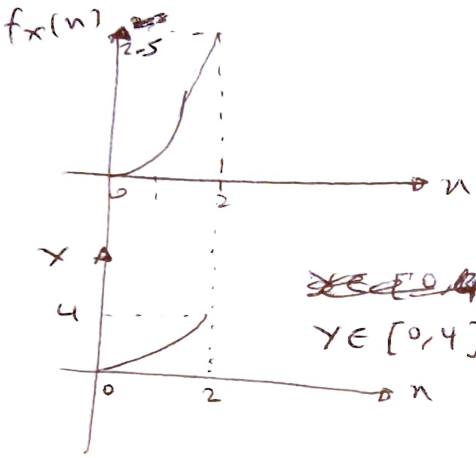
$\boxed{1.}$

$$f_X(x) = \begin{cases} \frac{5}{32} x^4 & x \in (0,2] \\ 0 & \text{else.} \end{cases}$$

$Y = X^2$

(i) <u>Using CDF formula</u>



$$F_X(x=\alpha) = \begin{cases} 0 & \alpha \le 0 \\ \frac{1}{32}\alpha^5 & \alpha \in (0,2] \\ 1 & \alpha > 2 \end{cases}$$

$Y \in [0,4)$

$F_Y(y=\alpha) = P(y \le \alpha)$

or $F_Y(y=\alpha) = P(x^2 \le \alpha)$

or $F_Y(y=\alpha) = P(x \le \sqrt{\alpha})$

$$\text{or } F_Y(y=\alpha) = F_X(\sqrt{\alpha}) = \begin{cases} 0 & \sqrt{\alpha} \le 0 \\ \frac{1}{32}(\sqrt{\alpha})^5 & \sqrt{\alpha} \in (0,2] \\ 1 & \sqrt{\alpha} > 2 \end{cases}$$

$\boxed{1a.}$

$$\boxed{\text{or } F_Y(y=\alpha) = \begin{cases} 0 & \alpha \le 0 \\ \frac{1}{32}\alpha^{\frac{5}{2}} & \alpha \in (0,4] \\ 1 & \alpha > 4 \end{cases}}$$

Ans

1 b.

$$f_Y(y) = \frac{d}{dy}\left(F_Y(y=\alpha)\right)\Big|_{y=\alpha} = \begin{cases} 0 & y < 0 \\ \frac{5}{64} y^{\frac{3}{2}} & y \in [0, 4] \\ 0 & y > 4 \end{cases}$$

Ans

(ii) Using Transformative ("cute") formula:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

or $f_Y(y) = \frac{d}{dx}\left(F_X(g^{-1}(y))\right) \cdot \left|\frac{dx}{dy}\right|$

Here since $y = g(x) = x^2$,

$g^{-1}(y) = \sqrt{y}$.

$\cancel{\frac{dy}{dx} = \frac{d}{dx}(\sqrt{y}) =}$

$$\frac{dx}{dy} = \frac{d}{dy}(\sqrt{y}) = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{1}{2\sqrt{y}}\right|$$

$$f_Y(y) = \begin{cases} 0 & \sqrt{y} \le 0 \\ \frac{5}{32}(\sqrt{y})^4 \cdot \left|\frac{1}{2\sqrt{y}}\right| & \sqrt{y} \in (0, 2] \\ 0 & \sqrt{y} > 2 \end{cases}$$

1 b (again!)

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} 0 & y \le 0 \\ \frac{5}{64} y^{\frac{3}{2}} & y \in (0, 4] \\ 0 & y > 4 \end{cases}}$$ Ans (Again).

(i) Using $f_Y(y)$

$$E[Y] = \int_{y=0}^{y=4} y\, f_Y(y)\, dy$$

$$\Rightarrow E(Y) = \int_{y=0}^{4} y \times \frac{5}{64} y^{\frac{3}{2}}\, dy$$

$$\Rightarrow E(Y) = \frac{5}{64} \cdot y^{\frac{7}{2}} \cdot \frac{2}{7} \Big|_{y=0}^{y=4}$$

$$\Rightarrow E[Y] = \frac{10}{64 \times 7} \cdot 4^{\frac{7}{2}}$$

1c

$$\Rightarrow \boxed{E[Y] = \frac{20}{7}}$$ Ans

(ii) Using Law of The Unconscious Statistician (LOTUS):

$$E[Y] = E[X^2] = \int_{x=0}^{x=2} x^2 \cdot f_X(x)\, dx$$

$$\Rightarrow E[Y] = \int_{x=0}^{x=2} x^2 \cdot \frac{5}{32} x^4\, dx$$

$$\Rightarrow E(Y) = \frac{5}{32} \cdot \frac{x^7}{7} \Big|_{x=0}^{x=2}$$

1c (again!)

$$\Rightarrow \boxed{E(Y) = \frac{20}{7}}$$ Ans (again!)

2.

Roll # 1 $\xrightarrow{\text{Roll again?}}$ Roll # 2

$X = \{1,2,3,4,5,6\}$ $Y = \{1,2,3,4,5,6\}$

Let **A** be the event that you cash in on the first die throw.

~~E(W) = F~~

~~E(W) = A E(X) + $\bar{A}$ E(Y)~~

~~W = X F $\bar{X}$~~

then $\boxed{W = AX + \bar{A}Y}$ (1)

~~X = Cashing in on the first throw.~~

In order to Maximize W, we need to optimize the event (algorithm) A.

Intuitively, we would want to throw again if we get 'lower' values of X and hold back from throwing again if we get a high value of Y.

1 For quantifying A, we can utilize a variable
2 g (for good throw) which stands for
3 the ~~top~~ ~~base~~ number of allowed 'high-valued'
4 throws allowed.
5
6 } g

Here, $g = 2$, i.e. hold back upon getting the highest 2 values of X : 5 or 6.

In such a case, $E(X \mid g=2) = \dfrac{5+6}{2} = 5.5$

TABLE 1: $E(X|g)$ vs g.

g	$X \mid g$	$E(X \mid g)$
1	{6}	6
2	{6, 5}	5.5
3	{6, 5, 4}	5
4	{6, 5, 4, 3}	4.5
5	{6, 5, 4, 3, 2}	4
6	{6, 5, 4, 3, 2, 1}	3.5

From the Table 1, we can devise $E(X|g)$ as a simple linear function of g:

$$\boxed{E(X|g) = 6.5 - \frac{g}{2}} \quad (2)$$

Taking expectation on both sides of (1):

$$E(W) = E(AX) + E(\bar{A}Y)$$

and using g to quantify event A:

~~get~~

or $$\boxed{E(W) = E(X|g) \cdot P(g) + E(Y|\bar{g}) \cdot P(\bar{g})} \quad (3)$$

where $X|g$ = Value of X given that only the highest g values of X are allowed for holding back from throwing again.

$P(g)$ = Probability of obtaining any of the highest g values on the first die throw

$X|\bar{g}$ = given that the none of the highest g values were landed on the first throw, the value landed on the second throw.

It may be noted that ~~g~~ ~~that~~ E(Y) is not dependent on the value of g or $\bar{g}$. So $E(Y|\bar{g}) = E(Y)$.

So $E(W) = \left(6.5 - \frac{1}{2}g\right)\left(\frac{g}{6}\right) + \left(\frac{1+2+3+4+5+6}{6}\right)\left(1 - \frac{g}{6}\right)$

or $\boxed{E(W) = -\frac{g^2}{12} + \frac{g}{2} + 3.5}$ (4)

$\arg\left(\text{Max } E(W)\right) = \arg\left(\frac{d}{dg} E(W) = 0\right)$

or $g_{Max} = \arg\left(-\frac{g}{6} + \frac{1}{2} = 0\right)$

or $\boxed{g_{Max} = 3}$ (5)

~~which was~~

Putting g_{Max} into (4):

$\text{Max. } E(W) = \frac{-3^2}{12} + \frac{3}{2} + 3.5$

or $\boxed{\text{Max. } E(W) = 4.25}$ Ans.

∴ An expected \$4.25 can be obtained by throwing again only if the first throw lands 1, 2 or 3.

Ans

$\boxed{3.}$

outcomes	i	$p(i)$	X_i winning #
T	1	$(1/2)$	2^0
HT	2	$(1/2)^2$	2^1
HHT	3	$(1/2)^3$	2^2
⋮	⋮	⋮	⋮
$\underbrace{H\ldots H}_{(n-1)}T$	n	$(1/2)^n$	2^{n-1}

$$E(X) = \sum_{i=1}^{\infty} X_i\, p(i)$$

$$\Rightarrow E(X) = \sum_{i=1}^{\infty} 2^{i-1} \cdot \left(\tfrac{1}{2}\right)^i$$

$$\Rightarrow E(X) = \sum_{i=1}^{\infty} 2^{-1}$$

3(a)

$$\Rightarrow \boxed{E(X_i) \to \infty}$$

$$P(X > 65) = P(K > \log_2(65) + 1) \quad \text{as } X = 2^{K-1}.$$

$$\Rightarrow P(X>65) = P(K > 7)$$

$$\Rightarrow P(X>65) = 1 - P(K \le 7)$$

$$\Rightarrow P(X>65) = 1 - \sum_{k=1}^{k=7} \left(\tfrac{1}{2}\right)^k$$

$$\Rightarrow P(X>65) = 1 - \frac{\left(\frac{1}{2}\right)^1 \cdot \left(1 - \left(\frac{1}{2}\right)^7\right)}{1 - \frac{1}{2}}$$

$$\Rightarrow P(X>65) = 1 - \left(1 - \left(\tfrac{1}{2}\right)^7\right)$$

3(b)

$$\Rightarrow \boxed{P(X>65) = \left(\tfrac{1}{2}\right)^7} \quad \text{Ans}$$

~~E(Y)=~~

If $X_{Max} = 2^{30}$, then the player should not bet after

$k = 31$ trials.

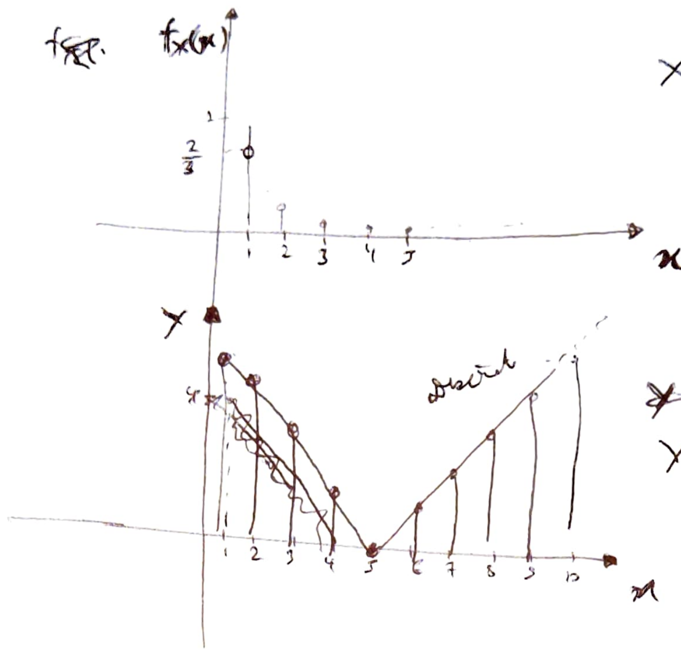
$$E(Y) = \sum_{k=1}^{30} 2^{k-1} \cdot \left(\tfrac{1}{2}\right)^k + \$\, 2^k \cdot \left(\tfrac{1}{2}\right)^k \cdot 1$$

we win irrespective of the outcome on the 31st trial.
If its a TAIL, the game ends and I take home $\$2^{30}$.
If not, I would still have won the $\$2^{30}$ and forfeit the game after the 31st trial.

or
$$\boxed{E(Y) = \left(\tfrac{1}{2}\right) \times 30 = \cancel{15} \; +1 = 16}$$

$\boxed{4.}$ $X \sim$ geometric$\left(\frac{1}{3}\right)$ $\Rightarrow f_X(x) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{x-1}$ $x \geq 1, 2, 3, \ldots$

$Y = |X-5|$



$X \in \{1, 2, 3, 4 \ldots \infty\}$

~~$Y \in [0, \infty)$~~

$Y \in \{0, 1, 2 \ldots \infty\}$.

$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \frac{2}{3} & x = 1 \\ \frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3} & x = 2 \\ \sum_{i=1}^{i=n} \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{i-1} & x = n \end{cases} = \frac{2}{3} \cdot \frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}$

$\frac{2}{3} \cdot \frac{\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}}$

Y	X	$F_Y(y)$
0	5	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4$
1	4, 6	
2	3, 7	
3	2, 8	
4	1, 9	
5	10	

$\boxed{4.}$ Ans,

Y_i	X_{i_y}	$pmf(Y_i)$
0	5	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4$
1	4,6	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^5$
2	3,7	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^6$
3	2,8	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^1 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^7$
4	1,9	$\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8$
5	10	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^9$
⋮		
N	N+5	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{N+4}$

$F_Y(y) = P(|x-5| \leq \alpha)$

$F_Y(y) = P(\underset{x \geq 5}{x \leq \alpha+5}) + P(5-x < \alpha)$ $x \in [1, 5)$ 1, 2, 3, 4.

$F_Y(y) = P(\underset{x \geq 5}{x \leq \alpha+5}) + P(x > 5-\alpha)$.

$F_Y(y) = \underset{x \geq 5 \text{ else } 0}{F_X(\alpha+5)} + (1 - \underset{x = 1,2,3,4 \text{ else } 0}{F_X(5-\alpha)})$.

$F_X(y) = \Big\{$ $\quad x = 1, 2, 3, 4$

5.

5.

$$p_X(k) = \begin{cases} 0.5 & k=1 \\ 0.3 & k=2 \\ 0.2 & k=3 \\ 0 & \text{otherwise} \end{cases}$$

5(a)

$\boxed{E(X) = 0.5 + 0.3\times 2 + 0.2\times 3 = 1.7}$ Ans

$E(X^2) = 0.5 + 0.3\times 4 + 0.2\times 9 = 3.5$

$V(X) = E(X^2) - [E(X)]^2 = 3.5 - 1.7^2 =$

5(b)(i)

$\boxed{V(X) = 0.61}$ Ans

5b(ii)

$\boxed{\sigma(X) = \sqrt{0.61}}$ Ans

$Y = \dfrac{2}{X}$ $\quad E(Y) = 0.5\times\dfrac{2}{1} + 0.3\times\dfrac{2}{2} + 0.2\times\dfrac{2}{3}$

$\therefore E(Y) = 1 + 0.3 + \dfrac{0.4}{3}$

5(c)

$\therefore \boxed{E(Y) = 1.4333}$ Ans

6.

$$\text{Exp} \begin{cases} H \text{ — 20 MORE flips} & \text{Event } \#1 = H \\ T \text{ — 10 MORE flips} & \text{Event } \#1 = T \end{cases}$$

$X = \#$ heads in ALL flips (including first one).

(a) $P(X=5) = P(X=5 \mid \#1=H) \cdot P(\#1=H) + P(X=5 \mid \#1=T) \cdot P(\#1=T)$

6 (a)

$$\boxed{P(X=5) = {}^{20}C_4 \cdot \left(\tfrac{1}{2}\right)^{20} \cdot \left(\tfrac{1}{2}\right) + {}^{10}C_5 \left(\tfrac{1}{2}\right)^{10} \cdot \left(\tfrac{1}{2}\right) \qquad P(X=5) = 0.1254}$$ Ans

(b) $P(\#1=H \mid X=5) = \dfrac{P(X=5 \mid \#1=H) \cdot P(\#1=H)}{P(X=5)}$

$P(\#1=H \mid X=5) = \dfrac{\left\{{}^{20}C_4 \cdot \left(\tfrac{1}{2}\right)^{20}\right\} \cdot \left(\tfrac{1}{2}\right)}{0.1254}$

6 (b)

$$\boxed{P(\#1=H \mid X=5) = 0.0184}$$ Ans

(c) $P(\#\text{Last}=H \mid X=5) = \dfrac{P(X=5 \mid \#\text{Last}=H) \cdot P(\#\text{Last}=H)}{P(X=5)}$

$P(\#\text{Last}=H \mid X=5) = \dfrac{\left[P(X=5 \mid \#\text{Last}=H, \#1=H) \cdot P(\#1=H) + P(X=5 \mid \#\text{Last}=H, \#1=T) \cdot P(\#1=T)\right] \cdot \left(\tfrac{1}{2}\right)}{0.1254}$

$P(\#\text{Last}=H \mid X=5) = \dfrac{\left[{}^{19}C_3 \cdot \left(\tfrac{1}{2}\right)^{19} \left(\tfrac{1}{2}\right) + {}^{9}C_4 \left(\tfrac{1}{2}\right)^{9} \cdot \left(\tfrac{1}{2}\right)\right] \left(\tfrac{1}{2}\right)}{0.1254}$

6 (c)

$$\boxed{P(\#\text{Last}=H \mid X=5) = 0.4945}$$ Ans

$\boxed{2 | 2}$ given A ~~and~~ is ALMOST INDEPENDENT of B if

$$P(A|B) \in [P(A) - 0.01, P(A) + 0.01]$$

(a) if given A is 'AI' of B, or $\boxed{P(A|B) \in [P(A) - 0.01, P(A) + 0.01]}$ ②
check if $\bar{A}$ is also 'AI' of B.

we know that $P(A|B) + P(\bar{A}|B) = 1$

or $P(\bar{A}|B) = 1 - P(A|B)$

or $P(\bar{A}|B) = 1 - [P(A) - 0.01, P(A) + 0.01]$

or $P(\bar{A}|B) = [1 - P(A) - 0.01, 1 - P(A) + 0.01]$

or $P(\bar{A}|B) = [P(\bar{A}) - 0.01, P(\bar{A}) + 0.01]$

$\boxed{2|2(a)}$ ~~P(A~~

$\Rightarrow$ $\boxed{\bar{A} \text{ is 'AI' of } B}$

$\boxed{\text{if } A \text{ is 'AI' of } B}$ Ans.

(b) given A is 'AI' of B ~~i.e. P(A|B) ∈ [P(A) − 0.01, P(A)...~~

i.e. $\boxed{P(A|B) \in [P(A) - 0.01, P(A) + 0.01]}$ ①

check if B is also 'AI' of A

i.e. $\boxed{P(B|A) \in [P(B) - 0.01, P(B) + 0.01]}$ ②

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

$$\text{or } P(B|A) = \frac{[P(A) - 0.01, P(A) + 0.01] \, P(B)}{P(A)}$$

2 | 2.2

or $P(B|A) = \left[P(B) - 0.01\underbrace{\frac{P(B)}{P(A)}}, P(B) + 0.01\frac{P(B)}{P(A)}\right]$

For B to be 'AI' of A too,

we require that $\frac{P(B)}{P(A)} \le 1$

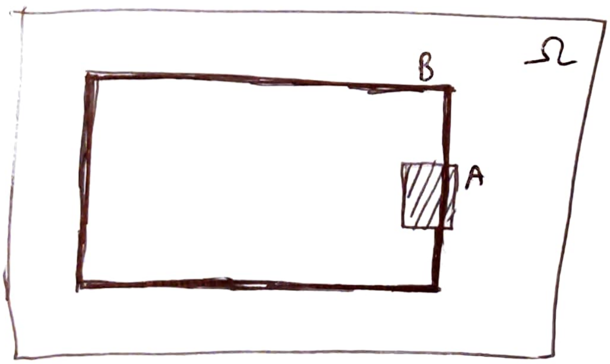
which is NOT a ~~general~~ or ~~true~~ correct ~~assumption~~ inequality in general.

2/2 (b) ∴ The the statement is False;

Ans

i.e. ~~A~~ given A is 'AI' of B $\not\Rightarrow$ B is 'AI' of A

Counterexample



Here B □

A ▨

$P(B) = 0.75$

Let's take ~~B ≈ ...~~

~~P(A)~~

~~Say P(A) = P(A)~~

Assume that 1% of A is outside B.

In that case P(A)

By construction/hypothesis:

$P(B) = 0.75$

$P(A) =$ some smaller value

$P(A\bar{B}) = 0.01P(A)$

$P(A|B) =$

$P(A|B) = \frac{P(AB)}{P(B)}$

$\therefore P(A|B) = \frac{0.99P(A)}{P(B)}$

$\therefore P(A|B) = \frac{0.99P(A)}{0.75}$

$P(B|A) = \frac{P(AB)}{P(A)} = \frac{0.99P(A)}{P(A)}$

$\therefore P(B|A) = 0.99$

$\notin [P(B) - 0.01, P(B) + 0.01]$