EE 507 Random Processes
Homework 01

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Problem 1

You pick two cards at random (without replacement) from a standard deck of 52 cards. Please answer the following questions:

a. What is the probability that both cards show the same value (e.g. both are 4's or both are Kings.)

b. Given that the two cards have the same color, what is the probability that both cards show the same value? Is this probability larger or smaller than the probability in part a? Conceptually, why does this make sense?

c. Are the events that the two cards have the same color and the two cards have the same value independent?

Solution

Experiment S: Two cards are drawn at random without replacement from a standard deck of 52 cards.
Let event A = $\{$"Both cards are of the same value"$\}$.
Let event B = $\{$"Both cards are of the same colour"$\}$.
So AB = $\{$"Both cards are of the same value and of the same colour"$\}$

a.

$$\begin{aligned} P(A) &= \frac{52 * 3}{52 * 51} \\ \text{or, } P(A) &= \frac{1}{17} \\ \text{or, } P(A) &\approx 0.0588 \end{aligned} \tag{1}$$

b.

$$P(A|B) = \frac{P(AB)}{P(B)} \tag{2}$$

$$\text{But, } P(B) = \frac{52 * 25}{52 * 51}$$

$$\text{or, } P(B) = \frac{25}{51}$$

$$\text{or, } P(B) \approx 0.4902 \tag{3}$$

$$\text{or, } P(AB) = \frac{52 * 1}{52 * 51}$$

$$\text{or, } P(AB) = \frac{1}{51}$$

$$\text{or, } P(AB) \approx 0.0196 \tag{4}$$

Thus, Using eqs. (1) to (4), we get:

$$P(A|B) = \frac{1}{25}$$

$$\text{or, } P(A|B) = 0.040 \tag{5}$$

This probability is smaller (approximately two-thirds of) than the probability computed in part a). Conceptually, this makes sense as the be restricting the outcomes to only those cards with the same colour, the total number of outcomes were reduced by around half ($52 * 25$ from $52 * 51$) but the number of successful outcomes reduced to a third of the original ($52 * 1$ from $52 * 3$).

c. In part b), it was seen that the knowledge of event B happening altered our knowledge (decreased the probability of happening) of event A (eq. (5) compared to eq. (1)). This violates the condition of independence of events A and B as $P(A|B) \neq P(A)$.

Problem 2

a. If two events A and B are independent, is it necessarily true that A and $\bar{B}$ are independent Please prove or give a counterexample.

b. If events A and B are independent, and events A and C are independent, is it always true that events B and C are independent? Please prove or give a counterexample.

c. If two events are disjoint, are they always, sometimes, or never independent? Please explain.

Solution

a. From Law of Total Probability, we know that:

$$\text{Given that, } P(AB) = P(A)P(B) \tag{6}$$

$$\text{To check if: } P(A\bar{B}) = P(A)P(\bar{B}) \tag{7}$$

$$A = AB + A\bar{B} \tag{8}$$

$$\begin{aligned}
\text{or, } P(A) &= P(AB) + P(A\bar{B}) \\
\text{or, } P(A) &= P(A)P(B) + P(A\bar{B}) \\
\text{or, } P(A) - P(A)P(B) &= P(A\bar{B}) \\
\text{or, } P(A)1 - P(B) &= P(A\bar{B}) \\
\text{or, } P(A)P(\bar{B}) &= P(A\bar{B})
\end{aligned}$$

which is exactly what we set to prove i.e. eq. (7).
$\implies$ If A and B are independent, then so are A and $\bar{B}$.
Hence Proved.😄

b. No. Assume the trivial cases $B = C$ or $B = \bar{C}$. Even when $P(A|B) = P(A)$ and $P(A|C) = P(C)$ as given, $P(B|C) = 1 \neq P(B)$ in the first case and $P(B|C) = 0 \neq P(B)$ in the second.

c. Disregarding the trivial case when at least one of the events is impossible, two disjoint events can NEVER be independent. Conceptually this makes sense, as say, if events A and B with non-zero probabilities are completely disjoint, then we could say that $B \in \bar{A}$, thus $P(A|B) = 0 \neq P(A)$ and vice versa $P(B|A) = 0 \neq P(B)$, which basically violates the condition of independence.

Problem 3

Solution

Problem 4

Solution

Problem 5

Solution

Problem 6

Solution

Problem 7

Solution