Decentralized Data Fusion with Probabilisticly Conservative Ellipsoidal Intersection

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Introduction

The basic method for "combining" probability distribution functions (PDFs) from multiple sensors is to use Bayes' rule, which (roughly) involves multiplying the input PDFs together and then normalizing. This approach, however, leads to one of the central problems in enabling distributed data fusion networks: the "track-to-track correlation" problem, because Bayes' rule assumes that the input distributions to the fusion center are independent, however in a distributed data fusion technique, correlations are introduced both by "feedback", or by different "tracks" of data coming from a single node to the node performing fusion.
Addressing this correlation problem has been the focus of significant efforts in the past, however in this paper we focus on data fusion techniques that yield conservative probability distributions regardless of the correlation on the inputs (correlation agnostic). In previous works considering correlation-agnostic fusion of Gaussian PDFs, the covariance of the fusion technique's output was determined solely by the covariances of the input distribution. The primary contribution of this paper is a technique for considering both the covariance and mean information of the input distributions when performing data fusion. The mean information is used to (probabilisticly) bound the amount of common information that can be present.

Probablistic Constraint (PC)

Proposition 1. *If multiple Gaussian PDFs express uncertainty about a single estimate, the differences in their means are due solely to the presence of noise.*

Corollary 2. *A difference of means between two inputs implies the presence of independent information in the inputs.*

Based on Proposition 1 and assuming $\mu_{a\backslash c}$, $\mu_{b\backslash c}$, and μ_c are estimates of the same quantity, then the differences in mean value should be due solely to noise, i.e. different estimates of the quantity being estimated. This proposition can be expressed as a null hypothesis test

$$H_0 : \mu_{a\backslash c} - \mu_{b\backslash c} = 0 \tag{1}$$
$$H_A : \mu_{a\backslash c} - \mu_{b\backslash c} \neq 0 \tag{2}$$

where the alternative hypothesis is that $\mu_{a\backslash c}$ and $\mu_{b\backslash c}$ are estimates of different quantities.

Definition 3. *An estimate of common information is* probablisticly admissible *if there is not sufficient evidence to reject the null hypothesis that they are estimates of the same quantity.*

By constraining the common information estimates to probabilistically admissible estimates, the estimated quantity of common information will be reduced, leading to fusion results that are less conservative than previous techniques.
For Gaussian distributions, the null hypothesis test can be expressed as a threshold on the Mahalanobis distance

$$\Psi(C_c) = (\mu_{a\backslash c} - \mu_{b\backslash c})^\top (C_{a\backslash c} + C_{b\backslash c})^{-1} (\mu_{a\backslash c} - \mu_{b\backslash c}) \tag{3}$$

where $C_{a\backslash c}$, $C_{b\backslash c}$, $\mu_{a\backslash c}$, and $\mu_{b\backslash c}$ are all functions of C_c. The threshold value γ is derived from a user selected level of significance $\alpha = p(\Psi \geq \gamma | H_0)$, or the probability of rejecting the null hypothesis when the null hypothesis was actually true.

References

[1] Joris Sijs and Mircea Lazar. State fusion with unknown correlation: Ellipsoidal intersection. *Automatica*, 48(8):1874–1878, 2012.

[2] Muhammad Abu Bakr and Sukhan Lee. Distributed multisensor data fusion under unknown correlation and data inconsistency. *Sensors*, 17(11):2472, 2017.

Ellipsoidal Intersection

To create a fusion algorithm that considers probabilistic constraints on the common information, we will be modifying the Ellipsoidal Intersection (EI) algorithm[1]. To review, the EI algorithm attempts to solve the optimization problem

$$\begin{aligned} C_c = \operatorname*{argmin}_{\Upsilon} |\Upsilon| \\ s.t.\ x^\top(\Upsilon - C_a)x \geq 0\,\forall x. \\ x^\top(\Upsilon - C_b)x \geq 0\,\forall x. \end{aligned} \tag{4}$$

where $|\Upsilon|$ represents the determinant of the matrix and the constraints express the admissible constraint of Definition 3.
To find the matrix that has the minimum determinant, given the constraints, ellipsoidal geometry is used to derive the closed form expression

$$\begin{aligned} D_{C_c} = \begin{cases} \max(D_2, 1) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \\ C_c = E_a D_a^{0.5} Q_b D_{C_c} Q_b^T D_a^{0.5} E_a^T + \epsilon I \end{aligned} \tag{5}$$

where ϵ is a small numerical constant used for numerical stability when computing the mean and

$$\begin{aligned} C_a = E_a D_a E_a^T \\ Q_b D_b Q_b^T = D_a^{-0.5} E_a^T C_b E_a D_a^{-0.5} \end{aligned} \tag{6}$$

Probablistic Consistent Fusion

The complete probabilisticly constrained fusion technique can be described using the following optimization problem:

$$C_c = C_{c,EI} + SS^T \tag{7}$$

$$\begin{aligned} S := \operatorname*{argmin}_{\Upsilon} |\Upsilon| \\ s.t.\ \Psi(C_{c,EI} + \Upsilon\Upsilon^T) \leq \gamma \end{aligned} \tag{8}$$

where Υ is a lower triangular matrix, γ is the chi-square threshold selected by the user for a problem of dimension N, $C_{c,EI}$ is the common covariance computed by ellipsoidal intersection, and $\Psi(...)$ is described by (3). This optimization problem finds the smallest matrix SS^T (where small is defined by its trace) required to make $C_{c_{PC}}$ meet the probabilistic constraint.

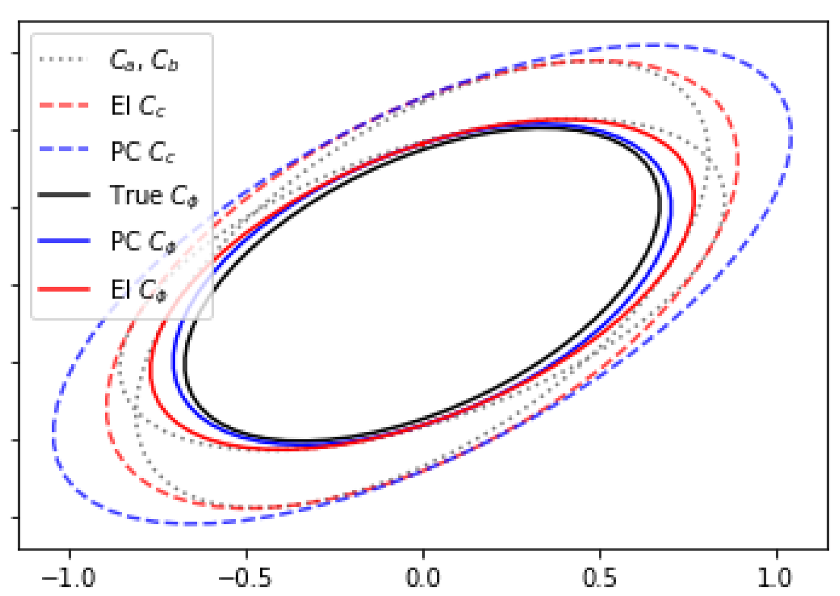

Figure 5: Comparison of PC (blue) and EI (red) fusion approaches. Light gray dotted ellipses denote the input covariances and dark black the optimal fusion result. Note that PC has a larger estimated common covariance (blue-dash) than EI (red-dash) leading to a smaller fused covariance output from PC(blue solid) than EI (red solid).

Figure 5 demonstrates the influence of the additional constraint (based on Definition 3), on the predicted common information. Notice how the increase in the common covariance-caused by the addition of SS^T, leads to a fused estimate much closer to the truth. Note that the estimated common covariance of the PC method is larger, meaning it estimates less common information between the two inputs.

Intuition behind probabilistic constraint

Input Distribution Covariances are the same: In Figure 2, we considered inputs of (1, 1) and (-1, 1). Note that the EI independent μ are very large and do not meet Definition 3($\Psi = 100.5 \geq \gamma$). However by constraining C_c using the probabilistic constraint, the independent distributions will be much closer together ($\Psi = 6.6348$) making the independent distributions *probabilistically admissible* (Definition 3). Note that when the PC is added: (1) the estimated independent covariances are smaller, (2) the independent means are much closer, (3) and the fused result has a much smaller covariance (.77 vs 1) - shown in Figure 2.

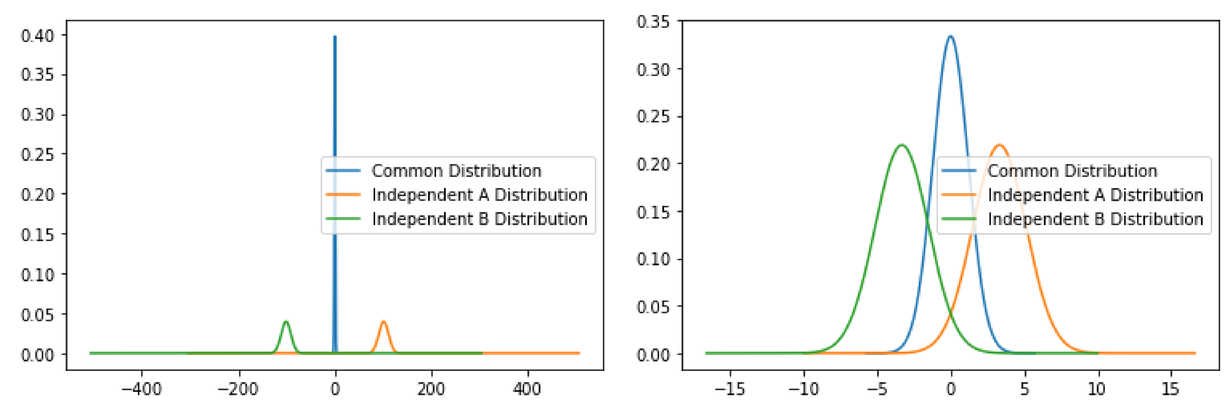

Figure 1: Visualization of the estimated 1-dimensional independent distributions both without (a) and with (b) the probabilistic constraints. Input distributions have covariances of 1 and means of 1 and -1. Note the significantly different ranges on the x axis between the subfigures.

Input Distribution Covariances are different: This concept is demonstrated in Figure 2, where p_b is held constant at $(1, 1)$ and p_a has a constant initial mean equal to -1. Notice in Figure 2, we plot, as a function of C_a (the x-axis), what the estimated independent mean values are both without (the solid, outside lines) and with (the interior, dashed lines) the probabilistically admissible constraint enforced. Note that as C_a approaches 1, the mean values asymptotically go to $\pm\infty$ without the probabilistically admissible constraint. (The graph is truncated in the y direction for readability.)

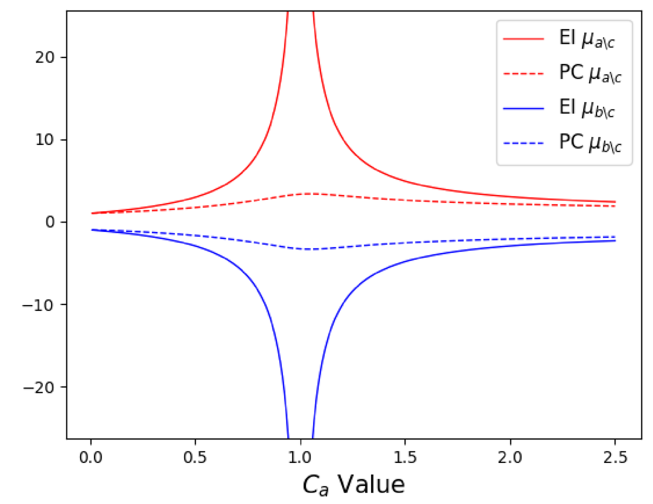

Figure 2: Effect of probabilistic constraint (PC) on the independent mean estimates (subfigure a). Other values are $\mu_a = 1$ and $p_b = (1, 1)$.

Simulations/Results

To evaluate the effectiveness of the probabilisticly constrained fusion algorithm, we evaluate two different scenarios in this section: fusion of two input distributions (by comparing both the output covariance and mean squared error with and without the probablistic constraint) and we look at a distributed estimation scenario where both EI and PC are used till network convergence.
A. Two input distributions: To evaluate PC performance on two input distributions we ran Monte Carlo simulations over 35 different sets of 2 input distributions. To evaluate the effect of how similar the input distributions are, we varied the degrees of freedom of the Inverse Wishart distribution used to generate the covariances, and the metrics used were (1) the determinant of the output (fused) covariance and (2) the mean squared error (MSE) of the fused means. The results of the Monte Carlo simulations are summarized in Table 1, and note that using the PC leads to a significant reduction in the size of the fused covariance determinant, an indicator that it is less conservative.
Figure 3 also shows a box and whisker plot showing the percentage decrease in determinate using PC fusion at $\alpha = .01$ compared with EI fusion for the same monte-carlo simulations described previously. This chart demonstrates that (1) PC fusion is always as good as and often better than EI fusion (no negative percentages) and (2) as the input covariances become more similar, the advantage of PC fusion over EI fusion is amplified.

Table 1: Monte-carlo inverse wishart simulation result comparison between EI and PC for different degrees of freedom.

	PC, α =0.05	**PC, α =0.01**	**EI**
	df=2		
Determinate	0.7229	0.7418	0.7946
% better than EI	67.76%	67.22%	
MSE	1.927	1.9442	2.019
% better than EI	16.26%	0.5829%	
	df=10		
Determinate	0.415	0.435	0.485
% better than EI	14.47%	10.28%	
MSE	0.574	0.588	0.640
% better than EI	10.32%	8.09%	
	df=100		
Determinate	0.537	0.565	0.651
% better than EI	17.62%	13.24%	
MSE	0.672	0.684	0.867
% better than EI	22.46%	21.15%	

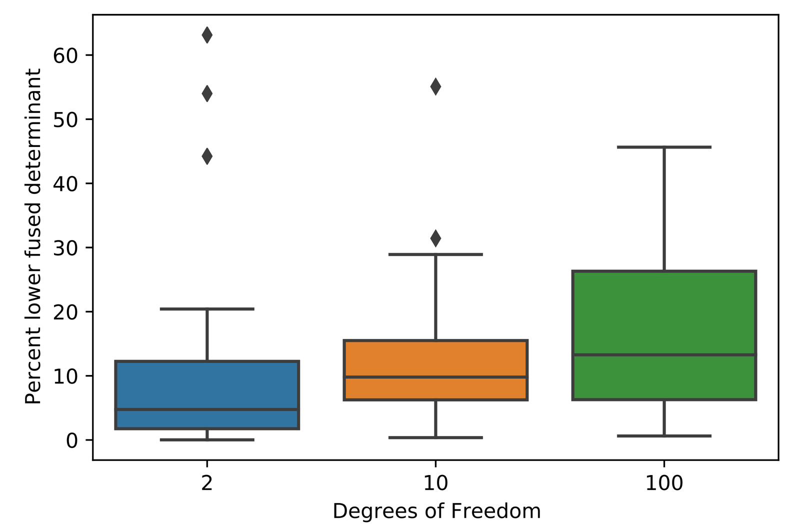

Figure 3: Decrease in fused covariance determinate (y-axis) using PC ($\alpha = 0.01$) as opposed to EI fusion at different inverse wishart degrees of freedom (x-axis). Each degree of freedom had 35 runs of data.

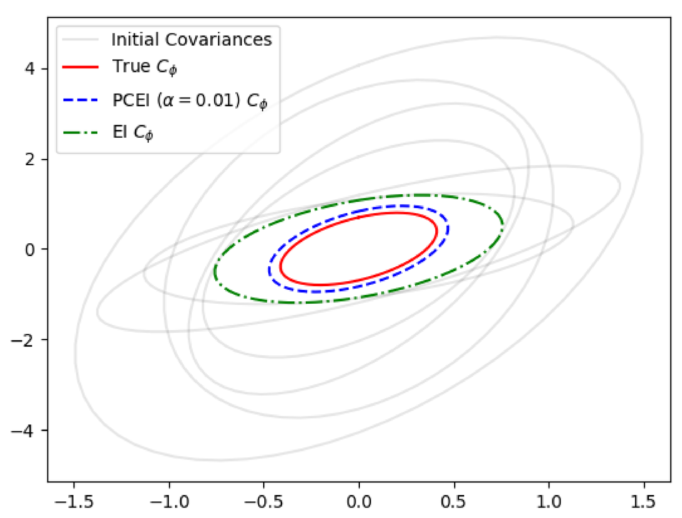

Figure 4: Covariance results of decentralized data fusion using probabilisticly constrained (PC) and ellipsodal intersection (EI) approaches, with the input covariances and centralized fusion covariance also shown for comparison

B. Decentralized Network Convergence: To demonstrate the performance of PC fusion in a distributed network, we simulated a network of six nodes (each with an independent Gaussian distribution) with a randomly generated topology. For a single iteration, the EI or PC fusion was applied to each edge in the graph, replacing the estimate in that node with the fused (between both ends of the edge) distribution. We ran this procedure 25 times to ensure convergance of the network. In Figure 4, we show the resulting covariances from a centralized (true) fusion technique, EI, and PC. As shown, adding the probabilistic constraints leads to a fused covariance that is much less conservative than the traditional EI technique.