T(\mathbf{A}^\top \cdot \mathbf{y})$, the proof of $\Gamma_S(\mathbf{b})^\top \cdot \gamma^S_T(\mathbf{y}) = \mathbf{b}^\top \cdot \mathbf{y}$ is similar.

Pick arbitrary $L \in \mathrm{Orbits}_S(C)$. Using Lemma 2.23-vi pick $c \in L$ supported by $S \cup T$.

$$\begin{aligned}
&(\gamma^S_T(\mathbf{A}^\top \cdot \mathbf{y}))(L) \\
&= \frac{1}{|\mathrm{Aut}(T)|} \cdot \left(\sum_{\pi \in \mathrm{Aut}(T)} \pi(\mathbf{A}^\top \cdot \mathbf{y}) \right)(c) && \text{(Lemma 7.56)} \\
&= \frac{1}{|\mathrm{Aut}(T)|} \cdot \left(\sum_{\pi \in \mathrm{Aut}(T)} \pi(\mathbf{A}^\top) \cdot \pi(\mathbf{y}) \right)(c) && \text{(Item Lemma 2.25-i)} \\
&= \frac{1}{|\mathrm{Aut}(T)|} \cdot \left(\sum_{\pi \in \mathrm{Aut}(T)} \mathbf{A}^\top \cdot \pi(\mathbf{y}) \right)(c) && \text{($\mathbf{A}$ is S-supported)} \\
&= \left(\mathbf{A}^\top \cdot \frac{1}{|\mathrm{Aut}(T)|} \cdot \left(\sum_{\pi \in \mathrm{Aut}(T)} \pi(\mathbf{y}) \right) \right)(c) \\
&= \sum_{b \in B} \mathbf{A}^\top(c, b) \cdot \left(\frac{1}{|\mathrm{Aut}(T)|} \cdot \left(\sum_{\pi \in \mathrm{Aut}(T)} \pi(\mathbf{y}) \right) \right)(b) \\
&= \sum_{b \in B} \mathbf{A}^\top(c, b) \cdot (\gamma^S_T(\mathbf{y}))(\mathrm{orbit}_S(b)) && \text{(Lemma 7.56)} \\
&= \sum_{K \in \mathrm{Orbits}_S(B)} (\gamma^S_T(\mathbf{y}))(K) \cdot \sum_{b \in K} \mathbf{A}(b, c) && \text{(Lemma 7.56)} \\
&= \sum_{K \in \mathrm{Orbits}_S(B)} (\gamma^S_T(\mathbf{y}))(K) \cdot \Gamma_S(\mathbf{A})(K, L) && \text{(Definition 7.25)} \\
&= (\Gamma_S(\mathbf{A})^\top \cdot \gamma^S_T(\mathbf{y}))(L)
\end{aligned}$$

This finishes the proof. ∎

Proof of Lemma 7.35

This lemma follows from Lemmas 7.53, 7.55 and 7.57 in the same way that Lemma 7.29 follows from Lemmas 7.43, 7.47, 7.49 and 7.50. Note that we do not need a counterpart of Lemma 7.47 since unlike in the proof Lemma 7.29, where we had to show that $\delta_T^S(\mathbf{x})$ is supported by $S \cup T$, in Lemma 7.35 we do not have to prove any similar assertion regarding $\gamma_T^S(\mathbf{y})$. ■

7.11 Do orbit-finite linear programs approximate large finite linear programs?

In this section we try to address the question raised in § 1.2.2. Let $\mathcal{U}$ be an orbit-finite $B \times C$-linear program supported by some finite set $S \subseteq_{\mathrm{FIN}} \mathbb{A}$. For any finite set $T \subseteq_{\mathrm{FIN}} \mathbb{A} \setminus S$ let $B_{S\cup T}$ and $C_{S\cup T}$ be the subsets of elements respectively of B and C, which are supported by $S \cup T$. Removing all variables indexed by elements outside $C_{S\cup T}$ and considering only the equalities indexed by $B_{S\cup T}$ we get a finite linear program. Call this $\mathcal{U}_T$. Note that the function $T \mapsto \mathcal{U}_T$ is S-supported. We say $\mathcal{U}$ is a *good approximation* if there exists $N \in \mathbb{N}$ such that for all $T \subseteq_{\mathrm{FIN}} \mathbb{A} \setminus S$ of size at least N, the linear programs $\mathcal{U}_T$ and $\mathcal{U}$ are equivalent (Definition 7.30).

Orbit-finite systems are not always good approximations even if we ignore the objective function and focus on just the solvability of the constraints. We illustrate this with the following example.

Example 7.58. Let $\star$ be an equivariant element, i.e. $\pi(\star) = \star$ for all π in $\mathrm{Aut}(\mathbb{A})$. Let $\mathcal{U}$ be the following system of inequalities with variables $\{\mathbf{x}(\alpha) \ : \ \alpha \in \mathbb{A}\}$:

$$\begin{aligned} \sum_{\alpha \in \mathbb{A}\setminus\{\beta\}} \mathbf{x}(\alpha) &\leqslant 1 \quad (\beta \in \mathbb{A}) \\ \sum_{\alpha \in \mathbb{A}} \mathbf{x}(\alpha) &> 1\,. \end{aligned} \tag{7.32}$$

We prove that $\mathcal{U}_T$ is solvable for every $T \subseteq_{\mathrm{FIN}} \mathbb{A}$ of size at least 2, but $\mathcal{U}$ is unsolvable, even of we allow orbit-infinite solutions.

Pick $T \subseteq_{\mathrm{FIN}} \mathbb{A}$ of size at least 2. The system $\mathcal{U}_T$ contains the following inequalities:

$$\begin{aligned} \sum_{\alpha \in T\setminus\{\beta\}} \mathbf{x}(\alpha) &\leqslant 1 \quad (\beta \in T) \\ \sum_{\alpha \in T} \mathbf{x}(\alpha) &> 1\,. \end{aligned}$$

The vector solution $\mathbf{x}_T$ which assigns the value $\frac{1}{|T|-1}$ to all the variables is a solution of $\mathcal{U}_T$.

Now we show unsolvability of $\mathcal{U}$. Assume otherwise. Let $\mathbf{y} : \mathbb{A} \to \mathbb{R}$ be a solution of $\mathcal{U}$. The sum $\sum_{\alpha \in \mathbb{A}} \mathbf{y}(\alpha)$ is well defined. Hence for any $n \geqslant 1$ there exists $\beta_n \in \mathbb{A}$ such that $\mathbf{y}(\beta_n) \leqslant \frac{1}{n}$. This implies for every $n \geqslant 1$

$$\sum_{\alpha \in \mathbb{A}} \mathbf{y}(\alpha) \;=\; \left(\sum_{\alpha \in \mathbb{A} \setminus \{\beta_n\}} \mathbf{y}(\alpha) \right) + \mathbf{y}(\beta_n) \;\leqslant\; 1 + \frac{1}{n}\,.$$

Since the above is true for all n, we conclude

$$\sum_{\alpha \in \mathbb{A}} \mathbf{y}(\alpha) \leqslant 1\,.$$

This contradicts the fact the $\mathbf{y}$ is a solution of $\mathcal{U}$. ◀

The situation improves if we restrict our attention to orbit-finite linear programs which are either column-finite or row-finite.

Theorem 7.59. *Let $\mathcal{U}$ be an orbit-finite $B \times C$-linear program supported by some $S \subseteq_{\text{FIN}} \mathbb{A}$ and of atom-dimension d. If $\mathcal{U}$ is either column-finite or row-finite, then for every $T \subseteq_{\text{FIN}} \mathbb{A} \setminus S$ of size at least d, the linear programs $\mathcal{U}_T$ and $\mathcal{U}$ are equivalent.*

The remainder of this section is devoted to proving the above theorem.

Proof of Theorem 7.59

Pick an arbitrary orbit-finite column-finite linear program $\mathcal{U}$. Then its dual $\mathcal{U}^\top$ is an arbitrary orbit-finite row-finite linear program. Let $S \subseteq_{\text{FIN}} \mathbb{A}$ be the support of $\mathcal{U}$ and d be its atom-dimension. Then S and d are also respectively the support and atom-dimension of $\mathcal{U}^\top$. Pick $T \subseteq_{\text{FIN}} \mathbb{A} \setminus S$ of size at least d. Observe that $(\mathcal{U}_T)^\top = (\mathcal{U}^\top)_T$. Hence we can write $\mathcal{U}_T^\top$ without any ambiguity. To prove Theorem 7.59 we show that the linear programs $\mathcal{U}$ and $\mathcal{U}_T$ are equivalent, and so are the linear programs $\mathcal{U}^\top$ and $\mathcal{U}_T^\top$.

WLOG assume $\mathcal{U}$ is a maximisation problem, then $\mathcal{U}$ and $\mathcal{U}^\top$ can respectively be written as

$$\begin{array}{llcll}
\text{maximise} & \mathbf{c}^\top \cdot \mathbf{x} & \qquad & \text{minimise} & \mathbf{b}^\top \cdot \mathbf{y} \\
\text{subject to} & \mathbf{A} \cdot \mathbf{x} \leqslant \mathbf{b} & & \text{subject to} & \mathbf{A}^\top \cdot \mathbf{y} \geqslant \mathbf{c} \\
& \mathbf{x} \geqslant \mathbf{0} & & & \mathbf{y} \geqslant \mathbf{0} \\
& \mathbf{x} \in \text{Lin}(C) & & & \mathbf{y} \in \text{Lin}(B)
\end{array}$$

for some orbit-finite sets B and C, column-finite matrix $\mathbf{A} \in \text{Lin}(B \times C)$, vectors $\mathbf{b} \in \text{FinLin}(B)$ and $\mathbf{c} \in \text{Lin}(C)$, all of them supported by S. Define $B_{S \cup T}$ and

$C_{S\cup T}$ to be the subsets of respectively B and C containing elements which are supported by $S \cup T$:

$$\begin{aligned} B_{S\cup T} &= \{b \in B \ : \ \text{support}\,(b) \subseteq S \cup T\} \\ C_{S\cup T} &= \{c \in C \ : \ \text{support}\,(c) \subseteq S \cup T\} \ . \end{aligned}$$

Let $\mathbf{A}_{S\cup T}$, $\mathbf{b}_{S\cup T}$ and $\mathbf{c}_{S\cup T}$ respectively be the restrictions of $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ to $B_{S\cup T} \times C_{S\cup T}$, $B_{S\cup T}$ and $C_{S\cup T}$. $(\mathbf{A}_{S\cup T})^{\top} = (\mathbf{A}^{\top})_{S\cup T}$, $(\mathbf{b}_{S\cup T})^{\top} = (\mathbf{b}^{\top})_{S\cup T}$, and $(\mathbf{c}_{S\cup T})^{\top} = (\mathbf{c}^{\top})_{S\cup T}$. Hence we can write $\mathbf{A}^{\top}_{S\cup T}$, $\mathbf{b}^{\top}_{S\cup T}$, $\mathbf{c}^{\top}_{S\cup T}$ without any confusion. Now, $\mathcal{U}_T$ and $\mathcal{U}_T^{\top}$ can be written as

$$\begin{array}{llcll} \text{maximise} & \mathbf{c}^{\top}_{S\cup T} \cdot \mathbf{u} & \qquad & \text{minimise} & \mathbf{b}^{\top}_{S\cup T} \cdot \mathbf{v} \\ \text{subject to} & \mathbf{A}_{S\cup T} \cdot \mathbf{u} \leqslant \mathbf{b}_{S\cup T} & & \text{subject to} & \mathbf{A}^{\top}_{S\cup T} \cdot \mathbf{v} \geqslant \mathbf{c}_{S\cup T} \\ & \mathbf{u} \geqslant \mathbf{0} & & & \mathbf{v} \geqslant \mathbf{0} \\ & \mathbf{u} \in \mathrm{Lin}(C_{S\cup T}) & & & \mathbf{v} \in \mathrm{Lin}(B_{S\cup T}) \end{array}$$

Since $\mathcal{U}_T$ and $\mathcal{U}_T^{\top}$ are finite linear programs, they have optimal solutions when their optimum is finite. The same is true for $\mathcal{U}$ and $\mathcal{U}^{\top}$ due to Theorem 7.12. Hence for proving the equivalences between $\mathcal{U}$ and $\mathcal{U}_T$, and between $\mathcal{U}^{\top}$ and $\mathcal{U}_T^{\top}$, we just have to show that $\mathcal{U}$ and $\mathcal{U}_T$ have the same optimum, and likewise for $\mathcal{U}^{\top}$ and $\mathcal{U}_T^{\top}$.

The proof for $\mathcal{U}$ and $\mathcal{U}_T$ is split into the following two lemmas.

Lemma 7.60. *The optimum of $\mathcal{U}$ is larger than or equal to the optimum of $\mathcal{U}_T$.*

Lemma 7.61. *The optimum of $\mathcal{U}$ is smaller than or equal to the optimum of $\mathcal{U}_T$.*

And the proof for $\mathcal{U}^{\top}$ and $\mathcal{U}_T^{\top}$ is split into the following two lemmas.

Lemma 7.62. *The optimum of $\mathcal{U}^{\top}$ is smaller than or equal to the optimum of $\mathcal{U}_T^{\top}$.*

Lemma 7.63. *The optimum of $\mathcal{U}^{\top}$ is bigger than or equal to the optimum of $\mathcal{U}_T^{\top}$.*

The remainder of the section is devoted to proving these lemmas.

Proof of Lemma 7.60

Pick a solution $\mathbf{u}'$ of $\mathcal{U}_T$. We find a solution $\mathbf{x}'$ of $\mathcal{U}$ such that

$$\mathbf{c}^{\top} \cdot \mathbf{x}' = \mathbf{c}^{\top}_{S\cup T} \cdot \mathbf{u}' \ .$$

Define $\mathbf{x}' : C \to \mathbb{R}$ by extending $\mathbf{u}'$ to C by assigning 0 to elements outside $C_{S\cup T}$:

$$\mathbf{x}'(c) = \begin{cases} \mathbf{u}'(c) & \text{if } c \in C_{S\cup T} \\ 0 & \text{otherwise.} \end{cases}$$

Immediately from the definition of $\mathbf{x}'$ we get

$$\mathbf{c}^{\mathsf{T}} \cdot \mathbf{x}' = \mathbf{c}_{S \cup T}^{\mathsf{T}} \cdot \mathbf{u}' .$$

We just need to prove that $\mathbf{x}'$ is a solution of $\mathcal{U}$. The vector $\mathbf{x}'$ is finite and hence also orbit-finite (Lemma 2.47). It is also non-negative. Hence we just have to prove that $\mathbf{A} \cdot \mathbf{x}' \leqslant \mathbf{b}$, equivalently $(\mathbf{A} \cdot \mathbf{x}')(b) \leqslant \mathbf{b}(b)$ for all $b \in B$. We split the proof into two cases depending on whether b is inside $B_{S \cup T}$ or not.

Case 1 ($b \notin B_{S \cup T}$): Since $\mathbf{b}$ is a finite vector supported by S and the set B is also supported by S, using Lemma 2.47 we conclude that $\mathbf{b}(b) = 0$. We finish the proof for this case by showing $(\mathbf{A} \cdot \mathbf{x}')(b) = 0$. The matrix $\mathbf{A}$ is supported by S and the vector $\mathbf{x}$ is supported by $S \cup T$, hence the product $\mathbf{A} \cdot \mathbf{x}'$ is also supported by $S \cup T$. Since $\mathbf{A}$ is column-finite and $\mathbf{x}'$ is a finite vector, using Lemma 7.9 we conclude that $\mathbf{A} \cdot \mathbf{x}'$ is also a finite vector. Again using Lemma 2.47 we get $(\mathbf{A} \cdot \mathbf{x}')(b) = 0$.

Case 2 ($b \in B_{S \cup T}$): Using the definition of matrix multiplication we get

$$(\mathbf{A} \cdot \mathbf{x}')(b) = \sum_{c \in C} \mathbf{A}(b, c) \cdot \mathbf{x}'(c) .$$

Since $\mathbf{x}'$ is 0 outside $C_{S \cup T}$ we have

$$\sum_{c \in C} \mathbf{A}(b, c) \cdot \mathbf{x}'(c) = \sum_{c \in C_{S \cup T}} \mathbf{A}(b, c) \cdot \mathbf{x}'(c) .$$

Using the fact $b \in B_{S \cup T}$ and the definitions of $\mathbf{A}_{S \cup T}$ and $\mathbf{x}'$ we conclude

$$\sum_{c \in C_{S \cup T}} \mathbf{A}(b, c) \cdot \mathbf{x}'(c) = \sum_{c \in C_{S \cup T}} \mathbf{A}_{S \cup T}(c) \cdot \mathbf{u}'(c) .$$

Since $\mathbf{u}'$ is a solution of $\mathcal{U}_T$

$$\sum_{c \in C_{S \cup T}} \mathbf{A}_{S \cup T}(b, c) \cdot \mathbf{u}'(c) \leqslant \mathbf{b}_{S \cup T}(b)$$

The vector $\mathbf{b}_{S \cup T}$ is the restriction of $\mathbf{b}$ to $B_{S \cup T}$. Hence

$$\mathbf{b}_{S \cup T}(b) = \mathbf{b}(b) .$$

Combining the above equalities and inequalities we get

$$(\mathbf{A} \cdot \mathbf{x}')(b) \leqslant \mathbf{b}(b) .$$

■

Proof of Lemma 7.61

Pick a solution $\mathbf{x}''$ of $\mathcal{U}$. We find a solution $\mathbf{u}'$ of $\mathcal{U}_T$ such that

$$\mathbf{c}^\top \cdot \mathbf{x}'' \leqslant \mathbf{c}_{S\cup T}^\top \cdot \mathbf{u}' .$$

Using Theorem 7.16 we conclude existence of an $(S \cup T)$-supported finite solution $\mathbf{x}'$ of $\mathcal{U}$ such that

$$\mathbf{c}^\top \cdot \mathbf{x}' \geqslant \mathbf{c}^\top \cdot \mathbf{x}'' .$$

Let $\mathbf{u}'$ be the restriction of $\mathbf{x}'$ to $C_{S\cup T}$. Lemma 2.47 implies that $\mathbf{x}'$ is 0 outside $C_{S\cup T}$. Hence $\mathbf{c}^\top \cdot \mathbf{x}' = \mathbf{c}_{S\cup T}^\top \cdot \mathbf{u}'$. It remains to prove that $\mathbf{u}'$ is a solution of $\mathcal{U}_T$.

The vector $\mathbf{u}'$ being the restriction of the non-negative vector $\mathbf{x}'$ is also non-negative. Hence to show $\mathbf{u}'$ is a solution of $\mathcal{U}_T$ we have to prove $(\mathbf{A}_{S\cup T} \cdot \mathbf{u}') \leqslant \mathbf{b}_{S\cup T}$.

Pick $b \in B_{S\cup T}$. We need to show $(\mathbf{A}_{S\cup T} \cdot \mathbf{u}')(b) \leqslant \mathbf{b}_{S\cup T}(b)$. Since $\mathbf{x}'$ is 0 outside $C_{S\cup T}$ and $\mathbf{x}'$ is a solution of $\mathcal{U}$ we have,

$$\begin{aligned}
\mathbf{b}_{S\cup T}(b) &= \mathbf{b}(b) \\
&\geqslant (\mathbf{A} \cdot \mathbf{x}')(b) \\
&= \sum_{c\in C} \mathbf{A}(b,c) \cdot \mathbf{x}'(c) \\
&= \sum_{c\in C_{S\cup T}} \mathbf{A}(b,c) \cdot \mathbf{x}'(c) \\
&= \sum_{c\in C_{S\cup T}} \mathbf{A}_{S\cup T}(b,c) \cdot \mathbf{u}'(c) = (\mathbf{A}_{S\cup T} \cdot \mathbf{u}')(b)
\end{aligned}$$

This finishes the proof. ∎

Proof of Lemma 7.62

remove Γ_S

Recall Corollary 7.36 which says that $\Gamma_S(\mathcal{U})$ and $\Gamma_S(\mathcal{U})^\top$ are equivalent. Hence to show that the optimum of $\mathcal{U}^\top$ is smaller than or equal to the optimum of $\mathcal{U}_T^\top$, it is enough to prove that for any solution $\mathbf{v}'$ of $\mathcal{U}_T^\top$ there exists a solution $\mathbf{z}'$ of $\Gamma_S(\mathcal{U})^\top$ such that

$$\Gamma_S(\mathbf{b})^\top \cdot \mathbf{z}' = \mathbf{b}_{S\cup T}^\top \cdot \mathbf{v}' .$$

Pick a solution $\mathbf{v}'$ of $\mathcal{U}_T^\top$. Extend $\mathbf{v}'$ to a $\mathbf{y}' : B \to \mathbb{R}$ by assigning 0 outside $B_{S\cup T}$:

$$\mathbf{y}'(b) = \begin{cases} \mathbf{v}'(b) & \text{if } b \in B_{S\cup T} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbf{A}$ is a column-finite matrix, $\mathbf{A}^\top \cdot \mathbf{y}'$ is well-defined (Lemma 7.9). Define $\mathbf{c}' : C \to \mathbb{R}$ as

$$\mathbf{c}'(c) = \begin{cases} \mathbf{c}(c) & \text{if } c \in C_{S \cup T} \\ (\mathbf{A}^\top \cdot \mathbf{y}')(c) & \text{otherwise.} \end{cases}$$

Claim 7.62.1. $\mathbf{A}^\top \cdot \mathbf{y}' \geqslant \mathbf{c}'$

Claim 7.62.2. $\gamma^S_T(\mathbf{c}')^\top = \Gamma_S(\mathbf{c}^\top)$.

Before proving the above claims we show that they imply $\mathbf{z}' = \gamma^S_T(\mathbf{y}')$ is a solution of $\Gamma_S(\mathcal{U})^\top$ such that

$$\Gamma_S(\mathbf{b})^\top \cdot \gamma^S_T(\mathbf{y}') = \mathbf{b}^\top_{S \cup T} \cdot \mathbf{v}' \,.$$

The vector $\mathbf{v}'$ is non-negative and hence $\mathbf{y}'$ is also so. Using the definition of γ^S_T we conclude $\gamma^S_T(\mathbf{y}')$ is non-negative as well. Now

$$\begin{aligned} & \Gamma_S(\mathbf{A})^\top \cdot \gamma^S_T(\mathbf{y}') \\ &= \gamma^S_T(\mathbf{A}^\top \cdot \mathbf{y}') && \text{Lemma 7.57} \\ &\geqslant \gamma^S_T(\mathbf{c}') && \text{Lemma 7.53 and Claim 7.62.1} \\ &= \Gamma_S(\mathbf{c}^\top)^\top && \text{Claim 7.62.2.} \end{aligned}$$

Also, using Lemma 7.57 we get

$$\Gamma_S(\mathbf{b})^\top \cdot \gamma^S_T(\mathbf{y}') = \mathbf{b}^\top \cdot \mathbf{y}' \,.$$

Since $\mathbf{y}'$ is 0 outside $B_{S \cup T}$ and agrees with $\mathbf{v}'$ inside $B_{S \cup T}$ we have

$$\mathbf{b}^\top \cdot \mathbf{y}' = \sum_{b \in B} \mathbf{b}(b) \cdot \mathbf{y}'(b) = \sum_{b \in B_{S \cup T}} \mathbf{b}_{S \cup T}(b) \cdot \mathbf{v}'(b) = \mathbf{b}^\top_{S \cup T} \cdot \mathbf{v}' \,.$$

As a consequence we get

$$\Gamma_S(\mathbf{b})^\top \cdot \gamma^S_T(\mathbf{y}') = \mathbf{b}^\top_{S \cup T} \cdot \mathbf{v}' \,.$$

This finishes the proof of Lemma 7.62 modulo the proofs of Claims 7.62.1 and 7.62.2, which we do now.

Proof of Claim 7.62.1. Pick $c \in C$. We show $(\mathbf{A}^\top \cdot \mathbf{y}')(c) \geqslant \mathbf{c}'(c)$. We split the proof into two cases depending on whether c is inside or outside $C_{S \cup T}$.

Case 1:($c \notin C_{S \cup T}$) The proof of this case follows from the definition of the vector $\mathbf{c}'$.

Case 1:($c \in C_{S\cup T}$) Since $\mathbf{v}'$ is a solution of $\mathcal{U}_T^{\mathsf{T}}$ we have

$$(\mathbf{A}_{S\cup T}^{\mathsf{T}} \cdot \mathbf{v}')(c) \geqslant \mathbf{c}_{S\cup T}(c) = \mathbf{c}'(c) \ .$$

To finish the proof for this case we show that

$$(\mathbf{A}_{S\cup T}^{\mathsf{T}} \cdot \mathbf{v}')(c) = (\mathbf{A}^{\mathsf{T}} \cdot \mathbf{y}')(c)$$

By definition of matrix multiplication

$$(\mathbf{A}^{\mathsf{T}} \cdot \mathbf{y}')(c) = \mathbf{A}^{\mathsf{T}}(c,-) \cdot \mathbf{y}' \ .$$

Using Lemma 2.49-ii we conclude that the vector $\mathbf{A}^{\mathsf{T}}(c,-)$ is supported by $S \cup \operatorname{support}(c) \subseteq S \cup T$. The matrix $\mathbf{A}^{\mathsf{T}}$ is row-finite. Hence Lemma 2.47 implies that $\mathbf{A}^{\mathsf{T}}(c,b) \neq 0$ only if $b \in B_{S\cup T}$. Hence

$$\mathbf{A}^{\mathsf{T}}(c,-) \cdot \mathbf{y}' = \sum_{b \in B_{S\cup T}} \mathbf{A}^{\mathsf{T}}(c,b) \cdot \mathbf{y}'(b) \ .$$

Using the definitions of $\mathbf{A}_{S\cup T}$ and $\mathbf{y}'$ we rewrite the RHS of the above sum as

$$\sum_{b \in B_{S\cup T}} \mathbf{A}^{\mathsf{T}}(c,b) \cdot \mathbf{y}'(b) = \sum_{b \in B_{S\cup T}} \mathbf{A}_{S\cup T}^{\mathsf{T}}(c,b) \cdot \mathbf{v}'(b) \ .$$

Clearly

$$\sum_{b \in B_{S\cup T}} \mathbf{A}_{S\cup T}^{\mathsf{T}}(c,b) \cdot \mathbf{v}'(b) = (\mathbf{A}_{S\cup T}^{\mathsf{T}} \cdot \mathbf{v}')(b)$$

Using the above inequalities we conclude that

$$(\mathbf{A}_{S\cup T}^{\mathsf{T}} \cdot \mathbf{v}')(c) = (\mathbf{A}^{\mathsf{T}} \cdot \mathbf{y}')(c)$$

and finish the proof of this claim. □

Proof of Claim 7.62.2. The vectors $\mathbf{c}$ and $\mathbf{c}'$ agree on $C_{S\cup T}$. Since the function γ_T^S only looks at values inside $C_{S\cup T}$, we have

$$\gamma_T^S(\mathbf{c}') = \gamma_T^S(\mathbf{c}) \ .$$

The vector $\mathbf{c}$ is supported by S. Hence Lemma 7.55 implies

$$\gamma_T^S(\mathbf{c})^{\mathsf{T}} = \Gamma_S(\mathbf{c}^{\mathsf{T}}) \ .$$

Combining the above two equalities we get $\gamma_T^S(\mathbf{c}')^{\mathsf{T}} = \Gamma_S(\mathbf{c}^{\mathsf{T}})$. □

With the proofs of Claims 7.62.1 and 7.62.2, the proof of Lemma 7.62 is also complete. ■

Proof of Lemma 7.63

Pick a solution $\mathbf{y}'$ of $\mathcal{U}^\top$. We construct a solution $\mathbf{v}'$ of $\mathcal{U}_T^\top$ such that

$$\mathbf{b}^\top \cdot \mathbf{y}' = \mathbf{b}_{S\cup T}^\top \cdot \mathbf{v}' .$$

Define $\mathbf{v}'$ to be simply the restriction of $\mathbf{y}'$ to $B_{S\cup T}$. Then $\mathbf{v}'$ is non-negative since $\mathbf{y}'$ is also so. Since $\mathbf{b}$ is finite vector supported by S, Lemma 2.47 implies $\mathbf{b}$ is 0 outside $B_{S\cup T}$. Hence

$$\mathbf{b}^\top \cdot \mathbf{y}' = \sum_{b\in B_{S\cup T}} \mathbf{b}(b) \cdot \mathbf{y}'(b) = \sum_{b\in B_{S\cup T}} \mathbf{b}_{S\cup T}(b) \cdot \mathbf{v}'(b) = \mathbf{b}_{S\cup T}^\top \cdot \mathbf{v}' .$$

Hence we just have to show that $\mathbf{v}'$ is a solution of $\mathcal{U}_T^\top$.

Since $\mathbf{y}'$ is non-negative, so is $\mathbf{v}'$. We have to show $\mathbf{A}_{S\cup T}^\top \cdot \mathbf{v}' \geqslant \mathbf{c}_{S\cup T}$. Pick arbitrary $c \in C_{S\cup T}$. We show $(\mathbf{A}_{S\cup T}^\top \cdot \mathbf{v}')(c) \geqslant \mathbf{c}_{S\cup T}(c)$. Lemma 2.49-ii implies the row $\mathbf{A}^\top(c,-)$ is supported by $\operatorname{support}(\mathbf{A}) \cup \operatorname{support}(c) \subseteq S \cup T$. The matrix $\mathbf{A}^\top$ is row-finite. Hence Lemma 2.47 the row $\mathbf{A}^\top(c,-)$ is 0 outside $B_{S\cup T}$. Hence

$$\begin{aligned}
(\mathbf{A}^\top \cdot \mathbf{y}')(c) &= \sum_{b\in B_{S\cup T}} \mathbf{A}^\top(c,b) \cdot \mathbf{y}'(b) \\
&= \sum_{b\in B_{S\cup T}} \mathbf{A}_{S\cup T}^\top(c,b) \cdot \mathbf{v}'(b) \\
&= (\mathbf{A}_{S\cup T}^\top \cdot \mathbf{v}')(c) .
\end{aligned}$$

Since $\mathbf{y}'$ is a solution of $\mathcal{U}^\top$ we have $(\mathbf{A}^\top \cdot \mathbf{y}')(c) \geqslant \mathbf{c}(c) = \mathbf{c}_{S\cup T}(c)$, which finishes the proof. ■

Now that we have proven Lemmas 7.60 to 7.63, the proof of Theorem 7.59 is completed. ■

Chapter 8

Final Remarks

In this thesis, we have studied solvability of orbit-finite systems of equations and inequalities. Other than the results, there are a few takeaways which we present in this final chapter of the thesis.

As pointed out in Remark 2.22, we focused on equality atoms. The final goal of this line of research would be to answer the following:

Question 8.1. *For which structures of atoms, solvability of orbit-finite systems of equations and inequalities is decidable?*

While this is a hard question, this thesis gives a roadmap for proving decidability of systems of equations. The first step would be to extend the orbit-finite basis theorem (Theorem 3.1) and use it to reduce solvability to finitary solvability (extension of Theorem 4.3), and the second would be to solve finitary solvability.

The first step also raises another fundamental classification question:

Question 8.2. *For which structures of atoms $\mathbb{X}$, the vector spaces of orbit-finite functions from orbit-finite sets to some field $\mathbb{F}$ have orbit-finite bases?*

The recent work [31] provides a positive answer to the above question for $\mathbb{X}$ a *stable* structure or a dense linear order. A natural next step is to extend Theorem 4.3 to these structures. In [31] it has also been conjectured that the class of structures for which the answer to Question 8.2 is positive is exactly the class of NIP structures. Example 6.9 in [6] shows that the answer to the above question is negative when $\mathbb{X}$ is the structure of *Rado graph*.

What is this, add citation

The second step, i.e. finitary solvability of orbit-finite systems of equations has also been solved for several cases. In Chapter 4 we proved its decidability for equality atoms, by reducing it to order equivariant finitary solvability and showing the latter is decidable. As mentioned in Remark 4.35, finitary solvability with ordered atoms can also be reduced to order equivariant finitary solvability.

For this decidability result we recycled the proof of [6, Theorem 4.8] which implies that both equality and ordered atoms satisfy the *Noetherian property for orbit-finitely generated vector spaces*: for an equivariant orbit-finite set B, any increasing sequence of equivariant subspaces of $\mathrm{FinLin}(B)$ stabilises. We believe this strategy can work in general. That is, given a proof of the Noetherian property of some structure of atoms $\mathbb{X}$, we can convert it to an algorithm for deciding finitary solvability for orbit-finite systems of equations with atoms $\mathbb{X}$. In this context we recall the following conjecture presented in [6].

Conjecture 8.3. *A structures of atoms $\mathbb{X}$ satisfies the Noetherian property for orbit-finitely generated vector spaces if and only if it is oligomorphic.*

Another interesting thing to notice regarding finitary solvability is the reappearance of the orbit summation function in different avatars, (Definitions 4.25 and 7.21) and how these functions reduce finitary solvability to solvability of a finite system (Theorem 4.29 and Corollary 7.31). It is possible that this result can be extended to the general oligomorphic case.

We also mention [16, Theorem 12] [1] which says that structure of atoms satisfying a natural well-quasi-ordering property ([16, Definition 6] satisfies *Noetherian property for equivariant polynomial ideals* ([16, Property 4]). The well-quasi-ordering property implies oligomorphicity and the Noetherian property for equivariant polynomial ideals implies the Noetherian property for orbit-finitely generated vector spaces. Hence this result partially confirms Conjecture 8.3. In our ongoing work with Aliaume Lopez we have proven decidability of the equivariant ideal membership problem for the subclass of atoms satisfying the above mentioned well-quasi ordering property, which implies decidability of finitary solvability of orbit-finite systems of linear equations, which reinforces the belief that for any structure of atoms, Noetherian property for orbit-finitely generated vector spaces implies decidability of orbit-finite systems of linear equations.

Our technique in § 5.5 for solving systems of inequalities is more difficult (if at all possible) to extend to other atoms. This is simply because for most structures of atoms $\mathbb{X}$, the number of $\{T\}$-orbits of $\mathbb{X}$ increases with T. We encourage the reader to verify this for ordered atoms and graph atoms (Remark 2.22). Having said that, we believe that this technique can be helpful in solving questions regarding data Petri nets. A similar technique has already been used to decide continuous reachability for data Petri nets ([17]). For attacking the problem of solving orbit-finite systems of linear inequalities with a wide class of atoms, we probably need to gain more intuition by working on specific structures of atoms. Solving the problem for ordered atoms would be a natural next step.

[1] This work was done during the doctoral studies of the author but is not part of this thesis.

There are still a few important topics related to orbit-finite systems of equations and inequalities which we have not covered.

Firstly, we did not discuss unrestricted solvability (§ 2.4.1), i.e. solvability without the assumption of orbit-finiteness of solutions. This leads to the following open question:

Question 8.4. *Is unrestricted solvability of orbit-finite systems of equations and inequalities decidable?*

Secondly, we have not discussed equivariant/finitely supported subspaces (§ 2.2.1) in detail. These are studied in [6] and [36]. In [6] the authors have proven that they are spanned by an orbit-finite set, and in [36] the author has given a formula to compute the length of orbit-finitely generated vector spaces.[2] However, none of these results can be applied to compute a spanning set of the kernel of an orbit-finite matrix or, equivalently, a spanning set of the subspace of solutions for an orbit-finite system of homogeneous linear equations. We state this as an open question.

Question 8.5. *Prove computability of spanning sets of kernels of an orbit-finite matrix.*

Finally, we also have not discussed orbit-finitely generated cones and solution sets of orbit-finite sets of linear inequalities. For atom-dimension 1 these objects are studied in [26]. **Arka[4]:** Acknowledgements: We are indebted to Damian Niwiński for posing the questions about duality in orbit-finite linear programming. **Arka[5]:** Last comment

[2]We have described this result in § 2.2.1

Bibliography

[1] Albert Atserias, Anuj Dawar, and Joanna Fijalkow. On the power of symmetric linear programs. *J. ACM*, 68(4):26:1–26:35, 2021.

[2] Pascal Baumann, Eren Keskin, Roland Meyer, and Georg Zetzsche. Separability in Büchi VASS and singly non-linear systems of inequalities. In Karl Bringmann, Martin Grohe, Gabriele Puppis, and Ola Svensson, editors, *51st International Colloquium on Automata, Languages, and Programming, ICALP 2024, July 8-12, 2024, Tallinn, Estonia*, volume 297 of *LIPIcs*, pages 126:1–126:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2024.

[3] Mikołaj Bojańczyk. *An Automata Toolbox*. Available at `https://www.mimuw.edu.pl/~bojan/papers/toolbox.pdf`.

[4] Mikołaj Bojańczyk. Data monoids. In *Proc. STACS 2011*, volume 9 of *LIPIcs*, pages 105–116, 2011.

[5] Mikołaj Bojańczyk. *Slightly Infinite Sets*. September 2019.

[6] Mikołaj Bojańczyk, Joanna Fijalkow, Bartek Klin, and Joshua Moerman. Orbit-finite-dimensional vector spaces and weighted register automata. *TheoretiCS*, 3, 2024.

[7] Mikołaj Bojańczyk, Bartek Klin, and Sławomir Lasota. Automata with group actions. In *Proc. LICS 2011*, pages 355–364, 2011.

[8] Mikołaj Bojańczyk, Bartek Klin, and Sławomir Lasota. Automata theory in nominal sets. *Log. Methods Comput. Sci.*, 10(3), 2014.

[9] Mikołaj Bojańczyk, Bartek Klin, Sławomir Lasota, and Szymon Toruńczyk. Turing machines with atoms. In *Proc. LICS'13*, pages 183–192, 2013.

[10] Mikołaj Bojańczyk and Szymon Toruńczyk. Imperative Programming in Sets with Atoms. In *Proc. FSTTCS 2012*, volume 18, pages 4–15, 2012.

[11] Mikołaj Bojańczyk and Szymon Toruńczyk. On computability and tractability for infinite sets. In Anuj Dawar and Erich Grädel, editors, *Proc. LICS 2018*, pages 145–154. ACM, 2018.

[12] Lorenzo Clemente and Sławomir Lasota. Reachability analysis of first-order definable pushdown systems. In Stephan Kreutzer, editor, *Proc. CSL 2015*, volume 41 of *LIPIcs*, pages 244–259, 2015.

[13] D.E Cohen. On the laws of a metabelian variety. *Journal of Algebra*, 5(3):267–273, 1967.

[14] Richard G. Cooke. *Infinite Matrices and Seqeuence Spaces*. Dover Publications, July 2014.

[15] Wojciech Czerwinski and Łukasz Orlikowski. Reachability in vector addition systems is ackermann-complete. In *62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2021, Denver, CO, USA, February 7-10, 2022*, pages 1229–1240. IEEE, 2021.

[16] Arka Ghosh and Sławomir Lasota. Equivariant ideals of polynomials. In Pawel Sobociński, Ugo Dal Lago, and Javier Esparza, editors, *Proceedings of the 39th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2024, Tallinn, Estonia, July 8-11, 2024*, pages 38:1–38:14. ACM, 2024.

[17] Utkarsh Gupta, Preey Shah, S. Akshay, and Piotr Hofman. Continuous reachability for unordered data Petri nets is in ptime. In Mikołaj Bojańczyk and Alex Simpson, editors, *Proc. FOSSACS 2019*, volume 11425 of *Lecture Notes in Computer Science*, pages 260–276. Springer, 2019.

[18] Piotr Hofman and Sławomir Lasota. Linear equations with ordered data. In Sven Schewe and Lijun Zhang, editors, *29th International Conference on Concurrency Theory, CONCUR 2018, September 4-7, 2018, Beijing, China*, volume 118 of *LIPIcs*, pages 24:1–24:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.

[19] Piotr Hofman, Sławomir Lasota, Ranko Lazic, Jérôme Leroux, Sylvain Schmitz, and Patrick Totzke. Coverability trees for Petri nets with unordered data. In Bart Jacobs and Christof Löding, editors, *Foundations of Software Science and Computation Structures - 19th International Conference, FOSSACS 2016, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2016, Eindhoven, The Netherlands, April 2-8, 2016, Proceedings*, volume 9634 of *Lecture Notes in Computer Science*, pages 445–461. Springer, 2016.

[20] Piotr Hofman, Jérôme Leroux, and Patrick Totzke. Linear combinations of unordered data vectors. In *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*, pages 1–11. IEEE Computer Society, 2017.

[21] Piotr Hofman and Jakub Rózycki. Linear equations for unordered data vectors in $[\mathcal{D}]^k \to \mathbb{Z}^d$. *Log. Methods Comput. Sci.*, 18(4), 2022.

[22] Khadijeh Keshvardoost, Bartek Klin, Sławomir Lasota, Joanna Fijalkow, and Szymon Toruńczyk. Definable isomorphism problem. *Log. Methods Comput. Sci.*, 15(4), 2019.

[23] Bartek Klin, Eryk Kopczynski, Joanna Fijalkow, and Szymon Toruńczyk. Locally finite constraint satisfaction problems. In *30th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2015, Kyoto, Japan, July 6-10, 2015*, pages 475–486. IEEE Computer Society, 2015.

[24] Bartek Klin, Sławomir Lasota, Joanna Fijalkow, and Szymon Toruńczyk. Turing machines with atoms, constraint satisfaction problems, and descriptive complexity. In Thomas A. Henzinger and Dale Miller, editors, *Proc. CSL-LICS 2014*, pages 58:1–58:10. ACM, 2014.

[25] Bartek Klin, Sławomir Lasota, Joanna Fijalkow, and Szymon Toruńczyk. Homomorphism problems for first-order definable structures. In Akash Lal, S. Akshay, Saket Saurabh, and Sandeep Sen, editors, *Proc. FSTTCS 2016*, volume 65 of *LIPIcs*, pages 14:1–14:15, 2016.

[26] Dinh V. Le and Tim Römer. Theorems of Carathéodory, Minkowski–Weyl, and Gordan up to symmetry. *SIAM Journal on Applied Algebra and Geometry*, 7(1):291–310, 2023.

[27] Jérôme Leroux. The reachability problem for Petri nets is not primitive recursive. In *62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2021, Denver, CO, USA, February 7-10, 2022*, pages 1241–1252. IEEE, 2021.

[28] Jérôme Leroux and Sylvain Schmitz. Reachability in vector addition systems is primitive-recursive in fixed dimension. In *34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019, Vancouver, BC, Canada, June 24-27, 2019*, pages 1–13. IEEE, 2019.

[29] Christos H. Papadimitriou and Kenneth Steiglitz. *Combinatorial Optimization : Algorithms and Complexity*. Dover Publications, 1998.

[30] A. M. Pitts. *Nominal Sets: Names and Symmetry in Computer Science*, volume 57 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2013.

[31] Michał R. Przybyłek. A note on stone-Čech compactification in ZFA. *arXiv-preprint*, 2024.

[32] H. Edwin Romeijn, Robert L. Smith, and ames C. Bean. Duality in infinite dimensional linear programming. *Mathematical Programming*, 93, 1992.

[33] Walter Rudin. *Real and complex analysis, 3rd ed.* McGraw-Hill, Inc., USA, 1987.

[34] Arne Storjohann. A fast+practical+deterministic algorithm for triangularizing integer matrices. 1996.

[35] Gilbert Strang. *Linear Algebra and Its Applications*. Brooks Cole, 4th edition, July 2005.

[36] Jingjie Yang. Equivariant subspaces of orbit-finite dimensional vector spaces, 2023. Available at `https://www.cs.ox.ac.uk/files/14521/masters-thesis.pdf`.

[37] Jingjie Yang. Personal communication, 2024.