Normal Forms and Imperfections

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Example 1

Let's begin with this equation:

$$\frac{dx}{dt} = -x(x^2 - 2x - \mu) \tag{1}$$

which has equilibrium points at

$$x_0 = 0$$
$$x_0 = 1 + \sqrt{1 + \mu}$$
$$x_0 = 1 - \sqrt{1 + \mu}$$

such that:

$$\frac{dx_0}{dt} = -x_0(x_0^2 - 2x_0 - \mu) = 0 \tag{2}$$

Perturbing x around an equilibrium point x_0 by a small value $\tilde{x}$, i.e. using $x = x_0 + \tilde{x}$, we can rewrite equation (1) as:

$$\begin{aligned}\frac{dx}{dt} &= \frac{d(x_0 + \tilde{x})}{dt} \\ \frac{dx}{dt} &= \frac{dx_0}{dt} + \frac{d\tilde{x}}{dt} \\ \frac{dx}{dt} &= \frac{d\tilde{x}}{dt}\end{aligned} \tag{3}$$

and putting $x = x_0 + \tilde{x}$ and $\mu = \mu_0 + \tilde{\mu}$ in the RHS of equation (1), we get:

$$\begin{aligned}\frac{d\tilde{x}}{dt} &= -(x_0 + \tilde{x})\{(x_0 + \tilde{x})^2 - 2(x_0 + \tilde{x}) - (\mu_0 + \tilde{\mu})\} \\ \frac{d\tilde{x}}{dt} &= -(x_0 + \tilde{x})\{x_0^2 + 2x_0\tilde{x} + \tilde{x}^2 - 2x_0 - 2\tilde{x} - \mu_0 - \tilde{\mu}\}\end{aligned} \tag{4}$$

Re-arranging equation (4) as a polynomial in $\tilde{x}$, we get:

$$
\begin{aligned}
\frac{d\tilde{x}}{dt} &= -(\tilde{x} + x_0)\{\tilde{x}^2 + (2x_0 - 2)\tilde{x} + (x_0^2 - 2x_0 - \mu_0 - \tilde{\mu})\} \\
\frac{d\tilde{x}}{dt} &= -\{\tilde{x}^3 + (3x_0 - 2)\tilde{x}^2 + (3x_0^2 - 4x_0 - \mu_0 - \tilde{\mu})\tilde{x} \\
&\quad + (x_0^3 - 2x_0^2 - x_0\mu_0 - x_0\tilde{\mu})\}
\end{aligned} \tag{5}
$$

We now check for the stability of equation (1) around the point $(0, 0)$. We do this by making $x = 0 + \tilde{x}$ and $\mu = 0 + \tilde{\mu}$, which is equivalent to making $x_0 = 0$ and $\mu_0 = 0$ in equation (5). Thus, the equation with the substitutions becomes:

$$
\begin{aligned}
\frac{d\tilde{x}}{dt} &= -\{\tilde{x}^3 + (-2)\tilde{x}^2 + (-\tilde{\mu})\tilde{x} + (0)\} \\
\frac{d\tilde{x}}{dt} &= -\tilde{x}^3 + 2\tilde{x}^2 + \tilde{\mu}\tilde{x}
\end{aligned} \tag{6}
$$

We can neglect the first cubic term in equation (6) and arrive at the equation of a transcritical bifurcation:

$$
\frac{d\tilde{x}}{dt} = \tilde{\mu}\tilde{x} + 2\tilde{x}^2 \tag{7}
$$

We can similarly check for the stability of the dynamical system around the point $(1, -1)$, by putting $x_0 = 1$ and $\mu_0 = -1$ in equation (5):

$$
\begin{aligned}
\frac{d\tilde{x}}{dt} &= -\{\tilde{x}^3 + (1)\tilde{x}^2 + (-\tilde{\mu})\tilde{x} + (-\tilde{\mu})\} \\
\frac{d\tilde{x}}{dt} &= -\tilde{x}^3 - \tilde{x}^2 + \tilde{\mu}\tilde{x} + \tilde{\mu}
\end{aligned} \tag{8}
$$

and this is where my confusion begins. The biggest term here is $\tilde{\mu}$, and does not match with what the lecture says.