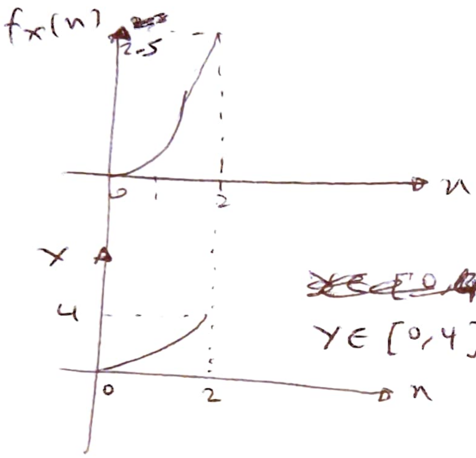
$\boxed{1.}$

$$f_X(x) = \begin{cases} \frac{5}{32} x^4 & x \in (0,2] \\ 0 & \text{else.} \end{cases}$$

$Y = X^2$

(i) <u>Using CDF formula</u>



$$F_X(x=\alpha) = \begin{cases} 0 & \alpha \le 0 \\ \frac{1}{32}\alpha^5 & \alpha \in (0,2] \\ 1 & \alpha > 2 \end{cases}$$

$Y \in [0,4)$

$F_Y(y=\alpha) = P(Y \le \alpha)$

or $F_Y(y=\alpha) = P(x^2 \le \alpha)$

or $F_Y(y=\alpha) = P(x \le \sqrt{\alpha})$

or $$F_Y(y=\alpha) = F_X(\sqrt{\alpha}) = \begin{cases} 0 & , \sqrt{\alpha} \le 0 \\ \frac{1}{32}(\sqrt{\alpha})^5 & \sqrt{\alpha} \in (0,2] \\ 1 & \sqrt{\alpha} > 2 \end{cases}$$

$\boxed{1a.}$

or $$\boxed{F_Y(y=\alpha) = \begin{cases} 0 & \alpha \le 0 \\ \frac{1}{32}\alpha^{\frac{5}{2}} & \alpha \in (0,4] \\ 1 & \alpha > 4 \end{cases}}$$

Ans

1 b.

$$f_Y(y) = \frac{d}{dy}\left(F_Y(y=\alpha)\right)\Big|_{y=\alpha} = \begin{cases} 0 & y < 0 \\ \frac{5}{64} y^{\frac{3}{2}} & y \in [0,4] \\ 0 & y > 4 \end{cases}$$

Ans

(ii) Using Transformative ("cute") formula:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

$$\text{or } f_Y(y) = \frac{d}{dx}\left(F_X(g^{-1}(y))\right) \cdot \left|\frac{dx}{dy}\right|$$

Here since $\cancel{g(x)} =$ $y = g(x) = x^2$,

$g^{-1}(y) = \sqrt{y}$.

$\cancel{\frac{dy}{dx} = \frac{d}{dx}(\sqrt{y}) =}$

$$\frac{dx}{dy} = \frac{d}{dy}(\sqrt{y}) = \frac{1}{2\sqrt{y}}$$

$$\text{or } f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{1}{2\sqrt{y}}\right|$$

$$\text{or } f_Y(y) = \begin{cases} 0 & \sqrt{y} \le 0 \\ \frac{5}{32}(\sqrt{y})^4 \cdot \left|\frac{1}{2\sqrt{y}}\right| & \cancel{\sqrt{y} \in (0,2]}\ \sqrt{y} \in (0,2] \\ 0 & \sqrt{y} > 2 \end{cases}$$

1 b (again!)

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} 0 & y \le 0 \\ \frac{5}{64} y^{\frac{3}{2}} & y \in (0, 4] \\ 0 & y > 4 \end{cases}}$$ Ans (Again).

(i) Using $f_Y(y)$

$$E[Y] = \int_{y=0}^{y=4} y\, f_Y(y)\, dy$$

$$\Rightarrow E(Y) = \int_{y=0}^{4} y \times \frac{5}{64} y^{\frac{3}{2}}\, dy$$

$$\Rightarrow E(Y) = \frac{5}{64} \cdot y^{\frac{7}{2}} \cdot \frac{2}{7} \Big|_{y=0}^{y=4}$$

$$\Rightarrow E(Y) = \frac{10}{64 \times 7} \cdot 4^{\frac{7}{2}}$$

1c

$$\Rightarrow \boxed{E(Y) = \frac{20}{7}}$$ Ans

(ii) Using Law of The Unconscious Statistician (LOTUS):

$$E(Y) = E[X^2] = \int_{x=0}^{x=2} x^2 \cdot f_X(x)\, dx$$

$$\Rightarrow E(Y) = \int_{x=0}^{x=2} x^2 \cdot \frac{5}{32} x^4\, dx$$

$$\Rightarrow E(Y) = \frac{5}{32} \cdot \frac{x^7}{7} \Big|_{x=0}^{x=2}$$

1c (again!)

$$\Rightarrow \boxed{E(Y) = \frac{20}{7}}$$ Ans (again!)

2.

Roll #1 $\xrightarrow{\text{Roll again?}}$ Roll #2

$X = \{1,2,3,4,5,6\}$ $\quad$ $Y = \{1,2,3,4,5,6\}$

Let A be the event that you cash in on the first die throw.

~~E[W] = F~~

$\boxed{\cancel{E(W) = A E(X) + \bar{A} E(Y)}}$

~~W = X F X~~

then $\boxed{W = AX + \bar{A}Y}$ (1)

~~X Cashing in on the first throw~~.

In order to maximize W, we need to optimize the event (algorithm) A.

Intuitively, we would want to throw again if we get 'lower' values of X and hold back from throwing again if we get a high value of Y.

For quantifying A, we can utilize a variable g (for good throw) which stands for the number of allowed 'high-valued' throws allowed.

1
2
3
4
5 $\}$ g
6

Here, $g=2$, i.e. hold back upon getting the highest 2 values of X : 5 or 6. In such a case,

$E(X|g=2) = \frac{5+6}{2} = 5.5$

TABLE 1: $E(X|g)$ vs g.

g	$X\|g$	$E(X\|g)$
1	$\{6\}$	6
2	$\{6,5\}$	5.5
3	$\{6,5,4\}$	5
4	$\{6,5,4,3\}$	4.5
5	$\{6,5,4,3,2\}$	4
6	$\{6,5,4,3,2,1\}$	3.5

From the Table 1, we can devise $E(X|g)$ as a simple linear function of g:

$$\boxed{E(X|g) = 6.5 - \frac{g}{2}} \quad (2)$$

Taking expectation on both sides of (1):

$$E(W) = E(AX) + E(\bar{A}Y)$$

and using g to quantify event A:

~~get~~ or

$$\boxed{E(W) = E(X|g) \cdot P(g) + E(Y|\bar{g}) \cdot P(\bar{g})} \quad (3)$$

where $X|g$ = Value of X given that only the highest g values of X are allowed for holding back from throwing again.

$P(g)$ = Probability of obtaining any of the highest g values on the first die throw

$X|\bar{g}$ = given that none of the highest g values were landed on the first throw, the value landed on the second throw.

It may be noted that ~~g~~ ~~is not~~ $E(Y)$ is not dependent on the value of g or $\bar{g}$. So $E(Y|\bar{g}) = E(Y)$.

So $E(W) = \left(6.5 - \frac{1}{2}g\right)\left(\frac{g}{6}\right) + \left(\frac{1+2+3+4+5+6}{6}\right)\left(1 - \frac{g}{6}\right)$

or $\boxed{E(W) = -\frac{g^2}{12} + \frac{g}{2} + 3.5}$ (4)

$\arg\left(\text{Max } E(W)\right) = \arg\left(\frac{d}{dg} E(W) = 0\right)$

or $g_{\text{Max}} = \arg\left(-\frac{g}{6} + \frac{1}{2} = 0\right)$

or $\boxed{g_{\text{Max}} = 3}$ (5)

~~which real~~

Putting g_{Max} into (4):

$\text{Max. } E(W) = -\frac{3^2}{12} + \frac{3}{2} + 3.5$

or $\boxed{\text{Max. } E(W) = 4.25}$ Ans.

∴ An expected \$4.25 can be obtained by throwing again only if the first throw lands 1, 2 or 3.

Ans

—— × —— × —— × ——

3.1

3\.

outcomes

	i	$p(i)$	X_i winning #
T	1	$(1/2)$	2^0
HT	2	$(1/2)^2$	2^1
HHT	3	$(1/2)^3$	2^2
⋮		⋮	⋮
$\underbrace{H\ldots H}_{(n-1)}T$	n	$(1/2)^n$	2^{n-1}

$$E(X) = \sum_{i=1}^{\infty} X_i\, p(i)$$

$$\Rightarrow E(X) = \sum_{i=1}^{\infty} 2^{i-1}\cdot\left(\frac{1}{2}\right)^i$$

$$\Rightarrow E(X) = \sum_{i=1}^{\infty} 2^{-1}$$

3(a)

$$\Rightarrow \boxed{E(X_i) \to \infty}$$

$$P(X > 65) = P(K > \log_2(65) + 1) \quad \text{as} \quad X = 2^{K-1}.$$

$$\Rightarrow P(X>65) = P(K > 7)$$

$$\Rightarrow P(X>65) = 1 - P(K \leq 7)$$

$$\Rightarrow P(X>65) = 1 - \sum_{k=1}^{k=7} \left(\frac{1}{2}\right)^k$$

$$\Rightarrow P(X>65) = 1 - \frac{\left(\frac{1}{2}\right)^1 \cdot \left(1 - \left(\frac{1}{2}\right)^7\right)}{1-\frac{1}{2}}$$

$$\Rightarrow P(X>65) = 1 - \left(1 - \left(\frac{1}{2}\right)^7\right)$$

3(b)

$$\Rightarrow \boxed{P(X>65) = \left(\frac{1}{2}\right)^7} \quad \text{Ans}$$

3.2

~~E[Y] =~~

If $X_{Max} = 2^{30}$, then the player should not bet after

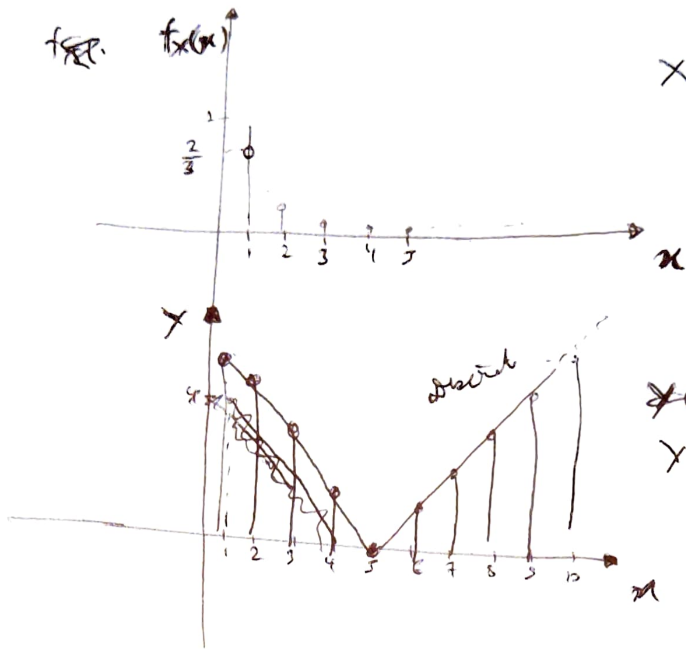
$K = 31$ trials.

$$E(Y) = \sum_{k=1}^{31} 2^{k-1} \cdot \left(\frac{1}{2}\right)^k$$

or $$\boxed{E(Y) = \left(\frac{1}{2}\right) \times 31 = 15.5}$$

$\boxed{4.}$ $X \sim \text{geometric}\left(\frac{1}{3}\right) \Rightarrow f_X(x) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{x-1} \quad x \geq 1, 2, 3, \ldots$

$Y = |X - 5|$



$X \in \{1, 2, 3, 4 \ldots \infty\}$

~~$Y \in [0, \infty)$~~

$Y \in \{0, 1, 2, \ldots \infty\}$.

$\boxed{4.}$ Ans.

Y_i	X_{i_y}	$pmf(Y_i)$
0	5	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4$
1	4, 6	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^5$
2	3, 7	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^6$
3	2, 8	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^1 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^7$
4	1, 9	$\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8$
5	10	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^9$
$\vdots$		
N	N+5	$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{N+4}$

$F_X(y) = \Bigg\{ \quad x = 1, 2, 3, 4$

5.

5.

$$p_X(k) = \begin{cases} 0.5 & k=1 \\ 0.3 & k=2 \\ 0.2 & k=3 \\ 0 & \text{otherwise} \end{cases}$$

5(a)

$E(X) = 0.5 + 0.3 \times 2 + 0.2 \times 3 = 1.7$ Ans

$E(X^2) = 0.5 + 0.3 \times 4 + 0.2 \times 9 = 3.5$

$V(X) = E(X^2) - [E(X)]^2 = 3.5 - 1.7^2 =$

5(b)(i)

$V(X) = 0.61$ Ans

5b(ii)

$\sigma(X) = \sqrt{0.61}$ Ans

$Y = \frac{2}{X}$ $\quad E(Y) = 0.5 \times \frac{2}{1} + 0.3 \times \frac{2}{2} + 0.2 \times \frac{2}{3}$

$\therefore E(Y) = 1 + 0.3 + \frac{0.4}{3}$

5(c)

$\therefore E(Y) = 1.4333$ Ans
