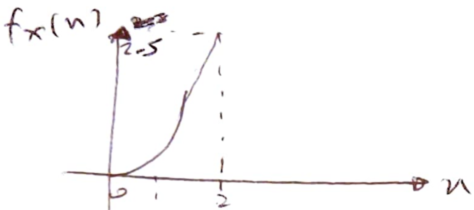
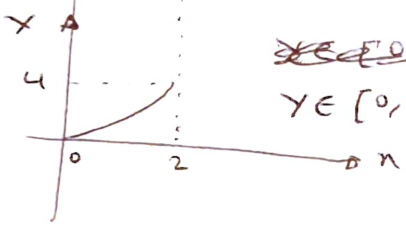
$\boxed{1.}$

$$f_X(x) = \begin{cases} \frac{5}{32} x^4 & x \in (0,2] \\ 0 & \text{else.} \end{cases}$$

$Y = X^2$





$\cancel{x \in [0,4]}$

$Y \in [0,4)$

(i) Using CDF formula

$$F_X(x=\alpha) = \begin{cases} 0 & \alpha \le 0 \\ \frac{1}{32} \alpha^5 & \alpha \in (0,2] \\ 1 & \alpha > 2 \end{cases}$$

$F_Y(y=\alpha) = P(y \le \alpha)$

or $F_Y(y=\alpha) = P(x^2 \le \alpha)$

or $F_Y(y=\alpha) = P(x \le \sqrt{\alpha})$

or $F_Y(y=\alpha) = F_X(\sqrt{\alpha}) = \begin{cases} 0 & , \sqrt{\alpha} \le 0 \\ \frac{1}{32}(\sqrt{\alpha})^5 & \sqrt{\alpha} \in (0,2] \\ 1 & \sqrt{\alpha} > 2 \end{cases}$

$\boxed{1a.}$

or $$\boxed{F_Y(y=\alpha) = \begin{cases} 0 & \alpha \le 0 \\ \frac{1}{32} \alpha^{\frac{5}{2}} & \alpha \in (0,4] \\ 1 & \alpha > 4 \end{cases}}$$

Ans

1 b.

1.2

$$f_Y(y) = \frac{d}{dy}\left(F_Y(y=\alpha)\right)\Big|_{y=\alpha} = \begin{cases} 0 & y < 0 \\ \frac{5}{64} y^{\frac{3}{2}} & y \in [0,4] \\ 0 & y > 4 \end{cases}$$

Ans

(ii) Using Transformative ("cute") formula:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

or $$f_Y(y) = \frac{d}{dx}\left(F_X(g^{-1}(y))\right) \cdot \left|\frac{dx}{dy}\right|$$

Here since ~~g(x)~~ $y = g(x) = x^2$,

$$g^{-1}(y) = \sqrt{y}.$$

$$\cancel{\frac{dy}{dx} = \frac{d}{dx}(\sqrt{y}) =}$$

$$\frac{dx}{dy} = \frac{d}{dy}(\sqrt{y}) = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{1}{2\sqrt{y}}\right|$$

$$f_Y(y) = \begin{cases} 0 & \sqrt{y} \le 0 \\ \frac{5}{32}(\sqrt{y})^4 \cdot \left|\frac{1}{2\sqrt{y}}\right| & \sqrt{y} \in (0,2] \\ 0 & \sqrt{y} > 2 \end{cases}$$

1 b (again!)

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} 0 & y \le 0 \\ \frac{5}{64} y^{\frac{3}{2}} & y \in (0, 4] \\ 0 & y > 4 \end{cases}}$$

Ans (Again).

(i) Using $f_Y(y)$

$$E[Y] = \int_{y=0}^{y=4} y\, f_Y(y)\, dy$$

$$\Rightarrow E(Y) = \int_{y=0}^{4} y \times \frac{5}{64} y^{\frac{3}{2}}\, dy$$

$$\Rightarrow E(Y) = \frac{5}{64} \cdot y^{\frac{7}{2}} \cdot \frac{2}{7} \Big|_{y=0}^{y=4}$$

$$\Rightarrow E(Y) = \frac{10}{64 \times 7} \cdot 4^{\frac{7}{2}}$$

1c

$$\Rightarrow \boxed{E(Y) = \frac{20}{7}}$$ Ans

(ii) Using Law of The Unconscious Statistician (LOTUS):

$$E(Y) = E(X^2) = \int_{x=0}^{x=2} x^2 \cdot f_X(x)\, dx$$

$$\Rightarrow E(Y) = \int_{x=0}^{x=2} x^2 \cdot \frac{5}{32} x^4\, dx$$

$$\Rightarrow E(Y) = \frac{5}{32} \cdot \frac{x^7}{7} \Big|_{x=0}^{x=2}$$

1c (again!)

$$\Rightarrow \boxed{E(Y) = \frac{20}{7}}$$ Ans (again!)

2.

Roll # 1 $\xrightarrow{\text{Roll again?}}$ Roll # 2

$X = \{1,2,3,4,5,6\}$ $Y = \{1,2,3,4,5,6\}$

Let A be the event that you cash in on the first die throw.

~~E[W] = F~~

$\cancel{E(W) = A\,E(X) + \bar{A}\,E(Y)}$

~~W = X F X~~

then $\boxed{W = AX + \bar{A}Y}$ (1)

~~X cashing in on the first throw.~~

In order to maximize W, we need to optimize the event (algorithm) A.

Intuitively, we would want to throw again if we get 'lower' values of X and hold back from throwing again if we get a high value of Y.

1 For quantifying A, we can utilize a variable
2 g (for good throw) which stands for
3 the ~~top~~ ~~base~~ number of allowed 'high-valued'
4 throws allowed.
5 }
6 } g

Here, $g=2$, i.e. hold back upon getting the highest 2 values of X: 5 or 6. In such a case, $E(X|g=2) = \frac{5+6}{2} = 5.5$

TABLE 1: $E(X|g)$ vs g.

g	$X\|g$	$E(X\|g)$
1	{6}	6
2	{6,5}	5.5
3	{6,5,4}	5
4	{6,5,4,3}	4.5
5	{6,5,4,3,2}	4
6	{6,5,4,3,2,1}	3.5

From the Table 1, we can devise $E(X|g)$ as a simple linear function of g:

$$\boxed{E(X|g) = 6.5 - \frac{g}{2}} \quad (2)$$

Taking Expectation on both sides of (1):

$$E(W) = E(AX) + E(\bar{A}Y)$$

and using g to quantify event A:

get

or $$\boxed{E(W) = E(X|g).P(g) + E(Y|\bar{g}).P(\bar{g})} \quad (3)$$

where $X|g$ = Value of X given that only the highest g values of X are allowed for holding back from throwing again.

$P(g)$ = Probability of obtaining any of the highest g values on the first die throw

$X|\bar{g}$ = given that none of the highest g values were landed on the first throw, the value landed on the second throw.

It may be noted that ~~that~~ $E(Y)$ is not dependent on the value of g or $\bar{g}$. So $E(Y|\bar{g}) = E(Y)$.

So $E(W) = \left(6.5 - \frac{1}{2}g\right)\left(\frac{g}{6}\right) + \left(\frac{1+2+3+4+5+6}{6}\right)\left(1 - \frac{g}{6}\right)$

or $\boxed{E(W) \equiv -\frac{g^2}{12} + \frac{g}{2} + 3.5}$ (4)

$\arg\left(\operatorname{Max} E(W)\right) = \arg\left(\frac{d}{dg} E(W) = 0\right)$

or $g_{Max} = \arg\left(-\frac{g}{6} + \frac{1}{2} = 0\right)$

or $\boxed{g_{Max} = 3}$ (5)

~~which is~~

Putting g_{Max} into (4):

$\operatorname{Max.} E(W) = \frac{-3^2}{12} + \frac{3}{2} + 3.5$

or $\boxed{\operatorname{Max.} E(W) = 4.25}$ Ans.

∴ An expected \$4.25 can be obtained by throwing again only if the first throw lands 1, 2 or 3.

Ans