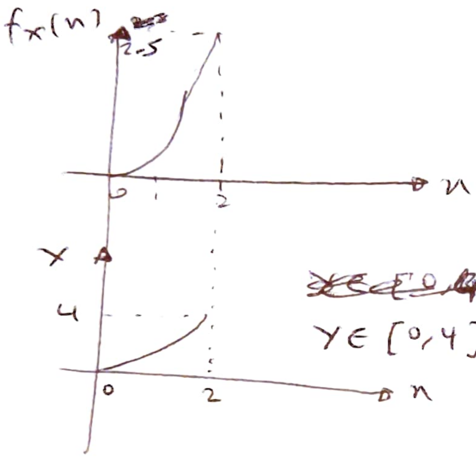
$\boxed{1.}$

$$f_x(n) = \begin{cases} \frac{5}{32} n^4 & n \in (0,2] \\ 0 & \text{else.} \end{cases}$$

$Y = X^2$



(i) Using CDF formula

$$F_x(n=\alpha) = \begin{cases} 0 & \alpha \le 0 \\ \frac{1}{32} \alpha^5 & \alpha \in (0,2] \\ 1 & \alpha > 2 \end{cases}$$

~~$X \in [0,4]$~~

$Y \in [0,4)$

~~Example P(~~

$F_Y(y=\alpha) = P(y \le \alpha)$

or $F_Y(y=\alpha) = P(n^2 \le \alpha)$

or $F_Y(y=\alpha) = P(n \le \sqrt{\alpha})$

or $$F_Y(y=\alpha) = F_x(\sqrt{\alpha}) = \begin{cases} 0 & \sqrt{\alpha} \le 0 \\ \frac{1}{32}(\sqrt{\alpha})^5 & \sqrt{\alpha} \in (0,2] \\ 1 & \sqrt{\alpha} > 2 \end{cases}$$

$\boxed{1a.}$

or $$\boxed{F_Y(y=\alpha) = \begin{cases} 0 & \alpha \le 0 \\ \frac{1}{32} \alpha^{\frac{5}{2}} & \alpha \in (0,4] \\ 1 & \alpha > 4 \end{cases}}$$

Ans

1 b.

$$f_Y(y) = \frac{d}{dy}\left(F_Y(y=\alpha)\right)\Big|_{y=\alpha} = \begin{cases} 0 & y<0 \\ \frac{5}{64}y^{\frac{3}{2}} & y\in[0,4] \\ 0 & y>4 \end{cases}$$

Ans

(ii) Using Transformative ("cute") formula:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(g^{-1}(y))$$

or $f_Y(y) = \frac{d}{dx}\left(F_X(g^{-1}(y))\right) \cdot \left|\frac{dx}{dy}\right|$

Here since $\cancel{g(x)} =$ $y = g(x) = x^2$,

$g^{-1}(y) = \sqrt{y}$.

$\cancel{\frac{dy}{dx} = \frac{d}{dx}(\sqrt{y}) =}$

$$\frac{dx}{dy} = \frac{d}{dy}(\sqrt{y}) = \frac{1}{2\sqrt{y}}$$

or $f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{1}{2\sqrt{y}}\right|$

or $$f_Y(y) = \begin{cases} 0 & \sqrt{y} \le 0 \\ \frac{5}{32}(\sqrt{y})^4 \cdot \left|\frac{1}{2\sqrt{y}}\right| & \sqrt{y}\in(0,2] \\ 0 & \sqrt{y}>2 \end{cases}$$

1 b (again!)

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} 0 & y \le 0 \\ \frac{5}{64} y^{\frac{3}{2}} & y \in (0, 4] \\ 0 & y > 4 \end{cases}}$$ Ans (Again).

(i) Using $f_Y(y)$

$$E[Y] = \int_{y=0}^{y=4} y \, f_Y(y) \, dy$$

$$\Rightarrow E(Y) = \int_{y=0}^{4} y \times \frac{5}{64} y^{\frac{3}{2}} \, dy$$

$$\Rightarrow E(Y) = \frac{5}{64} \cdot y^{\frac{7}{2}} \cdot \frac{2}{7} \Big|_{y=0}^{y=4}$$

$$\Rightarrow E(Y) = \frac{10}{64 \times 7} \cdot 4^{\frac{7}{2}}$$

1c

$$\Rightarrow \boxed{E(Y) = \frac{20}{7}}$$ Ans

(ii) Using Law of The Unconscious Statistician (LOTUS):

$$E(Y) = E[X^2] = \int_{x=0}^{x=2} x^2 \cdot f_X(x) \, dx$$

$$\Rightarrow E(Y) = \int_{x=0}^{x=2} x^2 \cdot \frac{5}{32} x^4 \, dx$$

$$\Rightarrow E(Y) = \frac{5}{32} \cdot \frac{x^7}{7} \Big|_{x=0}^{x=2}$$

1c (again!)

$$\Rightarrow \boxed{E(Y) = \frac{20}{7}}$$ Ans (again!)

2.

Roll # 1 $\xrightarrow{\text{Roll again?}}$ Roll # 2

$X = \{1, 2, 3, 4, 5, 6\}$ $\quad$ $Y = \{1, 2, 3, 4, 5, 6\}$

Let A be the event that you cash in on the first die throw.

~~E[W] = F~~

~~$E(W) = A E(X) + \bar{A} E(Y)$~~

~~$W = X \cdot F \cdot X$~~

then $\boxed{W = AX + \bar{A}Y}$ (1)

~~X = Cashing in on the first throw~~.

In order to Maximize W, we need to optimize the event (algorithm) A.

Intuitively, we would want to throw again if we get 'lower' values of X and hold back from throwing again if we get a high value of Y.

1 For quantifying A, we can utilize a variable

2 g (for good throw) which stands for

3 the ~~top~~ ~~base~~ number of allowed 'high-valued'

4 throws allowed.

5

6 $\}$ g

Here, g=2, i.e. hold back upon getting the highest 2 values of X: 5 or 6.

In such a case, $E(X|g=2) = \frac{5+6}{2} = 5.5$

TABLE 1: $E(X|g)$ vs g.

g	$X\|g$	$E(X\|g)$
1	{6}	6
2	{6,5}	5.5
3	{6, 5, 4}	5
4	{6,5,4,3}	4.5
5	{6,5,4,3,2}	4
6	{6,5,4,3,2,1}	3.5

From the Table 1, we can devise $E(X|g)$ as a simple linear function of g:

$$\boxed{E(X|g) = 6.5 - \frac{g}{2}} \quad (2)$$

Taking Expectation on both sides of (1):

$$E(W) = E(AX) + E(\bar{A}Y)$$

and using g to quantify event A:

or get

$$\boxed{E(W) = E(X|g) . P(g) + E(Y|\bar{g}) . P(\bar{g})} \quad (3)$$

where $X|g$ = Value of X given that only the highest g values of X are allowed for holding back from throwing again.

$P(g)$ = Probability of obtaining any of the highest g values on the first die throw

$X|\bar{g}$ = given that none of the highest g values were landed on the first throw, the value landed on the second throw.

It may be noted that ~~g ~~ ~~that~~ $E(Y)$ is not dependent on the value of g or $\bar{g}$. So $E(Y|\bar{g}) = E(Y)$.

So $E(W) = \left(6.5 - \frac{1}{2}g\right)\left(\frac{g}{6}\right) + \left(\frac{1+2+3+4+5+6}{6}\right)\left(1 - \frac{g}{6}\right)$

or $\boxed{E(W) = -\frac{g^2}{12} + \frac{g}{2} + 3.5}$ (4)

$\arg\left(\text{Max } E(W)\right) = \arg\left(\frac{d}{dg}E(W) = 0\right)$

or $g_{Max} = \arg\left(-\frac{g}{6} + \frac{1}{2} = 0\right)$

or $\boxed{g_{Max} = 3}$ (5)

~~which gives~~

Putting g_{Max} into (4):

$\text{Max. } E(W) = \frac{-3^2}{12} + \frac{3}{2} + 3.5$

or $\boxed{\text{Max. } E(W) = 4.25}$ Ans.

∴ An expected \$4.25 can be obtained by throwing again only if the first throw lands 1, 2 or 3.

Ans

———— X ———— X ———— X ————

3.1

3.

outcomes	i	$p(i)$	X_i winnings #
T	1	$(1/2)$	2^0
HT	2	$(1/2)^2$	2^1
HHT	3	$(1/2)^3$	2^2
⋮		⋮	⋮
$\underbrace{H \dots H}_{(n-1)} T$	n	$(1/2)^n$	2^{n-1}

$$E(X) = \sum_{i=1}^{\infty} X_i \, p(i)$$

$$\text{or } E(X) = \sum_{i=1}^{\infty} 2^{i-1} \cdot \left(\frac{1}{2}\right)^i$$

$$\text{or } E(X) = \sum_{i=1}^{\infty} 2^{-1}$$

3(a)

$$\text{or } \boxed{E(X_i) \to \infty}$$

$$P(X > 65) = P(K > \log_2(65) + 1) \quad \text{as } X = 2^{K-1}.$$

$$\text{or } P(X>65) = P(K > 7)$$

$$\text{or } P(X>65) = 1 - P(K \leq 7)$$

$$\text{or } P(X>65) = 1 - \sum_{k=1}^{k=7} \left(\frac{1}{2}\right)^k$$

$$\text{or } P(X>65) = 1 - \frac{\left(\frac{1}{2}\right)^1 \cdot \left(1 - \left(\frac{1}{2}\right)^7\right)}{1 - \frac{1}{2}}$$

$$\text{or } P(X>65) = 1 - \left(1 - \left(\frac{1}{2}\right)^7\right)$$

3(b)

$$\text{or } \boxed{P(X>65) = \left(\frac{1}{2}\right)^7} \quad \underline{\text{Ans}}$$

~~E(X)=~~

If $X_{Max} = 2^{30}$, then the player should not bet after $k = 31$ trials.

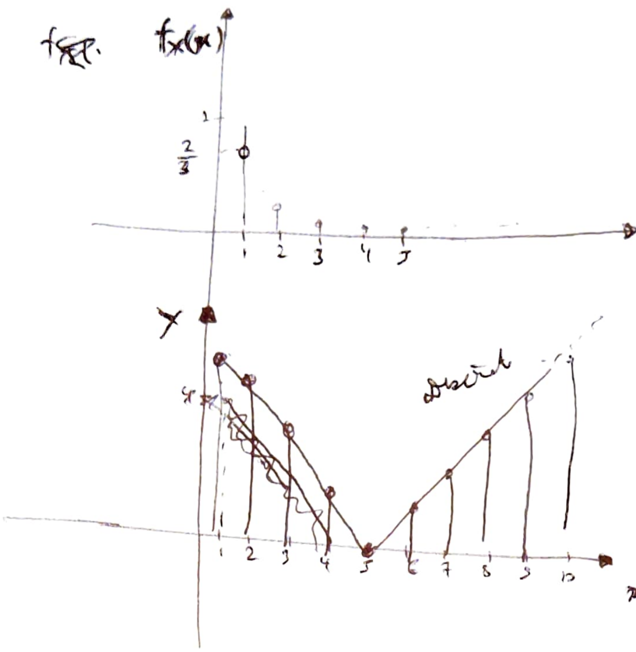
$$E(Y) = \sum_{k=1}^{30} 2^{k-1} \cdot \left(\frac{1}{2}\right)^k + \$ 2^k \cdot \left(\frac{1}{2}\right)^k \cdot 1$$

we win irrespective of the outcome on the 31st trial.
If its a ~~T~~ H, ~~then~~ the game ~~ends~~ ~~and~~ ends and I take home $\$2^{30}$.
If not, I would still have won the $\$2^{30}$ and forfeit the game. after the 31st trial.

or $\boxed{E(Y) = \left(\frac{1}{2}\right) \times 30 = \cancel{15} \quad +1 = 16}$

$\boxed{4.}$ $X \sim$ geometric$\left(\frac{1}{3}\right)$ $\Rightarrow f_X(x) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{x-1}$ $x \geq 1, 2, 3, \ldots$

$Y = |X-5|$



$X \in \{1, 2, 3, 4 \ldots \infty)\}$

~~$Y \in [0, \infty)$~~

$Y \in \{0, 1, 2, \ldots \infty)\}$.

$\boxed{4.}$ Ans

Y_i	X_{i_y}	pmf(Y_i)
0	5	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4$
1	4,6	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^5$
2	3,7	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^6$
3	2,8	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^1 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^7$
4	1,9	$\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8$
5	10	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^9$
$\vdots$		
N	N+5	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{N+4}$

~~$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \frac{2}{3} & x = 1 \\ \frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3} & x = 2 \\ \sum_{i=1}^{n} \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{i-1} & x = n \end{cases} = \frac{\frac{2}{3}\cdot\left(1 - \left(\frac{1}{3}\right)^n\right)}{1 - \frac{1}{3}} = \frac{2}{3}\, 1 - \left(\frac{1}{3}\right)^n$~~

~~$F_Y(y) = P(|x-5| \leq \alpha)$~~

~~$F_Y(y) = P(x \leq \alpha + 5) + P(5 - x < \alpha)$~~

~~$F_Y(y) = P(x \leq \alpha + 5) + P(x > 5 - \alpha)$~~

~~$F_Y(y) = F_X(\alpha + 5) + (1 - F_X(5 - \alpha))$~~ $x \geq 5$ else 0, $x = 1, 2, 3, 4$ else 0.

$F_X(y) = \begin{cases} & x = 1, 2, 3, 4 \end{cases}$

5.

5.

$$p_X(k) = \begin{cases} 0.5 & k=1 \\ 0.3 & k=2 \\ 0.2 & k=3 \\ 0 & \text{otherwise} \end{cases}$$

5(a)

$E(X) = 0.5 + 0.3\times 2 + 0.2\times 3 = 1.7$ Ans

$E(X^2) = 0.5 + 0.3\times 4 + 0.2\times 9 = 3.5$

$V(X) = E(X^2) - [E(X)]^2 = 3.5 - 1.7^2 =$

5(b)(i)

$V(X) = 0.61$ Ans

5b(ii)

$\sigma(X) = \sqrt{0.61}$ Ans

$Y = \frac{2}{X}$ $\quad E(Y) = 0.5\times\frac{2}{1} + 0.3\times\frac{2}{2} + 0.2\times\frac{2}{3}$

$E(Y) = 1 + 0.3 + \frac{0.4}{3}$

5(c)

$E(Y) = 1.4333$ Ans

6.

Exp — H — 20 MORE flips Event #1 = H

Exp — T — 10 MORE flips Event #1 = T

X = # heads in ALL flips (including first one).

(a) $P(X=5) = P(X=5 \mid \#1=H) \cdot P(\#1=H) + P(X=5 \mid \#1=T) \cdot P(\#1=T)$

6 (a)

$$\boxed{\begin{aligned} P(X=5) &= {}^{20}C_4 \cdot \left(\tfrac{1}{2}\right)^{20} \cdot \left(\tfrac{1}{2}\right) + {}^{10}C_5 \left(\tfrac{1}{2}\right)^{10} \cdot \left(\tfrac{1}{2}\right) \\ P(X=5) &= 0.1254 \end{aligned}}$$ Ans

(b) $P(\#1=H \mid X=5) = \dfrac{P(X=5 \mid \#1=H) \cdot P(\#1=H)}{P(X=5)}$

$P(\#1=H \mid X=5) = \dfrac{\left\{ {}^{20}C_4 \cdot \left(\frac{1}{2}\right)^{20} \right\} \cdot \left(\frac{1}{2}\right)}{0.1254}$

6 (b)

$\boxed{P(\#1=H \mid X=5) = 0.0184}$ Ans

(c) $P(\#\text{Last}=H \mid X=5) = \dfrac{P(X=5 \mid \#\text{Last}=H) \cdot P(\#\text{Last}=H)}{P(X=5)}$

$P(\#\text{Last}=H \mid X=5) = \dfrac{\left[P(X=5 \mid \#\text{Last}=H, \#1=H) \cdot P(\#1=H) + P(X=5 \mid \#\text{Last}=H, \#1=T) \cdot P(\#1=T) \right] \cdot \left(\frac{1}{2}\right)}{0.1254}$

$P(\#\text{Last}=H \mid X=5) = \dfrac{\left[{}^{19}C_3 \cdot \left(\frac{1}{2}\right)^{19} \left(\frac{1}{2}\right) + {}^{9}C_4 \left(\frac{1}{2}\right)^{9} \cdot \left(\frac{1}{2}\right) \right] \left(\frac{1}{2}\right)}{0.1254}$

6 (c)

$\boxed{P(\#\text{Last}=H \mid X=5) = 0.4945}$ Ans

1.7.

If $\boxed{P(AB) = P(A).P(B)}$ (1)

check if $\boxed{P(AB|C) = P(A|C).P(B|C)}$ (?)

1/7

FALSE Ans

Counter example: (found on stack overflow).

Exp: Flip 2 coins. ~~(as a repeated)~~

Event A: Coin 1 is H

Event B: Coin 2 is H

Event C: Both coins show the same face.

$P(AB) = \frac{1}{4}$ $P(A).P(B) = \frac{1}{4}$ i.e. A and B are independent events

Ω_A, Ω_B:

TT	TH
HT	HH

Ω_C:

TT	HH

$P(A|C) = \frac{1}{2}$ $P(B|C) = \frac{1}{2}$

But $P(AB|C) = \frac{1}{2} \neq P(A|C).P(B|C)$.

$\boxed{1|8}$ given $\boxed{P(A+B) = P(A) + P(B)}$ (1) $\Rightarrow \boxed{P(AB) = 0 \;\; A \cap B = \phi}$ (1 a).

$\cancel{\Rightarrow P(AB}$

Check if $\boxed{P(A+B \mid C) = P(A \mid C) + P(B \mid C)}$ (?).

~~C is a subset of~~

$\boxed{C \subseteq \Omega}$

if $A \cap B$ in Ω ~~...~~ does not exist

then ~~it~~ $A \cap B$ does not exist in any subset

of Ω, such as C, too.

So $P(AB \mid C) = 0$ too, like $P(AB) = 0$ as ~~(already seen in)~~ (as seen in (1a))

So $P(A+B \mid C) = P(A \mid C) + P(B \mid C) - P(AB \mid C)$

But $P(AB \mid C) = 0$

$\boxed{1|8}$

So $\boxed{P(A+B \mid C) = P(A \mid C) + P(B \mid C)}$ Ans

Hence Proved.

The statement is TRUE!

02/1

X(W)	outcomes included (W)	~~P(X(W))~~ P(X = X(W))	~~CDF~~ $F_X(x)$
1	□	0.1	0.1
2	C	0.1	0.2
4	D	0.1	0.3
7	A,B	0.7	1.0

~~R = {B, C, D}~~

Event	outcomes included	P(event)
Q	{A, B}	0.7
R	{B, C, D}	0.5
S	{D, □}	0.2
T	{D}	0.1

02/1 (a). Check if $P(RS) = P(R).P(S)$

$RS = \{D\}$ $P(RS) = 0.1$.

$P(R).P(S) = 0.5 \times 0.2 = 0.1$

02/1(a)

$\Rightarrow$ $\boxed{\text{R and S are independent}}$ Ans

(b). Given $Z = Q + RS$, find ~~cd~~ ~~Fx(x)~~ CDF, PMF and PDF of x.

$Z = Q + RS$

$\therefore Z = \{A, B\} + \{B, C, D\} \& \{D, □\}$

$\therefore Z = \{A, B\} + \{D\}$

$\therefore \boxed{Z = \{A, B, D\}}$ $\Rightarrow$ $\boxed{P(Z) = 0.8}$

02/1.2

~~P(X|Z) = P(Z|X)~~.

given Z, we can generate the table for P(X), CDF(X).

$$P(X|Z) = \frac{P(XZ)}{P(Z)}$$

02/1(b)

X(w)	P(X(w)|Z) = PMF(X(w)|Z)	CDF(X(w)|Z)
4	0.1/0.8 = 1/8	1/8
7	0.7/0.8 = 7/8	1.0

Ans

and $\text{pmf}(X|Z) = \frac{1}{8}\delta(x-4) + \frac{7}{8}\delta(x-7)$.

(c) $E(X) = 1\times 0.1 + 2\times 0.1 + 4\times 0.1 + 7\times 0.7 = 5.6$

02/1(c)(i)

$\boxed{E(X) = 5.6}$ Ans

$E(X|Z) = 4\times\frac{1}{8} + 7\times\frac{7}{8} = \frac{53}{8} = 6.625$

02/1(c)(ii)

$\boxed{E(X|Z) = 6.625}$ Ans

(d) by check if $(R|X\le 5)$ and $(S|X\le 5)$ are independent?

$P(R|X\le 5) = P(\{C, D\}) = 0.2$

$P(S|X\le 5) = P(\{D, \square\}) = 0.2$

$P(RS|X\le 5) = P(\{D\}) = 0.1$

R: $P(R|X\le 5).P(S|X\le 5) \neq P(RS|X\le 5)$

02/1(d)

$\therefore$ $R|X\le 5$ and $S|X\le 5$ are NOT independent.

Ans

(e) given P(outcomes) are ~~unknown~~ unknown but only P(α) and P(T) known.
check if we can find pmf(X|Z).

$P(\alpha) = P(\{A, B\}) = P(X=7)$ ✓ (known)

$P(T) = P(\{\{D\}\}) = P(X=4)$ ✓ (known).

02 | 1(e)

$\because$ $X|Z = P(\{A, B, D\}) = P(X=7) + P(X=4)$

We can still find PMF(X|Z).

Ans.

$\boxed{2|\cancel{1}2}$ given A ~~and~~ is ALMOST INDEPENDENT of B if

$$P(A|B) \in [P(A) - 0.01, P(A) + 0.01]$$

(a) if given A is 'AI' of B, or $\boxed{P(A|B) \in [P(A)-0.01, P(A)+0.01]}$ ②
check if $\bar{A}$ is also 'AI' of B.

we know that $P(A|B) + P(\bar{A}|B) = 1$

or $P(\bar{A}|B) = 1 - P(A|B)$

or $P(\bar{A}|B) = 1 - [P(A)-0.01, P(A)+0.01]$

or $P(\bar{A}|B) = [1-P(A) - 0.01, 1 - P(A) + 0.01]$

or $P(\bar{A}|B) = [P(\bar{A}) - 0.01, P(\bar{A}) + 0.01]$

$\boxed{2|2(a)}$ ~~P(A~~

$\Rightarrow$ $\boxed{\bar{A} \text{ is 'AI' of } B}$

$\boxed{\text{if } A \text{ is 'AI' of } B}$ Ans

(b) given A is 'AI' of B ~~i.e.~~ ~~P(A|B) ∈~~ [~~P(A)~~ - 0.01, ~~P(A)...~~

i.e. $\boxed{P(A|B) \in [P(A)-0.01, P(A)+0.01]}$ ①

check if B is also 'AI' of A

i.e. $\boxed{P(B|A) \in [P(B)-0.01, P(B)+0.01]}$ ②

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

or $$P(B|A) = \frac{[P(A) - 0.01, P(A) + 0.01]\, P(B)}{P(A)}$$

or $P(B|A) = \left[P(B) - \underbrace{0.01\frac{P(B)}{P(A)}}, P(B) + 0.01\frac{P(B)}{P(A)} \right]$

For B to be 'AI' of A too,

we require that $\frac{P(B)}{P(A)} \leq 1$

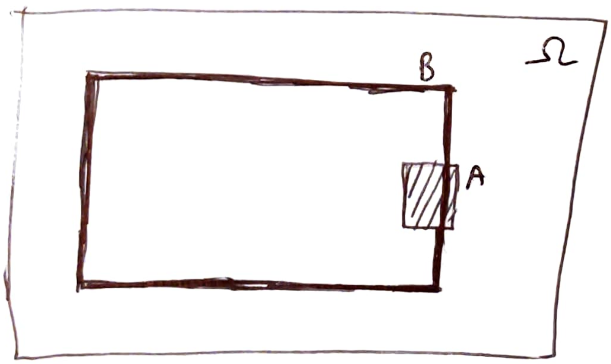
which is NOT a ~~general~~ e ~~true~~ a correct ~~assumption~~ inequality in general.

2|~~2~~ (b) $\therefore$ The the statement is False;

i.e. ~~A~~ given A is 'AI' of B $\not\Rightarrow$ B is 'AI' of A

Ans

Counterexample



Here B □

A ▨

$P(B) = 0.75$

Let's take ~~P~~ ≈ ~~a~~ ~~P(B)~~

~~P(A)~~

~~Say P(A) = P(A)~~

Assume that 1% of A is outside B.

In that case P(A)

By construction/hypothesis
$$\begin{cases} P(B) = 0.75 \\ P(A) = \text{some smaller value} \\ P(A\bar{B}) = 0.01 P(A) \end{cases}$$

$P(A|B) =$

$P(A|B) = \frac{P(AB)}{P(B)}$

$\therefore P(A|B) = \frac{0.99 P(A)}{P(B)}$

$\therefore P(A|B) = \frac{0.99 P(A)}{0.75}$

$P(B)$

$P(B|A) = \frac{P(AB)}{P(A)} = \frac{0.99 P(A)}{P(A)}$

$\therefore P(B|A) = 0.99$

$\notin [P(B) - 0.01, P(B) + 0.01]$

02/10

02/10

given A, B, C are pairwise independent, check if

PT $\boxed{P(ABC) = P(A).P(B).P(C)}$?

~~Starting with the LHS of ? :~~

~~$P(ABC) = P(AB|C).P(C)$~~

~~or $P(ABC) =$~~

False!

Example obtained from Stack Overflow. ☺

Let an Exp. S = Toss two coins.

Let Event A : Heads on first coin

B : Heads on second coin.

C : Same face on both coins.

$\boxed{\Omega = \cancel{TH}\ TT\ \ TH \\ HT\ \ HH}$

$P(\underset{\{HT, HH\}}{A}) = \frac{1}{2}$ $\quad P(\underset{\{TH, HH\}}{B}) = \frac{1}{2}$ $\quad P(\underset{\{TT, HH\}}{C}) = \frac{1}{2}$

$P(A\underset{\{HH\}}{B}) = \frac{1}{4} = P(A).P(B)$

$P(B\underset{\{HH\}}{C}) = \frac{1}{4} = P(B).P(C).$ $\quad$ ⇒ Pairwise independent.

$P(C\underset{\{HH\}}{A}) = \frac{1}{4} = P(C).P(A)$

But $P(A\underset{\{HH\}}{B}C) = \frac{1}{4} \neq P(A).P(B).P(C) = \frac{1}{8}$

02/10 (a)

Pairwise independence of A, B, C does NOT ⇒ imply $P(ABC) = P(A).P(B).P(C)$. Ans

02/10.2

(b) Check if $P(A|B) + P(A|\bar{B}) \leq 1$ (?)

We know that:

$$P(A|B) \cdot P(B) + P(A|\bar{B}) \cdot P(\bar{B}) = P(A)$$

$$\frac{P(AB)}{P(B)} + \frac{P(A\bar{B})}{P(\bar{B})}$$

$$\frac{P(AB)}{P(B)} + \frac{P(A) - P(AB)}{1 - P(B)}$$

Let A and B be independent..

$P(A) + P(A) = 2P(A)$

let $P(A) = 2/3$

$P(B) = 1/2$

$P(AB) = 1/3$

$\frac{0.3}{0.5} + \frac{0.3}{0.5} = 1.2$ $\quad P(A\bar{B}) = 1/2$

$P(A|B) = 2/3 \quad P(A|\bar{B}) = 2/3$

$\frac{A \cap A}{A \cap (B)} + \frac{A \cap (A - A \cap (B))}{1 - A \cap (B)}$

(?) is FALSE!

Imagine an experiment S: Throwing a die (fair, 6-sided).

Event A: $\# < 5$ i.e. $A = \{1, 2, 3, 4\}$

Event B: # is Even i.e. $B = \{2, 4, 6\}$

$\Rightarrow \bar{B} = \{1, 3, 5\}$

$AB = \{2, 4\}$ $\quad A\bar{B} = \{1, 3\}$

$P(A|B) + P(A|\bar{B}) = \frac{P(AB)}{P(B)} + \frac{P(A\bar{B})}{P(\bar{B})}$

$\Rightarrow P(A|B) + P(A|\bar{B}) = \frac{2/6}{3/6} + \frac{2/6}{3/6} = \frac{4}{3} > 1$

02/10(b)

$\therefore$ $\boxed{P(A|B) + P(A|\bar{B}) \leq 1 \text{ is FALSE!}}$ Ans

$\boxed{02/10.3}$

(c) check if $\boxed{P(A+B+C) = P(A) + P(\bar{A}B) + P(\bar{A}\bar{B}C)}$ (?)

~~Let's start with~~

Let's start with the event of the LHS of (?), Let's call it E:

~~As~~ $E = A+B+C$

or $E = A + (B+C)$

or $E = A + \bar{A}(B+C)$

or $E = A + \bar{A}(B + \bar{B}C)$

or $E = A + \bar{A}B + \bar{A}\bar{B}C$

or $P(E) = P(A) + P(\bar{A}B) + P(\bar{A}\bar{B}C)$ (Ans. 3)

$\boxed{02/10(c)}$

or $\boxed{P(A+B+C) = P(A) + P(\bar{A}B) + P(\bar{A}\bar{B}C)}$ is True! Ans

Hence Proved! ☺
