Saddle-node Bifurcation: Checking the Stability of Equilibrium Points

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April 28, 2022

Below is the equation for the Saddle-node Bifurcation:

$$\frac{dx}{dt} = \mu - x^2 \tag{1}$$

which has equilibrium points at

$$x_0 = +\sqrt{\mu}$$
$$x_0 = -\sqrt{\mu}$$

such that:

$$\frac{dx_0}{dt} = \mu - x_0^2 = 0 \tag{2}$$

Perturbing x around an equilibrium point x_0 by a small value $\tilde{x}$, i.e. using $x = x_0 + \tilde{x}$, we can rewrite equation (1) as:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d(x_0 + \tilde{x})}{dt} \\
\frac{dx}{dt} &= \frac{dx_0}{dt} + \frac{d\tilde{x}}{dt} \\
\frac{dx}{dt} &= \frac{d\tilde{x}}{dt}
\end{aligned} \tag{3}$$

and putting $x = x_0 + \tilde{x}$ in the RHS of equation (1), we get:

$$\begin{aligned}
\frac{d\tilde{x}}{dt} &= \mu - (x_0 + \tilde{x})^2 \\
\frac{d\tilde{x}}{dt} &= \mu - x_0^2 - 2x_0\tilde{x} - \tilde{x}^2
\end{aligned} \tag{4}$$

Using equation (1) and equation (2) in equation (4) and neglecting the small last term in the RHS, we get:

$$\frac{d\tilde{x}}{dt} = -2x_0\tilde{x} \tag{5}$$

which is a first order differential equation with constant coefficients whose solution can be expressed as:

$$\tilde{x}(t) = x(0)\exp\left(-2x_0 t\right) \tag{6}$$

Thus the solution expressed in equation (6) is an exponentially decaying stable one if x_0 is positive (in this case $x_0 = +\sqrt{\mu}$) but an exponentially increasing unstable one if x_0 is negative (in this case $x_0 = -\sqrt{\mu}$).