Problem 3

Consider any uncertain experiment, and let A, B, and C be three events defined for the experiment. Please prove the following equalities or inequalities. In doing so, you may use the three axioms, any equalities/inequalities developed in class, standard set theory and algebra concepts, and earlier parts of the problem.

a. $P(A\bar{B}) = P(A) - P(AB)$

b. If A is contained in B, then $P(A) \leq P(B)$.

c. $P(A + B) + P(\bar{A}) + P(\bar{B}) - P(\bar{A} + \bar{B}) = 1$

d. $P(AB) + P(\bar{A}C) + P(\bar{B}\bar{C}) \leq 1$

   Please also give an example showing that the bound can be achieved.

Solution

a. To Prove: $P(A\bar{B}) = P(A) - P(AB)$

   Proof.

   From the Law of Total Probability, we know that

   $$P(A) = P(A\bar{B}) + P(AB) \tag{1}$$

   Rearranging **??**, we get:

   $$P(A\bar{B}) = P(A) - P(AB) \tag{2}$$

   Hence Proved. 😄

   □

b. To Prove: If A is contained in B, then $P(A) \leq P(B)$

   Proof.

   To Prove: if $A \subseteq B$, then $P(A) \leq P(B)$

   From the Law of Total Probability, we know that

   $$P(B) = P(B\bar{A}) + P(BA) \tag{3}$$

   $$A \subseteq B \implies A \cap B = A \implies P(BA) = P(A)$$

   $$\text{or,} \quad P(B) = P(B\bar{A}) + P(A)$$

   But, by Axiom 1 of Probability

   $$P(B\bar{A}) \geq 0$$

   $$\implies P(A) \leq P(B)$$

   Hence Proved. 😄

   □

c. To Prove: $P(A+B)+P(\bar{A})+P(\bar{B})-P(\bar{A}+\bar{B})=1$

Proof.

$$
\begin{aligned}
LHS &= \{P(A)+P(B)-P(AB)\}+P(\bar{A})+P(\bar{B}) && \text{(Expanding the } LHS\text{)}\\
&\quad -\{P(\bar{A})+P(\bar{B})-P(\bar{A}\bar{B})\}\\
&= P(A)+P(B)-P(AB)+P(\bar{A}\bar{B})\\
&= P(A)-P(AB)+P(\bar{A}\bar{B})+P(B) && \text{(Rearranging the equation)}\\
&= P(A\bar{B})+P(\bar{A}\bar{B})+P(B) && \text{(Using \textbf{??})}\\
&= P(\bar{B})+P(B) && \text{(Using \textbf{??})}\\
&= 1 && \text{Hence Proved 😄}
\end{aligned}
$$

□

d. To Prove: $P(AB)+P(\bar{A}C)+P(\bar{B}\bar{C})\leq 1$
Also give an example for the equality case.

Proof.

$$
\begin{aligned}
AB\cap\bar{A}C\cap\bar{B}\bar{C} &= \phi && \text{(Pairwise Disjoint Sets)}\\
\implies P(AB)+P(\bar{A}C)+P(\bar{B}\bar{C}) &= P(AB+\bar{A}C+\bar{B}\bar{C}) && \text{(Axiom 3)}\\
&\leq 1 && \text{(Axiom 1)}\\
& && \text{Hence Proved 😄}
\end{aligned}
$$

□

For the special case of equality

$$P(AB)+P(\bar{A}C)+P(\bar{B}\bar{C})=1 \tag{4}$$

We may equate the constituent events of **??** to trivial identities in order to potentially reverse engineer the relationships between the three events A, B and C:

Let

$$AB+\bar{A}C+\bar{B}\bar{C}=A+\bar{A} \tag{5}$$

Making comparisons, we get:

$$
\begin{aligned}
AB &= A\\
\implies A &\subseteq B \qquad (6)\\
\bar{A}C &= \bar{A}\\
\implies \bar{A} &\subseteq C \qquad (7)\\
\bar{B}\bar{C} &= 0\\
\implies \bar{B}\cap\bar{C} &= \phi\\
\implies B\cup C &= 1 \qquad (8)
\end{aligned}
$$

?? gives two cases: Either A and C are disjoint events which exhaustively represent the sample space (i.e. $A \cup C = \phi$ and $A + C = \Omega$) or C encompasses the whole sample space and A is merely a subset of it (i.e. $C = \Omega$ and $A \subseteq C$).

One of the solutions satisfying **??**, **??** and **??** is $A = B$ and $A \cap C = \phi$. fig. 1 represents this instance of the relationship between A, B and C.

Another solution satisfying **??**, **??** and **??** is $A \subseteq B \subseteq C$, represented by fig. 2.

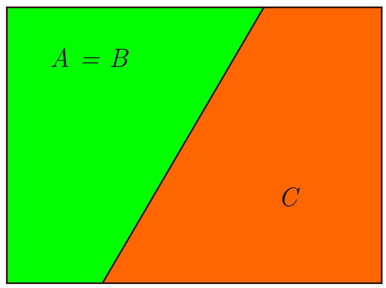

Figure 1: Instance 1 which satisfies the equality criterion.

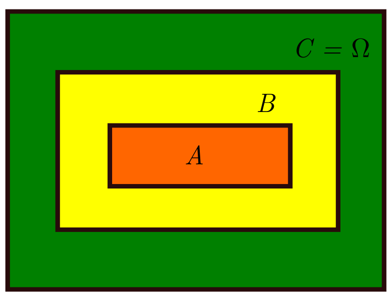

Figure 2: Instance 2 which satisfies the equality criterion.

Let

$$AB + \bar{A}C + \bar{B}\bar{C} = C + \bar{C} \tag{9}$$

Making comparisons, we get:

$$\begin{aligned} \bar{A}C &= C \\ \implies A &\subseteq \bar{C} \quad (10) \\ \bar{B}\bar{C} &= \bar{C} \\ \implies B &\subseteq C \quad (11) \\ AB &= 0 \\ \implies A \cap B &= \phi \quad (12) \end{aligned}$$

A solution satisfying **??**, **??** and **??** is $A \subseteq B \subseteq C$, represented by fig. 3.

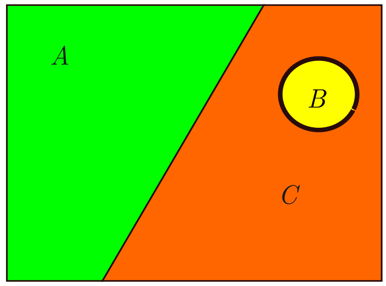

Figure 3: Instance 3 which satisfies the equality criterion.