Normal Forms and Imperfections

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Let's begin with this equation:

$$\frac{dx}{dt} = -x(x^2 - 2x - \mu) \tag{1}$$

which has equilibrium points at

$$\begin{aligned} x_0 &= 0 \\ x_0 &= 1 + \sqrt{1+\mu} \\ x_0 &= 1 - \sqrt{1+\mu} \end{aligned}$$

such that:

$$\frac{dx_0}{dt} = -x_0(x_0^2 - 2x_0 - \mu) = 0 \tag{2}$$

Perturbing x around an equilibrium point x_0 by a small value $\tilde{x}$, i.e. using $x = x_0 + \tilde{x}$, we can rewrite equation (1) as:

$$\begin{aligned} \frac{dx}{dt} &= \frac{d(x_0 + \tilde{x})}{dt} \\ \frac{dx}{dt} &= \frac{dx_0}{dt} + \frac{d\tilde{x}}{dt} \\ \frac{dx}{dt} &= \frac{d\tilde{x}}{dt} \end{aligned} \tag{3}$$

and putting $x = x_0 + \tilde{x}$ and $\mu = \mu_0 + \tilde{\mu}$ in the RHS of equation (1), we get:

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= -(x_0 + \tilde{x})\{(x_0 + \tilde{x})^2 - 2(x_0 + \tilde{x}) - (\mu_0 + \tilde{\mu})\} \\ \frac{d\tilde{x}}{dt} &= -(x_0 + \tilde{x})\{x_0^2 + 2x_0\tilde{x} + \tilde{x}^2 - 2x_0 - 2\tilde{x} - \mu_0 - \tilde{\mu}\} \end{aligned} \tag{4}$$

Re-arranging equation (4) as a polynomial in $\tilde{x}$, we get:

$$\frac{d\tilde{x}}{dt} = -(\tilde{x} + x_0)\{\tilde{x}^2 + (2x_0 - 2)\tilde{x} + (x_0^2 - 2x_0 - \mu_0 - \tilde{\mu})\}$$
$$\frac{d\tilde{x}}{dt} = -\{\tilde{x}^3 + (3x_0 - 2)\tilde{x}^2 + (3x_0^2 - 4x_0 - \mu_0 - \tilde{\mu})\tilde{x} + (x_0^3 - 2x_0^2 - x_0\mu_0 - x_0\tilde{\mu})\} \tag{5}$$

Using equation (1) and equation (2) in equation (4) and neglecting the small last term in the RHS, we get:

$$\frac{d\tilde{x}}{dt} = \tag{6}$$

which is a first order differential equation with constant coefficients whose solution can be expressed as:

$$\tilde{x}(t) = \tag{7}$$

Thus the solution expressed in equation (7) is an exponentially decaying stable one if x_0 is positive (in this case $x_0 =$) but an exponentially increasing unstable one if x_0 is negative (in this case $x_0 =$).