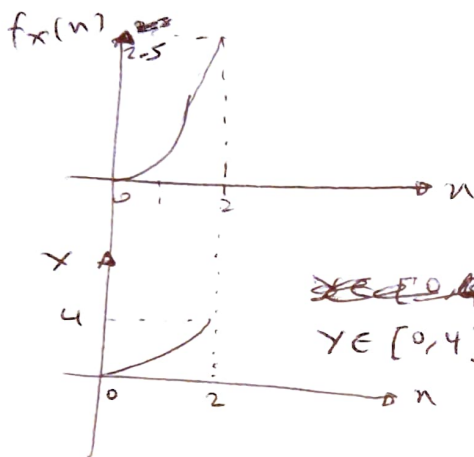


1.

$$f_X(n) = \begin{cases} \frac{5}{32} n^4 & n \in (0, 2] \\ 0 & \text{else.} \end{cases}$$

$$Y = X^2$$



(i) Using CDF formula

$$F_X(n=\alpha) = \begin{cases} 0 & \alpha \leq 0 \\ \frac{1}{32} \alpha^5 & \alpha \in (0, 2] \\ 1 & \alpha > 2 \end{cases}$$

~~Example 4~~

$$F_Y(y=\alpha) = P(Y \leq \alpha)$$

or  $F_Y(y=\alpha) = P(X^2 \leq \alpha) \Rightarrow$

or  $F_Y(y=\alpha) = P(X \leq \sqrt{\alpha})$

or  $F_Y(y=\alpha) = F_X(\sqrt{\alpha}) = \begin{cases} 0 & \sqrt{\alpha} \leq 0 \\ \frac{1}{32} (\sqrt{\alpha})^5 & \sqrt{\alpha} \in (0, 2] \\ 1 & \sqrt{\alpha} > 2 \end{cases}$

1a.

or  $F_Y(y=\alpha) = \begin{cases} 0 & \alpha \leq 0 \\ \frac{1}{32} \alpha^{\frac{5}{2}} & \alpha \in (0, 4] \\ 1 & \alpha > 4 \end{cases}$

Ans

$$f_X(y) = \left. \frac{d}{dy} (F_X(y=x)) \right|_{y=x} =$$

$$= \begin{cases} 0 & y < 0 \\ \frac{5}{64} y^{\frac{3}{2}} & y \in [0, 4] \\ 0 & y > 4 \end{cases}$$

(ii) Using Transformation ('cute') formula:

Ans

$$f_X(y) = \frac{d}{dy} F_X(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

$$= \frac{d}{dn} (F_X(g^{-1}(y))) \cdot \left| \frac{dn}{dy} \right|$$

Here since  ~~$g(x) = y$~~   $y = g(n) = n^2$ ,

$$g^{-1}(y) = \sqrt{y}.$$

$$\frac{dy}{dn} = \frac{d}{dn} (n^2) = 2n$$

$$\frac{dn}{dy} = \frac{d}{dy} (\sqrt{y}) = \frac{1}{2\sqrt{y}}$$

$$f_X(y) = \left. f_X(g^{-1}(y)) \cdot \left| \frac{1}{2\sqrt{y}} \right| \right|_{y=x}$$

$$= \begin{cases} 0 & \sqrt{y} \leq 0 \\ \frac{5}{32} (\sqrt{y})^4 \cdot \left| \frac{1}{2\sqrt{y}} \right| & \sqrt{y} \in (0, 2] \\ 0 & \sqrt{y} > 2 \end{cases}$$

1b (again!)

1-3

$$f_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{5}{64} y^{\frac{3}{2}} & y \in (0, 4] \\ 0 & y > 4 \end{cases}$$

Ans (Again).

(i) Using  $f_Y(y)$

$$E[Y] = \int_{y=0}^{y=4} y f_Y(y) dy$$

$$E[Y] = \int_{y=0}^4 y \times \frac{5}{64} y^{\frac{3}{2}} dy$$

$$E[Y] = \left. \frac{5}{64} \cdot y^{\frac{7}{2}} \cdot \frac{2}{\frac{7}{2}} \right|_{y=0}^{y=4}$$

$$E[Y] = \frac{10}{64 \times 7} \cdot 4^{\frac{7}{2}}$$

(1c)

$$E[Y] = \frac{20}{7} \quad \underline{\underline{\text{Ans}}}$$

(ii) using Law of The Unconscious Statistician (LOTUS):

$$E[Y] = E[X^2] = \int_{n=0}^{n=2} x^2 \cdot f_X(n) dn$$

$$E[Y] = \int_{n=0}^{n=2} n^2 \cdot \frac{5}{32} n^4 dn$$

$$E[Y] = \left. \frac{5}{32} \cdot \frac{n^7}{7} \right|_{n=0}^{n=2}$$

$$E[Y] = \frac{20}{7} \quad \underline{\underline{\text{Ans (again!)}}}$$

2.

Roll #1

$$X = \{1, 2, 3, 4, 5, 6\}$$

Roll again?

Roll #2

$$Y = \{1, 2, 3, 4, 5, 6\}$$

Let  $A$  be the event that you cash in on the first die throw.

$$E(W) = E$$

$$E(W) = E(AE(X) + \bar{A}E(Y))$$

$$W = AX + \bar{A}Y$$

then

$$W = AX + \bar{A}Y \quad (1)$$

~~Cashing in on the first throw.~~

In order to maximize  $W$ , we need to optimize the event (algorithm)  $A$ .

Intuitively, we would want to throw again if we get 'lower' values of  $X$  and hold back from throwing again if we get a high value of  $Y$ .

For quantifying  $A$ , we can utilize a variable  $g$  (for good throw) which stands for the ~~top~~ ~~low~~ number of allowed 'high-valued' throws allowed.

TABLE 1:  $E(X|g)$  vs  $g$ .

$g$	$X g$	$E(X g)$
1	{6}	6
2	{6, 5}	5.5
3	{6, 5, 4}	5
4	{6, 5, 4, 3}	4.5
5	{6, 5, 4, 3, 2}	4
6	{6, 5, 4, 3, 2, 1}	3.5

Here,  $g=2$ ,

i.e.

hold back upon getting

the highest 2

values of  $X$ : 5 or 6.

In such a case,

$$E(X|g=2) = \frac{5+6}{2} = 5.5$$

From the Table 1, we can derive  $E(X|g)$  as a simple linear function of  $g$ :

$$E(X|g) = 6.5 - \frac{g}{2} \quad (2)$$

Taking expectation on both sides of (1):

$$E(W) = E(AX) + E(\bar{A}X)$$

and using  $g$  to quantify event  $A$ :

$$E(W) = E(X|g) \cdot P(g) + E(X|\bar{g}) \cdot P(\bar{g}) \quad (3)$$

where  $X|g$  = value of  $X$  given that only the highest  $g$  values of  $X$  are allowed for holding back from throwing again.

$P(g)$  = Probability of obtaining any of the highest  $g$  values on the first die throw.

$X|\bar{g}$  = gives that the <sup>none of the</sup> highest  $g$  values were landed on the first throw, the value landed on the second throw..

It may be noted that ~~that~~ ~~that~~ ~~that~~  $E(Y)$  is not dependent on the value of  $g$  or  $\bar{g}$ . So  $E(Y|\bar{g}) = E(Y)$ .

So  $E(W) = \left(6.5 - \frac{1}{2}g\right)\left(\frac{g}{6}\right) + \left(\frac{1+2+3+4+5+6}{6}\right)\left(1 - \frac{g}{6}\right)$

$E(W) \equiv -\frac{g^2}{12} + \frac{g}{2} + 3.5$  (4)

arg (Max  $E(W)$ ) = arg  $\left(\frac{d}{dg} E(W) = 0\right)$

$g_{Max} = \arg\left(-\frac{g}{6} + \frac{1}{2} = 0\right)$

$g_{Max} = 3$  (5)

~~which is~~

Putting  $g_{Max}$  into (4):

Max.  $E(W) = -\frac{3^2}{12} + \frac{3}{2} + 3.5$

Max.  $E(W) = 4.25$  Ans.

An expected \$4.25 can be obtained by  
thereby again <sup>only</sup> the first three lands 1, 2 or 3.





3.

outcomes	$i$	$P(X_i)$	$X_i$
T	1	$(1/2)$	$2^0$
HT	2	$(1/2)^2$	$2^1$
HHT	3	$(1/2)^3$	$2^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\underbrace{H \dots H T}_{(n-1)}$	$n$	$(1/2)^n$	$2^{n-1}$

$$E(X) = \sum_{i=1}^{\infty} P(X_i) \cdot X_i \cdot P(X_i)$$

$$\Rightarrow E(X) = \sum_{i=1}^{\infty} 2^{i-1} \cdot \left(\frac{1}{2}\right)^i$$

$$\Rightarrow E(X) = \sum_{i=1}^{\infty} 2^{-1}$$

3(a)

$$\Rightarrow E(X) \rightarrow \infty$$

~~Ex~~ ~~$E(X)$~~ 

$$P(X > 65) = P(K > \log_2(65) + 1)$$

$$\Leftrightarrow X = 2^{K-2}$$

$$\Rightarrow P(X > 65) = P(K > 7)$$

$$\Rightarrow P(X > 65) = 1 - P(K \leq 7)$$

$$\Rightarrow P(X > 65) = 1 - \sum_{k=2}^{K=7} \left(\frac{1}{2}\right)^k$$

$$\Rightarrow P(X > 65) = 1 - \frac{\left(\frac{1}{2}\right)^1 \cdot \left(1 - \left(\frac{1}{2}\right)^7\right)}{1 - \frac{1}{2}}$$

$$\Rightarrow P(X > 65) = 1 - \left(1 - \left(\frac{1}{2}\right)^7\right)$$

3(b)

$$\Rightarrow P(X > 65) = \left(\frac{1}{2}\right)^7 \quad \underline{\text{Ans}}$$

~~E(X)~~ =

If  $X_{\text{max}} = 2^{30}$ , then the player should not bet after

$k = 31$  trials.

$$E(X) = \sum_{k=1}^{30} 2^{k-1} \cdot \left(\frac{1}{2}\right)^k + 2^{30} \cdot \left(\frac{1}{2}\right)^{30} \cdot 1$$

$$E(X) = \left(\frac{1}{2}\right) \times 30 + 1 = 16$$

↓  
We win  
independent of the  
outcome on the 31st  
trial.

If it's a  $\frac{1}{2}$ , then  
the game ends and I take  
home  $2^{30}$ .

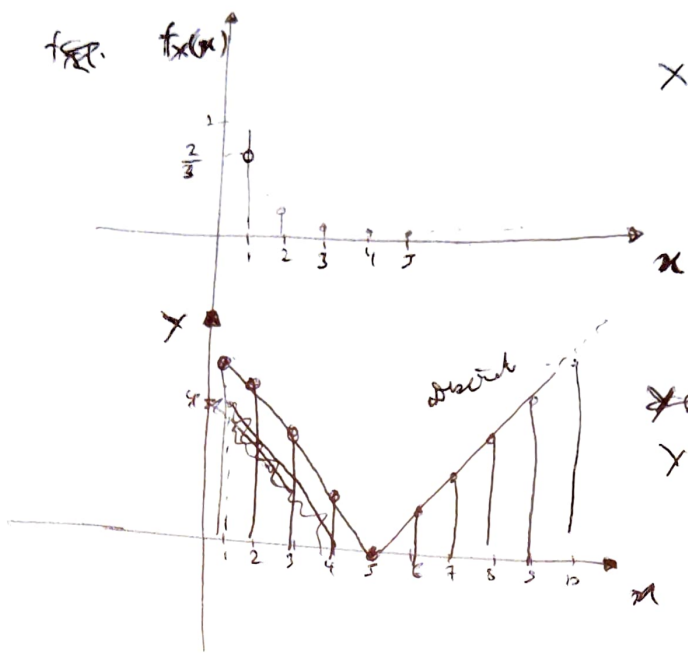
If not, I would  
still have won the  
 $2^{30}$  and forfeit  
the game after  
the 31st trial.

————— X ————— X ————— X —————



4.  $X \sim \text{geometric}(\frac{1}{3}) \Rightarrow f_X(n) = (\frac{2}{3})(\frac{1}{3})^{n-1} \quad n \geq 1, 2, 3, \dots$

$Y = |X - 5|$



$X \in \{1, 2, 3, 4, \dots, \infty\}$

$X \in [0, \infty)$

$Y \in \{0, 1, 2, \dots, \infty\}$

$F_X(n) = 0 \quad n \leq 0$

$\frac{2}{3} \quad n=1$

$\frac{2}{3} + \frac{2}{3} \cdot \frac{1}{3} \quad n=2$

$\sum_{i=1}^n (\frac{2}{3})(\frac{1}{3})^{i-1} \quad n=n$

$\frac{2}{3} \cdot (1 - (\frac{1}{3})^{n+1})$

$1 - \frac{1}{3}$

$X$	$X_i$	$F_X(n)$
0	5	$(\frac{2}{3})(\frac{1}{3})^4$
1	4,6	$(\frac{2}{3})(\frac{1}{3})^5$
2	3,7	$(\frac{2}{3})(\frac{1}{3})^6$
3	2,8	$(\frac{2}{3})(\frac{1}{3})^7$
4	1,9	$(\frac{2}{3})(\frac{1}{3})^8$
5	0,10	$(\frac{2}{3})(\frac{1}{3})^9$

4.1

$Y_i$	$X_i$	$\text{pmf}(Y_i)$
0	5	$(\frac{2}{3})(\frac{1}{3})^4$
1	4,6	$(\frac{2}{3})(\frac{1}{3})^3 + (\frac{2}{3})(\frac{1}{3})^5$
2	3,7	$(\frac{2}{3})(\frac{1}{3})^2 + (\frac{2}{3})(\frac{1}{3})^6$
3	2,8	$(\frac{2}{3})(\frac{1}{3})^1 + (\frac{2}{3})(\frac{1}{3})^7$
4	1,9	$(\frac{2}{3}) + (\frac{2}{3})(\frac{1}{3})^8$
5	0,10	$(\frac{2}{3})(\frac{1}{3})^9$
...	...	...
N	N+5	$(\frac{2}{3})(\frac{1}{3})^{N+4}$

$F_Y(y) = P(|X-5| \leq y)$

$F_Y(y) = P(n \leq \alpha+5) + P(5-n \leq \alpha)$

$F_Y(y) = P(n \leq \alpha+5) + P(n > 5-\alpha)$

$F_Y(y) = P_X(\alpha+5) + (1 - F_X(5-\alpha))$

$$F_X(4) =$$

$$n = 1, 2, 3, 4$$

5.

5.)

$$P_X(k) = \begin{cases} 0.5 & k=1 \\ 0.3 & k=2 \\ 0.2 & k=3 \\ 0 & \text{otherwise} \end{cases}$$

5(a)

$$E(X) = 0.5 + 0.3 \times 2 + 0.2 \times 3 = 1.7$$

A<sub>1</sub>

$$E(X^2) = 0.5 + 0.3 \times 4 + 0.2 \times 9 = 3.5$$

$$V(X) = E(X^2) - (E(X))^2 = 3.5 - 1.7^2 =$$

5(b)

$$V(X) = 0.61$$

A<sub>2</sub>

5(b)

$$\sigma(X) = \sqrt{0.61}$$

A<sub>2</sub>

$$Y = \frac{2}{X}$$

$$E(Y) = 0.5 \times \frac{2}{1} + 0.3 \times \frac{2}{2} + 0.2 \times \frac{2}{3}$$

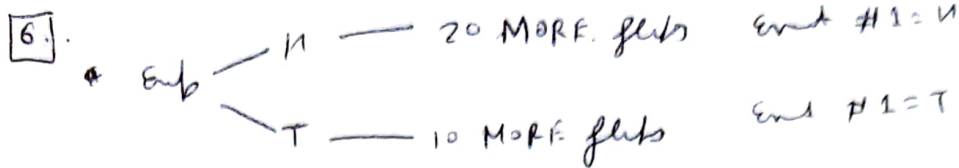
$$E(Y) = 1 + 0.3 + \frac{0.4}{3}$$

5(c)

$$E(Y) =$$

$$1.4333$$

A<sub>1</sub>



#  $X = \# \text{ heads in ALL flips (including first one)}$ .

$$a) P(X=5) = P(X=5 | \#1=H) \cdot P(\#1=H) + P(X=5 | \#1=T) \cdot P(\#1=T)$$

6(a)

$$P(X=5) = {}^{20}C_4 \cdot \left(\frac{1}{2}\right)^{20} \cdot \left(\frac{1}{2}\right) + {}^{10}C_5 \cdot \left(\frac{1}{2}\right)^{10} \cdot \left(\frac{1}{2}\right)$$

$$= 0.1254 \quad \underline{Ans}$$

$$b) P(\#1=H | X=5) = \frac{P(X=5 | \#1=H) \cdot P(\#1=H)}{P(X=5)}$$

$$P(\#1=H | X=5) = \frac{{}^{20}C_4 \cdot \left(\frac{1}{2}\right)^{20} \cdot \left(\frac{1}{2}\right)}{0.1254}$$

6(b)

$$P(\#1=H | X=5) = 0.0184 \quad \underline{Ans}$$

$$c) P(\#Z_{end}=H | X=5) = \frac{P(X=5 | \#Z_{end}=H) \cdot P(\#Z_{end}=H)}{P(X=5)}$$

$$P(\#Z_{end}=H | X=5) = \frac{P(X=5 | \#Z_{end}=H, \#1=H) \cdot P(\#1=H) + P(X=5 | \#Z_{end}=H, \#1=T) \cdot P(\#1=T)}{0.1254} \cdot \left(\frac{1}{2}\right)$$

$$P(\#Z_{end}=H | X=5) = \left[ {}^{19}C_3 \cdot \left(\frac{1}{2}\right)^{19} \cdot \left(\frac{1}{2}\right) + {}^9C_4 \cdot \left(\frac{1}{2}\right)^9 \cdot \left(\frac{1}{2}\right) \right] \left(\frac{1}{2}\right)$$

$$P(\#Z_{end}=H | X=5) = 0.4945 \quad \underline{Ans}$$