EE 507 Random Processes
Homework 02

Aryan Ritwajeet Jha

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Contents

Problem 1

You pick two cards at random (without replacement) from a standard deck of 52 cards. Please answer the following questions:

a. What is the probability that both cards show the same value (e.g. both are 4's or both are Kings.)

b. Given that the two cards have the same color, what is the probability that both cards show the same value? Is this probability larger or smaller than the probability in part a? Conceptually, why does this make sense?

c. Are the events that the two cards have the same color and the two cards have the same value independent?

Solution

Experiment S: Two cards are drawn at random without replacement from a standard deck of 52 cards.
Let event A = {"Both cards are of the same value"}.
Let event B = {"Both cards are of the same colour"}.
So AB = {"Both cards are of the same value and of the same colour"}

a.

$$\begin{aligned} P(A) &= \frac{52*3}{52*51} \\ \text{or, } P(A) &= \frac{1}{17} \\ \text{or, } P(A) &\approx 0.0588 \end{aligned} \tag{1}$$

b.

$$P(A|B) = \frac{P(AB)}{P(B)} \tag{2}$$

$$\text{But, } P(B) = \frac{52 * 25}{52 * 51}$$
$$\text{or, } P(B) = \frac{25}{51}$$
$$\text{or, } P(B) \approx 0.4902 \tag{3}$$
$$\text{or, } P(AB) = \frac{52 * 1}{52 * 51}$$
$$\text{or, } P(AB) = \frac{1}{51}$$
$$\text{or, } P(AB) \approx 0.0196 \tag{4}$$

Thus, Using Eqs. (1) to (4), we get:

$$P(A|B) = \frac{1}{25}$$
$$\text{or, } P(A|B) = 0.040 \tag{5}$$

This probability is smaller (approximately two-thirds of) than the probability computed in part a). Conceptually, this makes sense as the be restricting the outcomes to only those cards with the same colour, the total number of outcomes was reduced by around half ($52 * 25$ from $52 * 51$) but the number of successful outcomes was reduced to a third of the original ($52 * 1$ from $52 * 3$).

c. In part b), it was seen that the knowledge of event B happening altered our knowledge (decreased the probability of happening) of event A (Eq. (5) compared to Eq. (1)). This violates the condition of independence of events A and B as $P(A|B) \neq P(A)$.

Problem 2

2a. If two events A and B are independent, is it necessarily true that A and $\bar{B}$ are independent Please prove or give a counterexample.

2b. If events A and B are independent, and events A and C are independent, is it always true that events B and C are independent? Please prove or give a counterexample.

2c. If two events are disjoint, are they always, sometimes, or never independent? Please explain.

Solution

2a. From Law of Total Probability, we know that:

$$\text{Given that, } P(AB) = P(A)P(B) \tag{6}$$
$$\text{To check if: } P(A\bar{B}) = P(A)P(\bar{B}) \tag{7}$$
$$A = AB + A\bar{B} \tag{8}$$
$$\begin{aligned} \text{or, } P(A) &= P(AB) + P(A\bar{B}) \\ \text{or, } P(A) &= P(A)P(B) + P(A\bar{B}) \\ \text{or, } P(A) - P(A)P(B) &= P(A\bar{B}) \\ \text{or, } P(A)1 - P(B) &= P(A\bar{B}) \\ \text{or, } P(A)P(\bar{B}) &= P(A\bar{B}) \end{aligned}$$

which is exactly what we set to prove i.e. Eq. (7).
$\implies$ If A and B are independent, then so are A and $\bar{B}$.
Hence Proved.😄

2b. No. Assume the trivial cases $B = C$ or $B = \bar{C}$. Even when $P(A|B) = P(A)$ and $P(A|C) = P(C)$ as given, $P(B|C) = 1 \neq P(B)$ in the first case and $P(B|C) = 0 \neq P(B)$ in the second.

2c. Disregarding the trivial case when at least one of the events is impossible, two disjoint events can NEVER be independent. Conceptually this makes sense, as say, if events A and B with non-zero probabilities are completely disjoint, then we could say that $B \in \bar{A}$, thus $P(A|B) = 0 \neq P(A)$ and vice versa $P(B|A) = 0 \neq P(B)$, which basically violates the condition of independence.

Problem 3

A randomly selected student in an engineering class id diligent with probability 0.3, and lazy with probability 0.7. A diligent student completes all of her homework with probability 0.9 and does not do so with probability 0.1. Meanwhile, a lazy student completes all homework with probability 0.5 and does not do so with probability 0.5. A student who completes all homework receives an A grade with probability 0.6, and receives a B grade with probability 0.4 (irrespective of whether the student was diligent or lazy). A student who doesn't complete all homework receives an A grade with probability 0.3, a B grade with probability 0.4 and an F grade with probability 0.3 (again, irrespective of whether the student was diligent or lazy).
Please answer the following questions.

3a. What is the probability that a randomly selected student is diligent, completes all their homework and receives an A grade?

3b. What is the probability that the student completes all of their homework?

3c. What is the probability that the student receives an A grade?

3d. Given that a randomly selected student completed all their homework, what is the probability that the student was lazy?

3e. Given that a randomly-selected student received an A grade, what is the probability that the student is lazy?

3f. Is the event that a student passes the course (receives an A or B grade) independent of the event that the student is lazy?

3g. Is the event that a student receives a B grade independent of the event that the student is lazy?

Solution

Experiment S: A student is randomly selected from an engineering class.

Let, Event D = {"Selected student is Diligent"}
$\Longrightarrow$ Event $\bar{D}$ = {"Selected student is Lazy"}
Let, Event H = {"Selected student did all of their homework"}
$\Longrightarrow$ Event $\bar{H}$ = {"Selected student did NOT do all of their homework"}
Let, Event A = {"Selected student receives an A grade."}
Let, Event B = {"Selected student receives a B grade."}
Let, Event F = {"Selected student receives an F grade."}☹

For reference, Fig. 1 is a graph which displays all possible variations of a randomly chosen student (or in terms of probability theory all events with non-zero probabilities).

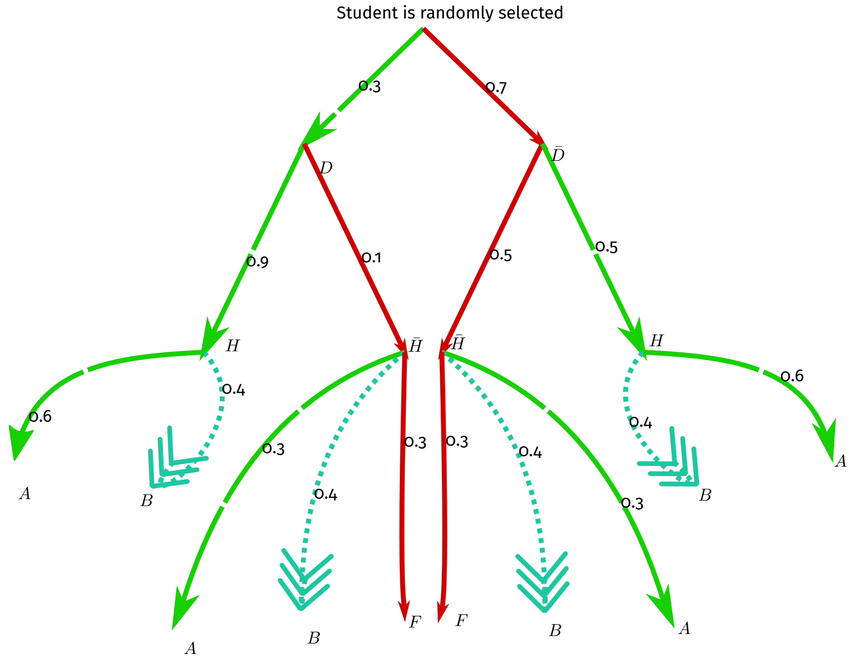

Figure 1: Decision Tree representing all possible types of students selected.

3a.

Using conditional probability theorem:

$$
\begin{aligned}
&P(DHA) = P(A|HD)P(H|D)P(D) \\
\text{or, } &P(DHA) = (0.6)(0.9)(0.3) \\
\text{or, } &P(DHA) = 0.162
\end{aligned}
\tag{9}
$$

3b.

Using Law of Total Probability:

$$
\begin{aligned}
P(H) &= P(H|D)P(D) \\
\text{or, } P(H) &= P(H|D)P(D) + P(H|\bar{D})P(\bar{D}) \\
\text{or, } P(H) &= (0.9)(0.3) + (0.5)(0.7) \\
\text{or, } P(H) &= 0.62
\end{aligned} \tag{10}
$$

3c.

Using Law of Total Probability:

$$
\begin{aligned}
P(A) = &P(A|\bar{D}\bar{H})P(\bar{D}\bar{H}) + P(A|\bar{D}H)P(\bar{D}H) \\
&+ P(A|D\bar{H})P(D\bar{H}) + P(A|DH)P(DH)
\end{aligned}
$$

Using $P(DH) = P(H|D)P(D)$ and so on for other terms:

$$
\begin{aligned}
\text{or, } P(A) = &(0.3)(0.7)(0.5) + (0.6)(0.7)(0.5) \\
&+ (0.3)(0.3)(0.1) + (0.6)(0.3)(0.9) \\
\text{or, } P(A) = &0.486
\end{aligned} \tag{11}
$$

3d.

Using Bayes' rule:

$$
P(\bar{D}|H) = \frac{P(H|\bar{D})P(\bar{D})}{P(H)} \tag{12}
$$

Using Eq. (10) in Eq. (12):

$$
\begin{aligned}
\text{or, } P(\bar{D}|H) &= \frac{(0.5)(0.7)}{0.62} \\
\text{or, } P(\bar{D}|H) &\approx 0.5645
\end{aligned} \tag{13}
$$

3e.

Using Bayes' rule:

$$
P(\bar{D}|A) = \frac{P(A|\bar{D})P(\bar{D})}{P(A)} \tag{14}
$$

Using Law of Total Probability to expand $P(A|\bar{D})$:

$$
\text{or, } P(\bar{D}|A) = \frac{\left\{P(A|\bar{H}\bar{D})P(\bar{H}|\bar{D}) + P(A|\bar{D}H)P(H|\bar{D})\right\}P(\bar{D})}{P(A)} \tag{15}
$$

Using Eq. (11) in Eq. (15):

$$\text{or, } P(\bar{D}|A) = \frac{\Big\{(0.3)(0.5) + (0.6)(0.5)\Big\}(0.7)}{0.486}$$
$$\text{or, } P(\bar{D}|A) \approx 0.6481 \tag{16}$$

3f.

Check if:

$$P(\{A + B\}|\bar{D}) = P(\{A + B\})? \tag{17}$$

Let event $Pass$ = {"Student obtained A or B grade."}

Or,

$$Pass = A + B \tag{18}$$

Starting from the LHS of Eq. (17), using Eq. (24) and the Law of Total Probability:

$$P(Pass|\bar{D}) = P(Pass|\bar{D}\bar{H})P(\bar{H}|\bar{D}) + P(Pass|\bar{D}H)P(H|\bar{D})$$
$$\text{or, } P(Pass|\bar{D}) = (0.7)(0.5) + (1.0)(0.5)$$
$$\text{or, } P(Pass|\bar{D}) = 0.85 \tag{19}$$

Now starting with the RHS of Eq. (17) and using the Law of Total Probability:

$$\begin{aligned} P(Pass) &= P(Pass|\bar{D})P(\bar{D}) + P(Pass|D)P(D) \\ \text{or, } P(Pass) &= P(Pass|\bar{D})P(\bar{D}) \\ &\quad + \Big\{P(Pass|D\bar{H})P(\bar{H}|D) \\ &\quad\quad + P(Pass|DH)P(H|D)\Big\}P(D) \end{aligned} \tag{20}$$

Using Eq. (19) in Eq. (24):

$$\begin{aligned} P(Pass) &= (0.85)(0.7) + \Big\{(0.7)(0.1) \\ &\quad + (1.0)(0.9)\Big\}(0.3) \\ \text{or, } P(Pass) &= 0.886 \end{aligned} \tag{21}$$

Since $P(Pass|\bar{D})$ in Eq. (19) and $P(Pass)$ in Eq. (21) are not equal, therefore the events of a student passing and the student being lazy are NOT independent events. 😐

3g.

Check if:

$$P(B|\bar{D}) = P(B)? \tag{22}$$

Or,

Starting from the LHS of Eq. (22), and using the Law of Total Probability:

$$\begin{aligned} P(B|\bar{D}) &= P(B|\bar{D}\bar{H})P(\bar{H}|\bar{D}) + P(B|\bar{D}H)P(H|\bar{D}) \\ \text{or, } P(B|\bar{D}) &= (0.4)(0.5) + (0.4)(0.5) \\ \text{or, } P(B\bar{D}) &= 0.4 \end{aligned} \tag{23}$$

Now starting with the RHS of Eq. (22) and using the Law of Total Probability:

$$P(B) = P(B|\bar{H})P(\bar{H}) + P(B|H)P(H) \tag{24}$$

$$\begin{aligned} \text{or, } P(B) &= (0.4)P(\bar{H}) + (0.4)P(H) \\ \text{or, } P(B) &= 0.4 \end{aligned} \tag{25}$$

Since $P(B|\bar{D})$ in Eq. (23) and $P(B)$ in Eq. (25) are equal, therefore the events of a student getting a B grade and the student being lazy are independent events. ☺

3h.

Using Bayes' rule on $P(H|\{\bar{D}+A\})$:

$$\begin{aligned} P(H|\{\bar{D}+A\}) &= \frac{P(\{\bar{D}+A\}|H)P(H)}{P(\{\bar{D}+A\})} \\ \text{or, } P(H|\{\bar{D}+A\}) &= \frac{P(\{\bar{D}+DA\}|H)P(H)}{P(\{\bar{D}+DA\})} \end{aligned}$$

Using Axiom 1 of Probability:

$$\begin{aligned} P(H|\{\bar{D}+A\}) &= \frac{\Big\{P(\bar{D}|H) + P(DA|H)\Big\}P(H)}{\{P(\bar{D}) + P(DA)\}} \\ \text{or, } P(H|\{\bar{D}+A\}) &= \frac{\Big\{P(\bar{D}|H) + \frac{P(DAH)}{P(H)}\Big\}P(H)}{\{P(\bar{D}) + P(DAH) + P(DA\bar{H})\}} \\ \text{or, } P(H|\{\bar{D}+A\}) &= \frac{\Big\{P(\bar{D}|H) + \frac{P(DAH)}{P(H)}\Big\}P(H)}{\{P(\bar{D}) + P(DAH) + P(A|D\bar{H})P(\bar{H}|D)P(D)\}} \end{aligned} \tag{26}$$

Using Eqs. (9) to (10) and (13) in Eq. (26):

$$P(H|\{\bar{D}+A\}) = \frac{\left\{0.5645 + \frac{0.162}{0.62}\right\}(0.62)}{\{0.7 + 0.162 + (0.3)(0.1)(0.3)\}}$$

or, $P(H|\{\bar{D}+A\}) \approx 0.5878$ (27)

Problem 4

An experiment has three outcomes $A, B,$ and C, which have probabilities $P(A) = 0.5, P(B) = 0.3$ and $P(C) = 0.2$. You repeat this experiment 100 times, independently. Please answer the following questions.

4a. What is the probability that outcome A occurs on exactly 50 trials.

4b. What is the probability that the outcome A occurs on 50 trials, outcome B occurs on 25 trials, and outcome C occurs on 25 trials?

4c. Given that the outcome A occurred on exactly 60 trials, what is the probability that A occurred on the first trial? How about B?

Solution

4a. It should be noted that outcomes are always independent events, therefore we can multiply the probabilities of individual outcomes in a repeated experiment.

$$P(\#A = 50) = \binom{100}{50}(0.5)^{50}\binom{50}{50}(1-0.5)^{50}$$
$$\text{or, } P(\#A = 50) = \binom{100}{50}(0.5)^{100}$$
$$\text{or, } P(\#A = 50) \approx 0.0796$$

4b.

$$P(\#A = 50, \#B = 25, \#C = 25) = \binom{100}{50}(0.5)^{50}\binom{50}{25}(0.3)^{25}\binom{25}{25}(0.2)^{25}$$
$$\text{or, } P(\#A = 50, \#B = 25, \#C = 25) = \binom{100}{50}(0.5)^{50}\binom{50}{25}(0.3)^{25}(0.2)^{25}$$
$$\text{or, } P(\#A = 50, \#B = 25, \#C = 25) \approx 0.00322$$

4c.

$$P(\text{Trial}\#1 = A|\#A = 60) = \frac{P(\text{Trial}\#1 = A, \#A = 60)}{P(\#A = 60)}$$
$$\text{or, } P(\text{Trial}\#1 = A|\#A = 60) = \frac{\binom{1}{1}(0.5)^{1}\binom{99}{59}(0.5)^{59}\binom{40}{40}(1-0.5)^{40}}{\binom{100}{60}(0.5)^{60}\binom{40}{40}(1-0.5)^{40}}$$
$$\text{or, } P(\text{Trial}\#1 = A|\#A = 60) = \frac{\binom{99}{59}(0.5)^{100}}{\binom{100}{60}(0.5)^{100}}$$
$$\text{or, } P(\text{Trial}\#1 = A|\#A = 60) = 0.6$$

Doing the same for outcome B:

$$
\begin{aligned}
P(\text{Trial}\#1 = B|\#A = 60) &= \frac{P(\text{Trial}\#1 = B, \#A = 60)}{P(\#A = 60)} \\
\text{or, } P(\text{Trial}\#1 = B|\#A = 60) &= \frac{\binom{1}{1}(0.3)^1\binom{99}{60}(0.5)^{60}\binom{39}{39}(1-0.5)^{39}}{\binom{100}{60}(0.5)^{60}\binom{40}{40}(1-0.5)^{40}} \\
\text{or, } P(\text{Trial}\#1 = B|\#A = 60) &= \frac{\binom{99}{60}(0.5)^{99}}{\binom{100}{60}(0.5)^{100}} \\
\text{or, } P(\text{Trial}\#1 = B|\#A = 60) &= \frac{(40)(0.3)}{(100)(0.5)} \\
\text{or, } P(\text{Trial}\#1 = B|\#A = 60) &= 0.24
\end{aligned}
$$

Conceptually, in a 'universe' (a subset of all possible events constrained by one or more specified events happening) where A is likely 60 times out of 100, it is logical for any specific trial to have the same constrained probability ($\frac{60}{100} = 0.6$ in this case). Similarly B will have a probability $0.3 * \left\{1 - \frac{60}{100}\right\} = 0.24$ and C will have a probability of $0.2 * \left\{1 - \frac{60}{100}\right\} = 0.16$

Problem 5

5a. Please define a random variable, and explain why the concept is important.

5b. Please define the notion of a cumulative distribution function (CDF).

Solution

5a. A Random Variable is an injective (one-one) mapping from the outcomes of an experiment to countable numbers. Random Variables are a good choice to classify outcomes due to the following reasons:

- Experiments which have a continuous, and therefore infinite, set of outcomes, can only be represented by random variables, and not nominal names, practically speaking.
- Random Variables can be used to efficiently trim the set of all possible outcomes into manageable bins of events.
- Random Variables often carry more meaning than a nominal name for an outcome in most physical/simulated experiments. For example to measure the height of a population, it makes sense to assign every possible height (with an appropriate level of quantization) as an outcome itself. If two random variables appear 'close' in value, then the two variables may also be conveying outcomes in the physical/simulated world which are also close to each other. For example $H = 164cm$ and $H = 165cm$ as random variables for describing the distribution in an adult human population are outcomes which can be considered in proximity to one another (i.e. both the outcomes classify adults of almost similar heights), whereas they are 'far-apart' from $H = 190cm$, both in random variable value and in physical meaning of the outcome they are representing.

5b. CDF$_x(\alpha) = F_X(\alpha) = P(x \leq \alpha)$ for a random variable X is a convenient way to represent the probability of the set of all outcomes which are below a certain threshold α. This definition can then be extrapolated to query for random variables in a specific range, like $P_X(\alpha_1 < x < \alpha_2)$
Taking the same example as above of an experiment of sampling an adult human and measuring their height, it does not make sense to seek humans who have an exact height of, say $161.327cm$. Rather, the experimenter is usually interested in range queries, such as the probability that a randomly selected adult human has a height between $160cm$ and $165cm$, in which case the value can be conveniently calculated by computing the CDFs at the two heights and taking their difference. None the less, an exact pdf can always be computed by taking the derivative of the CDF wrt the random variable at the desired value.

Problem 6

An experiment has four outcomes A, B, C and D which have probabilities $P(A) = 0.4$, $P(B) = 0.3$, $P(C) = 0.2$, and $P(D) = 0.1$. We define a random varialbe x as follows: $X(A) = 1$, $X(B) = 5$, $X(C) = 2$, $X(D) = 1$. Also, we define the event $Z = \{A, B, C\}$. Please answer the following questions:

6a. Find and plot the CDF of X.

6b. What is the probability that $2 < X < 6$?

6c. Find and plot the probability mass function (pmf) of X.

6d. Find the CDF of X given Z has occurred.

6e. What is the probability of Z, given that $X > 1.5$?

Solution

Refer to Table 1 for the calculations.

Table 1: Random Variable vs pmf and CDF values for the experiment.

X	outcomes	pmf$_X(\alpha)$	$F_X(\alpha)$
1	A, D	0.5	0.5
2	C	0.2	0.7
5	B	0.3	1.0

6a.

$$\text{CDF}_X(\alpha) = F_X(\alpha) = \begin{cases} 0 & \text{if } \alpha < 1 \\ 0.5 & \text{if } \alpha \in [1, 2) \\ 0.7 & \text{if } \alpha \in [2, 5) \\ 1.0 & \text{if } \alpha \geq 5 \end{cases}$$

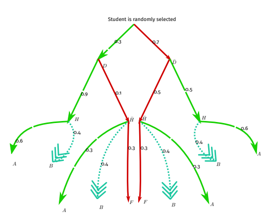

Figure 2: CDF plot of the experiment.

6b.

$$
\begin{aligned}
&P_X(x \in (2,6)) = F_X(x \to 6^-) - F_X(x \to 2^+) \\
\text{or, } &P_X(x \in (2,6)) = 1.0 - 0.7 \\
\text{or, } &P_X(x \in (2,6)) = 0.3
\end{aligned}
$$

6c. Refer to Table 2 for the values.

Table 2: Probability Mass Function (pmf) of the experiment.

X	PMF$_X(x)$
1	0.5
2	0.2
3	0.3

6d. To compute: $F_{X|Z}(x|z)$
But $Z = \bar{D}$.
So we can compute the CDF for outcomes A, B and C given that D has NOT occurred. The same has been done in Table 3.

Table 3: Modified pmf and CDF values of the experiment given that event Z has occurred.

X_Z	outcome	pmf$_{X|Z}(\alpha)$	$F_{X|Z}(\alpha)$
1	A	$\frac{0.4}{1-0.1} = \frac{4}{9}$	$\frac{4}{9}$
2	C	$\frac{0.2}{1-0.1} = \frac{2}{9}$	$\frac{6}{9}$
5	B	$\frac{0.3}{1-0.1} = \frac{3}{9}$	1

$$
\text{CDF}_{X|Z}(\alpha) = F_{X|Z}(\alpha) = \begin{cases} 0 & \text{if } \alpha < 1 \\ \frac{4}{9} & \text{if } \alpha \in [1,2) \\ \frac{6}{9} & \text{if } \alpha \in [2,5) \\ 1.0 & \text{if } \alpha \geq 5 \end{cases}
$$

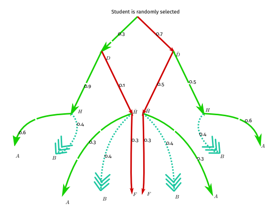

Figure 3: Modified CDF plot of the experiment given that event Z has occurred..

6e.

$$P(Z|X>1.5) = \frac{P(X>1.5|Z)P(Z)}{P(X>1.5)}$$

$$\text{or, } P(Z|X>1.5) = \frac{(1-F_{X|Z}(1.5))(0.9)}{1-F_X(1.5)}$$

$$\text{or, } P(Z|X>1.5) = \frac{\left\{1-\frac{4}{9}\right\}(0.9)}{1-0.5}$$

$$\text{or, } P(Z|X>1.5) = 1$$

☺ Of course.. $\{X>1.5\} = \{B, C\} \in Z$

.

Problem 7

You throw a dart at a circular dartboard with unit radius, and have an equal probability of hitting each point on the dartboard. Let R be the distance of the dart from the centre of the dartboard. Please answer the following questions.

7a. Please argue that R is a random variable.

7b. Please find the CDF of R.

7c. What is the probability that $0.2 < R < 0.5$?

7d. Please find the probability density function (pdf) of R.

7e. Now let Q be the vertical distance of the dart from the bottom of the dartboard. Please find the pdf and CDF of Q.

Solution

Refer to Fig. 4 for the calculations.

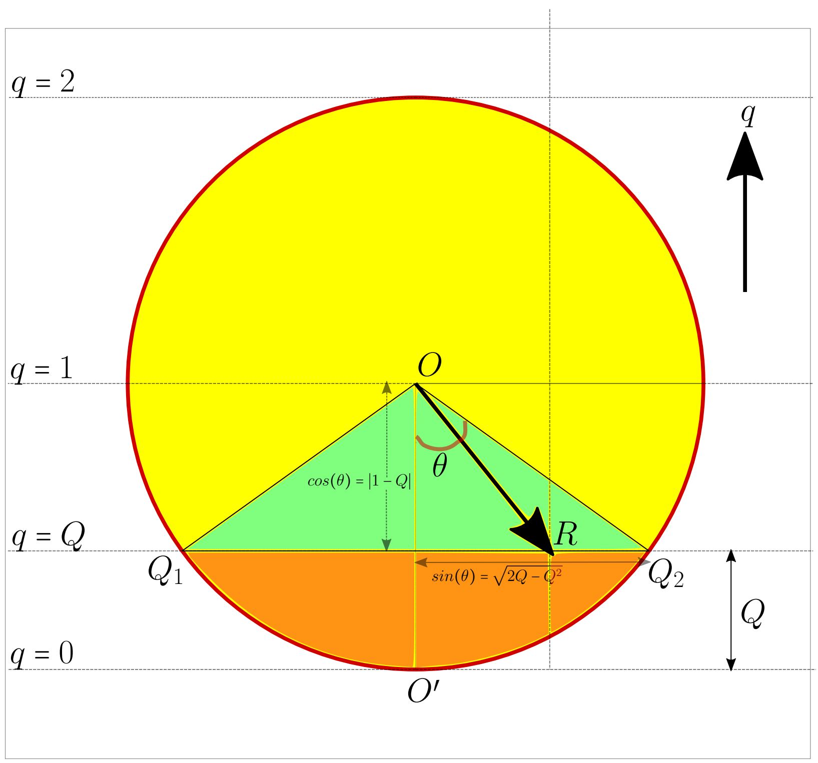

Figure 4: Dartboard

7a. Given that R is the distance from the centre of the dartboard it can be used to represent every possible experiment outcome with a number. In other words, R is an injective (one-one) mapping from the set of all of outcomes to a set of numbers. Also R as a random variable follows a set of trivial rules accounting for impossible events, via $P_R(r \to -\infty) = 0$ (a dart cannot be thrown at a location at a negative distance from the centre) and $P_R(r \to \infty) = 0$ (possibility of dart being thrown out of the dart board has been rules out as per the problem statement).

7b. $\text{CDF}_R(\alpha) = P_R(r < \alpha)$ is simply the area of the circle whose centre coincides with the dart board's centre and whose radius is R.

$$\text{CDF}_R(\alpha) = \begin{cases} 0 & \text{if } \alpha < 0 \\ \frac{\pi\alpha^2}{\pi 1^2} = \alpha^2 & \text{if } \alpha \in [0,1) \\ 1 & \text{if } \alpha \geq 1 \end{cases}$$

7c.

$$\begin{aligned} P_R(r \in (0.2, 0.5)) &= \text{CDF}_R(0.5) - \text{CDF}_R(0.2) \\ \text{or, } P_R(r \in (0.2, 0.5)) &= \pi 0.25 - \pi 0.04 \\ \text{or, } P_R(r \in (0.2, 0.5)) &\approx 0.6597 \end{aligned}$$

7d. pdf is the derivative of CDF wrt the random variable.

$$f_R(\alpha) = \frac{dF_R(\alpha)}{d\alpha} = \begin{cases} 0 & \text{if } \alpha < 0 \\ 2\alpha & \text{if } \alpha \in [0,1) \\ 0 & \text{if } \alpha \geq 1 \end{cases}$$

7e. Note: Refer to Fig. 4 for notation and labels.
The CDF $F_Q(q)$ of Q would be the normalized area of the segment Q_1Q_2.

Let θ be the angle between the two radius long arms extending from the centre O of the dartboard to point O' where the $q = 0$ tangent touches the dartboard, and to the point Q_2, the point lying on the circle which has the same ordinate Q as outcome (hit location) R and lies on the same side as R. It is easier to compute the area of the segment Q_1Q_2 using random variable Θ and later converting it back to the request Q random variable.

$$\begin{aligned} F_\Theta(\theta) &= \frac{\text{Area(Segment } Q_1Q_2)}{\text{Area(Circle } Q_1OQ_2)} \\ \text{or, } F_\Theta(\theta) &= \frac{\text{Area(Sector } Q_1OQ_2) - \text{Area}(\Delta Q_1OQ_2)}{\text{Area(Circle } Q_1OQ_2)} \\ \text{or, } F_\Theta(\theta) &= \frac{\frac{2\theta}{2\pi}\pi 1^2 - \cos(\theta)\sin(\theta)}{\pi 1^2} \\ \text{or, } F_\Theta(\theta) &= \frac{\theta - 0.5\sin(2\theta)}{\pi} \text{ for } \theta \in [0,\pi]. \end{aligned} \tag{28}$$

The pdf $f_Q(q)$ may be computed, by taking the derivative of $F_Q(q)$ wrt dummy variable q.

$$\begin{aligned} f_\Theta(\theta) &= \frac{dF_Q(\theta)}{d\theta} \\ \text{or, } f_\Theta(\theta) &= \frac{1-\cos(2\theta)}{\pi} = \frac{2\sin^2\theta}{\pi} \text{ for } \theta \in [0,\pi] \end{aligned} \tag{29}$$

It is easy to convert θ in Eqs. (28) and (29) to q using the equation $q = \arccos(\theta)$:

$$\text{Thus, } F_Q(q) = \frac{\arccos(1-q) - \sqrt{2q-q^2}(1-q)}{\pi} \text{ for } q \in [0,2) \tag{30}$$

$$\begin{aligned} &= 0 \quad \text{for } q < 0 \\ &= 1 \quad \text{for } q \geq 2 \end{aligned}$$

$$\text{and, } f_Q(q) = \frac{2(2q-q^2)}{\pi} \text{ for } q \in [0,2) \tag{31}$$

$$\begin{aligned} &= 0 \text{ for } q < 0 \\ &= 0 \text{ for } q \geq 2 \end{aligned}$$