EE 507 Random Processes
Homework 02

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Contents

Problem 1

You pick two cards at random (without replacement) from a standard deck of 52 cards. Please answer the following questions:

a. What is the probability that both cards show the same value (e.g. both are 4's or both are Kings.)

b. Given that the two cards have the same color, what is the probability that both cards show the same value? Is this probability larger or smaller than the probability in part a? Conceptually, why does this make sense?

c. Are the events that the two cards have the same color and the two cards have the same value independent?

Solution

Experiment S: Two cards are drawn at random without replacement from a standard deck of 52 cards.
Let event A = $\{$"Both cards are of the same value"$\}$.
Let event B = $\{$"Both cards are of the same colour"$\}$.
So AB = $\{$"Both cards are of the same value and of the same colour"$\}$

a.

$$\begin{aligned} P(A) &= \frac{52 * 3}{52 * 51} \\ \text{or, } P(A) &= \frac{1}{17} \\ \text{or, } P(A) &\approx 0.0588 \end{aligned} \tag{1}$$

b.

$$P(A|B) = \frac{P(AB)}{P(B)} \tag{2}$$

$$\text{But, } P(B) = \frac{52 * 25}{52 * 51}$$

$$\text{or, } P(B) = \frac{25}{51}$$

$$\text{or, } P(B) \approx 0.4902 \tag{3}$$

$$\text{or, } P(AB) = \frac{52 * 1}{52 * 51}$$

$$\text{or, } P(AB) = \frac{1}{51}$$

$$\text{or, } P(AB) \approx 0.0196 \tag{4}$$

Thus, Using Eqs. (1) to (4), we get:

$$P(A|B) = \frac{1}{25}$$

$$\text{or, } P(A|B) = 0.040 \tag{5}$$

This probability is smaller (approximately two-thirds of) than the probability computed in part a). Conceptually, this makes sense as the be restricting the outcomes to only those cards with the same colour, the total number of outcomes was reduced by around half ($52 * 25$ from $52 * 51$) but the number of successful outcomes was reduced to a third of the original ($52 * 1$ from $52 * 3$).

c. In part b), it was seen that the knowledge of event B happening altered our knowledge (decreased the probability of happening) of event A (Eq. (5) compared to Eq. (1)). This violates the condition of independence of events A and B as $P(A|B) \neq P(A)$.

Problem 2

a. If two events A and B are independent, is it necessarily true that A and $\bar{B}$ are independent Please prove or give a counterexample.

b. If events A and B are independent, and events A and C are independent, is it always true that events B and C are independent? Please prove or give a counterexample.

c. If two events are disjoint, are they always, sometimes, or never independent? Please explain.

Solution

a. From Law of Total Probability, we know that:

$$\text{Given that, } P(AB) = P(A)P(B) \tag{6}$$
$$\text{To check if: } P(A\bar{B}) = P(A)P(\bar{B}) \tag{7}$$
$$A = AB + A\bar{B} \tag{8}$$
$$\begin{aligned}
\text{or, } P(A) &= P(AB) + P(A\bar{B}) \\
\text{or, } P(A) &= P(A)P(B) + P(A\bar{B}) \\
\text{or, } P(A) - P(A)P(B) &= P(A\bar{B}) \\
\text{or, } P(A)1 - P(B) &= P(A\bar{B}) \\
\text{or, } P(A)P(\bar{B}) &= P(A\bar{B})
\end{aligned}$$

which is exactly what we set to prove i.e. Eq. (7).
$\implies$ If A and B are independent, then so are A and $\bar{B}$.
Hence Proved.😄

b. No. Assume the trivial cases $B = C$ or $B = \bar{C}$. Even when $P(A|B) = P(A)$ and $P(A|C) = P(C)$ as given, $P(B|C) = 1 \neq P(B)$ in the first case and $P(B|C) = 0 \neq P(B)$ in the second.

c. Disregarding the trivial case when at least one of the events is impossible, two disjoint events can NEVER be independent. Conceptually this makes sense, as say, if events A and B with non-zero probabilities are completely disjoint, then we could say that $B \in \bar{A}$, thus $P(A|B) = 0 \neq P(A)$ and vice versa $P(B|A) = 0 \neq P(B)$, which basically violates the condition of independence.

Problem 3

A randomly selected student in an engineering class id diligent with probability 0.3, and lazy with probability 0.7. A diligent student completes all of her homework with probability 0.9 and does not do so with probability 0.1. Meanwhile, a lazy student completes all homework with probability 0.5 and does not do so with probability 0.5. A student who completes all homework receives an A grade with probability 0.6, and receives a B grade with probability 0.4 (irrespective of whether the student was diligent or lazy). A student who doesn't complete all homework receives an A grade with probability 0.3, a B grade withe probability 0.4 and an F grade with probability 0.3 (again, irrespective of whether the student was diligent or lazy).
Please answer the following questions.

a. What is the probability that a randomly selected student is diligent, completes all their homework and receives an A grade?

b. What is the probability that the student completes all of their homework?

c. What is the probability that the student receives an A grade?

d. Given that a randomly selected student completed all their homework, what is the probability that the student was lazy?

e. Given that a randomly-selected student received an A grade, what is the probability that the student is lazy?

f. Is the event that a student passes the course (receives an A or B grade) independent of the event that the student is lazy?

g. Is the event that a student receives a B grade independent of the event that the student is lazy?

Solution

Experiment S: A student is randomly selected from an engineering class.

Let, Event D = $\{$"Selected student is Diligent"$\}$

$\implies$ Event $\bar{D}$ = $\{$"Selected student is Lazy"$\}$

Let, Event H = $\{$"Selected student did all of their homework"$\}$

$\implies$ Event $\bar{H}$ = $\{$"Selected student did NOT do all of their homework"$\}$

Let, Event A = $\{$"Selected student receives an A grade."$\}$

Let, Event B = $\{$"Selected student receives a B grade."$\}$

Let, Event F = $\{$"Selected student receives an F grade."$\}$☹

For reference, Fig. 1 is a graph which displays all possible variations of a randomly chosen student (or in terms of probability theory all events with non-zero probabilities).

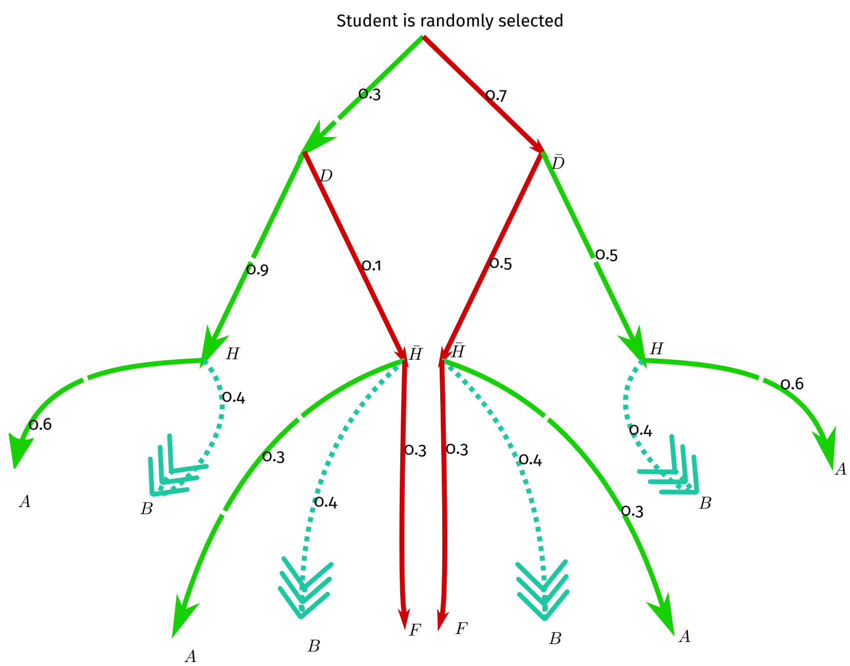

Figure 1: Decision Tree representing all possible types of students selected.

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a.

Using conditional probability theorem:

$$\begin{aligned} P(DHA) &= P(A|HD)P(H|D)P(D) \\ \text{or, } P(DHA) &= (0.6)(0.9)(0.3) \\ \text{or, } P(DHA) &= 0.162 \end{aligned} \tag{9}$$

b.

Using Law of Total Probability:

$$
\begin{aligned}
& P(H) = P(H|D)P(D) \\
\text{or, } & P(H) = P(H|D)P(D) + P(H|\bar{D})P(\bar{D}) \\
\text{or, } & P(H) = (0.9)(0.3) + (0.5)(0.7) \\
\text{or, } & P(H) = 0.62
\end{aligned} \tag{10}
$$

c.

Using Law of Total Probability:

$$
\begin{aligned}
P(A) = & P(A|\bar{D}\bar{H})P(\bar{D}\bar{H}) + P(A|\bar{D}H)P(\bar{D}H) \\
& + P(A|D\bar{H})P(D\bar{H}) + P(A|DH)P(DH)
\end{aligned}
$$

Using $P(DH) = P(H|D)P(D)$ and so on for other terms:

$$
\begin{aligned}
\text{or, } P(A) = & (0.3)(0.7)(0.5) + (0.6)(0.7)(0.5) \\
& + (0.3)(0.3)(0.1) + (0.6)(0.3)(0.9) \\
\text{or, } P(A) = & 0.486
\end{aligned} \tag{11}
$$

d.

Using Bayes' rule:

$$
P(\bar{D}|H) = \frac{P(H|\bar{D})P(\bar{D})}{P(H)} \tag{12}
$$

Using Eq. (10) in Eq. (12):

$$
\begin{aligned}
\text{or, } & P(\bar{D}|H) = \frac{(0.5)(0.7)}{0.62} \\
\text{or, } & P(\bar{D}|H) \approx 0.5645
\end{aligned} \tag{13}
$$

e.

Using Bayes' rule:

$$
P(\bar{D}|A) = \frac{P(A|\bar{D})P(\bar{D})}{P(A)} \tag{14}
$$

Using Law of Total Probability to expand $P(A|\bar{D})$:

$$
\text{or, } P(\bar{D}|A) = \frac{\Big\{P(A|\bar{H}\bar{D})P(\bar{H}|\bar{D}) + P(A|\bar{D}H)P(H|\bar{D})\Big\}P(\bar{D})}{P(A)} \tag{15}
$$

Using Eq. (11) in Eq. (15):

$$\text{or, } P(\bar{D}|A) = \frac{\Big\{(0.3)(0.5) + (0.6)(0.5)\Big\}(0.7)}{0.486}$$

$$\text{or, } P(\bar{D}|A) \approx 0.6481 \tag{16}$$

Problem 4

Solution

Problem 5

Solution

Problem 6

Solution

Problem 7

Solution