EE 507 Random Processes
Homework 01

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Problem 1

You pick two cards at random (without replacement) from a standard deck of 52 cards. Please answer the following questions:

a. What is the probability that both cards show the same value (e.g. both are 4's or both are Kings.)

b. Given that the two cards have the same color, what is the probability that both cards show the same value? Is this probability larger or smaller than the probability in part a? Conceptually, why does this make sense?

c. Are the events that the two cards have the same color and the two cards have the same value independent?

Solution

Experiment S: Two cards are drawn at random without replacement from a standard deck of 52 cards.
Let event $A = \{$"Both cards are of the same value"$\}$.
Let event $B = \{$"Both cards are of the same colour"$\}$.
So $AB = \{$"Both cards are of the same value and of the same colour"$\}$

a.

$$\begin{aligned} P(A) &= \frac{52 * 3}{52 * 51} \\ \text{or, } P(A) &= \frac{1}{17} \\ \text{or, } P(A) &\approx 0.0588 \end{aligned} \tag{1}$$

b.

$$\begin{aligned}
P(A|B) &= \frac{P(AB)}{P(B)} &\quad (2)\\
\text{But, } P(B) &= \frac{52*25}{52*51}\\
\text{or, } P(B) &= \frac{25}{51}\\
\text{or, } P(B) &\approx 0.4902 &\quad (3)\\
\text{or, } P(AB) &= \frac{52*1}{52*51}\\
\text{or, } P(AB) &= \frac{1}{51}\\
\text{or, } P(AB) &\approx 0.0196 &\quad (4)
\end{aligned}$$

Thus, Using eqs. (1) to (4), we get:

$$\begin{aligned}
P(A|B) &= \frac{1}{25}\\
\text{or, } P(A|B) &= 0.040 &\quad (5)
\end{aligned}$$

This probability is smaller (approximately two-thirds of) than the probability computed in part a). Conceptually, this makes sense as the be restricting the outcomes to only those cards with the same colour, the total number of outcomes were reduced by around half ($52*25$ from $52*51$) but the number of successful outcomes reduced to a third of the original ($52*1$ from $52*3$).

c. In part b), it was seen that the knowledge of event B happening altered our knowledge (decreased the probability of happening) of event A (eq. (5) compared to eq. (1)). This violates the condition of independence of events A and B as $P(A|B) \neq P(A)$.

Problem 2

a. Please define the following terms:

   (i) Probability
   (ii) Outcome
   (iii) Event
   (iv) Sample Space
   (v) Axiom

b. Please describe, in words, the three axioms of probability.

c. It's important in defining an experiment to have the correct level of granularity (detail). Please give an example in which the granularity is too low, and one in which the granularity is too high.

Solution

a.
   - **Outcome**
     A possible result of an experiment. Only one outcome is reached at the end of an experiment, i.e. Outcomes are always disjoint and never overlap.
   - **Sample Space**
     The set of all possible outcomes of an experiment is called the sample space.
   - **Event**
     For an experiment, an event is uniquely defined via the set of permissible outcomes as part of itself. In set theory, an event E is the subset of the power set $\wp$ of the sample space Ω of an experiment.
   - **Probability**
     The probability of an event is defined as the ratio of the odds of that event occurring to the odds of all possible events occurring at the end of an experiment.
   - **Axiom**
     A basic rule which cannot be proven but only assumed. Axioms when combined with other axioms, together can be used as a standalone set of rules to prove all subsequent results forming the base of a mathematical area.

b. The three axioms of probability are:

   Axiom 1. *For an event A as part of an experiment, the probability* $P(A) \geq 0$

Probability of an event can never be less than zero. At the least the event can be an impossible event, whose probability would then be zero.

Axiom 2. *For the sample space Ω of an experiment, the probability $P(\Omega) = 1$.*

The set of all possible outcomes of an experiment is called the sample space. Since the sample space encompasses every possibility of an experiment, the probability of getting an outcome which is part of the sample space is 1, i.e. it is a certain event.

Axiom 3. *If events A and B are disjoint, i.e. $A \cap B = \phi$, then $P(A + B) = P(A) + P(B)$.*

If two events do not have any overlapping outcomes, then they are mutually exclusive events and thus their union is simply the sum of their individual possible outcomes.

c. Let's say the experiment is to determine whether a particular candidate is able to pass the course EE 507. Let's label this experiment E and its outcomes as 'Pass' and 'Fail'. We wish to determine P('Pass'):

- **Too Granular**: All of the candidate's grades for the previous six years of their university education. The candidate's relationship with their classmates and the instructor. The total time the candidate allotted to study for their real-time exams, the number of pages the candidate practised before the examination day. The breakfast the candidate had on the examination day and the duration of sound sleep the night before.
- **Good Granularity**: The candidate has taken a similar course in their previous degree and performed well. The candidate likes studying hard maths and spends a decent amount of their time in solving the given assignments.
- **Low Granularity**: The candidate comes from country XYZ, the candidate topped their English exams in Class 5. The candidate likes the number 507.

Problem 3

Consider any uncertain experiment, and let A, B, and C be three events defined for the experiment. Please prove the following equalities or inequalities. In doing so, you may use the three axioms, any equalities/inequalities developed in class, standard set theory and algebra concepts, and earlier parts of the problem.

a. $P(A\bar{B}) = P(A) - P(AB)$

b. If A is contained in B, then $P(A) \leq P(B)$.

c. $P(A+B) + P(\bar{A}) + P(\bar{B}) - P(\bar{A}+\bar{B}) = 1$

d. $P(AB) + P(\bar{A}C) + P(\bar{B}\bar{C}) \leq 1$

   Please also give an example showing that the bound can be achieved.

Solution

Problem 4

Consider an experiment that has four outcomes, A, B, C, and $\square$.

a. How many events can be defined for this experiment? What are they?

b. What is the sample space for the experiment?

c. For this part, assume that $P(\{A, B, \square\}) = 0.8$, $P(\{A, C\}) = 0.4$ and $P(\{A, \square\}) = 0.4$.
   Please find the probabilities of all events defined for this experiment.

d. For this part, assume that the probabilities $P(\{A, B, \square\})$ and $P(\{A, C\})$ are known, and that we can measure the probability of exactly one more event. Please find all other events such that the knowledge of the three events' probabilities allows us to compute the probabilities of all events.

Solution

Let the event be called E_4.
Given $\Omega = \{A, B, C, \square\}$

a. # Events = $2^4 = 16$. They are the elements of the powerset $\wp$ made from Ω

$$\begin{aligned}\wp = \{&\{\phi\}, \{A\} \cdots \{\square\},\\ &\{A, B\}, \cdots \{C, \square\},\\ &\{A, B, C\}, \cdots \{B, C, \square\},\\ &\{A, B, C, \square\}\}\end{aligned}$$

   Refer to table 1 for the full list.

b. Sample Space $\Omega = \{A, B, C, \square\}$

c. Since A, B, C, and $\square$ are all outcomes, they are disjoint events. Axiom 3 may be used to compute the probabilities of their unions.

$$\begin{aligned} P(\{A, B, \square\}) &= 0.8 \\ \implies P(C) &= 0.2 && \text{(Axiom 2)} \\ P(\{A, C\}) &= 0.4 \\ \implies P(A) &= 0.2 \\ P(\{A, \square\}) &= 0.4 \\ \implies P(\square) &= 0.2 \\ \implies P(B) &= 0.4 \end{aligned}$$

   Refer to table 1 for the event-wise probabilities.

Table 1: Probabilities of all events of Experiment E_4. The 'Sufficient?' column is meant for part d. of the problem where knowledge of the event's probability value when combined with two other given probability values is sufficient for determining the probabilities of all possible events of E_4.

$S.No.$	$Event$	$Probability$	Sufficient?
1.	ϕ	0	No
2.	A	0.2	No
3.	B	0.4	Yes
4.	C	0.2	No
5.	$\square$	0.2	Yes
6.	$\{A, B\}$	0.6	Yes
7.	$\{A, C\}$	0.4	No
8.	$\{A, \square\}$	0.4	Yes
9.	$\{B, C\}$	0.6	Yes
10.	$\{B, \square\}$	0.6	No
11.	$\{C, \square\}$	0.4	Yes
12.	$\{A, B, C\}$	0.8	Yes
13.	$\{A, B, \square\}$	0.8	No
14.	$\{A, C, \square\}$	0.6	Yes
15.	$\{B, C, \square\}$	0.8	No
16.	$\{A, B, C, \square\}$	1.0	No

d. On similar lines as the previous part c. of this problem, since we are able to find out the values of $P(A)$ and $P(C)$ from the given values, we are only interested in knowing the values of either $P(B)$ or $P(\square)$. Only events which are supersets of unions of both B and $\square$ outcomes are unhelpful in determining their individual values. Refer to the 'Sufficient?' column in table 1.

Problem 5

You toss three fair coins. What is the probability that at least two show heads? Also, what is the probability that at least two show heads, given that the number of heads showing is even?

Solution

Let E be the experiment of tossing three fair coins. The sample space for E is
Ω = {TTT, TTH, THT, THH, HTT, HTH, HHT, HHH}.
Let Event A: {At least two heads}
or $A = \{THH, HTH, HHT, HHH\}$

$\implies P(A) = \frac{4}{8}$

Let Event B: {Even # of heads}
or $B = \{TTT, THH, HTH, HHT\}$

$\implies P(B) = \frac{4}{8}$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$
$A \cap B = \{THH, HTH, HHT\}$

$P(A|B) = \frac{\frac{3}{8}}{\frac{4}{8}} = \frac{3}{4}$

Problem 6

Solution

Problem 7

Let's say an experiment has three outcomes.

a. How many events can be defined for this experiment?

Now we say we repeat the experiment twice (once more), independently.

b. How many outcomes does the combined experiment have?

c. How may events can be defined for the combined experiment?

d. How many events for the combined experiment can not be derived from the cartesian products for the individual experiments? Please give an example.

Solution

Let the experiment be called E_1 which has the three outcomes be A, B and C i.e. Ω = {A, B, C}

a. # Events = 2^n where n is the cardinality of the sample space Ω.
Here $n = 3$ and therefore # Events = $2^3 = 8$.

Let the combined experiment be called E_2. The outcomes of this combined experiment will be represented by the new sample space Ω_2 = {AA, AB, AC, BA, BB, BC, CA, CB, CC}.

b. # Outcomes = Cardinality ($n_2 = n^2$) of the sample space Ω_2.
Here $n_2 = 3^2 = 9$.

c. # Events = 2^{n_2}
Here $n_2 = 9$ and therefore # Events = $2^9 = 512$.

d. # Events of the combined experiment which can be defined from the cartesian products of the individual experiments = $2^n * 2^n$ = 2^{2n} = 2^6.
Total # Events of the combined experiment = 2^{n^2} = 2^9.
Therefore, # Events which cannot be defined from the cartesian products of the individual experiments = $2^{n^2} - 2^{2n}$ = $2^9 - 2^6$ = $512 - 64$ = 448.

Examples for an event of E_2 which can be formed from the cartesian products of the events of E_1:

$$\{A\} \times \{A\} = \{AA\}$$
$$\{A, B, C\} \times \{A\} = \{AA, BA, CA\}$$
$$\{A, C\} \times \{B\} = \{AB, CB\}$$

Examples for events of E_2 which can NOT be formed from the cartesian products of the events of E_1:

$$\{AB, BC, CA\}$$
$$\{AA, CB\}$$
$$\{BC, CB\}$$
