EE 507 HW05

ARYAN RITWAJEET JHA

$\boxed{1.}$ (a) Check if $\boxed{f_X(x) \cdot f_Y(y) \cdot f_Z(z) = f_{X,Y,Z}(x,y,z)}$ ⓞ

implies that ~~the three~~ X and Y are independent,

i.e. check if ⓞ implies $\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$ ⓘ

~~given~~ Integrating both sides of ⓞ w.r.t. ~~z~~ z:

$$\int_{\forall z} f_X(x) \cdot f_Y(y) \cdot f_Z(z)\, dz = \int_{\forall z} f_{X,Y,Z}(x,y,z)\, dz$$

or $$f_X(x) f_Y(y) \int_{\forall z} f_Z(z) dz = f_{X,Y}(x,y)$$ (marginal pdf of X, Y)

or $$f_X(x) \cdot f_Y(y) \cdot 1 = f_{X,Y}(x,y)$$

or $$\boxed{f_X(x) f_Y(y) = f_{X,Y}(x,y)}$$ which is ⓘ which is what we set to check for!

$\boxed{1(a)}$ ∴ $\boxed{f_X(x) \cdot f_Y(y) f_Z(z) = f_{X,Y,Z}(x,y,z) \Rightarrow X \text{ and } Y \text{ are independent}}$

Ans

1(b) check if $\boxed{P(X) P(Y) P(Z) = P(X,Y,Z)}$ ⓞ implies that X and Y are independent i.e. $\boxed{P(X) \cdot P(Y) = P(X,Y)}$ ⓘ where X, Y, Z are events of a probabilistic experiment.

No.

Counterexample:

~~Let experiment~~

In an experiment, ~~let~~ we are picking a number from 1 to 8 randomly: Each number has equal probability of being picked. This experiment is like throwing a fair 8-sided dice.

Now let us define three events:

$X = \{1, 2, 3, 4\} \Rightarrow P(X) = \frac{1}{2}$ ~~$P(X, Y)$~~

$Y = \{1, 3, 4, 5\} \Rightarrow P(Y) = \frac{1}{2}$

$Z = \{1, 6, 7, 8\} \Rightarrow P(Z) = \frac{1}{2}$

$X \cap Y \cap Z = \{1\} \Rightarrow P(X \cap Y \cap Z) = \frac{1}{8}$

~~So $P(X) \cdot P(Y)$~~

So equation ⓪ is followed by events X, Y, Z, as

$$P(X) \cdot P(Y) \cdot P(Z) = \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{8} = P(X, Y, Z)$$

But $P(X) \cdot P(Y) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$P(X \cap Y) = P(\{1, 3, 4\}) = \frac{3}{8}$

X and Y are NOT independent.

In fact, $P(Y) \cdot P(Z) = \frac{1}{4}$

$P(Y \cap Z) = P(\{1\}) = \frac{1}{8}$

Y and Z are NOT independent either

And $P(X) \cdot P(Z) = \frac{1}{4}$

$P(X \cap Z) = P(\{1\}) = \frac{1}{8}$

X and Z are NOT independent either.

$\boxed{1(b)}$

$\therefore$ The analogous event $P(X) \cdot P(Y) \cdot P(Z) = P(XYZ)$ does NOT imply independence of X and Y.

Ans

2.1

2. 2(a) $X, Y \sim \mathcal{N}(\mu_X = 0, \mu_Y = 0, \sigma_X^2 = 1, \sigma_Y^2 = 4, \rho = 0.5)$

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{1-\rho^2}\,\sigma_X\sqrt{2\pi}\,\sigma_Y\sqrt{2\pi}} e^{\underbrace{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right\}}}$$

Only component (say $g(x,y)$) dependent on x and y.

Taking out the component $g(x,y)$ and putting the given values of $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ and ρ_{XY}:

$$g(x,y) = \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - \frac{2(x)(y)}{(1)(2)}$$

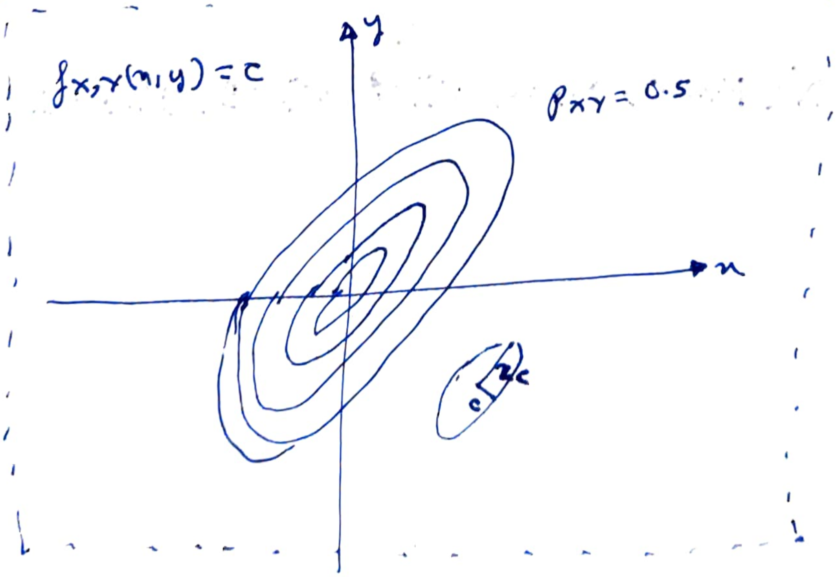
$\Rightarrow f_{X,Y}(x,y) \rightarrow c$ i.e. is constant for $g(x,y) \rightarrow c$ i.e. is constant

$\Rightarrow$ Contour of $f_{X,Y}(x,y)$ is

$$g(x,y) = \left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 - \cancel{2}\,xy = c$$

which is the equation for an ellipse, tilted and with positive slope.

2(a)



Ans

2(b)

$$\boxed{\begin{aligned} f_X(x) &= \mathcal{N}(x; 0, 1^2) = \frac{1}{1.\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{1}\right)^2\right\}} \quad \forall x \\ f_Y(y) &= \mathcal{N}(y, 0, 2^2) = \frac{1}{2.\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y}{2}\right)^2\right\}} \quad \forall y \end{aligned}}$$

Ans

2(c)

$$f_{Y|X}(y|X=x) = \frac{f_{X,Y}(X=x, Y)}{f_X(X=x)} \quad \forall x, y$$

or
$$f_{Y|X}(y|X=x) = \frac{\dfrac{1}{\sqrt{\left(1-\frac{1}{2^2}\right)}\, 1.\sqrt{2\pi}\,.\,2.\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2^2}}\cdot\left\{\left(\frac{x}{1}\right)^2+\left(\frac{y}{2}\right)^2 - 2\times\frac{1}{2}\times\frac{x}{1}\times\frac{y}{2}\right\}} \quad \cancel{\forall x, y}}{\dfrac{1}{1.\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x}{1}\right)^2\right\}}}$$

$\forall (x, y)$

or
$$f_{Y|X}(y|X=x) = \frac{1}{\sqrt{3}\,.\,\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\left\{\left(\frac{x}{\sqrt{3}}\right)^2+\left(\frac{y}{\sqrt{3}}\right)^2 - 2.1.\left(\frac{x}{\sqrt{3}}\right)\left(\frac{y}{\sqrt{3}}\right)\right\}}$$

$\forall (x, y)$

2(c)

or $\cancel{\boxed{f_{Y|X}(y|X=x) = \sqrt{2\pi}\,\mathcal{N}(\mu_X=0, \mu_Y=0, \sigma_X^2=3, \sigma_Y^2=3, \rho_{XY}=1)}}$ $\forall (x,y)$ Ans

or $\cancel{f_{Y|X}(y|X=x) = \mathcal{N}_y(\mu_Y = \cancel{2x}, \sigma_Y^2 = 3)} \ \forall y$

or $f_{Y|X}(y|X=x) = \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{y-x}{\sqrt{3}}\right)^2\right\}} \quad \forall y$

2(c)

or $\boxed{f_{Y|X}(y|X=x) = \mathcal{N}(y, \mu_Y = x, \sigma_Y^2 = 3) \quad \forall y}$ Ans

PTO.

~~2(d) Find x s.t. $E[Y \mid X=x] = -2$.~~

~~From 2(c), we know that $Y \mid X=x$ is fully correlated to $X=x$. ($\rho_{Y|X, x} = 1$).~~

~~$\because$ Y and X have the same support/domain of $x \in \mathbb{R}$ and $y \in \mathbb{R}$,~~

~~$\therefore$ $E_Y[Y \mid X=x] = -2 \equiv E_X[X=x] = -2$~~

~~But $E_X[X=x] = x$~~

~~$\boxed{2(d)}$ $\Rightarrow \boxed{x = -2}$ Ans~~

2(d) Find x s.t. $E[Y \mid X=x] = -2$

From 2(c), we know that $Y \mid X=x$ is a gaussian with mean x.

So $E[Y \mid X=x] = -2 \Rightarrow \boxed{x=-2}$ Ans

2(e) $Z = X + Y - 1$

Z is also a gaussian, we need to only compute μ_Z and σ_Z^2 to find $f_Z(z)$:

$E(Z) = E(X) + E(Y) - E(1)$

or $\mu_Z = \mu_X + \mu_Y - 1$

or $\mu_Z = 0 + 0 - 2$

or $\boxed{\mu_Z = -1}$

$\sigma_Z^2 = E\left[\cancel{X - \mu_Z} \{(X + Y - 1) - E[X + Y - 1]\}^2 \right]$

or $\sigma_Z^2 = E\left[\{ (X - \mu_X) + (Y - \mu_Y) + (-1 - E(-1)) \}^2 \right]$

or $\sigma_Z^2 = E\left[\{ (X - \mu_X) + (Y - \mu_Y) \}^2 \right]$

or $\sigma_Z^2 = E[(X - \mu_X)^2] + E[(Y - \mu_Y)^2] + \cancel{2\rho_{XY}}$ $2E[(X - \mu_X)(Y - \mu_Y)]$

or $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_Z^2 = 1^2 + 2^2 + 2 \times 0.5 \times 1 \times 2$

or $\boxed{\sigma_Z^2 = 7}$

2(e)

$\therefore \boxed{Z \sim \mathcal{N}(z, \mu_Z = -1, \sigma_Z^2 = 7)}$ Ans

2(f) $\boxed{2.5}$

$Z = 2X + 3Y$ $\qquad$ $W = X - Y$

$E(Z) = 2\mu_X + 3\mu_Y$ $\qquad$ $E(W) = \mu_X - \mu_Y$

2(f)(i)

or $\boxed{\mu_Z = 0}$ Ans

2(f)(iii)

$\boxed{\mu_W = 0}$ Ans

$\sigma_Z^2 = E\left[\{(2X+3Y) - (2\mu_X + 3\mu_Y)\}^2\right]$

or $\sigma_Z^2 = E\left[\{2(X-\mu_X)\}^2 + \{3(Y-\mu_Y)\}^2 + 2.2.3.(X-\mu_X)(Y-\mu_Y)\right]$

or $\sigma_Z^2 = 4\sigma_X^2 + 9\sigma_Y^2 + 12\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_Z^2 = 4.1^2 + 9.2^2 + 12.(0.5).1.2$

or $\sigma_Z^2 = 4 + 36 + 12$

2(f)(ii)

or $\boxed{\sigma_Z^2 = 52}$ Ans

$\sigma_W^2 = E\left[\{(X-Y) - (\mu_X - \mu_Y)\}^2\right]$

or $\sigma_W^2 = E\left[\{X-\mu_X\}^2 + \{Y-\mu_Y\}^2 - \{2.1.1.(X-\mu_X)(Y-\mu_Y)\}\right]$

or $\sigma_W^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y$

or $\sigma_W^2 = 1^2 + 2^2 - 2.(0.5).1.2$

or $\sigma_W^2 = 1 + 4 - 2$

2(f)(iv)

$\boxed{\sigma_W^2 = 3}$ Ans

2(g)

$Cov(Z,W) = E\left[(Z-\mu_Z)(W-\mu_W)\right]$

or $Cov(Z,W) = E\left[(2X+3Y-2\mu_X-3\mu_Y)(X-Y-\mu_X-(-\mu_Y))\right]$

or $Cov(Z,W) = E\left[\{2(X-\mu_X)+3(Y-\mu_Y)\}\{1(X-\mu_X)-1(Y-\mu_Y)\}\right]$

or $Cov(Z,W) = E\left[2(X-\mu_X)^2 - 3(Y-\mu_Y)^2 + 1(X-\mu_X)(Y-\mu_Y)\right]$

or $Cov(Z,W) = 2\sigma_X^2 - 3\sigma_Y^2 + \rho_{XY}\sigma_X\sigma_Y$

or $Cov(Z,W) = 2.1^2 - 3.2^2 + (0.5).1.2$

or $Cov(Z,W) = 2 - 12 + 1$

2(f)(v)

or $\boxed{Cov(Z, W) = -9}$ Ans

$$\boxed{\rho_{Z,W} = \frac{Cov(Z,W)}{\sigma_Z \sigma_W} = \frac{-9}{\sqrt{52}\cdot\sqrt{3}} \approx -0.7206}$$

2(f)(vi)

$$\boxed{f_{Z,W}(z,w) = \mathcal{N}(\mu_Z = 0, \mu_W = 0, \sigma_Z^2 = 52, \sigma_W^2 = 3, \rho_{Z,W} = -0.7206)}$$

Ans

2(g) $R = aX + bY \Rightarrow \sigma_R^2 = a^2 \cdot \sigma_X^2 + b^2 \sigma_Y^2$

$$\text{or } \sigma_R^2 = a^2 \cdot 1^2 + b^2 \cdot 2^2$$

$$\text{or } \sigma_R^2 = a^2 + 4b^2$$

$Cov(R, Y) = 0$ is a sufficient condition for R and Y to be independent, as both are gaussian.

$$Cov(R,Y) = E[(aX + bY - a\mu_X - b\mu_Y)(Y - \mu_Y)]$$

$$\text{or } Cov(R,Y) = E[a(X-\mu_X)(Y-\mu_Y) + b(Y-\mu_Y)^2]$$

$$\text{or } Cov(R,Y) = a\,\rho_{X,Y}\,\sigma_X \sigma_Y + b\,\sigma_Y^2$$

$$\text{or } Cov(R,Y) = a(0.5)\cdot 1\cdot 2 + b\cdot 2^2$$

$$\text{or } Cov(R,Y) = a + 4b$$

2(g)

For R, Y to be independent distributions, $a + 4b = 0$

we can use any set of Real numbers to do so,

say $a = -4, b = 1$

Ans

2(b) $Q = aX$

given $p(a=2) = 0.5$ $\quad f(Q \mid a=2) = \cancel{2} \, f_X(X=2x)$

$p(a=1) = 0.5$ $\quad f(Q \mid a=1) = f_X(X=x)$

$$Q \xrightarrow{0.5} Q = X$$

$$Q \xrightarrow{0.5} Q = 2X$$

$$f_Q(q) = f_{Q|a}(Q \mid a=1) \cdot P(a=1) + f_{Q|a}(Q \mid a=2) \cdot P(a=2) \quad \text{(LoTP)}$$

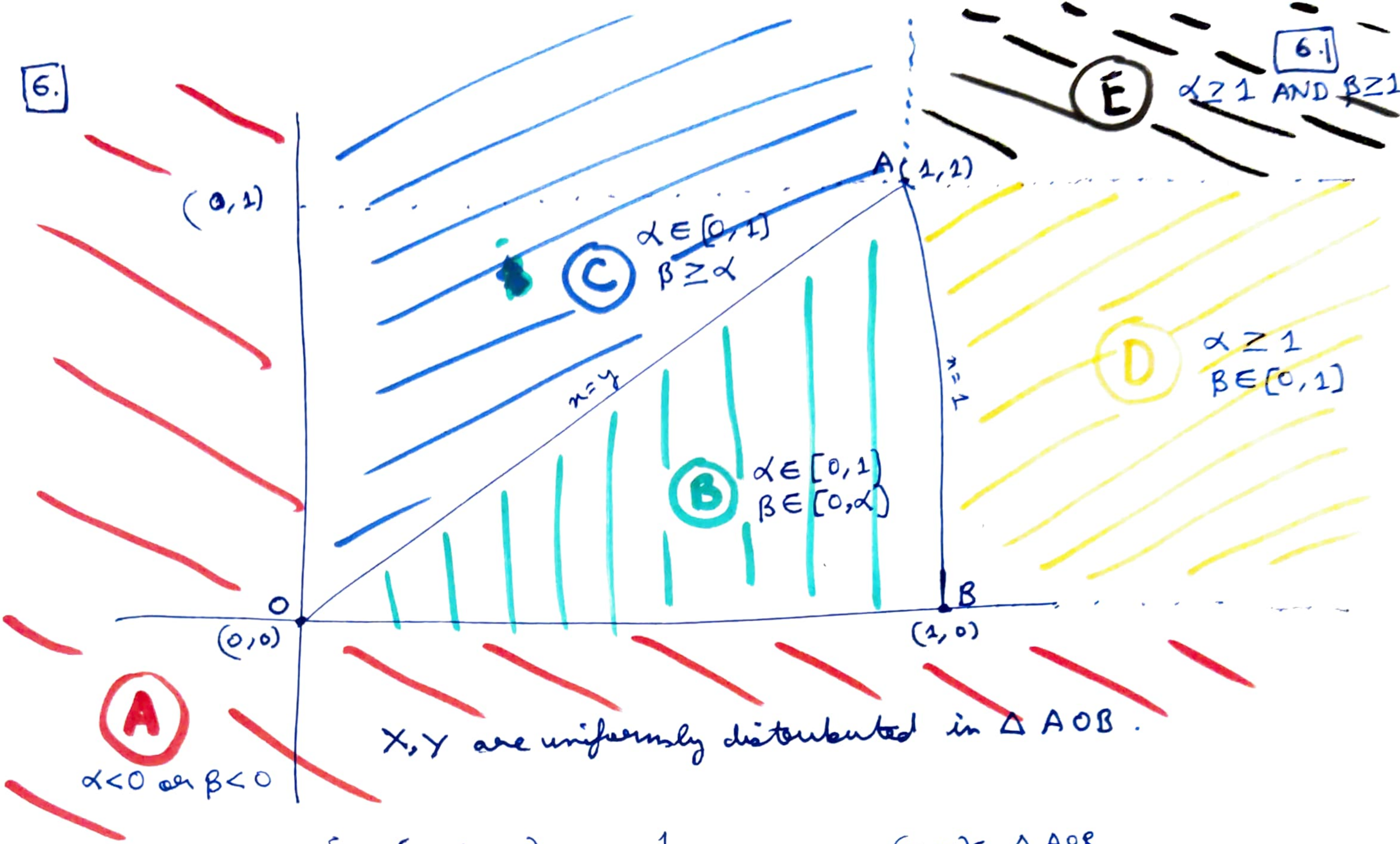
or $$f_Q(q) = 0.5\, \mathcal{N}(q, \mu_q = \mu_x, \sigma_q^2 = \sigma_x^2) + 0.5\, \mathcal{N}(q, \mu_q = 2\mu_x, \sigma_q^2 = 2^2 \sigma_x^2)$$

2(b)

or $$\boxed{f_Q(q) = 0.5\, \mathcal{N}(q, \mu_q = 0, \sigma_q^2 = 1) + 0.5\, \mathcal{N}(q, \mu_q = 0, \sigma_q^2 = 4)}$$ Ans

———— X ———— X ———— X ———— X ————

6.



X, Y are uniformly distributed in Δ AOB.

So $f_{X,Y}(x,y) = \dfrac{1}{\text{Area of } \Delta AOB}$ $\quad (x,y) \in \Delta AOB$

or $f_{X,Y}(x,y) = \dfrac{1}{\frac{1}{2} \times 1 \times 1}$ $\quad y \in [0,1]$ or $x \in [y,1]$

6(a)

or $f_{X,Y}(x,y) = \begin{cases} 2 & y \in [0,1],\ x \in [y,1] \\ 0 & \text{else} \end{cases}$

Ans

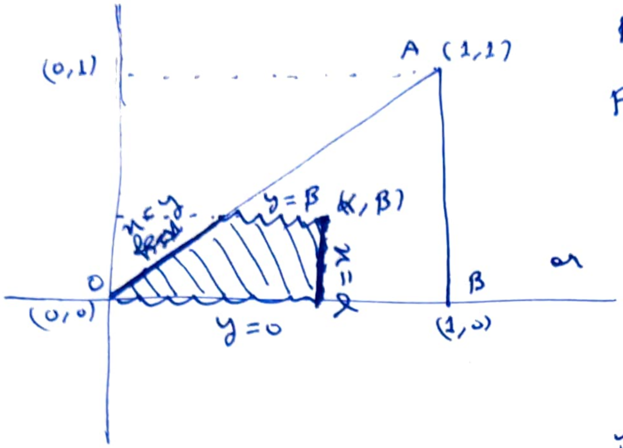
6(b) (Calculations on next pages.)

$$F_{X,Y}(x=\alpha, y=\beta) = \begin{cases} 0 & \alpha < 0 \text{ or } \beta < 0 \quad \text{A} \\ 2\alpha\beta - \beta^2 & \alpha \in [0,1],\ \beta \in [0,\alpha] \quad \text{B} \\ \alpha^2 & \alpha \in [0,1],\ \beta \geq \alpha \quad \text{C} \\ 2\beta - \beta^2 & \alpha \geq 1,\ \beta \in [0,\alpha] \quad \text{D} \\ 1 & \alpha \geq 1,\ \beta \geq 1 \quad \text{E} \end{cases}$$

Ans

For (α,β) in B:

i.e. $\alpha \in [0,1]$, $\beta \in [0,\alpha]$



$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\,dx\,dy \quad (\alpha,\beta)\in B$$

or $$F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\beta}\int_{x=y}^{x=\alpha} 2\,dx\,dy \quad (\alpha,\beta)\in B$$

or $$F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2x\Big|_y^\alpha dy \quad (\alpha,\beta)\in B$$

or $$F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2(\alpha-y)\,dy \quad (\alpha,\beta)\in B$$

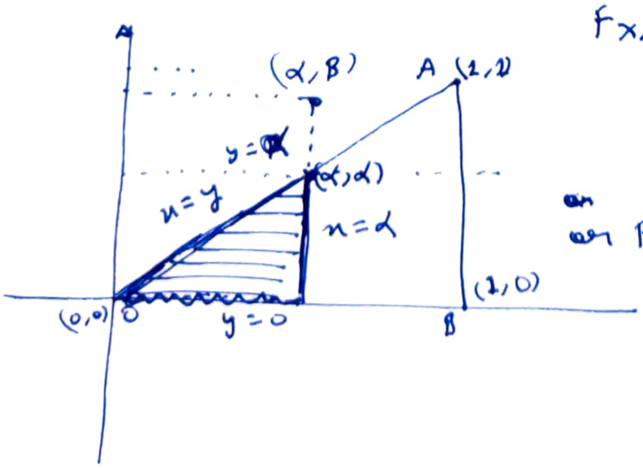
or $$F_{X,Y}(\alpha,\beta) = 2\alpha y - y^2\Big|_0^\beta \quad (\alpha,\beta)\in B$$

or $$\boxed{F_{X,Y}(\alpha,\beta) = 2\alpha\beta - \beta^2 \quad (\alpha,\beta)\in B}$$

or $\alpha \in [0,1)$, $\beta \in [0,\alpha]$

For (α,β) in C:

i.e. $\alpha \in [0,1]$, $\beta \ge \alpha$



$$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\,dx\,dy \quad (\alpha,\beta)\in C$$

or $$F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha}\int_{x=y}^{x=\alpha} 2\,dx\,dy \quad (\alpha,\beta)\in C$$

or $F_{X,Y}(x=\alpha, y=\beta) = \int_{y=0}^{y=\alpha} 2(\alpha - y)\,dy \qquad (\alpha,\beta) \in$ C

or $F_{X,Y}(x=\alpha, y=\beta) = 2\alpha^2 - \alpha^2 \qquad (\alpha,\beta) \in$ C

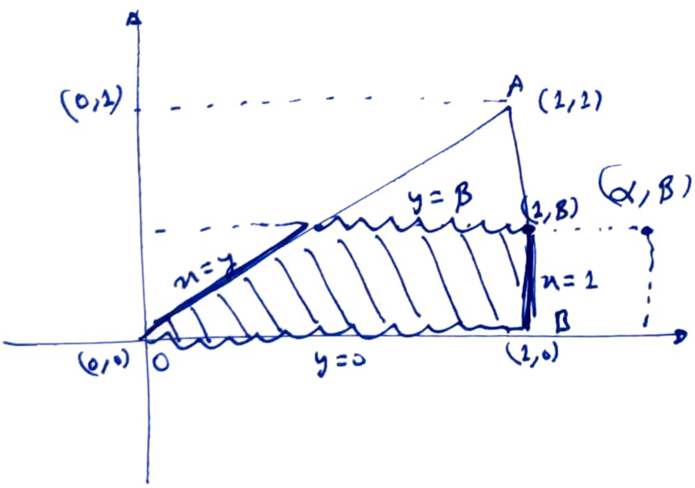
or $\boxed{F_{X,Y}(x=\alpha, y=\beta) = \alpha^2 \qquad (\alpha,\beta) \in \text{C} \quad \text{or} \quad \alpha \in [0,1],\ \beta \geq \alpha}$

Area of the shaded Δ.
* $f_{X,Y}(x,y)$ in the region.

For (α,β) in D

i.e. $\alpha \geq 1$, $\beta \in [0,1]$

$F_{X,Y}(x=\alpha, y=\beta) = \iint_{\text{shaded region}} 2\,dx\,dy \qquad (\alpha,\beta) \in$ D



or $F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} \int_{x=y}^{x=1} 2\,dx\,dy \qquad (\alpha,\beta) \in$ D

or $F_{X,Y}(\alpha,\beta) = \int_{y=0}^{y=\beta} 2(1-y)\,dy \qquad (\alpha,\beta) \in$ D

or $F_{X,Y}(\alpha,\beta) = 2y - y^2 \Big|_0^\beta \qquad (\alpha,\beta) \in$ D

or $\boxed{F_{X,Y}(\alpha,\beta) = 2\beta - \beta^2 \qquad (\alpha,\beta) \in \text{D} \quad \text{or} \quad \alpha \geq 1,\ \beta \in [0,1)}$

$$f_X(x) = \int_{\forall y} f_{X,Y}(x,y)\,dy$$

or $$f_X(x) = \begin{cases} 0 & x<0 \\ \int_{y=0}^{y=x} 2\,dy & x \in [0,1) \\ 0 & x \geq 1 \end{cases}$$

6(c)

or $$\boxed{f_X(x) = \begin{cases} 0 & x<0 \\ 2x & x \in [0,1) \\ 0 & x>1 \end{cases}}$$ Ans

$$f_Y(y) = \int_{\forall x} f_{X,Y}(x,y)\,dx$$

or $$f_Y(y) = \begin{cases} 0 & y<0 \\ \int_{x=y}^{x=1} 2\,dx & y \in [0,1] \\ 0 & y>1 \end{cases}$$

6(d)

or $$\boxed{f_Y(y) = \begin{cases} 0 & y<0 \\ 2(1-y) & y \in [0,1) \\ 0 & y>1 \end{cases}}$$ Ans

$$f_{X|Y}(x|Y=y) = \frac{f_{X,Y}(x,y=y)}{f_Y(y)}$$

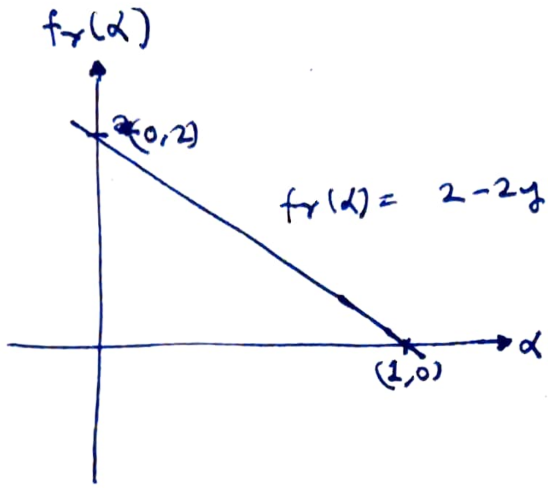
6(e)

or

$$f_{X|Y}(x|Y=y) = \begin{cases} \text{not defined} & y < 0 \\ \dfrac{1}{1-y} & y \in (0,1),\ x \in (y,1) \\ \text{not defined} & y \geq 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Ans

8.



8 (a) $\hat{Y}_{MP} = \arg\max_{\alpha} \left(f_Y(\alpha) \right) \Big|_{\alpha \in [0,1]}$

or $\hat{Y}_{MP} = \arg\max_{\alpha} \left(2-2\alpha \right) \Big|_{\alpha \in [0,1]}$

8(a)

or $\boxed{\hat{Y}_{MP} = 0}$ Ans for which $f_Y(\alpha=0) = 2$

Ans

8(b) $\hat{Y}_{MMSE} = \arg\min_{\alpha} \left[E[(Y-\alpha)^2] \right]$

We know that $\hat{Y}_{MMSE} = \mu_Y = E(Y)$

or $\hat{Y}_{MMSE} = \int_{y=0}^{y=1} y \cdot f_Y(y)\, dy$

or $\hat{y}_{MMSE} = \int_{y=0}^{y=1} y \cdot (2-2y)\, dy$

or $\hat{y}_{MMSE} = \int_{y=0}^{y=1} 2y - 2y^2\, dy$

or $\hat{y}_{MMSE} = y^2 - \frac{2}{3}y^3 \Big|_{y=0}^{y=1}$

8(b)

or $\boxed{\hat{y}_{MMSE} = \frac{1}{3}}$ Ans

PTO

None the less, let us derive $\hat{Y}_{MMSE}$ using only first principles.

$$\hat{y}_{MMSE} = \arg\min_{\alpha} \left[E\left[(Y-\alpha)^2\right] \right]$$

or $$\hat{y}_{MMSE} = \arg\min_{\alpha} \left[\int_{y=0}^{y=1} (y-\alpha)^2 \cdot (2-2y)\, dy \right]$$

or $$\hat{y}_{MMSE} = \arg\min_{\alpha} \left[2\int_{y=0}^{y=1} \{y^2 - 2\alpha y + \alpha^2\}\{1-y\}\, dy \right]$$

or $$\hat{y}_{MMSE} = \arg\min_{\alpha} \left[2\int_{y=0}^{y=1} \{-y^3 + (1+2\alpha)y^2 - (\alpha^2+2\alpha)y + \alpha^2\} dy \right]$$

or $$\hat{y}_{MMSE} = \arg\min_{\alpha} \left[2 \left[-\frac{y^4}{4} + \frac{(1+2\alpha)}{3}y^3 - \frac{(\alpha^2+2\alpha)}{2} y^2 + \alpha^2 y \right]_{y=0}^{y=1} \right]$$

or $$\hat{y}_{MMSE} = \arg\min_{\alpha} \left[2 \left[-\frac{1}{4} + \frac{(1+2\alpha)}{3} - \frac{(\alpha^2+2\alpha)}{2} + \alpha^2 \right] \right]$$

or $$\hat{y}_{MMSE} = \arg\min_{\alpha} \left[2 \left[\frac{\alpha^2}{2} - \frac{1}{3}\alpha + \frac{1}{12} \right] \right]$$

or $$\hat{y}_{MMSE} = \arg\min_{\alpha} \left[\left(\alpha - \frac{1}{3}\right)^2 + \frac{1}{18} \right]$$

8(b)

or $$\boxed{\hat{y}_{MMSE} = \frac{1}{3} \quad \text{with error} = \frac{1}{18}}$$ Ans

8(c)

$$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(E(|y-\alpha|)\right) = \underset{\alpha}{\arg\min}\left(\int_{y=0}^{y=\alpha} (\alpha-y)(2-2y)\,dy + \int_{y=\alpha}^{y=1} (y-\alpha)(2-2y)\,dy \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(2\int_{y=0}^{y=\alpha} (y-\alpha)(y-1)\,dy + 2\int_{y=1}^{y=\alpha} (y-\alpha)(y-1)\,dy \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(2\int_{y=0}^{y=\alpha} \{y^2-(\alpha+1)y+\alpha\}\,dy + 2\int_{y=1}^{y=\alpha} \{y^2-(\alpha+1)y+\alpha\}\,dy \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(4\times\left[\frac{y^3}{3} - \frac{(\alpha+1)}{2}y^2 + \alpha y\right]\Bigg|_{y=\alpha} - 2\left[\frac{y^3}{3} - \frac{(\alpha+1)}{2}y^2 + \alpha y\right]\Bigg|_{y=1} - 2\left[\frac{y^3}{3} - \frac{(\alpha+1)}{2}y^2 + \alpha y\right]\Bigg|_{y=0} \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(4\left[\frac{\alpha^3}{3} - \frac{(\alpha+1)}{2}\alpha^2 + \alpha^2\right] - 2\left[\frac{1}{3} - \frac{(\alpha+1)}{2} + \alpha\right] \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left(4\left[-\frac{\alpha^3}{6} + \frac{\alpha^2}{2}\right] - 2\left[\frac{\alpha}{2} - \frac{1}{6}\right] \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg\min}\left[-\frac{4}{6}\alpha^3 + 2\alpha^2 - \alpha + \frac{1}{3} \right]$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg}\left(\frac{d}{d\alpha}\left[-\frac{2}{3}\alpha^3 + 2\alpha^2 - \alpha + \frac{1}{3} \right] = 0 \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg}\left(-2\alpha^2 + 4\alpha - 1 = 0 \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg}\left(-2(\alpha-1)^2 + 1 = 0 \right)$$

or $$\hat{y}_{MMAE} = \underset{\alpha}{\arg}\left((\alpha-1)^2 = \frac{1}{2} \right)$$

or $$\hat{y}_{MMAE} = 1 - \frac{1}{\sqrt{2}}$$

8(c) or $$\boxed{\hat{y}_{MMAE} \approx 0.293}$$ Ans