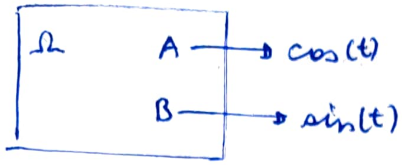
2. 2.1




| $\omega$ | $P(\omega)$ | $X(\omega,t)$ |
|---|---|---|
| A | 0.5 | $\cos(t)$ |
| B | 0.5 | $\sin(t)$ |

$$f_X(x(t)) = \sum_{\omega \in \Omega} \left[ P(\omega) \cdot \delta(x - X(\omega,t)) \right]$$

2(a)

or $$\boxed{f_X(x(t)) = 0.5\,\delta(x(t) - \cos(t)) + 0.5\,\delta(x(t) - \sin(t))}$$ Ans

$$f_{X_1,X_2}(x(t_1) = x_1, x(t_2) = x_2) = \,?$$

$$(x_1, x_2) = \begin{cases} (\cos(t_1), \cos(t_2)) & \text{w.p. } 0.5 \\ (\sin(t_1), \sin(t_2)) & \text{w.p. } 0.5 \end{cases}$$

2(b)

$$\boxed{f_{X_1,X_2}(x_1,x_2) = 0.5\,\delta(x_1 - \cos(t_1))\,\delta(x_2 - \cos(t_2)) + 0.5\,\delta(x_1 - \sin(t_1))\,\delta(x_2 - \sin(t_2))}$$ Ans

2(c)

$$\boxed{f_{X_1 \dots X_n}(x_1 \dots x_n) = 0.5 \prod_{i=1}^{n} \delta(x_i - \cos(t_i)) + 0.5 \prod_{i=1}^{n} \delta(x_i - \sin(t_i))}$$ Ans

2(d)(i)

$$\boxed{E[X(t)] = 0.5\cos(t) + 0.5\sin(t)}$$ Ans

$$R_{xx}(t_1, t_2) = E[X(t_1)X(t_2)]$$

or $R_{xx}(t_1, t_2) = E\left\{ \begin{array}{l} 0.5\cos(t_1)\cos(t_2) \\ + 0.5\sin(t_1)\sin(t_2) \end{array} \right\}$

2(d)(ii)

or $R_{xx}(t_1, t_2) = 0.5\cos(t_1 - t_2)$

or $R_{xx}(\tau) = 0.5\cos(\tau)$

Ans

——— X ——— X ——— X ———

3.

~~fxk~~

$f_{x_1,x_2,x_3\ldots x_n}(x_1,x_2\ldots x_n) = f_{x_1}(x_1) \cdot f_{x_2}(x_2) \cdot f_{x_n}(x_n)$

as the instances are independent.

3(a)

$$\boxed{f_{x_1\ldots x_n} = \prod_{i=1}^{n} \{0.4\,\delta(x_i-1) + 0.6\,\delta(x_i)\}}$$ Ans

$E[X[k]] = 0.4\times1 + 0.6\times0$

3(b)(i)

or $\boxed{E[X(k)] = 0.4}$ Ans

$R_{xx}[k,\tau] = E[X(k).X(\tau)]$

| $X(k)$ | $X(\tau)$ | $P(X(k).X(\tau))$ | $X(k).X(\tau)$ |
|---|---|---|---|
| 0 | 0 | $0.6^2$ | 0 |
| 0 | 1 | $(0.6)(0.4)$ | 0 |
| 1 | 0 | $(0.4)(0.6)$ | 0 |
| 1 | 1 | $(0.4)^2$ | 1 |

$R_{xx}(k,\tau) = (0.4)^2 \cdot 1$ ~~= 0~~

3(b)(ii)

or $\boxed{R_{xx}(k,\tau) = 0.16}$ Ans

3.2

3(b)(iii)

$C_{xx}(k,\tau) = R_{xx}(k,\tau) - E[x(k)]\,E[x(\tau)]$

or $C_{xx}(k,\tau) = 0.16 - (0.4)(0.4)$

3(b)(iii)

or $\boxed{C_{xx}(k,\tau) = 0}$ Ans

$$Y[k] = \sum_{i=0}^{k-1} x[i] \qquad k = 1, 2, 3, \ldots$$

$$Y[k] = \begin{cases} 0 & \text{w.p. } \binom{k}{0}(0.6)^{k-0}(0.4)^0 \\ 1 & \text{w.p. } \binom{k}{1}(0.6)^{k-1}(0.4)^1 \\ 2 & \text{w.p. } \binom{k}{2}(0.6)^{k-2}(0.4)^2 \\ \vdots & \vdots \\ k & \text{w.p. } \binom{k}{k}(0.6)^{k-k}(0.4)^k \end{cases}$$

3(c)

$$f_Y(y) = \sum_{y_k=0}^{k} \left[\binom{k}{y_k}(0.6)^{k-y_k}(0.4)^{y_k}\,\delta(y - y_k)\right]$$

where $y_k = Y[k]$

Ans

$$f_{Y_1, Y_2 \ldots Y_n}(Y[k_1]=y_1, Y(k_2)=y_2 \ldots Y[k_n]=y_n] = ?$$

$$\boxed{f_{Y_1 \ldots Y_n} = f_{Y_1}[Y[k_1]=y_1) \cdot f_{Y_2|Y_1}(Y[k_2]=y_2 | y_1) \ldots \ldots f_{Y_n|Y_{n-1}}(Y[k_n]=y_n | y_{n-1}]} \quad (0)$$

Let us define some notation for convenience of writing:

| | | | | | | |
|---|---|---|---|---|---|---|
| discrete time value | $k_1 \xrightarrow{\Delta k_{21}} k_2 \xrightarrow{\Delta k_{32}} k_3$ | | | $\ldots$ | $k_{n-1} \xrightarrow{\Delta k_{n(n-1)}} k_n$ | |
| value of $Y$ | $y_1 \xrightarrow[\Delta y_{21}]{} y_2 \xrightarrow[\Delta y_{32}]{} y_3$ | | | $\ldots$ | $y_{n-1} \xrightarrow[\Delta y_{(n)(n-1)}]{} y_n$ | |
| instance # | 1 | 2 | 3 | $\ldots$ | $n-1$ | $n$ |

Difference b/w two instances $(k_i, k_j)$ is expressed as $\Delta k_{ji} = k_j - k_i > 0 \quad \forall\, j > i$

Similarly, difference b/w two values of $y_k$'s at instances $i$ and $j$ is expressed as

$$\Delta y_{ji} = Y[k_j] - Y[k_i] = y_j - y_i \geq 0 \;\forall\, j > i$$

Going back to (0):

~~$f_{Y_1 \ldots Y_n} = {}^{k_1}C_{y_1} \; {}^{\Delta k_{21}}C_{\Delta y_{21}}$~~

$$f_{Y_1 \ldots Y_n} = \binom{k_1}{y_1}(0.6)^{k_1 - y_1}(0.4)^{y_1} \cdot \binom{\Delta k_{21}}{\Delta y_{21}}(0.6)^{\Delta k_{21} - \Delta y_{21}} \cdot (0.4)^{\Delta y_{21}}$$

$$\ldots \binom{\Delta k_{n(n-1)}}{\Delta y_{n(n-1)}}(0.6)^{\Delta k_{n(n-1)} - \Delta y_{n(n-1)}} \cdot (0.4)^{\Delta y_{n(n-1)}}$$

$$\ldots \delta(Y[k_1]=y_1) \cdot \delta(Y(k_2)=y_2) \ldots \delta(Y(k_n)=y_n)$$

$\boxed{3(d)}$ $\boxed{3.4}$

or
$$\boxed{f_{Y_1 \ldots Y_n} = \binom{k_1}{y_1}\binom{\Delta k_{21}}{\Delta y_{21}} \cdots \binom{\Delta k_{n(n-1)}}{\Delta y_{n(n-1)}} \cdot (0.6)^{k_n - y_n} \cdot (0.4)^{y_n} \cdot \delta(Y[k_1] - y_1) \cdot \delta(Y[k_2] - y_2) \cdots \delta(Y[k_n] - y_n)}$$

where $y_1 \le y_2 \le y_3 \ldots \le y_n$

$k_1 < k_2 < k_3 \ldots < k_n$

$\Delta y_{ji} = Y[K_j] - Y[K_i] \ge 0 \ \forall \ j > i$

$\Delta k_{ji} = k_j - k_i > 0 \ \forall \ j > i$

Ans

$\boxed{3(e)}$

3(e)
$$\boxed{E[Y(K)] = \sum_{y=0}^{K} \left[ {}^{K}C_y \ (0.6)^{K-y} (0.4)^{y} \cdot y \right]}$$
Ans

3(f)
$$R_{YY}(K,T) = E[Y[K] \cdot Y(T)]$$

~~or $R_{YY}(K,T)$~~

and
$$C_{YY}(K,T) = R_{YY}(K,T) - E[Y(K)] E[Y(T)]$$

$$R_{YY}(K,T) = \sum_{y_K=0}^{y_K=K} \sum_{\Delta y=0}^{\Delta y=\Delta K} \left[ y_K (y_K + \Delta y) \binom{K}{y_K} \binom{\Delta K}{\Delta y} (0.6)^{K+\Delta K - (y_K + \Delta y)} \cdot (0.4)^{(y_K + \Delta y)} \right]$$

$$C_{YY}(K,T) = \sum_{y_K=0}^{y_K=K} \sum_{\Delta y=0}^{\Delta y=\Delta K} \left[ \left\{ y_K (y_K + \Delta y) \binom{K}{y_K} \binom{\Delta K}{\Delta y} (0.6)^{(K+\Delta K - (y_K+\Delta y))} \cdot (0.4)^{(y_K+\Delta y)} \right\} - \left\{ \left( {}^{K}C_{y_K} (0.6)^{K-y_K} (0.4)^{y_K} \cdot y_K \right) \cdot \binom{\Delta K}{\Delta y} (0.6)^{\Delta K - \Delta y} (0.4) \right. \right.$$

3.5

3.5

| $Y[k]$ | $Y[\tau]$ | $Y[k]\cdot Y[\tau]$ | $f_{Y(k),Y(\tau)} = f_{Y(k)} \cdot f_{Y(\tau)\mid Y(k)}(Y(\tau)\mid Y(k))$ |
|---|---|---|---|
| 0 | 0 | 0 | |
| 0 | 1 | 0 | $\binom{k}{y_k}(0.6)^{k-y_k}\cdot(0.4)^{y_k} \cdot \sum_{\Delta y=0}^{\Delta y=\Delta k}(0.6)^{\Delta k-\Delta y}\cdot(0.4)^{\Delta y}$ |
| ⋮ | ⋮ | ⋮ | $y_k=0$ |
| 0 | $\Delta k$ | 0 | $\forall\, y_k = 0 \ldots k$ |
| 1 | 1 | 1 | |
| 1 | 2 | 2 | |
| ⋮ | ⋮ | ⋮ | |
| 1 | $1+\Delta k$ | $1(1+\Delta k)$ | |
| ⋮ | ⋮ | ⋮ | |
| k | k | $k\cdot k$ | |
| k | k+1 | $k(k+1)$ | |
| ⋮ | ⋮ | ⋮ | |
| k | $k+\Delta k$ | $k(k+\Delta k)$ | |

3(ii)

$$\boxed{R_{XX}(k, k+\Delta k) = \sum_{y_k=0}^{y_k=k}\sum_{\Delta y=0}^{\Delta y=\Delta k}\left\{(y_k)(y_k+\Delta y)\binom{k}{y_k}\binom{\Delta k}{\Delta y}\cdot(0.6)^{\{(k+\Delta k)-(y+\Delta y)\}}\cdot(0.4)^{\{(y+\Delta y)\}}\right\}}$$

where,
$k+\Delta k=\tau$ & $y_k = Y(k)$ $y_k+\Delta y = Y(\tau)$ $\Delta k > 0$ $\Delta y \geq 0$ All are $\in Z^+$
$(\Delta y \leq \Delta k)$

Ans

3.6

3(e)(iii)

$$C_{xx}(k, k+\Delta k) = R_{xx}(k, k+\Delta k) - \left[\sum_{y_k=0}^{y_k=k} \left\{ \binom{k}{y_k} (0.6)^{k-y_k} \cdot (0.4)^{y_k} \cdot y_k \right\}\right] \cdot \left[\sum_{\Delta y_k=0}^{\Delta y_k=\Delta k} \left\{ \binom{\Delta k}{\Delta y} (0.6)^{\Delta k-\Delta y} (0.4)^{\Delta y} \cdot \Delta y \right\}\right]$$

where $k+\Delta k = \tau$, $\Delta k \geq 0$,

$y_k = Y(k)$, $y_k + \Delta y = Y[\tau]$, $\cancel{\Delta y \geq 0}$ $\Delta y \in [0, \Delta k]$

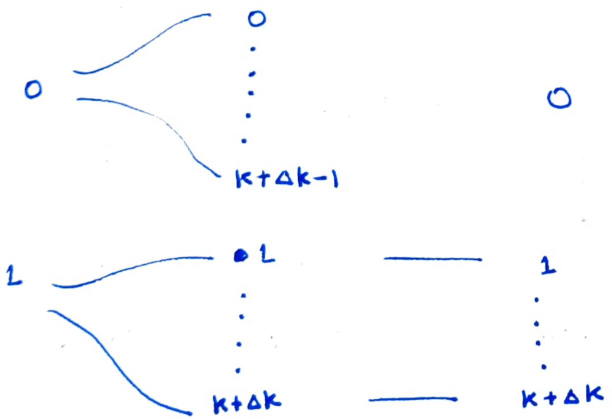
All variables $\in Z^+$

Ans

3(f) $R_{XY}(k, \tau) = E[X(k)Y(\tau)] \leq$

$X[k]$ $Y[k+\Delta k]$ $X(k)Y(k+\Delta k)$ $f_{X,Y}(X[k]=x, Y[k+\Delta k])$

$= f_{Y_{k+\Delta k}|X_k, Y_k}(Y_{k+\Delta k} | Y_k, X_k) \cdot P(X_k) \cdot f_{Y_k}$




Don't care as Not needed for E $R_{XY}$.

$$\sum^{k} \binom{k}{y_k} \cdot (0.6)^{k-y_k} \cdot (0.4)^{y_k} \cdot (0.4) \cdot \binom{\Delta k - 1}{\Delta y - 1} (0.4)^{\Delta y - 1} (0.6)^{\Delta k - \Delta y}$$

3(f)

3.7

$$R_{XY}(k, \tau) = \sum_{y_k=0}^{y_k=k} \sum_{\Delta y=1}^{\Delta y=\Delta k} \left\{ \binom{k}{y_k} \binom{\Delta k-1}{\Delta y-1} \cdot (0.6)^{\{(k+\Delta k)-(y_k+\Delta y)\}} \cdot (0.4)^{\{(y_k+\Delta y)\}} \cdot (y_k+\Delta y) \cdot 1 \right\}$$

where,

$k+\Delta k = \tau$, $\Delta k > 0$,

$y_k = Y[k]$,

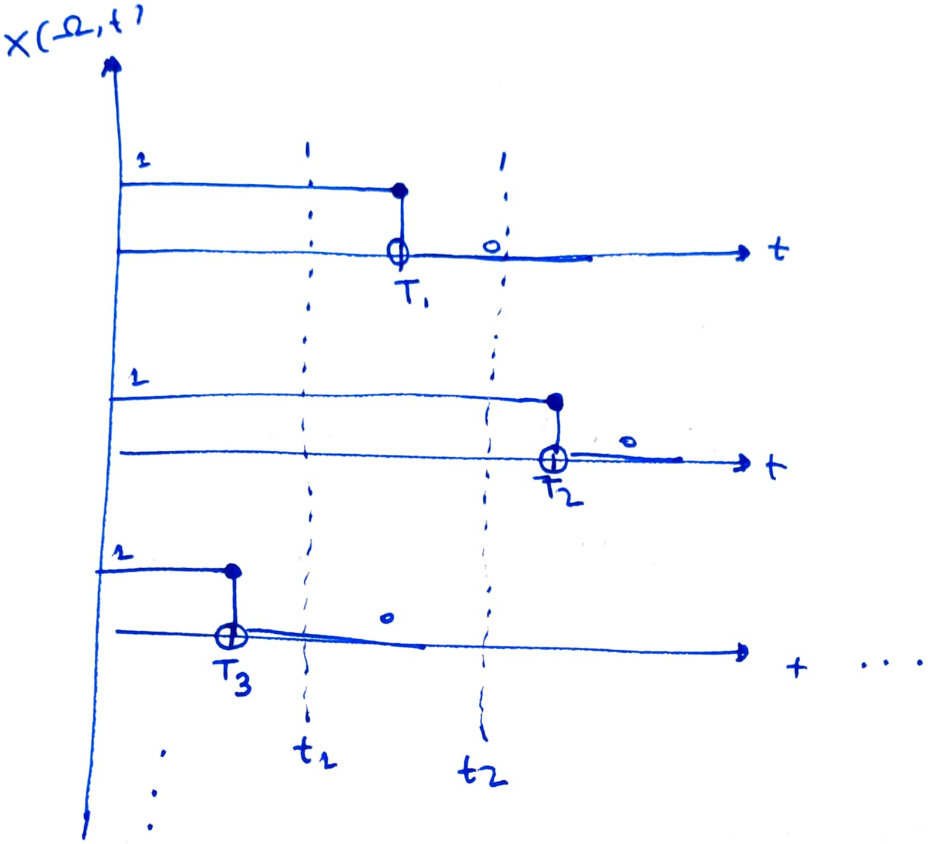
$y_k + \Delta y = Y[\tau]$, $\Delta y \in [0, \Delta k]$

All variables are $\in Z^+$

Ans

$$T \sim \exp(1)$$

$$X(T,t) = \begin{cases} 1 & t \leq T \\ 0 & t > T \end{cases}$$




$$f_{x_1,x_2}(x_1,x_2) = ?$$

$$(x_1,x_2) = \begin{cases} (1,1) & (t_2 < T) \\ (1,0) & \cancel{t_2}\,(T \in (t_1,t_2)) \\ (0,0) & t_1 > T \end{cases} \quad \text{where } t_1 < t_2$$

or

$$pmf(x_1,x_2) = \begin{cases} P(T > t_2), & (x_1,x_2) = (1,1) \\ P(T \in (t_1,t_2)), & (x_1,x_2) = (1,0) \\ P(T < t_1), & (x_1,x_2) = (0,0) \end{cases}$$

$\boxed{4.2}$

or $$pmf(x_1,x_2) = \begin{cases} \displaystyle\int_{T=t_2}^{\infty} e^{-T}\,dT & (x_1,x_2) = (1,1) \\ \displaystyle\int_{T=t_1}^{T=t_2} e^{-T}\,dT & (x_1,x_2) = (1,0) \\ \displaystyle\int_{T=0}^{T=t_1} e^{-T}\,dT & (x_1,x_2) = (0,0) \end{cases}$$

or $$pmf(x_1,x_2) = \begin{cases} e^{-t_2} & (x_1,x_2) = (1,1) \\ e^{-t_1} - e^{-t_2} & (x_1,x_2) = (1,0) \\ 1 - e^{-t_1} & (x_1,x_2) = (0,0) \end{cases}$$

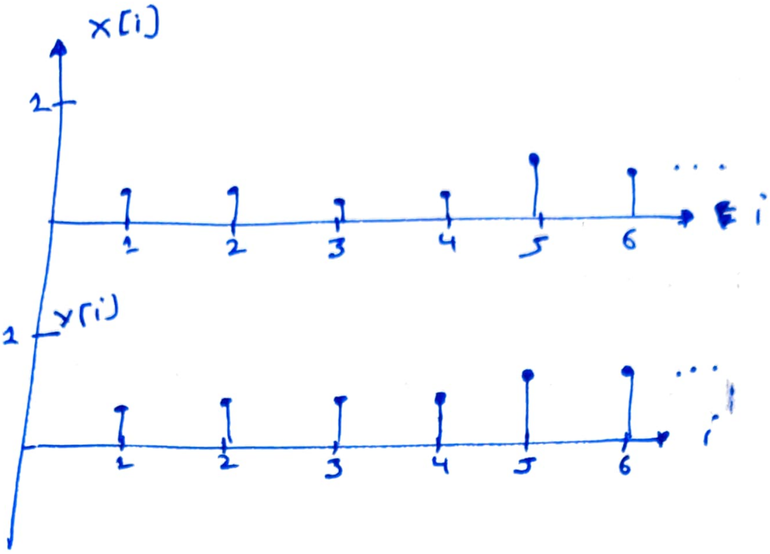
$\boxed{4}$

$$\Rightarrow \boxed{\begin{aligned} & f_{X_1,X_2}(X(t_1)=x_1, X(t_2)=x_2) \\ &= (e^{-t_2})\,\delta(x(t_1)-1)\,\delta(x(t_2)-1) \\ &\quad + (e^{-t_1} - e^{-t_2})\,\delta(x(t_1)-1).\,\delta(x(t_2)) \\ &\quad + (1 - e^{-t_1})\,\delta(x(t_1)).\,\delta(x(t_2)) \end{aligned}}$$

Ans

— x — x — x —

$\boxed{5.}$




$X[K] \sim \text{unif}(0,1) \quad \forall K$

$\Rightarrow \lim_{K\to\infty} X[K] \sim \text{unif}(0,1)$ i.e. NOT a constant value.

$\Rightarrow \lim_{K\to\infty} P\left(|X[K] = \cancel{K}\,\pi| \le \varepsilon\right) = 1$

is NOT possible.

$\boxed{5(a)^{(i)}}$

$\Rightarrow$ $\boxed{X \text{ D.N.C. in Probability.!}}$ Ans

$\boxed{5(a)^{(ii)}}$

$\boxed{X \text{ D.N.C. in mean square sense !}}$ Ans

as mean square sense convergence is a subset of convergence in probability.

$f_Y(Y[i]=y) = ?$

use CDF method:

$$F_Y(y) = P(Y[i] \le y)$$

or $F_Y(y) = P(X[1] \le y) \cdot P(X[1] \le y] \ldots\ldots P(X[i] \le y)$

or $F_Y(y) = y^i$

$$\Rightarrow \boxed{f_Y(Y[i]=y) = i\,y^{i-1}} = \begin{cases} 0 & y \in [0,1) \\ 1 & y = 1 \end{cases}$$

~~$F_Y(Y[i]) \le$~~

as $\boxed{\lim_{i\to\infty} Y[i] = \begin{cases} 1 & \text{w.p. } 1 \\ 0 & \text{w.p. } 0 \end{cases}}$ $\Rightarrow E[Y[i]] = 1$ as $i\to\infty$, $E[Y[i]^2] = 1$ as $i\to\infty$

5(b)(i)

$\Rightarrow$ $\boxed{Y \text{ converges in Probability to } \bar{y} = 1}$ Ans

$$\lim_{i\to\infty} E[(Y_{(i)}-\bar{y})^2] = \lim_{i\to\infty}\left(E[Y_{(i)}^2] + \bar{y}^2 - 2\bar{y}\,E[Y(i)]\right)$$

or $\lim_{i\to\infty} E[(Y[i]-\bar{y})^2) = 1 + \bar{y}^2 - 2\bar{y}$

or $\Rightarrow \lim_{i\to\infty} E[(Y[i]-\bar{y})^2) = 0$ if $\bar{y} = 1$

$\Rightarrow$ $\boxed{Y \text{ converges in mean square sense to } \bar{y} = 1}$ Ans

———— X ———— X ———— X ————