1.1

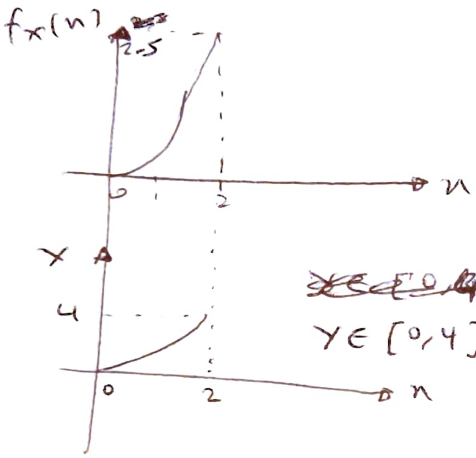
$\boxed{1.}$

$$f_X(x) = \begin{cases} \frac{5}{32} x^4 & x \in (0,2] \\ 0 & \text{else.} \end{cases}$$

$$Y = X^2$$

(i) <u>Using CDF formula</u>

$$F_X(x=\alpha) = \begin{cases} 0 & \alpha \le 0 \\ \frac{1}{32} \alpha^5 & \alpha \in (0,2] \\ 1 & \alpha > 2 \end{cases}$$



~~$X \in [0,4]$~~

$Y \in [0,4)$

~~Example P(~~

$$F_Y(y=\alpha) = P(y \le \alpha)$$

or $F_Y(y=\alpha) = P(x^2 \le \alpha)$

or $F_Y(y=\alpha) = P(x \le \sqrt{\alpha})$

or
$$F_Y(y=\alpha) = F_X(\sqrt{\alpha}) = \begin{cases} 0 & \sqrt{\alpha} \le 0 \\ \frac{1}{32} (\sqrt{\alpha})^5 & \sqrt{\alpha} \in (0,2] \\ 1 & \sqrt{\alpha} > 2 \end{cases}$$

$\boxed{1a.}$

or
$$\boxed{F_Y(y=\alpha) = \begin{cases} 0 & \alpha \le 0 \\ \frac{1}{32} \alpha^{\frac{5}{2}} & \alpha \in (0,4] \\ 1 & \alpha > 4 \end{cases}}$$

Ans

1 b.

$$f_Y(y) = \frac{d}{dy}\left(F_Y(y=\alpha)\right)\Big|_{y=\alpha} = \begin{cases} 0 & y < 0 \\ \frac{5}{64} y^{\frac{3}{2}} & y \in [0,4] \\ 0 & y > 4 \end{cases}$$

Ans

(ii) Using Transformative ("cute") formula:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

or $$f_Y(y) = \frac{d}{dx}\left(F_X(g^{-1}(y))\right) \cdot \left|\frac{dx}{dy}\right|$$

Here since $\cancel{g(x)} = y = g(x) = x^2$,

$$g^{-1}(y) = \sqrt{y}.$$

$$\cancel{\frac{dy}{dx} = \frac{d}{dx}(\sqrt{y}) =}$$

$$\frac{dx}{dy} = \frac{d}{dy}(\sqrt{y}) = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{1}{2\sqrt{y}}\right|$$

$$f_Y(y) = \begin{cases} 0 & \sqrt{y} \le 0 \\ \frac{5}{32}(\sqrt{y})^4 \cdot \left|\frac{1}{2\sqrt{y}}\right| & \cancel{\sqrt{y} \in (0,2]} \;\; \sqrt{y} \in (0,2] \\ 0 & \sqrt{y} > 2 \end{cases}$$

1 b (again!)

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} 0 & y \le 0 \\ \frac{5}{64} y^{\frac{3}{2}} & y \in (0, 4] \\ 0 & y > 4 \end{cases}}$$

Ans (Again).

(i) Using $f_Y(y)$

$$E[Y] = \int_{y=0}^{y=4} y\, f_Y(y)\, dy$$

$$\Rightarrow E(Y) = \int_{y=0}^{4} y \times \frac{5}{64} y^{\frac{3}{2}}\, dy$$

$$\Rightarrow E(Y) = \frac{5}{64} \cdot y^{\frac{7}{2}} \cdot \frac{2}{7} \Big|_{y=0}^{y=4}$$

$$\Rightarrow E(Y) = \frac{10}{64 \times 7} \cdot 4^{\frac{7}{2}}$$

1c

$$\Rightarrow \boxed{E(Y) = \frac{20}{7}}$$ Ans

(ii) Using Law of The Unconscious Statistician (LOTUS):

$$E(Y) = E(X^2) = \int_{x=0}^{x=2} x^2 \cdot f_X(x)\, dx$$

$$\Rightarrow E(Y) = \int_{x=0}^{x=2} x^2 \cdot \frac{5}{32} x^4\, dx$$

$$\Rightarrow E(Y) = \frac{5}{32} \cdot \frac{x^7}{7} \Big|_{x=0}^{x=2}$$

1c (again!)

$$\Rightarrow \boxed{E(Y) = \frac{20}{7}}$$ Ans (again!)

2.

Roll # 1 $\xrightarrow{\text{Roll again?}}$ Roll # 2

$X = \{1, 2, 3, 4, 5, 6\}$ $\qquad$ $Y = \{1, 2, 3, 4, 5, 6\}$

Let A be the event that you cash in on the first die throw.

~~$E[W] = F$~~

~~$E(W) = A E(X) + \bar{A} E(Y)$~~

~~$W = X F X$~~

then $\boxed{W = AX + \bar{A} Y}$ (1)

~~X = cashing in on the first throw.~~

In order to maximize W, we need to optimize the event (algorithm) A.

Intuitively, we would want to throw again if we get 'lower' values of X and hold back from throwing again if we get a high value of Y.

1 $\qquad$ For quantifying A, we can utilize a variable

2 $\qquad$ g (for good throw) which stands for

3 $\qquad$ the ~~top~~ ~~base~~ number of allowed 'high-valued'

4 $\qquad$ throws allowed.

5

6

$\left.\begin{matrix}5\\6\end{matrix}\right\} g$

Here, $g = 2$, i.e. hold back upon getting the highest 2 values of X: 5 or 6.

In such a case, $E(X \mid g=2) = \dfrac{5+6}{2} = 5.5$

TABLE 1: $E(X|g)$ vs g.

g	$X \mid g$	$E(X\mid g)$
1	$\{6\}$	6
2	$\{6, 5\}$	5.5
3	$\{6, 5, 4\}$	5
4	$\{6, 5, 4, 3\}$	4.5
5	$\{6, 5, 4, 3, 2\}$	4
6	$\{6, 5, 4, 3, 2, 1\}$	3.5

From the Table 1, we can devise $E(X|g)$ as a simple linear function of g:

$$\boxed{E(X|g) = 6.5 - \frac{g}{2}} \quad (2)$$

Taking Expectation on both sides of (1):

$$E(W) = E(AX) + E(\bar{A}Y)$$

and using g to quantify event A:

~~get~~

or $$\boxed{E(W) = E(X|g).P(g) + E(Y|\bar{g}).P(\bar{g})} \quad (3)$$

where $X|g$ = Value of X given that only the highest g values of X are allowed for holding back from throwing again.

$P(g)$ = Probability of obtaining any of the highest g values on the first die throw

$X|\bar{g}$ = given that the none of the highest g values were landed on the first throw, the value landed on the second throw.

It may be noted that ~~g~~ ~~that~~ E(Y) is not dependent on the value of g or $\bar{g}$. So $E(Y|\bar{g}) = E(Y)$.

$\therefore$ So $E(W) = \left(6.5 - \frac{1}{2}g\right)\left(\frac{g}{6}\right) + \left(\frac{1+2+3+4+5+6}{6}\right)\left(1 - \frac{g}{6}\right)$

or $\boxed{E(W) \equiv -\frac{g^2}{12} + \frac{g}{2} + 3.5}$ (4)

$\arg(\text{Max } E(W)) = \arg\left(\frac{d}{dg}E(W) = 0\right)$

or $g_{Max} = \arg\left(-\frac{g}{6} + \frac{1}{2} = 0\right)$

or $\boxed{g_{Max} = 3}$ (5)

~~which said~~

Putting g_{Max} into (4):

$\text{Max. } E(W) = \frac{-3^2}{12} + \frac{3}{2} + 3.5$

or $\boxed{\text{Max. } E(W) = 4.25}$ Ans.

$\therefore$ An expected \$4.25 can be obtained by throwing again only if the first throw lands 1, 2 or 3.

Ans

———— × ———— × ———— × ————

3.1

3.

outcomes	i	$p(i)$	X_i winning #
T	1	$(1/2)$	2^0
HT	2	$(1/2)^2$	2^1
HHT	3	$(1/2)^3$	2^2
⋮		⋮	⋮
$\underbrace{H\dots H}_{(n-1)}T$	n	$(1/2)^n$	2^{n-1}

$$E(X) = \sum_{i=1}^{\infty} X_i\, p(i)$$

$$\text{or } E(X) = \sum_{i=1}^{\infty} 2^{i-1}\cdot\left(\tfrac{1}{2}\right)^i$$

$$\text{or } E(X) = \sum_{i=1}^{\infty} 2^{-1}$$

3(a)

$$\text{or } \boxed{E(X_i) \rightarrow \infty}$$

$$P(X>65) = P(K > \log_2(65)+1) \quad \text{as } X = 2^{K-1}.$$

$$\text{or } P(X>65) = P(K>7)$$

$$\text{or } P(X>65) = 1 - P(K\le 7)$$

$$\text{or } P(X>65) = 1 - \sum_{k=1}^{k=7} \left(\tfrac{1}{2}\right)^k$$

$$\text{or } P(X>65) = 1 - \frac{\left(\frac{1}{2}\right)^1\cdot\left(1-\left(\frac{1}{2}\right)^7\right)}{1-\frac{1}{2}}$$

$$\text{or } P(X>65) = 1 - \left(1-\left(\tfrac{1}{2}\right)^7\right)$$

3(b)

$$\text{or } \boxed{P(X>65) = \left(\tfrac{1}{2}\right)^7}$$ Ans

~~E(Y) =~~

If $X_{Max} = 2^{30}$, then the player should not bet after

$k = 31$ trials.

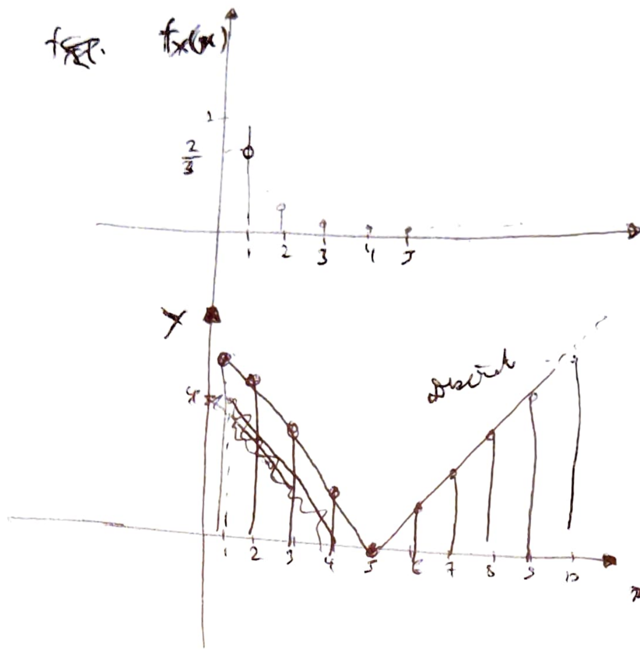
$$E(Y) = \sum_{k=1}^{30} 2^{k-1} \cdot \left(\frac{1}{2}\right)^{k} + \$ 2^{k} \cdot \left(\frac{1}{2}\right)^{k} \cdot 1$$

we win irrespective of the outcome on the 31st trial.
If its a T/H, the game ends and I take home $\$2^{30}$.
If not, I would still have won the $\$2^{30}$ and forfeit the game after the 31st trial.

or $\boxed{E(Y) = \left(\frac{1}{2}\right) \times 30 = \cancel{15} + 1 = 16}$

4. $X \sim$ geometric$\left(\frac{1}{3}\right)$ $\Rightarrow f_X(x) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{x-1}$ $\quad x \geq 1, 2, 3, \ldots$

$Y = |X-5|$



$X \in \{1, 2, 3, 4 \ldots \infty\}$

~~$Y \in [0, \infty)$~~

$Y \in \{0, 1, 2 \ldots \infty)\}$.

~~$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \frac{2}{3} & x = 1 \\ \frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3} & x = 2 \\ \sum_{i=1}^{i=n} \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{i-1} & x = n \end{cases} = \frac{2}{3} \cdot \frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}$~~

4. Ans.

Y_i	X_{i_y}	pmf(Y_i)
0	5	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4$
1	4, 6	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^5$
2	3, 7	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^6$
3	2, 8	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^1 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^7$
4	1, 9	$\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8$
5	10	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^9$
⋮		
N	N+5	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{N+4}$

~~$F_Y(y) = P(|x-5| \leq \alpha)$~~

~~$F_Y(y) = P(x \leq \alpha+5) + P(5-x < \alpha)$~~

~~$F_Y(y) = P(x \leq \alpha+5) + P(x > 5-\alpha)$~~

~~$F_Y(y) = F_X(\alpha+5) + (1 - F_X(5-\alpha))$ $\quad x \geq 5$ else 0, $\quad x = 1, 2, 3, 4$. else 0.~~

or $F_X(y) = \begin{cases} & x = 1, 2, 3, 4 \end{cases}$

5.

5.)

$$p_X(k) = \begin{cases} 0.5 & k=1 \\ 0.3 & k=2 \\ 0.2 & k=3 \\ 0 & \text{otherwise} \end{cases}$$

5(a))

$$E(X) = 0.5 + 0.3\times 2 + 0.2\times 3 = 1.7$$ Ans

$$E(X^2) = 0.5 + 0.3\times 4 + 0.2\times 9 = 3.5$$

$$V(X) = E(X^2) - [E(X)]^2 = 3.5 - 1.7^2 =$$

5(b)(i)

$$V(X) = 0.61$$ Ans

5b(ii)

$$\sigma(X) = \sqrt{0.61}$$ Ans

$$Y = \frac{2}{X} \qquad E(Y) = 0.5\times\frac{2}{1} + 0.3\times\frac{2}{2} + 0.2\times\frac{2}{3}$$

$$E(Y) = 1 + 0.3 + \frac{0.4}{3}$$

5(c)

$$E(Y) = 1.4333$$ Ans

6.

Exp — H — 20 MORE flips   Event #1 = H

Exp — T — 10 MORE flips   Event #1 = T

X = # heads in ALL flips (including first one).

(a) $P(X=5) = P(X=5 \mid \#1=H) \cdot P(\#1=H) + P(X=5 \mid \#1=T) \cdot P(\#1=T)$

6(a)

$$\boxed{\begin{aligned} P(X=5) &= {}^{20}C_4 \cdot \left(\tfrac{1}{2}\right)^{20} \cdot \left(\tfrac{1}{2}\right) + {}^{10}C_5 \left(\tfrac{1}{2}\right)^{10} \cdot \left(\tfrac{1}{2}\right) \\ P(X=5) &= 0.1254 \end{aligned}}$$

Ans

(b) $P(\#1=H \mid X=5) = \dfrac{P(X=5 \mid \#1=H) \cdot P(\#1=H)}{P(X=5)}$

$P(\#1=H \mid X=5) = \dfrac{\left\{ {}^{20}C_4 \cdot \left(\frac{1}{2}\right)^{20} \right\} \cdot \left(\frac{1}{2}\right)}{0.1254}$

6(b)

$\boxed{P(\#1=H \mid X=5) = 0.0184}$ Ans

(c) $P(\#\text{Last}=H \mid X=5) = \dfrac{P(X=5 \mid \#\text{Last}=H) \cdot P(\#\text{Last}=H)}{P(X=5)}$

$P(\#\text{Last}=H \mid X=5) = \dfrac{\left[P(X=5 \mid \#\text{Last}=H, \#1=H) \cdot P(\#1=H) + P(X=5 \mid \#\text{Last}=H, \#1=T) \cdot P(\#1=T) \right] \cdot \left(\frac{1}{2}\right)}{0.1254}$

$P(\#\text{Last}=H \mid X=5) = \dfrac{\left[{}^{19}C_3 \cdot \left(\frac{1}{2}\right)^{19} \cdot \left(\frac{1}{2}\right) + {}^{9}C_4 \left(\frac{1}{2}\right)^{9} \cdot \left(\frac{1}{2}\right) \right] \left(\frac{1}{2}\right)}{0.1254}$

6(c)

$\boxed{P(\#\text{Last}=H \mid X=5) = 0.4945}$ Ans

1.7.

If $\boxed{P(AB) = P(A).P(B)}$ (1)

Check if $\boxed{P(AB|C) = P(A|C).P(B|C)}$ (?)

1/7

FALSE Ans

Counter example: (found on stack overflow).

exp: Flip 2 coins. ~~naked (as a repeated~~

Event A: Coin 1 is H

Event B: Coin 2 is H

Event C: Both coins show the same face.

$P(AB) = \frac{1}{4}$ $P(A).P(B) = \frac{1}{4}$ i.e. A and B are independent events

Ω_A, Ω_B:

TT	TH
HT	HH

Ω_C:

TT	HH

$P(A|C) = \frac{1}{2}$ $P(B|C) = \frac{1}{2}$

But $P(AB|C) = \frac{1}{2} \neq P(A|C).P(B|C)$.

1|8.1

1|8 given $\boxed{P(A+B) = P(A) + P(B)}$ (1) $\Rightarrow$ $\boxed{P(AB) = 0 \quad A \cap B = \phi}$ (1a).

~~⇒ P(AB~~

Check if $\boxed{P(A+B \mid C) = P(A \mid C) + P(B \mid C)}$. (?).

~~C is a subset~~ of

$\boxed{C \subseteq \Omega}$.

if $A \cap B$ in Ω ~~C~~ does not exist

then ~~it~~ $A \cap B$ does not exist in any subset

of Ω, such as C, too.

So $P(AB \mid C) = 0$ too, like $P(AB) = 0$ ~~as similarly seen~~ ~~fro~~
(seen (1a))

So $P(A+B \mid C) = P(A \mid C) + P(B \mid C) - P(AB \mid C)$

But $P(AB \mid C) = 0$

1|8
So $\boxed{P(A+B \mid C) = P(A \mid C) + P(B \mid C)}$ An

Hence Proved.

The statement is TRUE!

X —— X —— X

2|2 given A ~~and~~ is ALMOST INDEPENDENT of B if

$$P(A|B) \in [P(A)-0.01,\ P(A)+0.01]$$

(a) if given A is 'AI' of B, or $\boxed{P(A|B) \in [P(A)-0.01,\ P(A)+0.01]}$ (2)
check if $\bar{A}$ is also 'AI' of B.

we know that $P(A|B) + P(\bar{A}|B) = 1$

or $P(\bar{A}|B) = 1 - P(A|B)$

or $P(\bar{A}|B) = 1 - [P(A)-0.01,\ P(A)+0.01]$

or $P(\bar{A}|B) = [1-P(A)-0.01,\ 1-P(A)+0.01]$

or $P(\bar{A}|B) = [P(\bar{A})-0.01,\ P(\bar{A})+0.01]$

2|2(a) ~~P(A~~

$\Rightarrow$ $\boxed{\bar{A} \text{ is 'AI' of } B}$
$\boxed{\text{if } A \text{ is 'AI' of } B}$ Ans.

(b) given A is 'AI' of B ~~i.e. P(A|B) ∈ [P(A)-0.01, P(A)...~~

i.e. $\boxed{P(A|B) \in [P(A)-0.01,\ P(A)+0.01]}$ (1)

check if B is also 'AI' of A

i.e. $\boxed{P(B|A) \in [P(B)-0.01,\ P(B)+0.01]}$ (2)

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

$$\text{or } P(B|A) = \frac{[P(A)-0.01,\ P(A)+0.01]\ P(B)}{P(A)}$$

2|2.2

or $P(B|A) = \left[P(B) - \underbrace{0.01\frac{P(B)}{P(A)}}_{\downarrow},\ P(B) + 0.01\frac{P(B)}{P(A)}\right]$

For B to be 'AI' of A too,

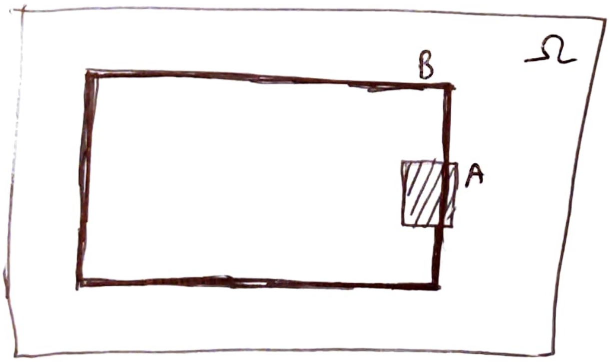
we require that $\frac{P(B)}{P(A)} \le 1$

which is NOT a ~~general~~ or ~~true~~ correct ~~assumption~~ inequality in general.

2/2 (b) $\therefore$ The ~~the~~ statement is False; Ans

i.e. ~~A~~ given A is 'AI' of B $\not\Rightarrow$ B is 'AI' of A

Counterexample



Here B $\square$

A ▨

$P(B) = 0.75$

Let's take ~~B ≈ ...~~

~~P(AB)~~ ~~...~~

~~Say P(A) = P(A)~~

Assume that 1% of A is outside B.

In that case P(A)

By construction/hypothesis
$$\begin{cases} P(B) = 0.75 \\ P(A) = \text{some smaller value} \\ P(A\bar{B}) = 0.01\,P(A) \end{cases}$$

$P(A|B) =$

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$\therefore P(A|B) = \frac{0.99\,P(A)}{P(B)}$$

$$\therefore P(A|B) = \frac{0.99\,P(A)}{0.75}$$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{0.99\,P(A)}{P(A)}$$

$$\therefore P(B|A) = 0.99$$

$$\notin \left[P(B) - 0.01,\ P(B) + 0.01\right]$$