Exercise 1.1

Question 1:

Determine whether each of the following relations are reflexive, symmetric and transitive:

(i)Relation R in the set $A = \{1, 2, 3 \ldots 13, 14\}$ defined as

$R = \{(x, y): 3x - y = 0\}$

(ii) Relation R in the set **N** of natural numbers defined as

$R = \{(x, y): y = x + 5 \text{ and } x < 4\}$

(iii) Relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ as

R = {(*x*, *y*): *y* is divisible by *x*}

(iv) Relation R in the set **Z** of all integers defined as

R = {(*x*, *y*): $x - y$ is as integer}

(v) Relation R in the set *A* of human beings in a town at a particular time given by

(a) R = {(*x*, *y*): *x* and *y* work at the same place}

(b) R = {(*x*, *y*): *x* and *y* live in the same locality}

(c) R = {(*x*, *y*): *x* is exactly 7 cm taller than *y*}

(d) R = {(*x*, *y*): *x* is wife of *y*}

(e) R = {(*x*, *y*): *x* is father of *y*}

Answer

(i) $A = \{1, 2, 3 \ldots 13, 14\}$

$R = \{(x, y): 3x - y = 0\}$

$\therefore R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$

R is not reflexive since $(1, 1), (2, 2) \ldots (14, 14) \notin R$.

Also, R is not symmetric as $(1, 3) \in R$, but $(3, 1) \notin R$. $[3(3) - 1 \neq 0]$

Also, R is not transitive as $(1, 3), (3, 9) \in R$, but $(1, 9) \notin R$.

$[3(1) - 9 \neq 0]$

Hence, R is neither reflexive, nor symmetric, nor transitive.

(ii) $R = \{(x, y): y = x + 5 \text{ and } x < 4\} = \{(1, 6), (2, 7), (3, 8)\}$

It is seen that $(1, 1) \notin R$.

$\therefore R$ is not reflexive.

$(1, 6) \in R$

But,

$(1, 6) \notin$ R.

$\therefore$R is not symmetric.

Now, since there is no pair in R such that (x, y) and $(y, z) \in$ R so, we need not look for the orderd pair (x,z) in R.

$\therefore$ R is transitive

Hence, R is neither reflexive, nor symmetric but it is transitive.

(iii) $A = \{1, 2, 3, 4, 5, 6\}$

R = $\{(x, y)$: y is divisible by $x\}$

We know that any number (x) is divisible by itself.

$\Rightarrow (x, x) \in$R

$\therefore$R is reflexive.

Now,

$(2, 4) \in$R [as 4 is divisible by 2]

But,

$(4, 2) \notin$ R. [as 2 is not divisible by 4]

$\therefore$R is not symmetric.

Let $(x, y), (y, z) \in$ R. Then, y is divisible by x and z is divisible by y.

$\therefore z$ is divisible by x.

$\Rightarrow (x, z) \in$R

$\therefore$R is transitive.

Hence, R is reflexive and transitive but not symmetric.

(iv) R = $\{(x, y)$: $x - y$ is an integer$\}$

Now, for every $x \in \mathbf{Z}$, $(x, x) \in$R as $x - x = 0$ is an integer.

$\therefore$R is reflexive.

Now, for every $x, y \in \mathbf{Z}$ if $(x, y) \in$ R, then $x - y$ is an integer.

$\Rightarrow -(x - y)$ is also an integer.

$\Rightarrow (y - x)$ is an integer.

$\therefore (y, x) \in$ R

$\therefore$R is symmetric.

Now,

Let (x, y) and $(y, z) \in$R, where $x, y, z \in \mathbf{Z}$.

$\Rightarrow (x - y)$ and $(y - z)$ are integers.

$\Rightarrow x - z = (x - y) + (y - z)$ is an integer.

$\therefore (x, z) \in R$

$\therefore$R is transitive.

Hence, R is reflexive, symmetric, and transitive.

(v) (a) R = {(x, y): x and y work at the same place}

$\Rightarrow (x, x) \in R$

$\therefore$ R is reflexive.

If $(x, y) \in R$, then x and y work at the same place.

$\Rightarrow$ y and x work at the same place.

$\Rightarrow (y, x) \in R$.

$\therefore$R is symmetric.

Now, let $(x, y), (y, z) \in R$

$\Rightarrow$ x and y work at the same place and y and z work at the same place.

$\Rightarrow$ x and z work at the same place.

$\Rightarrow (x, z) \in R$

$\therefore$ R is transitive.

Hence, R is reflexive, symmetric, and transitive.

(b) R = {(x, y): x and y live in the same locality}

Clearly $(x, x) \in R$ as x and x is the same human being.

$\therefore$ R is reflexive.

If $(x, y) \in R$, then x and y live in the same locality.

$\Rightarrow$ y and x live in the same locality.

$\Rightarrow (y, x) \in R$

$\therefore$R is symmetric.

Now, let $(x, y) \in R$ and $(y, z) \in R$.

$\Rightarrow$ x and y live in the same locality and y and z live in the same locality.

$\Rightarrow$ x and z live in the same locality.

$\Rightarrow (x, z) \in R$

$\therefore$ R is transitive.

Hence, R is reflexive, symmetric, and transitive.

(c) R = {(x, y): x is exactly 7 cm taller than y}

Now,

$(x, x) \notin R$

Since human being x cannot be taller than himself.

∴R is not reflexive.

Now, let $(x, y) \in$R.

⇒ x is exactly 7 cm taller than y.

Then, y is not taller than x.

∴ $(y, x) \notin$R

Indeed if x is exactly 7 cm taller than y, then y is exactly 7 cm shorter than x.

∴R is not symmetric.

Now,

Let $(x, y), (y, z) \in$ R.

⇒ x is exactly 7 cm taller than y and y is exactly 7 cm taller than z.

⇒ x is exactly 14 cm taller than z .

∴$(x, z) \notin$R

∴ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(d) R = $\{(x, y)$: x is the wife of $y\}$

Now,

$(x, x) \notin$ R

Since x cannot be the wife of herself.

∴R is not reflexive.

Now, let $(x, y) \in$ R

⇒ x is the wife of y.

Clearly y is not the wife of x.

∴$(y, x) \notin$ R

Indeed if x is the wife of y, then y is the husband of x.

∴ R is not transitive.

Let $(x, y), (y, z) \in$ R

⇒ x is the wife of y and y is the wife of z.

This case is not possible. Also, this does not imply that x is the wife of z.

∴$(x, z) \notin$ R

∴R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(e) R = $\{(x, y)$: x is the father of $y\}$

$(x, x) \notin$ R

As x cannot be the father of himself.

$\therefore$R is not reflexive.

Now, let $(x, y) \in$R.

$\Rightarrow$ x is the father of y.

$\Rightarrow$ y cannot be the father of y.

Indeed, y is the son or the daughter of y.

$\therefore(y, x) \notin$ R

$\therefore$ R is not symmetric.

Now, let $(x, y) \in$ R and $(y, z) \in$ R.

$\Rightarrow$ x is the father of y and y is the father of z.

$\Rightarrow$ x is not the father of z.

Indeed x is the grandfather of z.

$\therefore$ $(x, z) \notin$ R

$\therefore$R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 2:

Show that the relation R in the set **R** of real numbers, defined as

R = $\{(a, b): a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.

Answer

R = $\{(a, b): a \leq b^2\}$

It can be observed that $\left(\frac{1}{2}, \frac{1}{2}\right) \notin \mathbf{R}$, since $\frac{1}{2} > \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

$\therefore$R is not reflexive.

Now, $(1, 4) \in$ R as $1 < 4^2$

But, 4 is not less than 1^2.

$\therefore(4, 1) \notin$ R

$\therefore$R is not symmetric.

Now,

$(3, 2), (2, 1.5) \in$ R

(as $3 < 2^2 = 4$ and $2 < (1.5)^2 = 2.25$)

But, $3 > (1.5)^2 = 2.25$

∴$(3, 1.5) \notin R$

∴ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 3:

Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as

R = $\{(a, b): b = a + 1\}$ is reflexive, symmetric or transitive.

Answer

Let $A = \{1, 2, 3, 4, 5, 6\}$.

A relation R is defined on set A as:

R = $\{(a, b): b = a + 1\}$

∴R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}

We can find $(a, a) \notin R$, where $a \in A$.

For instance,

$(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) \notin R$

∴R is not reflexive.

It can be observed that $(1, 2) \in R$, but $(2, 1) \notin R$.

∴R is not symmetric.

Now, $(1, 2), (2, 3) \in \mathbf{R}$

But,

$(1, 3) \notin R$

∴R is not transitive

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 4:

Show that the relation R in **R** defined as R = $\{(a, b): a \leq b\}$, is reflexive and transitive but not symmetric.

Answer

R = $\{(a, b); a \leq b\}$

Clearly $(a, a) \in R$ as $a = a$.

∴R is reflexive.

Now,

$(2, 4) \in R$ (as $2 < 4$)

But, $(4, 2) \notin R$ as 4 is greater than 2.

$\therefore$ R is not symmetric.

Now, let $(a, b), (b, c) \in R$.

Then,

$a \le b$ and $b \le c$

$\Rightarrow a \le c$

$\Rightarrow (a, c) \in R$

$\therefore$R is transitive.

Hence,R is reflexive and transitive but not symmetric.

Question 5:

Check whether the relation R in **R** defined as $R = \{(a, b): a \le b^3\}$ is reflexive, symmetric or transitive.

Answer

$R = \{(a, b): a \le b^3\}$

It is observed that $\left(\frac{1}{2}, \frac{1}{2}\right) \notin R$ as $\frac{1}{2} > \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

$\therefore$ R is not reflexive.

Now,

$(1, 2) \in R$ (as $1 < 2^3 = 8$)

But,

$(2, 1) \notin R$ (as $2^3 > 1$)

$\therefore$ R is not symmetric.

We have $\left(3, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{6}{5}\right) \in R$ as $3 < \left(\frac{3}{2}\right)^3$ and $\frac{3}{2} < \left(\frac{6}{5}\right)^3$.

But $\left(3, \frac{6}{5}\right) \notin R$ as $3 > \left(\frac{6}{5}\right)^3$.

$\therefore$ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 6:

Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric but neither reflexive nor transitive.

Answer

Let A = {1, 2, 3}.

A relation R on A is defined as R = {(1, 2), (2, 1)}.

It is seen that (1, 1), (2, 2), (3, 3) $\notin$R.

$\therefore$ R is not reflexive.

Now, as (1, 2) $\in$ R and (2, 1) $\in$ R, then R is symmetric.

Now, (1, 2) and (2, 1) $\in$ R

However,

(1, 1) $\notin$ R

$\therefore$ R is not transitive.

Hence, R is symmetric but neither reflexive nor transitive.

Question 7:

Show that the relation R in the set A of all the books in a library of a college, given by R = {(x, y): x and y have same number of pages} is an equivalence relation.

Answer

Set A is the set of all books in the library of a college.

R = {x, y): x and y have the same number of pages}

Now, R is reflexive since (x, x) $\in$ R as x and x has the same number of pages.

Let (x, y) $\in$ R $\Rightarrow$ x and y have the same number of pages.

$\Rightarrow$ y and x have the same number of pages.

$\Rightarrow$ (y, x) $\in$ R

$\therefore$R is symmetric.

Now, let (x, y) $\in$R and (y, z) $\in$ R.

$\Rightarrow$ x and y and have the same number of pages and y and z have the same number of pages.

$\Rightarrow$ x and z have the same number of pages.

$\Rightarrow$ (x, z) $\in$ R

$\therefore$R is transitive.

Hence, R is an equivalence relation.

Question 8:

Show that the relation R in the set A = {1, 2, 3, 4, 5} given by

$\mathrm{R}=\{(a,\ b):|a-b|\ \text{is even}\}$, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of 2, 4}.

Answer

A = {1, 2, 3, 4, 5}

$\mathrm{R}=\{(a,\ b):|a-b|\ \text{is even}\}$

It is clear that for any element $a \in A$, we have $|a-a|=0$ (which is even).

∴R is reflexive.

Let $(a, b) \in$ R.

$\Rightarrow |a-b|$ is even.

$\Rightarrow |-(a-b)|=|b-a|$ is also even.

$\Rightarrow (b,\ a)\in \mathrm{R}$

∴R is symmetric.

Now, let $(a, b) \in$ R and $(b, c) \in$ R.

$\Rightarrow |a-b|$ is even and $|b-c|$ is even.

$\Rightarrow (a-b)$ is even and $(b-c)$ is even.

$\Rightarrow (a-c)=(a-b)+(b-c)$ is even. [Sum of two even integers is even]

$\Rightarrow |a-c|$ is even.

$\Rightarrow (a, c) \in$ R

∴R is transitive.

Hence, R is an equivalence relation.

Now, all elements of the set {1, 2, 3} are related to each other as all the elements of this subset are odd. Thus, the modulus of the difference between any two elements will be even.

Similarly, all elements of the set {2, 4} are related to each other as all the elements of this subset are even.

Also, no element of the subset {1, 3, 5} can be related to any element of {2, 4} as all elements of {1, 3, 5} are odd and all elements of {2, 4} are even. Thus, the modulus of the difference between the two elements (from each of these two subsets) will not be even.

Question 9:

Show that each of the relation R in the set $\mathrm{A}=\{x \in \mathbf{Z}: 0 \leq x \leq 12\}$, given by

(i) $\mathrm{R}=\{(a,\ b):|a-b| \text{ is a multiple of } 4\}$

(ii) $\mathrm{R}=\{(a,\ b): a=b\}$

is an equivalence relation. Find the set of all elements related to 1 in each case.

Answer

$A=\{x \in \mathbf{Z}: 0 \leq x \leq 12\}=\{0,\ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

(i) $\mathrm{R}=\{(a,\ b):|a-b| \text{ is a multiple of } 4\}$

For any element a ∈A, we have (a, a) ∈ R as $|a-a|=0$ is a multiple of 4.

∴R is reflexive.

Now, let (a, b) ∈ R ⇒ $|a-b|$ is a multiple of 4.

$\Rightarrow|-(a-b)|=|b-a|$ is a multiple of 4.

⇒ (b, a) ∈ R

∴R is symmetric.

Now, let (a, b), (b, c) ∈ R.

$\Rightarrow|a-b|$ is a multiple of 4 and $|b-c|$ is a multiple of 4.

$\Rightarrow(a-b)$ is a multiple of 4 and $(b-c)$ is a multiple of 4.

$\Rightarrow(a-c)=(a-b)+(b-c)$ is a multiple of 4.

$\Rightarrow|a-c|$ is a multiple of 4.

⇒ (a, c) ∈R

∴ R is transitive.

Hence, R is an equivalence relation.

The set of elements related to 1 is {1, 5, 9} since

$|1-1| = 0$ is a multiple of 4,

$|5-1| = 4$ is a multiple of 4, and

$|9-1| = 8$ is a multiple of 4.

(ii) R = {(a, b): a = b}

For any element a ∈A, we have (a, a) ∈ R, since a = a.

∴R is reflexive.

Now, let (a, b) ∈ R.

⇒ a = b

⇒ b = a

⇒ (b, a) ∈ R

∴R is symmetric.

Now, let (a, b) ∈ R and (b, c) ∈ R.

⇒ a = b and b = c

⇒ a = c

⇒ (a, c) ∈ R

∴ R is transitive.

Hence, R is an equivalence relation.

The elements in R that are related to 1 will be those elements from set A which are equal to 1.

Hence, the set of elements related to 1 is {1}.

Question 10:

Given an example of a relation. Which is

(i) Symmetric but neither reflexive nor transitive.

(ii) Transitive but neither reflexive nor symmetric.

(iii) Reflexive and symmetric but not transitive.

(iv) Reflexive and transitive but not symmetric.

(v) Symmetric and transitive but not reflexive.

Answer

(i) Let A = {5, 6, 7}.

Define a relation R on A as R = {(5, 6), (6, 5)}.

Relation R is not reflexive as (5, 5), (6, 6), (7, 7) ∉ R.

Now, as (5, 6) ∈ R and also (6, 5) ∈ R, R is symmetric.

⇒ (5, 6), (6, 5) ∈ R, but (5, 5) ∉ R

∴R is not transitive.

Hence, relation R is symmetric but not reflexive or transitive.

(ii)Consider a relation R in **R** defined as:

R = {(a, b): $a < b$}

For any a ∈ R, we have (a, a) ∉ R since a cannot be strictly less than a itself. In fact, a = a.

∴ R is not reflexive.

Now,

(1, 2) ∈ R (as 1 < 2)

But, 2 is not less than 1.

∴ (2, 1) ∉ R

∴ R is not symmetric.

Now, let (a, b), (b, c) ∈ R.

⇒ $a < b$ and $b < c$

⇒ $a < c$

⇒ (a, c) ∈ R

∴R is transitive.

Hence, relation R is transitive but not reflexive and symmetric.

(iii)Let A = {4, 6, 8}.

Define a relation R on A as:

A = {(4, 4), (6, 6), (8, 8), (4, 6), (6, 4), (6, 8), (8, 6)}

Relation R is reflexive since for every $a \in A$, (a, a) ∈R i.e., (4, 4), (6, 6), (8, 8)} ∈ R.

Relation R is symmetric since (a, b) ∈ R ⇒ (b, a) ∈ R for all a, b ∈ R.

Relation R is not transitive since (4, 6), (6, 8) ∈ R, but (4, 8) ∉ R.

Hence, relation R is reflexive and symmetric but not transitive.

(iv) Define a relation R in **R** as:

R = {a, b): $a^3 \geq b^3$}

Clearly (a, a) ∈ R as $a^3 = a^3$.

∴R is reflexive.

Now,

(2, 1) ∈ R (as $2^3 \geq 1^3$)

But,

$(1, 2) \notin R$ (as $1^3 < 2^3$)

∴ R is not symmetric.

Now,

Let $(a, b), (b, c) \in R$.

$\Rightarrow a^3 \geq b^3$ and $b^3 \geq c^3$

$\Rightarrow a^3 \geq c^3$

$\Rightarrow (a, c) \in R$

∴R is transitive.

Hence, relation R is reflexive and transitive but not symmetric.

(v) Let $A = \{-5, -6\}$.

Define a relation R on A as:

$R = \{(-5, -6), (-6, -5), (-5, -5)\}$

Relation R is not reflexive as $(-6, -6) \notin R$.

Relation R is symmetric as $(-5, -6) \in R$ and $(-6, -5\}\in R$.

It is seen that $(-5, -6), (-6, -5) \in R$. Also, $(-5, -5) \in R$.

∴The relation R is transitive.

Hence, relation R is symmetric and transitive but not reflexive.

Question 11:

Show that the relation R in the set A of points in a plane given by R = {(P, Q): distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all point related to a point P ≠ (0, 0) is the circle passing through P with origin as centre.

Answer

R = {(P, Q): distance of point P from the origin is the same as the distance of point Q from the origin}

Clearly, $(P, P) \in R$ since the distance of point P from the origin is always the same as the distance of the same point P from the origin.

∴R is reflexive.

Now,

Let $(P, Q) \in R$.

⇒ The distance of point P from the origin is the same as the distance of point Q from the origin.

⇒ The distance of point Q from the origin is the same as the distance of point P from the origin.

⇒ (Q, P) ∈ R

∴R is symmetric.

Now,

Let (P, Q), (Q, S) ∈ R.

⇒ The distance of points P and Q from the origin is the same and also, the distance of points Q and S from the origin is the same.

⇒ The distance of points P and S from the origin is the same.

⇒ (P, S) ∈ R

∴R is transitive.

Therefore, R is an equivalence relation.

The set of all points related to P ≠ (0, 0) will be those points whose distance from the origin is the same as the distance of point P from the origin.

In other words, if O (0, 0) is the origin and OP = k, then the set of all points related to P is at a distance of k from the origin.

Hence, this set of points forms a circle with the centre as the origin and this circle passes through point P.

Question 12:

Show that the relation R defined in the set A of all triangles as R = $\{(T_1, T_2)$: T_1 is similar to $T_2\}$, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

Answer

R = $\{(T_1, T_2)$: T_1 is similar to $T_2\}$

R is reflexive since every triangle is similar to itself.

Further, if (T_1, T_2) ∈ R, then T_1 is similar to $T_{2.}$

⇒ T_2 is similar to $T_{1.}$

⇒ (T_2, T_1) ∈R

∴R is symmetric.

Now,

Let $(T_1, T_2), (T_2, T_3) \in$ R.

$\Rightarrow T_1$ is similar to T_2 and T_2 is similar to T_3.

$\Rightarrow T_1$ is similar to T_3.

$\Rightarrow (T_1, T_3) \in$ R

$\therefore$ R is transitive.

Thus, R is an equivalence relation.

Now, we can observe that:

$$\frac{3}{6}=\frac{4}{8}=\frac{5}{10}\left(=\frac{1}{2}\right)$$

$\therefore$The corresponding sides of triangles T_1 and T_3 are in the same ratio.

Then, triangle T_1 is similar to triangle T_3.

Hence, T_1 is related to T_3.

Question 13:

Show that the relation R defined in the set A of all polygons as R = $\{(P_1, P_2)$: P_1 and P_2 have same number of sides$\}$, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

Answer

R = $\{(P_1, P_2)$: P_1 and P_2 have same the number of sides$\}$

R is reflexive since $(P_1, P_1) \in$ R as the same polygon has the same number of sides with itself.

Let $(P_1, P_2) \in$ R.

$\Rightarrow P_1$ and P_2 have the same number of sides.

$\Rightarrow P_2$ and P_1 have the same number of sides.

$\Rightarrow (P_2, P_1) \in$ R

$\therefore$R is symmetric.

Now,

Let $(P_1, P_2), (P_2, P_3) \in$ R.

$\Rightarrow P_1$ and P_2 have the same number of sides. Also, P_2 and P_3 have the same number of sides.

$\Rightarrow P_1$ and P_3 have the same number of sides.

$\Rightarrow (P_1, P_3) \in$ R

∴R is transitive.

Hence, R is an equivalence relation.

The elements in *A* related to the right-angled triangle (*T*) with sides 3, 4, and 5 are those polygons which have 3 sides (since *T* is a polygon with 3 sides).

Hence, the set of all elements in *A* related to triangle *T* is the set of all triangles.

Question 14:

Let L be the set of all lines in XY plane and R be the relation in L defined as R = $\{(L_1, L_2)$: L_1 is parallel to $L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

Answer

R = $\{(L_1, L_2)$: L_1 is parallel to $L_2\}$

R is reflexive as any line L_1 is parallel to itself i.e., $(L_1, L_1) \in$ R.

Now,

Let $(L_1, L_2) \in$ R.

$\Rightarrow L_1$ is parallel to L_2.

$\Rightarrow L_2$ is parallel to L_1.

$\Rightarrow (L_2, L_1) \in$ R

∴ R is symmetric.

Now,

Let $(L_1, L_2), (L_2, L_3) \in$R.

$\Rightarrow L_1$ is parallel to L_2. Also, L_2 is parallel to L_3.

$\Rightarrow L_1$ is parallel to L_3.

∴R is transitive.

Hence, R is an equivalence relation.

The set of all lines related to the line $y = 2x + 4$ is the set of all lines that are parallel to the line $y = 2x + 4$.

Slope of line $y = 2x + 4$ is $m = 2$

It is known that parallel lines have the same slopes.

The line parallel to the given line is of the form $y = 2x + c$, where $c \in \mathbf{R}$.

Hence, the set of all lines related to the given line is given by $y = 2x + c$, where $c \in \mathbf{R}$.

Question 15:

Let R be the relation in the set {1, 2, 3, 4} given by R = {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}. Choose the correct answer.

(A) R is reflexive and symmetric but not transitive.

(B) R is reflexive and transitive but not symmetric.

(C) R is symmetric and transitive but not reflexive.

(D) R is an equivalence relation.

Answer

R = {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}

It is seen that $(a, a) \in$ **R,** for every $a \in \{1, 2, 3, 4\}$.

$\therefore$ R is reflexive.

It is seen that $(1, 2) \in R$, but $(2, 1) \notin R$.

$\therefore$R is not symmetric.

Also, it is observed that $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in \{1, 2, 3, 4\}$.

$\therefore$ R is transitive.

Hence, R is reflexive and transitive but not symmetric.

The correct answer is B.

Question 16:

Let R be the relation in the set **N** given by $R = \{(a, b): a = b - 2, b > 6\}$. Choose the correct answer.

(A) $(2, 4) \in R$ (B) $(3, 8) \in R$ (C) $(6, 8) \in R$ (D) $(8, 7) \in R$

Answer

$R = \{(a, b): a = b - 2, b > 6\}$

Now, since $b > 6$, $(2, 4) \notin R$

Also, as $3 \neq 8 - 2$, $(3, 8) \notin R$

And, as $8 \neq 7 - 2$

$\therefore (8, 7) \notin R$

Now, consider (6, 8).

We have $8 > 6$ and also, $6 = 8 - 2$.

$\therefore (6, 8) \in R$

The correct answer is C.

Exercise 1.2

Question 1:

Show that the function *f*: **R**$_*$ → **R**$_*$ defined by $f(x)=\frac{1}{x}$ is one-one and onto, where **R**$_*$ is the set of all non-zero real numbers. Is the result true, if the domain **R**$_*$ is replaced by **N** with co-domain being same as **R**$_*$?

Answer

It is given that *f*: **R**$_*$ → **R**$_*$ is defined by $f(x)=\frac{1}{x}$.

One-one:

$$f(x)=f(y)$$
$$\Rightarrow \frac{1}{x}=\frac{1}{y}$$
$$\Rightarrow x=y$$

∴*f* is one-one.

Onto:

It is clear that for *y*∈ **R**$_*$, there exists $x=\frac{1}{y}\in \mathrm{R}_*$ $(\text{Exists as } y\neq 0)$ such that

$$f(x)=\frac{1}{\left(\frac{1}{y}\right)}=y.$$

∴*f* is onto.

Thus, the given function (*f*) is one-one and onto.

Now, consider function *g*: **N** → **R**$_*$defined by

$$g(x)=\frac{1}{x}.$$

We have,

$$g(x_1)=g(x_2)\Rightarrow \frac{1}{x_1}=\frac{1}{x_2}\Rightarrow x_1=x_2$$

∴*g* is one-one.

Further, it is clear that g is not onto as for $1.2 \in \mathbf{R}_*$ there does not exit any x in **N** such that $g(x) = \frac{1}{1.2}$.

Hence, function g is one-one but not onto.

Question 2:

Check the injectivity and surjectivity of the following functions:

(i) f: $\mathbf{N} \to \mathbf{N}$ given by $f(x) = x^2$

(ii) f: $\mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^2$

(iii) f: $\mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$

(iv) f: $\mathbf{N} \to \mathbf{N}$ given by $f(x) = x^3$

(v) f: $\mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^3$

Answer

(i) f: $\mathbf{N} \to \mathbf{N}$ is given by,

$f(x) = x^2$

It is seen that for $x, y \in \mathbf{N}$, $f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in \mathbf{N}$. But, there does not exist any x in **N** such that $f(x) = x^2 = 2$.

$\therefore f$ is not surjective.

Hence, function f is injective but not surjective.

(ii) f: $\mathbf{Z} \to \mathbf{Z}$ is given by,

$f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

$\therefore f$ is not injective.

Now, $-2 \in \mathbf{Z}$. But, there does not exist any element $x \in \mathbf{Z}$ such that $f(x) = x^2 = -2$.

$\therefore f$ is not surjective.

Hence, function f is neither injective nor surjective.

(iii) f: $\mathbf{R} \to \mathbf{R}$ is given by,

$f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

$\therefore f$ is not injective.

Now, $-2 \in \mathbf{R}$. But, there does not exist any element $x \in \mathbf{R}$ such that $f(x) = x^2 = -2$.

∴ f is not surjective.

Hence, function f is neither injective nor surjective.

(iv) f: **N** → **N** given by,

$f(x) = x^3$

It is seen that for x, $y \in$ **N**, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

∴ f is injective.

Now, 2 ∈ **N**. But, there does not exist any element x in domain **N** such that $f(x) = x^3 = 2$.

∴ f is not surjective

Hence, function f is injective but not surjective.

(v) f: **Z** → **Z** is given by,

$f(x) = x^3$

It is seen that for x, $y \in$ **Z**, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

∴ f is injective.

Now, 2 ∈ **Z**. But, there does not exist any element x in domain **Z** such that $f(x) = x^3 = 2$.

∴ f is not surjective.

Hence, function f is injective but not surjective.

Question 3:

Prove that the Greatest Integer Function f: **R** → **R** given by $f(x) = [x]$, is neither one-once nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

Answer

f: **R** → **R** is given by,

$f(x) = [x]$

It is seen that $f(1.2) = [1.2] = 1$, $f(1.9) = [1.9] = 1$.

∴ $f(1.2) = f(1.9)$, but $1.2 \neq 1.9$.

∴ f is not one-one.

Now, consider 0.7 ∈ **R**.

It is known that $f(x) = [x]$ is always an integer. Thus, there does not exist any element $x \in$ **R** such that $f(x) = 0.7$.

∴ f is not onto.

Hence, the greatest integer function is neither one-one nor onto.

Question 4:

Show that the Modulus Function *f*: **R** → **R** given by $f(x)=|x|$, is neither one-one nor onto, where $|x|$ is *x*, if *x* is positive or 0 and $|x|$ is − *x*, if *x* is negative.

Answer

f: **R** → **R** is given by,

$$f(x)=|x|=\begin{cases}x, & \text{if } x\geq 0\\ -x, & \text{if } x<0\end{cases}$$

It is seen that $f(-1)=|-1|=1, f(1)=|1|=1$.

∴*f*(−1) = *f*(1), but −1 ≠ 1.

∴ *f* is not one-one.

Now, consider −1 ∈ **R**.

It is known that *f*(*x*) = $|x|$ is always non-negative. Thus, there does not exist any element *x* in domain **R** such that *f*(*x*) = $|x|$ = −1.

∴ *f* is not onto.

Hence, the modulus function is neither one-one nor onto.

Question 5:

Show that the Signum Function *f*: **R** → **R**, given by

$$f(x)=\begin{cases}1, & \text{if } x>0\\ 0, & \text{if } x=0\\ -1, & \text{if } x<0\end{cases}$$

is neither one-one nor onto.

Answer

f: **R** → **R** is given by,

$$f(x)=\begin{cases}1, & \text{if } x>0\\ 0, & \text{if } x=0\\ -1, & \text{if } x<0\end{cases}$$

It is seen that *f*(1) = *f*(2) = 1, but 1 ≠ 2.

∴*f* is not one-one.

Now, as $f(x)$ takes only 3 values (1, 0, or −1) for the element −2 in co-domain **R**, there does not exist any x in domain **R** such that $f(x) = -2$.

∴ f is not onto.

Hence, the signum function is neither one-one nor onto.

Question 6:

Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Show that f is one-one.

Answer

It is given that $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$.

$f: A \to B$ is defined as $f = \{(1, 4), (2, 5), (3, 6)\}$.

∴ $f(1) = 4$, $f(2) = 5$, $f(3) = 6$

It is seen that the images of distinct elements of A under f are distinct.

Hence, function f is one-one.

Question 7:

In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i) $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3 - 4x$

(ii) $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1 + x^2$

Answer

(i) $f: \mathbf{R} \to \mathbf{R}$ is defined as $f(x) = 3 - 4x$.

Let $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) = f(x_2)$.

$\Rightarrow 3 - 4x_1 = 3 - 4x_2$

$\Rightarrow -4x_1 = -4x_2$

$\Rightarrow x_1 = x_2$

∴ f is one-one.

For any real number (y) in **R**, there exists $\frac{3-y}{4}$ in **R** such that

$f\left(\frac{3-y}{4}\right) = 3 - 4\left(\frac{3-y}{4}\right) = y.$

∴f is onto.

Hence, f is bijective.

(ii) f: **R** → **R** is defined as

$$f(x) = 1 + x^2.$$

Let $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) = f(x_2)$.

$$\Rightarrow 1 + x_1^2 = 1 + x_2^2$$
$$\Rightarrow x_1^2 = x_2^2$$
$$\Rightarrow x_1 = \pm x_2$$

$\therefore f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$.

For instance,

$$f(1) = f(-1) = 2$$

∴ f is not one-one.

Consider an element −2 in co-domain **R**.

It is seen that $f(x) = 1 + x^2$ is positive for all $x \in$ **R**.

Thus, there does not exist any x in domain **R** such that $f(x) = -2$.

∴ f is not onto.

Hence, f is neither one-one nor onto.

Question 8:

Let A and B be sets. Show that f: $A \times B \to B \times A$ such that $(a, b) = (b, a)$ is bijective function.

Answer

f: $A \times B \to B \times A$ is defined as $f(a, b) = (b, a)$.

Let $(a_1, b_1), (a_2, b_2) \in \mathrm{A} \times \mathrm{B}$ such that $f(a_1, b_1) = f(a_2, b_2)$.

$$\Rightarrow (b_1, a_1) = (b_2, a_2)$$
$$\Rightarrow b_1 = b_2 \text{ and } a_1 = a_2$$
$$\Rightarrow (a_1, b_1) = (a_2, b_2)$$

∴ f is one-one.

Now, let $(b, a) \in B \times A$ be any element.

Then, there exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$. [By definition of f]

∴ *f* is onto.

Hence, *f* is bijective.

Question 9:

Let *f*: **N** → **N** be defined by $f(n)=\begin{cases}\dfrac{n+1}{2}, & \text{if } n \text{ is odd}\\ \dfrac{n}{2}, & \text{if } n \text{ is even}\end{cases}$ for all $n \in \mathbf{N}$.

State whether the function f is bijective. Justify your answer.

Answer

$$f(n)=\begin{cases}\dfrac{n+1}{2}, & \text{if } n \text{ is odd}\\ \dfrac{n}{2}, & \text{if } n \text{ is even}\end{cases} \quad \text{for all } n \in \mathbf{N}.$$

It can be observed that:

$$f(1)=\frac{1+1}{2}=1 \text{ and } f(2)=\frac{2}{2}=1 \qquad [\text{By definition of } f]$$

$$\therefore f(1)=f(2), \text{ where } 1 \neq 2.$$

∴ *f* is not one-one.

Consider a natural number (*n*) in co-domain **N**.

Case **I:** *n* is odd

∴*n* = 2*r* + 1 for some *r* ∈ **N.** Then, there exists 4*r* + 1∈**N** such that

$$f(4r+1)=\frac{4r+1+1}{2}=2r+1.$$

Case **II:** *n* is even

∴*n* = 2*r* for some *r* ∈ **N.** Then,there exists 4*r* ∈**N** such that $f(4r)=\dfrac{4r}{2}=2r$.

∴ *f* is onto.

Hence, *f* is not a bijective function.

Question 10:

Let A = **R** − {3} and B = **R** − {1}. Consider the function *f*: A → B defined by

$f(x)=\left(\frac{x-2}{x-3}\right)$. Is *f* one-one and onto? Justify your answer.

Answer

A = **R** − {3}, B = **R** − {1}

f: A → B is defined as $f(x)=\left(\frac{x-2}{x-3}\right)$.

Let $x,\ y \in \text{A}$ such that $f(x)=f(y)$.

$$\Rightarrow \frac{x-2}{x-3}=\frac{y-2}{y-3}$$
$$\Rightarrow (x-2)(y-3)=(y-2)(x-3)$$
$$\Rightarrow xy-3x-2y+6=xy-3y-2x+6$$
$$\Rightarrow -3x-2y=-3y-2x$$
$$\Rightarrow 3x-2x=3y-2y$$
$$\Rightarrow x=y$$

∴ *f* is one-one.

Let *y* ∈B = **R** − {1}. Then, *y* ≠ 1.

The function *f* is onto if there exists *x* ∈A such that *f*(*x*) = *y.*

Now,

$$f(x)=y$$
$$\Rightarrow \frac{x-2}{x-3}=y$$
$$\Rightarrow x-2=xy-3y$$
$$\Rightarrow x(1-y)=-3y+2$$
$$\Rightarrow x=\frac{2-3y}{1-y}\in \text{A} \qquad [y\neq 1]$$

Thus, for any *y* ∈ B, there exists $\frac{2-3y}{1-y}\in \text{A}$ such that

$$f\left(\frac{2-3y}{1-y}\right)=\frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3}=\frac{2-3y-2+2y}{2-3y-3+3y}=\frac{-y}{-1}=y.$$

$\therefore f$ is onto.

Hence, function *f* is one-one and onto.

Question 11:

Let *f*: **R** → **R** be defined as $f(x) = x^4$. Choose the correct answer.

(A) *f* is one-one onto (B) *f* is many-one onto

(C) *f* is one-one but not onto (D) *f* is neither one-one nor onto

Answer

f: **R** → **R** is defined as $f(x)=x^4$.

Let $x, y \in$ **R** such that $f(x) = f(y)$.

$$\Rightarrow x^4 = y^4$$
$$\Rightarrow x = \pm y$$

$\therefore f(x_1)=f(x_2)$ does not imply that $x_1 = x_2$.

For instance,

$$f(1)=f(-1)=1$$

∴ *f* is not one-one.

Consider an element 2 in co-domain **R**. It is clear that there does not exist any *x* in domain **R** such that $f(x) = 2$.

∴ *f* is not onto.

Hence, function *f* is neither one-one nor onto.

The correct answer is D.

Question 12:

Let *f*: **R** → **R** be defined as $f(x) = 3x$. Choose the correct answer.

(A) *f* is one-one onto (B) *f* is many-one onto

(C) *f* is one-one but not onto (D) *f* is neither one-one nor onto

Answer

f: **R** → **R** is defined as $f(x) = 3x$.

Let $x, y \in$ **R** such that $f(x) = f(y)$.

$\Rightarrow 3x = 3y$

$\Rightarrow x = y$

$\therefore f$ is one-one.

Also, for any real number (y) in co-domain **R**, there exists $\frac{y}{3}$ in **R** such that

$f\left(\frac{y}{3}\right) = 3\left(\frac{y}{3}\right) = y$.

$\therefore f$ is onto.

Hence, function *f* is one-one and onto.

The correct answer is A.

Exercise 1.3

Question 1:

Let f: {1, 3, 4} → {1, 2, 5} and g: {1, 2, 5} → {1, 3} be given by f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}. Write down gof.

Answer

The functions f: {1, 3, 4} → {1, 2, 5} and g: {1, 2, 5} → {1, 3} are defined as f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}.

$$gof(1) = g(f(1)) = g(2) = 3 \qquad [f(1) = 2 \text{ and } g(2) = 3]$$
$$gof(3) = g(f(3)) = g(5) = 1 \qquad [f(3) = 5 \text{ and } g(5) = 1]$$
$$gof(4) = g(f(4)) = g(1) = 3 \qquad [f(4) = 1 \text{ and } g(1) = 3]$$
$$\therefore gof = \{(1,3),(3,1),(4,3)\}$$

Question 2:

Let f, g and h be functions from **R** to **R**. Show that

$$(f+g)oh = foh + goh$$
$$(f \cdot g)oh = (foh)\cdot(goh)$$

Answer

To prove:

$$(f+g)oh = foh + goh$$

Consider:

$$((f+g)oh)(x)$$
$$=(f+g)(h(x))$$
$$= f(h(x)) + g(h(x))$$
$$=(foh)(x)+(goh)(x)$$
$$=\{(foh)+(goh)\}(x)$$
$$\therefore ((f+g)oh)(x) = \{(foh)+(goh)\}(x) \qquad \forall x \in \mathbf{R}$$

Hence, $(f+g)oh = foh + goh$.

To prove:

$(f \cdot g)oh = (foh)\cdot(goh)$

Consider:

$((f \cdot g)oh)(x)$

$= (f \cdot g)(h(x))$

$= f(h(x)).g(h(x))$

$= (foh)(x).(goh)(x)$

$= \{(foh).(goh)\}(x)$

$\therefore ((f \cdot g)oh)(x) = \{(foh).(goh)\}(x)\ \forall x \in \mathrm{R}$

Hence, $(f \cdot g)oh = (foh)\cdot(goh)$.

Question 3:

Find *gof* and *fog*, if

(i) $f(x) = |x|$ and $g(x) = |5x-2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

Answer

(i) $f(x) = |x|$ and $g(x) = |5x-2|$

$\therefore (gof)(x) = g(f(x)) = g(|x|) = |5|x|-2|$

$(fog)(x) = f(g(x)) = f(|5x-2|) = ||5x-2|| = |5x-2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

$\therefore (gof)(x) = g(f(x)) = g(8x^3) = (8x^3)^{\frac{1}{3}} = 2x$

$(fog)(x) = f(g(x)) = f\left(x^{\frac{1}{3}}\right) = 8\left(x^{\frac{1}{3}}\right)^3 = 8x$

Question 4:

If $f(x)=\frac{(4x+3)}{(6x-4)}$, $x\neq\frac{2}{3}$, show that *f* o *f*(*x*) = *x*, for all $x\neq\frac{2}{3}$. What is the inverse of *f*?

Answer

It is given that $f(x)=\frac{(4x+3)}{(6x-4)}$, $x\neq\frac{2}{3}$.

$$(fof)(x)=f(f(x))=f\left(\frac{4x+3}{6x-4}\right)$$

$$=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}=\frac{16x+12+18x-12}{24x+18-24x+16}=\frac{34x}{34}=x$$

Therefore, $fof(x)=x$, for all $x\neq\frac{2}{3}$.

$\Rightarrow fof=\mathrm{I}$

Hence, the given function *f* is invertible and the inverse of *f* is *f* itself.

Question 5:

State with reason whether following functions have inverse

(i) *f*: {1, 2, 3, 4} → {10} with

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

(ii) *g*: {5, 6, 7, 8} → {1, 2, 3, 4} with

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

(iii) *h*: {2, 3, 4, 5} → {7, 9, 11, 13} with

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

Answer

(i) *f*: {1, 2, 3, 4} → {10}defined as:

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

From the given definition of *f*, we can see that *f* is a many one function as: *f*(1) = *f*(2) = *f*(3) = *f*(4) = 10

∴*f* is not one-one.

Hence, function *f* does not have an inverse.

(ii) *g*: {5, 6, 7, 8} → {1, 2, 3, 4} defined as:

$g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

From the given definition of *g*, it is seen that *g* is a many one function as: $g(5) = g(7) = 4$.

∴*g* is not one-one,

Hence, function g does not have an inverse.

(iii) $h: \{2, 3, 4, 5\} \to \{7, 9, 11, 13\}$ defined as:

$h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

It is seen that all distinct elements of the set {2, 3, 4, 5} have distinct images under *h*.

∴Function *h* is one-one.

Also, *h* is onto since for every element *y* of the set {7, 9, 11, 13}, there exists an element *x* in the set {2, 3, 4, 5}such that $h(x) = y$.

Thus, *h* is a one-one and onto function. Hence, *h* has an inverse.

Question 6:

Show that $f: [-1, 1] \to \mathbf{R}$, given by $f(x) = \frac{x}{(x+2)}$ is one-one. Find the inverse of the function $f: [-1, 1] \to$ Range *f*.

(Hint: For $y \in$ Range *f*, $y = f(x) = \frac{x}{x+2}$, for some *x* in $[-1, 1]$, i.e., $x = \frac{2y}{(1-y)}$)

Answer

$f: [-1, 1] \to R$ is given as $f(x) = \frac{x}{(x+2)}$.

Let $f(x) = f(y)$.

$$\Rightarrow \frac{x}{x+2} = \frac{y}{y+2}$$
$$\Rightarrow xy + 2x = xy + 2y$$
$$\Rightarrow 2x = 2y$$
$$\Rightarrow x = y$$

∴ *f* is a one-one function.

It is clear that $f: [-1, 1] \to$ Range *f* is onto.

∴ $f: [-1, 1] \to$ Range *f* is one-one and onto and therefore, the inverse of the function:

f: [−1, 1] → Range *f* exists.

Let *g*: Range *f* → [−1, 1] be the inverse of *f*.

Let *y* be an arbitrary element of range *f*.

Since *f*: [−1, 1] → Range *f* is onto, we have:

$$y = f(x) \text{ for same } x \in [-1, 1]$$

$$\Rightarrow y = \frac{x}{x+2}$$

$$\Rightarrow xy + 2y = x$$

$$\Rightarrow x(1-y) = 2y$$

$$\Rightarrow x = \frac{2y}{1-y},\ y \neq 1$$

Now, let us define *g*: Range *f* → [−1, 1] as

$$g(y) = \frac{2y}{1-y},\ y \neq 1.$$

$$\text{Now, } (gof)(x) = g(f(x)) = g\left(\frac{x}{x+2}\right) = \frac{2\left(\frac{x}{x+2}\right)}{1-\frac{x}{x+2}} = \frac{2x}{x+2-x} = \frac{2x}{2} = x$$

$$(fog)(y) = f(g(y)) = f\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2} = \frac{2y}{2y+2-2y} = \frac{2y}{2} = y$$

∴*gof* = $I_{[-1,\ 1]}$ and *fog* = $I_{\text{Range} f}$

∴ $f^{-1} = g$

$$\Rightarrow f^{-1}(y) = \frac{2y}{1-y}, y \neq 1$$

Question 7:

Consider *f*: **R** → **R** given by *f*(*x*) = 4*x* + 3. Show that *f* is invertible. Find the inverse of *f*.

Answer

f: **R** → **R** is given by,

f(*x*) = 4*x* + 3

One-one:

Let $f(x) = f(y)$.

$$\Rightarrow 4x+3=4y+3$$

$$\Rightarrow 4x=4y$$

$$\Rightarrow x=y$$

∴ f is a one-one function.

Onto:

For $y \in \mathbf{R}$, let $y = 4x + 3$.

$$\Rightarrow x=\frac{y-3}{4}\in\mathbf{R}$$

Therefore, for any $y \in \mathbf{R}$, there exists $x=\frac{y-3}{4}\in\mathbf{R}$ such that

$$f(x)=f\left(\frac{y-3}{4}\right)=4\left(\frac{y-3}{4}\right)+3=y.$$

∴ f is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define g: $\mathbf{R}\to\mathbf{R}$ by $g(y) = (y-3)/4$.

$$\text{Now}, (gof)(x)=g(f(x))=g(4x+3)=\frac{(4x+3)-3}{4}=x$$

$$(fo\,g)(y)=f(g(y))=f\left(\frac{y-3}{4}\right)=4\left(\frac{y-3}{4}\right)+3=y-3+3=y$$

$$\therefore\ gof=fog=I_{\mathbf{R}}$$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y)=g(y)=\frac{y-3}{4}.$$

Question 8:

Consider f: $\mathbf{R}_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of given f by $f^{-1}(y)=\sqrt{y-4}$, where $\mathbf{R}_+$ is the set of all non-negative real numbers.

Answer

f: $\mathbf{R}_+ \to [4, \infty)$ is given as $f(x) = x^2 + 4$.

One-one:

Let $f(x) = f(y)$.

$$\Rightarrow x^2 + 4 = y^2 + 4$$
$$\Rightarrow x^2 = y^2$$
$$\Rightarrow x = y \qquad [\text{as } x = y \in \mathbf{R}_+]$$

$\therefore f$ is a one-one function.

Onto:

For $y \in [4, \infty)$, let $y = x^2 + 4$.

$$\Rightarrow x^2 = y - 4 \geq 0 \qquad [\text{as } y \geq 4]$$
$$\Rightarrow x = \sqrt{y-4} \geq 0$$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \sqrt{y-4} \in \mathbf{R}$ such that

$$f(x) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = y - 4 + 4 = y.$$

$\therefore f$ is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define g: $[4, \infty) \rightarrow \mathbf{R}_+$ by,

$$g(y) = \sqrt{y-4}$$

$$\text{Now, } gof(x) = g(f(x)) = g\left(x^2 + 4\right) = \sqrt{\left(x^2+4\right) - 4} = \sqrt{x^2} = x$$

$$\text{And, } fog(y) = f(g(y)) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = (y-4) + 4 = y$$

$$\therefore gof = fog = I_{\mathbf{R}+}$$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \sqrt{y-4}.$$

Question 9:

Consider f: $\mathbf{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with

$$f^{-1}(y)=\left(\frac{\left(\sqrt{y+6}\right)-1}{3}\right).$$

Answer

f: $\mathbf{R}_+ \to [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Let y be an arbitrary element of $[-5, \infty)$.

Let $y = 9x^2 + 6x - 5$.

$$\Rightarrow y=(3x+1)^2-1-5=(3x+1)^2-6$$

$$\Rightarrow (3x+1)^2=y+6$$

$$\Rightarrow 3x+1=\sqrt{y+6} \qquad [\text{as } y\geq -5 \Rightarrow y+6>0]$$

$$\Rightarrow x=\frac{\sqrt{y+6}-1}{3}$$

$\therefore f$ is onto, thereby range $f = [-5, \infty)$.

Let us define g: $[-5, \infty) \to \mathbf{R}_+$ as $g(y) = \frac{\sqrt{y+6}-1}{3}$.

We now have:

$$(gof)(x) = g(f(x)) = g(9x^2+6x-5)$$
$$= g\left((3x+1)^2-6\right)$$
$$= \frac{\sqrt{(3x+1)^2-6+6}-1}{3}$$
$$= \frac{3x+1-1}{3} = x$$

$$\text{And,}(fog)(y) = f(g(y)) = f\left(\frac{\sqrt{y+6}-1}{3}\right)$$
$$= \left[3\left(\frac{\sqrt{y+6}-1}{3}\right)+1\right]^2-6$$
$$= \left(\sqrt{y+6}\right)^2-6 = y+6-6 = y$$

$\therefore gof = \mathrm{I}_{\mathbf{R}_+}$ and $fog = \mathrm{I}_{[-5,\infty)}$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{\sqrt{y+6}-1}{3}.$$

Question 10:

Let f: $X \to Y$ be an invertible function. Show that f has unique inverse.

(Hint: suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y$,

$fog_1(y) = \mathrm{I}_Y(y) = fog_2(y)$. Use one-one ness of f).

Answer

Let f: $X \to Y$ be an invertible function.

Also, suppose f has two inverses (say g_1 and g_2).

Then, for all $y \in Y$, we have:

$$fog_1(y) = I_Y(y) = fog_2(y)$$
$$\Rightarrow f(g_1(y)) = f(g_2(y))$$
$$\Rightarrow g_1(y) = g_2(y) \qquad [f \text{ is invertible} \Rightarrow f \text{ is one-one}]$$
$$\Rightarrow g_1 = g_2 \qquad [g \text{ is one-one}]$$

Hence, *f* has a unique inverse.

Question 11:

Consider *f*: {1, 2, 3} → {*a*, *b*, *c*} given by *f*(1) = *a*, *f*(2) = *b* and *f*(3) = *c*. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Answer

Function *f*: {1, 2, 3} → {*a*, *b*, *c*} is given by,

f(1) = *a*, *f*(2) = *b,* and *f*(3) = *c*

If we define *g*: {*a*, *b*, *c*} → {1, 2, 3} as *g*(*a*) = 1, *g*(*b*) = 2, *g*(*c*) = 3, then we have:

$$(fog)(a) = f(g(a)) = f(1) = a$$
$$(fog)(b) = f(g(b)) = f(2) = b$$
$$(fog)(c) = f(g(c)) = f(3) = c$$

And,

$$(gof)(1) = g(f(1)) = g(a) = 1$$
$$(gof)(2) = g(f(2)) = g(b) = 2$$
$$(gof)(3) = g(f(3)) = g(c) = 3$$

∴ $gof = I_X$ and $fog = I_Y$, where *X* = {1, 2, 3} and *Y*= {*a*, *b*, *c*}.

Thus, the inverse of *f* exists and f^{-1} **=** *g*.

∴f^{-1}: {*a*, *b*, *c*} → {1, 2, 3} is given by,

$f^{-1}(a) = 1$, $f^{-1}(b) = 2$, $f^{-1}(c) = 3$

Let us now find the inverse of f^{-1} i.e., find the inverse of *g*.

If we define *h*: {1, 2, 3} → {*a*, *b*, *c*} as

h(1) = *a*, *h*(2) = *b*, *h*(3) = *c*, then we have:

$$(goh)(1) = g(h(1)) = g(a) = 1$$
$$(goh)(2) = g(h(2)) = g(b) = 2$$
$$(goh)(3) = g(h(3)) = g(c) = 3$$

And,

$$(hog)(a) = h(g(a)) = h(1) = a$$
$$(hog)(b) = h(g(b)) = h(2) = b$$
$$(hog)(c) = h(g(c)) = h(3) = c$$

$\therefore goh = I_X$ and $hog = I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.

It can be noted that $h = f$.

Hence, $(f^{-1})^{-1} = f$.

Question 12:

Let $f: X \to Y$ be an invertible function. Show that the inverse of f^{-1} is f, i.e.,
$(f^{-1})^{-1} = f$.

Answer

Let $f: X \to Y$ be an invertible function.

Then, there exists a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$.

Here, $f^{-1} = g$.

Now, $gof = I_X$ and $fog = I_Y$

$\Rightarrow f^{-1}of = I_X$ and $fof^{-1} = I_Y$

Hence, $f^{-1}: Y \to X$ is invertible and f is the inverse of f^{-1}

i.e., $(f^{-1})^{-1} = f$.

Question 13:

If f: **R** → **R** be given by $f(x) = (3 - x^3)^{\frac{1}{3}}$, then $fof(x)$ is

(A) $\frac{1}{x^3}$ (B) x^3 (C) x (D) $(3 - x^3)$

Answer

f: **R** → **R** is given as $f(x)=\left(3-x^3\right)^{\frac{1}{3}}$.

$$f(x)=\left(3-x^3\right)^{\frac{1}{3}}$$

$$\therefore fof(x)=f(f(x))=f\left(\left(3-x^3\right)^{\frac{1}{3}}\right)=\left[3-\left(\left(3-x^3\right)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$=\left[3-\left(3-x^3\right)\right]^{\frac{1}{3}}=\left(x^3\right)^{\frac{1}{3}}=x$$

$$\therefore fof(x)=x$$

The correct answer is C.

Question 14:

Let $f:\mathbf{R}-\left\{-\frac{4}{3}\right\}\to\mathbf{R}$ be a function defined as $f(x)=\frac{4x}{3x+4}$. The inverse of *f* is map *g*:

Range $f\to\mathbf{R}-\left\{-\frac{4}{3}\right\}$ given by

(A) $g(y)=\frac{3y}{3-4y}$ (B) $g(y)=\frac{4y}{4-3y}$

(C) $g(y)=\frac{4y}{3-4y}$ (D) $g(y)=\frac{3y}{4-3y}$

Answer

It is given that $f:\mathbf{R}-\left\{-\frac{4}{3}\right\}\to\mathbf{R}$ is defined as $f(x)=\frac{4x}{3x+4}$.

Let *y* be an arbitrary element of Range *f*.

Then, there exists $x\in\mathbf{R}-\left\{-\frac{4}{3}\right\}$ such that $y=f(x)$.

$$\Rightarrow y=\frac{4x}{3x+4}$$
$$\Rightarrow 3xy+4y=4x$$
$$\Rightarrow x(4-3y)=4y$$
$$\Rightarrow x=\frac{4y}{4-3y}$$

Let us define *g*: Range $f\to\mathbf{R}-\left\{-\frac{4}{3}\right\}$ as $g(y)=\frac{4y}{4-3y}$.

Now, $(g \circ f)(x) = g(f(x)) = g\left(\frac{4x}{3x+4}\right)$

$$= \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)} = \frac{16x}{12x+16-12x} = \frac{16x}{16} = x$$

$$\text{And,} (f \circ g)(y) = f(g(y)) = f\left(\frac{4y}{4-3y}\right)$$

$$= \frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4} = \frac{16y}{12y+16-12y} = \frac{16y}{16} = y$$

$\therefore\ gof = \mathrm{I}_{\mathbf{R}-\left\{-\frac{4}{3}\right\}}$ and $fog = \mathrm{I}_{\text{Range}\,f}$

Thus, *g* is the inverse of *f* i.e., $f^{-1} = g$.

Hence, the inverse of *f* is the map *g*: Range $f \to \mathbf{R}-\left\{-\frac{4}{3}\right\}$, which is given by

$g(y) = \frac{4y}{4-3y}$.

The correct answer is B.

Exercise 1.4

Question 1:

Determine whether or not each of the definition of given below gives a binary operation.

In the event that * is not a binary operation, give justification for this.

(i) On $\mathbf{Z}^+$, define * by $a * b = a - b$

(ii) On $\mathbf{Z}^+$, define * by $a * b = ab$

(iii) On $\mathbf{R}$, define * by $a * b = ab^2$

(iv) On $\mathbf{Z}^+$, define * by $a * b = |a - b|$

(v) On $\mathbf{Z}^+$, define * by $a * b = a$

Answer

(i) On $\mathbf{Z}^+$, * is defined by $a * b = a - b$.

It is not a binary operation as the image of (1, 2) under * is $1 * 2 = 1 - 2$

$= -1 \notin \mathbf{Z}^+$.

(ii) On $\mathbf{Z}^+$, * is defined by $a * b = ab$.

It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element ab in $\mathbf{Z}^+$.

This means that * carries each pair (a, b) to a unique element $a * b = ab$ in $\mathbf{Z}^+$.

Therefore, * is a binary operation.

(iii) On $\mathbf{R}$, * is defined by $a * b = ab^2$.

It is seen that for each $a, b \in \mathbf{R}$, there is a unique element ab^2 in $\mathbf{R}$.

This means that * carries each pair (a, b) to a unique element $a * b = ab^2$ in $\mathbf{R}$.

Therefore, * is a binary operation.

(iv) On $\mathbf{Z}^+$, * is defined by $a * b = |a - b|$.

It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element $|a - b|$ in $\mathbf{Z}^+$.

This means that * carries each pair (a, b) to a unique element $a * b =$

$|a - b|$ in $\mathbf{Z}^+$.

Therefore, * is a binary operation.

(v) On $\mathbf{Z}^+$, * is defined by $a * b = a$.

* carries each pair (a, b) to a unique element $a * b = a$ in $\mathbf{Z}^+$.

Therefore, * is a binary operation.

Question 2:

For each binary operation * defined below, determine whether * is commutative or associative.

(i) On **Z**, define $a * b = a - b$

(ii) On **Q**, define $a * b = ab + 1$

(iii) On **Q**, define $a * b = \frac{ab}{2}$

(iv) On $\mathbf{Z}^+$, define $a * b = 2^{ab}$

(v) On $\mathbf{Z}^+$, define $a * b = a^b$

(vi) On **R** − {−1}, define $a * b = \frac{a}{b+1}$

Answer

(i) On **Z**, * is defined by $a * b = a - b$.

It can be observed that $1 * 2 = 1 - 2 = 1$ and $2 * 1 = 2 - 1 = 1$.

$\therefore 1 * 2 \neq 2 * 1$; where $1, 2 \in \mathbf{Z}$

Hence, the operation * is not commutative.

Also we have:

$(1 * 2) * 3 = (1 - 2) * 3 = -1 * 3 = -1 - 3 = -4$

$1 * (2 * 3) = 1 * (2 - 3) = 1 * -1 = 1 - (-1) = 2$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$; where $1, 2, 3 \in \mathbf{Z}$

Hence, the operation * is not associative.

(ii) On **Q**, * is defined by $a * b = ab + 1$.

It is known that:

$ab = ba$ ☐ $a, b \in \mathbf{Q}$

$\Rightarrow ab + 1 = ba + 1$ ☐ $a, b \in \mathbf{Q}$

$\Rightarrow a * b = a * b$ ☐ $a, b \in \mathbf{Q}$

Therefore, the operation * is commutative.

It can be observed that:

$(1 * 2) * 3 = (1 \times 2 + 1) * 3 = 3 * 3 = 3 \times 3 + 1 = 10$

$1 * (2 * 3) = 1 * (2 \times 3 + 1) = 1 * 7 = 1 \times 7 + 1 = 8$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$; where $1, 2, 3 \in \mathbf{Q}$

Therefore, the operation * is not associative.

(iii) On **Q**, * is defined by $a * b = \frac{ab}{2}$.

It is known that:

ab = *ba* ☐ *a, b* ∈ **Q**

⇒ $\frac{ab}{2} = \frac{ba}{2}$ ☐ *a, b* ∈ **Q**

⇒ *a* * *b* = *b* * *a* ☐ *a, b* ∈ **Q**

Therefore, the operation * is commutative.

For all *a, b*, *c* ∈ **Q**, we have:

$$(a*b)*c = \left(\frac{ab}{2}\right)*c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4}$$

$$a*(b*c) = a*\left(\frac{bc}{2}\right) = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$$

$\therefore (a*b)*c = a*(b*c)$

Therefore, the operation * is associative.

(iv) On $\mathbf{Z}^+$, * is defined by $a * b = 2^{ab}$.

It is known that:

ab = *ba* ☐ *a, b* ∈ $\mathbf{Z}^+$

⇒ $2^{ab} = 2^{ba}$ ☐ *a, b* ∈ $\mathbf{Z}^+$

⇒ *a* * *b* = *b* * *a* ☐ *a, b* ∈ $\mathbf{Z}^+$

Therefore, the operation * is commutative.

It can be observed that:

$$(1*2)*3 = 2^{(1\times2)}*3 = 4*3 = 2^{4\times3} = 2^{12}$$

$$1*(2*3) = 1*2^{2\times3} = 1*2^{6} = 1*64 = 2^{64}$$

∴(1 * 2) * 3 ≠ 1 * (2 * 3) ; where 1, 2, 3 ∈ $\mathbf{Z}^+$

Therefore, the operation * is not associative.

(v) On $\mathbf{Z}^+$, * is defined by $a * b = a^b$.

It can be observed that:

$1*2 = 1^2 = 1$ and $2*1 = 2^1 = 2$

$\therefore$ 1 * 2 ≠ 2 * 1 ; where 1, 2 ∈ $\mathbf{Z}^+$

Therefore, the operation * is not commutative.

It can also be observed that:

$$(2*3)*4 = 2^3 * 4 = 8*4 = 8^4 = \left(2^3\right)^4 = 2^{12}$$

$$2*(3*4) = 2*3^4 = 2*81 = 2^{81}$$

$\therefore$(2 * 3) * 4 ≠ 2 * (3 * 4) ; where 2, 3, 4 ∈ $\mathbf{Z}^+$

Therefore, the operation * is not associative.

(vi) On **R**, * − {−1} is defined by $a*b = \frac{a}{b+1}$.

It can be observed that $1*2 = \frac{1}{2+1} = \frac{1}{3}$ and $2*1 = \frac{2}{1+1} = \frac{2}{2} = 1$.

$\therefore$1 * 2 ≠ 2 * 1 ; where 1, 2 ∈ **R** − {−1}

Therefore, the operation * is not commutative.

It can also be observed that:

$$(1*2)*3 = \frac{1}{3}*3 = \frac{\frac{1}{3}}{3+1} = \frac{1}{12}$$

$$1*(2*3) = 1*\frac{2}{3+1} = 1*\frac{2}{4} = 1*\frac{1}{2} = \frac{1}{\frac{1}{2}+1} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$\therefore$ (1 * 2) * 3 ≠ 1 * (2 * 3) ; where 1, 2, 3 ∈ **R** − {−1}

Therefore, the operation * is not associative.

Question 3:

Consider the binary operation ∨ on the set {1, 2, 3, 4, 5} defined by *a* ∨*b* = min {*a*, *b*}. Write the operation table of the operation∨.

Answer

The binary operation ∨ on the set {1, 2, 3, 4, 5} is defined as *a* ∨ *b* = min {*a*, *b*}

☐ *a*, *b* ∈ {1, 2, 3, 4, 5}.

Thus, the operation table for the given operation ∨ can be given as:

∨	1	2	3	4	5

1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Question 4:

Consider a binary operation * on the set {1, 2, 3, 4, 5} given by the following multiplication table.

(i) Compute (2 * 3) * 4 and 2 * (3 * 4)

(ii) Is * commutative?

(iii) Compute (2 * 3) * (4 * 5).

(Hint: use the following table)

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Answer

(i) (2 * 3) * 4 = 1 * 4 = 1

2 * (3 * 4) = 2 * 1 = 1

(ii) For every $a, b \in\{1, 2, 3, 4, 5\}$, we have $a * b = b * a$. Therefore, the operation * is commutative.

(iii) (2 * 3) = 1 and (4 * 5) = 1

$\therefore (2 * 3) * (4 * 5) = 1 * 1 = 1$

Question 5:

Let*′ be the binary operation on the set {1, 2, 3, 4, 5} defined by *a* *′ *b* = H.C.F. of *a* and *b*. Is the operation *′ same as the operation * defined in Exercise 4 above? Justify your answer.

Answer

The binary operation *′ on the set {1, 2, 3 4, 5} is defined as *a* *′ *b* = H.C.F of *a* and *b*.

The operation table for the operation *′ can be given as:

*′	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

We observe that the operation tables for the operations * and *′ are the same.

Thus, the operation *′ is same as the operation*.

Question 6:

Let * be the binary operation on **N** given by *a* * *b* = L.C.M. of *a* and *b*. Find

(i) 5 * 7, 20 * 16 (ii) Is * commutative?

(iii) Is * associative? (iv) Find the identity of * in **N**

(v) Which elements of **N** are invertible for the operation *?

Answer

The binary operation * on **N** is defined as *a* * *b* = L.C.M. of *a* and *b*.

(i) 5 * 7 = L.C.M. of 5 and 7 = 35

20 * 16 = L.C.M of 20 and 16 = 80

(ii) It is known that:

L.C.M of *a* and *b* = L.C.M of *b* and *a* ☐ *a, b* ∈ **N**.

$\therefore a * b = b * a$

Thus, the operation * is commutative.

(iii) For $a, b, c \in$ **N**, we have:

$(a * b) * c$ = (L.C.M of a and b) $* c$ = LCM of a, b, and c

$a * (b * c) = a *$ (LCM of b and c) = L.C.M of a, b, and c

$\therefore (a * b) * c = a * (b * c)$

Thus, the operation * is associative.

(iv) It is known that:

L.C.M. of a and 1 = a = L.C.M. 1 and a ☐ $a \in$ **N**

$\Rightarrow a * 1 = a = 1 * a$ ☐ $a \in$ **N**

Thus, 1 is the identity of * in **N**.

(v) An element a in **N** is invertible with respect to the operation * if there exists an element b in **N**, such that $a * b = e = b * a$.

Here, $e = 1$

This means that:

L.C.M of a and b = 1 = L.C.M of b and a

This case is possible only when a and b are equal to 1.

Thus, 1 is the only invertible element of **N** with respect to the operation *.

Question 7:

Is * defined on the set {1, 2, 3, 4, 5} by $a * b$ = L.C.M. of a and b a binary operation? Justify your answer.

Answer

The operation * on the set A = {1, 2, 3, 4, 5} is defined as

$a * b$ = L.C.M. of a and b.

Then, the operation table for the given operation * can be given as:

*	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	6	3	12	15

4	4	4	12	4	20
5	5	10	15	20	5

It can be observed from the obtained table that:

3 * 2 = 2 * 3 = 6 ∉ A, 5 * 2 = 2 * 5 = 10 ∉ A, 3 * 4 = 4 * 3 = 12 ∉ A

3 * 5 = 5 * 3 = 15 ∉ A, 4 * 5 = 5 * 4 = 20 ∉ A

Hence, the given operation * is not a binary operation.

Question 8:

Let * be the binary operation on **N** defined by *a* * *b* = H.C.F. of *a* and *b*. Is * commutative? Is * associative? Does there exist identity for this binary operation on **N**?

Answer

The binary operation * on **N** is defined as:

a * *b* = H.C.F. of *a* and *b*

It is known that:

H.C.F. of *a* and *b* = H.C.F. of *b* and *a* ☐ *a, b* ∈ **N**.

∴*a* * *b* = *b* * *a*

Thus, the operation * is commutative.

For *a*, *b*, *c* ∈ **N**, we have:

(*a* * *b*)* *c* = (H.C.F. of *a* and *b*) * *c* = H.C.F. of *a*, *b*, and *c*

a *(*b* * *c*)= *a* *(H.C.F. of *b* and *c*) = H.C.F. of *a*, *b*, and *c*

∴(*a* * *b*) * *c* = *a* * (*b* * *c*)

Thus, the operation * is associative.

Now, an element *e* ∈ **N** will be the identity for the operation * if *a* * *e* = *a* = e* *a* ∀ *a* ∈ **N**.

But this relation is not true for any *a* ∈ **N.**

Thus, the operation * does not have any identity in **N.**

Question 9:

Let * be a binary operation on the set **Q** of rational numbers as follows:

(i) $a * b = a - b$ (ii) $a * b = a^2 + b^2$

(iii) $a * b = a + ab$ (iv) $a * b = (a - b)^2$

(v) $a*b=\frac{ab}{4}$ (vi) $a * b = ab^2$

Find which of the binary operations are commutative and which are associative.
Answer

(i) On **Q**, the operation * is defined as $a * b = a - b$.

It can be observed that:

$\frac{1}{2}*\frac{1}{3}=\frac{1}{2}-\frac{1}{3}=\frac{3-2}{6}=\frac{1}{6}$ and $\frac{1}{3}*\frac{1}{2}=\frac{1}{3}-\frac{1}{2}=\frac{2-3}{6}=\frac{-1}{6}$

$\therefore \frac{1}{2}*\frac{1}{3}\neq\frac{1}{3}*\frac{1}{2}$; where $\frac{1}{2},\ \frac{1}{3}\in\mathbf{Q}$

Thus, the operation * is not commutative.

It can also be observed that:

$$\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}=\left(\frac{1}{2}-\frac{1}{3}\right)*\frac{1}{4}=\frac{1}{6}*\frac{1}{4}=\frac{1}{6}-\frac{1}{4}=\frac{2-3}{12}=\frac{-1}{12}$$

$$\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)=\frac{1}{2}*\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{1}{2}*\frac{1}{12}=\frac{1}{2}-\frac{1}{12}=\frac{6-1}{12}=\frac{5}{12}$$

$$\therefore\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}\neq\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)\text{ ; where }\frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}\in\mathbf{Q}$$

Thus, the operation * is not associative.

(ii) On **Q**, the operation * is defined as $a * b = a^2 + b^2$.

For $a, b \in$ **Q**, we have:

$$a*b=a^2+b^2=b^2+a^2=b*a$$

$\therefore a * b = b * a$

Thus, the operation * is commutative.

It can be observed that:

$$(1*2)*3=\left(1^2+2^2\right)*3=(1+4)*4=5*4=5^2+4^2=41$$

$$1*(2*3)=1*\left(2^2+3^2\right)=1*(4+9)=1*13=1^2+13^2=169$$

$$\therefore (1*2)*3\neq 1*(2*3)\text{ ; where }1,2,3\in\mathbf{Q}$$

Thus, ,the operation * is not associative.

(iii) On **Q**, the operation * is defined as $a * b = a + ab$.

It can be observed that:

$$1*2=1+1\times 2=1+2=3$$
$$2*1=2+2\times 1=2+2=4$$

$$\therefore 1*2\neq 2*1 \text{ ; where } 1, 2 \in \mathbf{Q}$$

Thus, the operation * is not commutative.

It can also be observed that:

$$(1*2)*3=(1+1\times 2)*3=3*3=3+3\times 3=3+9=12$$
$$1*(2*3)=1*(2+2\times 3)=1*8=1+1\times 8=9$$

$$\therefore (1*2)*3\neq 1*(2*3) \text{ ; where } 1, 2, 3 \in \mathbf{Q}$$

Thus, the operation * is not associative.

(iv) On **Q**, the operation * is defined by $a * b = (a - b)^2$.

For a, $b \in$ **Q**, we have:

$a * b = (a - b)^2$

$b * a = (b - a)^2 = [-(a - b)]^2 = (a - b)^2$

$\therefore a * b = b * a$

Thus, the operation * is commutative.

It can be observed that:

$$(1*2)*3=(1-2)^2*3=(-1)^2*3=1*3=(1-3)^2=(-2)^2=4$$
$$1*(2*3)=1*(2-3)^2=1*(-1)^2=1*1=(1-1)^2=0$$

$$\therefore (1*2)*3\neq 1*(2*3) \text{ ; where } 1, 2, 3 \in \mathbf{Q}$$

Thus, the operation * is not associative.

(v) On **Q**, the operation * is defined as $a*b=\frac{ab}{4}$.

For a, $b \in$ **Q**, we have:

$$a*b=\frac{ab}{4}=\frac{ba}{4}=b*a$$

$\therefore a * b = b * a$

Thus, the operation * is commutative.

For *a, b, c* ∈ **Q**, we have:

$$(a*b)*c=\frac{ab}{4}*c=\frac{\frac{ab}{4}\cdot c}{4}=\frac{abc}{16}$$

$$a*(b*c)=a*\frac{bc}{4}=\frac{a\cdot\frac{bc}{4}}{4}=\frac{abc}{16}$$

$\therefore (a * b) * c = a * (b * c)$

Thus, the operation * is associative.

(vi) On **Q**, the operation * is defined as $a * b = ab^2$

It can be observed that:

$$\frac{1}{2}*\frac{1}{3}=\frac{1}{2}\cdot\left(\frac{1}{3}\right)^2=\frac{1}{2}\cdot\frac{1}{9}=\frac{1}{18}$$

$$\frac{1}{3}*\frac{1}{2}=\frac{1}{3}\cdot\left(\frac{1}{2}\right)^2=\frac{1}{3}\cdot\frac{1}{4}=\frac{1}{12}$$

$$\therefore \frac{1}{2}*\frac{1}{3}\neq\frac{1}{3}*\frac{1}{2}\text{ ; where }\frac{1}{2},\frac{1}{3}\in\mathbf{Q}$$

Thus, the operation * is not commutative.

It can also be observed that:

$$\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}=\left[\frac{1}{2}\cdot\left(\frac{1}{3}\right)^2\right]*\frac{1}{4}=\frac{1}{18}*\frac{1}{4}=\frac{1}{18}\cdot\left(\frac{1}{4}\right)^2=\frac{1}{18\times16}$$

$$\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)=\frac{1}{2}*\left[\frac{1}{3}\cdot\left(\frac{1}{4}\right)^2\right]=\frac{1}{2}*\frac{1}{48}=\frac{1}{2}\cdot\left(\frac{1}{48}\right)^2=\frac{1}{2\times(48)^2}$$

$$\therefore \left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}\neq\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)\text{ ; where }\frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}\in\mathbf{Q}$$

Thus, the operation * is not associative.

Hence, the operations defined in (ii), (iv), (v) are commutative and the operation defined in (v) is associative.

Question 10:

Find which of the operations given above has identity.

Answer

An element $e \in \mathbf{Q}$ will be the identity element for the operation * if

$a * e = a = e * a,\ \forall a \in \mathbf{Q}$.

However, there is no such element $e \in \mathbf{Q}$ with respect to each of the six operations satisfying the above condition.

Thus, none of the six operations has identity.

Question 11:

Let A = $\mathbf{N} \times \mathbf{N}$ and * be the binary operation on A defined by

$(a, b) * (c, d) = (a + c, b + d)$

Show that * is commutative and associative. Find the identity element for * on A, if any.

Answer

A = $\mathbf{N} \times \mathbf{N}$

* is a binary operation on A and is defined by:

$(a, b) * (c, d) = (a + c, b + d)$

Let $(a, b), (c, d) \in$ A

Then, $a, b, c, d \in \mathbf{N}$

We have:

$(a, b) * (c, d) = (a + c, b + d)$

$(c, d) * (a, b) = (c + a, d + b) = (a + c, b + d)$

[Addition is commutative in the set of natural numbers]

$\therefore (a, b) * (c, d) = (c, d) * (a, b)$

Therefore, the operation * is commutative.

Now, let $(a, b), (c, d), (e, f) \in$A

Then, $a, b, c, d, e, f \in \mathbf{N}$

We have:

$$\left((a,b)*(c,d)\right)*(e,f)=(a+c,\ b+d)*(e,f)=(a+c+e,\ b+d+f)$$

$$(a,b)*((c,d)*(e,f))=(a,b)*(c+e,\ d+f)=(a+c+e,\ b+d+f)$$

$$\therefore ((a,b)*(c,d))*(e,f)=(a,b)*((c,d)*(e,f))$$

Therefore, the operation * is associative.

An element $e=(e_1,e_2)\in A$ will be an identity element for the operation * if $a*e=a=e*a\ \forall\ a=(a_1,a_2)\in A$, i.e., $(a_1+e_1,a_2+e_2)=(a_1,a_2)=(e_1+a_1,e_2+a_2)$, which is not true for any element in A.

Therefore, the operation * does not have any identity element.

Question 12:

State whether the following statements are true or false. Justify.

(i) For an arbitrary binary operation * on a set **N**, *a* * *a* = *a* $\forall$ *a* * **N**.

(ii) If * is a commutative binary operation on **N**, then *a* * (*b* * *c*) = (*c* * *b*) * *a*

Answer

(i) Define an operation * on **N** as:

a * *b* = *a* + *b* $\forall$ *a, b* ∈ **N**

Then, in particular, for *b* = *a* = 3, we have:

3 * 3 = 3 + 3 = 6 ≠ 3

Therefore, statement (i) is false.

(ii) R.H.S. = (*c* * *b*) * *a*

= (*b* * *c*) * *a* [* is commutative]

= *a* * (*b* * *c*) [Again, as * is commutative]

= L.H.S.

∴ *a* * (*b* * *c*) = (*c* * *b*) * *a*

Therefore, statement (ii) is true.

Question 13:

Consider a binary operation * on **N** defined as $a * b = a^3 + b^3$. Choose the correct answer.

(A) Is * both associative and commutative?

(B) Is * commutative but not associative?

(C) Is * associative but not commutative?

(D) Is * neither commutative nor associative?

Answer

On **N**, the operation * is defined as $a * b = a^3 + b^3$.

For, a, b, $\in$ **N**, we have:

$a * b = a^3 + b^3 = b^3 + a^3 = b * a$ [Addition is commutative in **N**]

Therefore, the operation * is commutative.

It can be observed that:

$$(1*2)*3 = (1^3 + 2^3)*3 = 9*3 = 9^3 + 3^3 = 729 + 27 = 756$$

$$1*(2*3) = 1*(2^3 + 3^3) = 1*(8+27) = 1\times 35 = 1^3 + 35^3 = 1 + (35)^3 = 1 + 42875 = 42876$$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$; where 1, 2, 3 $\in$ **N**

Therefore, the operation * is not associative.

Hence, the operation * is commutative, but not associative. Thus, the correct answer is B.

Miscellaneous Questions:

Question 1:

Let f: $\mathbf{R} \rightarrow \mathbf{R}$ be defined as $f(x) = 10x + 7$. Find the function g: $\mathbf{R} \rightarrow \mathbf{R}$ such that $g \circ f = f \circ g = 1_{\mathbf{R}}$.

Answer

It is given that f: $\mathbf{R} \rightarrow \mathbf{R}$ is defined as $f(x) = 10x + 7$.

One-one:

Let $f(x) = f(y)$, where $x, y \in \mathbf{R}$.

$\Rightarrow 10x + 7 = 10y + 7$

$\Rightarrow x = y$

$\therefore f$ is a one-one function.

Onto:

For $y \in \mathbf{R}$, let $y = 10x + 7$.

$$\Rightarrow x = \frac{y-7}{10} \in \mathbf{R}$$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \frac{y-7}{10} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y.$$

$\therefore f$ is onto.

Therefore, f is one-one and onto.

Thus, f is an invertible function.

Let us define g: $\mathbf{R} \rightarrow \mathbf{R}$ as $g(y) = \frac{y-7}{10}$.

Now, we have:

$$gof(x) = g(f(x)) = g(10x+7) = \frac{(10x+7)-7}{10} = \frac{10x}{10} = 10$$

And,

$$fog(y) = f(g(y)) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$$\therefore gof = \mathrm{I}_{\mathbf{R}} \text{ and } fog = \mathrm{I}_{\mathbf{R}}$$

Hence, the required function g: **R** → **R** is defined as $g(y) = \dfrac{y-7}{10}$.

Question 2:

Let f: W → W be defined as $f(n) = n - 1$, if is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.

Answer

It is given that:

f: W → W is defined as $f(n) = \begin{cases} n-1, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$

One-one:

Let $f(n) = f(m)$.

It can be observed that if n is odd and m is even, then we will have $n - 1 = m + 1$.

$\Rightarrow n - m = 2$

However, this is impossible.

Similarly, the possibility of n being even and m being odd can also be ignored under a similar argument.

∴Both n and m must be either odd or even.

Now, if both n and m are odd, then we have:

$f(n) = f(m) \Rightarrow n - 1 = m - 1 \Rightarrow n = m$

Again, if both n and m are even, then we have:

$f(n) = f(m) \Rightarrow n + 1 = m + 1 \Rightarrow n = m$

∴f is one-one.

It is clear that any odd number $2r + 1$ in co-domain **N** is the image of $2r$ in domain **N** and any even number $2r$ in co-domain **N** is the image of $2r + 1$ in domain **N**.

∴f is onto.

Hence, f is an invertible function.

Let us define g: W → W as:

$$g(m) = \begin{cases} m+1, & \text{if } m \text{ is even} \\ m-1, & \text{if } m \text{ is odd} \end{cases}$$

Now, when n is odd:

$$gof(n) = g(f(n)) = g(n-1) = n-1+1 = n$$

And, when n is even:

$$gof(n) = g(f(n)) = g(n+1) = n+1-1 = n$$

Similarly, when m is odd:

$$fog(m) = f(g(m)) = f(m-1) = m-1+1 = m$$

When m is even:

$$fog(m) = f(g(m)) = f(m+1) = m+1-1 = m$$

$\therefore$ $\therefore gof = I_W$ and $fog = I_W$

Thus, f is invertible and the inverse of f is given by $f^{-1} = g$, which is the same as f.

Hence, the inverse of f is f itself.

Question 3:

If f: **R** → **R** is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

Answer

It is given that f: **R** → **R** is defined as $f(x) = x^2 - 3x + 2$.

$$\begin{aligned} f(f(x)) &= f\left(x^2-3x+2\right) \\ &= \left(x^2-3x+2\right)^2 - 3\left(x^2-3x+2\right)+2 \\ &= x^4+9x^2+4-6x^3-12x+4x^2-3x^2+9x-6+2 \\ &= x^4-6x^3+10x^2-3x \end{aligned}$$

Question 4:

Show that function f: **R** → $\{x \in \mathbf{R}: -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in \mathbf{R}$ is one-one and onto function.

Answer

It is given that f: **R** → $\{x \in \mathbf{R}: -1 < x < 1\}$ is defined as $f(x) = \frac{x}{1+|x|}$, $x \in \mathbf{R}$.

Suppose $f(x) = f(y)$, where $x, y \in \mathbf{R}$.

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$$

It can be observed that if *x* is positive and *y* is negative, then we have:

$$\frac{x}{1+x} = \frac{y}{1-y} \Rightarrow 2xy = x - y$$

Since *x* is positive and *y* is negative:

$x > y \Rightarrow x - y > 0$

But, $2xy$ is negative.

Then, $2xy \neq x - y$.

Thus, the case of *x* being positive and *y* being negative can be ruled out.

Under a similar argument, *x* being negative and *y* being positive can also be ruled out

$\therefore$ *x* and *y* have to be either positive or negative.

When *x* and *y* are both positive, we have:

$$f(x) = f(y) \Rightarrow \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$$

When *x* and *y* are both negative, we have:

$$f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y} \Rightarrow x - xy = y - yx \Rightarrow x = y$$

$\therefore$ *f* is one-one.

Now, let $y \in \mathbf{R}$ such that $-1 < y < 1$.

If *y* is negative, then there exists $x = \frac{y}{1+v} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y.$$

If *y* is positive, then there exists $x = \frac{y}{1-y} \in \mathbf{R}$ such that

$$f(x)=f\left(\frac{y}{1-y}\right)=\frac{\left(\frac{y}{1-y}\right)}{1+\left|\left(\frac{y}{1-y}\right)\right|}=\frac{\frac{y}{1-y}}{1+\frac{y}{1-y}}=\frac{y}{1-y+y}=y.$$

∴ *f* is onto.

Hence, *f* is one-one and onto.

Question 5:

Show that the function *f*: **R** → **R** given by $f(x) = x^3$ is injective.

Answer

f: **R** → **R** is given as $f(x) = x^3$.

Suppose $f(x) = f(y)$, where $x, y \in$ **R.**

⇒ $x^3 = y^3$ … (1)

Now, we need to show that $x = y$.

Suppose $x \neq y$, their cubes will also not be equal.

$\Rightarrow x^3 \neq y^3$

However, this will be a contradiction to (1).

∴ $x = y$

Hence, *f* is injective.

Question 6:

Give examples of two functions *f*: **N** → **Z** and *g*: **Z** → **Z** such that *g* o *f* is injective but *g* is not injective.

(Hint: Consider $f(x) = x$ and $g(x) = |x|$)

Answer

Define *f*: **N** → **Z** as $f(x) = x$ and *g*: **Z** → **Z** as $g(x) = |x|$.

We first show that *g* is not injective.

It can be observed that:

$g(-1) = |-1| = 1$

$g(1) = |1| = 1$

$\therefore$ $g(-1) = g(1)$, but $-1 \neq 1$.

$\therefore$ g is not injective.

Now, *gof*: **N** → **Z** is defined as $gof(x) = g(f(x)) = g(x) = |x|$.

Let x, $y \in$ **N** such that $gof(x) = gof(y)$.

$\Rightarrow |x| = |y|$

Since x and $y \in$ **N,** both are positive.

$\therefore |x| = |y| \Rightarrow x = y$

Hence, *gof* is injective

Question 7:

Given examples of two functions *f*: **N** → **N** and *g*: **N** → **N** such that *gof* is onto but *f* is not onto.

(Hint: Consider $f(x) = x + 1$ and $g(x) = \begin{cases} x-1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$

Answer

Define *f*: **N** → **N** by,

$f(x) = x + 1$

And, *g*: **N** → **N** by,

$$g(x) = \begin{cases} x-1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$$

We first show that *g* is not onto.

For this, consider element 1 in co-domain **N**. It is clear that this element is not an image of any of the elements in domain **N**.

$\therefore$ *f* is not onto.

Now, *gof*: **N** → **N** is defined by,

$$\begin{aligned} gof(x) = g(f(x)) = g(x+1) &= (x+1)-1 \qquad [x \in \mathbf{N} \Rightarrow (x+1) > 1] \\ &= x \end{aligned}$$

Then, it is clear that for $y \in$ **N**, there exists $x = y \in$ **N** such that $gof(x) = y$.

Hence, *gof* is onto.

Question 8:

Given a non empty set X, consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows:

For subsets A, B in P(X), ARB if and only if $A \subset B$. Is R an equivalence relation on P(X)?

Justify you answer:

Answer

Since every set is a subset of itself, ARA for all $A \in$ P(X).

$\therefore$R is reflexive.

Let AR$B \Rightarrow A \subset B$.

This cannot be implied to $B \subset A$.

For instance, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then it cannot be implied that B is related to A.

$\therefore$ R is not symmetric.

Further, if ARB and BRC, then $A \subset B$ and $B \subset C$.

$\Rightarrow A \subset C$

$\Rightarrow A$RC

$\therefore$ R is transitive.

Hence, R is not an equivalence relation since it is not symmetric.

Question 9:

Given a non-empty set X, consider the binary operation $*$: P(X) × P(X) → P(X) given by $A * B = A \cap B$ ☐ A, B in P(X) is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in P(X) with respect to the operation$*$.

Answer

It is given that $*: \mathrm{P}(X) \times \mathrm{P}(X) \rightarrow \mathrm{P}(X)$ is defined as $A * B = A \cap B \ \forall \ A, B \in \mathrm{P}(X)$.

We know that $A \cap X = A = X \cap A \ \forall \ A \in \mathrm{P}(X)$.

$\Rightarrow A * X = A = X * A \ \forall \ A \in \mathrm{P}(X)$

Thus, X is the identity element for the given binary operation $*$.

Now, an element $A \in \mathrm{P}(X)$ is invertible if there exists $B \in \mathrm{P}(X)$ such that

$A * B = X = B * A$. (As X is the identity element)

i.e.,

$A \cap B = X = B \cap A$

This case is possible only when $A = X = B$.

Thus, X is the only invertible element in P(X) with respect to the given operation*.

Hence, the given result is proved.

Question 10:

Find the number of all onto functions from the set {1, 2, 3, … , n) to itself.

Answer

Onto functions from the set {1, 2, 3, … ,n} to itself is simply a permutation on n symbols 1, 2, …, n.

Thus, the total number of onto maps from {1, 2, … , n} to itself is the same as the total number of permutations on n symbols 1, 2, …, n, which is n.

Question 11:

Let S = {a, b, c} and T = {1, 2, 3}. Find F^{-1} of the following functions F from S to T, if it exists.

(i) F = {(a, 3), (b, 2), (c, 1)} (ii) F = {(a, 2), (b, 1), (c, 1)}

Answer

S = {a, b, c}, T = {1, 2, 3}

(i) F: $S \rightarrow T$ is defined as:

F = {(a, 3), (b, 2), (c, 1)}

⇒ F (a) = 3, F (b) = 2, F(c) = 1

Therefore, F^{-1}: $T \rightarrow S$ is given by

F^{-1} = {(3, a), (2, b), (1, c)}.

(ii) F: $S \rightarrow T$ is defined as:

F = {(a, 2), (b, 1), (c, 1)}

Since F (b) = F (c) = 1, F is not one-one.

Hence, F is not invertible i.e., F^{-1} does not exist.

Question 12:

Consider the binary operations*: **R** ×**R** → and o: **R** × **R** → **R** defined as $a*b=|a-b|$ and *a* o *b* = *a*, □*a*, *b* ∈ **R**. Show that * is commutative but not associative, o is associative but not commutative. Further, show that □*a*, *b*, *c* ∈ **R**, *a**(*b* o *c*) = (*a* * *b*) o (*a* * *c*). [If it is so, we say that the operation * distributes over the operation o]. Does o distribute over *? Justify your answer.

Answer

It is given that *: **R** ×**R** → and o: **R** × **R** → **R** isdefined as

$a*b=|a-b|$ and *a* o *b* = *a*, □*a*, *b* ∈ **R**.

For *a*, *b* ∈ **R**, we have:

$a*b=|a-b|$

$b*a=|b-a|=|-(a-b)|=|a-b|$

∴*a* * *b* = *b* * *a*

∴ The operation * is commutative.

It can be observed that,

$(1*2)*3=(|1-2|)*3=1*3=|1-3|=2$

$1*(2*3)=1*(|2-3|)=1*1=|1-1|=0$

$\therefore (1*2)*3\neq 1*(2*3)$ (where $1, 2, 3 \in \mathbf{R}$)

∴The operation * is not associative.

Now, consider the operation o:

It can be observed that 1 o 2 = 1 and 2 o 1 = 2.

∴1 o 2 ≠ 2 o 1 (where 1, 2 ∈ **R**)

∴The operation o is not commutative.

Let *a*, *b*, *c* ∈ **R**. Then, we have:

(*a* o *b*) o *c* = *a* o *c* = *a*

a o (*b* o *c*) = *a* o *b* = *a*

⇒ *a* o *b*) o c = *a* o (*b* o *c*)

∴ The operation o is associative.

Now, let *a*, *b*, *c* ∈ **R**, then we have:

a * (*b* o *c*) = a * *b* = $|a-b|$

(*a* * *b*) o (*a* * *c*) = $(|a-b|)\text{o}(|a-c|)=|a-b|$

Hence, *a* * (*b* o *c*) = (*a* * *b*) o (*a* * *c*).

Now,

1 o (2 * 3) = $1\text{ o}(|2-3|)=1\text{ o }1=1$

(1 o 2) * (1 o 3) = 1 * 1 = $|1-1|=0$

∴1 o (2 * 3) ≠ (1 o 2) * (1 o 3) (where 1, 2, 3 ∈ **R**)

∴ The operation o does not distribute over *.

Question 13:

Given a non-empty set *X*, let *: P(*X*) × P(*X*) → P(*X*) be defined as *A* * *B* = (*A* − *B*) ∪ (*B* − *A*), ☐ *A*, *B* ∈ P(*X*). Show that the empty set *Φ* is the identity for the operation * and all the elements *A* of P(*X*) are invertible with $A^{-1} = A$. (Hint: (*A* − *Φ*) ∪ (*Φ* − *A*) = *A* and (*A* − *A*) ∪ (*A* − *A*) = *A* * *A* = *Φ*).

Answer

It is given that *: P(*X*) × P(*X*) → P(*X*) is defined as

A * *B* = (*A* − *B*) ∪ (*B* − *A*) ☐ *A*, *B* ∈ P(*X*).

Let *A* ∈ P(*X*). Then, we have:

A * *Φ* = (*A* − *Φ*) ∪ (*Φ* − *A*) = *A* ∪ *Φ* = *A*

Φ * *A* = (*Φ* − *A*) ∪ (*A* − *Φ*) = *Φ* ∪ *A* = *A*

∴*A* * *Φ* = *A* = *Φ* * *A*. ☐ *A* ∈ P(*X*)

Thus, *Φ* is the identity element for the given operation*.

Now, an element *A* ∈ P(*X*) will be invertible if there exists *B* ∈ P(*X*) such that

A * *B* = *Φ* = *B* * *A*. (As Φ is the identity element)

Now, we observed that $A*A=(A-A)\cup(A-A)=\phi\cup\phi=\phi\ \forall A\in \text{P}(X)$.

Hence, all the elements *A* of P(*X*) are invertible with $A^{-1} = A$.

Question 14:

Define a binary operation *on the set {0, 1, 2, 3, 4, 5} as

$$a*b=\begin{cases}a+b, & \text{if } a+b<6\\ a+b-6 & \text{if } a+b\geq 6\end{cases}$$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$ being the inverse of a.

Answer

Let $X = \{0, 1, 2, 3, 4, 5\}$.

The operation * on X is defined as:

$$a*b=\begin{cases}a+b & \text{if } a+b<6\\ a+b-6 & \text{if } a+b\geq 6\end{cases}$$

An element $e \in X$ is the identity element for the operation *, if $a*e=a=e*a\ \forall a\in X$.

For $a \in X$, we observed that:

$$a*0=a+0=a \qquad [a\in X\Rightarrow a+0<6]$$

$$0*a=0+a=a \qquad [a\in X\Rightarrow 0+a<6]$$

$$\therefore a*0=a=0*a\ \forall a\in X$$

Thus, 0 is the identity element for the given operation *.

An element $a \in X$ is invertible if there exists $b \in X$ such that $a * b = 0 = b * a$.

$$\text{i.e.,}\begin{cases}a+b=0=b+a, & \text{if } a+b<6\\ a+b-6=0=b+a-6, & \text{if } a+b\geq 6\end{cases}$$

i.e.,

$a = -b$ or $b = 6 - a$

But, $X = \{0, 1, 2, 3, 4, 5\}$ and $a, b \in X$. Then, $a \neq -b$.

$\therefore b = 6 - a$ is the inverse of a □ $a \in X$.

Hence, the inverse of an element $a \in X$, $a \neq 0$ is $6 - a$ i.e., $a^{-1} = 6 - a$.

Question 15:

Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g: A \to B$ be functions defined by $f(x) = x^2 - x, x \in A$ and $g(x)=2\left|x-\frac{1}{2}\right|-1,\ x\in A$. Are f and g equal?

Justify your answer. (Hint: One may note that two function $f: A \to B$ and $g: A \to B$ such that $f(a) = g(a)$ □$a \in A$, are called equal functions).

Answer

It is given that $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$.

Also, it is given that $f, g: A \to B$ are defined by $f(x) = x^2 - x$, $x \in A$ and

$$g(x) = 2\left|x - \frac{1}{2}\right| - 1,\ x \in A$$
.

It is observed that:

$$f(-1) = (-1)^2 - (-1) = 1 + 1 = 2$$

$$g(-1) = 2\left|(-1) - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$$

$$\Rightarrow f(-1) = g(-1)$$

$$f(0) = (0)^2 - 0 = 0$$

$$g(0) = 2\left|0 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$

$$\Rightarrow f(0) = g(0)$$

$$f(1) = (1)^2 - 1 = 1 - 1 = 0$$

$$g(1) = 2\left|1 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$

$$\Rightarrow f(1) = g(1)$$

$$f(2) = (2)^2 - 2 = 4 - 2 = 2$$

$$g(2) = 2\left|2 - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$$

$$\Rightarrow f(2) = g(2)$$

$$\therefore f(a) = g(a)\ \forall a \in A$$

Hence, the functions f and g are equal.

Question 16:

Let $A = \{1, 2, 3\}$. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1 (B) 2 (C) 3 (D) 4

Answer

The given set is A = {1, 2, 3}.

The smallest relation containing (1, 2) and (1, 3) which is reflexive and symmetric, but not transitive is given by:

R = {(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (3, 1)}

This is because relation R is reflexive as (1, 1), (2, 2), (3, 3) ∈ R.

Relation R is symmetric since (1, 2), (2, 1) ∈R and (1, 3), (3, 1) ∈R.

But relation R is not transitive as (3, 1), (1, 2) ∈ R, but (3, 2) ∉ R.

Now, if we add any two pairs (3, 2) and (2, 3) (or both) to relation R, then relation R will become transitive.

Hence, the total number of desired relations is one.

The correct answer is A.

Question 17:

Let *A* = {1, 2, 3}. Then number of equivalence relations containing (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4

Answer

It is given that *A* = {1, 2, 3}.

The smallest equivalence relation containing (1, 2) is given by,

R_1 = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}

Now, we are left with only four pairs i.e., (2, 3), (3, 2), (1, 3), and (3, 1).

If we odd any one pair [say (2, 3)] to R_1, then for symmetry we must add (3, 2). Also, for transitivity we are required to add (1, 3) and (3, 1).

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing (1, 2) is two.

The correct answer is B.

Question 18:

Let *f*: **R** → **R** be the Signum Function defined as

$$f(x)=\begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$$

and *g*: **R** → **R** be the Greatest Integer Function given by *g*(*x*) = [*x*], where [*x*] is greatest integer less than or equal to *x*. Then does *fog* and *gof* coincide in (0, 1]?

Answer

It is given that,

f: **R** → **R** is defined as $f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$

Also, *g*: **R** → **R** is defined as *g*(*x*) = [*x*], where [*x*] is the greatest integer less than or equal to *x*.

Now, let *x* ∈ (0, 1].

Then, we have:

[*x*] = 1 if *x* = 1 and [*x*] = 0 if 0 < *x* < 1.

$$\therefore fog(x)=f(g(x))=f([x])=\begin{cases}f(1), & \text{if } x=1\\ f(0), & \text{if } x\in(0,1)\end{cases}=\begin{cases}1, & \text{if } x=1\\ 0, & \text{if } x\in(0,1)\end{cases}$$

$$\begin{aligned}gof(x)&=g(f(x))\\&=g(1) \qquad [x>0]\\&=[1]=1\end{aligned}$$

Thus, when *x* ∈ (0, 1), we have *fog*(*x*) = 0and *gof* (*x*) = 1.

Hence, *fog* and *gof* do not coincide in (0, 1].

Question 19:

Number of binary operations on the set {*a*, *b*} are

(A) 10 (B) 16 (C) 20 (D) 8

Answer

A binary operation * on {*a*, *b*} is a function from {*a*, *b*} × {*a*, *b*} → {*a*, *b*}

i.e., * is a function from {(*a*, *a*), (*a*, *b*), (*b*, *a*), (*b*, *b*)} → {*a*, *b*}.

Hence, the total number of binary operations on the set {*a*, *b*} is 2^4 i.e., 16.

The correct answer is B.

Exercise 2.1

Question 1:

Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$

Answer

Let $\sin^{-1}\left(-\frac{1}{2}\right) = y.$ Then $\sin y = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right).$

We know that the range of the principal value branch of $\sin^{-1}$ is

$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}.$

Therefore, the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}.$

Question 2:

Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Answer

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y.$ Then, $\cos y = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right).$

We know that the range of the principal value branch of $\cos^{-1}$ is

$[0,\pi]$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

Therefore, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

Question 3:

Find the principal value of $\text{cosec}^{-1}(2)$

Answer

Let $\operatorname{cosec}^{-1}(2) = y$. Then, $\operatorname{cosec} y = 2 = \operatorname{cosec}\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$.

Therefore, the principal value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

Question 4:

Find the principal value of $\tan^{-1}\left(-\sqrt{3}\right)$

Answer

$$\text{Let } \tan^{-1}\left(-\sqrt{3}\right) = y. \text{ Then, } \tan y = -\sqrt{3} = -\tan\frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right).$$

We know that the range of the principal value branch of $\tan^{-1}$ is

$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{3}\right)$ is $-\sqrt{3}$.

Therefore, the principal value of $\tan^{-1}\left(\sqrt{3}\right)$ is $-\frac{\pi}{3}$.

Question 5:

Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$

Answer

$$\text{Let } \cos^{-1}\left(-\frac{1}{2}\right) = y. \text{ Then, } \cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi-\frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right).$$

We know that the range of the principal value branch of $\cos^{-1}$ is

$[0,\pi]$ and $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

Question 6:

Find the principal value of $\tan^{-1}(-1)$

Answer

Let $\tan^{-1}(-1) = y$. Then, $\tan y = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{4}\right) = -1$.

Therefore, the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

Question 7:

Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Answer

Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$. Then, $\sec y = \frac{2}{\sqrt{3}} = \sec\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\sec^{-1}$ is

$[0, \pi] - \left\{\frac{\pi}{2}\right\}$ and $\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$.

Therefore, the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$.

Question 8:

Find the principal value of $\cot^{-1}\left(\sqrt{3}\right)$

Answer

Let $\cot^{-1}\left(\sqrt{3}\right) = y$. Then, $\cot y = \sqrt{3} = \cot\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\cot^{-1}$ is (0,π) and

$$\cot\left(\frac{\pi}{6}\right)=\sqrt{3}.$$

Therefore, the principal value of $\cot^{-1}\left(\sqrt{3}\right)$ is $\frac{\pi}{6}.$

Question 9:

Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Answer

$$\text{Let } \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=y. \text{ Then, } \cos y=-\frac{1}{\sqrt{2}}=-\cos\left(\frac{\pi}{4}\right)=\cos\left(\pi-\frac{\pi}{4}\right)=\cos\left(\frac{3\pi}{4}\right).$$

We know that the range of the principal value branch of $\cos^{-1}$ is [0,π] and

$$\cos\left(\frac{3\pi}{4}\right)=-\frac{1}{\sqrt{2}}.$$

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}.$

Question 10:

Find the principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

Answer

$$\text{Let } \operatorname{cosec}^{-1}\left(-\sqrt{2}\right)=y. \text{ Then, } \operatorname{cosec} y=-\sqrt{2}=-\operatorname{cosec}\left(\frac{\pi}{4}\right)=\operatorname{cosec}\left(-\frac{\pi}{4}\right).$$

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is

$$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\} \text{ and } \operatorname{cosec}\left(-\frac{\pi}{4}\right)=-\sqrt{2}.$$

Therefore, the principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$ is $-\frac{\pi}{4}.$

Question 11:

Find the value of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

Answer

Let $\tan^{-1}(1)=x$. Then, $\tan x = 1 = \tan\frac{\pi}{4}$.

$\therefore \tan^{-1}(1)=\frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right)=y$. Then, $\cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right)=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\left(\frac{2\pi}{3}\right)$.

$\therefore \cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right)=z$. Then, $\sin z = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right)=\sin\left(-\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

$\therefore \tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

$=\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}$

$=\frac{3\pi+8\pi-2\pi}{12}=\frac{9\pi}{12}=\frac{3\pi}{4}$

Question 12:

Find the value of $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

Answer

Let $\cos^{-1}\left(\frac{1}{2}\right)=x$. Then, $\cos x=\frac{1}{2}=\cos\left(\frac{\pi}{3}\right)$.

$\therefore \cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$

Let $\sin^{-1}\left(\frac{1}{2}\right)=y$. Then, $\sin y=\frac{1}{2}=\sin\left(\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

$\therefore \cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}+\frac{2\pi}{6}=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$

Question 13:

Find the value of if $\sin^{-1} x = y$, then

(A) $0\le y\le\pi$ **(B)** $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$

(C) $0<y<\pi$ **(D)** $-\frac{\pi}{2}<y<\frac{\pi}{2}$

Answer

It is given that $\sin^{-1} x = y$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Therefore, $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$.

Question 14:

Find the value of $\tan^{-1}\sqrt{3}-\sec^{-1}(-2)$ is equal to

(A) π (**B**) $-\frac{\pi}{3}$ (**C**) $\frac{\pi}{3}$ (**D**) $\frac{2\pi}{3}$

Answer

Let $\tan^{-1}\sqrt{3} = x$. Then, $\tan x = \sqrt{3} = \tan\frac{\pi}{3}$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Let $\sec^{-1}(-2) = y$. Then, $\sec y = -2 = -\sec\left(\frac{\pi}{3}\right) = \sec\left(\pi - \frac{\pi}{3}\right) = \sec\frac{2\pi}{3}$.

We know that the range of the principal value branch of $\sec^{-1}$ is $[0,\pi] - \left\{\frac{\pi}{2}\right\}$.

$\therefore \sec^{-1}(-2) = \frac{2\pi}{3}$

Hence, $\tan^{-1}\left(\sqrt{3}\right) - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$

Exercise 2.2

Question 1:

Prove $3\sin^{-1}x = \sin^{-1}\left(3x - 4x^3\right),\ x \in \left[-\frac{1}{2},\ \frac{1}{2}\right]$

Answer

To prove: $3\sin^{-1}x = \sin^{-1}\left(3x - 4x^3\right),\ x \in \left[-\frac{1}{2},\ \frac{1}{2}\right]$

Let x = sinθ. Then, $\sin^{-1}x = \theta.$

We have,

R.H.S. = $\sin^{-1}\left(3x - 4x^3\right) = \sin^{-1}\left(3\sin\theta - 4\sin^3\theta\right)$

$= \sin^{-1}\left(\sin 3\theta\right)$

= 3θ

$= 3\sin^{-1}x$

= L.H.S.

Question 2:

Prove $3\cos^{-1}x = \cos^{-1}\left(4x^3 - 3x\right),\ x \in \left[\frac{1}{2},\ 1\right]$

Answer

To prove: $3\cos^{-1}x = \cos^{-1}\left(4x^3 - 3x\right),\ x \in \left[\frac{1}{2},\ 1\right]$

Let x = cosθ. Then, $\cos^{-1}x = \theta$.

We have,

$$\begin{aligned}\text{R.H.S.} &= \cos^{-1}\left(4x^3-3x\right)\\ &= \cos^{-1}\left(4\cos^3\theta-3\cos\theta\right)\\ &= \cos^{-1}\left(\cos 3\theta\right)\\ &= 3\theta\\ &= 3\cos^{-1}x\\ &= \text{L.H.S.}\end{aligned}$$

Question 3:

Prove $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

Answer

To prove: $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

$$\text{L.H.S.} = \tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}$$

$$= \tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\cdot\frac{7}{24}} \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{\frac{48+77}{11\times 24}}{\frac{11\times 24-14}{11\times 24}}$$

$$= \tan^{-1}\frac{48+77}{264-14}=\tan^{-1}\frac{125}{250}=\tan^{-1}\frac{1}{2}=\text{R.H.S.}$$

Question 4:

Prove $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Answer

To prove: $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

$$\text{L.H.S.} = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{2\cdot\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} + \tan^{-1}\frac{1}{7} \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{1}{\left(\frac{3}{4}\right)} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{\left(\frac{28+3}{21}\right)}{\left(\frac{21-4}{21}\right)}$$

$$= \tan^{-1}\frac{31}{17} = \text{R.H.S.}$$

Question 5:

Write the function in the simplest form:

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x},\ x \neq 0$$

Answer

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$$

Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$$\therefore \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$$

Question 6:

Write the function in the simplest form:

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}},\ |x|>1$$

Answer

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}},\ |x|>1$$

Put x = cosec $\theta \Rightarrow \theta$ = cosec^{-1} x

$$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}}=\tan^{-1}\frac{1}{\sqrt{\operatorname{cos}ec^2\theta-1}}$$

$$=\tan^{-1}\left(\frac{1}{\cot\theta}\right)=\tan^{-1}(\tan\theta)$$

$$=\theta=\operatorname{cosec}^{-1}x=\frac{\pi}{2}-\sec^{-1}x \qquad \left[\operatorname{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2}\right]$$

Question 7:

Write the function in the simplest form:

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),\ x<\pi$$

Answer

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),\ x<\pi$$

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$=\tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)=\tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$=\frac{x}{2}$$

Question 8:

Write the function in the simplest form:

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right),\ 0<x<\pi$$

Answer

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$$

$$=\tan^{-1}\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$=\tan^{-1}(1)-\tan^{-1}(\tan x) \qquad \left[\tan^{-1}\frac{x-y}{1-xy}=\tan^{-1}x-\tan^{-1}y\right]$$

$$=\frac{\pi}{4}-x$$

Question 9:

Write the function in the simplest form:

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}},\ |x|<a$$

Answer

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$$

$$\text{Put } x=a\sin\theta \Rightarrow \frac{x}{a}=\sin\theta \Rightarrow \theta=\sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \tan^{-1}\frac{x}{\sqrt{a^2-x^2}}=\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)=\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

$$=\tan^{-1}(\tan\theta)=\theta=\sin^{-1}\frac{x}{a}$$

Question 10:

Write the function in the simplest form:

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right),\ a>0;\ \frac{-a}{\sqrt{3}}\le x\le\frac{a}{\sqrt{3}}$$

Answer

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$$

$$\text{Put } x=a\tan\theta \Rightarrow \frac{x}{a}=\tan\theta \Rightarrow \theta=\tan^{-1}\frac{x}{a}$$

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)=\tan^{-1}\left(\frac{3a^2\cdot a\tan\theta-a^3\tan^3\theta}{a^3-3a\cdot a^2\tan^2\theta}\right)$$

$$=\tan^{-1}\left(\frac{3a^3\tan\theta-a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right)$$

$$=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$=\tan^{-1}(\tan 3\theta)$$

$$=3\theta$$

$$=3\tan^{-1}\frac{x}{a}$$

Question 11:

Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

Answer

Let $\sin^{-1}\frac{1}{2}=x$. Then, $\sin x=\frac{1}{2}=\sin\left(\frac{\pi}{6}\right)$.

$$\therefore \sin^{-1}\frac{1}{2}=\frac{\pi}{6}$$

$$\therefore \tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]=\tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[2\cos\frac{\pi}{3}\right]=\tan^{-1}\left[2\times\frac{1}{2}\right]$$

$$=\tan^{-1}1=\frac{\pi}{4}$$

Question 12:

Find the value of $\cot\left(\tan^{-1} a+\cot^{-1} a\right)$

Answer

$$\cot\left(\tan^{-1} a+\cot^{-1} a\right)$$

$$=\cot\left(\frac{\pi}{2}\right) \qquad \left[\tan^{-1} x+\cot^{-1} x=\frac{\pi}{2}\right]$$

$$=0$$

Question 13:

Find the value of $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right],\ |x|<1,\ y>0 \text{ and } xy<1$

Answer

Let $x = \tan\theta$. Then, $\theta = \tan^{-1} x$.

$$\therefore \sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}\left(\sin 2\theta\right)=2\theta=2\tan^{-1} x$$

Let $y = \tan\Phi$. Then, $\Phi = \tan^{-1} y$.

$$\therefore \cos^{-1}\frac{1-y^2}{1+y^2}=\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)=\cos^{-1}\left(\cos 2\phi\right)=2\phi=2\tan^{-1} y$$

$$\therefore \tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$=\tan\frac{1}{2}\left[2\tan^{-1} x+2\tan^{-1} y\right]$$

$$=\tan\left[\tan^{-1} x+\tan^{-1} y\right]$$

$$=\tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\frac{x+y}{1-xy}$$

Question 14:

If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of *x*.

Answer

$$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$$

$$\Rightarrow \sin\left(\sin^{-1}\frac{1}{5}\right)\cos\left(\cos^{-1}x\right)+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\left[\sin(A+B)=\sin A\cos B+\cos A\sin B\right]$$

$$\Rightarrow \frac{1}{5}\times x+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\Rightarrow \frac{x}{5}+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1 \qquad ...(1)$$

Now, let $\sin^{-1}\frac{1}{5}=y$.

Then, $\sin y=\frac{1}{5}\Rightarrow \cos y=\sqrt{1-\left(\frac{1}{5}\right)^2}=\frac{2\sqrt{6}}{5}\Rightarrow y=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right)$.

$$\therefore \sin^{-1}\frac{1}{5}=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right) \qquad ...(2)$$

Let $\cos^{-1}x=z$.

Then, $\cos z=x\Rightarrow \sin z=\sqrt{1-x^2}\Rightarrow z=\sin^{-1}\left(\sqrt{1-x^2}\right)$.

$$\therefore \cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}\right) \qquad ...(3)$$

From (1), (2), and (3) we have:

$$\frac{x}{5}+\cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right)\cdot\sin\left(\sin^{-1}\sqrt{1-x^2}\right)=1$$

$$\Rightarrow \frac{x}{5}+\frac{2\sqrt{6}}{5}\cdot\sqrt{1-x^2}=1$$

$$\Rightarrow x+2\sqrt{6}\sqrt{1-x^2}=5$$

$$\Rightarrow 2\sqrt{6}\sqrt{1-x^2}=5-x$$

On squaring both sides, we get:

$$(4)(6)(1-x^2)=25+x^2-10x$$
$$\Rightarrow 24-24x^2=25+x^2-10x$$
$$\Rightarrow 25x^2-10x+1=0$$
$$\Rightarrow (5x-1)^2=0$$
$$\Rightarrow (5x-1)=0$$
$$\Rightarrow x=\frac{1}{5}$$

Hence, the value of *x* is $\frac{1}{5}$.

Question 15:

If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$, then find the value of *x*.

Answer

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4} \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)(x+2)+(x+1)(x-2)}{(x+2)(x-2)-(x-1)(x+1)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x^2-4}{-3}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan\left[\tan^{-1}\frac{4-2x^2}{3}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{4-2x^2}{3}=1$$

$$\Rightarrow 4-2x^2=3$$

$$\Rightarrow 2x^2=4-3=1$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

Hence, the value of x is $\pm\frac{1}{\sqrt{2}}$.

Question 16:

Find the values of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

Answer

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

We know that $\sin^{-1}(\sin x) = x$ if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal value branch of $\sin^{-1}x$.

Here, $\frac{2\pi}{3} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ can be written as:

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{3}\right)\right] = \sin^{-1}\left(\sin\frac{\pi}{3}\right) \text{ where } \frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

Question 17:

Find the values of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

Answer

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$$

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{3\pi}{4} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ can be written as:

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left[-\tan\left(\frac{-3\pi}{4}\right)\right] = \tan^{-1}\left[-\tan\left(\pi - \frac{\pi}{4}\right)\right]$$

$$= \tan^{-1}\left[-\tan\frac{\pi}{4}\right] = \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] \textit{ where } -\frac{\pi}{4} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right]=\frac{-\pi}{4}$$

Question 18:

Find the values of $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

Answer

Let $\sin^{-1}\frac{3}{5}=x$. Then, $\sin x=\frac{3}{5}\Rightarrow \cos x=\sqrt{1-\sin^2 x}=\frac{4}{5}\Rightarrow \sec x=\frac{5}{4}.$

$$\therefore \tan x=\sqrt{\sec^2 x-1}=\sqrt{\frac{25}{16}-1}=\frac{3}{4}$$

$$\therefore x=\tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4} \qquad ...(i)$$

$$\text{Now, } \cot^{-1}\frac{3}{2}=\tan^{-1}\frac{2}{3} \qquad ...(ii) \qquad \left[\tan^{-1}\frac{1}{x}=\cot^{-1}x\right]$$

$$\text{Hence, } \tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$$

$$=\tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right) \qquad [\text{Using (i) and (ii)}]$$

$$=\tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right) \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$=\tan\left(\tan^{-1}\frac{9+8}{12-6}\right)$$

$$=\tan\left(\tan^{-1}\frac{17}{6}\right)=\frac{17}{6}$$

Question 19:

Find the values of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$ **(B)** $\frac{5\pi}{6}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{6}$

Answer

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{7\pi}{6} \notin x \in [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{-7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right] \quad [\cos(2\pi + x) = \cos x]$$

$$= \cos^{-1}\left[\cos\frac{5\pi}{6}\right] \text{ where } \frac{5\pi}{6} \in [0, \pi]$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$$

The correct answer is B.

Question 20:

Find the values of $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** 1

Answer

Let $\sin^{-1}\left(\frac{-1}{2}\right) = x$. Then, $\sin x = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right)$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$$

$$\therefore \sin\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{3\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

The correct answer is D.

Miscellaneous Solutions

Question 1:

Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

Answer

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{13\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right] = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right], \text{ where } \frac{\pi}{6} \in [0, \pi].$$

$$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right] = \frac{\pi}{6}$$

Question 2:

Find the value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Answer

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{7\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ can be written as:

$$\tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left[\tan\left(2\pi-\frac{5\pi}{6}\right)\right] \qquad \left[\tan(2\pi-x)=-\tan x\right]$$

$$=\tan^{-1}\left[-\tan\left(\frac{5\pi}{6}\right)\right]=\tan^{-1}\left[\tan\left(-\frac{5\pi}{6}\right)\right]=\tan^{-1}\left[\tan\left(\pi-\frac{5\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{6}\right)\right],\text{ where }\frac{\pi}{6}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left(\tan\frac{\pi}{6}\right)=\frac{\pi}{6}$$

Question 3:

Prove $2\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{24}{7}$

Answer

$$\text{Let }\sin^{-1}\frac{3}{5}=x.\text{ Then, }\sin x=\frac{3}{5}.$$

$$\Rightarrow \cos x=\sqrt{1-\left(\frac{3}{5}\right)^2}=\frac{4}{5}$$

$$\therefore \tan x=\frac{3}{4}$$

$$\therefore x=\tan^{-1}\frac{3}{4}\Rightarrow \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$$

Now, we have:

$$\text{L.H.S.} = 2\sin^{-1}\frac{3}{5} = 2\tan^{-1}\frac{3}{4}$$

$$= \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right) \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{16-9}{16}}\right) = \tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right)$$

$$= \tan^{-1}\frac{24}{7} = \text{R.H.S.}$$

Question 4:

Prove $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{77}{36}$

Answer

Let $\sin^{-1}\frac{8}{17}=x$. Then, $\sin x=\frac{8}{17}\Rightarrow \cos x=\sqrt{1-\left(\frac{8}{17}\right)^2}=\sqrt{\frac{225}{289}}=\frac{15}{17}$.

$\therefore \tan x=\frac{8}{15}\Rightarrow x=\tan^{-1}\frac{8}{15}$

$\therefore \sin^{-1}\frac{8}{17}=\tan^{-1}\frac{8}{15} \qquad ...(1)$

Now, let $\sin^{-1}\frac{3}{5}=y$. Then, $\sin y=\frac{3}{5}\Rightarrow \cos y=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$.

$\therefore \tan y=\frac{3}{4}\Rightarrow y=\tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4} \qquad ...(2)$

Now, we have:

$$\text{L.H.S.} = \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$$

$$= \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{3}{4} \qquad \left[\text{Using } (1) \text{ and } (2)\right]$$

$$= \tan^{-1}\frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15} \times \frac{3}{4}}$$

$$= \tan^{-1}\left(\frac{32+45}{60-24}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{77}{36} = \text{R.H.S.}$$

Question 5:

Prove $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

Answer

Let $\cos^{-1}\frac{4}{5}=x$. Then, $\cos x=\frac{4}{5}\Rightarrow \sin x=\sqrt{1-\left(\frac{4}{5}\right)^2}=\frac{3}{5}$.

$\therefore \tan x=\frac{3}{4}\Rightarrow x=\tan^{-1}\frac{3}{4}$

$\therefore \cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4} \qquad ...(1)$

Now, let $\cos^{-1}\frac{12}{13}=y$. Then, $\cos y=\frac{12}{13}\Rightarrow \sin y=\frac{5}{13}$.

$\therefore \tan y=\frac{5}{12}\Rightarrow y=\tan^{-1}\frac{5}{12}$

$\therefore \cos^{-1}\frac{12}{13}=\tan^{-1}\frac{5}{12} \qquad ...(2)$

Let $\cos^{-1}\frac{33}{65}=z$. Then, $\cos z=\frac{33}{65}\Rightarrow \sin z=\frac{56}{65}$.

$\therefore \tan z=\frac{56}{33}\Rightarrow z=\tan^{-1}\frac{56}{33}$

$\therefore \cos^{-1}\frac{33}{65}=\tan^{-1}\frac{56}{33} \qquad ...(3)$

Now, we will prove that:

$$\begin{aligned}
\text{L.H.S.} &= \cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}\\
&= \tan^{-1}\frac{3}{4}+\tan^{-1}\frac{5}{12} && \left[\text{Using } (1) \text{ and } (2)\right]\\
&= \tan^{-1}\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}.\frac{5}{12}} && \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]\\
&= \tan^{-1}\frac{36+20}{48-15}\\
&= \tan^{-1}\frac{56}{33}\\
&= \tan^{-1}\frac{56}{33} && \left[\text{by } (3)\right]\\
&= \text{R.H.S.}
\end{aligned}$$

Question 6:

Prove $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Answer

Let $\sin^{-1}\frac{3}{5}=x$. Then, $\sin x=\frac{3}{5}\Rightarrow \cos x=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$.

$\therefore \tan x=\frac{3}{4}\Rightarrow x=\tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4} \qquad ...(1)$

Now, let $\cos^{-1}\frac{12}{13}=y$. Then, $\cos y=\frac{12}{13}\Rightarrow \sin y=\frac{5}{13}$.

$\therefore \tan y=\frac{5}{12}\Rightarrow y=\tan^{-1}\frac{5}{12}$

$\therefore \cos^{-1}\frac{12}{13}=\tan^{-1}\frac{5}{12} \qquad ...(2)$

Let $\sin^{-1}\frac{56}{65}=z$. Then, $\sin z=\frac{56}{65}\Rightarrow \cos z=\frac{33}{65}$.

$\therefore \tan z=\frac{56}{33}\Rightarrow z=\tan^{-1}\frac{56}{33}$

$\therefore \sin^{-1}\frac{56}{65}=\tan^{-1}\frac{56}{33} \qquad ...(3)$

Now, we have:

$$\text{L.H.S.} = \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5}$$

$$= \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{3}{4} \qquad \left[\text{Using } (1) \text{ and } (2)\right]$$

$$= \tan^{-1}\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}.\frac{3}{4}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{20+36}{48-15}$$

$$= \tan^{-1}\frac{56}{33}$$

$$= \sin^{-1}\frac{56}{65} = \text{R.H.S.} \qquad \left[\text{Using } (3)\right]$$

Question 7:

Prove $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

Answer

$$\text{Let } \sin^{-1}\frac{5}{13} = x. \text{ Then, } \sin x = \frac{5}{13} \Rightarrow \cos x = \frac{12}{13}.$$

$$\therefore \tan x = \frac{5}{12} \Rightarrow x = \tan^{-1}\frac{5}{12}$$

$$\therefore \sin^{-1}\frac{5}{13} = \tan^{-1}\frac{5}{12} \qquad ...(1)$$

$$\text{Let } \cos^{-1}\frac{3}{5} = y. \text{ Then, } \cos y = \frac{3}{5} \Rightarrow \sin y = \frac{4}{5}.$$

$$\therefore \tan y = \frac{4}{3} \Rightarrow y = \tan^{-1}\frac{4}{3}$$

$$\therefore \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{4}{3} \qquad ...(2)$$

Using (1) and (2), we have

$$\text{R.H.S.} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$$

$$= \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{4}{3}$$

$$= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{15+48}{36-20}\right)$$

$$= \tan^{-1}\frac{63}{16}$$

$$= \text{L.H.S.}$$

Question 8:

Prove $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Answer

$$\text{L.H.S.} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{7+5}{35-1}\right) + \tan^{-1}\left(\frac{8+3}{24-1}\right)$$

$$= \tan^{-1}\frac{12}{34} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

$$= \tan^{-1}\left(\frac{138+187}{391-66}\right)$$

$$= \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}1$$

$$= \frac{\pi}{4} = \text{R.H.S.}$$

Question 9:

Prove $$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),\ x \in [0,\ 1]$$

Answer

Let $x = \tan^2\theta$. Then, $\sqrt{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$.

$$\therefore \frac{1-x}{1+x} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$$

Now, we have:

$$\text{R.H.S.} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta) = \frac{1}{2}\times 2\theta = \theta = \tan^{-1}\sqrt{x} = \text{L.H.S.}$$

Question 10:

Prove $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2},\ x\in\left(0,\ \frac{\pi}{4}\right)$

Answer

Consider $\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2} \qquad \text{(by rationalizing)}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x-1+\sin x}$$

$$=\frac{2\left(1+\sqrt{1-\sin^2 x}\right)}{2\sin x}=\frac{1+\cos x}{\sin x}=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

$$\therefore \text{L.H.S.}=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}=\text{R.H.S.}$$

Question 11:

Prove $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ -\frac{1}{\sqrt{2}}\le x\le 1$ [**Hint:** put $x = \cos 2\theta$]

Answer

Put $x = \cos 2\theta$ so that $\theta = \frac{1}{2}\cos^{-1}x$. Then, we have:

$$\text{L.H.S.} = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right) = \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= \tan^{-1}1 - \tan^{-1}(\tan\theta) \qquad \left[\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y\right]$$

$$= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x = \text{R.H.S.}$$

Question 12:

Prove $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Answer

$$\text{L.H.S.} = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}$$
$$= \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right)$$
$$= \frac{9}{4}\left(\cos^{-1}\frac{1}{3}\right) \qquad . \qquad ....(1) \quad \left[\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

Now, let $\cos^{-1}\frac{1}{3} = x$. Then, $\cos x = \frac{1}{3} \Rightarrow \sin x = \sqrt{1-\left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$.

$$\therefore x = \sin^{-1}\frac{2\sqrt{2}}{3} \Rightarrow \cos^{-1}\frac{1}{3} = \sin^{-1}\frac{2\sqrt{2}}{3}$$
$$\therefore \text{L.H.S.} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3} = \text{R.H.S.}$$

Question 13:

Solve $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

Answer

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$
$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x) \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$
$$\Rightarrow \frac{2\cos x}{1-\cos^2 x} = 2\operatorname{cosec} x$$
$$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$$
$$\Rightarrow \cos x = \sin x$$
$$\Rightarrow \tan x = 1$$
$$\therefore x = \frac{\pi}{4}$$

Question 14:

Solve $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x > 0)$

Answer

$$\tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x \qquad \left[\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \frac{\pi}{4}=\frac{3}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}x=\frac{\pi}{6}$$

$$\Rightarrow x=\tan\frac{\pi}{6}$$

$$\therefore x=\frac{1}{\sqrt{3}}$$

Question 15:

Solve $\sin\left(\tan^{-1}x\right),\ |x|<1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ (**B**) $\frac{1}{\sqrt{1-x^2}}$ (**C**) $\frac{1}{\sqrt{1+x^2}}$ (**D**) $\frac{x}{\sqrt{1+x^2}}$

Answer

Let $\tan^{-1}x = y$. Then, $\tan y = x \Rightarrow \sin y=\frac{x}{\sqrt{1+x^2}}$.

$$\therefore y=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\Rightarrow \tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$$\therefore \sin\left(\tan^{-1}x\right)=\sin\left(\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)=\frac{x}{\sqrt{1+x^2}}$$

The correct answer is D.

Question 16:

Solve $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$**,** then x is equal to

(A) $0, \frac{1}{2}$ **(B)** $1, \frac{1}{2}$ **(C)** 0 **(D)** $\frac{1}{2}$

Answer

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow -2\sin^{-1}x = \frac{\pi}{2} - \sin^{-1}(1-x)$$

$$\Rightarrow -2\sin^{-1}x = \cos^{-1}(1-x) \qquad ...(1)$$

Let $\sin^{-1}x = \theta \Rightarrow \sin\theta = x \Rightarrow \cos\theta = \sqrt{1-x^2}$.

$$\therefore \theta = \cos^{-1}\left(\sqrt{1-x^2}\right)$$

$$\therefore \sin^{-1}x = \cos^{-1}\left(\sqrt{1-x^2}\right)$$

Therefore, from equation (1), we have

$$-2\cos^{-1}\left(\sqrt{1-x^2}\right) = \cos^{-1}(1-x)$$

Put $x = \sin y$. Then, we have:

$$-2\cos^{-1}\left(\sqrt{1-\sin^2 y}\right) = \cos^{-1}(1-\sin y)$$

$$\Rightarrow -2\cos^{-1}(\cos y) = \cos^{-1}(1-\sin y)$$

$$\Rightarrow -2y = \cos^{-1}(1-\sin y)$$

$$\Rightarrow 1-\sin y = \cos(-2y) = \cos 2y$$

$$\Rightarrow 1-\sin y = 1-2\sin^2 y$$

$$\Rightarrow 2\sin^2 y - \sin y = 0$$

$$\Rightarrow \sin y(2\sin y - 1) = 0$$

$$\Rightarrow \sin y = 0 \text{ or } \frac{1}{2}$$

$$\therefore x = 0 \text{ or } x = \frac{1}{2}$$

But, when $x = \frac{1}{2}$, it can be observed that:

$$\text{L.H.S.} = \sin^{-1}\left(1-\frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2}$$
$$= \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2}$$
$$= -\sin^{-1}\frac{1}{2}$$
$$= -\frac{\pi}{6} \neq \frac{\pi}{2} \neq \text{R.H.S.}$$

$\therefore x = \frac{1}{2}$ is not the solution of the given equation.

Thus, x = 0.

Hence, the correct answer is **C**.

Question 17:

Solve $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$ **(B).** $\frac{\pi}{3}$ **(C)** $\frac{\pi}{4}$ **(D)** $\frac{-3\pi}{4}$

Answer

$$\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\frac{x-y}{x+y}$$

$$=\tan^{-1}\left[\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right]$$

$$\left[\tan^{-1}y-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}\right]$$

$$=\tan^{-1}\left[\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{\frac{y(x+y)+x(x-y)}{y(x+y)}}\right]$$

$$=\tan^{-1}\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right)$$

$$=\tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)=\tan^{-1}1=\frac{\pi}{4}$$

Hence, the correct answer is **C**.

Exercise 3.1

Question 1:

In the matrix $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$, write:

(i) The order of the matrix (ii) The number of elements,

(iii) Write the elements a_{13}, a_{21}, a_{33}, a_{24}, a_{23}

Answer

(i) In the given matrix, the number of rows is 3 and the number of columns is 4.
Therefore, the order of the matrix is 3 × 4.

(ii) Since the order of the matrix is 3 × 4, there are 3 × 4 = 12 elements in it.

(iii) $a_{13} = 19$, $a_{21} = 35$, $a_{33} = -5$, $a_{24} = 12$, $a_{23} = \frac{5}{2}$

Question 2:

If a matrix has 24 elements, what are the possible order it can have? What, if it has 13 elements?

Answer

We know that if a matrix is of the order $m \times n$, it has mn elements. Thus, to find all the possible orders of a matrix having 24 elements, we have to find all the ordered pairs of natural numbers whose product is 24.

The ordered pairs are: (1, 24), (24, 1), (2, 12), (12, 2), (3, 8), (8, 3), (4, 6), and (6, 4)

Hence, the possible orders of a matrix having 24 elements are:

1 × 24, 24 × 1, 2 × 12, 12 × 2, 3 × 8, 8 × 3, 4 × 6, and 6 × 4

(1, 13) and (13, 1) are the ordered pairs of natural numbers whose product is 13.

Hence, the possible orders of a matrix having 13 elements are 1 × 13 and 13 × 1.

Question 3:

If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?

Answer

We know that if a matrix is of the order *m* × *n*, it has *mn* elements. Thus, to find all the possible orders of a matrix having 18 elements, we have to find all the ordered pairs of natural numbers whose product is 18.

The ordered pairs are: (1, 18), (18, 1), (2, 9), (9, 2), (3, 6,), and (6, 3)

Hence, the possible orders of a matrix having 18 elements are:

1 × 18, 18 × 1, 2 × 9, 9 × 2, 3 × 6, and 6 × 3

(1, 5) and (5, 1) are the ordered pairs of natural numbers whose product is 5.

Hence, the possible orders of a matrix having 5 elements are 1 × 5 and 5 × 1.

Question 5:

Construct a 3 × 4 matrix, whose elements are given by

(i) $a_{ij} = \frac{1}{2}\left|-3i + j\right|$ (ii) $a_{ij} = 2i - j$

Answer

In general, a 3 × 4 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

(i) $a_{ij} = \frac{1}{2}\left|-3i + j\right|, \ i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4$

$$\therefore a_{11} = \frac{1}{2}|-3\times 1+1| = \frac{1}{2}|-3+1| = \frac{1}{2}|-2| = \frac{2}{2} = 1$$
$$a_{21} = \frac{1}{2}|-3\times 2+1| = \frac{1}{2}|-6+1| = \frac{1}{2}|-5| = \frac{5}{2}$$
$$a_{31} = \frac{1}{2}|-3\times 3+1| = \frac{1}{2}|-9+1| = \frac{1}{2}|-8| = \frac{8}{2} = 4$$

$$a_{12} = \frac{1}{2}|-3\times 1+2| = \frac{1}{2}|-3+2| = \frac{1}{2}|-1| = \frac{1}{2}$$
$$a_{22} = \frac{1}{2}|-3\times 2+2| = \frac{1}{2}|-6+2| = \frac{1}{2}|-4| = \frac{4}{2} = 2$$
$$a_{32} = \frac{1}{2}|-3\times 3+2| = \frac{1}{2}|-9+2| = \frac{1}{2}|-7| = \frac{7}{2}$$

$$a_{13} = \frac{1}{2}|-3\times 1+3| = \frac{1}{2}|-3+3| = 0$$
$$a_{23} = \frac{1}{2}|-3\times 2+3| = \frac{1}{2}|-6+3| = \frac{1}{2}|-3| = \frac{3}{2}$$
$$a_{33} = \frac{1}{2}|-3\times 3+3| = \frac{1}{2}|-9+3| = \frac{1}{2}|-6| = \frac{6}{2} = 3$$

$$a_{14} = \frac{1}{2}|-3\times 1+4| = \frac{1}{2}|-3+4| = \frac{1}{2}|1| = \frac{1}{2}$$
$$a_{24} = \frac{1}{2}|-3\times 2+4| = \frac{1}{2}|-6+4| = \frac{1}{2}|-2| = \frac{2}{2} = 1$$
$$a_{34} = \frac{1}{2}|-3\times 3+4| = \frac{1}{2}|-9+4| = \frac{1}{2}|-5| = \frac{5}{2}$$

Therefore, the required matrix is $A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$

(ii) $a_{ij} = 2i - j,\ i = 1,2,3$ and $j = 1,2,3,4$

$$\therefore a_{11} = 2\times1-1 = 2-1 = 1$$
$$a_{21} = 2\times2-1 = 4-1 = 3$$
$$a_{31} = 2\times3-1 = 6-1 = 5$$

$$a_{12} = 2\times1-2 = 2-2 = 0$$
$$a_{22} = 2\times2-2 = 4-2 = 2$$
$$a_{32} = 2\times3-2 = 6-2 = 4$$

$$a_{13} = 2\times1-3 = 2-3 = -1$$
$$a_{23} = 2\times2-3 = 4-3 = 1$$
$$a_{33} = 2\times3-3 = 6-3 = 3$$

$$a_{14} = 2\times1-4 = 2-4 = -2$$
$$a_{24} = 2\times2-4 = 4-4 = 0$$
$$a_{34} = 2\times3-4 = 6-4 = 2$$

Therefore, the required matrix is $A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}$

Question 6:

Find the value of *x*, *y*, and *z* from the following equation:

(i) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$

(iii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

Answer

(i) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$

As the given matrices are equal, their corresponding elements are also equal.

Comparing the corresponding elements, we get:

$x = 1$, $y = 4$, and $z = 3$

(ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$

As the given matrices are equal, their corresponding elements are also equal.

Comparing the corresponding elements, we get:

$x + y = 6$, $xy = 8$, $5 + z = 5$

Now, $5 + z = 5 \Rightarrow z = 0$

We know that:

$(x - y)^2 = (x + y)^2 - 4xy$

$\Rightarrow (x - y)^2 = 36 - 32 = 4$

$\Rightarrow x - y = \pm 2$

Now, when $x - y = 2$ and $x + y = 6$, we get $x = 4$ and $y = 2$

When $x - y = -2$ and $x + y = 6$, we get $x = 2$ and $y = 4$

$\therefore x = 4$, $y = 2$, and $z = 0$ or $x = 2$, $y = 4$, and $z = 0$

(iii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

As the two matrices are equal, their corresponding elements are also equal.

Comparing the corresponding elements, we get:

$x + y + z = 9$ … (1)

$x + z = 5$ … (2)

$y + z = 7$ … (3)

From (1) and (2), we have:

$y + 5 = 9$

$\Rightarrow y = 4$

Then, from (3), we have:

$4 + z = 7$

$\Rightarrow z = 3$

$\therefore x + z = 5$

$\Rightarrow x = 2$

$\therefore x = 2$, $y = 4$, and $z = 3$

Question 7:

Find the value of *a*, *b*, *c*, and *d* from the equation:

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

Answer

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

As the two matrices are equal, their corresponding elements are also equal.

Comparing the corresponding elements, we get:

$a - b = -1$ … (1)

$2a - b = 0$ … (2)

$2a + c = 5$ … (3)

$3c + d = 13$ … (4)

From (2), we have:

$b = 2a$

Then, from (1), we have:

$a - 2a = -1$

$\Rightarrow a = 1$

$\Rightarrow b = 2$

Now, from (3), we have:

$2 \times 1 + c = 5$

$\Rightarrow c = 3$

From (4) we have:

$3 \times 3 + d = 13$

$\Rightarrow 9 + d = 13 \Rightarrow d = 4$

$\therefore a = 1$, $b = 2$, $c = 3$, and $d = 4$

Question 8:

$A = \left[a_{ij}\right]_{m \times n}$ is a square matrix, if

(A) $m < n$

(B) $m > n$

(C) $m = n$

(D) None of these

Answer

The correct answer is C.

It is known that a given matrix is said to be a square matrix if the number of rows is equal to the number of columns.

Therefore, $A = \left[a_{ij}\right]_{m\times n}$ is a square matrix, if $m = n$.

Question 9:

Which of the given values of x and y make the following pair of matrices equal

$$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix} = \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$$

(A) $x = \frac{-1}{3}, y = 7$

(B) Not possible to find

(C) $y = 7, x = \frac{-2}{3}$

(D) $x = \frac{-1}{3}, y = \frac{-2}{3}$

Answer

The correct answer is B.

It is given that $\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix} = \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$

Equating the corresponding elements, we get:

$$3x + 7 = 0 \Rightarrow x = -\frac{7}{3}$$
$$5 = y - 2 \Rightarrow y = 7$$
$$y + 1 = 8 \Rightarrow y = 7$$
$$2 - 3x = 4 \Rightarrow x = -\frac{2}{3}$$

We find that on comparing the corresponding elements of the two matrices, we get two different values of x, which is not possible.

Hence, it is not possible to find the values of x and y for which the given matrices are equal.

Question 10:

The number of all possible matrices of order 3×3 with each entry 0 or 1 is:

(A) 27

(B) 18

(C) 81

(D) 512

Answer

The correct answer is D.

The given matrix of the order 3×3 has 9 elements and each of these elements can be either 0 or 1.

Now, each of the 9 elements can be filled in two possible ways.

Therefore, by the multiplication principle, the required number of possible matrices is $2^9 = 512$

Exercise 3.2

Question 1:

Let $A=\begin{bmatrix}2 & 4\\ 3 & 2\end{bmatrix}, B=\begin{bmatrix}1 & 3\\ -2 & 5\end{bmatrix}, C=\begin{bmatrix}-2 & 5\\ 3 & 4\end{bmatrix}$

Find each of the following

(i) $A+B$ (ii) $A-B$ (iii) $3A-C$

(iv) AB (v) BA

Answer

(i) $A+B=\begin{bmatrix}2 & 4\\ 3 & 2\end{bmatrix}+\begin{bmatrix}1 & 3\\ -2 & 5\end{bmatrix}=\begin{bmatrix}2+1 & 4+3\\ 3-2 & 2+5\end{bmatrix}=\begin{bmatrix}3 & 7\\ 1 & 7\end{bmatrix}$

(ii) $A-B=\begin{bmatrix}2 & 4\\ 3 & 2\end{bmatrix}-\begin{bmatrix}1 & 3\\ -2 & 5\end{bmatrix}=\begin{bmatrix}2-1 & 4-3\\ 3-(-2) & 2-5\end{bmatrix}=\begin{bmatrix}1 & 1\\ 5 & -3\end{bmatrix}$

(iii) $3A-C=3\begin{bmatrix}2 & 4\\ 3 & 2\end{bmatrix}-\begin{bmatrix}-2 & 5\\ 3 & 4\end{bmatrix}$

$$=\begin{bmatrix}3\times 2 & 3\times 4\\ 3\times 3 & 3\times 2\end{bmatrix}-\begin{bmatrix}-2 & 5\\ 3 & 4\end{bmatrix}$$

$$=\begin{bmatrix}6 & 12\\ 9 & 6\end{bmatrix}-\begin{bmatrix}-2 & 5\\ 3 & 4\end{bmatrix}$$

$$=\begin{bmatrix}6+2 & 12-5\\ 9-3 & 6-4\end{bmatrix}$$

$$=\begin{bmatrix}8 & 7\\ 6 & 2\end{bmatrix}$$

(iv) Matrix *A* has 2 columns. This number is equal to the number of rows in matrix *B*. Therefore, *AB* is defined as:

$$AB=\begin{bmatrix}2 & 4\\ 3 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\ -2 & 5\end{bmatrix}=\begin{bmatrix}2(1)+4(-2) & 2(3)+4(5)\\ 3(1)+2(-2) & 3(3)+2(5)\end{bmatrix}$$

$$=\begin{bmatrix}2-8 & 6+20\\ 3-4 & 9+10\end{bmatrix}=\begin{bmatrix}-6 & 26\\ -1 & 19\end{bmatrix}$$

(v) Matrix *B* has 2 columns. This number is equal to the number of rows in matrix *A*.

Therefore, *BA* is defined as:

$$BA=\begin{bmatrix}1 & 3\\ -2 & 5\end{bmatrix}\begin{bmatrix}2 & 4\\ 3 & 2\end{bmatrix}=\begin{bmatrix}1(2)+3(3) & 1(4)+3(2)\\ -2(2)+5(3) & -2(4)+5(2)\end{bmatrix}$$

$$=\begin{bmatrix}2+9 & 4+6\\ -4+15 & -8+10\end{bmatrix}=\begin{bmatrix}11 & 10\\ 11 & 2\end{bmatrix}$$

Question 2:

Compute the following:

(i) $\begin{bmatrix}a & b\\ -b & a\end{bmatrix}+\begin{bmatrix}a & b\\ b & a\end{bmatrix}$ (ii) $\begin{bmatrix}a^2+b^2 & b^2+c^2\\ a^2+c^2 & a^2+b^2\end{bmatrix}+\begin{bmatrix}2ab & 2bc\\ -2ac & -2ab\end{bmatrix}$

(iii) $\begin{bmatrix}-1 & 4 & -6\\ 8 & 5 & 16\\ 2 & 8 & 5\end{bmatrix}+\begin{bmatrix}12 & 7 & 6\\ 8 & 0 & 5\\ 3 & 2 & 4\end{bmatrix}$

(v) $\begin{bmatrix}\cos^2 x & \sin^2 x\\ \sin^2 x & \cos^2 x\end{bmatrix}+\begin{bmatrix}\sin^2 x & \cos^2 x\\ \cos^2 x & \sin^2 x\end{bmatrix}$

Answer

(i) $\begin{bmatrix}a & b\\ -b & a\end{bmatrix}+\begin{bmatrix}a & b\\ b & a\end{bmatrix}=\begin{bmatrix}a+a & b+b\\ -b+b & a+a\end{bmatrix}=\begin{bmatrix}2a & 2b\\ 0 & 2a\end{bmatrix}$

(ii) $\begin{bmatrix}a^2+b^2 & b^2+c^2\\ a^2+c^2 & a^2+b^2\end{bmatrix}+\begin{bmatrix}2ab & 2bc\\ -2ac & -2ab\end{bmatrix}$

$$=\begin{bmatrix} a^2+b^2+2ab & b^2+c^2+2bc \\ a^2+c^2-2ac & a^2+b^2-2ab \end{bmatrix}$$

$$=\begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$$

(iii) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix}+\begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$

$$=\begin{bmatrix} -1+12 & 4+7 & -6+6 \\ 8+8 & 5+0 & 16+5 \\ 2+3 & 8+2 & 5+4 \end{bmatrix}$$

$$=\begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}$$

(iv) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix}+\begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$

$$=\begin{bmatrix} \cos^2 x+\sin^2 x & \sin^2 x+\cos^2 x \\ \sin^2 x+\cos^2 x & \cos^2 x+\sin^2 x \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \left(\because \sin^2 x+\cos^2 x=1\right)$$

Question 3:

Compute the indicated products

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

(ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

Answer

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$$= \begin{bmatrix} a(a)+b(b) & a(-b)+b(a) \\ -b(a)+a(b) & -b(-b)+a(a) \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b^2 & -ab+ab \\ -ab+ab & b^2+a^2 \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$$

(ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1(2) & 1(3) & 1(4) \\ 2(2) & 2(3) & 2(4) \\ 3(2) & 3(3) & 3(4) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

$$=\begin{bmatrix} 1(1)-2(2) & 1(2)-2(3) & 1(3)-2(1) \\ 2(1)+3(2) & 2(2)+3(3) & 2(3)+3(1) \end{bmatrix}$$

$$=\begin{bmatrix} 1-4 & 2-6 & 3-2 \\ 2+6 & 4+9 & 6+3 \end{bmatrix}=\begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$$

(iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

$$=\begin{bmatrix} 2(1)+3(0)+4(3) & 2(-3)+3(2)+4(0) & 2(5)+3(4)+4(5) \\ 3(1)+4(0)+5(3) & 3(-3)+4(2)+5(0) & 3(5)+4(4)+5(5) \\ 4(1)+5(0)+6(3) & 4(-3)+5(2)+6(0) & 4(5)+5(4)+6(5) \end{bmatrix}$$

$$=\begin{bmatrix} 2+0+12 & -6+6+0 & 10+12+20 \\ 3+0+15 & -9+8+0 & 15+16+25 \\ 4+0+18 & -12+10+0 & 20+20+30 \end{bmatrix}=\begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$$

(v) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

$$=\begin{bmatrix} 2(1)+1(-1) & 2(0)+1(2) & 2(1)+1(1) \\ 3(1)+2(-1) & 3(0)+2(2) & 3(1)+2(1) \\ -1(1)+1(-1) & -1(0)+1(2) & -1(1)+1(1) \end{bmatrix}$$

$$=\begin{bmatrix} 2-1 & 0+2 & 2+1 \\ 3-2 & 0+4 & 3+2 \\ -1-1 & 0+2 & -1+1 \end{bmatrix}=\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}$$

(vi) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

$$=\begin{bmatrix} 3(2)-1(1)+3(3) & 3(-3)-1(0)+3(1) \\ -1(2)+0(1)+2(3) & -1(-3)+0(0)+2(1) \end{bmatrix}$$

$$=\begin{bmatrix} 6-1+9 & -9-0+3 \\ -2+0+6 & 3+0+2 \end{bmatrix}=\begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$$

Question 4:

If $A=\begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$, and $C=\begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$, then

compute $(A+B)$ and $(B-C)$. Also, verify that $A+(B-C)=(A+B)-C$

Answer

$$A+B=\begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}+\begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 1+3 & 2-1 & -3+2 \\ 5+4 & 0+2 & 2+5 \\ 1+2 & -1+0 & 1+3 \end{bmatrix}=\begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$$

$$B-C=\begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}-\begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 3-4 & -1-1 & 2-2 \\ 4-0 & 2-3 & 5-2 \\ 2-1 & 0-(-2) & 3-3 \end{bmatrix}=\begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$A+(B-C)=\begin{bmatrix}1 & 2 & -3\\ 5 & 0 & 2\\ 1 & -1 & 1\end{bmatrix}+\begin{bmatrix}-1 & -2 & 0\\ 4 & -1 & 3\\ 1 & 2 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1+(-1) & 2+(-2) & -3+0\\ 5+4 & 0+(-1) & 2+3\\ 1+1 & -1+2 & 1+0\end{bmatrix}=\begin{bmatrix}0 & 0 & -3\\ 9 & -1 & 5\\ 2 & 1 & 1\end{bmatrix}$$

$$(A+B)-C=\begin{bmatrix}4 & 1 & -1\\ 9 & 2 & 7\\ 3 & -1 & 4\end{bmatrix}-\begin{bmatrix}4 & 1 & 2\\ 0 & 3 & 2\\ 1 & -2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}4-4 & 1-1 & -1-2\\ 9-0 & 2-3 & 7-2\\ 3-1 & -1-(-2) & 4-3\end{bmatrix}=\begin{bmatrix}0 & 0 & -3\\ 9 & -1 & 5\\ 2 & 1 & 1\end{bmatrix}$$

Hence, we have verified that $A+(B-C)=(A+B)-C$.

Question 5:

If $A=\begin{bmatrix}\frac{2}{3} & 1 & \frac{5}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3}\\ \frac{7}{3} & 2 & \frac{2}{3}\end{bmatrix}$ and $B=\begin{bmatrix}\frac{2}{5} & \frac{3}{5} & 1\\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5}\\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5}\end{bmatrix}$ then compute $3A-5B$.

Answer

$$3A-5B=3\begin{bmatrix}\frac{2}{3} & 1 & \frac{5}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3}\\ \frac{7}{3} & 2 & \frac{2}{3}\end{bmatrix}-5\begin{bmatrix}\frac{2}{5} & \frac{3}{5} & 1\\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5}\\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5}\end{bmatrix}$$

$$=\begin{bmatrix}2 & 3 & 5\\ 1 & 2 & 4\\ 7 & 6 & 2\end{bmatrix}-\begin{bmatrix}2 & 3 & 5\\ 1 & 2 & 4\\ 7 & 6 & 2\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Question 6:

Simplify $\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$

Answer

$$\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ -\sin\theta\cos\theta & \cos^2\theta\end{bmatrix}+\begin{bmatrix}\sin^2\theta & -\sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\sin\theta\cos\theta\\ -\sin\theta\cos\theta+\sin\theta\cos\theta & \cos^2\theta+\sin^2\theta\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\qquad\left(\because\cos^2\theta+\sin^2\theta=1\right)$$

Question 7:

Find *X* and *Y*, if

(i) $X+Y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}$ and $X-Y=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$

(ii) $2X+3Y=\begin{bmatrix}2 & 3\\ 4 & 0\end{bmatrix}$ and $3X+2Y=\begin{bmatrix}2 & -2\\ -1 & 5\end{bmatrix}$

Answer

(i) $X+Y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}$ ...(1)

$X-Y=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$ ...(2)

Adding equations (1) and (2), we get:

$$2X=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}+\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}=\begin{bmatrix}7+3 & 0+0\\ 2+0 & 5+3\end{bmatrix}=\begin{bmatrix}10 & 0\\ 2 & 8\end{bmatrix}$$

$$\therefore X=\frac{1}{2}\begin{bmatrix}10 & 0\\ 2 & 8\end{bmatrix}=\begin{bmatrix}5 & 0\\ 1 & 4\end{bmatrix}$$

$$\text{Now, } X+Y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}5 & 0\\ 1 & 4\end{bmatrix}+Y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}$$

$$\Rightarrow Y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}-\begin{bmatrix}5 & 0\\ 1 & 4\end{bmatrix}$$

$$\Rightarrow Y=\begin{bmatrix}7-5 & 0-0\\ 2-1 & 5-4\end{bmatrix}$$

$$\therefore Y=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}$$

(ii) $2X+3Y=\begin{bmatrix}2 & 3\\ 4 & 0\end{bmatrix}$ ...(3)

$3X+2Y=\begin{bmatrix}2 & -2\\ -1 & 5\end{bmatrix}$ ...(4)

Multiplying equation (3) with (2), we get:

$$2(2X+3Y)=2\begin{bmatrix}2 & 3\\ 4 & 0\end{bmatrix}$$

$$\Rightarrow 4X+6Y=\begin{bmatrix}4 & 6\\ 8 & 0\end{bmatrix} \qquad ...(5)$$

Multiplying equation (4) with (3), we get:

$$3(3X+2Y)=3\begin{bmatrix}2 & -2\\ -1 & 5\end{bmatrix}$$

$$\Rightarrow 9X+6Y=\begin{bmatrix}6 & -6\\ -3 & 15\end{bmatrix} \qquad ...(6)$$

From (5) and (6), we have:

$$(4X+6Y)-(9X+6Y)=\begin{bmatrix}4 & 6\\ 8 & 0\end{bmatrix}-\begin{bmatrix}6 & -6\\ -3 & 15\end{bmatrix}$$

$$\Rightarrow -5X=\begin{bmatrix}4-6 & 6-(-6)\\ 8-(-3) & 0-15\end{bmatrix}=\begin{bmatrix}-2 & 12\\ 11 & -15\end{bmatrix}$$

$$\therefore X=-\frac{1}{5}\begin{bmatrix}-2 & 12\\ 11 & -15\end{bmatrix}=\begin{bmatrix}\frac{2}{5} & -\frac{12}{5}\\ -\frac{11}{5} & 3\end{bmatrix}$$

Now, $2X+3Y=\begin{bmatrix}2 & 3\\ 4 & 0\end{bmatrix}$

$$\Rightarrow 2\begin{bmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{bmatrix} + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{4}{5} & -\frac{24}{5} \\ -\frac{22}{5} & 6 \end{bmatrix} + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$$

$$\Rightarrow 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} & -\frac{24}{5} \\ -\frac{22}{5} & 6 \end{bmatrix}$$

$$\Rightarrow 3Y = \begin{bmatrix} 2-\frac{4}{5} & 3+\frac{24}{5} \\ 4+\frac{22}{5} & 0-6 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{39}{5} \\ \frac{42}{5} & -6 \end{bmatrix}$$

$$\therefore Y = \frac{1}{3}\begin{bmatrix} \frac{6}{5} & \frac{39}{5} \\ \frac{42}{5} & -6 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$$

Question 8:

Find *X*, if $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

Answer

$$2X+Y=\begin{bmatrix}1 & 0\\ -3 & 2\end{bmatrix}$$

$$\Rightarrow 2X+\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}=\begin{bmatrix}1 & 0\\ -3 & 2\end{bmatrix}$$

$$\Rightarrow 2X=\begin{bmatrix}1 & 0\\ -3 & 2\end{bmatrix}-\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}=\begin{bmatrix}1-3 & 0-2\\ -3-1 & 2-4\end{bmatrix}$$

$$\Rightarrow 2X=\begin{bmatrix}-2 & -2\\ -4 & -2\end{bmatrix}$$

$$\therefore X=\frac{1}{2}\begin{bmatrix}-2 & -2\\ -4 & -2\end{bmatrix}=\begin{bmatrix}-1 & -1\\ -2 & -1\end{bmatrix}$$

Question 9:

Find x and y, if $2\begin{bmatrix}1 & 3\\ 0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$

Answer

$$2\begin{bmatrix}1 & 3\\ 0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}2 & 6\\ 0 & 2x\end{bmatrix}+\begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}2+y & 6\\ 1 & 2x+2\end{bmatrix}=\begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$$

Comparing the corresponding elements of these two matrices, we have:

$$2+y=5$$
$$\Rightarrow y=3$$

$$2x+2=8$$
$$\Rightarrow x=3$$

$\therefore x = 3$ and $y = 3$

Question 10:

Solve the equation for *x*, *y*, *z* and *t* if

$$2\begin{bmatrix} x & z \\ y & t \end{bmatrix}+3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}=3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

Answer

$$2\begin{bmatrix} x & z \\ y & t \end{bmatrix}+3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}=3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix}+\begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix}=\begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+3 & 2z-3 \\ 2y & 2t+6 \end{bmatrix}=\begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

Comparing the corresponding elements of these two matrices, we get:

$$2x+3=9$$
$$\Rightarrow 2x=6$$
$$\Rightarrow x=3$$

$$2y=12$$
$$\Rightarrow y=6$$

$$2z-3=15$$
$$\Rightarrow 2z=18$$
$$\Rightarrow z=9$$

$$2t+6=18$$
$$\Rightarrow 2t=12$$
$$\Rightarrow t=6$$

$$\therefore x=3,\ y=6,\ z=9,\ \text{and } t=6$$

Question 11:

If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix}+y\begin{bmatrix} -1 \\ 1 \end{bmatrix}=\begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find values of *x* and *y*.

Answer

$$x\begin{bmatrix}2\\3\end{bmatrix}+y\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}10\\5\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}2x\\3x\end{bmatrix}+\begin{bmatrix}-y\\y\end{bmatrix}=\begin{bmatrix}10\\5\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}2x-y\\3x+y\end{bmatrix}=\begin{bmatrix}10\\5\end{bmatrix}$$

Comparing the corresponding elements of these two matrices, we get:

$2x - y = 10$ and $3x + y = 5$

Adding these two equations, we have:

$5x = 15$

$\Rightarrow x = 3$

Now, $3x + y = 5$

$\Rightarrow y = 5 - 3x$

$\Rightarrow y = 5 - 9 = -4$

$\therefore x = 3$ and $y = -4$

Question 12:

Given $3\begin{bmatrix}x & y\\ z & w\end{bmatrix}=\begin{bmatrix}x & 6\\ -1 & 2w\end{bmatrix}+\begin{bmatrix}4 & x+y\\ z+w & 3\end{bmatrix}$, find the values of x, y, z and w.

Answer

$$3\begin{bmatrix}x & y\\ z & w\end{bmatrix}=\begin{bmatrix}x & 6\\ -1 & 2w\end{bmatrix}+\begin{bmatrix}4 & x+y\\ z+w & 3\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}3x & 3y\\ 3z & 3w\end{bmatrix}=\begin{bmatrix}x+4 & 6+x+y\\ -1+z+w & 2w+3\end{bmatrix}$$

Comparing the corresponding elements of these two matrices, we get:

$3x = x + 4$
$\Rightarrow 2x = 4$
$\Rightarrow x = 2$

$3y = 6 + x + y$
$\Rightarrow 2y = 6 + x = 6 + 2 = 8$
$\Rightarrow y = 4$

$3w = 2w + 3$
$\Rightarrow w = 3$

$3z = -1 + z + w$
$\Rightarrow 2z = -1 + w = -1 + 3 = 2$
$\Rightarrow z = 1$

$\therefore x = 2,\ y = 4,\ z = 1,\ \text{and } w = 3$

Question 13:

If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $F(x)F(y) = F(x+y)$.

Answer

$$F(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}, F(y)=\begin{bmatrix}\cos y & -\sin y & 0\\ \sin y & \cos y & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$F(x+y)=\begin{bmatrix}\cos(x+y) & -\sin(x+y) & 0\\ \sin(x+y) & \cos(x+y) & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$F(x)F(y)$$
$$=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos y & -\sin y & 0\\ \sin y & \cos y & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}\cos x\cos y-\sin x\sin y+0 & -\cos x\sin y-\sin x\cos y+0 & 0\\ \sin x\cos y+\cos x\sin y+0 & -\sin x\sin y+\cos x\cos y+0 & 0\\ 0 & 0 & 0\end{bmatrix}$$
$$=\begin{bmatrix}\cos(x+y) & -\sin(x+y) & 0\\ \sin(x+y) & \cos(x+y) & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$=F(x+y)$$

$$\therefore F(x)F(y)=F(x+y)$$

Question 14:

Show that

(i) $\begin{bmatrix}5 & -1\\ 6 & 7\end{bmatrix}\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}\neq\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & -1\\ 6 & 7\end{bmatrix}$

(ii) $\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 2 & 3 & 4\end{bmatrix}\neq\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 2 & 3 & 4\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}$

Answer

(i) $\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

$$=\begin{bmatrix} 5(2)-1(3) & 5(1)-1(4) \\ 6(2)+7(3) & 6(1)+7(4) \end{bmatrix}$$

$$=\begin{bmatrix} 10-3 & 5-4 \\ 12+21 & 6+28 \end{bmatrix}=\begin{bmatrix} 7 & 1 \\ 33 & 34 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$$

$$=\begin{bmatrix} 2(5)+1(6) & 2(-1)+1(7) \\ 3(5)+4(6) & 3(-1)+4(7) \end{bmatrix}$$

$$=\begin{bmatrix} 10+6 & -2+7 \\ 15+24 & -3+28 \end{bmatrix}=\begin{bmatrix} 16 & 5 \\ 39 & 25 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$$

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

$$=\begin{bmatrix} 1(-1)+2(0)+3(2) & 1(1)+2(-1)+3(3) & 1(0)+2(1)+3(4) \\ 0(-1)+1(0)+0(2) & 0(1)+1(-1)+0(3) & 0(0)+1(1)+0(4) \\ 1(-1)+1(0)+0(2) & 1(1)+1(-1)+0(3) & 1(0)+1(1)+0(4) \end{bmatrix}$$

$$=\begin{bmatrix} 5 & 8 & 14 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1(1)+1(0)+0(1) & -1(2)+1(1)+0(1) & -1(3)+1(0)+0(0) \\ 0(1)+(-1)(0)+1(1) & 0(2)+(-1)(1)+1(1) & 0(3)+(-1)(0)+1(0) \\ 2(1)+3(0)+4(1) & 2(2)+3(1)+4(1) & 2(3)+3(0)+4(0) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -3 \\ 1 & 0 & 0 \\ 6 & 11 & 6 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Question 15:

Find $A^2 - 5A + 6I$ if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Answer

We have $A^2 = A \times A$

$$A^2 = AA = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2(2)+0(2)+1(1) & 2(0)+0(1)+1(-1) & 2(1)+0(3)+1(0) \\ 2(2)+1(2)+3(1) & 2(0)+1(1)+3(-1) & 2(1)+1(3)+3(0) \\ 1(2)+(-1)(2)+0(1) & 1(0)+(-1)(1)+0(-1) & 1(1)+(-1)(3)+0(0) \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\therefore A^2 - 5A + 6I$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10 & -1-0 & 2-5 \\ 9-10 & -2-5 & 5-15 \\ 0-5 & -1+5 & -2-0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -5+6 & -1+0 & -3+0 \\ -1+0 & -7+6 & -10+0 \\ -5+0 & 4+0 & -2+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

Question 16:

If $A=\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$, prove that $A^3-6A^2+7A+2I=O$

Answer

$$A^2=AA=\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+0+4 & 0+0+0 & 2+0+6\\ 0+0+2 & 0+4+0 & 0+2+3\\ 2+0+6 & 0+0+0 & 4+0+9\end{bmatrix}=\begin{bmatrix}5 & 0 & 8\\ 2 & 4 & 5\\ 8 & 0 & 13\end{bmatrix}$$

Now $A^3=A^2\cdot A$

$$=\begin{bmatrix}5 & 0 & 8\\ 2 & 4 & 5\\ 8 & 0 & 13\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}5+0+16 & 0+0+0 & 10+0+24\\ 2+0+10 & 0+8+0 & 4+4+15\\ 8+0+26 & 0+0+0 & 16+0+39\end{bmatrix}$$

$$=\begin{bmatrix}21 & 0 & 34\\ 12 & 8 & 23\\ 34 & 0 & 55\end{bmatrix}$$

$\therefore A^3-6A^2+7A+2I$

$$=\begin{bmatrix}21 & 0 & 34\\ 12 & 8 & 23\\ 34 & 0 & 55\end{bmatrix}-6\begin{bmatrix}5 & 0 & 8\\ 2 & 4 & 5\\ 8 & 0 & 13\end{bmatrix}+7\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}+2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}21 & 0 & 34\\12 & 8 & 23\\34 & 0 & 55\end{bmatrix}-\begin{bmatrix}30 & 0 & 48\\12 & 24 & 30\\48 & 0 & 78\end{bmatrix}+\begin{bmatrix}7 & 0 & 14\\0 & 14 & 7\\14 & 0 & 21\end{bmatrix}+\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}$$

$$=\begin{bmatrix}21+7+2 & 0+0+0 & 34+14+0\\12+0+0 & 8+14+2 & 23+7+0\\34+14+0 & 0+0+0 & 55+21+2\end{bmatrix}-\begin{bmatrix}30 & 0 & 48\\12 & 24 & 30\\48 & 0 & 78\end{bmatrix}$$

$$=\begin{bmatrix}30 & 0 & 48\\12 & 24 & 30\\48 & 0 & 78\end{bmatrix}-\begin{bmatrix}30 & 0 & 48\\12 & 24 & 30\\48 & 0 & 78\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}=O$$

$$\therefore A^3-6A^2+7A+2I=O$$

Question 17:

If $A=\begin{bmatrix}3 & -2\\4 & -2\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, find k so that $A^2=kA-2I$

Answer

$$A^2=A\cdot A=\begin{bmatrix}3 & -2\\4 & -2\end{bmatrix}\begin{bmatrix}3 & -2\\4 & -2\end{bmatrix}$$

$$=\begin{bmatrix}3(3)+(-2)(4) & 3(-2)+(-2)(-2)\\4(3)+(-2)(4) & 4(-2)+(-2)(-2)\end{bmatrix}=\begin{bmatrix}1 & -2\\4 & -4\end{bmatrix}$$

$$\text{Now } A^2=kA-2I$$

$$\Rightarrow\begin{bmatrix}1 & -2\\4 & -4\end{bmatrix}=k\begin{bmatrix}3 & -2\\4 & -2\end{bmatrix}-2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}1 & -2\\4 & -4\end{bmatrix}=\begin{bmatrix}3k & -2k\\4k & -2k\end{bmatrix}-\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}1 & -2\\4 & -4\end{bmatrix}=\begin{bmatrix}3k-2 & -2k\\4k & -2k-2\end{bmatrix}$$

Comparing the corresponding elements, we have:

$3k-2=1$

$\Rightarrow 3k=3$

$\Rightarrow k=1$

Thus, the value of *k* is 1.

Question 18:

If $A=\begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that

$I+A=(I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Answer

On the L.H.S.

$$I + A$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix} \quad \ldots(1)$$

On the R.H.S.

$$(I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix} \right) \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha + \sin\alpha\tan\frac{\alpha}{2} & -\sin\alpha + \cos\alpha\tan\frac{\alpha}{2} \\ -\cos\alpha\tan\frac{\alpha}{2} + \sin\alpha & \sin\alpha\tan\frac{\alpha}{2} + \cos\alpha \end{bmatrix} \quad \ldots(2)$$

$$=\begin{bmatrix} 1-2\sin^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\tan\frac{\alpha}{2} & -2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\left(2\cos^2\frac{\alpha}{2}-1\right)\tan\frac{\alpha}{2} \\ -\left(2\cos^2\frac{\alpha}{2}-1\right)\tan\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} & 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\tan\frac{\alpha}{2}+1-2\sin^2\frac{\alpha}{2} \end{bmatrix}$$

$$=\begin{bmatrix} 1-2\sin^2\frac{\alpha}{2}+2\sin^2\frac{\alpha}{2} & -2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}-\tan\frac{\alpha}{2} \\ -2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\tan\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} & 2\sin^2\frac{\alpha}{2}+1-2\sin^2\frac{\alpha}{2} \end{bmatrix}$$

$$=\begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

Thus, from (1) and (2), we get L.H.S. = R.H.S.

Question 19:

A trust fund has Rs 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs 30,000 among the two types of bonds. If the trust fund must obtain an annual total interest of:

(a) Rs 1,800 (b) Rs 2,000

Answer

(a) Let Rs x be invested in the first bond. Then, the sum of money invested in the second bond will be Rs $(30000 - x)$.

It is given that the first bond pays 5% interest per year and the second bond pays 7% interest per year.

Therefore, in order to obtain an annual total interest of Rs 1800, we have:

$$\begin{bmatrix} x & (30000-x) \end{bmatrix}\begin{bmatrix} \frac{5}{100} \\ \frac{7}{100} \end{bmatrix}=1800 \qquad \left[\text{S.I. for 1 year} = \frac{\text{Principal}\times\text{Rate}}{100}\right]$$

$$\Rightarrow \frac{5x}{100}+\frac{7(30000-x)}{100}=1800$$
$$\Rightarrow 5x+210000-7x=180000$$
$$\Rightarrow 210000-2x=180000$$
$$\Rightarrow 2x=210000-180000$$
$$\Rightarrow 2x=30000$$
$$\Rightarrow x=15000$$

Thus, in order to obtain an annual total interest of Rs 1800, the trust fund should invest Rs 15000 in the first bond and the remaining Rs 15000 in the second bond.

(b) Let Rs *x* be invested in the first bond. Then, the sum of money invested in the second bond will be Rs (30000 − *x*).

Therefore, in order to obtain an annual total interest of Rs 2000, we have:

$$\begin{bmatrix} x & (30000-x) \end{bmatrix}\begin{bmatrix} \frac{5}{100} \\ \frac{7}{100} \end{bmatrix}=2000$$

$$\Rightarrow \frac{5x}{100}+\frac{7(30000-x)}{100}=2000$$
$$\Rightarrow 5x+210000-7x=200000$$
$$\Rightarrow 210000-2x=200000$$
$$\Rightarrow 2x=210000-200000$$
$$\Rightarrow 2x=10000$$
$$\Rightarrow x=5000$$

Thus, in order to obtain an annual total interest of Rs 2000, the trust fund should invest Rs 5000 in the first bond and the remaining Rs 25000 in the second bond.

Question 20:

The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are Rs 80, Rs 60 and Rs 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

Answer

The bookshop has 10 dozen chemistry books, 8 dozen physics books, and 10 dozen economics books.

The selling prices of a chemistry book, a physics book, and an economics book are respectively given as Rs 80, Rs 60, and Rs 40.

The total amount of money that will be received from the sale of all these books can be represented in the form of a matrix as:

$$12\begin{bmatrix}10 & 8 & 10\end{bmatrix}\begin{bmatrix}80\\60\\40\end{bmatrix}$$

$$\begin{aligned}&=12\left[10\times80+8\times60+10\times40\right]\\&=12\left(800+480+400\right)\\&=12\left(1680\right)\\&=20160\end{aligned}$$

Thus, the bookshop will receive Rs 20160 from the sale of all these books.

Question 21:

Assume *X*, *Y*, *Z*, *W* and *P* are matrices of order $2\times n, 3\times k, 2\times p,\ n\times3$, and $p\times k$ respectively. The restriction on *n*, *k* and *p* so that $PY+WY$ will be defined are:

A. *k* = 3, *p* = *n*

B. *k* is arbitrary, *p* = 2

C. *p* is arbitrary, *k* = 3

D. *k* = 2, *p* = 3

Answer

Matrices *P* and *Y* are of the orders *p* × *k* and 3 × *k* respectively.

Therefore, matrix *PY* will be defined if *k* = 3. Consequently, *PY* will be of the order *p* × *k*.

Matrices *W* and *Y* are of the orders *n* × 3 and 3 × *k* respectively.

Since the number of columns in W is equal to the number of rows in Y, matrix WY is well-defined and is of the order $n \times k$.

Matrices PY and WY can be added only when their orders are the same.

However, PY is of the order $p \times k$ and WY is of the order $n \times k$. Therefore, we must have $p = n$.

Thus, $k = 3$ and $p = n$ are the restrictions on n, k, and p so that $PY + WY$ will be defined.

Question 22:

Assume X, Y, Z, W and P are matrices of order $2 \times n, 3 \times k, 2 \times p, n \times 3$, and $p \times k$ respectively. If $n = p$, then the order of the matrix $7X - 5Z$ is

A $p \times 2$ **B** $2 \times n$ **C** $n \times 3$ **D** $p \times n$

Answer

The correct answer is B.

Matrix X is of the order $2 \times n$.

Therefore, matrix $7X$ is also of the same order.

Matrix Z is of the order $2 \times p$, i.e., $2 \times n$ [Since $n = p$]

Therefore, matrix $5Z$ is also of the same order.

Now, both the matrices $7X$ and $5Z$ are of the order $2 \times n$.

Thus, matrix $7X - 5Z$ is well-defined and is of the order $2 \times n$.

Exercise 3.3

Question 1:

Find the transpose of each of the following matrices:

(i) $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$

Answer

(i) Let $A = \begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$, then $A^{T} = \begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$

(ii) Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, then $A^{T} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

(iii) Let $A = \begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$, then $A^{T} = \begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}$

Question 2:

If $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, then verify that

(i) $(A+B)' = A' + B'$

(ii) $(A-B)' = A' - B'$

Answer

We have:

$$A' = \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix}, B' = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix}$$

(i) $$A+B = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 & -2 \\ 6 & 9 & 9 \\ -1 & 4 & 2 \end{bmatrix}$$

$$\therefore (A+B)' = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix}$$

$$A' + B' = \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix}$$

Hence, we have verified that $(A+B)' = A' + B'$

(ii) $$A-B = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 8 \\ 4 & 5 & 9 \\ -3 & -2 & 0 \end{bmatrix}$$

$$\therefore (A-B)' = \begin{bmatrix} 3 & 4 & -3 \\ 1 & 5 & -2 \\ 8 & 9 & 0 \end{bmatrix}$$

$$A' - B' = \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -3 \\ 1 & 5 & -2 \\ 8 & 9 & 0 \end{bmatrix}$$

Hence, we have verified that $(A-B)' = A' - B'$.

Question 3:

If $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then verify that

(i) $(A+B)' = A' + B'$

(ii) $(A-B)' = A' - B'$

Answer

(i) It is known that $A = (A')'$

Therefore, we have:

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

$$B' = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 4 & 4 \end{bmatrix}$$

$$\therefore (A+B)' = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

$$A' + B' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

Thus, we have verified that $(A+B)' = A' + B'$.

(ii) $A-B=\begin{bmatrix}3 & -1 & 0\\ 4 & 2 & 1\end{bmatrix}-\begin{bmatrix}-1 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}=\begin{bmatrix}4 & -3 & -1\\ 3 & 0 & -2\end{bmatrix}$

$$\therefore (A-B)'=\begin{bmatrix}4 & 3\\ -3 & 0\\ -1 & -2\end{bmatrix}$$

$$A'-B'=\begin{bmatrix}3 & 4\\ -1 & 2\\ 0 & 1\end{bmatrix}-\begin{bmatrix}-1 & 1\\ 2 & 2\\ 1 & 3\end{bmatrix}=\begin{bmatrix}4 & 3\\ -3 & 0\\ -1 & -2\end{bmatrix}$$

Thus, we have verified that $(A-B)'=A'-B'$.

Question 4:

If $A'=\begin{bmatrix}-2 & 3\\ 1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 0\\ 1 & 2\end{bmatrix}$, then find $(A+2B)'$

Answer

We know that $A=(A')'$

$$\therefore A=\begin{bmatrix}-2 & 1\\ 3 & 2\end{bmatrix}$$

$$\therefore A+2B=\begin{bmatrix}-2 & 1\\ 3 & 2\end{bmatrix}+2\begin{bmatrix}-1 & 0\\ 1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 3 & 2\end{bmatrix}+\begin{bmatrix}-2 & 0\\ 2 & 4\end{bmatrix}=\begin{bmatrix}-4 & 1\\ 5 & 6\end{bmatrix}$$

$$\therefore (A+2B)'=\begin{bmatrix}-4 & 5\\ 1 & 6\end{bmatrix}$$

Question 5:

For the matrices *A* and *B*, verify that $(AB)' = B'A'$ where

(i) $A=\begin{bmatrix}1\\-4\\3\end{bmatrix}, B=\begin{bmatrix}-1 & 2 & 1\end{bmatrix}$

(ii) $A=\begin{bmatrix}0\\1\\2\end{bmatrix}, B=\begin{bmatrix}1 & 5 & 7\end{bmatrix}$

Answer

(i) $AB=\begin{bmatrix}1\\-4\\3\end{bmatrix}\begin{bmatrix}-1 & 2 & 1\end{bmatrix}=\begin{bmatrix}-1 & 2 & 1\\4 & -8 & -4\\-3 & 6 & 3\end{bmatrix}$

$$\therefore (AB)'=\begin{bmatrix}-1 & 4 & -3\\2 & -8 & 6\\1 & -4 & 3\end{bmatrix}$$

Now, $A'=\begin{bmatrix}1 & -4 & 3\end{bmatrix}, B'=\begin{bmatrix}-1\\2\\1\end{bmatrix}$

$$\therefore B'A'=\begin{bmatrix}-1\\2\\1\end{bmatrix}\begin{bmatrix}1 & -4 & 3\end{bmatrix}=\begin{bmatrix}-1 & 4 & -3\\2 & -8 & 6\\1 & -4 & 3\end{bmatrix}$$

Hence, we have verified that $(AB)'=B'A'$.

(ii) $AB=\begin{bmatrix}0\\1\\2\end{bmatrix}\begin{bmatrix}1 & 5 & 7\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\1 & 5 & 7\\2 & 10 & 14\end{bmatrix}$

$\therefore (AB)'=\begin{bmatrix}0 & 1 & 2\\0 & 5 & 10\\0 & 7 & 14\end{bmatrix}$

Now, $A'=\begin{bmatrix}0 & 1 & 2\end{bmatrix}, B'=\begin{bmatrix}1\\5\\7\end{bmatrix}$

$\therefore B'A'=\begin{bmatrix}1\\5\\7\end{bmatrix}\begin{bmatrix}0 & 1 & 2\end{bmatrix}=\begin{bmatrix}0 & 1 & 2\\0 & 5 & 10\\0 & 7 & 14\end{bmatrix}$

Hence, we have verified that $(AB)'=B'A'$.

Question 6:

If (i) $A=\begin{bmatrix}\cos\alpha & \sin\alpha\\-\sin\alpha & \cos\alpha\end{bmatrix}$, then verify that $A'A=I$

(ii) $A=\begin{bmatrix}\sin\alpha & \cos\alpha\\-\cos\alpha & \sin\alpha\end{bmatrix}$, then verify that $A'A=I$

Answer

(i)

$$A=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$$

$$\therefore A'=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

$$A'A=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$$
$$=\begin{bmatrix}(\cos\alpha)(\cos\alpha)+(-\sin\alpha)(-\sin\alpha) & (\cos\alpha)(\sin\alpha)+(-\sin\alpha)(\cos\alpha)\\ (\sin\alpha)(\cos\alpha)+(\cos\alpha)(-\sin\alpha) & (\sin\alpha)(\sin\alpha)+(\cos\alpha)(\cos\alpha)\end{bmatrix}$$
$$=\begin{bmatrix}\cos^2\alpha+\sin^2\alpha & \sin\alpha\cos\alpha-\sin\alpha\cos\alpha\\ \sin\alpha\cos\alpha-\sin\alpha\cos\alpha & \sin^2\alpha+\cos^2\alpha\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

Hence, we have verified that $A'A=I$.

(ii) $$A=\begin{bmatrix}\sin\alpha & \cos\alpha\\ -\cos\alpha & \sin\alpha\end{bmatrix}$$

$$\therefore A'=\begin{bmatrix}\sin\alpha & -\cos\alpha\\ \cos\alpha & \sin\alpha\end{bmatrix}$$

$$A'A=\begin{bmatrix}\sin\alpha & -\cos\alpha\\ \cos\alpha & \sin\alpha\end{bmatrix}\begin{bmatrix}\sin\alpha & \cos\alpha\\ -\cos\alpha & \sin\alpha\end{bmatrix}$$

$$\begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix}\begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$$
$$=\begin{bmatrix} (\sin\alpha)(\sin\alpha)+(-\cos\alpha)(-\cos\alpha) & (\sin\alpha)(\cos\alpha)+(-\cos\alpha)(\sin\alpha) \\ (\cos\alpha)(\sin\alpha)+(\sin\alpha)(-\cos\alpha) & (\cos\alpha)(\cos\alpha)+(\sin\alpha)(\sin\alpha) \end{bmatrix}$$
$$=\begin{bmatrix} \sin^2\alpha+\cos^2\alpha & \sin\alpha\cos\alpha-\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha-\sin\alpha\cos\alpha & \cos^2\alpha+\sin^2\alpha \end{bmatrix}$$
$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

Hence, we have verified that $A'A=I$.

Question 7:

(i) Show that the matrix $A=\begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ is a symmetric matrix

(ii) Show that the matrix $A=\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ is a skew symmetric matrix

Answer

(i) We have:

$$A'=\begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}=A$$

$\therefore A'=A$

Hence, A is a symmetric matrix.

(ii) We have:

$$A' = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = -A$$

$$\therefore A' = -A$$

Hence, A is a skew-symmetric matrix.

Question 8:

For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that

(i) $(A + A')$ is a symmetric matrix

(ii) $(A - A')$ is a skew symmetric matrix

Answer

$$A' = \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix}$$

(i) $A + A' = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}$

$$\therefore (A + A')' = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix} = A + A'$$

Hence, $(A + A')$ is a symmetric matrix.

(ii) $A - A' = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$(A - A')' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -(A - A')$$

Hence, $(A - A')$ is a skew-symmetric matrix.

Question 9:

Find $\frac{1}{2}(A+A')$ and $\frac{1}{2}(A-A')$, when $A=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}$

Answer

The given matrix is $A=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}$

$$\therefore A'=\begin{bmatrix}0 & -a & -b\\ a & 0 & -c\\ b & c & 0\end{bmatrix}$$

$$A+A'=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}+\begin{bmatrix}0 & -a & -b\\ a & 0 & -c\\ b & c & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\therefore \frac{1}{2}(A+A')=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\text{Now, } A-A'=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}-\begin{bmatrix}0 & -a & -b\\ a & 0 & -c\\ b & c & 0\end{bmatrix}=\begin{bmatrix}0 & 2a & 2b\\ -2a & 0 & 2c\\ -2b & -2c & 0\end{bmatrix}$$

$$\therefore \frac{1}{2}(A-A')=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}$$

Question 10:

Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

(i) $\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}$

(ii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$

Answer

(i) Let $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$, then $A' = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$

$$\text{Now, } A + A' = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix}$$

$$\text{Let } P = \frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}$$

$$\text{Now, } P' = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} = P$$

Thus, $P = \frac{1}{2}(A + A')$ is a symmetric matrix.

$$\text{Now, } A - A' = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

$$\text{Let } Q = \frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$\text{Now, } Q' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = -Q$$

Thus, $Q = \frac{1}{2}(A - A')$ is a skew-symmetric matrix.

Representing *A* as the sum of *P* and *Q*:

$$P + Q = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = A$$

(ii) Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$, then $A' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$$\text{Now, } A + A' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix}$$

$$\text{Let } P = \frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\text{Now, } P' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = P$$

Thus, $P = \frac{1}{2}(A + A')$ is a symmetric matrix.

$$\text{Now, } A - A' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Let } Q = \frac{1}{2}(A - A') = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Now, } Q' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -Q$$

Thus, $Q = \frac{1}{2}(A - A')$ is a skew-symmetric matrix.

Representing *A* as the sum of *P* and *Q*:

$$P + Q = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = A$$

(iii) $\text{Let } A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

$$\text{Now, } A + A' = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$$

Let $P=\frac{1}{2}\left(A+A'\right)=\frac{1}{2}\begin{bmatrix}6 & 1 & -5\\ 1 & -4 & -4\\ -5 & -4 & 4\end{bmatrix}=\begin{bmatrix}3 & \frac{1}{2} & -\frac{5}{2}\\ \frac{1}{2} & -2 & -2\\ -\frac{5}{2} & -2 & 2\end{bmatrix}$

Now, $P'=\begin{bmatrix}3 & \frac{1}{2} & -\frac{5}{2}\\ \frac{1}{2} & -2 & -2\\ -\frac{5}{2} & -2 & 2\end{bmatrix}=P$

Thus, $P=\frac{1}{2}\left(A+A'\right)$ is a symmetric matrix.

Now, $A-A'=\begin{bmatrix}3 & 3 & -1\\ -2 & -2 & 1\\ -4 & -5 & 2\end{bmatrix}-\begin{bmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{bmatrix}=\begin{bmatrix}0 & 5 & 3\\ -5 & 0 & 6\\ -3 & -6 & 0\end{bmatrix}$

Let $Q=\frac{1}{2}\left(A-A'\right)=\frac{1}{2}\begin{bmatrix}0 & 5 & 3\\ -5 & 0 & 6\\ -3 & -6 & 0\end{bmatrix}=\begin{bmatrix}0 & \frac{5}{2} & \frac{3}{2}\\ -\frac{5}{2} & 0 & 3\\ -\frac{3}{2} & -3 & 0\end{bmatrix}$

Now, $Q'=\begin{bmatrix}0 & -\frac{5}{2} & -\frac{3}{2}\\ \frac{5}{2} & 0 & -3\\ \frac{3}{2} & 3 & 0\end{bmatrix}=-Q$

Thus, $Q=\frac{1}{2}\left(A-A'\right)$ is a skew-symmetric matrix.

Representing *A* as the sum of *P* and *Q*:

$$P+Q=\begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix}+\begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 3 \\ -\frac{3}{2} & -3 & 0 \end{bmatrix}=\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}=A$$

(iv) Let $A=\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$, then $A'=\begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}$

$$\text{Now } A+A'=\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}+\begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\text{Let } P=\frac{1}{2}(A+A')=\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\text{Now, } P'=\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}=P$$

Thus, $P=\frac{1}{2}(A+A')$ is a symmetric matrix.

$$\text{Now, } A-A'=\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}-\begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}=\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}$$

$$\text{Let } Q=\frac{1}{2}(A-A')=\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

$$\text{Now, } Q'=\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}=-Q$$

Thus, $Q=\frac{1}{2}(A-A')$ is a skew-symmetric matrix.

Representing *A* as the sum of *P* and *Q*:

$$P+Q=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}+\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}=\begin{bmatrix}1 & 5\\ -1 & 2\end{bmatrix}=A$$

Question 11:

If A, B are symmetric matrices of same order, then $AB - BA$ is a

A. Skew symmetric matrix **B.** Symmetric matrix

C. Zero matrix **D.** Identity matrix

Answer

The correct answer is A.

A and B are symmetric matrices, therefore, we have:

$$A' = A \text{ and } B' = B \qquad \ldots(1)$$

$$\begin{aligned}\text{Consider } (AB-BA)' &= (AB)' - (BA)' && \left[(A-B)' = A' - B'\right]\\ &= B'A' - A'B' && \left[(AB)' = B'A'\right]\\ &= BA - AB && \left[\text{by } (1)\right]\\ &= -(AB-BA)\end{aligned}$$

$$\therefore (AB-BA)' = -(AB-BA)$$

Thus, $(AB - BA)$ is a skew-symmetric matrix.

Question 12:

If $A=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$, then $A+A'=I$, if the value of α is

A. $\frac{\pi}{6}$ **B.** $\frac{\pi}{3}$

C. π **D.** $\frac{3\pi}{2}$

Answer

The correct answer is B.

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\Rightarrow A' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Now, $A + A' = I$

$$\therefore \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Comparing the corresponding elements of the two matrices, we have:

$$2\cos\alpha = 1$$

$$\Rightarrow \cos\alpha = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\therefore \alpha = \frac{\pi}{3}$$

Exercise 3.4

Question 1:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

Answer

Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

We know that $A = IA$

$$\therefore \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A \qquad (R_2 \to R_2 - 2R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} A \qquad \left(R_2 \to \frac{1}{5} R_2\right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} A \qquad (R_1 \to R_1 + R_2)$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Question 2:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A \qquad (R_1 \to R_1 - R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} A \qquad (R_2 \to R_2 - R_1)$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Question 3:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A \qquad (\mathrm{R}_2 \to \mathrm{R}_2 - 2\mathrm{R}_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} A \qquad (\mathrm{R}_1 \to \mathrm{R}_1 - 3\mathrm{R}_2)$$

$$\therefore A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

Question 4:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{2} \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A \qquad \left(R_1 \to \frac{1}{2} R_1\right)$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{5}{2} & 1 \end{bmatrix} A \qquad (R_2 \to R_2 - 5R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ -\frac{5}{2} & 1 \end{bmatrix} A \qquad (R_1 \to R_1 + 3R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} A \qquad (R_2 \to -2R_1)$$

$$\therefore A^{-1} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

Question 5:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A \qquad \left(R_1 \to \frac{1}{2} R_1 \right)$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{7}{2} & 1 \end{bmatrix} A \qquad \left(R_2 \to R_2 - 7R_1 \right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -\frac{7}{2} & 1 \end{bmatrix} A \qquad \left(R_1 \to R_1 - R_2 \right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} A \qquad \left(R_2 \to 2R_2 \right)$$

$$\therefore A^{-1} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$$

Question 6:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{5}{2} \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A \qquad \left(R_1 \to \frac{1}{2} R_1\right)$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{5}{2} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} A \qquad \left(R_2 \to R_2 - R_1\right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -\frac{1}{2} & 1 \end{bmatrix} A \qquad \left(R_1 \to R_2 - 5R_2\right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} A \qquad \left(R_2 \to 2R_2\right)$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Question 7:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

We know that $A = AI$

$$\therefore \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad (C_1 \to C_1 - 2C_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \qquad (C_2 \to C_2 - C_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \qquad (C_1 \to C_1 - C_2)$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

Question 8:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A \qquad (R_1 \to R_1 - R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} A \qquad (R_2 \to R_2 - 3R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} A \qquad (R_1 \to R_1 - R_2)$$

$$\therefore A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$$

Question 9:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A \qquad (R_1 \to R_1 - R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} A \qquad (R_2 \to R_2 - 2R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix} A \qquad (R_1 \to R_1 - 3R_2)$$

$$\therefore A^{-1} = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$$

Question 10:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$$

We know that $A = AI$

$$\therefore \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad (C_1 \to C_1 + 2C_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = A \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \qquad (C_2 \to C_2 + C_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} \qquad \left(C_2 \to \frac{1}{2} C_2\right)$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$$

Question 11:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$$

We know that $A = AI$

$$\therefore \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \qquad (C_2 \to C_2 + 3C_1)$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix} \qquad (C_1 \to C_1 - C_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} \qquad \left(C_1 \to \frac{1}{2} C_1\right)$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Question 12:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 1 \end{bmatrix} A \qquad \left(R_1 \to \frac{1}{6} R_1\right)$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ \frac{1}{3} & 1 \end{bmatrix} A \qquad \left(R_2 \to R_2 + 2R_1\right)$$

Now, in the above equation, we can see all the zeros in the second row of the matrix on the L.H.S.

Therefore, A^{-1} does not exist.

Question 13:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Answer

$$\text{Let } A=\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}A$$

$$\Rightarrow \begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}A \qquad (R_1 \to R_1 + R_2)$$

$$\Rightarrow \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}A \qquad (R_2 \to R_2 + R_1)$$

$$\Rightarrow \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}A \qquad (R_1 \to R_1 + R_2)$$

$$\therefore A^{-1}=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$$

Question 14:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix}2 & 1\\ 4 & 2\end{bmatrix}$$

Answer

$$\text{Let } A=\begin{bmatrix}2 & 1\\ 4 & 2\end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix}2 & 1\\ 4 & 2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}A$$

Applying $R_1 \to R_1 - \frac{1}{2}R_2$, we have:

$$\begin{bmatrix}0 & 0\\ 4 & 2\end{bmatrix}=\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & 1\end{bmatrix}A$$

Now, in the above equation, we can see all the zeros in the first row of the matrix on the L.H.S.

Therefore, A^{-1} does not exist.

Question 16:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 - 2R_1$, we have:

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + 3R_3$ and $R_2 \to R_2 + 8R_3$, we have:

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 21 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -13 & 1 & 8 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 + R_2$, we have:

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 21 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -13 & 1 & 8 \\ -15 & 1 & 9 \end{bmatrix} A$$

Applying $R_3 \to \frac{1}{25} R_3$, we have:

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 21 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -13 & 1 & 8 \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A$$

Applying $R_1 \to R_1 - 10R_3$, and $R_2 \to R_2 - 21R_3$, we have:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$$

Question 17:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

We know that *A* = *IA*

$$\therefore \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to \frac{1}{2} R_1$, we have:

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 5R_1$, we have:

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 - R_2$, we have:

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ \frac{5}{2} & -1 & 1 \end{bmatrix} A$$

Applying $R_3 \to 2R_3$, we have:

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ 5 & -2 & 2 \end{bmatrix} A$$

Applying $R_1 \to R_1 + \frac{1}{2}R_3$, and $R_2 \to R_2 - \frac{5}{2}R_3$, we have:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Question 18:

Matrices *A* and *B* will be inverse of each other only if

A. $AB = BA$

C. $AB = 0$, $BA = I$

B. $AB = BA = 0$

D. $AB = BA = I$

Answer

Answer: D

We know that if A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that $AB = BA = I$, then B is said to be the inverse of A. In this case, it is clear that A is the inverse of B.

Thus, matrices A and B will be inverses of each other only if $AB = BA = I$.

Miscellaneous Solutions

Question 1:

Let $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, show that $(aI+bA)^n = a^nI + na^{n-1}bA$, where I is the identity matrix of order 2 and $n \in \mathbf{N}$

Answer

It is given that $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

To show: $\quad \mathrm{P}(n):(aI+bA)^n = a^nI + na^{n-1}bA,\ n\in\mathbf{N}$

We shall prove the result by using the principle of mathematical induction.

For n = 1, we have:

$\mathrm{P}(1):(aI+bA) = aI + ba^0A = aI + bA$

Therefore, the result is true for n = 1.

Let the result be true for n = k.

That is,

$\mathrm{P}(k):(aI+bA)^k = a^kI + ka^{k-1}bA$

Now, we prove that the result is true for n = k + 1.

Consider

$$\begin{aligned}(aI+bA)^{k+1} &= (aI+bA)^k(aI+bA)\\ &= \left(a^kI+ka^{k-1}bA\right)(aI+bA)\\ &= a^{k+1}I + ka^kbAI + a^kbIA + ka^{k-1}b^2A^2\\ &= a^{k+1}I + (k+1)a^kbA + ka^{k-1}b^2A^2 \qquad ...(1)\end{aligned}$$

Now, $A^2 = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = O$

From (1), we have:

$$(aI+bA)^{k+1} = a^{k+1}I + (k+1)a^k bA + O$$
$$= a^{k+1}I + (k+1)a^k bA$$

Therefore, the result is true for $n = k + 1$.

Thus, by the principle of mathematical induction, we have:

$$(aI+bA)^n = a^n I + na^{n-1}bA \text{ where } A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, n \in \mathbf{N}$$

Question 2:

If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}, n \in \mathbf{N}$

Answer

It is given that $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

To show: $\quad \mathrm{P}(n): A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}, n \in \mathbf{N}$

We shall prove the result by using the principle of mathematical induction.

For $n = 1$, we have:

$$P(1): \begin{bmatrix} 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \end{bmatrix} = \begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = A$$

Therefore, the result is true for $n = 1$.

Let the result be true for $n = k$.

That is $P(k)$: $A^k = \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$

Now, we prove that the result is true for $n = k + 1$.

$$\text{Now, } A^{k+1} = A \cdot A^k$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} \\ 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} \\ 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3^{(k+1)-1} & 3^{(k+1)-1} & 3^{(k+1)-1} \\ 3^{(k+1)-1} & 3^{(k+1)-1} & 3^{(k+1)-1} \\ 3^{(k+1)-1} & 3^{(k+1)-1} & 3^{(k+1)-1} \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$.

Thus by the principle of mathematical induction, we have:

$$A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}, \ n \in \mathbf{N}$$

Question 3:

If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ where n is any positive integer

Answer

It is given that $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$\text{To prove: } \quad \mathrm{P}(n): A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}, \ n \in \mathbf{N}$$

We shall prove the result by using the principle of mathematical induction.

For $n = 1$, we have:

$$P(1):\ A^1=\begin{bmatrix}1+2 & -4\\ 1 & 1-2\end{bmatrix}=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=A$$

Therefore, the result is true for $n = 1$.

Let the result be true for $n = k$.

That is,

$$P(k):\ A^k=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix},\ n\in\mathbf{N}$$

Now, we prove that the result is true for $n = k + 1$.

Consider

$$A^{k+1}=A^k\cdot A$$

$$=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$$
$$=\begin{bmatrix}3(1+2k)-4k & -4(1+2k)+4k\\ 3k+1-2k & -4k-1(1-2k)\end{bmatrix}$$
$$=\begin{bmatrix}3+6k-4k & -4-8k+4k\\ 3k+1-2k & -4k-1+2k\end{bmatrix}$$
$$=\begin{bmatrix}3+2k & -4-4k\\ 1+k & -1-2k\end{bmatrix}$$
$$=\begin{bmatrix}1+2(k+1) & -4(k+1)\\ 1+k & 1-2(k+1)\end{bmatrix}$$

Therefore, the result is true for $n = k + 1$.

Thus, by the principle of mathematical induction, we have:

$$A^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix},\ n\in\mathbf{N}$$

Question 4:

If A and B are symmetric matrices, prove that $AB - BA$ is a skew symmetric matrix.

Answer

It is given that A and B are symmetric matrices. Therefore, we have:

$A' = A$ and $B' = B$ ...(1)

$$\begin{aligned}\text{Now, } (AB-BA)' &= (AB)' - (BA)' && \left[(A-B)' = A' - B'\right]\\ &= B'A' - A'B' && \left[(AB)' = B'A'\right]\\ &= BA - AB && \left[\text{Using (1)}\right]\\ &= -(AB-BA)\end{aligned}$$

$$\therefore (AB-BA)' = -(AB-BA)$$

Thus, (*AB* − *BA*) is a skew-symmetric matrix.

Question 5:

Show that the matrix $B'AB$ is symmetric or skew symmetric according as *A* is symmetric or skew symmetric.

Answer

We suppose that *A* is a symmetric matrix, then $A' = A$ … (1)

Consider

$$\begin{aligned}(B'AB)' &= \{B'(AB)\}'\\ &= (AB)'(B')' && \left[(AB)' = B'A'\right]\\ &= B'A'(B) && \left[(B')' = B\right]\\ &= B'(A'B)\\ &= B'(AB) && \left[\text{Using (1)}\right]\end{aligned}$$

$$\therefore (B'AB)' = B'AB$$

Thus, if *A* is a symmetric matrix, then $B'AB$ is a symmetric matrix.

Now, we suppose that *A* is a skew-symmetric matrix.

Then, $A' = -A$

Consider

$$\begin{aligned}(B'AB)' &= \left[B'(AB)\right]' = (AB)'(B')' \\ &= (B'A')B = B'(-A)B \\ &= -B'AB\end{aligned}$$

$$\therefore (B'AB)' = -B'AB$$

Thus, if *A* is a skew-symmetric matrix, then $B'AB$ is a skew-symmetric matrix.

Hence, if A is a symmetric or skew-symmetric matrix, then $B'AB$ is a symmetric or skew-symmetric matrix accordingly.

Question 6:

Solve system of linear equations, using matrix method.

$$2x - y = -2$$

$$3x + 4y = 3$$

Answer

The given system of equations can be written in the form of *AX* = *B*, where

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Now,

$$|A| = 8 + 3 = 11 \neq 0$$

Thus, *A* is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1}=\frac{1}{|A|}adjA=\frac{1}{11}\begin{bmatrix}4 & 1\\ -3 & 2\end{bmatrix}$$

$$\therefore X=A^{-1}B=\frac{1}{11}\begin{bmatrix}4 & 1\\ -3 & 2\end{bmatrix}\begin{bmatrix}-2\\ 3\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-8+3\\ 6+6\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-5\\ 12\end{bmatrix}=\begin{bmatrix}-\frac{5}{11}\\ \frac{12}{11}\end{bmatrix}$$

Hence, $x=\frac{-5}{11}$ and $y=\frac{12}{11}$.

Question 7:

For what values of x, $\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 2 & 0 & 1\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}0\\ 2\\ x\end{bmatrix}=O$?

Answer

We have:

$$\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 2 & 0 & 1\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}0\\ 2\\ x\end{bmatrix}=O$$

$$\Rightarrow\begin{bmatrix}1+4+1 & 2+0+0 & 0+2+2\end{bmatrix}\begin{bmatrix}0\\ 2\\ x\end{bmatrix}=O$$

$$\Rightarrow\begin{bmatrix}6 & 2 & 4\end{bmatrix}\begin{bmatrix}0\\ 2\\ x\end{bmatrix}=O$$

$$\Rightarrow\left[6(0)+2(2)+4(x)\right]=O$$

$$\Rightarrow[4+4x]=[0]$$

$\therefore 4+4x=0$

$\Rightarrow x=-1$

Thus, the required value of x is -1.

Question 8:

If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, show that $A^2-5A+7I=O$

Answer

It is given that $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$

$$\therefore A^2=A\cdot A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}3(3)+1(-1) & 3(1)+1(2)\\ -1(3)+2(-1) & -1(1)+2(2)\end{bmatrix}$$

$$=\begin{bmatrix}9-1 & 3+2\\ -3-2 & -1+4\end{bmatrix}=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}$$

$$\therefore \text{L.H.S.}=A^2-5A+7I$$

$$=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}-5\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}+7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}-\begin{bmatrix}15 & 5\\ -5 & 10\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$$

$$=\begin{bmatrix}-7 & 0\\ 0 & -7\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$=O=\text{R.H.S.}$$

$$\therefore A^2-5A+7I=O$$

Question 9:

Find x, if $\begin{bmatrix} x & -5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$

Answer

We have:

$$\begin{bmatrix} x & -5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} x+0-2 & 0-10+0 & 2x-5-3 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} x-2 & -10 & 2x-8 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$$

$$\Rightarrow \left[x(x-2)-40+2x-8 \right] = O$$

$$\Rightarrow \left[x^2-2x-40+2x-8 \right] = [0]$$

$$\Rightarrow \left[x^2-48 \right] = [0]$$

$$\therefore x^2-48=0$$

$$\Rightarrow x^2=48$$

$$\Rightarrow x=\pm 4\sqrt{3}$$

Question 10:

A manufacturer produces three products *x*, *y*, *z* which he sells in two markets.

Annual sales are indicated below:

Market	**Products**		
I	10000	2000	18000
II	6000	20000	8000

(a) If unit sale prices of *x*, *y* and *z* are Rs 2.50, Rs 1.50 and Rs 1.00, respectively, find the total revenue in each market with the help of matrix algebra.

(b) If the unit costs of the above three commodities are Rs 2.00, Rs 1.00 and 50 paise respectively. Find the gross profit.

Answer

(a) The unit sale prices of *x*, *y*, and *z* are respectively given as Rs 2.50, Rs 1.50, and Rs 1.00.

Consequently, the total revenue in market **I** can be represented in the form of a matrix as:

$$\begin{bmatrix}10000 & 2000 & 18000\end{bmatrix}\begin{bmatrix}2.50\\1.50\\1.00\end{bmatrix}$$

$$=10000\times2.50+2000\times1.50+18000\times1.00$$

$$=25000+3000+18000$$

$$=46000$$

The total revenue in market **II** can be represented in the form of a matrix as:

$$\begin{bmatrix}6000 & 20000 & 8000\end{bmatrix}\begin{bmatrix}2.50\\1.50\\1.00\end{bmatrix}$$

$$=6000\times2.50+20000\times1.50+8000\times1.00$$

$$=15000+30000+8000$$

$$=53000$$

Therefore, the total revenue in market **I** isRs 46000 and the same in market **II** isRs 53000.

(b) The unit cost prices of *x*, *y*, and *z* are respectively given as Rs 2.00, Rs 1.00, and 50 paise.

Consequently, the total cost prices of all the products in market **I** can be represented in the form of a matrix as:

$$\begin{bmatrix}10000 & 2000 & 18000\end{bmatrix}\begin{bmatrix}2.00\\1.00\\0.50\end{bmatrix}$$

$$=10000\times2.00+2000\times1.00+18000\times0.50$$

$$=20000+2000+9000$$

$$=31000$$

Since the total revenue in market **I** isRs 46000, the gross profit in this marketis (Rs 46000 − Rs 31000) Rs 15000.

The total cost prices of all the products in market **II** can be represented in the form of a matrix as:

$$\begin{bmatrix} 6000 & 20000 & 8000 \end{bmatrix}\begin{bmatrix} 2.00 \\ 1.00 \\ 0.50 \end{bmatrix}$$

$$= 6000 \times 2.00 + 20000 \times 1.00 + 8000 \times 0.50$$

$$= 12000 + 20000 + 4000$$

$$= \text{Rs } 36000$$

Since the total revenue in market **II** isRs 53000, the gross profit in this market is (Rs 53000 − Rs 36000) Rs 17000.

Question 11:

Find the matrix X so that $X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

Answer

It is given that:

$$X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

The matrix given on the R.H.S. of the equation is a 2 × 3 matrix and the one given on the L.H.S. of the equation is a 2 × 3 matrix. Therefore, X has to be a 2 × 2 matrix.

Now, let $X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Therefore, we have:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+4c & 2a+5c & 3a+6c \\ b+4d & 2b+5d & 3b+6d \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

Equating the corresponding elements of the two matrices, we have:

$a+4c=-7, \quad 2a+5c=-8, \quad 3a+6c=-9$
$b+4d=2, \quad 2b+5d=4, \quad 3b+6d=6$

Now, $a+4c=-7 \Rightarrow a=-7-4c$

$$\therefore 2a+5c=-8 \Rightarrow -14-8c+5c=-8$$
$$\Rightarrow -3c=6$$
$$\Rightarrow c=-2$$

$$\therefore a=-7-4(-2)=-7+8=1$$

Now, $b+4d=2 \Rightarrow b=2-4d$

$$\therefore 2b+5d=4 \Rightarrow 4-8d+5d=4$$
$$\Rightarrow -3d=0$$
$$\Rightarrow d=0$$

$$\therefore b=2-4(0)=2$$

Thus, $a = 1$, $b = 2$, $c = -2$, $d = 0$

Hence, the required matrix X is $\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$.

Question 12:

If A and B are square matrices of the same order such that $AB = BA$, then prove by induction that $AB^n = B^n A$. Further, prove that $(AB)^n = A^n B^n$ for all $n \in \mathbf{N}$

Answer

A and B are square matrices of the same order such that $AB = BA$.

To prove: $\quad \mathrm{P}(n): AB^n = B^n A, \ n \in \mathbf{N}$

For $n = 1$, we have:

$\mathrm{P}(1): AB = BA \qquad [\text{Given}]$

$\Rightarrow AB^1 = B^1 A$

Therefore, the result is true for $n = 1$.

Let the result be true for $n = k$.

$$P(k): AB^k = B^k A \qquad ...(1)$$

Now, we prove that the result is true for $n = k + 1$.

$$\begin{aligned} AB^{k+1} &= AB^k \cdot B \\ &= \left(B^k A\right) B && \left[\text{By } (1)\right] \\ &= B^k (AB) && [\text{Associative law}] \\ &= B^k (BA) && \left[AB = BA \ (\text{Given})\right] \\ &= \left(B^k B\right) A && [\text{Associative law}] \\ &= B^{k+1} A \end{aligned}$$

Therefore, the result is true for $n = k + 1$.

Thus, by the principle of mathematical induction, we have $AB^n = B^n A,\ n \in \mathbf{N}$.

Now, we prove that $(AB)^n = A^n B^n$ for all $n \in \mathbf{N}$

For $n = 1$, we have:

$$(AB)^1 = A^1 B^1 = AB$$

Therefore, the result is true for $n = 1$.

Let the result be true for $n = k$.

$$(AB)^k = A^k B^k \qquad ...(2)$$

Now, we prove that the result is true for $n = k + 1$.

$$\begin{aligned} (AB)^{k+1} &= (AB)^k \cdot (AB) \\ &= \left(A^k B^k\right) \cdot (AB) && \left[\text{By } (2)\right] \\ &= A^k \left(B^k A\right) B && [\text{Associative law}] \\ &= A^k \left(AB^k\right) B && \left[AB^n = B^n A \text{ for all } n \in \mathbf{N}\right] \\ &= \left(A^k A\right) \cdot \left(B^k B\right) && [\text{Associative law}] \\ &= A^{k+1} B^{k+1} \end{aligned}$$

Therefore, the result is true for $n = k + 1$.

Thus, by the principle of mathematical induction, we have $(AB)^n = A^n B^n$, for all natural numbers.

Question 13:

Choose the correct answer in the following questions:

If $A=\begin{bmatrix}\alpha & \beta \\ \gamma & -\alpha\end{bmatrix}$ is such that $A^2 = I$ then

A. $1+\alpha^2+\beta\gamma=0$

B. $1-\alpha^2+\beta\gamma=0$

C. $1-\alpha^2-\beta\gamma=0$

D. $1+\alpha^2-\beta\gamma=0$

Answer

Answer: C

$$A=\begin{bmatrix}\alpha & \beta \\ \gamma & -\alpha\end{bmatrix}$$

$$\begin{aligned}\therefore A^2 = A\cdot A &= \begin{bmatrix}\alpha & \beta \\ \gamma & -\alpha\end{bmatrix}\begin{bmatrix}\alpha & \beta \\ \gamma & -\alpha\end{bmatrix}\\ &= \begin{bmatrix}\alpha^2+\beta\gamma & \alpha\beta-\alpha\beta \\ \alpha\gamma-\alpha\gamma & \beta\gamma+\alpha^2\end{bmatrix}\\ &= \begin{bmatrix}\alpha^2+\beta\gamma & 0 \\ 0 & \beta\gamma+\alpha^2\end{bmatrix}\end{aligned}$$

$$\text{Now, } A^2 = I \Rightarrow \begin{bmatrix}\alpha^2+\beta\gamma & 0 \\ 0 & \beta\gamma+\alpha^2\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

On comparing the corresponding elements, we have:

$$\alpha^2 + \beta\gamma = 1$$
$$\Rightarrow \alpha^2 + \beta\gamma - 1 = 0$$
$$\Rightarrow 1 - \alpha^2 - \beta\gamma = 0$$

Question 14:

If the matrix *A* is both symmetric and skew symmetric, then

A. *A* is a diagonal matrix

B. *A* is a zero matrix

C. *A* is a square matrix

D. None of these

Answer

Answer: B

If *A* is both symmetric and skew-symmetric matrix, then we should have

$$A' = A \text{ and } A' = -A$$
$$\Rightarrow A = -A$$
$$\Rightarrow A + A = O$$
$$\Rightarrow 2A = O$$
$$\Rightarrow A = O$$

Therefore, *A* is a zero matrix.

Question 15:

If *A* is square matrix such that $A^2 = A,$ then $(I + A)^3 - 7A$ is equal to

A. *A* **B.** *I* − *A* **C.** *I* **D.** 3*A*

Answer

Answer: C

$$
\begin{aligned}
(I+A)^3-7A &= I^3+A^3+3I^2A+3A^2I-7A \\
&= I+A^3+3A+3A^2-7A \\
&= I+A^2\cdot A+3A+3A-7A \qquad \left[A^2=A\right] \\
&= I+A\cdot A-A \\
&= I+A^2-A \\
&= I+A-A \\
&= I
\end{aligned}
$$

$\therefore (I+A)^3-7A=I$

Exercise 4.1

Question 1:

Evaluate the determinants in Exercises 1 and 2.

$$\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$$

Answer

$\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2(-1) - 4(-5) = -2 + 20 = 18$

Question 2:

Evaluate the determinants in Exercises 1 and 2.

(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Answer

(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = (\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta) = \cos^2\theta + \sin^2\theta = 1$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

$= (x^2 - x + 1)(x + 1) - (x - 1)(x + 1)$

$= x^3 - x^2 + x + x^2 - x + 1 - (x^2 - 1)$

$= x^3 + 1 - x^2 + 1$

$= x^3 - x^2 + 2$

Question 3:

If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$

Answer

The given matrix is $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$.

$$\therefore 2A = 2\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$$

$$\therefore \text{L.H.S.} = |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 2\times4 - 4\times8 = 8 - 32 = -24$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 1\times2 - 2\times4 = 2 - 8 = -6$$

$$\therefore \text{R.H.S.} = 4|A| = 4\times(-6) = -24$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

Question 4:

If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$.

Answer

The given matrix is $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

It can be observed that in the first column, two entries are zero. Thus, we expand along the first column (C_1) for easier calculation.

$$|A| = 1\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 1(4-0) - 0 + 0 = 4$$

$$\therefore 27|A| = 27(4) = 108 \qquad \text{...(i)}$$

$$\text{Now, } 3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$$

$$\therefore |3A| = 3\begin{vmatrix} 3 & 6 \\ 0 & 12 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 \\ 0 & 12 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 3 & 6 \end{vmatrix}$$

$$= 3(36-0) = 3(36) = 108 \qquad \text{...(ii)}$$

From equations (i) and (ii), we have:

$$|3A| = 27|A|$$

Hence, the given result is proved.

Question 5:

Evaluate the determinants

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (iii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(ii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{bmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$

Answer

(i) Let $A = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$.

It can be observed that in the second row, two entries are zero. Thus, we expand along the second row for easier calculation.

$$|A| = -0\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 0\begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} = (-15+3) = -12$$

(ii) Let $A = \begin{bmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$.

By expanding along the first row, we have:

$$|A| = 3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 3(1+6) + 4(1+4) + 5(3-2)$$

$$= 3(7) + 4(5) + 5(1)$$

$$= 21 + 20 + 5 = 46$$

(iii) Let $A=\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$.

By expanding along the first row, we have:

$$\begin{aligned} |A| &= 0\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix} \\ &= 0-1(0-6)+2(-3-0) \\ &= -1(-6)+2(-3) \\ &= 6-6=0 \end{aligned}$$

(iv) Let $A=\begin{bmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$.

By expanding along the first column, we have:

$$\begin{aligned} |A| &= 2\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} - 0\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 3\begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} \\ &= 2(0-5)-0+3(1+4) \\ &= -10+15=5 \end{aligned}$$

Question 6:

If $A=\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|\mathrm{A}|$.

Answer

Let $A=\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$.

By expanding along the first row, we have:

$$|A| = 1\begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$$

$$= 1(-9+12) - 1(-18+15) - 2(8-5)$$

$$= 1(3) - 1(-3) - 2(3)$$

$$= 3 + 3 - 6$$

$$= 6 - 6$$

$$= 0$$

Question 7:

Find values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Answer

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$$\Rightarrow 2\times1 - 5\times4 = 2x\times x - 6\times4$$

$$\Rightarrow 2 - 20 = 2x^2 - 24$$

$$\Rightarrow 2x^2 = 6$$

$$\Rightarrow x^2 = 3$$

$$\Rightarrow x = \pm\sqrt{3}$$

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

$$\Rightarrow 2\times5 - 3\times4 = x\times5 - 3\times2x$$

$$\Rightarrow 10 - 12 = 5x - 6x$$

$$\Rightarrow -2 = -x$$

$$\Rightarrow x = 2$$

Question 8:

If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

(A) 6 (B) ±6 (C) −6 (D) 0

Answer

Answer: B

$$\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$$

$$\Rightarrow x^2 - 36 = 36 - 36$$

$$\Rightarrow x^2 - 36 = 0$$

$$\Rightarrow x^2 = 36$$

$$\Rightarrow x = \pm 6$$

Hence, the correct answer is B.

Exercise 4.2

Question 1:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$$

Answer

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix} = 0+0=0$$

[Here, the two columns of the determinants are identical]

Question 2:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Answer

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$, we have:

$$\Delta = \begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ -(a-c) & -(b-a) & -(c-b) \end{vmatrix}$$

$$= -\begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ a-c & b-a & c-b \end{vmatrix}$$

Here, the two rows R_1 and R_3 are identical.

$\therefore \Delta = 0$.

Question 3:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$$

Answer

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = \begin{vmatrix} 2 & 7 & 63+2 \\ 3 & 8 & 72+3 \\ 5 & 9 & 81+5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix} + \begin{vmatrix} 2 & 7 & 2 \\ 3 & 8 & 3 \\ 5 & 9 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 9(7) \\ 3 & 8 & 9(8) \\ 5 & 9 & 9(9) \end{vmatrix} + 0 \qquad \left[\text{Two columns are identical}\right]$$

$$= 9\begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix}$$

$$= 0 \qquad \left[\text{Two columns are identical}\right]$$

Question 4:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Answer

$$\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

By applying $C_3 \to C_3 + C_2$, we have:

$$\Delta = \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$$

Here, two columns C_1 and C_3 are proportional.

$\therefore \Delta = 0$.

Question 5:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Answer

$$\Delta = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix} + \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

$$= \Delta_1 + \Delta_2 \text{ (say)} \qquad \ldots(1)$$

$$\text{Now, } \Delta_1 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$, we have:

$$\Delta_1 = \begin{vmatrix} b+c & q+r & y+z \\ c & r & z \\ a & p & x \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$, we have:

$$\Delta_1 = \begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

Applying $R_1 \leftrightarrow R_3$ and $R_2 \leftrightarrow R_3$, we have:

$$\Delta_1 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \qquad ...(2)$$

$$\Delta_2 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$, we have:

$$\Delta_2 = \begin{vmatrix} c & r & z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, we have:

$$\Delta_2 = \begin{vmatrix} c & r & z \\ a & p & x \\ b & q & y \end{vmatrix}$$

Applying $R_1 \leftrightarrow R_2$ and $R_2 \leftrightarrow R_3$, we have:

$$\Delta_2 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \qquad ...(3)$$

From (1), (2), and (3), we have:

$$\Delta = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Hence, the given result is proved.

Question 6:

By using properties of determinants, show that:

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Answer

We have,

$$\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Applying $R_1 \to cR_1$, we have:

$$\Delta = \frac{1}{c}\begin{vmatrix} 0 & ac & -bc \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Applying $R_1 \to R_1 - bR_2$, we have:

$$\Delta = \frac{1}{c}\begin{vmatrix} ab & ac & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

$$= \frac{a}{c}\begin{vmatrix} b & c & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Here, the two rows R_1 and R_3 are identical.

$\therefore \Delta = 0$.

Question 7:

By using properties of determinants, show that:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Answer

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \qquad \left[\text{Taking out factors } a,\ b,\ c \text{ from } R_1,\ R_2, \text{ and } R_3\right]$$

$$= a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \qquad \left[\text{Taking out factors } a,\ b,\ c \text{ from } C_1,\ C_2, \text{ and } C_3\right]$$

Applying $R_2 \rightarrow R_2 + R_1$ and $R_3 \rightarrow R_3 + R_1$, we have:

$$\Delta = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= a^2b^2c^2(-1)\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= -a^2b^2c^2(0-4) = 4a^2b^2c^2$$

Question 8:

By using properties of determinants, show that:

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Answer

(i) Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we have:

$$\Delta = \begin{vmatrix} 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$=(c-a)(b-c)\begin{vmatrix} 0 & -1 & -a-c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$, we have:

$$\Delta = (b-c)(c-a)\begin{vmatrix} 0 & 0 & -a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$=(a-b)(b-c)(c-a)\begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = (a-b)(b-c)(c-a)\begin{vmatrix} 0 & -1 \\ 1 & b+c \end{vmatrix} = (a-b)(b-c)(c-a)$$

Hence, the given result is proved.

(ii) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$.

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we have:

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a^3-c^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ (a-c)(a^2+ac+c^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$= (c-a)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & c \\ -(a^2+ac+c^2) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we have:

$$\Delta = (c-a)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ (b^2-a^2)+(bc-ac) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$= (b-c)(c-a)(a-b)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -(a+b+c) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -1 & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = (a-b)(b-c)(c-a)(a+b+c)(-1)\begin{vmatrix} 0 & 1 \\ 1 & c \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

Hence, the given result is proved.

Question 9:

By using properties of determinants, show that:

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Answer

$$\text{Let } \Delta = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}.$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = \begin{vmatrix} x & x^2 & yz \\ y-x & y^2-x^2 & zx-yz \\ z-x & z^2-x^2 & xy-yz \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & yz \\ -(x-y) & -(x-y)(x+y) & z(x-y) \\ (z-x) & (z-x)(z+x) & -y(z-x) \end{vmatrix}$$

$$= (x-y)(z-x)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 1 & z+x & -y \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$, we have:

$$\Delta = (x-y)(z-x)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & z-y & z-y \end{vmatrix}$$

$$= (x-y)(z-x)(z-y)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & 1 & 1 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta=\left[(x-y)(z-x)(z-y)\right]\left[(-1)\begin{vmatrix}x & yz\\ -1 & z\end{vmatrix}+1\begin{vmatrix}x & x^2\\ -1 & -x-y\end{vmatrix}\right]$$
$$=(x-y)(z-x)(z-y)\left[(-xz-yz)+(-x^2-xy+x^2)\right]$$
$$=-(x-y)(z-x)(z-y)(xy+yz+zx)$$
$$=(x-y)(y-z)(z-x)(xy+yz+zx)$$

Hence, the given result is proved.

Question 10:

By using properties of determinants, show that:

(i) $\begin{vmatrix}x+4 & 2x & 2x\\ 2x & x+4 & 2x\\ 2x & 2x & x+4\end{vmatrix}=(5x+4)(4-x)^2$

(ii) $\begin{vmatrix}y+k & y & y\\ y & y+k & y\\ y & y & y+k\end{vmatrix}=k^2(3y+k)$

Answer

(i) $\Delta=\begin{vmatrix}x+4 & 2x & 2x\\ 2x & x+4 & 2x\\ 2x & 2x & x+4\end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta=\begin{vmatrix}5x+4 & 5x+4 & 5x+4\\ 2x & x+4 & 2x\\ 2x & 2x & x+4\end{vmatrix}$$
$$=(5x+4)\begin{vmatrix}1 & 1 & 1\\ 2x & x+4 & 2x\\ 2x & 2x & x+4\end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we have:

$$\Delta=(5x+4)\begin{vmatrix}1 & 0 & 0\\ 2x & -x+4 & 0\\ 2x & 0 & -x+4\end{vmatrix}$$

$$=(5x+4)(4-x)(4-x)\begin{vmatrix}1 & 0 & 0\\ 2x & 1 & 0\\ 2x & 0 & 1\end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta=(5x+4)(4-x)^2\begin{vmatrix}1 & 0\\ 2x & 1\end{vmatrix}$$

$$=(5x+4)(4-x)^2$$

Hence, the given result is proved.

(ii) $\Delta=\begin{vmatrix}y+k & y & y\\ y & y+k & y\\ y & y & y+k\end{vmatrix}$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we have:

$$\Delta=\begin{vmatrix}3y+k & 3y+k & 3y+k\\ y & y+k & y\\ y & y & y+k\end{vmatrix}$$

$$=(3y+k)\begin{vmatrix}1 & 1 & 1\\ y & y+k & y\\ y & y & y+k\end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we have:

$$\Delta=(3y+k)\begin{vmatrix}1 & 0 & 0\\ y & k & 0\\ y & 0 & k\end{vmatrix}$$

$$=k^2(3y+k)\begin{vmatrix}1 & 0 & 0\\ y & 1 & 0\\ y & 0 & 1\end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = k^2(3y+k)\begin{vmatrix} 1 & 0 \\ y & 1 \end{vmatrix} = k^2(3y+k)$$

Hence, the given result is proved.

Question 11:

By using properties of determinants, show that:

(i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

Answer

(i) $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we have:

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^3\begin{vmatrix} 1 & 0 & 0 \\ 2b & -1 & 0 \\ 2c & 0 & -1 \end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = (a+b+c)^3(-1)(-1) = (a+b+c)^3$$

Hence, the given result is proved.

(ii) $$\Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we have:

$$\Delta = \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)^3\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = 2(x+y+z)^3(1)(1-0) = 2(x+y+z)^3$$

Hence, the given result is proved.

Question 12:

By using properties of determinants, show that:

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Answer

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$= \left(1+x+x^2\right)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = \left(1+x+x^2\right)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix}$$

$$= \left(1+x+x^2\right)(1-x)(1-x)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix}$$

$$= \left(1-x^3\right)(1-x)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = \left(1-x^3\right)(1-x)(1)\begin{vmatrix} 1+x & x \\ -x & 1 \end{vmatrix}$$

$$= \left(1-x^3\right)(1-x)\left(1+x+x^2\right)$$

$$= \left(1-x^3\right)\left(1-x^3\right)$$

$$= \left(1-x^3\right)^2$$

Hence, the given result is proved.

Question 13:

By using properties of determinants, show that:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$$

Answer

$$\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Applying $R_1 \to R_1 + bR_3$ and $R_2 \to R_2 - aR_3$, we have:

$$\Delta = \begin{vmatrix} 1+a^2+b^2 & 0 & -b\left(1+a^2+b^2\right) \\ 0 & 1+a^2+b^2 & a\left(1+a^2+b^2\right) \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \left(1+a^2+b^2\right)^2 \begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = \left(1+a^2+b^2\right)^2 \left[(1)\begin{vmatrix} 1 & a \\ -2a & 1-a^2-b^2 \end{vmatrix} - b\begin{vmatrix} 0 & 1 \\ 2b & -2a \end{vmatrix}\right]$$

$$= \left(1+a^2+b^2\right)^2 \left[1-a^2-b^2+2a^2-b(-2b)\right]$$

$$= \left(1+a^2+b^2\right)^2 \left(1+a^2+b^2\right)$$

$$= \left(1+a^2+b^2\right)^3$$

Question 14:

By using properties of determinants, show that:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Answer

$$\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Taking out common factors *a*, *b*, and *c* from R_1, R_2, and R_3 respectively, we have:

$$\Delta = abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix}$$

Applying $C_1 \to aC_1$, $C_2 \to bC_2$, and $C_3 \to cC_3$, we have:

$$\Delta = abc\times\frac{1}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = -1\begin{vmatrix} b^2 & c^2 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} a^2+1 & b^2 \\ -1 & 1 \end{vmatrix}$$

$$= -1\left(-c^2\right) + \left(a^2+1+b^2\right) = 1+a^2+b^2+c^2$$

Hence, the given result is proved.

Question 15:

Choose the correct answer.

Let A be a square matrix of order 3 × 3, then $|kA|$ is equal to

A. $k|A|$ **B.** $k^2|A|$ **C.** $k^3|A|$ **D.** $3k|A|$

Answer

Answer: C

A is a square matrix of order 3 × 3.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

$$\text{Then, } kA = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix}.$$

$$\therefore |kA| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix}$$

$$= k^3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad (\text{Taking out common factors } k \text{ from each row})$$

$$= k^3|A|$$

$$\therefore |kA| = k^3|A|$$

Hence, the correct answer is C.

Question 16:

Which of the following is correct?

A. Determinant is a square matrix.

B. Determinant is a number associated to a matrix.

C. Determinant is a number associated to a square matrix.

D. None of these

Answer

Answer: C

We know that to every square matrix, $A=[aij]$ of order *n*. We can associate a number called the determinant of square matrix *A*, where $aij=(i,j)^{\text{th}}$ element of *A*.

Thus, the determinant is a number associated to a square matrix.

Hence, the correct answer is C.

Exercise 4.3

Question 1:

Find area of the triangle with vertices at the point given in each of the following:

(i) (1, 0), (6, 0), (4, 3) (ii) (2, 7), (1, 1), (10, 8)

(iii) (−2, −3), (3, 2), (−1, −8)

Answer

(i) The area of the triangle with vertices (1, 0), (6, 0), (4, 3) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(0-3) - 0(6-4) + 1(18-0)\right]$$

$$= \frac{1}{2}\left[-3+18\right] = \frac{15}{2} \text{ square units}$$

(ii) The area of the triangle with vertices (2, 7), (1, 1), (10, 8) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[2(1-8) - 7(1-10) + 1(8-10)\right]$$

$$= \frac{1}{2}\left[2(-7) - 7(-9) + 1(-2)\right]$$

$$= \frac{1}{2}\left[-14+63-2\right] = \frac{1}{2}\left[-16+63\right]$$

$$= \frac{47}{2} \text{ square units}$$

(iii) The area of the triangle with vertices (−2, −3), (3, 2), (−1, −8)

is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix}-2 & -3 & 1\\ 3 & 2 & 1\\ -1 & -8 & 1\end{vmatrix}$$

$$=\frac{1}{2}\left[-2(2+8)+3(3+1)+1(-24+2)\right]$$

$$=\frac{1}{2}\left[-2(10)+3(4)+1(-22)\right]$$

$$=\frac{1}{2}\left[-20+12-22\right]$$

$$=-\frac{30}{2}=-15$$

Hence, the area of the triangle is $|-15|=15$ square units.

Question 2:

Show that points

$A(a,b+c), B(b,c+a), C(c,a+b)$ are collinear

Answer

Area of ΔABC is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix}a & b+c & 1\\ b & c+a & 1\\ c & a+b & 1\end{vmatrix}$$

$$=\frac{1}{2}\begin{vmatrix}a & b+c & 1\\ b-a & a-b & 0\\ c-a & a-c & 0\end{vmatrix}\quad (\text{Applying } R_2\to R_2-R_1 \text{ and } R_3\to R_3-R_1)$$

$$=\frac{1}{2}(a-b)(c-a)\begin{vmatrix}a & b+c & 1\\ -1 & 1 & 0\\ 1 & -1 & 0\end{vmatrix}$$

$$=\frac{1}{2}(a-b)(c-a)\begin{vmatrix}a & b+c & 1\\ -1 & 1 & 0\\ 0 & 0 & 0\end{vmatrix}\quad (\text{Applying } R_3\to R_3+R_2)$$

$$=0 \qquad (\text{All elements of } R_3 \text{ are } 0)$$

Thus, the area of the triangle formed by points A, B, and C is zero.

Hence, the points A, B, and C are collinear.

Question 3:

Find values of k if area of triangle is 4 square units and vertices are

(i) $(k, 0), (4, 0), (0, 2)$ (ii) $(-2, 0), (0, 4), (0, k)$

Answer

We know that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is the absolute value of the determinant (Δ), where

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

It is given that the area of triangle is 4 square units.

$\therefore \Delta = \pm 4.$

(i) The area of the triangle with vertices $(k, 0), (4, 0), (0, 2)$ is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[k(0-2)-0(4-0)+1(8-0)\right]$$

$$= \frac{1}{2}\left[-2k+8\right] = -k+4$$

$\therefore -k + 4 = \pm 4$

When $-k + 4 = -4$, $k = 8$.

When $-k + 4 = 4$, $k = 0$.

Hence, $k = 0, 8$.

(ii) The area of the triangle with vertices (−2, 0), (0, 4), (0, *k*) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-2(4-k)\right]$$

$$= k-4$$

$\therefore k - 4 = \pm 4$

When $k - 4 = -4$, $k = 0$.

When $k - 4 = 4$, $k = 8$.

Hence, $k = 0, 8$.

Question 4:

(i) Find equation of line joining (1, 2) and (3, 6) using determinants

(ii) Find equation of line joining (3, 1) and (9, 3) using determinants

Answer

(i) Let P (x, y) be any point on the line joining points A (1, 2) and B (3, 6). Then, the points A, B, and P are collinear. Therefore, the area of triangle ABP will be zero.

$$\therefore \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}\left[1(6-y)-2(3-x)+1(3y-6x)\right]=0$$

$$\Rightarrow 6-y-6+2x+3y-6x=0$$

$$\Rightarrow 2y-4x=0$$

$$\Rightarrow y=2x$$

Hence, the equation of the line joining the given points is $y = 2x$.

(ii) Let P (x, y) be any point on the line joining points A (3, 1) and

B (9, 3). Then, the points A, B, and P are collinear. Therefore, the area of triangle ABP will be zero.

$$\therefore \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}\left[3(3-y)-1(9-x)+1(9y-3x)\right]=0$$

$$\Rightarrow 9-3y-9+x+9y-3x=0$$

$$\Rightarrow 6y-2x=0$$

$$\Rightarrow x-3y=0$$

Hence, the equation of the line joining the given points is $x - 3y = 0$.

Question 5:

If area of triangle is 35 square units with vertices (2, −6), (5, 4), and (k, 4). Then k is

A. 12 **B.** −2 **C.** −12, −2 **D.** 12, −2

Answer

Answer: D

The area of the triangle with vertices (2, −6), (5, 4), and (k, 4) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left[2(4-4)+6(5-k)+1(20-4k)\right]$$

$$=\frac{1}{2}\left[30-6k+20-4k\right]$$

$$=\frac{1}{2}\left[50-10k\right]$$

$$=25-5k$$

It is given that the area of the triangle is ±35.

Therefore, we have:

$$\Rightarrow 25-5k=\pm 35$$

$$\Rightarrow 5(5-k)=\pm 35$$

$$\Rightarrow 5-k=\pm 7$$

When $5 - k = -7$, $k = 5 + 7 = 12$.

When $5 - k = 7$, $k = 5 - 7 = -2$.

Hence, $k = 12, -2$.

The correct answer is D.

Exercise 4.4

Question 1:

Write Minors and Cofactors of the elements of following determinants:

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Answer

(i) The given determinant is $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$.

Minor of element a_{ij} is M_{ij}.

$\therefore M_{11}$ = minor of element $a_{11} = 3$

M_{12} = minor of element $a_{12} = 0$

M_{21} = minor of element $a_{21} = -4$

M_{22} = minor of element $a_{22} = 2$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

(ii) The given determinant is $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$.

Minor of element a_{ij} is M_{ij}.

$\therefore M_{11}$ = minor of element $a_{11} = d$

M_{12} = minor of element $a_{12} = b$

M_{21} = minor of element $a_{21} = c$

M_{22} = minor of element $a_{22} = a$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (d) = d$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (b) = -b$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (c) = -c$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (a) = a$

Question 2:

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Answer

(i) The given determinant is $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$.

By the definition of minors and cofactors, we have:

M_{11} = minor of a_{11}= $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

M_{12} = minor of a_{12}= $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

M_{13} = minor of a_{13} = $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

M_{21} = minor of a_{21} = $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

M_{22} = minor of a_{22} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

M_{23} = minor of a_{23} = $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

M_{31} = minor of a_{31}= $\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$

M_{32} = minor of a_{32} = $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

M_{33} = minor of a_{33} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

A_{11} = cofactor of a_{11}= $(-1)^{1+1}$ M_{11} = 1

A_{12} = cofactor of a_{12} = $(-1)^{1+2}$ M_{12} = 0

A_{13} = cofactor of a_{13} = $(-1)^{1+3}$ M_{13} = 0

A_{21} = cofactor of a_{21} = $(-1)^{2+1}$ M_{21} = 0

A_{22} = cofactor of a_{22} = $(-1)^{2+2}$ M_{22} = 1

A_{23} = cofactor of a_{23} = $(-1)^{2+3}$ M_{23} = 0

A_{31} = cofactor of a_{31} = $(-1)^{3+1}$ M_{31} = 0

A_{32} = cofactor of a_{32} = $(-1)^{3+2}$ M_{32} = 0

A_{33} = cofactor of a_{33} = $(-1)^{3+3}$ M_{33} = 1

(ii) The given determinant is $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$.

By definition of minors and cofactors, we have:

M_{11} = minor of a_{11}= $\begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 + 1 = 11$

M_{12} = minor of a_{12}= $\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$

M_{13} = minor of a_{13} = $\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$

M_{21} = minor of a_{21} = $\begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4$

M_{22} = minor of a_{22} = $\begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

M_{23} = minor of a_{23} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

M_{31} = minor of a_{31}= $\begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20$

M_{32} = minor of a_{32} = $\begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13$

M_{33} = minor of a_{33} = $\begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5$

A_{11} = cofactor of a_{11}= $(-1)^{1+1}$ M_{11} = 11

A_{12} = cofactor of a_{12} = $(-1)^{1+2}$ M_{12} = −6

A_{13} = cofactor of a_{13} = $(-1)^{1+3}$ M_{13} = 3

A_{21} = cofactor of a_{21} = $(-1)^{2+1}$ M_{21} = 4

A_{22} = cofactor of a_{22} = $(-1)^{2+2}$ M_{22} = 2

A_{23} = cofactor of a_{23} = $(-1)^{2+3}$ M_{23} = −1

A_{31} = cofactor of a_{31} = $(-1)^{3+1}$ M_{31} = −20

A_{32} = cofactor of a_{32} = $(-1)^{3+2}$ M_{32} = 13

A_{33} = cofactor of a_{33} = $(-1)^{3+3}$ M_{33} = 5

Question 3:

Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

Answer

The given determinant is $\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

We have:

$M_{21} = \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = 9 - 16 = -7$

$\therefore A_{21}$ = cofactor of $a_{21} = (-1)^{2+1} M_{21} = 7$

$M_{22} = \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$

$\therefore A_{22}$ = cofactor of $a_{22} = (-1)^{2+2} M_{22} = 7$

$M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$

$\therefore A_{23}$ = cofactor of $a_{23} = (-1)^{2+3} M_{23} = -7$

We know that Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$\therefore \Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = 2(7) + 0(7) + 1(-7) = 14 - 7 = 7$

Question 4:

Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Answer

The given determinant is $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

We have:

$M_{13} = \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$

$M_{23} = \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = z - x$

$M_{33} = \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$

$\therefore A_{13}$ = cofactor of $a_{13} = (-1)^{1+3} M_{13} = (z - y)$

A_{23} = cofactor of $a_{23} = (-1)^{2+3} M_{23} = -(z - x) = (x - z)$

A_{33} = cofactor of $a_{33} = (-1)^{3+3} M_{33} = (y - x)$

We know that Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$$\therefore \Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$
$$= yz(z-y) + zx(x-z) + xy(y-x)$$
$$= yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y$$
$$= (x^2z - y^2z) + (yz^2 - xz^2) + (xy^2 - x^2y)$$
$$= z(x^2 - y^2) + z^2(y-x) + xy(y-x)$$
$$= z(x-y)(x+y) + z^2(y-x) + xy(y-x)$$
$$= (x-y)\left[zx + zy - z^2 - xy\right]$$
$$= (x-y)\left[z(x-z) + y(z-x)\right]$$
$$= (x-y)(z-x)[-z+y]$$
$$= (x-y)(y-z)(z-x)$$

Hence, $\Delta = (x-y)(y-z)(z-x)$.

Question 5:

For the matrices *A* and *B*, verify that (*AB*)′ = $B'A'$ where

(i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$

Answer

(i) $AB=\begin{bmatrix}1\\-4\\3\end{bmatrix}\begin{bmatrix}-1 & 2 & 1\end{bmatrix}=\begin{bmatrix}-1 & 2 & 1\\4 & -8 & -4\\-3 & 6 & 3\end{bmatrix}$

$$\therefore (AB)'=\begin{bmatrix}-1 & 4 & -3\\2 & -8 & 6\\1 & -4 & 3\end{bmatrix}$$

Now, $A'=\begin{bmatrix}1 & -4 & 3\end{bmatrix}, B'=\begin{bmatrix}-1\\2\\1\end{bmatrix}$

$$\therefore B'A'=\begin{bmatrix}-1\\2\\1\end{bmatrix}\begin{bmatrix}1 & -4 & 3\end{bmatrix}=\begin{bmatrix}-1 & 4 & -3\\2 & -8 & 6\\1 & -4 & 3\end{bmatrix}$$

Hence, we have verified that $(AB)'=B'A'$.

(ii) $AB=\begin{bmatrix}0\\1\\2\end{bmatrix}\begin{bmatrix}1 & 5 & 7\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\1 & 5 & 7\\2 & 10 & 14\end{bmatrix}$

$$\therefore (AB)'=\begin{bmatrix}0 & 1 & 2\\0 & 5 & 10\\0 & 7 & 14\end{bmatrix}$$

Now, $A'=\begin{bmatrix}0 & 1 & 2\end{bmatrix}, B'=\begin{bmatrix}1\\5\\7\end{bmatrix}$

$$\therefore B'A'=\begin{bmatrix}1\\5\\7\end{bmatrix}\begin{bmatrix}0 & 1 & 2\end{bmatrix}=\begin{bmatrix}0 & 1 & 2\\0 & 5 & 10\\0 & 7 & 14\end{bmatrix}$$

Hence, we have verified that $(AB)'=B'A'$.

Exercise 4.5

Question 1:

Find adjoint of each of the matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Answer

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

We have,

$A_{11} = 4,\ A_{12} = -3,\ A_{21} = -2,\ A_{22} = 1$

$$\therefore adjA = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Question 2:

Find adjoint of each of the matrices.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

Answer

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$.

We have,

$$A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$$

$$A_{12} = -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2 + 10) = -12$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$$

$$A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1-0) = 1$$

$$A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1+4 = 5$$

$$A_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0-2) = 2$$

$$A_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5-6 = -11$$

$$A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1$$

$$A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3+2 = 5$$

$$\text{Hence, } adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}.$$

Question 3:

Verify *A* (*adj A*) = (*adj A*) *A* = $|A| I$.

$$\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

Answer

$A=\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}$

we have,

$|A|=-12-(-12)=-12+12=0$

$\therefore |A|I=0\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

Now,

$A_{11}=-6, A_{12}=4, A_{21}=-3, A_{22}=2$

$\therefore adjA=\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}$

Now,

$$A(adjA)=\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}$$
$$=\begin{bmatrix}-12+12 & -6+6\\ 24-24 & 12-12\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Also, $(adjA)A=\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}$
$$=\begin{bmatrix}-12+12 & -18+18\\ 8-8 & 12-12\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Hence, $A(adjA)=(adjA)A=|A|I.$

Question 4:

Verify *A* (*adj A*) = (*adj A*) *A* = $|A|I$.

$\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$

Answer

$$A=\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$$

$$|A|=1(0-0)+1(9+2)+2(0-0)=11$$

$$\therefore |A|I=11\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

Now,

$$A_{11}=0, A_{12}=-(9+2)=-11, A_{13}=0$$

$$A_{21}=-(-3-0)=3, A_{22}=3-2=1, A_{23}=-(0+1)=-1$$

$$A_{31}=2-0=2, A_{32}=-(-2-6)=8, A_{33}=0+3=3$$

$$\therefore adjA = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

Now,

$$A(adjA) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0+0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Also,

$$(adjA)\cdot A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0+0+0 & 0+2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Hence, $A(adjA) = (adjA)A = |A|I.$

Question 6:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$$

Answer

Let $A=\begin{bmatrix}-1 & 5\\ -3 & 2\end{bmatrix}$.

we have,

$|A|=-2+15=13$

Now,

$A_{11}=2, A_{12}=3, A_{21}=-5, A_{22}=-1$

$\therefore adjA=\begin{bmatrix}2 & -5\\ 3 & -1\end{bmatrix}$

$\therefore A^{-1}=\frac{1}{|A|}adjA=\frac{1}{13}\begin{bmatrix}2 & -5\\ 3 & -1\end{bmatrix}$

Question 7:

Find the inverse of each of the matrices (if it exists).

$\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 4\\ 0 & 0 & 5\end{bmatrix}$

Answer

Let $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 4\\ 0 & 0 & 5\end{bmatrix}$.

We have,

$|A|=1(10-0)-2(0-0)+3(0-0)=10$

Now,

$A_{11}=10-0=10, A_{12}=-(0-0)=0, A_{13}=0-0=0$

$A_{21}=-(10-0)=-10, A_{22}=5-0=5, A_{23}=-(0-0)=0$

$A_{31}=8-6=2, A_{32}=-(4-0)=-4, A_{33}=2-0=2$

$$\therefore adjA = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore \; A^{-1} = \frac{1}{|A|} adjA = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

Question 8:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$$

Answer

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$.

We have,

$|A| = 1(-3-0) - 0 + 0 = -3$

Now,

$A_{11} = -3 - 0 = -3, A_{12} = -(-3-0) = 3, A_{13} = 6 - 15 = -9$

$A_{21} = -(0-0) = 0, A_{22} = -1 - 0 = -1, A_{23} = -(2-0) = -2$

$A_{31} = 0 - 0 = 0, A_{32} = -(0-0) = 0, A_{33} = 3 - 0 = 3$

$$\therefore adjA = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$\therefore \; A^{-1} = \frac{1}{|A|} adjA = -\frac{1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

Question 9:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

Answer

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$.

We have,

$$\begin{aligned} |A| &= 2(-1-0)-1(4-0)+3(8-7) \\ &= 2(-1)-1(4)+3(1) \\ &= -2-4+3 \\ &= -3 \end{aligned}$$

Now,

$A_{11} = -1-0 = -1, A_{12} = -(4-0) = -4, A_{13} = 8-7 = 1$

$A_{21} = -(1-6) = 5, A_{22} = 2+21 = 23, A_{23} = -(4+7) = -11$

$A_{31} = 0+3 = 3, A_{32} = -(0-12) = 12, A_{33} = -2-4 = -6$

$$\therefore adjA = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = -\frac{1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

Question 10:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}.$$

Answer

Let $A=\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}$.

By expanding along C_1, we have:

$|A|=1(8-6)-0+3(3-4)=2-3=-1$

Now,

$A_{11}=8-6=2, A_{12}=-(0+9)=-9, A_{13}=0-6=-6$

$A_{21}=-(-4+4)=0, A_{22}=4-6=-2, A_{23}=-(-2+3)=-1$

$A_{31}=3-4=-1, A_{32}=-(-3-0)=3, A_{33}=2-0=2$

$$\therefore adjA=\begin{bmatrix}2 & 0 & -1\\ -9 & -2 & 3\\ -6 & -1 & 2\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}adjA=-\begin{bmatrix}2 & 0 & -1\\ -9 & -2 & 3\\ -6 & -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$$

Question 11:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & \sin\alpha & -\cos\alpha\end{bmatrix}$$

Answer

Let $A=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & \sin\alpha & -\cos\alpha\end{bmatrix}$.

We have,

$|A|=1\left(-\cos^2\alpha-\sin^2\alpha\right)=-\left(\cos^2\alpha+\sin^2\alpha\right)=-1$

Now,

$A_{11}=-\cos^2\alpha-\sin^2\alpha=-1, A_{12}=0, A_{13}=0$

$A_{21}=0, A_{22}=-\cos\alpha, A_{23}=-\sin\alpha$

$A_{31}=0, A_{32}=-\sin\alpha, A_{33}=\cos\alpha$

$\therefore adjA=\begin{bmatrix}-1 & 0 & 0\\ 0 & -\cos\alpha & -\sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}$

$\therefore\ A^{-1}=\frac{1}{|A|}\cdot adjA=-\begin{bmatrix}-1 & 0 & 0\\ 0 & -\cos\alpha & -\sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & \sin\alpha & -\cos\alpha\end{bmatrix}$

Question 12:

Let $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$. Verify that $(AB)^{-1}=B^{-1}A^{-1}$

Answer

Let $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$.

We have,

$|A|=15-14=1$

Now,

$A_{11}=5, A_{12}=-2, A_{21}=-7, A_{22}=3$

$\therefore adjA=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$

$\therefore A^{-1}=\frac{1}{|A|}\cdot adjA=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$

Now, let $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$.

We have,

$|B|=54-56=-2$

$\therefore adjB=\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}$

$\therefore B^{-1}=\frac{1}{|B|}adjB=-\frac{1}{2}\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}=\begin{bmatrix}-\frac{9}{2} & 4\\ \frac{7}{2} & -3\end{bmatrix}$

Now,

$$B^{-1}A^{-1}=\begin{bmatrix}-\frac{9}{2} & 4\\ \frac{7}{2} & -3\end{bmatrix}\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{45}{2}-8 & \frac{63}{2}+12\\ \frac{35}{2}+6 & -\frac{49}{2}-9\end{bmatrix}=\begin{bmatrix}-\frac{61}{2} & \frac{87}{2}\\ \frac{47}{2} & -\frac{67}{2}\end{bmatrix} \quad ...(1)$$

Then,

$$AB=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$$

$$=\begin{bmatrix}18+49 & 24+63\\ 12+35 & 16+45\end{bmatrix}$$

$$=\begin{bmatrix}67 & 87\\ 47 & 61\end{bmatrix}$$

Therefore,we have $|AB| = 67\times 61 - 87\times 47 = 4087 - 4089 = -2.$

Also,

$$adj\left(AB\right)=\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix}$$

$$\therefore \left(AB\right)^{-1}=\frac{1}{|AB|}adj\left(AB\right)=-\frac{1}{2}\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{61}{2} & \frac{87}{2}\\ \frac{47}{2} & -\frac{67}{2}\end{bmatrix} \quad ...(2)$$

From (1) and (2), we have:

$(AB)^{-1} = B^{-1}A^{-1}$

Hence, the given result is proved.

Question 13:

If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, show that $A^2-5A+7I=O$. Hence find A^{-1}.

Answer

$$A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$$

$$A^2=A\cdot A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}9-1 & 3+2\\ -3-2 & -1+4\end{bmatrix}=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}$$

$$\therefore A^2-5A+7I$$

$$=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}-5\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}+7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}-\begin{bmatrix}15 & 5\\ -5 & 10\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$$

$$=\begin{bmatrix}-7 & 0\\ 0 & -7\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Hence, $A^2-5A+7I=O.$

$$\therefore A\cdot A-5A=-7I$$

$$\Rightarrow A\cdot A\left(A^{-1}\right)-5AA^{-1}=-7IA^{-1} \qquad \left[\text{Post-multiplying by } A^{-1} \text{ as } |A|\neq 0\right]$$

$$\Rightarrow A\left(AA^{-1}\right)-5I=-7A^{-1}$$

$$\Rightarrow AI-5I=-7A^{-1}$$

$$\Rightarrow A^{-1}=-\frac{1}{7}(A-5I)$$

$$\Rightarrow A^{-1}=\frac{1}{7}(5I-A)$$

$$=\frac{1}{7}\left(\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}-\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\right)=\frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$$

Question 14:

For the matrix $A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$, find the numbers *a* and *b* such that $A^2 + aA + bI = O$.

Answer

$$A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$$

$$\therefore A^2=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}9+2 & 6+2\\ 3+1 & 2+1\end{bmatrix}=\begin{bmatrix}11 & 8\\ 4 & 3\end{bmatrix}$$

Now,

$$A^2+aA+bI=O$$

$$\Rightarrow (AA)A^{-1}+aAA^{-1}+bIA^{-1}=O \qquad \left[\text{Post-multiplying by } A^{-1} \text{ as } |A|\neq 0\right]$$

$$\Rightarrow A\left(AA^{-1}\right)+aI+b\left(IA^{-1}\right)=O$$

$$\Rightarrow AI+aI+bA^{-1}=O$$

$$\Rightarrow A+aI=-bA^{-1}$$

$$\Rightarrow A^{-1}=-\frac{1}{b}(A+aI)$$

Now,

$$A^{-1}=\frac{1}{|A|}adjA=\frac{1}{1}\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}$$

We have:

$$\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=-\frac{1}{b}\left(\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}+\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}\right)=-\frac{1}{b}\begin{bmatrix}3+a & 2\\ 1 & 1+a\end{bmatrix}=\begin{bmatrix}\frac{-3-a}{b} & -\frac{2}{b}\\ -\frac{1}{b} & \frac{-1-a}{b}\end{bmatrix}$$

Comparing the corresponding elements of the two matrices, we have:

$$-\frac{1}{b}=-1\Rightarrow b=1$$

$$\frac{-3-a}{b}=1\Rightarrow -3-a=1\Rightarrow a=-4$$

Hence, −4 and 1 are the required values of *a* and *b* respectively.

Question 15:

For the matrix $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$ show that $A^3 - 6A^2 + 5A + 11\,I = O$. Hence, find $A^{-1.}$

Answer

$$A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$A^2=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+1+2 & 1+2-1 & 1-3+3\\ 1+2-6 & 1+4+3 & 1-6-9\\ 2-1+6 & 2-2-3 & 2+3+9\end{bmatrix}=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}$$

$$A^3=A^2\cdot A=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}4+2+2 & 4+4-1 & 4-6+3\\ -3+8-28 & -3+16+14 & -3-24-42\\ 7-3+28 & 7-6-14 & 7+9+42\end{bmatrix}$$

$$=\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}$$

$\therefore A^3-6A^2+5A+11I$

$$=\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}-6\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}+5\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}+11\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}-\begin{bmatrix}24 & 12 & 6\\ -18 & 48 & -84\\ 42 & -18 & 84\end{bmatrix}+\begin{bmatrix}5 & 5 & 5\\ 5 & 10 & -15\\ 10 & -5 & 15\end{bmatrix}+\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

$$=\begin{bmatrix}24 & 12 & 6\\ -18 & 48 & -84\\ 42 & -18 & 84\end{bmatrix}-\begin{bmatrix}24 & 12 & 6\\ -18 & 48 & -84\\ 42 & -18 & 84\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=O$$

Thus, $A^3-6A^2+5A+11I=O$.

Now,

$A^3-6A^2+5A+11I=O$

$\Rightarrow (AAA)A^{-1}-6(AA)A^{-1}+5AA^{-1}+11IA^{-1}=0$ $\quad$ [Post-multiplying by A^{-1} as $|A|\neq 0$]

$\Rightarrow AA(AA^{-1})-6A(AA^{-1})+5(AA^{-1})=-11(IA^{-1})$

$\Rightarrow A^2-6A+5I=-11A^{-1}$

$\Rightarrow A^{-1}=-\frac{1}{11}(A^2-6A+5I) \qquad \ldots(1)$

Now,

$A^2 - 6A + 5I$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 & 1 \\ -3 & 13 & -14 \\ 7 & -3 & 19 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

From equation (1), we have:

$$A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

Question 16:

If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}

Answer

$$A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$A^2=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}4+1+1 & -2-2-1 & 2+1+2\\ -2-2-1 & 1+4+1 & -1-2-2\\ 2+1+2 & -1-2-2 & 1+1+4\end{bmatrix}$$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$$

$$A^3=A^2A=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}12+5+5 & -6-10-5 & 6+5+10\\ -10-6-5 & 5+12+5 & -5-6-10\\ 10+5+6 & -5-10-6 & 5+5+12\end{bmatrix}$$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}$$

Now,

$A^3-6A^2+9A-4I$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-6\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}+9\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}-4\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-\begin{bmatrix}36 & -30 & 30\\ -30 & 36 & -30\\ 30 & -30 & 36\end{bmatrix}+\begin{bmatrix}18 & -9 & 9\\ -9 & 18 & -9\\ 9 & -9 & 18\end{bmatrix}-\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}40 & -30 & 30\\ -30 & 40 & -30\\ 30 & -30 & 40\end{bmatrix}-\begin{bmatrix}40 & -30 & 30\\ -30 & 40 & -30\\ 30 & -30 & 40\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\therefore A^3-6A^2+9A-4I=O$

Now,

$A^3-6A^2+9A-4I=O$

$\Rightarrow (AAA)A^{-1}-6(AA)A^{-1}+9AA^{-1}-4IA^{-1}=O$ $\quad\left[\text{Post-multiplying by } A^{-1} \text{ as } |A|\neq 0\right]$

$\Rightarrow AA\left(AA^{-1}\right)-6A\left(AA^{-1}\right)+9\left(AA^{-1}\right)=4\left(IA^{-1}\right)$

$\Rightarrow AAI-6AI+9I=4A^{-1}$

$\Rightarrow A^2-6A+9I=4A^{-1}$

$\Rightarrow A^{-1}=\frac{1}{4}\left(A^2-6A+9I\right) \qquad \ldots(1)$

$A^2-6A+9I$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}-6\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}+9\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}-\begin{bmatrix}12 & -6 & 6\\ -6 & 12 & -6\\ 6 & -6 & 12\end{bmatrix}+\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$$=\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

From equation (1), we have:

$$A^{-1}=\frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

Question 17:

Let *A* be a nonsingular square matrix of order 3 × 3. Then $|adjA|$ is equal to

A. $|A|$ **B.** $|A|^2$ **C.** $|A|^3$ **D.** $3|A|$

Answer **B**

We know that,

$$(adjA)A=|A|I=\begin{bmatrix}|A| & 0 & 0\\ 0 & |A| & 0\\ 0 & 0 & |A|\end{bmatrix}$$

$$\Rightarrow |(adjA)A|=\begin{vmatrix}|A| & 0 & 0\\ 0 & |A| & 0\\ 0 & 0 & |A|\end{vmatrix}$$

$$\Rightarrow |adjA||A|=|A|^3\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=|A|^3(I)$$

$$\therefore |adjA|=|A|^2$$

Hence, the correct answer is B.

Question 18:

If *A* is an invertible matrix of order 2, then det (A^{-1}) is equal to

A. det (*A*) **B.** $\frac{1}{\det(A)}$ **C.** 1 **D.** 0

Answer

Since *A* is an invertible matrix, A^{-1} exists and $A^{-1}=\frac{1}{|A|}adjA.$

As matrix A is of order 2, let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then, $|A| = ad - bc$ and $adjA=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Now,

$$A^{-1}=\frac{1}{|A|}adjA=\begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$\therefore \left|A^{-1}\right|=\begin{vmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{vmatrix}=\frac{1}{|A|^2}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}=\frac{1}{|A|^2}(ad-bc)=\frac{1}{|A|^2}.|A|=\frac{1}{|A|}$$

$$\therefore \det\left(A^{-1}\right)=\frac{1}{\det(A)}$$

Hence, the correct answer is B.

Exercise 4.6

Question 1:

Examine the consistency of the system of equations.

$x + 2y = 2$

$2x + 3y = 3$

Answer

The given system of equations is:

$x + 2y = 2$

$2x + 3y = 3$

The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Now,

$$|A| = 1(3) - 2(2) = 3 - 4 = -1 \neq 0$$

∴ A is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 2:

Examine the consistency of the system of equations.

$2x - y = 5$

$x + y = 4$

Answer

The given system of equations is:

$2x - y = 5$

$x + y = 4$

The given system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix},\ X=\begin{bmatrix}x\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}5\\ 4\end{bmatrix}.$$

Now,

$$|A|=2(1)-(-1)(1)=2+1=3\neq 0$$

∴ *A* is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 3:

Examine the consistency of the system of equations.

$x + 3y = 5$

$2x + 6y = 8$

Answer

The given system of equations is:

$x + 3y = 5$

$2x + 6y = 8$

The given system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}1 & 3\\ 2 & 6\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix} \text{ and } B=\begin{bmatrix}5\\ 8\end{bmatrix}.$$

Now,

$$|A|=1(6)-3(2)=6-6=0$$

∴ *A* is a singular matrix.

Now,

$$(adjA) = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

$$(adjA)B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30-24 \\ -10+8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq O$$

Thus, the solution of the given system of equations does not exist. Hence, the system of equations is inconsistent.

Question 4:

Examine the consistency of the system of equations.

$x + y + z = 1$

$2x + 3y + 2z = 2$

$ax + ay + 2az = 4$

Answer

The given system of equations is:

$x + y + z = 1$

$2x + 3y + 2z = 2$

$ax + ay + 2az = 4$

This system of equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

Now,

$$|A| = 1(6a-2a) - 1(4a-2a) + 1(2a-3a)$$
$$= 4a - 2a - a = 4a - 3a = a \neq 0$$

$\therefore$ A is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 5:

Examine the consistency of the system of equations.

$3x - y - 2z = 2$

$2y - z = -1$

$3x - 5y = 3$

Answer

The given system of equations is:

$3x - y - 2z = 2$

$2y - z = -1$

$3x - 5y = 3$

This system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Now,

$$|A| = 3(0-5) - 0 + 3(1+4) = -15 + 15 = 0$$

$\therefore$ *A* is a singular matrix.

Now,

$$(adjA) = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$\therefore (adjA)B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10-10+15 \\ -6-6+9 \\ -12-12+18 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq O$$

Thus, the solution of the given system of equations does not exist. Hence, the system of equations is inconsistent.

Question 6:

Examine the consistency of the system of equations.

$5x - y + 4z = 5$

$2x + 3y + 5z = 2$

$5x - 2y + 6z = -1$

Answer

The given system of equations is:

$5x - y + 4z = 5$

$2x + 3y + 5z = 2$

$5x - 2y + 6z = -1$

This system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}.$$

Now,

$$\begin{aligned} |A| &= 5(18+10)+1(12-25)+4(-4-15) \\ &= 5(28)+1(-13)+4(-19) \\ &= 140-13-76 \\ &= 51 \neq 0 \end{aligned}$$

∴ A is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 7:

Solve system of linear equations, using matrix method.

$5x + 2y = 4$

$7x + 3y = 5$

Answer

The given system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}5 & 2\\ 7 & 3\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\ 5\end{bmatrix}.$$

Now, $|A| = 15-14 = 1 \neq 0.$

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1}=\frac{1}{|A|}(adjA)$$

$$\therefore A^{-1}=\begin{bmatrix}3 & -2\\ -7 & 5\end{bmatrix}$$

$$\therefore X = A^{-1}B=\begin{bmatrix}3 & -2\\ -7 & 5\end{bmatrix}\begin{bmatrix}4\\ 5\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}12-10\\ -28+25\end{bmatrix}=\begin{bmatrix}2\\ -3\end{bmatrix}$$

Hence, $x = 2$ and $y = -3$.

Question 8:

Solve system of linear equations, using matrix method.

$$2x - y = -2$$

$$3x + 4y = 3$$

Answer

The given system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix} \text{ and } B=\begin{bmatrix}-2\\ 3\end{bmatrix}.$$

Now,

$$|A| = 8+3 = 11 \neq 0$$

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} -\frac{5}{11} \\ \frac{12}{11} \end{bmatrix}$$

Hence, $x = \frac{-5}{11}$ and $y = \frac{12}{11}$.

Question 9:

Solve system of linear equations, using matrix method.

$4x - 3y = 3$

$3x - 5y = 7$

Answer

The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{and } B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Now,

$|A| = -20 + 9 = -11 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1}=\frac{1}{|A|}(adjA)=-\frac{1}{11}\begin{bmatrix}-5 & 3\\ -3 & 4\end{bmatrix}=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}$$

$$\therefore X=A^{-1}B=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}=\frac{1}{11}\begin{bmatrix}15-21\\ 9-28\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-6\\ -19\end{bmatrix}=\begin{bmatrix}-\frac{6}{11}\\ -\frac{19}{11}\end{bmatrix}$$

Hence, $x=\frac{-6}{11}$ and $y=\frac{-19}{11}$.

Question 10:

Solve system of linear equations, using matrix method.

5*x* + 2*y* = 3

3*x* + 2*y* = 5

Answer

The given system of equations can be written in the form of *AX* = *B*, where

$$A=\begin{bmatrix}5 & 2\\ 3 & 2\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix} \text{ and } B=\begin{bmatrix}3\\ 5\end{bmatrix}.$$

Now,

$|A|=10-6=4\neq 0$

Thus, *A* is non-singular. Therefore, its inverse exists.

Question 11:

Solve system of linear equations, using matrix method.

$$2x+y+z=1$$

$$x-2y-z=\frac{3}{2}$$

$$3y-5z=9$$

Answer

The given system of equations can be written in the form of *AX* = *B*, where

$$A=\begin{bmatrix}2 & 1 & 1\\ 1 & -2 & -1\\ 0 & 3 & -5\end{bmatrix},\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}1\\ \frac{3}{2}\\ 9\end{bmatrix}.$$

Now,

$$|A|=2(10+3)-1(-5-3)+0=2(13)-1(-8)=26+8=34\neq 0$$

Thus, *A* is non-singular. Therefore, its inverse exists.

Now, $A_{11}=13, A_{12}=5, A_{13}=3$

$A_{21}=8, A_{22}=-10, A_{23}=-6$

$A_{31}=1, A_{32}=3, A_{33}=-5$

$$\therefore\ A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{34}\begin{bmatrix}13 & 8 & 1\\ 5 & -10 & 3\\ 3 & -6 & -5\end{bmatrix}$$

$$\therefore\ X=A^{-1}B=\frac{1}{34}\begin{bmatrix}13 & 8 & 1\\ 5 & -10 & 3\\ 3 & -6 & -5\end{bmatrix}\begin{bmatrix}1\\ \frac{3}{2}\\ 9\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{34}\begin{bmatrix}13+12+9\\ 5-15+27\\ 3-9-45\end{bmatrix}$$

$$=\frac{1}{34}\begin{bmatrix}34\\ 17\\ -51\end{bmatrix}=\begin{bmatrix}1\\ \frac{1}{2}\\ -\frac{3}{2}\end{bmatrix}$$

Hence, $x=1, y=\frac{1}{2}$, and $z=-\frac{3}{2}$.

Question 12:

Solve system of linear equations, using matrix method.

$x - y + z = 4$

$2x + y - 3z = 0$

$x + y + z = 2$

Answer

The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

Now,

$$|A| = 1(1+3) + 1(2+3) + 1(2-1) = 4 + 5 + 1 = 10 \neq 0$$

Thus, A is non-singular. Therefore, its inverse exists.

$$\begin{aligned} \text{Now, } & A_{11} = 4, A_{12} = -5, A_{13} = 1 \\ & A_{21} = 2, A_{22} = 0, A_{23} = -2 \\ & A_{31} = 2, A_{32} = 5, A_{33} = 3 \end{aligned}$$

$$\therefore\ A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence, $x = 2, y = -1,$ and $z = 1$.

Question 13:

Solve system of linear equations, using matrix method.

$2x + 3y + 3z = 5$

$x - 2y + z = -4$

$3x - y - 2z = 3$

Answer

The given system of equations can be written in the form $AX = B$, where

$$A=\begin{bmatrix}2 & 3 & 3\\ 1 & -2 & 1\\ 3 & -1 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{and } B=\begin{bmatrix}5\\ -4\\ 3\end{bmatrix}.$$

Now,

$$|A| = 2(4+1)-3(-2-3)+3(-1+6) = 2(5)-3(-5)+3(5) = 10+15+15 = 40 \neq 0$$

Thus, A is non-singular. Therefore, its inverse exists.

$$\text{Now, } A_{11}=5, A_{12}=5, A_{13}=5$$
$$A_{21}=3, A_{22}=-13, A_{23}=11$$
$$A_{31}=9, A_{32}=1, A_{33}=-7$$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{40}\begin{bmatrix}5 & 3 & 9\\ 5 & -13 & 1\\ 5 & 11 & -7\end{bmatrix}$$

$$\therefore X=A^{-1}B=\frac{1}{40}\begin{bmatrix}5 & 3 & 9\\ 5 & -13 & 1\\ 5 & 11 & -7\end{bmatrix}\begin{bmatrix}5\\ -4\\ 3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{40}\begin{bmatrix}25-12+27\\ 25+52+3\\ 25-44-21\end{bmatrix}$$

$$=\frac{1}{40}\begin{bmatrix}40\\ 80\\ -40\end{bmatrix}$$

$$=\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$$

Hence, $x=1, y=2,$ and $z=-1.$

Question 14:

Solve system of linear equations, using matrix method.

$x - y + 2z = 7$

$3x + 4y - 5z = -5$

$2x - y + 3z = 12$

Answer

The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}.$$

Now,

$$|A| = 1(12-5)+1(9+10)+2(-3-8) = 7+19-22 = 4 \neq 0$$

Thus, A is non-singular. Therefore, its inverse exists.

$$\text{Now, } A_{11} = 7, A_{12} = -19, A_{13} = -11$$
$$A_{21} = 1, A_{22} = -1, A_{23} = -1$$
$$A_{31} = -3, A_{32} = 11, A_{33} = 7$$

$$\therefore\ A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Hence, $x = 2, y = 1, \text{and } z = 3.$

Question 15:

If $A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations

$$2x-3y+5z=11$$
$$3x+2y-4z=-5$$
$$x+y-2z=-3$$

Answer

$$A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$$

$$\therefore |A|=2(-4+4)+3(-6+4)+5(3-2)=0-6+5=-1\neq 0$$

Now, $A_{11}=0,\ A_{12}=2,\ A_{13}=1$

$A_{21}=-1,\ A_{22}=-9,\ A_{23}=-5$

$A_{31}=2,\ A_{32}=23,\ A_{33}=13$

$$\therefore\ A^{-1}=\frac{1}{|A|}(adjA)=-\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}=\begin{bmatrix}0 & 1 & -2\\ -2 & 9 & -23\\ -1 & 5 & -13\end{bmatrix} \qquad \ldots(1)$$

Now, the given system of equations can be written in the form of *AX* = *B*, where

$$A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}.$$

The solution of the system of equations is given by $X = A^{-1}B$.

$$X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \qquad \left[\text{Using } (1)\right]$$

$$= \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1,\ y = 2,$ and $z = 3$.

Question 16:

The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is Rs 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is Rs 70. Find cost of each item per kg by matrix method.

Answer

Let the cost of onions, wheat, and rice per kg be Rs *x*, Rs *y*,and Rs *z* respectively.

Then, the given situation can be represented by a system of equations as:

$$4x + 3y + 2z = 60$$
$$2x + 4y + 6z = 90$$
$$6x + 2y + 3z = 70$$

This system of equations can be written in the form of *AX* = *B*, where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}.$$

$$|A| = 4(12-12) - 3(6-36) + 2(4-24) = 0 + 90 - 40 = 50 \neq 0$$

Now, $A_{11} = 0, A_{12} = 30, A_{13} = -20$

$A_{21} = -5, A_{22} = 0, A_{23} = 10$

$A_{31} = 10, A_{32} = -20, A_{33} = 10$

$$\therefore adjA = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

Now,

$X = A^{-1} B$

$$\Rightarrow X = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0-450+700 \\ 1800+0-1400 \\ -1200+900+700 \end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\therefore x = 5, y = 8, \text{and } z = 8.$

Hence, the cost of onions is Rs 5 per kg, the cost of wheat is Rs 8 per kg, and the cost of rice is Rs 8 per kg.

Miscellaneous Solutions

Question 1:

Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

Answer

$$\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$$

$$= x\left(x^2 - 1\right) - \sin\theta\left(-x\sin\theta - \cos\theta\right) + \cos\theta\left(-\sin\theta + x\cos\theta\right)$$

$$= x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$$

$$= x^3 - x + x\left(\sin^2\theta + \cos^2\theta\right)$$

$$= x^3 - x + x$$

$$= x^3 \text{ (Independent of } \theta\text{)}$$

Hence, Δ is independent of θ.

Question 2:

Without expanding the determinant, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Answer

$$\text{L.H.S.} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix} \qquad [R_1 \to aR_1, R_2 \to bR_2, \text{and } R_3 \to cR_3]$$

$$= \frac{1}{abc}\cdot abc\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \qquad [\text{Taking out factor } abc \text{ from } C_3]$$

$$= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \qquad [\text{Applying } C_1 \leftrightarrow C_3 \text{ and } C_2 \leftrightarrow C_3]$$

$$= \text{R.H.S.}$$

Hence, the given result is proved.

Question 3:

Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Answer

$$\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = -\sin\alpha\left(-\sin\alpha\sin^2\beta - \cos^2\beta\sin\alpha\right) + \cos\alpha\left(\cos\alpha\cos^2\beta + \cos\alpha\sin^2\beta\right)$$
$$= \sin^2\alpha\left(\sin^2\beta + \cos^2\beta\right) + \cos^2\alpha\left(\cos^2\beta + \sin^2\beta\right)$$
$$= \sin^2\alpha(1) + \cos^2\alpha(1)$$
$$= 1$$

Question 4:

If a, b and c are real numbers, and determinant $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

Show that either $a + b + c = 0$ or $a = b = c$.

Answer

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = 2(a+b+c)(1)\left[(b-c)(c-b)-(b-a)(c-a)\right]$$
$$= 2(a+b+c)\left[-b^2-c^2+2bc-bc+ba+ac-a^2\right]$$
$$= 2(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right]$$

It is given that $\Delta = 0$.

$$(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right]=0$$

$\Rightarrow$ Either $a+b+c=0$, or $ab+bc+ca-a^2-b^2-c^2=0$.

Now,

$$ab+bc+ca-a^2-b^2-c^2=0$$
$$\Rightarrow -2ab-2bc-2ca+2a^2+2b^2+2c^2=0$$
$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$$
$$\Rightarrow (a-b)^2=(b-c)^2=(c-a)^2=0 \qquad \left[(a-b)^2,(b-c)^2,(c-a)^2 \text{ are non-negative}\right]$$
$$\Rightarrow (a-b)=(b-c)=(c-a)=0$$
$$\Rightarrow a=b=c$$

Hence, if $\Delta = 0$, then either $a + b + c = 0$ or $a = b = c$.

Question 5:

Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$

Answer

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get:

$$\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$(3x+a)\begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$$

Expanding along R_1, we have:

$$(3x+a)\left[1 \times a^2\right] = 0$$

$$\Rightarrow a^2(3x+a) = 0$$

But $a \neq 0$.

Therefore, we have:

$$3x + a = 0$$

$$\Rightarrow x = -\frac{a}{3}$$

Question 6:

Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Answer

$$\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking out common factors a, b, and c from $C_1, C_2,$ and C_3, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ b & b-c & -c \\ b-a & b & -a \end{vmatrix}$$

Applying $R_2 \to R_2 + R_1$, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b-a & b & -a \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 2b & 2b & 0 \end{vmatrix}$$

$$= 2ab^2c\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 1 & 1 & 0 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, we have:

$$\Delta = 2ab^2c\begin{vmatrix} a & c-a & a+c \\ a+b & -a & a \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = 2ab^2c\left[a(c-a)+a(a+c)\right]$$

$$= 2ab^2c\left[ac-a^2+a^2+ac\right]$$

$$= 2ab^2c(2ac)$$

$$= 4a^2b^2c^2$$

Hence, the given result is proved.

Question 8:

Let $A=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$ verify that

(i) $[adjA]^{-1}=adj\left(A^{-1}\right)$

(ii) $\left(A^{-1}\right)^{-1}=A$

Answer

$$A=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$$

$$\therefore |A|=1(15-1)+2(-10-1)+1(-2-3)=14-22-5=-13$$

$$\begin{aligned}\text{Now, } & A_{11}=14, A_{12}=11, A_{13}=-5\\ & A_{21}=11, A_{22}=4, A_{23}=-3\\ & A_{31}=-5, A_{32}=-3, A_{13}=-1\end{aligned}$$

$$\therefore adjA=\begin{bmatrix}14 & 11 & -5\\ 11 & 4 & -3\\ -5 & -3 & -1\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}(adjA)$$

$$=-\frac{1}{13}\begin{bmatrix}14 & 11 & -5\\ 11 & 4 & -3\\ -5 & -3 & -1\end{bmatrix}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}$$

(i)

$$\begin{aligned}|adjA| &= 14(-4-9)-11(-11-15)-5(-33+20)\\ &= 14(-13)-11(-26)-5(-13)\\ &= -182+286+65=169\end{aligned}$$

We have,

$$adj\left(adjA\right)=\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}$$

$$\therefore\left[adjA\right]^{-1}=\frac{1}{\left|adjA\right|}\left(adj\left(adjA\right)\right)$$

$$=\frac{1}{169}\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}$$

$$=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

$$\text{Now, } A^{-1}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}=\begin{bmatrix}-\frac{14}{13} & -\frac{11}{13} & \frac{5}{13}\\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13}\\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13}\end{bmatrix}$$

$$\therefore adj\left(A^{-1}\right)=\begin{bmatrix}-\frac{4}{169}-\frac{9}{169} & -\left(-\frac{11}{169}-\frac{15}{169}\right) & -\frac{33}{169}+\frac{20}{169}\\ -\left(-\frac{11}{169}-\frac{15}{169}\right) & -\frac{14}{169}-\frac{25}{169} & -\left(-\frac{42}{169}+\frac{55}{169}\right)\\ -\frac{33}{169}+\frac{20}{169} & -\left(-\frac{42}{169}+\frac{55}{169}\right) & \frac{56}{169}-\frac{121}{169}\end{bmatrix}$$

$$=\frac{1}{169}\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

Hence, $\left[adjA\right]^{-1}=adj\left(A^{-1}\right)$.

(ii)

We have shown that:

$$A^{-1}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}$$

And, $adjA^{-1}=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$

Now,

$$\left|A^{-1}\right|=\left(\frac{1}{13}\right)^3\left[-14\times(-13)+11\times(-26)+5\times(-13)\right]=\left(\frac{1}{13}\right)^3\times(-169)=-\frac{1}{13}$$

$$\therefore \left(A^{-1}\right)^{-1}=\frac{adjA^{-1}}{\left|A^{-1}\right|}=\frac{1}{\left(-\frac{1}{13}\right)}\times\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}=A$$

$$\therefore \left(A^{-1}\right)^{-1}=A$$

Question 9:

Evaluate $\begin{vmatrix}x & y & x+y\\ y & x+y & x\\ x+y & x & y\end{vmatrix}$

Answer

$$\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

$$= 2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = 2(x+y)\left[-x^2 + y(x-y)\right]$$

$$= -2(x+y)\left(x^2 + y^2 - yx\right)$$

$$= -2\left(x^3 + y^3\right)$$

Question 10:

Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Answer

$$\Delta=\begin{vmatrix}1 & x & y\\ 1 & x+y & y\\ 1 & x & x+y\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta=\begin{vmatrix}1 & x & y\\ 0 & y & 0\\ 0 & 0 & x\end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = 1(xy-0) = xy$$

Question 11:

Using properties of determinants, prove that:

$$\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ \beta & \beta^2 & \gamma+\alpha\\ \gamma & \gamma^2 & \alpha+\beta\end{vmatrix}=(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

Answer

$$\Delta=\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ \beta & \beta^2 & \gamma+\alpha\\ \gamma & \gamma^2 & \alpha+\beta\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta=\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ \beta-\alpha & \beta^2-\alpha^2 & \alpha-\beta\\ \gamma-\alpha & \gamma^2-\alpha^2 & \alpha-\gamma\end{vmatrix}$$

$$=(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ 1 & \beta+\alpha & -1\\ 1 & \gamma+\alpha & -1\end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we have:

$$\Delta=(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ 1 & \beta+\alpha & -1\\ 0 & \gamma-\beta & 0\end{vmatrix}$$

Expanding along R_3, we have:

$$\begin{aligned}\Delta&=(\beta-\alpha)(\gamma-\alpha)\left[-(\gamma-\beta)(-\alpha-\beta-\gamma)\right]\\&=(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)(\alpha+\beta+\gamma)\\&=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)\end{aligned}$$

Hence, the given result is proved.

Question 12:

Using properties of determinants, prove that:

$$\begin{vmatrix}x & x^2 & 1+px^3\\ y & y^2 & 1+py^3\\ z & z^2 & 1+pz^3\end{vmatrix}=(1+pxyz)(x-y)(y-z)(z-x)$$

Answer

$$\Delta=\begin{vmatrix}x & x^2 & 1+px^3\\ y & y^2 & 1+py^3\\ z & z^2 & 1+pz^3\end{vmatrix}$$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$, we have:

$$\Delta=\begin{vmatrix}x & x^2 & 1+px^3\\ y-x & y^2-x^2 & p(y^3-x^3)\\ z-x & z^2-x^2 & p(z^3-x^3)\end{vmatrix}$$

$$=(y-x)(z-x)\begin{vmatrix}x & x^2 & 1+px^3\\ 1 & y+x & p(y^2+x^2+xy)\\ 1 & z+x & p(z^2+x^2+xz)\end{vmatrix}$$

Applying $R_3\to R_3-R_2$, we have:

$$\Delta=(y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p(y^2+x^2+xy) \\ 0 & z-y & p(z-y)(x+y+z) \end{vmatrix}$$

$$=(y-x)(z-x)(z-y)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p(y^2+x^2+xy) \\ 0 & 1 & p(x+y+z) \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta=(x-y)(y-z)(z-x)\left[(-1)(p)(xy^2+x^3+x^2y)+1+px^3+p(x+y+z)(xy)\right]$$

$$=(x-y)(y-z)(z-x)\left[-pxy^2-px^3-px^2y+1+px^3+px^2y+pxy^2+pxyz\right]$$

$$=(x-y)(y-z)(z-x)(1+pxyz)$$

Hence, the given result is proved.

Question 13:

Using properties of determinants, prove that:

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}=3(a+b+c)(ab+bc+ca)$$

Answer

$$\Delta=\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Applying $C_1 \to C_1+C_2+C_3$, we have:

$$\Delta=\begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta=(a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta=(a+b+c)\left[(2b+a)(2c+a)-(a-b)(a-c)\right]$$

$$=(a+b+c)\left[4bc+2ab+2ac+a^2-a^2+ac+ba-bc\right]$$

$$=(a+b+c)(3ab+3bc+3ac)$$

$$=3(a+b+c)(ab+bc+ca)$$

Hence, the given result is proved.

Question 14:

Using properties of determinants, prove that:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}=1$$

Answer

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we have:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

Applying $R_3 \to R_3 - 3R_2$, we have:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = 1\begin{vmatrix} 1 & 2+p \\ 0 & 1 \end{vmatrix} = 1(1-0) = 1$$

Hence, the given result is proved.

Question 15:

Using properties of determinants, prove that:

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$$

Answer

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

$$= \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta-\sin\alpha\sin\delta \\ \sin\beta\sin\delta & \cos\beta\cos\delta & \cos\beta\cos\delta-\sin\beta\sin\delta \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta-\sin\gamma\sin\delta \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$, we have:

$$\Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \cos\alpha\cos\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta-\sin\alpha\sin\delta \\ \cos\beta\cos\delta & \cos\beta\cos\delta & \cos\beta\cos\delta-\sin\beta\sin\delta \\ \cos\gamma\cos\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta-\sin\gamma\sin\delta \end{vmatrix}$$

Here, two columns C_1 and C_2 are identical.

$\therefore \Delta = 0.$

Hence, the given result is proved.

Question 16:

Solve the system of the following equations

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

Answer

Let $\frac{1}{x}=p, \frac{1}{y}=q, \frac{1}{z}=r.$

Then the given system of equations is as follows:

$$2p+3q+10r=4$$

$$4p-6q+5r=1$$

$$6p+9q-20r=2$$

This system can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{bmatrix}, X=\begin{bmatrix}p\\ q\\ r\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}.$$

Now,

$$\begin{aligned}|A| &= 2(120-45)-3(-80-30)+10(36+36)\\ &=150+330+720\\ &=1200\end{aligned}$$

Thus, A is non-singular. Therefore, its inverse exists.

Now,

A_{11} = 75, A_{12} = 110, A_{13} = 72

A_{21} = 150, A_{22} = −100, A_{23} = 0

A_{31} = 75, A_{32} = 30, A_{33} = − 24

$$\therefore A^{-1}=\frac{1}{|A|}adjA$$

$$=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$$

Now,

$$X=A^{-1}B$$

$$\Rightarrow \begin{bmatrix}p\\ q\\ r\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$$

$$=\frac{1}{1200}\begin{bmatrix}300+150+150\\ 440-100+60\\ 288+0-48\end{bmatrix}$$

$$=\frac{1}{1200}\begin{bmatrix}600\\ 400\\ 240\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{3}\\ \frac{1}{5}\end{bmatrix}$$

$$\therefore p=\frac{1}{2}, q=\frac{1}{3}, \text{ and } r=\frac{1}{5}$$

Hence, $x=2, y=3,$ and $z=5$.

Question 17:

Choose the correct answer.

If *a*, *b*, *c*, are in A.P., then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

A. 0 **B.** 1 **C.** *x* **D.** 2*x*

Answer

Answer: A

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$$= \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+(a+c) \\ x+4 & x+5 & x+2c \end{vmatrix} \qquad (2b = a+c \text{ as } a, b, \text{and } c \text{ are in A.P.})$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$, we have:

$$\Delta = \begin{vmatrix} -1 & -1 & a-c \\ x+3 & x+4 & x+(a+c) \\ 1 & 1 & c-a \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$, we have:

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ x+3 & x+4 & x+a+c \\ 1 & 1 & c-a \end{vmatrix}$$

Here, all the elements of the first row (R_1) are zero.

Hence, we have $\Delta = 0$.

The correct answer is A.

Question 18:

Choose the correct answer.

If x, y, z are nonzero real numbers, then the inverse of matrix $A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

A. $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ **B.** $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

C. $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ **D.** $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer

Answer: A

$$A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$\therefore |A| = x(yz-0) = xyz \neq 0$$

$$\begin{aligned} \text{Now, } & A_{11}=yz, A_{12}=0, A_{13}=0 \\ & A_{21}=0, A_{22}=xz, A_{23}=0 \\ & A_{31}=0, A_{32}=0, A_{33}=xy \end{aligned}$$

$$\therefore adjA=\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}adjA$$

$$=\frac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$=\begin{bmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}=\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

The correct answer is A.

Question 19:

Choose the correct answer.

Let $A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$, then

A. Det (A) = 0
B. Det (A) ∈ (2, ∞)

C. Det (A) ∈ (2, 4)

D. Det (A)∈ [2, 4]

Answer

sAnswer: D

$$A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$$

$$\therefore |A| = 1\left(1+\sin^2\theta\right) - \sin\theta\left(-\sin\theta+\sin\theta\right) + 1\left(\sin^2\theta+1\right)$$

$$= 1+\sin^2\theta+\sin^2\theta+1$$

$$= 2+2\sin^2\theta$$

$$= 2\left(1+\sin^2\theta\right)$$

$$\text{Now, } 0 \le \theta \le 2\pi$$

$$\Rightarrow 0 \le \sin\theta \le 1$$

$$\Rightarrow 0 \le \sin^2\theta \le 1$$

$$\Rightarrow 1 \le 1+\sin^2\theta \le 2$$

$$\Rightarrow 2 \le 2\left(1+\sin^2\theta\right) \le 4$$

$$\therefore \text{Det}\left(A\right) \in \left[2,4\right]$$

The correct answer is D.

Exercise 5.1

Question 1:

Prove that the function $f(x)=5x-3$ is continuous at $x=0$, at $x=-3$ and at $x=5$.

Answer

The given function is $f(x)=5x-3$

At $x=0$, $f(0)=5\times0-3=3$

$\lim\limits_{x\to0} f(x)=\lim\limits_{x\to0}(5x-3)=5\times0-3=-3$

$\therefore \lim\limits_{x\to0} f(x)=f(0)$

Therefore, *f* is continuous at *x* = 0

At $x=-3$, $f(-3)=5\times(-3)-3=-18$

$\lim\limits_{x\to-3} f(x)=\lim\limits_{x\to-3}(5x-3)=5\times(-3)-3=-18$

$\therefore \lim\limits_{x\to-3} f(x)=f(-3)$

Therefore, *f* is continuous at *x* = −3

At $x=5$, $f(x)=f(5)=5\times5-3=25-3=22$

$\lim\limits_{x\to5} f(x)=\lim\limits_{x\to5}(5x-3)=5\times5-3=22$

$\therefore \lim\limits_{x\to5} f(x)=f(5)$

Therefore, *f* is continuous at *x* = 5

Question 2:

Examine the continuity of the function $f(x)=2x^2-1$ at $x=3$.

Answer

The given function is $f(x)=2x^2-1$

At $x=3$, $f(x)=f(3)=2\times3^2-1=17$

$\lim\limits_{x\to3} f(x)=\lim\limits_{x\to3}(2x^2-1)=2\times3^2-1=17$

$\therefore \lim\limits_{x\to3} f(x)=f(3)$

Thus, *f* is continuous at *x* = 3

Question 3:

Examine the following functions for continuity.

(a) $f(x)=x-5$ (b) $f(x)=\frac{1}{x-5}, x\neq 5$

(c) $f(x)=\frac{x^2-25}{x+5}, x\neq -5$ (d) $f(x)=|x-5|$

Answer

(a) The given function is $f(x)=x-5$

It is evident that *f* is defined at every real number *k* and its value at *k* is *k* − 5.

It is also observed that, $\lim\limits_{x\to k} f(x)=\lim\limits_{x\to k}(x-5)=k-5=f(k)$

$\therefore \lim\limits_{x\to k} f(x)=f(k)$

Hence, *f* is continuous at every real number and therefore, it is a continuous function.

(b) The given function is $f(x)=\frac{1}{x-5}, x\neq 5$

For any real number *k* ≠ 5, we obtain

$$\lim_{x\to k} f(x)=\lim_{x\to k}\frac{1}{x-5}=\frac{1}{k-5}$$

$$\text{Also, } f(k)=\frac{1}{k-5} \qquad (\text{As } k\neq 5)$$

$$\therefore \lim_{x\to k} f(x)=f(k)$$

Hence, *f* is continuous at every point in the domain of *f* and therefore, it is a continuous function.

(c) The given function is $f(x)=\frac{x^2-25}{x+5}, x\neq -5$

For any real number *c* ≠ −5, we obtain

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{x^2-25}{x+5} = \lim_{x\to c} \frac{(x+5)(x-5)}{x+5} = \lim_{x\to c}(x-5) = (c-5)$$

$$\text{Also, } f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5) \quad (\text{as } c \neq -5)$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Hence, *f* is continuous at every point in the domain of *f* and therefore, it is a continuous function.

(d) The given function is $f(x) = |x-5| = \begin{cases} 5-x, \text{ if } x < 5 \\ x-5, \text{ if } x \geq 5 \end{cases}$

This function *f* is defined at all points of the real line.

Let *c* be a point on a real line. Then, *c* < 5 or *c* = 5 or *c* > 5

Case I: *c* < 5

Then, *f* (*c*) = 5 − *c*

$$\lim_{x\to c} f(x) = \lim_{x\to c}(5-x) = 5-c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all real numbers less than 5.

Case II : *c* = 5

Then, $f(c) = f(5) = (5-5) = 0$

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5}(5-x) = (5-5) = 0$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5}(x-5) = 0$$

$$\therefore \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

Therefore, *f* is continuous at *x* = 5

Case III: *c* > 5

$$\text{Then, } f(c) = f(5) = c-5$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(x-5) = c-5$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all real numbers greater than 5.

Hence, *f* is continuous at every real number and therefore, it is a continuous function.

Question 4:

Prove that the function $f(x)=x^n$ is continuous at $x = n$, where n is a positive integer.

Answer

The given function is $f(x) = x^n$

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

$$\text{Then, } \lim_{x\to n} f(n)=\lim_{x\to n}\left(x^n\right)=n^n$$

$$\therefore \lim_{x\to n} f(x)=f(n)$$

Therefore, f is continuous at n, where n is a positive integer.

Question 5:

Is the function f defined by

$$f(x)=\begin{cases} x, & \text{if } x\le 1 \\ 5, & \text{if } x>1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Answer

The given function f is $f(x)=\begin{cases} x, & \text{if } x\le 1 \\ 5, & \text{if } x>1 \end{cases}$

At $x = 0$,

It is evident that f is defined at 0 and its value at 0 is 0.

$$\text{Then, } \lim_{x\to 0} f(x)= \lim_{x\to 0} x=0$$

$$\therefore \lim_{x\to 0} f(x)=f(0)$$

Therefore, f is continuous at $x = 0$

At $x = 1$,

f is defined at 1 and its value at 1 is 1.

The left hand limit of f at $x = 1$ is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} x=1$$

The right hand limit of f at $x = 1$ is,

$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+}(5) = 5$

$\therefore \lim\limits_{x\to1^-} f(x) \neq \lim\limits_{x\to1^+} f(x)$

Therefore, f is not continuous at $x = 1$

At $x = 2$,

f is defined at 2 and its value at 2 is 5.

Then, $\lim\limits_{x\to2} f(x) = \lim\limits_{x\to2}(5) = 5$

$\therefore \lim\limits_{x\to2} f(x) = f(2)$

Therefore, f is continuous at $x = 2$

Question 6:

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$$

Answer

$$f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$$

It is evident that the given function f is defined at all the points of the real line.

Let c be a point on the real line. Then, three cases arise.

(i) $c < 2$

(ii) $c > 2$

(iii) $c = 2$

Case (i) $c < 2$

Then, $f(c) = 2c + 3$

$\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c}(2x+3) = 2c+3$

$\therefore \lim\limits_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 2$

Case (ii) $c > 2$

Then, $f(c) = 2c - 3$

$\lim_{x \to c} f(x) = \lim_{x \to c}(2x - 3) = 2c - 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 2

Case (iii) *c* = 2

Then, the left hand limit of *f* at *x* = 2 is,

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-}(2x + 3) = 2 \times 2 + 3 = 7$

The right hand limit of *f* at *x* = 2 is,

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+}(2x - 3) = 2 \times 2 - 3 = 1$

It is observed that the left and right hand limit of *f* at *x* = 2 do not coincide.

Therefore, *f* is not continuous at *x* = 2

Hence, *x* = 2 is the only point of discontinuity of *f*.

Question 7:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} |x| + 3, \text{if } x \le -3 \\ -2x, \text{if } -3 < x < 3 \\ 6x + 2, \text{if } x \ge 3 \end{cases}$$

Answer

$$f(x) = \begin{cases} |x| + 3 = -x + 3, \text{ if } x \le -3 \\ -2x, \text{ if } -3 < x < 3 \\ 6x + 2, \text{ if } x \ge 3 \end{cases}$$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

If $c < -3$, then $f(c) = -c + 3$

$\lim_{x \to c} f(x) = \lim_{x \to c}(-x + 3) = -c + 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < -3$

Case II:

$$\text{If } c = -3, \text{ then } f(-3) = -(-3) + 3 = 6$$

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (-x + 3) = -(-3) + 3 = 6$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (-2x) = -2 \times (-3) = 6$$

$$\therefore \lim_{x \to -3} f(x) = f(-3)$$

Therefore, f is continuous at $x = -3$

Case III:

$$\text{If } -3 < c < 3, \text{ then } f(c) = -2c \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (-2x) = -2c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous in $(-3, 3)$.

Case IV:

If $c = 3$, then the left hand limit of f at $x = 3$ is,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x) = -2 \times 3 = -6$$

The right hand limit of f at $x = 3$ is,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2) = 6 \times 3 + 2 = 20$$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V:

$$\text{If } c > 3, \text{ then } f(c) = 6c + 2 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (6x + 2) = 6c + 2$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 3$

Hence, $x = 3$ is the only point of discontinuity of f.

Question 8:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}\frac{|x|}{x} & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

Answer

$$f(x)=\begin{cases}\frac{|x|}{x} & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

It is known that, $x<0\Rightarrow |x|=-x$ and $x>0\Rightarrow |x|=x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\frac{|x|}{x}=\frac{-x}{x}=-1 \text{ if } x<0\\ 0, \text{ if } x=0\\ \frac{|x|}{x}=\frac{x}{x}=1, \text{ if } x>0\end{cases}$$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

$$\text{If } c<0, \text{ then } f(c)=-1$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(-1)=-1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points $x < 0$

Case II:

If $c = 0$, then the left hand limit of *f* at $x = 0$ is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-1)=-1$$

The right hand limit of *f* at $x = 0$ is,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(1)=1$$

It is observed that the left and right hand limit of *f* at $x = 0$ do not coincide.

Therefore, *f* is not continuous at $x = 0$

Case III:

$\text{If } c > 0, \text{ then } f(c) = 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (1) = 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 0

Hence, *x* = 0 is the only point of discontinuity of *f*.

Question 9:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

Answer

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

It is known that, $x < 0 \Rightarrow |x| = -x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \dfrac{x}{|x|} = \dfrac{x}{-x} = -1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

$\Rightarrow f(x) = -1 \text{ for all } x \in \mathbf{R}$

Let *c* be any real number. Then, $\lim_{x \to c} f(x) = \lim_{x \to c} (-1) = -1$

Also, $f(c) = -1 = \lim_{x \to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Question 10:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x)=\begin{cases}x+1, & \text{if } x\ge 1\\ x^2+1, & \text{if } x<1\end{cases}$$

Answer

$$f(x)=\begin{cases}x+1, & \text{if } x\ge 1\\ x^2+1, & \text{if } x<1\end{cases}$$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

If $c<1$, then $f(c)=c^2+1$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^2+1\right)=c^2+1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* < 1

Case II:

If $c=1$, then $f(c)=f(1)=1+1=2$

The left hand limit of *f* at *x* = 1 is,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}\left(x^2+1\right)=1^2+1=2$

The right hand limit of *f* at *x* = 1 is,

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x+1)=1+1=2$

$\therefore \lim\limits_{x\to 1} f(x)=f(1)$

Therefore, *f* is continuous at *x* = 1

Case III:

If $c>1$, then $f(c)=c+1$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x+1)=c+1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Hence, the given function *f* has no point of discontinuity.

Question 11:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x)=\begin{cases}x^3-3, & \text{if } x\le 2\\ x^2+1, & \text{if } x>2\end{cases}$$

Answer

$$f(x)=\begin{cases}x^3-3, & \text{if } x\le 2\\ x^2+1, & \text{if } x>2\end{cases}$$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

$$\text{If } c<2, \text{ then } f(c)=c^3-3 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(x^3-3\right)=c^3-3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* < 2

Case II:

$$\text{If } c=2, \text{ then } f(c)=f(2)=2^3-3=5$$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}\left(x^3-3\right)=2^3-3=5$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\left(x^2+1\right)=2^2+1=5$$

$$\therefore \lim_{x\to 2} f(x)=f(2)$$

Therefore, *f* is continuous at *x* = 2

Case III:

$$\text{If } c>2, \text{ then } f(c)=c^2+1$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2+1\right)=c^2+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 2

Thus, the given function *f* is continuous at every point on the real line.

Hence, *f* has no point of discontinuity.

Question 12:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x)=\begin{cases} x^{10}-1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

Answer

$$f(x)=\begin{cases} x^{10}-1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

$$\text{If } c<1, \text{ then } f(c)=c^{10}-1 \text{ and } \lim_{x \to c} f(x)=\lim_{x \to c}\left(x^{10}-1\right)=c^{10}-1$$

$$\therefore \lim_{x \to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* < 1

Case II:

If *c* = 1, then the left hand limit of *f* at *x* = 1 is,

$$\lim_{x \to 1^-} f(x)=\lim_{x \to 1^-}\left(x^{10}-1\right)=1^{10}-1=1-1=0$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x \to 1^+} f(x)=\lim_{x \to 1^+}\left(x^2\right)=1^2=1$$

It is observed that the left and right hand limit of *f* at *x* = 1 do not coincide.

Therefore, *f* is not continuous at *x* = 1

Case III:

$$\text{If } c>1, \text{ then } f(c)=c^2$$

$$\lim_{x \to c} f(x)=\lim_{x \to c}\left(x^2\right)=c^2$$

$$\therefore \lim_{x \to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Thus, from the above observation, it can be concluded that *x* = 1 is the only point of discontinuity of *f*.

Question 13:

Is the function defined by

$$f(x)=\begin{cases}x+5, & \text{if } x\le 1\\ x-5, & \text{if } x>1\end{cases}$$

a continuous function?

Answer

The given function is $f(x)=\begin{cases}x+5, & \text{if } x\le 1\\ x-5, & \text{if } x>1\end{cases}$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

$$\text{If } c<1, \text{ then } f(c)=c+5 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x+5)=c+5$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* < 1

Case II:

$$\text{If } c=1, \text{ then } f(1)=1+5=6$$

The left hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x+5)=1+5=6$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x-5)=1-5=-4$$

It is observed that the left and right hand limit of *f* at *x* = 1 do not coincide.

Therefore, *f* is not continuous at *x* = 1

Case III:

$$\text{If } c>1, \text{ then } f(c)=c-5 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x-5)=c-5$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Thus, from the above observation, it can be concluded that *x* = 1 is the only point of discontinuity of *f*.

Question 14:

Discuss the continuity of the function *f*, where *f* is defined by

$$f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$$

Answer

The given function is $f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$

The given function is defined at all points of the interval [0, 10].

Let *c* be a point in the interval [0, 10].

Case I:

$$\text{If } 0\le c<1, \text{ then } f(c)=3 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(3)=3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous in the interval [0, 1).

Case II:

$$\text{If } c=1, \text{ then } f(3)=3$$

The left hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(3)=3$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(4)=4$$

It is observed that the left and right hand limits of *f* at *x* = 1 do not coincide.

Therefore, *f* is not continuous at *x* = 1

Case III:

$$\text{If } 1<c<3, \text{ then } f(c)=4 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(4)=4$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points of the interval (1, 3).

Case IV:

$$\text{If } c=3, \text{ then } f(c)=5$$

The left hand limit of *f* at *x* = 3 is,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(4) = 4$$

The right hand limit of *f* at *x* = 3 is,

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(5) = 5$$

It is observed that the left and right hand limits of *f* at *x* = 3 do not coincide.

Therefore, *f* is not continuous at *x* = 3

Case V:

$$\text{If } 3 < c \le 10, \text{ then } f(c) = 5 \text{ and } \lim_{x\to c} f(x) = \lim_{x\to c}(5) = 5$$

$$\lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points of the interval (3, 10].

Hence, *f* is not continuous at *x* = 1 and *x* = 3

Question 15:

Discuss the continuity of the function *f*, where *f* is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

Answer

The given function is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let *c* be a point on the real line.

Case I:

$$\text{If } c < 0, \text{ then } f(c) = 2c$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(2x) = 2c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* < 0

Case II:

$$\text{If } c = 0, \text{ then } f(c) = f(0) = 0$$

The left hand limit of *f* at *x* = 0 is,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x) = 2 \times 0 = 0$$

The right hand limit of *f* at *x* = 0 is,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (0) = 0$$

$$\therefore \lim_{x \to 0} f(x) = f(0)$$

Therefore, *f* is continuous at *x* = 0

Case III:

$$\text{If } 0 < c < 1, \text{ then } f(x) = 0 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (0) = 0$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points of the interval (0, 1).

Case IV:

$$\text{If } c = 1, \text{ then } f(c) = f(1) = 0$$

The left hand limit of *f* at *x* = 1 is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (0) = 0$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x) = 4 \times 1 = 4$$

It is observed that the left and right hand limits of *f* at *x* = 1 do not coincide.

Therefore, *f* is not continuous at *x* = 1

Case V:

$$\text{If } c < 1, \text{ then } f(c) = 4c \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (4x) = 4c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Hence, *f* is not continuous only at *x* = 1

Question 16:

Discuss the continuity of the function *f*, where *f* is defined by

$$f(x)=\begin{cases}-2, \text{ if } x\le -1\\ 2x, \text{ if } -1<x\le 1\\ 2, \text{ if } x>1\end{cases}$$

Answer

$$f(x)=\begin{cases}-2, \text{ if } x\le -1\\ 2x, \text{ if } -1<x\le 1\\ 2, \text{ if } x>1\end{cases}$$

The given function is defined at all points of the real line.

Let *c* be a point on the real line.

Case I:

If $c<-1$, then $f(c)=-2$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-2)=-2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points *x*, such that $x < -1$

Case II:

If $c=-1$, then $f(c)=f(-1)=-2$

The left hand limit of *f* at $x = -1$ is,

$\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^-}(-2)=-2$

The right hand limit of *f* at $x = -1$ is,

$\lim\limits_{x\to -1^+} f(x)=\lim\limits_{x\to -1^+}(2x)=2\times(-1)=-2$

$\therefore \lim\limits_{x\to -1} f(x)=f(-1)$

Therefore, *f* is continuous at $x = -1$

Case III:

If $-1<c<1$, then $f(c)=2c$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(2x)=2c$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points of the interval $(-1, 1)$.

Case IV:

$$\text{If } c = 1, \text{ then } f(c) = f(1) = 2\times1 = 2$$

The left hand limit of *f* at *x* = 1 is,

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(2x) = 2\times1 = 2$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} 2 = 2$$

$$\therefore \lim_{x\to1} f(x) = f(c)$$

Therefore, *f* is continuous at *x* = 2

Case V:

$$\text{If } c > 1, \text{ then } f(c) = 2 \text{ and } \lim_{x\to c} f(x) = \lim_{x\to c}(2) = 2$$

$$\lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Thus, from the above observations, it can be concluded that *f* is continuous at all points of the real line.

Question 17:

Find the relationship between *a* and *b* so that the function *f* defined by

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

is continuous at *x* = 3.

Answer

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

If *f* is continuous at *x* = 3, then

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) \qquad \ldots(1)$$

Also,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax+1) = 3a+1$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (bx+3) = 3b+3$$

$$f(3) = 3a+1$$

Therefore, from (1), we obtain

$$3a+1 = 3b+3 = 3a+1$$

$$\Rightarrow 3a+1 = 3b+3$$

$$\Rightarrow 3a = 3b+2$$

$$\Rightarrow a = b+\frac{2}{3}$$

Therefore, the required relationship is given by, $a = b+\frac{2}{3}$

Question 18:

For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2-2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$$

continuous at x = 0? What about continuity at x = 1?

Answer

The given function is $f(x) = \begin{cases} \lambda(x^2-2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$

If f is continuous at x = 0, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^-} \lambda(x^2-2x) = \lim_{x\to 0^+} (4x+1) = \lambda(0^2-2\times 0)$$

$$\Rightarrow \lambda(0^2-2\times 0) = 4\times 0+1 = 0$$

$$\Rightarrow 0 = 1 = 0, \text{ which is not possible}$$

Therefore, there is no value of λ for which f is continuous at x = 0

At $x = 1$,

$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$

$$\lim_{x\to 1}(4x+1) = 4\times 1+1 = 5$$

$$\therefore \lim_{x\to 1} f(x) = f(1)$$

Therefore, for any values of λ, f is continuous at $x = 1$

Question 19:

Show that the function defined by $g(x) = x-[x]$ is discontinuous at all integral point.

Here $[x]$ denotes the greatest integer less than or equal to x.

Answer

The given function is $g(x) = x-[x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$$g(n) = n-[n] = n-n = 0$$

The left hand limit of f at $x = n$ is,

$$\lim_{x\to n^-} g(x) = \lim_{x\to n^-}(x-[x]) = \lim_{x\to n^-}(x) - \lim_{x\to n^-}[x] = n-(n-1) = 1$$

The right hand limit of f at $x = n$ is,

$$\lim_{x\to n^+} g(x) = \lim_{x\to n^+}(x-[x]) = \lim_{x\to n^+}(x) - \lim_{x\to n^+}[x] = n-n = 0$$

It is observed that the left and right hand limits of f at $x = n$ do not coincide.

Therefore, f is not continuous at $x = n$

Hence, g is discontinuous at all integral points.

Question 20:

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at x = p?

Answer

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that *f* is defined at *x* = p

At $x=\pi,\ f(x)=f(\pi)=\pi^2-\sin\pi+5=\pi^2-0+5=\pi^2+5$

Consider $\lim\limits_{x\to\pi} f(x)=\lim\limits_{x\to\pi}\left(x^2-\sin x+5\right)$

Put $x=\pi+h$

If $x\to\pi$, *then it is evident that* $h\to 0$

$$\begin{aligned}
\therefore \lim_{x\to\pi} f(x) &= \lim_{x\to\pi}\left(x^2-\sin x+5\right)\\
&= \lim_{h\to 0}\left[(\pi+h)^2-\sin(\pi+h)+5\right]\\
&= \lim_{h\to 0}(\pi+h)^2-\lim_{h\to 0}\sin(\pi+h)+\lim_{h\to 0}5\\
&= (\pi+0)^2-\lim_{h\to 0}\left[\sin\pi\cosh+\cos\pi\sinh\right]+5\\
&= \pi^2-\lim_{h\to 0}\sin\pi\cosh-\lim_{h\to 0}\cos\pi\sinh+5\\
&= \pi^2-\sin\pi\cos 0-\cos\pi\sin 0+5\\
&= \pi^2-0\times 1-(-1)\times 0+5\\
&= \pi^2+5
\end{aligned}$$

$\therefore \lim\limits_{x\to\pi} f(x)=f(\pi)$

Therefore, the given function *f* is continuous at *x* = π

Question 21:

Discuss the continuity of the following functions.

(a) *f* (*x*) = sin *x* + cos *x*

(b) *f* (*x*) = sin *x* − cos *x*

(c) *f* (*x*) = sin *x* × cos x

Answer

It is known that if *g* and *h* are two continuous functions, then

$g+h,\ g-h,$ and $g.h$ are also continuous.

It has to proved first that *g* (*x*) = sin *x* and *h* (*x*) = cos *x* are continuous functions.

Let *g* (*x*) = sin *x*

It is evident that *g* (*x*) = sin *x* is defined for every real number.

Let *c* be a real number. Put *x* = *c* + *h*

If *x* → *c*, then *h* → 0

$$g(c) = \sin c$$

$$\begin{aligned}\lim_{x \to c} g(x) &= \lim_{x \to c} \sin x \\ &= \lim_{h \to 0} \sin(c+h) \\ &= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h] \\ &= \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h) \\ &= \sin c \cos 0 + \cos c \sin 0 \\ &= \sin c + 0 \\ &= \sin c\end{aligned}$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, *g* is a continuous function.

Let *h* (*x*) = cos *x*

It is evident that *h* (*x*) = cos *x* is defined for every real number.

Let *c* be a real number. Put *x* = *c* + *h*

If *x* → *c*, then *h* → 0

h (*c*) = cos *c*

$$\begin{aligned}\lim_{x \to c} h(x) &= \lim_{x \to c} \cos x \\ &= \lim_{h \to 0} \cos(c+h) \\ &= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h] \\ &= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c \times 1 - \sin c \times 0 \\ &= \cos c\end{aligned}$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, *h* is a continuous function.

Therefore, it can be concluded that

(a) *f* (*x*) = *g* (*x*) + *h* (*x*) = sin *x* + cos *x* is a continuous function

(b) *f* (*x*) = *g* (*x*) − *h* (*x*) = sin *x* − cos *x* is a continuous function

(c) *f* (*x*) = *g* (*x*) × *h* (*x*) = sin *x* × cos *x* is a continuous function

Question 22:

Discuss the continuity of the cosine, cosecant, secant and cotangent functions,

Answer

It is known that if g and h are two continuous functions, then

$$(i)\ \frac{h(x)}{g(x)},\ g(x)\neq 0 \text{ is continuous}$$

$$(ii)\ \frac{1}{g(x)},\ g(x)\neq 0 \text{ is continuous}$$

$$(iii)\ \frac{1}{h(x)},\ h(x)\neq 0 \text{ is continuous}$$

It has to be proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$$g(c) = \sin c$$

$$\begin{aligned}\lim_{x\to c} g(x) &= \lim_{x\to c} \sin x\\ &= \lim_{h\to 0} \sin(c+h)\\ &= \lim_{h\to 0}\left[\sin c\cos h + \cos c\sin h\right]\\ &= \lim_{h\to 0}(\sin c\cos h) + \lim_{h\to 0}(\cos c\sin h)\\ &= \sin c\cos 0 + \cos c\sin 0\\ &= \sin c + 0\\ &= \sin c\end{aligned}$$

$$\therefore\ \lim_{x\to c} g(x) = g(c)$$

Therefore, g is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c}\cos x\\ &= \lim_{h\to 0}\cos(c+h)\\ &= \lim_{h\to 0}\left[\cos c\cos h-\sin c\sin h\right]\\ &= \lim_{h\to 0}\cos c\cos h-\lim_{h\to 0}\sin c\sin h\\ &= \cos c\cos 0-\sin c\sin 0\\ &= \cos c\times 1-\sin c\times 0\\ &= \cos c\end{aligned}$$

$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is continuous function.

It can be concluded that,

$$\operatorname{cosec} x = \frac{1}{\sin x},\ \ \sin x \neq 0 \text{ is continuous}$$

$$\Rightarrow \operatorname{cosec} x,\ x \neq n\pi\ \ (n \in Z)\ \text{ is continuous}$$

Therefore, cosecant is continuous except at x = np, n Î **Z**

$$\sec x = \frac{1}{\cos x},\ \ \cos x \neq 0 \text{ is continuous}$$

$$\Rightarrow \sec x,\ x \neq (2n+1)\frac{\pi}{2}\ \ (n \in \mathbf{Z})\ \text{ is continuous}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2}\ \ (n \in \mathbf{Z})$

$$\cot x = \frac{\cos x}{\sin x},\ \ \sin x \neq 0 \text{ is continuous}$$

$$\Rightarrow \cot x,\ x \neq n\pi\ \ (n \in Z)\ \text{ is continuous}$$

Therefore, cotangent is continuous except at x = np, n Î **Z**

Question 23:

Find the points of discontinuity of f, where

$$f(x) = \begin{cases}\dfrac{\sin x}{x}, & \text{if } x<0\\ x+1, & \text{if } x\geq 0\end{cases}$$

Answer

$$f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{if } x<0\\ x+1, & \text{if } x\geq 0\end{cases}$$

It is evident that *f* is defined at all points of the real line.

Let *c* be a real number.

Case I:

$$\text{If } c<0, \text{ then } f(c)=\frac{\sin c}{c} \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(\frac{\sin x}{x}\right)=\frac{\sin c}{c}$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* < 0

Case II:

$$\text{If } c>0, \text{ then } f(c)=c+1 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 0

Case III:

$$\text{If } c=0, \text{ then } f(c)=f(0)=0+1=1$$

The left hand limit of *f* at *x* = 0 is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\frac{\sin x}{x}=1$$

The right hand limit of *f* at *x* = 0 is,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x+1)=1$$

$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

Therefore, *f* is continuous at *x* = 0

From the above observations, it can be concluded that *f* is continuous at all points of the real line.

Thus, *f* has no point of discontinuity.

Question 24:

Determine if *f* defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

is a continuous function?

Answer

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

It is evident that *f* is defined at all points of the real line.

Let *c* be a real number.

Case I:

$$\text{If } c\neq 0, \text{ then } f(c)=c^2\sin\frac{1}{c}$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2\sin\frac{1}{x}\right)=\left(\lim_{x\to c}x^2\right)\left(\lim_{x\to c}\sin\frac{1}{x}\right)=c^2\sin\frac{1}{c}$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points $x \neq 0$

Case II:

$$\text{If } c=0, \text{ then } f(0)=0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$$

It is known that, $-1 \le \sin\frac{1}{x} \le 1,\ x \ne 0$

$$\Rightarrow -x^2 \le \sin\frac{1}{x} \le x^2$$

$$\Rightarrow \lim_{x\to 0}\left(-x^2\right) \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le 0$$

$$\Rightarrow \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = 0$$

Similarly, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

Therefore, *f* is continuous at $x = 0$

From the above observations, it can be concluded that *f* is continuous at every point of the real line.

Thus, *f* is a continuous function.

Question 25:

Examine the continuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Answer

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1 & \text{if } x = 0 \end{cases}$$

It is evident that *f* is defined at all points of the real line.

Let *c* be a real number.

Case I:

$$\text{If } c \neq 0, \text{ then } f(c) = \sin c - \cos c$$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(\sin x - \cos x) = \sin c - \cos c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* ≠ 0

Case II:

$$\text{If } c = 0, \text{ then } f(0) = -1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Therefore, *f* is continuous at *x* = 0

From the above observations, it can be concluded that *f* is continuous at every point of the real line.

Thus, *f* is a continuous function.

Question 26:

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases} \qquad \text{at } x = \frac{\pi}{2}$$

Answer

$$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$$

The given function *f* is continuous at $x = \frac{\pi}{2}$, if *f* is defined at $x = \frac{\pi}{2}$ and if the value of the *f* at $x = \frac{\pi}{2}$ equals the limit of *f* at $x = \frac{\pi}{2}$.

It is evident that *f* is defined at $x=\frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right)=3$

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}$$

Put $x=\frac{\pi}{2}+h$

Then, $x\to\frac{\pi}{2}\Rightarrow h\to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=k\lim_{h\to 0}\frac{-\sin h}{-2h}=\frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h}=\frac{k}{2}.1=\frac{k}{2}$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2}=3$$

$$\Rightarrow k=6$$

Therefore, the required value of *k* is 6.

Question 27:

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x)=\begin{cases}kx^2, & \text{if } x\le 2\\ 3, & \text{if } x>2\end{cases} \qquad \text{at } x=2$$

Answer

The given function is $f(x)=\begin{cases}kx^2, & \text{if } x\le 2\\ 3, & \text{if } x>2\end{cases}$

The given function *f* is continuous at *x* = 2, if *f* is defined at *x* = 2 and if the value of *f* at *x* = 2 equals the limit of *f* at *x* = 2

It is evident that *f* is defined at *x* = 2 and $f(2)=k(2)^2=4k$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x\to 2^-} \left(kx^2\right) = \lim_{x\to 2^+} (3) = 4k$$

$$\Rightarrow k \times 2^2 = 3 = 4k$$

$$\Rightarrow 4k = 3 = 4k$$

$$\Rightarrow 4k = 3$$

$$\Rightarrow k = \frac{3}{4}$$

Therefore, the required value of k is $\frac{3}{4}$.

Question 28:

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases} \qquad \text{at } x = \pi$$

Answer

The given function is $f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

The given function *f* is continuous at *x* = p, if *f* is defined at *x* = p and if the value of *f* at *x* = p equals the limit of *f* at *x* = p

It is evident that *f* is defined at *x* = p and $f(\pi) = k\pi + 1$

$$\lim_{x\to \pi^-} f(x) = \lim_{x\to \pi^+} f(x) = f(\pi)$$

$$\Rightarrow \lim_{x\to \pi^-} (kx+1) = \lim_{x\to \pi^+} \cos x = k\pi + 1$$

$$\Rightarrow k\pi + 1 = \cos\pi = k\pi + 1$$

$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow k = -\frac{2}{\pi}$$

Therefore, the required value of k is $-\frac{2}{\pi}$.

Question 29:

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases} \qquad \text{at } x=5$$

Answer

$$f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases}$$

The given function *f* is continuous at *x* = 5, if *f* is defined at *x* = 5 and if the value of *f* at *x* = 5 equals the limit of *f* at *x* = 5

It is evident that *f* is defined at *x* = 5 and $f(5)=kx+1=5k+1$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$$
$$\Rightarrow \lim_{x\to 5^-}(kx+1)=\lim_{x\to 5^+}(3x-5)=5k+1$$
$$\Rightarrow 5k+1=15-5=5k+1$$
$$\Rightarrow 5k+1=10$$
$$\Rightarrow 5k=9$$
$$\Rightarrow k=\frac{9}{5}$$

Therefore, the required value of k is $\frac{9}{5}$.

Question 30:

Find the values of *a* and *b* such that the function defined by

$$f(x)=\begin{cases} 5, & \text{if } x\le 2 \\ ax+b, & \text{if } 2<x<10 \\ 21, & \text{if } x\ge 10 \end{cases}$$

is a continuous function.

Answer

$$f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$$

It is evident that the given function *f* is defined at all points of the real line.

If *f* is a continuous function, then *f* is continuous at all real numbers.

In particular, *f* is continuous at x = 2 and x = 10

Since *f* is continuous at x = 2, we obtain

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$
$$\Rightarrow \lim_{x\to 2^-}(5)=\lim_{x\to 2^+}(ax+b)=5$$
$$\Rightarrow 5=2a+b=5$$
$$\Rightarrow 2a+b=5 \qquad \ldots(1)$$

Since *f* is continuous at x = 10, we obtain

$$\lim_{x\to 10^-} f(x)=\lim_{x\to 10^+} f(x)=f(10)$$
$$\Rightarrow \lim_{x\to 10^-}(ax+b)=\lim_{x\to 10^+}(21)=21$$
$$\Rightarrow 10a+b=21=21$$
$$\Rightarrow 10a+b=21 \qquad \ldots(2)$$

On subtracting equation (1) from equation (2), we obtain

$8a = 16$

$\Rightarrow a = 2$

By putting a = 2 in equation (1), we obtain

$2 \times 2 + b = 5$

$\Rightarrow 4 + b = 5$

$\Rightarrow b = 1$

Therefore, the values of *a* and *b* for which *f* is a continuous function are 2 and 1 respectively.

Question 31:

Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Answer

The given function is $f(x) = \cos(x^2)$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = \cos x$ and $h(x) = x^2$

$$\left[\because (goh)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x)\right]$$

It has to be first proved that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that *g* is defined for every real number.

Let *c* be a real number.

Then, $g(c) = \cos c$

Put $x = c + h$

If $x \to c$, then $h \to 0$

$$\begin{aligned}
\lim_{x\to c} g(x) &= \lim_{x\to c} \cos x \\
&= \lim_{h\to 0} \cos(c+h) \\
&= \lim_{h\to 0}\left[\cos c \cos h - \sin c \sin h\right] \\
&= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c \times 1 - \sin c \times 0 \\
&= \cos c
\end{aligned}$$

$$\therefore \lim_{x\to c} g(x) = g(c)$$

Therefore, $g(x) = \cos x$ is continuous function.

$h(x) = x^2$

Clearly, *h* is defined for every real number.

Let *k* be a real number, then $h(k) = k^2$

$$\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$$

$$\therefore \lim_{x \to k} h(x) = h(k)$$

Therefore, *h* is a continuous function.

It is known that for real valued functions *g* and *h*,such that (*g* o *h*) is defined at *c*, if *g* is continuous at *c* and if *f* is continuous at *g* (*c*), then (*f* o *g*) is continuous at *c*.

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

Question 32:

Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Answer

The given function is $f(x) = |\cos x|$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

f = *g* o *h*, where $g(x) = |x|$ and $h(x) = \cos x$

$$\left[\because (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)\right]$$

It has to be first proved that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, g is continuous at all points x, such that $x < 0$

Case II:

$$\text{If } c > 0, \text{ then } g(c) = c \text{ and } \lim_{x \to c} g(x) = \lim_{x \to c} x = c$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, g is continuous at all points x, such that $x > 0$

Case III:

$$\text{If } c = 0, \text{ then } g(c) = g(0) = 0$$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$$

$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = g(0)$$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\begin{aligned}
\lim_{x \to c} h(x) &= \lim_{x \to c} \cos x \\
&= \lim_{h \to 0} \cos(c + h) \\
&= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c \times 1 - \sin c \times 0 \\
&= \cos c
\end{aligned}$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that for real valued functions g and h,such that (g o h) is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then (f o g) is continuous at c.

Therefore, $f(x)=(goh)(x)=g(h(x))=g(\cos x)=|\cos x|$ is a continuous function.

Question 33:

Examine that $\sin|x|$ is a continuous function.

Answer

Let $f(x)=\sin|x|$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

f = *g o h*, where $g(x)=|x|$ and $h(x)=\sin x$

$\left[\because (goh)(x)=g(h(x))=g(\sin x)=|\sin x|=f(x)\right]$

It has to be proved first that $g(x)=|x|$ and $h(x)=\sin x$ are continuous functions.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that *x* < 0

Case II:

If $c>0$, then $g(c)=c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that *x* > 0

Case III:

If $c=0$, then $g(c)=g(0)=0$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-}(-x) = 0$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}(x) = 0$$
$$\therefore \lim_{x\to 0^-} g(x) = \lim_{x\to 0^+}(x) = g(0)$$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = \sin x$

It is evident that $h(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + k$

If $x \to c$, then $k \to 0$

$h(c) = \sin c$

$$h(c) = \sin c$$
$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c}\sin x\\ &= \lim_{k\to 0}\sin(c+k)\\ &= \lim_{k\to 0}[\sin c\cos k + \cos c\sin k]\\ &= \lim_{k\to 0}(\sin c\cos k) + \lim_{h\to 0}(\cos c\sin k)\\ &= \sin c\cos 0 + \cos c\sin 0\\ &= \sin c + 0\\ &= \sin c\end{aligned}$$
$$\therefore \lim_{x\to c} h(x) = g(c)$$

Therefore, h is a continuous function.

It is known that for real valued functions g and h,such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.

Answer

The given function is $f(x) = |x| - |x+1|$

The two functions, *g* and *h*, are defined as

$$g(x)=|x| \text{ and } h(x)=|x+1|$$

Then, *f* = *g* − *h*

The continuity of *g* and *h* is examined first.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, *g* is continuous at all points *x*, such that *x* < 0

Case II:

If $c>0$, then $g(c)=c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, *g* is continuous at all points *x*, such that *x* > 0

Case III:

If $c=0$, then $g(c)=g(0)=0$

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$$

$$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x)=0$$

$$\therefore \lim_{x\to 0^-} g(x)=\lim_{x\to 0^+}(x)=g(0)$$

Therefore, *g* is continuous at *x* = 0

From the above three observations, it can be concluded that *g* is continuous at all points.

$h(x)=|x+1|$ can be written as

$$h(x)=\begin{cases}-(x+1), & \text{if, } x<-1\\ x+1, & \text{if } x\geq -1\end{cases}$$

Clearly, *h* is defined for every real number.

Let *c* be a real number.

Case I:

$$\text{If } c<-1, \text{ then } h(c)=-(c+1) \text{ and } \lim_{x\to c} h(x)=\lim_{x\to c}\left[-(x+1)\right]=-(c+1)$$

$$\therefore \lim_{x\to c} h(x)=h(c)$$

Therefore, *h* is continuous at all points *x*, such that *x* < −1

Case II:

$$\text{If } c>-1, \text{ then } h(c)=c+1 \text{ and } \lim_{x\to c} h(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} h(x)=h(c)$$

Therefore, *h* is continuous at all points *x*, such that *x* > −1

Case III:

$$\text{If } c=-1, \text{ then } h(c)=h(-1)=-1+1=0$$

$$\lim_{x\to -1^-} h(x)=\lim_{x\to -1^-}\left[-(x+1)\right]=-(-1+1)=0$$

$$\lim_{x\to -1^+} h(x)=\lim_{x\to -1^+}(x+1)=(-1+1)=0$$

$$\therefore \lim_{x\to -1^-} h(x)=\lim_{h\to -1^+} h(x)=h(-1)$$

Therefore, *h* is continuous at *x* = −1

From the above three observations, it can be concluded that *h* is continuous at all points of the real line.

g and *h* are continuous functions. Therefore, *f* = *g* − *h* is also a continuous function.

Therefore, *f* has no point of discontinuity.

Exercise 5.2

Question 1:

Differentiate the functions with respect to *x*.

$\sin\left(x^2+5\right)$

Answer

Let $f(x)=\sin\left(x^2+5\right)$, $u(x)=x^2+5$, and $v(t)=\sin t$

Then, $(vou)(x)=v\left(u(x)\right)=v\left(x^2+5\right)=\tan\left(x^2+5\right)=f(x)$

Thus, *f* is a composite of two functions.

Put $t=u(x)=x^2+5$

Then, we obtain

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos\left(x^2+5\right)$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(x^2+5\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)=2x+0=2x$$

Therefore, by chain rule, $\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos\left(x^2+5\right)\times 2x=2x\cos\left(x^2+5\right)$

Alternate method

$$\begin{aligned}\frac{d}{dx}\left[\sin\left(x^2+5\right)\right]&=\cos\left(x^2+5\right)\cdot\frac{d}{dx}\left(x^2+5\right)\\&=\cos\left(x^2+5\right)\cdot\left[\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)\right]\\&=\cos\left(x^2+5\right)\cdot[2x+0]\\&=2x\cos\left(x^2+5\right)\end{aligned}$$

Question 2:

Differentiate the functions with respect to *x*.

$\cos(\sin x)$

Answer

$\text{Let } f(x)=\cos(\sin x), u(x)=\sin x, \text{ and } v(t)=\cos t$

$\text{Then, } (vou)(x)=v(u(x))=v(\sin x)=\cos(\sin x)=f(x)$

Thus, *f* is a composite function of two functions.

Put $t = u(x) = \sin x$

$$\therefore \frac{dv}{dt}=\frac{d}{dt}[\cos t]=-\sin t=-\sin(\sin x)$$

$$\frac{dt}{dx}=\frac{d}{dx}(\sin x)=\cos x$$

By chain rule, $\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$

Alternate method

$$\frac{d}{dx}[\cos(\sin x)]=-\sin(\sin x)\cdot\frac{d}{dx}(\sin x)=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$$

Question 3:

Differentiate the functions with respect to *x*.

$\sin(ax+b)$

Answer

$\text{Let } f(x)=\sin(ax+b),\ u(x)=ax+b, \text{ and } v(t)=\sin t$

$\text{Then, } (vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=f(x)$

Thus, *f* is a composite function of two functions, *u* and *v*.

Put $t = u(x) = ax + b$

Therefore,

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$$

$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Hence, by chain rule, we obtain

$$\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos(ax+b)\cdot a=a\cos(ax+b)$$

Alternate method

$$\frac{d}{dx}\left[\sin(ax+b)\right]=\cos(ax+b).\frac{d}{dx}(ax+b)$$

$$=\cos(ax+b)\cdot\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$

$$=\cos(ax+b).(a+0)$$

$$=a\cos(ax+b)$$

Question 4:

Differentiate the functions with respect to *x*.

$$\sec\left(\tan\left(\sqrt{x}\right)\right)$$

Answer

Let $f(x)=\sec\left(\tan\sqrt{x}\right), u(x)=\sqrt{x}, v(t)=\tan t,$ and $w(s)=\sec s$

Then, $(wovou)(x)=w\left[v\left(u(x)\right)\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f(x)$

Thus, *f* is a composite function of three functions, *u, v*, and *w*.

Put $s=v(t)=\tan t$ and $t=u(x)=\sqrt{x}$

$$\text{Then, }\frac{dw}{ds}=\frac{d}{ds}(\sec s)=\sec s\tan s=\sec(\tan t).\tan(\tan t) \qquad [s=\tan t]$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right) \qquad \left[t=\sqrt{x}\right]$$

$$\frac{ds}{dt}=\frac{d}{dt}(\tan t)=\sec^2 t=\sec^2\sqrt{x}$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}\cdot x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we obtain

$$\frac{dt}{dx}=\frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$

$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right)\cdot\frac{d}{dx}\left(\tan\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Question 5:

Differentiate the functions with respect to *x*.

$$\frac{\sin\left(ax+b\right)}{\cos\left(cx+d\right)}$$

Answer

The given function is $f\left(x\right)=\frac{\sin\left(ax+b\right)}{\cos\left(cx+d\right)}=\frac{g\left(x\right)}{h\left(x\right)}$, where *g* (*x*) = sin (*ax* + *b*) and *h* (*x*) = cos (*cx* + *d*)

$$\therefore f'=\frac{g'h-gh'}{h^2}$$

Consider $g\left(x\right)=\sin\left(ax+b\right)$

Let $u\left(x\right)=ax+b, v\left(t\right)=\sin t$

Then, $\left(vou\right)\left(x\right)=v\left(u\left(x\right)\right)=v\left(ax+b\right)=\sin\left(ax+b\right)=g\left(x\right)$

$\therefore$ g is a composite function of two functions, u and v.

$$\text{Put } t = u(x) = ax + b$$

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax+b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Therefore, by chain rule, we obtain

$$g' = \frac{dg}{dx} = \frac{dv}{dt}\cdot\frac{dt}{dx} = \cos(ax+b)\cdot a = a\cos(ax+b)$$

$$\text{Consider } h(x) = \cos(cx+d)$$

$$\text{Let } p(x) = cx + d,\ q(y) = \cos y$$

$$\text{Then, } (qop)(x) = q(p(x)) = q(cx+d) = \cos(cx+d) = h(x)$$

$\therefore h$ is a composite function of two functions, p and q.

Put $y = p(x) = cx + d$

$$\frac{dq}{dy} = \frac{d}{dy}(\cos y) = -\sin y = -\sin(cx+d)$$

$$\frac{dy}{dx} = \frac{d}{dx}(cx+d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c$$

Therefore, by chain rule, we obtain

$$h' = \frac{dh}{dx} = \frac{dq}{dy}\cdot\frac{dy}{dx} = -\sin(cx+d)\times c = -c\sin(cx+d)$$

$$\therefore f' = \frac{a\cos(ax+b)\cdot\cos(cx+d)-\sin(ax+b)\{-c\sin(cx+d)\}}{[\cos(cx+d)]^2}$$

$$= \frac{a\cos(ax+b)}{\cos(cx+d)}+c\sin(ax+b)\cdot\frac{\sin(cx+d)}{\cos(cx+d)}\times\frac{1}{\cos(cx+d)}$$

$$= a\cos(ax+b)\sec(cx+d)+c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Question 6:

Differentiate the functions with respect to *x*.

$\cos x^3.\sin^2(x^5)$

Answer

The given function is $\cos x^3.\sin^2(x^5)$

$$\frac{d}{dx}\left[\cos x^3\cdot\sin^2(x^5)\right] = \sin^2(x^5)\times\frac{d}{dx}(\cos x^3)+\cos x^3\times\frac{d}{dx}\left[\sin^2(x^5)\right]$$

$$= \sin^2(x^5)\times(-\sin x^3)\times\frac{d}{dx}(x^3)+\cos x^3\times 2\sin(x^5)\cdot\frac{d}{dx}\left[\sin x^5\right]$$

$$= -\sin x^3\sin^2(x^5)\times 3x^2+2\sin x^5\cos x^3\cdot\cos x^5\times\frac{d}{dx}(x^5)$$

$$= -3x^2\sin x^3\cdot\sin^2(x^5)+2\sin x^5\cos x^5\cos x^3\cdot\times 5x^4$$

$$= 10x^4\sin x^5\cos x^5\cos x^3-3x^2\sin x^3\sin^2(x^5)$$

Question 7:

Differentiate the functions with respect to *x*.

$2\sqrt{\cot(x^2)}$

Answer

$$\frac{d}{dx}\left[2\sqrt{\cot\left(x^2\right)}\right]$$

$$= 2\cdot\frac{1}{2\sqrt{\cot\left(x^2\right)}}\times\frac{d}{dx}\left[\cot\left(x^2\right)\right]$$

$$= \sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times -\operatorname{cosec}^2\left(x^2\right)\times\frac{d}{dx}\left(x^2\right)$$

$$= -\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times\frac{1}{\sin^2\left(x^2\right)}\times(2x)$$

$$= \frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2}\,\sin x^2}$$

$$= \frac{-2\sqrt{2}\,x}{\sqrt{2\sin x^2\cos x^2}\,\sin x^2}$$

$$= \frac{-2\sqrt{2}\,x}{\sin x^2\sqrt{\sin 2x^2}}$$

Question 8:

Differentiate the functions with respect to *x*.

$\cos\left(\sqrt{x}\right)$

Answer

Let $f(x) = \cos\left(\sqrt{x}\right)$

Also, let $u(x) = \sqrt{x}$

And, $v(t) = \cos t$

$$\begin{aligned}\text{Then, } (vou)(x) &= v\left(u(x)\right)\\ &= v\left(\sqrt{x}\right)\\ &= \cos\sqrt{x}\\ &= f(x)\end{aligned}$$

Clearly, *f* is a composite function of two functions, *u* and *v*, such that

$t = u(x) = \sqrt{x}$

Then, $\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}x^{-\frac{1}{2}}$

$$=\frac{1}{2\sqrt{x}}$$

And, $\frac{dv}{dt}=\frac{d}{dt}(\cos t)=-\sin t$

$$=-\sin\left(\sqrt{x}\right)$$

By using chain rule, we obtain

$$\frac{dt}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}$$

$$=-\sin\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$

$$=-\frac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)$$

$$=-\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right]=-\sin\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=-\sin\left(\sqrt{x}\right)\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$=-\sin\sqrt{x}\times\frac{1}{2}x^{-\frac{1}{2}}$$

$$=\frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question 9:

Prove that the function *f* given by

$f(x)=|x-1|,\ x\in\mathbf{R}$ is notdifferentiable at x = 1.

Answer

The given function is $f(x)=|x-1|,\ x\in\mathbf{R}$

It is known that a function *f* is differentiable at a point *x* = *c* in its domain if both

$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at *x* = 1,

consider the left hand limit of *f* at *x* = 1

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{|1+h-1|-|1-1|}{h}$$

$$=\lim_{h\to 0^-}\frac{|h|-0}{h}=\lim_{h\to 0^-}\frac{-h}{h} \qquad (h<0\Rightarrow |h|=-h)$$

$$=-1$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{|1+h-1|-|1-1|}{h}$$

$$=\lim_{h\to 0^+}\frac{|h|-0}{h}=\lim_{h\to 0^+}\frac{h}{h} \qquad (h>0\Rightarrow |h|=h)$$

$$=1$$

Since the left and right hand limits of *f* at *x* = 1 are not equal, *f* is not differentiable at *x* = 1

Question 10:

Prove that the greatest integer function defined by $f(x)=[x], 0<x<3$ is not differentiable at *x* = 1 and *x* = 2.

Answer

The given function *f* is $f(x)=[x], 0<x<3$

It is known that a function *f* is differentiable at a point *x* = *c* in its domain if both

$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at *x* = 1, consider the left hand limit of *f* at *x* = 1

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^-}\frac{0-1}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^+}\frac{1-1}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of *f* at $x = 1$ are not equal, *f* is not differentiable at $x = 1$

To check the differentiability of the given function at $x = 2$, consider the left hand limit of *f* at $x = 2$

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^-}\frac{1-2}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^+}\frac{2-2}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of *f* at $x = 2$ are not equal, *f* is not differentiable at $x = 2$

Exercise 5.3

Question 1:

Find $\frac{dy}{dx}$:

$2x+3y=\sin x$

Answer

The given relationship is $2x+3y=\sin x$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(2x+3y)=\frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\cos x$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos x$$

$$\Rightarrow 3\frac{dy}{dx}=\cos x-2$$

$$\therefore \frac{dy}{dx}=\frac{\cos x-2}{3}$$

Question 2:

Find $\frac{dy}{dx}$

$2x+3y=\sin y$

Answer

The given relationship is $2x+3y=\sin y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(\sin y)$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \qquad \text{[By using chain rule]}$$

$$\Rightarrow 2=(\cos y-3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{2}{\cos y-3}$$

Question 3:

Find $\frac{dy}{dx}$

$ax+by^2=\cos y$

Answer

The given relationship is $ax+by^2=\cos y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(ax)+\frac{d}{dx}(by^2)=\frac{d}{dx}(\cos y)$$

$$\Rightarrow a+b\frac{d}{dx}(y^2)=\frac{d}{dx}(\cos y) \qquad ...(1)$$

Using chain rule, we obtain $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y)=-\sin y\frac{dy}{dx}$ $\qquad ...(2)$

From (1) and (2), we obtain

$$a+b\times 2y\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by+\sin y)\frac{dy}{dx}=-a$$

$$\therefore \frac{dy}{dx}=\frac{-a}{2by+\sin y}$$

Question 4:

Find $\frac{dy}{dx}$

$xy+y^2=\tan x+y$

Answer

The given relationship is $xy + y^2 = \tan x + y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(xy + y^2\right) = \frac{d}{dx}\left(\tan x + y\right)$$

$$\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}$$

$$\Rightarrow \left[y \cdot \frac{d}{dx}(x) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow y \cdot 1 + x.\frac{dy}{dx} + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{(x + 2y - 1)}$$

Question 5:

Find $\frac{dy}{dx}$

$x^2 + xy + y^2 = 100$

Answer

The given relationship is $x^2 + xy + y^2 = 100$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(x^2 + xy + y^2\right) = \frac{d}{dx}(100)$$

$$\Rightarrow \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(xy) + \frac{d}{dx}\left(y^2\right) = 0$$

$$\Rightarrow 2x+\left[y\cdot\frac{d}{dx}(x)+x\cdot\frac{dy}{dx}\right]+2y\frac{dy}{dx}=0 \qquad \left[\text{Using product rule and chain rule}\right]$$

$$\Rightarrow 2x+y\cdot 1+x\cdot\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\Rightarrow 2x+y+(x+2y)\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{2x+y}{x+2y}$$

Question 6:

Find $\frac{dy}{dx}$

$x^3+x^2y+xy^2+y^3=81$

Answer

The given relationship is $x^3+x^2y+xy^2+y^3=81$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(x^3+x^2y+xy^2+y^3\right)=\frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(x^2y\right)+\frac{d}{dx}\left(xy^2\right)+\frac{d}{dx}\left(y^3\right)=0$$

$$\Rightarrow 3x^2+\left[y\frac{d}{dx}\left(x^2\right)+x^2\frac{dy}{dx}\right]+\left[y^2\frac{d}{dx}(x)+x\frac{d}{dx}\left(y^2\right)\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow 3x^2+\left[y\cdot 2x+x^2\frac{dy}{dx}\right]+\left[y^2\cdot 1+x\cdot 2y\cdot\frac{dy}{dx}\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow \left(x^2+2xy+3y^2\right)\frac{dy}{dx}+\left(3x^2+2xy+y^2\right)=0$$

$$\therefore \frac{dy}{dx}=\frac{-\left(3x^2+2xy+y^2\right)}{\left(x^2+2xy+3y^2\right)}$$

Question 7:

Find $\frac{dy}{dx}$

$\sin^2 y+\cos xy=\pi$

Answer

The given relationship is $\sin^2 y + \cos xy = \pi$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin^2 y + \cos xy\right) = \frac{d}{dx}(\pi)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right) + \frac{d}{dx}(\cos xy) = 0 \qquad \text{...(1)}$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right) = 2\sin y \frac{d}{dx}(\sin y) = 2\sin y \cos y \frac{dy}{dx} \qquad \text{...(2)}$$

$$\frac{d}{dx}(\cos xy) = -\sin xy \frac{d}{dx}(xy) = -\sin xy\left[y\frac{d}{dx}(x) + x\frac{dy}{dx}\right]$$

$$= -\sin xy\left[y.1 + x\frac{dy}{dx}\right] = -y\sin xy - x\sin xy\frac{dy}{dx} \qquad \text{...(3)}$$

From (1), (2), and (3), we obtain

$$2\sin y\cos y\frac{dy}{dx} - y\sin xy - x\sin xy\frac{dy}{dx} = 0$$

$$\Rightarrow (2\sin y\cos y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow (\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\therefore \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Question 8:

Find $\frac{dy}{dx}$

$\sin^2 x + \cos^2 y = 1$

Answer

The given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin^2 x+\cos^2 y\right)=\frac{d}{dx}(1)$$
$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right)+\frac{d}{dx}\left(\cos^2 y\right)=0$$
$$\Rightarrow 2\sin x\cdot\frac{d}{dx}(\sin x)+2\cos y\cdot\frac{d}{dx}(\cos y)=0$$
$$\Rightarrow 2\sin x\cos x+2\cos y(-\sin y)\cdot\frac{dy}{dx}=0$$
$$\Rightarrow \sin 2x-\sin 2y\frac{dy}{dx}=0$$
$$\therefore \frac{dy}{dx}=\frac{\sin 2x}{\sin 2y}$$

Question 9:

Find $\frac{dy}{dx}$

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Answer

The given relationship is $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \sin y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \cos y\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad ...(1)$$

The function, $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$.

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{\left(1+x^2\right).\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$=\frac{\left(1+x^2\right).2-2x\cdot[0+2x]}{\left(1+x^2\right)^2}=\frac{2+2x^2-4x^2}{\left(1+x^2\right)^2}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2} \qquad ...(2)$$

Also, $\sin y=\frac{2x}{1+x^2}$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}=\sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}$$

$$=\sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}=\frac{1-x^2}{1+x^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{1-x^2}{1+x^2}\times\frac{dy}{dx}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+x^2}$$

Question 10:

Find $\frac{dy}{dx}$

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$$

Answer

The given relationship is $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \tan y=\frac{3x-x^3}{1-3x^2} \qquad ...(1)$$

It is known that, $\tan y=\dfrac{3\tan\frac{y}{3}-\tan^3\frac{y}{3}}{1-3\tan^2\frac{y}{3}}$ $\qquad ...(2)$

Comparing equations (1) and (2), we obtain

$$x=\tan\frac{y}{3}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}\cdot\frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}\cdot\frac{1}{3}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3}{\sec^2\frac{y}{3}}=\frac{3}{1+\tan^2\frac{y}{3}}$$

$$\therefore \frac{dy}{dx}=\frac{3}{1+x^2}$$

Question 11:

Find $\dfrac{dy}{dx}$

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0<x<1$$

Answer

The given relationship is,

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}}=\frac{1-x^2}{1+x^2}$$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$$\tan\frac{y}{2}=x$$

Differentiating this relationship with respect to *x*, we obtain

$$\sec^2\frac{y}{2}\cdot\frac{d}{dx}\left(\frac{y}{2}\right)=\frac{d}{dx}(x)$$

$$\Rightarrow \sec^2\frac{y}{2}\times\frac{1}{2}\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sec^2\frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+\tan^2\frac{y}{2}}$$

$$\therefore \frac{dy}{dx}=\frac{1}{1+x^2}$$

Question 12:

Find $\frac{dy}{dx}$

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ 0<x<1$$

Answer

The given relationship is $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y)=\cos y\cdot\frac{dy}{dx}$$

$$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$=\sqrt{\frac{\left(1+x^2\right)^2-\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}=\sqrt{\frac{4x^2}{\left(1+x^2\right)^2}}=\frac{2x}{1+x^2}$$

$$\therefore \frac{d}{dx}(\sin y)=\frac{2x}{1+x^2}\frac{dy}{dx} \qquad ...(2)$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)=\frac{\left(1+x^2\right)\cdot\left(1-x^2\right)'-\left(1-x^2\right)\cdot\left(1+x^2\right)'}{\left(1+x^2\right)^2} \qquad \text{[Using quotient rule]}$$

$$=\frac{\left(1+x^2\right)(-2x)-\left(1-x^2\right)\cdot(2x)}{\left(1+x^2\right)^2}$$

$$=\frac{-2x-2x^3-2x+2x^3}{\left(1+x^2\right)^2}$$

$$=\frac{-4x}{\left(1+x^2\right)^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{2x}{1+x^2}\frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Alternate method

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)\sin y=1-x^2$$

$$\Rightarrow \left(1+\sin y\right)x^2=1-\sin y$$

$$\Rightarrow x^2=\frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2=\frac{\left(\cos\frac{y}{2}-\sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2}+\sin\frac{y}{2}\right)^2}$$

$$\Rightarrow x=\frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$$

$$\Rightarrow x=\frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}$$

$$\Rightarrow x=\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\cdot\left[\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]$$

$$\Rightarrow 1=\sec^2\left(\frac{\pi}{4}-\frac{y}{2}\right)\cdot\frac{d}{dx}\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

$$\Rightarrow 1=\left[1+\tan^2\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]\cdot\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow 1=\left(1+x^2\right)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Question 13:

Find $\frac{dy}{dx}$

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right),-1<x<1$$

Answer

The given relationship is $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\cos y)=\frac{d}{dx}\cdot\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\sin y\cdot\frac{dy}{dx}=\frac{\left(1+x^2\right)\cdot\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y}\,\frac{dy}{dx} = \frac{\left(1+x^2\right)\times 2 - 2x\cdot 2x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx} = -\left[\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}\right]$$

$$\Rightarrow \sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}\cdot\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Question 14:

Find $\frac{dy}{dx}$

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right),\ -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

Answer

The given relationship is $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin y = 2x\sqrt{1-x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\cos y\frac{dy}{dx}=2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right)+\sqrt{1-x^2}\frac{dx}{dx}\right]$$

$$\Rightarrow\sqrt{1-\sin^2 y}\frac{dy}{dx}=2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right]$$

$$\Rightarrow\sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}\frac{dy}{dx}=2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\sqrt{1-4x^2\left(1-x^2\right)}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\sqrt{\left(1-2x^2\right)^2}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\left(1-2x^2\right)\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$$

Question 15:

Find $\frac{dy}{dx}$

$$y=\sec^{-1}\left(\frac{1}{2x^2-1}\right),0<x<\frac{1}{\sqrt{2}}$$

Answer

The given relationship is $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \sec y = \frac{1}{2x^2 - 1}$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2\cos^2 \frac{y}{2}$$

$$\Rightarrow x = \cos \frac{y}{2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos \frac{y}{2}\right)$$

$$\Rightarrow 1 = -\sin \frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{2} \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}} = \frac{-2}{\sqrt{1 - \cos^2 \frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$

Exercise 5.4

Question 1:

Differentiate the following w.r.t. *x*:

$$\frac{e^x}{\sin x}$$

Answer

Let $y = \dfrac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin x \frac{d}{dx}\left(e^x\right) - e^x \frac{d}{dx}(\sin x)}{\sin^2 x}\\ &= \frac{\sin x.\left(e^x\right) - e^x \cdot (\cos x)}{\sin^2 x}\\ &= \frac{e^x(\sin x - \cos x)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}\end{aligned}$$

Question 2:

Differentiate the following w.r.t. *x*:

$$e^{\sin^{-1} x}$$

Answer

Let $y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1}x}\right)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1}x} \cdot \frac{d}{dx}\left(\sin^{-1}x\right)$$

$$= e^{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}, x \in (-1,1)$$

Question 2:

Show that the function given by $f(x) = e^{2x}$ is strictly increasing on **R**.

Answer

Let x_1 and x_2 be any two numbers in **R**.

Then, we have:

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2 \Rightarrow e^{2x_1} < e^{2x_2} \Rightarrow f(x_1) < f(x_2)$$

Hence, f is strictly increasing on **R**.

Question 3:

Differentiate the following w.r.t. x:

e^{x^3}

Answer

Let $y = e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^3}\right) = e^{x^3} \cdot \frac{d}{dx}\left(x^3\right) = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$$

Question 4:

Differentiate the following w.r.t. x:

$\sin\left(\tan^{-1} e^{-x}\right)$

Answer

Let $y=\sin\left(\tan^{-1}e^{-x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin\left(\tan^{-1}e^{-x}\right)\right]$$

$$=\cos\left(\tan^{-1}e^{-x}\right)\cdot\frac{d}{dx}\left(\tan^{-1}e^{-x}\right)$$

$$=\cos\left(\tan^{-1}e^{-x}\right)\cdot\frac{1}{1+\left(e^{-x}\right)^2}\cdot\frac{d}{dx}\left(e^{-x}\right)$$

$$=\frac{\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}\cdot e^{-x}\cdot\frac{d}{dx}(-x)$$

$$=\frac{e^{-x}\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}\times(-1)$$

$$=\frac{-e^{-x}\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}$$

Question 5:

Differentiate the following w.r.t. *x*:

$\log\left(\cos e^{x}\right)$

Answer

Let $y=\log\left(\cos e^{x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(\cos e^{x}\right)\right]$$

$$=\frac{1}{\cos e^{x}}\cdot\frac{d}{dx}\left(\cos e^{x}\right)$$

$$=\frac{1}{\cos e^{x}}\cdot\left(-\sin e^{x}\right)\cdot\frac{d}{dx}\left(e^{x}\right)$$

$$=\frac{-\sin e^{x}}{\cos e^{x}}\cdot e^{x}$$

$$=-e^{x}\tan e^{x}, e^{x}\neq(2n+1)\frac{\pi}{2}, n\in\mathbf{N}$$

Question 6:

Differentiate the following w.r.t. *x*:

$e^x + e^{x^2} + \ldots + e^{x^5}$

Answer

$$\frac{d}{dx}\left(e^x + e^{x^2} + \ldots + e^{x^5}\right)$$

$$= \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}\left(e^{x^3}\right) + \frac{d}{dx}\left(e^{x^4}\right) + \frac{d}{dx}\left(e^{x^5}\right)$$

$$= e^x + \left[e^{x^2} \times \frac{d}{dx}\left(x^2\right)\right] + \left[e^{x^3} \cdot \frac{d}{dx}\left(x^3\right)\right] + \left[e^{x^4} \cdot \frac{d}{dx}\left(x^4\right)\right] + \left[e^{x^5} \cdot \frac{d}{dx}\left(x^5\right)\right]$$

$$= e^x + \left(e^{x^2} \times 2x\right) + \left(e^{x^3} \times 3x^2\right) + \left(e^{x^4} \times 4x^3\right) + \left(e^{x^5} \times 5x^4\right)$$

$$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$$

Question 7:

Differentiate the following w.r.t. *x*:

$\sqrt{e^{\sqrt{x}}}, x > 0$

Answer

Let $y = \sqrt{e^{\sqrt{x}}}$

Then, $y^2 = e^{\sqrt{x}}$

By differentiating this relationship with respect to *x*, we obtain

$$y^2 = e^{\sqrt{x}}$$

$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right) \qquad \text{[By applying the chain rule]}$$

$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{1}{2}.\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}},\ x > 0$$

Question 8:

Differentiate the following w.r.t. *x*:

$\log\left(\log x\right), x > 1$

Answer

Let $y = \log\left(\log x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\log x\right)\right]$$

$$= \frac{1}{\log x}\cdot\frac{d}{dx}\left(\log x\right)$$

$$= \frac{1}{\log x}\cdot\frac{1}{x}$$

$= \frac{1}{x\log x}$, $x > 1$

Question 9:

Differentiate the following w.r.t. *x*:

$\frac{\cos x}{\log x}, x > 0$

Answer

Let $y = \frac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x) \times \log x - \cos x \times \frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \frac{-\sin x \log x - \cos x \times \frac{1}{x}}{(\log x)^2}$$

$$= \frac{-[x \log x . \sin x + \cos x]}{x(\log x)^2}, x > 0$$

Question 10:

Differentiate the following w.r.t. *x*:

$\cos(\log x + e^x), x > 0$

Answer

Let $y = \cos(\log x + e^x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = -\sin(\log x + e^x) \cdot \frac{d}{dx}(\log x + e^x)$$

$$= -\sin(\log x + e^x) \cdot \left[\frac{d}{dx}(\log x) + \frac{d}{dx}(e^x)\right]$$

$$= -\sin(\log x + e^x) \cdot \left(\frac{1}{x} + e^x\right)$$

$$= -\left(\frac{1}{x} + e^x\right) \sin(\log x + e^x), x > 0$$

Exercise 5.5

Question 1:

Differentiate the function with respect to *x*.

$\cos x.\cos 2x.\cos 3x$

Answer

Let $y = \cos x.\cos 2x.\cos 3x$

Taking logarithm on both the sides, we obtain

$$\log y = \log\left(\cos x.\cos 2x.\cos 3x\right)$$
$$\Rightarrow \log y = \log\left(\cos x\right) + \log\left(\cos 2x\right) + \log\left(\cos 3x\right)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}\cdot\frac{d}{dx}\left(\cos x\right) + \frac{1}{\cos 2x}\cdot\frac{d}{dx}\left(\cos 2x\right) + \frac{1}{\cos 3x}\cdot\frac{d}{dx}\left(\cos 3x\right)$$
$$\Rightarrow \frac{dy}{dx} = y\left[-\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x}\cdot\frac{d}{dx}(2x) - \frac{\sin 3x}{\cos 3x}\cdot\frac{d}{dx}(3x)\right]$$
$$\therefore \frac{dy}{dx} = -\cos x.\cos 2x.\cos 3x\left[\tan x + 2\tan 2x + 3\tan 3x\right]$$

Question 2:

Differentiate the function with respect to *x*.

$$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Answer

$$\text{Let } y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Taking logarithm on both the sides, we obtain

$$\log y = \log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y = \frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\Rightarrow \log y = \frac{1}{2}\left[\log\{(x-1)(x-2)\} - \log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y = \frac{1}{2}\left[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\right]$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\begin{array}{l}\frac{1}{x-1}.\frac{d}{dx}(x-1) + \frac{1}{x-2}.\frac{d}{dx}(x-2) - \frac{1}{x-3}.\frac{d}{dx}(x-3) \\ -\frac{1}{x-4}.\frac{d}{dx}(x-4) - \frac{1}{x-5}.\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2}\left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

Question 3:

Differentiate the function with respect to *x*.

$$(\log x)^{\cos x}$$

Answer

$$\text{Let } y = (\log x)^{\cos x}$$

Taking logarithm on both the sides, we obtain

$$\log y = \cos x \cdot \log(\log x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\cos x)\times\log(\log x)+\cos x\times\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{1}{y}.\frac{dy}{dx}=-\sin x\log(\log x)+\cos x\times\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dy}{dx}=y\left[-\sin x\log(\log x)+\frac{\cos x}{\log x}\times\frac{1}{x}\right]$$

$$\therefore\frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x}{x\log x}-\sin x\log(\log x)\right]$$

Question 4:

Differentiate the function with respect to *x*.

$x^x-2^{\sin x}$

Answer

Let $y=x^x-2^{\sin x}$

Also, let $x^x=u$ and $2^{\sin x}=v$

$\therefore y=u-v$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}$$

$u = x^x$

Taking logarithm on both the sides, we obtain

$\log u=x\log x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\left[\frac{d}{dx}(x)\times\log x+x\times\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=u\left[1\times\log x+x\times\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^x(\log x+1)$$

$$\Rightarrow\frac{du}{dx}=x^x(1+\log x)$$

$v = 2^{\sin x}$

Taking logarithm on both the sides with respect to *x*, we obtain

$\log v = \sin x \cdot \log 2$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$

$$\therefore \frac{dy}{dx} = x^x(1+\log x) - 2^{\sin x} \cos x \log 2$$

Question 5:

Differentiate the function with respect to *x*.

$$(x+3)^2 . (x+4)^3 . (x+5)^4$$

Answer

$$\text{Let } y = (x+3)^2 . (x+4)^3 . (x+5)^4$$

Taking logarithm on both the sides, we obtain

$$\log y = \log(x+3)^2 + \log(x+4)^3 + \log(x+5)^4$$

$$\Rightarrow \log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y} . \frac{dy}{dx} = 2 \cdot \frac{1}{x+3} \cdot \frac{d}{dx}(x+3) + 3 \cdot \frac{1}{x+4} \cdot \frac{d}{dx}(x+4) + 4 \cdot \frac{1}{x+5} \cdot \frac{d}{dx}(x+5)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4 \cdot \left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4 \cdot \left[\frac{2(x+4)(x+5) + 3(x+3)(x+5) + 4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3 \cdot \left[2(x^2+9x+20) + 3(x^2+8x+15) + 4(x^2+7x+12)\right]$$

$$\therefore \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$$

Question 6:

Differentiate the function with respect to *x*.

$$\left(x+\frac{1}{x}\right)^{x}+x^{\left(1+\frac{1}{x}\right)}$$

Answer

$$\text{Let } y=\left(x+\frac{1}{x}\right)^{x}+x^{\left(1+\frac{1}{x}\right)}$$

$$\text{Also, let } u=\left(x+\frac{1}{x}\right)^{x} \text{ and } v=x^{\left(1+\frac{1}{x}\right)}$$

$$\therefore y=u+v$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$$\text{Then, } u=\left(x+\frac{1}{x}\right)^{x}$$

$$\Rightarrow \log u=\log\left(x+\frac{1}{x}\right)^{x}$$

$$\Rightarrow \log u=x\log\left(x+\frac{1}{x}\right)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow\frac{1}{u}\frac{du}{dx}=1\times\log\left(x+\frac{1}{x}\right)+x\times\frac{1}{\left(x+\frac{1}{x}\right)}\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow\frac{du}{dx}=u\left[\log\left(x+\frac{1}{x}\right)+\frac{x}{\left(x+\frac{1}{x}\right)}\times\left(1-\frac{1}{x^2}\right)\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{\left(x-\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]\qquad ...(2)$$

$$v=x^{\left(1+\frac{1}{x}\right)}$$

$$\Rightarrow\log v=\log\left[x^{\left(1+\frac{1}{x}\right)}\right]$$

$$\Rightarrow\log v=\left(1+\frac{1}{x}\right)\log x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x+\left(1+\frac{1}{x}\right)\cdot\frac{d}{dx}\log x$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\left(-\frac{1}{x^2}\right)\log x+\left(1+\frac{1}{x}\right)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{-\log x+x+1}{x^2}\right]$$

$$\Rightarrow\frac{dv}{dx}=x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)\qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)$$

Question 7:

Differentiate the function with respect to *x*.

$$(\log x)^x+x^{\log x}$$

Answer

Let $y=(\log x)^x+x^{\log x}$

Also, let $u=(\log x)^x$ and $v=x^{\log x}$

$\therefore y=u+v$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\qquad ...(1)$$

$u=(\log x)^x$

$$\Rightarrow\log u=\log\left[(\log x)^x\right]$$

$$\Rightarrow\log u=x\log(\log x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\log x)+x\cdot\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=u\left[1\times\log(\log x)+x\cdot\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{x}{\log x}\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\frac{\log(\log x)\cdot\log x+1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]\qquad ...(2)$$

$$v=x^{\log x}$$

$$\Rightarrow\log v=\log\left(x^{\log x}\right)$$

$$\Rightarrow\log v=\log x\log x=(\log x)^{2}$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\frac{d}{dx}\left[(\log x)^{2}\right]$$

$$\Rightarrow\frac{1}{v}.\frac{dv}{dx}=2(\log x)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dv}{dx}=2v(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x-1}\cdot\log x\qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]+2x^{\log x-1}\cdot\log x$$

Question 8:

Differentiate the function with respect to *x*.

$(\sin x)^x + \sin^{-1}\sqrt{x}$

Answer

Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Also, let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$$

$u = (\sin x)^x$

$\Rightarrow \log u = \log(\sin x)^x$

$\Rightarrow \log u = x\log(\sin x)$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)\times\log(\sin x) + x\times\frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1\cdot\log(\sin x) + x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\left[\log(\sin x) + \frac{x}{\sin x}\cdot\cos x\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x(x\cot x + \log\sin x) \qquad ...(2)$$

$v = \sin^{-1}\sqrt{x}$

Differentiating both sides with respect to x, we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (\sin x)^x(x\cot x + \log\sin x) + \frac{1}{2\sqrt{x-x^2}}$$

Question 9:

Differentiate the function with respect to x.

$x^{\sin x}+(\sin x)^{\cos x}$

Answer

Let $y=x^{\sin x}+(\sin x)^{\cos x}$

Also, let $u=x^{\sin x}$ and $v=(\sin x)^{\cos x}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad ...(1)$$

$u=x^{\sin x}$

$\Rightarrow \log u=\log\left(x^{\sin x}\right)$

$\Rightarrow \log u=\sin x\log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(\sin x)\cdot\log x+\sin x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\cos x\log x+\sin x\cdot\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{\sin x}\left[\cos x\log x+\frac{\sin x}{x}\right] \quad ...(2)$$

$v=(\sin x)^{\cos x}$

$\Rightarrow \log v=\log(\sin x)^{\cos x}$

$\Rightarrow \log v=\cos x\log(\sin x)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}(\cos x)\times\log(\sin x)+\cos x\times\frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow\frac{dv}{dx}=v\left[-\sin x.\log(\sin x)+\cos x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow\frac{dv}{dx}=(\sin x)^{\cos x}\left[-\sin x\log\sin x+\frac{\cos x}{\sin x}\cos x\right]$$

$$\Rightarrow\frac{dv}{dx}=(\sin x)^{\cos x}\left[-\sin x\log\sin x+\cot x\cos x\right]$$

$$\Rightarrow\frac{dv}{dx}=(\sin x)^{\cos x}\left[\cot x\cos x-\sin x\log\sin x\right] \quad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{\sin x}\left(\cos x\log x+\frac{\sin x}{x}\right)+(\sin x)^{\cos x}\left[\cos x\cot x-\sin x\log\sin x\right]$$

Question 10:

Differentiate the function with respect to *x*.

$$x^{x\cos x}+\frac{x^2+1}{x^2-1}$$

Answer

$$\text{Let } y=x^{x\cos x}+\frac{x^2+1}{x^2-1}$$

$$\text{Also, let } u=x^{x\cos x} \text{ and } v=\frac{x^2+1}{x^2-1}$$

$$\therefore y=u+v$$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad ...(1)$$

$$u=x^{x\cos x}$$

$$\Rightarrow\log u=\log\left(x^{x\cos x}\right)$$

$$\Rightarrow\log u=x\cos x\log x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)\cdot\cos x\cdot\log x + x\cdot\frac{d}{dx}(\cos x)\cdot\log x + x\cos x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[1\cdot\cos x\cdot\log x + x\cdot(-\sin x)\log x + x\cos x\cdot\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x}\left(\cos x\log x - x\sin x\log x + \cos x\right)$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x}\left[\cos x(1+\log x) - x\sin x\log x\right] \qquad ...(2)$$

$$v = \frac{x^2+1}{x^2-1}$$

$$\Rightarrow \log v = \log\left(x^2+1\right) - \log\left(x^2-1\right)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{2x}{x^2+1} - \frac{2x}{x^2-1}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{2x\left(x^2-1\right) - 2x\left(x^2+1\right)}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{x^2+1}{x^2-1}\times\left[\frac{-4x}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{-4x}{\left(x^2-1\right)^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{x\cos x}\left[\cos x(1+\log x) - x\sin x\log x\right] - \frac{4x}{\left(x^2-1\right)^2}$$

Question 11:

Differentiate the function with respect to *x*.

$$(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$$

Answer

Let $y=(x\cos x)^{x}+(x\sin x)^{\frac{1}{x}}$

Also, let $u=(x\cos x)^{x}$ and $v=(x\sin x)^{\frac{1}{x}}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=(x\cos x)^{x}$

$\Rightarrow \log u=\log(x\cos x)^{x}$

$\Rightarrow \log u=x\log(x\cos x)$

$\Rightarrow \log u=x[\log x+\log\cos x]$

$\Rightarrow \log u=x\log x+x\log\cos x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x\log x)+\frac{d}{dx}(x\log\cos x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\left\{\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)\right\}+\left\{\log\cos x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^{x}\left[\left(\log x\cdot 1+x\cdot\frac{1}{x}\right)+\left\{\log\cos x\cdot 1+x\cdot\frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^{x}\left[(\log x+1)+\left\{\log\cos x+\frac{x}{\cos x}\cdot(-\sin x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^{x}\left[(1+\log x)+(\log\cos x-x\tan x)\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^{x}\left[1-x\tan x+(\log x+\log\cos x)\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^{x}\left[1-x\tan x+\log(x\cos x)\right] \qquad ...(2)$$

$$v=(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v=\log(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v=\frac{1}{x}\log(x\sin x)$$

$$\Rightarrow \log v=\frac{1}{x}(\log x+\log\sin x)$$

$$\Rightarrow \log v=\frac{1}{x}\log x+\frac{1}{x}\log\sin x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{x}\log x\right)+\frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\left[\log x\cdot\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}\cdot\frac{d}{dx}(\log x)\right]+\left[\log(\sin x)\cdot\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}\cdot\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\left[\log x\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x}\cdot\frac{1}{x}\right]+\left[\log(\sin x)\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x}\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\frac{1}{x^2}(1-\log x)+\left[-\frac{\log(\sin x)}{x^2}+\frac{1}{x\sin x}\cdot\cos x\right]$$

$$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x}{x^2}+\frac{-\log(\sin x)+x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x-\log(\sin x)+x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x)+x\cot x}{x^2}\right] \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=(x\cos x)^x\left[1-x\tan x+\log(x\cos x)\right]+(x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x+1-\log(x\sin x)}{x^2}\right]$$

Question 12:

Find $\frac{dy}{dx}$ of function.

$x^y + y^x = 1$

Answer

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u + v = 1$

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad ...(1)$$

$$u = x^y$$

$$\Rightarrow \log u = \log\left(x^y\right)$$

$$\Rightarrow \log u = y \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \frac{dy}{dx} + y \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) \qquad ...(2)$$

$$v = y^x$$

$$\Rightarrow \log v = \log\left(y^x\right)$$

$$\Rightarrow \log v = x \log y$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx} = v\left(\log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$x^y\left(\log x\frac{dy}{dx}+\frac{y}{x}\right)+y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right)=0$$

$$\Rightarrow\left(x^y\log x+xy^{x-1}\right)\frac{dy}{dx}=-\left(yx^{y-1}+y^x\log y\right)$$

$$\therefore\frac{dy}{dx}=-\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$$

Question 13:

Find $\frac{dy}{dx}$ of function.

$y^x=x^y$

Answer

The given function is $y^x=x^y$

Taking logarithm on both the sides, we obtain

$x\log y=y\log x$

Differentiating both sides with respect to *x*, we obtain

$$\log y\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log y)=\log x\cdot\frac{d}{dx}(y)+y\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\log y\cdot1+x\cdot\frac{1}{y}\cdot\frac{dy}{dx}=\log x\cdot\frac{dy}{dx}+y\cdot\frac{1}{x}$$

$$\Rightarrow\log y+\frac{x}{y}\frac{dy}{dx}=\log x\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow\left(\frac{x}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}-\log y$$

$$\Rightarrow\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\therefore\frac{dy}{dx}=\frac{y}{x}\left(\frac{y-x\log y}{x-y\log x}\right)$$

Question 14:

Find $\frac{dy}{dx}$ of function.

$(\cos x)^y = (\cos y)^x$

Answer

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$y \log \cos x = x \log \cos y$

Differentiating both sides, we obtain

$$\log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log \cos x) = \log \cos y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) = \log \cos y \cdot 1 + x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx}(\cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + \frac{y}{\cos x} \cdot (-\sin x) = \log \cos y + \frac{x}{\cos y}(-\sin y) \cdot \frac{dy}{dx}$$

$$\Rightarrow \log \cos x \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \frac{dy}{dx}$$

$$\Rightarrow (\log \cos x + x \tan y)\frac{dy}{dx} = y \tan x + \log \cos y$$

$$\therefore \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$$

Question 15:

Find $\frac{dy}{dx}$ of function.

$xy = e^{(x-y)}$

Answer

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$$\log(xy) = \log\left(e^{x-y}\right)$$

$$\Rightarrow \log x + \log y = (x-y)\log e$$

$$\Rightarrow \log x + \log y = (x-y) \times 1$$

$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{d}{dx}(\log x)+\frac{d}{dx}(\log y)=\frac{d}{dx}(x)-\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}$$

$$\Rightarrow \left(1+\frac{1}{y}\right)\frac{dy}{dx}=1-\frac{1}{x}$$

$$\Rightarrow \left(\frac{y+1}{y}\right)\frac{dy}{dx}=\frac{x-1}{x}$$

$$\therefore \frac{dy}{dx}=\frac{y(x-1)}{x(y+1)}$$

Question 16:

Find the derivative of the function given by $f(x)=(1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)$ and hence find $f'(1)$.

Answer

The given relationship is $f(x)=(1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)$

Taking logarithm on both the sides, we obtain

$$\log f(x)=\log(1+x)+\log\left(1+x^2\right)+\log\left(1+x^4\right)+\log\left(1+x^8\right)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{f(x)}\cdot\frac{d}{dx}\left[f(x)\right]=\frac{d}{dx}\log(1+x)+\frac{d}{dx}\log\left(1+x^2\right)+\frac{d}{dx}\log\left(1+x^4\right)+\frac{d}{dx}\log\left(1+x^8\right)$$

$$\Rightarrow\frac{1}{f(x)}\cdot f'(x)=\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)+\frac{1}{1+x^2}\cdot\frac{d}{dx}\left(1+x^2\right)+\frac{1}{1+x^4}\cdot\frac{d}{dx}\left(1+x^4\right)+\frac{1}{1+x^8}\cdot\frac{d}{dx}\left(1+x^8\right)$$

$$\Rightarrow f'(x)=f(x)\left[\frac{1}{1+x}+\frac{1}{1+x^2}.2x+\frac{1}{1+x^4}\cdot4x^3+\frac{1}{1+x^8}\cdot8x^7\right]$$

$$\therefore f'(x)=(1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

$$\text{Hence, } f'(1)=(1+1)\left(1+1^2\right)\left(1+1^4\right)\left(1+1^8\right)\left[\frac{1}{1+1}+\frac{2\times1}{1+1^2}+\frac{4\times1^3}{1+1^4}+\frac{8\times1^7}{1+1^8}\right]$$

$$=2\times2\times2\times2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$=16\times\left(\frac{1+2+4+8}{2}\right)$$

$$=16\times\frac{15}{2}=120$$

Question 17:

Differentiate $\left(x^5-5x+8\right)\left(x^3+7x+9\right)$ in three ways mentioned below

(i) By using product rule.

(ii) By expanding the product to obtain a single polynomial.

(iii By logarithmic differentiation.

Do they all give the same answer?

Answer

(i) Let $y=\left(x^5-5x+8\right)\left(x^3+7x+9\right)$

Let $x^2-5x+8=u$ and $x^3+7x+9=v$

$\therefore y=uv$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}\cdot v+u\cdot\frac{dv}{dx} \qquad \text{(By using product rule)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(x^2-5x+8\right)\cdot\left(x^3+7x+9\right)+\left(x^2-5x+8\right)\cdot\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\Rightarrow \frac{dy}{dx}=\left(2x-5\right)\left(x^3+7x+9\right)+\left(x^2-5x+8\right)\left(3x^2+7\right)$$

$$\Rightarrow \frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+x^2\left(3x^2+7\right)-5x\left(3x^2+7\right)+8\left(3x^2+7\right)$$

$$\Rightarrow \frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4+7x^2\right)-15x^3-35x+24x^2+56$$

$$\therefore \frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

(ii) $y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$

$$=x^2\left(x^3+7x+9\right)-5x\left(x^3+7x+9\right)+8\left(x^3+7x+9\right)$$

$$=x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72$$

$$=x^5-5x^4+15x^3-26x^2+11x+72$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(x^5-5x^4+15x^3-26x^2+11x+72\right)$$

$$=\frac{d}{dx}\left(x^5\right)-5\frac{d}{dx}\left(x^4\right)+15\frac{d}{dx}\left(x^3\right)-26\frac{d}{dx}\left(x^2\right)+11\frac{d}{dx}\left(x\right)+\frac{d}{dx}(72)$$

$$=5x^4-5\times4x^3+15\times3x^2-26\times2x+11\times1+0$$

$$=5x^4-20x^3+45x^2-52x+11$$

(iii) $y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$

Taking logarithm on both the sides, we obtain

$$\log y=\log\left(x^2-5x+8\right)+\log\left(x^3+7x+9\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\log\left(x^2-5x+8\right)+\frac{d}{dx}\log\left(x^3+7x+9\right)$$

$$\Rightarrow\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}\cdot\frac{d}{dx}\left(x^2-5x+8\right)+\frac{1}{x^3+7x+9}\cdot\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\Rightarrow\frac{dy}{dx}=y\left[\frac{1}{x^2-5x+8}\times\left(2x-5\right)+\frac{1}{x^3+7x+9}\times\left(3x^2+7\right)\right]$$

$$\Rightarrow\frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\Rightarrow\frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{\left(2x-5\right)\left(x^3+7x+9\right)+\left(3x^2+7\right)\left(x^2-5x+8\right)}{\left(x^2-5x+8\right)\left(x^3+7x+9\right)}\right]$$

$$\Rightarrow\frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+3x^2\left(x^2-5x+8\right)+7\left(x^2-5x+8\right)$$

$$\Rightarrow\frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4-15x^3+24x^2\right)+\left(7x^2-35x+56\right)$$

$$\Rightarrow\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

From the above three observations, it can be concluded that all the results of $\frac{dy}{dx}$ are same.

Question 18:

If *u*, *v* and *w* are functions of *x*, then show that

$$\frac{d}{dx}\left(u.v.w\right)=\frac{du}{dx}v.w+u.\frac{dv}{dx}.w+u.v.\frac{dw}{dx}$$

in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Answer

Let $y=u.v.w=u.\left(v.w\right)$

By applying product rule, we obtain

$$\frac{dy}{dx}=\frac{du}{dx}.(v.w)+u.\frac{d}{dx}(v\cdot w)$$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}v\cdot w+u\left[\frac{dv}{dx}\cdot w+v\cdot\frac{dw}{dx}\right]\qquad\text{(Again applying product rule)}$$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}\cdot v\cdot w+u\cdot\frac{dv}{dx}\cdot w+u\cdot v\cdot\frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y=u.v.w$, we obtain

$$\log y=\log u+\log v+\log w$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\log u)+\frac{d}{dx}(\log v)+\frac{d}{dx}(\log w)$$

$$\Rightarrow\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow\frac{dy}{dx}=y\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow\frac{dy}{dx}=u.v.w.\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore\frac{dy}{dx}=\frac{du}{dx}\cdot v\cdot w+u\cdot\frac{dv}{dx}\cdot w+u\cdot v\cdot\frac{dw}{dx}$$

Exercise 5.6

Question 1:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = 2at^2$, $y = at^4$

Answer

The given equations are $x = 2at^2$ and $y = at^4$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}\left(2at^2\right) = 2a \cdot \frac{d}{dt}\left(t^2\right) = 2a \cdot 2t = 4at$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(at^4\right) = a \cdot \frac{d}{dt}\left(t^4\right) = a \cdot 4 \cdot t^3 = 4at^3$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

Question 2:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a\cos\theta$, $y = b\cos\theta$

Answer

The given equations are $x = a\cos\theta$ and $y = b\cos\theta$

$$\text{Then, } \frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos\theta\right) = a\left(-\sin\theta\right) = -a\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(b\cos\theta\right) = b\left(-\sin\theta\right) = -b\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

Question 3:

If *x* and *y* are connected parametrically by the equation, without eliminating the

parameter, find $\frac{dy}{dx}$.

$x = \sin t,\ y = \cos 2t$

Answer

The given equations are $x = \sin t$ and $y = \cos 2t$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t \cdot \frac{d}{dt}(2t) = -2\sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2 \cdot 2\sin t\cos t}{\cos t} = -4\sin t$$

Question 4:

If *x* and *y* are connected parametrically by the equation, without eliminating the

parameter, find $\frac{dy}{dx}$.

$x = 4t,\ y = \frac{4}{t}$

Answer

The given equations are $x = 4t \text{ and } y = \frac{4}{t}$

$$\frac{dx}{dt} = \frac{d}{dt}(4t) = 4$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4 \cdot \frac{d}{dt}\left(\frac{1}{t}\right) = 4 \cdot \left(\frac{-1}{t^2}\right) = \frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-4}{t^2}\right)}{4} = \frac{-1}{t^2}$$

Question 5:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$

Answer

The given equations are $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$

$$\text{Then, } \frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\theta - \cos 2\theta) = \frac{d}{d\theta}(\cos\theta) - \frac{d}{d\theta}(\cos 2\theta)$$
$$= -\sin\theta - (-2\sin 2\theta) = 2\sin 2\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta - \sin 2\theta) = \frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\sin 2\theta)$$
$$= \cos\theta - 2\cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$$

Question 6:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$

Answer

The given equations are $x = a(\theta - \sin\theta)$ and $y = a(1+\cos\theta)$

$$\text{Then,} \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\theta)\right] = a(1-\cos\theta)$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta)\right] = a\left[0+(-\sin\theta)\right] = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Question 7:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$$x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

Answer

The given equations are $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

$$\text{Then,} \frac{dx}{dt} = \frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$$

$$= \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt}\left(\sin^3 t\right) - \sin^3 t \cdot \frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t} \cdot 3\sin^2 t \cdot \frac{d}{dt}(\sin t) - \sin^3 t \times \frac{1}{2\sqrt{\cos 2t}} \cdot \frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t} \cdot \sin^2 t \cos t - \frac{\sin^3 t}{2\sqrt{\cos 2t}} . (-2\sin 2t)}{\cos 2t}$$

$$= \frac{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$= \frac{\sqrt{\cos 2t} . \frac{d}{dt}\left(\cos^3 t\right) - \cos^3 t \cdot \frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t} . 3\cos^2 t \cdot \frac{d}{dt}(\cos t) - \cos^3 t . \frac{1}{2\sqrt{\cos 2t}} \cdot \frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t} . \cos^2 t(-\sin t) - \cos^3 t \cdot \frac{1}{2\sqrt{\cos 2t}} . (-2\sin 2t)}{\cos 2t}$$

$$= \frac{-3\cos 2t . \cos^2 t . \sin t + \cos^3 t \sin 2t}{\cos 2t \cdot \sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t \sin 2t}{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t\left(2\sin t \cos t\right)}{3\cos 2t \sin^2 t \cos t + \sin^3 t\left(2\sin t \cos t\right)}$$

$$= \frac{\sin t \cos t\left[-3\cos 2t.\cos t + 2\cos^3 t\right]}{\sin t \cos t\left[3\cos 2t \sin t + 2\sin^3 t\right]}$$

$$= \frac{\left[-3\left(2\cos^2 t - 1\right)\cos t + 2\cos^3 t\right]}{\left[3\left(1 - 2\sin^2 t\right)\sin t + 2\sin^3 t\right]} \qquad \begin{bmatrix}\cos 2t = \left(2\cos^2 t - 1\right), \\ \cos 2t = \left(1 - 2\sin^2 t\right)\end{bmatrix}$$

$$= \frac{-4\cos^3 t + 3\cos t}{3\sin t - 4\sin^3 t}$$

$$= \frac{-\cos 3t}{\sin 3t} \qquad \begin{bmatrix}\cos 3t = 4\cos^3 t - 3\cos t, \\ \sin 3t = 3\sin t - 4\sin^3 t\end{bmatrix}$$

$$= -\cot 3t$$

Question 8:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = a\left(\cos t + \log \tan \frac{t}{2}\right),\ y = a\sin t$$

Answer

The given equations are $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$ and $y = a\sin t$

$$\text{Then, } \frac{dx}{dt} = a \cdot \left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log \tan \frac{t}{2} \right) \right]$$

$$= a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{d}{dt}\left(\tan \frac{t}{2} \right) \right]$$

$$= a\left[-\sin t + \cot \frac{t}{2} \cdot \sec^2 \frac{t}{2} \cdot \frac{d}{dt}\left(\frac{t}{2} \right) \right]$$

$$= a\left[-\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2} \right]$$

$$= a\left[-\sin t + \frac{1}{2\sin \frac{t}{2} \cos \frac{t}{2}} \right]$$

$$= a\left(-\sin t + \frac{1}{\sin t} \right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t} \right)$$

$$= a\frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a\frac{d}{dt}(\sin t) = a\cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)} = \frac{a\cos t}{\left(a\frac{\cos^2 t}{\sin t} \right)} = \frac{\sin t}{\cos t} = \tan t$$

Question 9:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a\sec\theta,\ y = b\tan\theta$

Answer

The given equations are $x = a\sec\theta$ and $y = b\tan\theta$

Then, $\dfrac{dx}{d\theta} = a\cdot\dfrac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$

$\dfrac{dy}{d\theta} = b\cdot\dfrac{d}{d\theta}(\tan\theta) = b\sec^2\theta$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\sec\theta\cot\theta = \frac{b\cos\theta}{a\cos\theta\sin\theta} = \frac{b}{a}\times\frac{1}{\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta$$

Question 10:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\dfrac{dy}{dx}$.

$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

Answer

The given equations are $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right] = a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$$

$$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta\cdot\frac{d}{d\theta}(\theta)\right\}\right]$$

$$= a[\cos\theta + \theta\sin\theta - \cos\theta]$$

$$= a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Question 11:

If $x=\sqrt{a^{\sin^{-1}t}},\ y=\sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx}=-\frac{y}{x}$

Answer

The given equations are $x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$\Rightarrow x=\left(a^{\sin^{-1}t}\right)^{\frac{1}{2}}$ and $y=\left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}$

$\Rightarrow x=a^{\frac{1}{2}\sin^{-1}t}$ and $y=a^{\frac{1}{2}\cos^{-1}t}$

Consider $x=a^{\frac{1}{2}\sin^{-1}t}$

Taking logarithm on both the sides, we obtain

$$\log x=\frac{1}{2}\sin^{-1}t\log a$$

$$\therefore \frac{1}{x}\cdot\frac{dx}{dt}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\sin^{-1}t\right)$$

$$\Rightarrow \frac{dx}{dt}=\frac{x}{2}\log a\cdot\frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt}=\frac{x\log a}{2\sqrt{1-t^2}}$$

Then, consider $y=a^{\frac{1}{2}\cos^{-1}t}$

Taking logarithm on both the sides, we obtain

$$\log y=\frac{1}{2}\cos^{-1}t\log a$$

$$\therefore \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\cos^{-1}t\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{y\log a}{2}\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{-y\log a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log a}{2\sqrt{1-t^2}}\right)} = -\frac{y}{x}.$$

Hence, proved.

Exercise 5.7

Question 1:

Find the second order derivatives of the function.

$x^2 + 3x + 2$

Answer

Let $y = x^2 + 3x + 2$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 2x + 3 + 0 = 2x + 3$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(2x+3) = \frac{d}{dx}(2x) + \frac{d}{dx}(3) = 2 + 0 = 2$$

Question 2:

Find the second order derivatives of the function.

x^{20}

Answer

Let $y = x^{20}$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{20}\right) = 20x^{19}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(20x^{19}\right) = 20\frac{d}{dx}\left(x^{19}\right) = 20 \cdot 19 \cdot x^{18} = 380x^{18}$$

Question 3:

Find the second order derivatives of the function.

$x \cdot \cos x$

Answer

Let $y = x \cdot \cos x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(x\cdot\cos x)=\cos x\cdot\frac{d}{dx}(x)+x\frac{d}{dx}(\cos x)=\cos x\cdot 1+x(-\sin x)=\cos x-x\sin x$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}[\cos x-x\sin x]=\frac{d}{dx}(\cos x)-\frac{d}{dx}(x\sin x)$$

$$=-\sin x-\left[\sin x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\sin x)\right]$$

$$=-\sin x-(\sin x+x\cos x)$$

$$=-(x\cos x+2\sin x)$$

Question 4:

Find the second order derivatives of the function.

$\log x$

Answer

Let $y=\log x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(\log x)=\frac{1}{x}$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$$

Question 5:

Find the second order derivatives of the function.

$x^3\log x$

Answer

Let $y=x^3\log x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[x^3\log x\right]=\log x\cdot\frac{d}{dx}\left(x^3\right)+x^3\cdot\frac{d}{dx}(\log x)$$
$$=\log x\cdot 3x^2+x^3\cdot\frac{1}{x}=\log x\cdot 3x^2+x^2$$
$$=x^2\left(1+3\log x\right)$$
$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[x^2\left(1+3\log x\right)\right]$$
$$=\left(1+3\log x\right)\cdot\frac{d}{dx}\left(x^2\right)+x^2\frac{d}{dx}\left(1+3\log x\right)$$
$$=\left(1+3\log x\right)\cdot 2x+x^2\cdot\frac{3}{x}$$
$$=2x+6x\log x+3x$$
$$=5x+6x\log x$$
$$=x\left(5+6\log x\right)$$

Question 6:

Find the second order derivatives of the function.

$e^x\sin 5x$

Answer

Let $y=e^x\sin 5x$

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^x\sin 5x\right)=\sin 5x\cdot\frac{d}{dx}\left(e^x\right)+e^x\frac{d}{dx}(\sin 5x)$$
$$=\sin 5x\cdot e^x+e^x\cdot\cos 5x\cdot\frac{d}{dx}(5x)=e^x\sin 5x+e^x\cos 5x\cdot 5$$
$$=e^x\left(\sin 5x+5\cos 5x\right)$$
$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[e^x\left(\sin 5x+5\cos 5x\right)\right]$$
$$=\left(\sin 5x+5\cos 5x\right)\cdot\frac{d}{dx}\left(e^x\right)+e^x\cdot\frac{d}{dx}\left(\sin 5x+5\cos 5x\right)$$
$$=\left(\sin 5x+5\cos 5x\right)e^x+e^x\left[\cos 5x\cdot\frac{d}{dx}(5x)+5(-\sin 5x)\cdot\frac{d}{dx}(5x)\right]$$
$$=e^x\left(\sin 5x+5\cos 5x\right)+e^x\left(5\cos 5x-25\sin 5x\right)$$

Then,
$$=e^x\left(10\cos 5x-24\sin 5x\right)=2e^x\left(5\cos 5x-12\sin 5x\right)$$

Question 7:

Find the second order derivatives of the function.

$e^{6x}\cos 3x$

Answer

Let $y = e^{6x}\cos 3x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{6x}\cdot\cos 3x\right) = \cos 3x\cdot\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\cdot\frac{d}{dx}\left(\cos 3x\right)$$

$$= \cos 3x\cdot e^{6x}\cdot\frac{d}{dx}(6x) + e^{6x}\cdot(-\sin 3x)\cdot\frac{d}{dx}(3x)$$

$$= 6e^{6x}\cos 3x - 3e^{6x}\sin 3x \qquad \ldots(1)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right) = 6\cdot\frac{d}{dx}\left(e^{6x}\cos 3x\right) - 3\cdot\frac{d}{dx}\left(e^{6x}\sin 3x\right)$$

$$= 6\cdot\left[6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right] - 3\cdot\left[\sin 3x\cdot\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\cdot\frac{d}{dx}(\sin 3x)\right] \qquad [\text{Using }(1)]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 3\left[\sin 3x\cdot e^{6x}\cdot 6 + e^{6x}\cdot\cos 3x\cdot 3\right]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x - 9e^{6x}\cos 3x$$

$$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$

$$= 9e^{6x}\left(3\cos 3x - 4\sin 3x\right)$$

Question 8:

Find the second order derivatives of the function.

$\tan^{-1} x$

Answer

Let $y = \tan^{-1} x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{1+x^2}\right)=\frac{d}{dx}\left(1+x^2\right)^{-1}=(-1)\cdot\left(1+x^2\right)^{-2}\cdot\frac{d}{dx}\left(1+x^2\right)$$

$$=\frac{-1}{\left(1+x^2\right)^2}\times 2x=\frac{-2x}{\left(1+x^2\right)^2}$$

Question 9:

Find the second order derivatives of the function.

$\log(\log x)$

Answer

Let $y=\log(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log(\log x)\right]=\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)=\frac{1}{x\log x}=(x\log x)^{-1}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[(x\log x)^{-1}\right]=(-1)\cdot(x\log x)^{-2}\cdot\frac{d}{dx}(x\log x)$$

$$=\frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)\right]$$

$$=\frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot 1+x\cdot\frac{1}{x}\right]=\frac{-(1+\log x)}{(x\log x)^2}$$

Question 10:

Find the second order derivatives of the function.

$\sin(\log x)$

Answer

Let $y=\sin(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin\left(\log x\right)\right]=\cos\left(\log x\right)\cdot\frac{d}{dx}\left(\log x\right)=\frac{\cos\left(\log x\right)}{x}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{\cos\left(\log x\right)}{x}\right]$$

$$=\frac{x\cdot\frac{d}{dx}\left[\cos\left(\log x\right)\right]-\cos\left(\log x\right)\cdot\frac{d}{dx}\left(x\right)}{x^2}$$

$$=\frac{x\cdot\left[-\sin\left(\log x\right)\cdot\frac{d}{dx}\left(\log x\right)\right]-\cos\left(\log x\right).1}{x^2}$$

$$=\frac{-x\sin\left(\log x\right).\frac{1}{x}-\cos\left(\log x\right)}{x^2}$$

$$=\frac{-\left[\sin\left(\log x\right)+\cos\left(\log x\right)\right]}{x^2}$$

Question 11:

If $y=5\cos x-3\sin x$, prove that $\frac{d^2y}{dx^2}+y=0$

Answer

It is given that, $y=5\cos x-3\sin x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x) - \frac{d}{dx}(3\sin x) = 5\frac{d}{dx}(\cos x) - 3\frac{d}{dx}(\sin x)$$
$$= 5(-\sin x) - 3\cos x = -(5\sin x + 3\cos x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[-(5\sin x + 3\cos x)\right]$$
$$= -\left[5\cdot\frac{d}{dx}(\sin x) + 3\cdot\frac{d}{dx}(\cos x)\right]$$
$$= -\left[5\cos x + 3(-\sin x)\right]$$
$$= -\left[5\cos x - 3\sin x\right]$$
$$= -y$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

Hence, proved.

Question 12:

If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of *y* alone.

Answer

It is given that, $y = \cos^{-1} x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}=-\left(1-x^2\right)^{\frac{-1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-\left(1-x^2\right)^{\frac{-1}{2}}\right]$$

$$=-\left(-\frac{1}{2}\right)\cdot\left(1-x^2\right)^{\frac{-3}{2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$=\frac{1}{2\sqrt{\left(1-x^2\right)^3}}\times(-2x)$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-x}{\sqrt{\left(1-x^2\right)^3}} \quad \ldots\text{(i)}$$

$y=\cos^{-1}x\Rightarrow x=\cos y$

Putting $x=\cos y$ in equation (i), we obtain

$$\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$=\frac{-\cos y}{\sin^3 y}$$

$$=\frac{-\cos y}{\sin y}\times\frac{1}{\sin^2 y}$$

$$\Rightarrow\frac{d^2y}{dx^2}=-\cot y\cdot\operatorname{cosec}^2 y$$

Question 13:

If $y=3\cos(\log x)+4\sin(\log x)$, show that $x^2y_2+xy_1+y=0$

Answer

It is given that, $y=3\cos(\log x)+4\sin(\log x)$

Then,

$$y_1 = 3 \cdot \frac{d}{dx}\left[\cos(\log x)\right] + 4 \cdot \frac{d}{dx}\left[\sin(\log x)\right]$$

$$= 3 \cdot \left[-\sin(\log x) \cdot \frac{d}{dx}(\log x)\right] + 4 \cdot \left[\cos(\log x) \cdot \frac{d}{dx}(\log x)\right]$$

$$\therefore y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} = \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

$$\therefore y_2 = \frac{d}{dx}\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right)$$

$$= \frac{x\left\{4\cos(\log x) - 3\sin(\log x)\right\}' - \left\{4\cos(\log x) - 3\sin(\log x)\right\}(x)'}{x^2}$$

$$= \frac{x\left[4\left\{\cos(\log x)\right\}' - 3\left\{\sin(\log x)\right\}'\right] - \left\{4\cos(\log x) - 3\sin(\log x)\right\}.1}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).(\log x)' - 3\cos(\log x).(\log x)'\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).\frac{1}{x} - 3\cos(\log x).\frac{1}{x}\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

$$\therefore x^2 y_2 + x y_1 + y$$

$$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right) + 3\cos(\log x) + 4\sin(\log x)$$

$$= -\sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) + 3\cos(\log x) + 4\sin(\log x)$$

$$= 0$$

Hence, proved.

Question 14:

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2 y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Answer

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A \cdot \frac{d}{dx}\left(e^{mx}\right) + B \cdot \frac{d}{dx}\left(e^{nx}\right) = A \cdot e^{mx} . \frac{d}{dx}(mx) + B \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(Ame^{mx} + Bne^{nx}\right) = Am \cdot \frac{d}{dx}\left(e^{mx}\right) + Bn \cdot \frac{d}{dx}\left(e^{nx}\right)$$

$$= Am \cdot e^{mx} \cdot \frac{d}{dx}(mx) + Bn \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx}$$

$$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n) \cdot \left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$

$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$$

$$= 0$$

Hence, proved.

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

Answer

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\frac{dy}{dx} = 500.\frac{d}{dx}\left(e^{7x}\right) + 600.\frac{d}{dx}\left(e^{-7x}\right)$$
$$= 500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) + 600 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)$$
$$= 3500e^{7x} - 4200e^{-7x}$$
$$\therefore \frac{d^2y}{dx^2} = 3500 \cdot \frac{d}{dx}\left(e^{7x}\right) - 4200.\frac{d}{dx}\left(e^{-7x}\right)$$
$$= 3500 \cdot e^{7x}.\frac{d}{dx}(7x) - 4200 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)$$
$$= 7 \times 3500 \cdot e^{7x} + 7 \times 4200 \cdot e^{-7x}$$
$$= 49 \times 500e^{7x} + 49 \times 600e^{-7x}$$
$$= 49\left(500e^{7x} + 600e^{-7x}\right)$$
$$= 49y$$

Hence, proved.

Question 16:

If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Answer

The given relationship is $e^y(x+1) = 1$

$$e^y(x+1) = 1$$
$$\Rightarrow e^y = \frac{1}{x+1}$$

Taking logarithm on both the sides, we obtain

$$y = \log\frac{1}{(x+1)}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right)=(x+1).\frac{-1}{(x+1)^2}=\frac{-1}{x+1}$$

$$\therefore \frac{d^2y}{dx^2}=-\frac{d}{dx}\left(\frac{1}{x+1}\right)=-\left(\frac{-1}{(x+1)^2}\right)=\frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{-1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Hence, proved.

Question 17:

If $y=\left(\tan^{-1}x\right)^2$, show that $\left(x^2+1\right)^2 y_2+2x\left(x^2+1\right)y_1=2$

Answer

The given relationship is $y=\left(\tan^{-1}x\right)^2$

Then,

$$y_1=2\tan^{-1}x\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$\Rightarrow y_1=2\tan^{-1}x.\frac{1}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)y_1=2\tan^{-1}x$$

Again differentiating with respect to x on both the sides, we obtain

$$\left(1+x^2\right)y_2+2xy_1=2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \left(1+x^2\right)^2 y_2+2x\left(1+x^2\right)y_1=2$$

Hence, proved.

Exercise 5.8

Question 1:

Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$

Answer

The given function, $f(x) = x^2 + 2x - 8$, being a polynomial function, is continuous in [−4, 2] and is differentiable in (−4, 2).

$$f(-4) = (-4)^2 + 2\times(-4) - 8 = 16 - 8 - 8 = 0$$
$$f(2) = (2)^2 + 2\times 2 - 8 = 4 + 4 - 8 = 0$$

∴ *f* (−4) = *f* (2) = 0

⇒ The value of *f* (*x*) at −4 and 2 coincides.

Rolle's Theorem states that there is a point *c* ∈ (−4, 2) such that $f'(c) = 0$

$$f(x) = x^2 + 2x - 8$$
$$\Rightarrow f'(x) = 2x + 2$$
$$\therefore f'(c) = 0$$
$$\Rightarrow 2c + 2 = 0$$
$$\Rightarrow c = -1, \text{ where } c = -1 \in (-4, 2)$$

Hence, Rolle's Theorem is verified for the given function.

Question 2:

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

Answer

By Rolle's Theorem, for a function $f:[a,\ b]\to\mathbf{R}$, if

(a) *f* is continuous on [*a*, *b*]

(b) *f* is differentiable on (*a*, *b*)

(c) *f* (*a*) = *f* (*b*)

then, there exists some *c* ∈ (*a*, *b*) such that $f'(c)=0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

It is evident that the given function *f* (*x*) is not continuous at every integral point.

In particular, *f*(*x*) is not continuous at *x* = 5 and *x* = 9

⇒ *f* (*x*) is not continuous in [5, 9].

$$\text{Also, } f(5)=[5]=5 \text{ and } f(9)=[9]=9$$
$$\therefore f(5)\neq f(9)$$

The differentiability of *f* in (5, 9) is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h \to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \to 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.
In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$

$\therefore f(-2) \neq f(2)$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h \to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \to 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

$f(1) = (1)^2 - 1 = 0$

$f(2) = (2)^2 - 1 = 3$

$\therefore f(1) \neq f(2)$

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=x^2-1$ for $x\in[1,\ 2]$.

Question 3:

If $f:[-5,5]\to\mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then

prove that $f(-5)\neq f(5)$.

Answer

It is given that $f:[-5,5]\to\mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) *f* is continuous on [−5, 5].

(b) *f* is differentiable on (−5, 5).

Therefore, by the Mean Value Theorem, there exists *c* ∈ (−5, 5) such that

$$f'(c)=\frac{f(5)-f(-5)}{5-(-5)}$$
$$\Rightarrow 10f'(c)=f(5)-f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c)\neq 0$$
$$\Rightarrow 10f'(c)\neq 0$$
$$\Rightarrow f(5)-f(-5)\neq 0$$
$$\Rightarrow f(5)\neq f(-5)$$

Hence, proved.

Question 4:

Verify Mean Value Theorem, if $f(x)=x^2-4x-3$ in the interval $[a,\ b]$, where

$a=1$ and $b=4$.

Answer

The given function is $f(x)=x^2-4x-3$

f, being a polynomial function, is continuous in [1, 4] and is differentiable in (1, 4) whose derivative is $2x-4$.

$$f(1)=1^2-4\times1-3=-6,\ f(4)=4^2-4\times4-3=-3$$

$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(1)}{4-1}=\frac{-3-(-6)}{3}=\frac{3}{3}=1$$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c)=1$

$$f'(c)=1$$
$$\Rightarrow 2c-4=1$$
$$\Rightarrow c=\frac{5}{2},\text{ where } c=\frac{5}{2}\in(1,\ 4)$$

Hence, Mean Value Theorem is verified for the given function.

Question 5:

Verify Mean Value Theorem, if $f(x)=x^3-5x^2-3x$ in the interval [*a*, *b*], where *a* = 1 and *b* = 3. Find all $c\in(1,3)$ for which $f'(c)=0$

Answer

The given function *f* is $f(x)=x^3-5x^2-3x$

f, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3) whose derivative is $3x^2-10x-3$.

$$f(1)=1^3-5\times1^2-3\times1=-7,\ f(3)=3^3-5\times3^2-3\times3=-27$$

$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(3)-f(1)}{3-1}=\frac{-27-(-7)}{3-1}=-10$$

Mean Value Theorem states that there exist a point $c \in (1, 3)$ such that $f'(c) = -10$

$$
\begin{aligned}
& f'(c) = -10 \\
& \Rightarrow 3c^2 - 10c - 3 = 10 \\
& \Rightarrow 3c^2 - 10c + 7 = 0 \\
& \Rightarrow 3c^2 - 3c - 7c + 7 = 0 \\
& \Rightarrow 3c(c-1) - 7(c-1) = 0 \\
& \Rightarrow (c-1)(3c-7) = 0 \\
& \Rightarrow c = 1,\ \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1,\ 3)
\end{aligned}
$$

Hence, Mean Value Theorem is verified for the given function and $c = \frac{7}{3} \in (1,\ 3)$ is the only point for which $f'(c) = 0$

Question 6:

Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Answer

Mean Value Theorem states that for a function $f : [a,\ \mathrm{b}] \to \mathbf{R}$, if

(a) f is continuous on $[a, b]$

(b) f is differentiable on (a, b)

then, there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

It is evident that the given function *f* (*x*) is not continuous at every integral point.

In particular, *f*(*x*) is not continuous at *x* = 5 and *x* = 9

⇒ *f* (*x*) is not continuous in [5, 9].

The differentiability of *f* in (5, 9) is checked as follows.

Let *n* be an integer such that *n* ∈ (5, 9).

The left hand limit of f at $x=n$ is,

$$\lim_{h\to0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to0^-}\frac{n-1-n}{h}=\lim_{h\to0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to0^+}\frac{n-n}{h}=\lim_{h\to0^+}0=0$$

Since the left and right hand limits of *f* at *x* = *n* are not equal, *f* is not differentiable at *x* = *n*

∴*f* is not differentiable in (5, 9).

It is observed that *f* does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function *f* (*x*) is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x \in [-2, 2]$.

(iii) $f(x)=x^2-1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x) = x^2 - 1 \text{ for } x \in [1,\ 2]$.

It can be proved as follows.

$f(1) = 1^2 - 1 = 0,\ f(2) = 2^2 - 1 = 3$

$\therefore \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{3 - 0}{1} = 3$

$f'(x) = 2x$

$\therefore f'(c) = 3$

$\Rightarrow 2c = 3$

$\Rightarrow c = \dfrac{3}{2} = 1.5, \text{ where } 1.5 \in [1, 2]$

Miscellaneous Solutions

Question 1:

$$\left(3x^2-9x+5\right)^9$$

Answer

$$\text{Let } y=\left(3x^2-9x+5\right)^9$$

Using chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(3x^2-9x+5\right)^9\\&=9\left(3x^2-9x+5\right)^8\cdot\frac{d}{dx}\left(3x^2-9x+5\right)\\&=9\left(3x^2-9x+5\right)^8\cdot\left(6x-9\right)\\&=9\left(3x^2-9x+5\right)^8\cdot3\left(2x-3\right)\\&=27\left(3x^2-9x+5\right)^8\left(2x-3\right)\end{aligned}$$

Question 2:

$$\sin^3 x+\cos^6 x$$

Answer

$$\text{Let } y=\sin^3 x+\cos^6 x$$

$$\begin{aligned}\therefore\frac{dy}{dx}&=\frac{d}{dx}\left(\sin^3 x\right)+\frac{d}{dx}\left(\cos^6 x\right)\\&=3\sin^2 x\cdot\frac{d}{dx}\left(\sin x\right)+6\cos^5 x\cdot\frac{d}{dx}\left(\cos x\right)\\&=3\sin^2 x\cdot\cos x+6\cos^5 x\cdot\left(-\sin x\right)\\&=3\sin x\cos x\left(\sin x-2\cos^4 x\right)\end{aligned}$$

Question 3:

$$\left(5x\right)^{3\cos 2x}$$

Answer

Let $y = (5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$\log y = 3\cos 2x \log 5x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\log 5x \cdot \frac{d}{dx}(\cos 2x) + \cos 2x \cdot \frac{d}{dx}(\log 5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\log 5x(-\sin 2x)\cdot\frac{d}{dx}(2x) + \cos 2x \cdot \frac{1}{5x}\cdot\frac{d}{dx}(5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[-2\sin 2x \log 5x + \frac{\cos 2x}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

$$\therefore \frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

Question 4:

$\sin^{-1}\left(x\sqrt{x}\right),\ 0 \le x \le 1$

Answer

Let $y = \sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}} \times \frac{d}{dx}\left(x\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-x^3}} \cdot \frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$

$$= \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2} \cdot x^{\frac{1}{2}}$$

$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

$$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Question 5:

$$\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, \quad -2 < x < 2$$

Answer

Let $y = \dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

By quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\,\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right) - \left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$= \frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{d}{dx}\left(\frac{x}{2}\right)\right] - \left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2\sqrt{2x+7}}\cdot\frac{d}{dx}(2x+7)}{2x+7}$$

$$= \frac{\sqrt{2x+7}\,\frac{-1}{\sqrt{4-x^2}} - \left(\cos^{-1}\frac{x}{2}\right)\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$= \frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)} - \frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$= -\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}} + \frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Question 6:

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$$

Answer

$$\text{Let } y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right] \quad ...(1)$$

$$\text{Then, } \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$=\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$=\frac{1+\cos x}{\sin x}$$

$$=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

Therefore, equation (1) becomes

$$y=\cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y=\frac{x}{2}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}$$

Question 7:

$(\log x)^{\log x}$, $x>1$

Answer

$\text{Let } y=(\log x)^{\log x}$

Taking logarithm on both the sides, we obtain

$$\log y = \log x \cdot \log(\log x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\log x \cdot \log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x).\frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x).\frac{1}{x} + \log x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

Question 8:

$\cos(a\cos x + b\sin x)$, for some constant *a* and *b*.

Answer

Let $y = \cos(a\cos x + b\sin x)$

By using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x) \cdot \frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x) \cdot \left[a(-\sin x) + b\cos x\right]$$

$$= (a\sin x - b\cos x) \cdot \sin(a\cos x + b\sin x)$$

Question 9:

$(\sin x - \cos x)^{(\sin x - \cos x)}, \ \frac{\pi}{4} < x < \frac{3\pi}{4}$

Answer

Let $y=(\sin x-\cos x)^{(\sin x-\cos x)}$

Taking logarithm on both the sides, we obtain

$$\log y=\log\left[(\sin x-\cos x)^{(\sin x-\cos x)}\right]$$

$$\Rightarrow \log y=(\sin x-\cos x)\cdot\log(\sin x-\cos x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left[(\sin x-\cos x)\log(\sin x-\cos x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\cdot\frac{d}{dx}(\sin x-\cos x)+(\sin x-\cos x)\cdot\frac{d}{dx}\log(\sin x-\cos x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\cdot(\cos x+\sin x)+(\sin x-\cos x)\cdot\frac{1}{(\sin x-\cos x)}\cdot\frac{d}{dx}(\sin x-\cos x)$$

$$\Rightarrow \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}\left[(\cos x+\sin x)\cdot\log(\sin x-\cos x)+(\cos x+\sin x)\right]$$

$$\therefore \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

Question 10:

$x^x+x^a+a^x+a^a$, for some fixed $a>0$ and $x>0$

Answer

Let $y=x^x+x^a+a^x+a^a$

Also, let $x^x=u,\ x^a=v,\ a^x=w,$ and $a^a=s$

$\therefore y=u+v+w+s$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\frac{ds}{dx} \qquad ...(1)$$

$u=x^x$

$\Rightarrow \log u=\log x^x$

$\Rightarrow \log u=x\log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{du}{dx}=u\left[\log x\cdot 1+x\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^x\left[\log x+1\right]=x^x(1+\log x) \quad ...(2)$$

$v=x^a$

$$\therefore\frac{dv}{dx}=\frac{d}{dx}\left(x^a\right)$$

$$\Rightarrow\frac{dv}{dx}=ax^{a-1} \quad ...(3)$$

$w=a^x$

$\Rightarrow\log w=\log a^x$

$\Rightarrow\log w=x\log a$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{w}\cdot\frac{dw}{dx}=\log a\cdot\frac{d}{dx}(x)$$

$$\Rightarrow\frac{dw}{dx}=w\log a$$

$$\Rightarrow\frac{dw}{dx}=a^x\log a \quad ...(4)$$

$s=a^a$

Since a is constant, a^a is also a constant.

$$\therefore\frac{ds}{dx}=0 \quad ...(5)$$

From (1), (2), (3), (4), and (5), we obtain

$$\frac{dy}{dx}=x^x(1+\log x)+ax^{a-1}+a^x\log a+0$$

$$=x^x(1+\log x)+ax^{a-1}+a^x\log a$$

Question 11:

$x^{x^2-3}+(x-3)^{x^2}$, for $x>3$

Answer

$\text{Let } y = x^{x^2-3}+(x-3)^{x^2}$

$\text{Also, let } u = x^{x^2-3} \text{ and } v=(x-3)^{x^2}$

$\therefore y = u+v$

Differentiating both sides with respect to *x,* we obtain

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u = x^{x^2-3}$

$\therefore \log u = \log\left(x^{x^2-3}\right)$

$\log u = \left(x^2-3\right)\log x$

Differentiating with respect to *x*, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\log x\cdot\frac{d}{dx}\left(x^2-3\right)+\left(x^2-3\right)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=\log x\cdot 2x+\left(x^2-3\right)\cdot\frac{1}{x}$$

$$\Rightarrow \frac{du}{dx}=x^{x^2-3}\cdot\left[\frac{x^2-3}{x}+2x\log x\right]$$

Also,

$v=(x-3)^{x^2}$

$\therefore \log v = \log(x-3)^{x^2}$

$\Rightarrow \log v = x^2\log(x-3)$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\log(x-3)\cdot\frac{d}{dx}\left(x^2\right)+x^2\cdot\frac{d}{dx}\left[\log(x-3)\right]$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\log(x-3)\cdot 2x+x^2\cdot\frac{1}{x-3}\cdot\frac{d}{dx}(x-3)$$

$$\Rightarrow\frac{dv}{dx}=v\left[2x\log(x-3)+\frac{x^2}{x-3}\cdot 1\right]$$

$$\Rightarrow\frac{dv}{dx}=(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Substituting the expressions of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx}=x^{x^2-3}\left[\frac{x^2-3}{x}+2x\log x\right]+(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Question 12:

Find $\frac{dy}{dx}$, if $y=12(1-\cos t), x=10(t-\sin t), -\frac{\pi}{2}<t<\frac{\pi}{2}$

Answer

It is given that, $y=12(1-\cos t), x=10(t-\sin t)$

$$\therefore\frac{dx}{dt}=\frac{d}{dt}\left[10(t-\sin t)\right]=10\cdot\frac{d}{dt}(t-\sin t)=10(1-\cos t)$$

$$\frac{dy}{dt}=\frac{d}{dt}\left[12(1-\cos t)\right]=12\cdot\frac{d}{dt}(1-\cos t)=12\cdot\left[0-(-\sin t)\right]=12\sin t$$

$$\therefore\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{12\sin t}{10(1-\cos t)}=\frac{12\cdot 2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{10\cdot 2\sin^2\frac{t}{2}}=\frac{6}{5}\cot\frac{t}{2}$$

Question 13:

Find $\frac{dy}{dx}$, if $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2},\ -1\le x\le 1$

Answer

It is given that, $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{x}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{2x\sqrt{1-x^2}}(-2x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx}=0$$

Question 14:

If $x\sqrt{1+y}+y\sqrt{1+x}=0$, for, $-1 < x < 1$, prove that

$$\frac{dy}{dx}=-\frac{1}{(1+x)^2}$$

Answer

It is given that,

$$x\sqrt{1+y}+y\sqrt{1+x}=0$$

$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides, we obtain

$x^2(1+y) = y^2(1+x)$

$\Rightarrow x^2 + x^2 y = y^2 + xy^2$

$\Rightarrow x^2 - y^2 = xy^2 - x^2 y$

$\Rightarrow x^2 - y^2 = xy(y-x)$

$\Rightarrow (x+y)(x-y) = xy(y-x)$

$\therefore x + y = -xy$

$\Rightarrow (1+x)y = -x$

$\Rightarrow y = \dfrac{-x}{(1+x)}$

Differentiating both sides with respect to *x*, we obtain

$y = \dfrac{-x}{(1+x)}$

$$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x)^2} = -\frac{(1+x)-x}{(1+x)^2} = -\frac{1}{(1+x)^2}$$

Hence, proved.

Question 15:

If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$ is a constant independent of *a* and *b*

Answer

It is given that, $(x-a)^2 + (y-b)^2 = c^2$

Differentiating both sides with respect to *x*, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}\left(c^2\right)$$

$$\Rightarrow 2(x-a)\cdot\frac{d}{dx}(x-a)+2(y-b)\cdot\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a)\cdot 1+2(y-b)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \qquad ...(1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$= -\left[\frac{(y-b)\cdot\frac{d}{dx}(x-a)-(x-a)\cdot\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad [\text{Using } (1)]$$

$$= -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\therefore \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}} = \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$= \frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}} = \frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

$= -c,$ which is constant and is independent of a and b

Hence, proved.

Question 16:

If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Answer

It is given that, $\cos y = x\cos(a+y)$

$$\therefore \frac{d}{dx}[\cos y] = \frac{d}{dx}\left[x\cos(a+y)\right]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y)\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}\left[\cos(a+y)\right]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y) + x\cdot\left[-\sin(a+y)\right]\frac{dy}{dx}$$

$$\Rightarrow \left[x\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y) \qquad ...(1)$$

Since $\cos y = x\cos(a+y)$, $x = \dfrac{\cos y}{\cos(a+y)}$

Then, equation (1) reduces to

$$\left[\frac{\cos y}{\cos(a+y)}\cdot\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow \left[\cos y\cdot\sin(a+y) - \sin y\cdot\cos(a+y)\right]\cdot\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = \cos^2(a+b)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+b)}{\sin a}$$

Hence, proved.

Question 17:

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$

Answer

It is given that, $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$$\therefore \frac{dx}{dt} = a\cdot\frac{d}{dt}(\cos t + t\sin t)$$

$$= a\left[-\sin t + \sin t\cdot\frac{d}{dx}(t) + t\cdot\frac{d}{dt}(\sin t)\right]$$

$$= a[-\sin t + \sin t + t\cos t] = at\cos t$$

$$\frac{dy}{dt} = a\cdot\frac{d}{dt}(\sin t - t\cos t)$$

$$= a\left[\cos t - \left\{\cos t\cdot\frac{d}{dt}(t) + t\cdot\frac{d}{dt}(\cos t)\right\}\right]$$

$$= a[\cos t - \{\cos t - t\sin t\}] = at\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\text{Then, } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t) = \sec^2 t\cdot\frac{dt}{dx}$$

$$= \sec^2 t\cdot\frac{1}{at\cos t} \qquad \left[\frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right]$$

$$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$$

Question 18:

If $f(x) = |x|^3$, show that $f''(x)$ exists for all real *x*, and find it.

Answer

It is known that, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Therefore, when $x \geq 0$, $f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

When $x < 0$, $f(x) = |x|^3 = (-x)^3 = -x^3$

In this case, $f'(x) = -3x^2$ and hence, $f''(x) = -6x$

Thus, for $f(x) = |x|^3$, $f''(x)$ exists for all real x and is given by,

$$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$$

Question 19:

Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

Answer

To prove: $P(n): \frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n

For $n = 1$,

$$P(1): \frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}$$

$\therefore$P(n) is true for n = 1

Let P(k) is true for some positive integer k.

That is, $P(k): \frac{d}{dx}(x^k) = kx^{k-1}$

It has to be proved that P(k + 1) is also true.

Consider $\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k)$

$$= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k) \qquad \text{[By applying product rule]}$$

$$= x^k \cdot 1 + x \cdot k \cdot x^{k-1}$$

$$= x^k + kx^k$$

$$= (k+1) \cdot x^k$$

$$= (k+1) \cdot x^{(k+1)-1}$$

Thus, P(k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, the statement P(n) is true for every positive integer n.

Hence, proved.

Question 20:

Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

Answer

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[\sin(A+B)\right]=\frac{d}{dx}(\sin A\cos B)+\frac{d}{dx}(\cos A\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B)=\cos B\cdot\frac{d}{dx}(\sin A)+\sin A\cdot\frac{d}{dx}(\cos B)$$
$$+\sin B\cdot\frac{d}{dx}(\cos A)+\cos A\cdot\frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B)=\cos B\cdot\cos A\frac{dA}{dx}+\sin A(-\sin B)\frac{dB}{dx}$$
$$+\sin B(-\sin A)\cdot\frac{dA}{dx}+\cos A\cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\cdot\left[\frac{dA}{dx}+\frac{dB}{dx}\right]=(\cos A\cos B-\sin A\sin B)\cdot\left[\frac{dA}{dx}+\frac{dB}{dx}\right]$$

$$\therefore \cos(A+B)=\cos A\cos B-\sin A\sin B$$

Question 22:

If $y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\Rightarrow y = (mc - nb) f(x) - (lc - na) g(x) + (lb - ma) h(x)$$

Then, $\frac{dy}{dx} = \frac{d}{dx}\left[(mc - nb) f(x)\right] - \frac{d}{dx}\left[(lc - na) g(x)\right] + \frac{d}{dx}\left[(lb - ma) h(x)\right]$

$$= (mc - nb) f'(x) - (lc - na) g'(x) + (lb - ma) h'(x)$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Thus, $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Question 23:

If $y = e^{a\cos^{-1} x}, -1 \le x \le 1$, show that $\left(1 - x^2\right)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} - a^2 y = 0$

Answer

It is given that, $y = e^{a\cos^{-1} x}$

Taking logarithm on both the sides, we obtain

$\log y = a\cos^{-1} x \log e$

$\log y = a\cos^{-1} x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = a \times \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\Rightarrow \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Again differentiating both sides with respect to x, we obtain

$$\left(\frac{dy}{dx}\right)^2 \frac{d}{dx}\left(1-x^2\right) + \left(1-x^2\right) \times \frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2\frac{d}{dx}\left(y^2\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + \left(1-x^2\right) \times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + \left(1-x^2\right) \times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow -x\frac{dy}{dx} + \left(1-x^2\right)\frac{d^2y}{dx^2} = a^2.y \qquad \left[\frac{dy}{dx} \neq 0\right]$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

Hence, proved.