7

Integrals

Short Answer Type Questions

Verify the following

Q. 1 $\int \frac{2x-1}{2x+3}\,dx = x - \log|(2x+3)^2| + C$

Sol. Let $$I = \int \frac{2x-1}{2x+3}\,dx = \int \frac{2x+3-3-1}{2x+3}dx$$

$$= \int 1dx - 4\int \frac{1}{2x+3}dx = x - \int \frac{4}{2\left(x+\frac{3}{2}\right)}\,dx$$

$$= x - 2\log + \left|\left(x+\frac{3}{2}\right)\right| C' = x - 2\log\left|\left(\frac{2x+3}{2}\right)\right| + C'$$

$$= x - 2\log|(2x+3)| + 2\log 2 + C' \qquad \left[\because \log\frac{m}{n} = \log m - \log n\right]$$

$$= x - \log|(2x+3)^2| + C \qquad [\because C = 2\log 2 + C']$$

Q. 2 $\int \frac{2x+3}{x^2+3x}\,dx = \log|x^2+3x| + C$

Sol. Let $$I = \int \frac{2x+3}{x^2+3x}dx$$

Put $$x^2 + 3x = t$$

$\Rightarrow$ $$(2x+3)\,dx = dt$$

$\therefore$ $$I = \int \frac{1}{t}\,dt = \log|t| + C$$

$$= \log|(x^2+3x)| + C$$

Q. 3 $\int \frac{(x^2+2)d}{x+1}x$

Thinking Process

First of all divided numerator by denominator, then use the formula $\int \frac{1}{x}dx = \log|x|$ *to get the solution.*

Sol. Let
$$I = \int \frac{x^2+2}{x+1}dx$$
$$= \int \left(x-1+\frac{3}{x+1}\right)dx$$
$$= \int (x-1)\,dx + 3\int \frac{1}{x+1}dx$$
$$= \frac{x^2}{2} - x + 3\log|(x+1)| + C$$

Q. 4 $\int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}dx$

Sol. Let
$$I = \int \left(\frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}\right)dx$$
$$= \int \left(\frac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}}\right)dx \qquad [\because a\log b = \log b^a]$$
$$= \int \left(\frac{x^6 - x^5}{x^4 - x^3}\right)dx \qquad [\because e^{\log x} = x]$$
$$= \int \left(\frac{x^3 - x^2}{x-1}\right)dx = \int \frac{x^2(x-1)}{x-1}dx$$
$$= \int x^2 dx = \frac{x^3}{3} + C$$

Q. 5 $\int \frac{(1+\cos x)}{x+\sin x}dx$

Sol. Consider that,
$$I = \int \frac{(1+\cos x)}{(x+\sin x)}dx$$
Let
$$x + \sin x = t \Rightarrow (1+\cos x)dx = dt$$
$\therefore$
$$I = \int \frac{1}{t}dt = \log|t| + C$$
$$= \log|(x+\sin x)| + C$$

Q. 6 $\int \frac{dx}{1+\cos x}$

Thinking Process

$\cos x = 2\cos^2 \frac{x}{2} - 1$ and also use formula i.e., $\int \sec^2 x = \tan x + C$ to solve the above problem.

Sol. Let
$$I = \int \frac{dx}{1+\cos x} = \int \frac{dx}{1+2\cos^2 \frac{x}{2} - 1}$$
$$= \frac{1}{2}\int \frac{1}{\cos^2 \frac{x}{2}} dx = \frac{1}{2}\int \sec^2 \frac{x}{2} dx$$
$$= \frac{1}{2} \cdot \tan\frac{x}{2} \cdot 2 + C = \tan\frac{x}{2} + C \qquad \left[\because \int \sec^2 x\, dx = \tan x\right]$$

Q. 7 $\int \tan^2 x \sec^4 x\, dx$

Thinking Process

Use the formula $\sec^2 x = 1 + \tan^2 x$ and put $\tan x = t$ to solve this problem.

Sol. Let $\quad I = \int \tan^2 x \sec^4 x\, dx$

Put $\quad \tan x = t \Rightarrow \sec^2 x dx = dt$

$\therefore \quad I = \int t^2(1+t^2)dt = \int (t^2 + t^4)dt$
$$= \frac{t^3}{3} + \frac{t^5}{5} + C = \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C$$

Q. 8 $\int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}} dx$

Sol. Let $\quad I = \int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}} dx = \int \frac{(\sin x + \cos x)}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}} dx$
$$= \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx = \int 1\, dx = x + C$$

Q. 9 $\int \sqrt{1+\sin x}\, dx$

Sol. Let $\quad I = \int \sqrt{1+\sin x}\, dx$
$$= \int \sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\, dx \qquad \left[\because \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1\right]$$
$$= \int \sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} dx = \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) dx$$
$$= -\cos\frac{x}{2} \cdot 2 + \sin\frac{x}{2} \cdot 2 + C = -2\cos\frac{x}{2} + 2\sin\frac{x}{2} + C$$

Q. 10 $\int \frac{x}{\sqrt{x}+1}dx$

Sol. Let $$I = \int \frac{x}{\sqrt{x}+1}dx$$

Put $$\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt$$

$\Rightarrow$ $$dx = 2\sqrt{x}\,dt$$

$\therefore$ $$I = 2\int\left(\frac{x\sqrt{x}}{t+1}\right)dt = 2\int\frac{t^2\cdot t}{t+1}dt = 2\int\frac{t^3}{t+1}dt$$

$$= 2\int\frac{t^3+1-1}{t+1}dt = 2\int\frac{(t+1)(t^2-t+1)}{t+1}dt - 2\int\frac{1}{t+1}dt$$

$$= 2\int(t^2-t+1)\,dt - 2\int\frac{1}{t+1}dt$$

$$= 2\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|(t+1)|\right] + C$$

$$= 2\left[\frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log|(\sqrt{x}+1)|\right] + C$$

Q. 11 $\int\sqrt{\frac{a+x}{a-x}}\,dx$

Thinking Process

Here, put $x = a\cos 2\theta$ and also use the formula i.e., $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$, to get the solution.

Sol. Let $$I = \int\sqrt{\frac{a+x}{a-x}}\,dx$$

Put $$x = a\cos 2\theta$$

$\Rightarrow$ $$dx = -a\cdot\sin 2\theta\cdot 2\cdot d\theta$$

$\therefore$ $$I = -2\int\sqrt{\frac{a+a\cos 2\theta}{a-a\cos 2\theta}}\cdot a\sin 2\theta\, d\theta$$

$$\left[\because \cos 2\theta = \frac{x}{a} \Rightarrow 2\theta = \cos^{-1}\frac{x}{a} \Rightarrow \theta = \frac{1}{2}\cos^{-1}\frac{x}{a}\right]$$

$$= -2a\int\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\sin 2\theta\, d\theta = -2a\int\sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}\sin 2\theta\, d\theta$$

$$= -2a\int\cot\theta\cdot\sin 2\theta\, d\theta = -2a\int\frac{\cos\theta}{\sin\theta}\cdot 2\sin\theta\cdot\cos\theta\, d\theta$$

$$= -4a\int\cos^2\theta\, d\theta = -2a\int(1+\cos 2\theta)\,d\theta$$

$$= -2a\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= -2a\left[\frac{1}{2}\cos^{-1}\frac{x}{a} + \frac{1}{2}\sqrt{1-\frac{x^2}{a^2}}\right] + C$$

$$= -a\left[\cos^{-1}\left(\frac{x}{a}\right) + \sqrt{1-\frac{x^2}{a^2}}\right] + C$$

Alternate Method

Let $$I = \int \sqrt{\frac{a+x}{a-x}}\, dx = \int \sqrt{\frac{(a+x)(a+x)}{(a-x)(a+x)}}\, dx$$

$$= \int \frac{(a+x)}{\sqrt{a^2 - x^2}} dx$$

$$I = \int \frac{a}{\sqrt{a^2 - x^2}} + \int \frac{x}{\sqrt{a^2 - x^2}} dx$$

$\therefore$ $$I = I_1 + I_2 \quad \ldots(i)$$

Now, $$I_1 = \int \frac{a}{\sqrt{a^2 - x^2}} = a\sin^{-1}\left(\frac{x}{a}\right) + C_1$$

and $$I_2 = \int \frac{x}{\sqrt{a^2 - x^2}} dx$$

Put $$a^2 - x^2 = t^2 \Rightarrow -2x\,dx = 2t\,dt$$

$\therefore$ $$I_2 = -\int \frac{t}{t} dte = -\int 1 dt$$

$$= -t + C_2 = -\sqrt{a^2 - x^2} + C_2$$

$\therefore$ $$I = a\sin^{-1}\left(\frac{x}{a}\right) + C_1 - \sqrt{a^2 - x^2} + C_2 \quad [\because t^2 = a^2 - x^2]$$

$$I = a\sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2} + C \quad [\because C = C_1 + C_2]$$

Q. 12 $\int \frac{x^{1/2}}{1 + x^{3/4}} dx$

Sol. Let $$I = \int \frac{x^{1/2}}{1 + x^{3/4}} dx$$

Put $$x = t^4 \Rightarrow dx = 4t^3 dt$$

$\therefore$ $$I = 4\int \frac{t^2(t^3)}{1+t^3} dt = 4\int \left(t^2 - \frac{t^2}{1+t^3}\right) dt$$

$$I = 4\int t^2 dt - 4\int \frac{t^2}{1+t^3} dt$$

$$I = I_1 - I_2$$

$$I_1 = 4\int t^2 dt = 4 \cdot \frac{t^3}{3} + C_1 = \frac{4}{3} x^{3/4} + C_1$$

Now, $$I_2 = 4\int \frac{t^2}{1+t^3} dt$$

Again, put $$1 + t^3 = z \Rightarrow 3t^2 dt = dz$$

$\Rightarrow$ $$t^2 dt = \frac{1}{3} dz = \frac{4}{3}\int \frac{1}{z} dz$$

$$= \frac{4}{3}\log|z| + C_2 = \frac{4}{3}\log|(1+t^3)| + C_2$$

$$= \frac{4}{3}\log|(1 + x^{3/4})| + C_2$$

$\therefore$ $$I = \frac{4}{3}x^{3/4} + C_1 - \frac{4}{3}\log|(1 + x^{3/4})| - C_2$$

$$= \frac{4}{3}x^{3/4} - \log|(1 + x^{3/4}) + C \quad [\because C = C_1 - C_2]$$

Q. 13 $\int \frac{\sqrt{1+x^2}}{x^4}dx$

Sol. Let $$I = \int \frac{\sqrt{1+x^2}}{x^4}dx = \int \frac{\sqrt{1+x^2}}{x} \cdot \frac{1}{x^3}dx$$
$$= \int \sqrt{\frac{1+x^2}{x^2}} \cdot \frac{1}{x^3}dx = \int \sqrt{\frac{1}{x^2}+1} \cdot \frac{1}{x^3}dx$$

Put $$1+\frac{1}{x^2} = t^2 \Rightarrow \frac{-2}{x^3}dx = 2t\,dt$$

$\Rightarrow$ $$-\frac{1}{x^3} = t\,dt$$

$\therefore$ $$I = -\int t^2 dt = -\frac{t^3}{3} + C = -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2} + C$$

Q. 14 $\int \frac{dx}{\sqrt{16-9x^2}}$

Thinking Process

First of all concert the expression in form of $\frac{1}{\sqrt{a^2-x^2}}$, *then use the formula,*

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

Sol. Let $$I = \int \frac{dx}{\sqrt{16-9x^2}} = \int \frac{dx}{\sqrt{(4)^2-(3x)^2}}dx \quad = \frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right) + C$$

Q. 15 $\int \frac{dt}{\sqrt{3t-2t^2}}$

Sol. Let $$I = \int \frac{dt}{\sqrt{3t-2t^2}} = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left(t^2-\frac{3}{2}t\right)}}$$
$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left[\left(t^2-2\cdot\frac{1}{2}\cdot\frac{3}{2}t\right)+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]}}$$
$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left[\left(t-\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]}}$$
$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2-\left(t-\frac{3}{4}\right)^2}}$$
$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{t-\frac{3}{4}}{\frac{3}{4}}\right) + C = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4t-3}{3}\right) + C$$

Q. 16 $\int \frac{3x-1}{\sqrt{x^2+9}} dx$

Thinking Process

First of all convert to the given integral into two parts, then by using formula i.e., $\int \frac{1}{\sqrt{a^2+x^2}} = \log|x+\sqrt{a^2+x^2}| + C$, *get the desired result.*

Sol. Let

$$I = \int \frac{3x-1}{\sqrt{x^2+9}} dx$$

$$I = \int \frac{3x}{\sqrt{x^2+9}} dx - \int \frac{1}{\sqrt{x^2+9}} dx$$

$$I = I_1 - I_2$$

Now, $$I_1 = \int \frac{3x}{\sqrt{x^2+9}}$$

Put $x^2 + 9 = t^2 \Rightarrow 2x\,dx = 2t\,dt \Rightarrow x\,dx = t\,dt$

$\therefore$ $$I_1 = 3\int \frac{t}{t} dt$$

$$= 3\int dt = 3t + C_1 = 3\sqrt{x^2+9} + C_1$$

and $$I_2 = \int \frac{1}{\sqrt{x^2+9}} dx = \int \frac{1}{\sqrt{x^2+(3)^2}} dx$$

$$= \log|x + \sqrt{x^2+9}| + C_2$$

$\therefore$ $$I = 3\sqrt{x^2+9} + C_1 - \log|x+\sqrt{x^2+9}| - C_2$$

$$= 3\sqrt{x^2+9} - \log|x+\sqrt{x^2+9}| + C \qquad [\because C = C_1 - C_2]$$

Q. 17 $\int \sqrt{5-2x+x^2} dx$

Thinking Process

First of all convert the given expression into $\sqrt{x^2+a^2}$ *form, then use the formula i.e.,* $\int \sqrt{x^2+a^2} dx = \frac{1}{2} x\sqrt{x^2+a^2} + \frac{a^2}{2} \log|x+\sqrt{x^2+a^2}| + C$.

Sol. Let

$$I = \int \sqrt{5-2x+x^2} dx = \int \sqrt{x^2-2x+1+4} dx$$

$$= \int \sqrt{(x-1)^2+(2)^2} dx = \int \sqrt{(2)^2+(x-1)^2} dx$$

$$= \frac{x-1}{2}\sqrt{2^2+(x-1)^2} + 2\log|x-1+\sqrt{2^2+(x-1)^2}| + C$$

$$= \frac{x-1}{2}\sqrt{5-2x+x^2} + 2\log|x-1+\sqrt{5-2x+x^2}| + C$$

Q. 18 $\int \frac{x}{x^4-1}dx$

Sol. Let $$I = \int \frac{x}{x^4-1}dx$$

Put $$x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{1}{2}dt$$

$\therefore$ $$I = \frac{1}{2}\int \frac{dt}{t^2-1} = \frac{1}{2}\cdot\frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C \qquad \left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

$$= \frac{1}{4}[\log|x^2-1| - \log|x^2+1|] + C$$

Q. 19 $\int \frac{x^2}{1-x^4}dx$

Thinking Process

Here, use $\int \frac{1}{1+x^2}dx = \tan^{-1}x + C$ *and* $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{1+x}{1-x}\right| + C$, *to solve this problem.*

Sol. Let $$I = \int \frac{x^2}{1-x^4}dx$$

$$= \int \frac{\left(\frac{1}{2}+\frac{x^2}{2}-\frac{1}{2}+\frac{x^2}{2}\right)}{(1-x^2)(1+x^2)}dx \qquad [\because a^2-b^2 = (a+b)(a-b)]$$

$$= \int \frac{\frac{1}{2}(1+x^2) - \frac{1}{2}(1-x^2)}{(1-x^2)(1+x^2)}dx$$

$$= \int \frac{\frac{1}{2}(1+x^2)}{(1-x^2)(1+x^2)}dx - \frac{1}{2}\int \frac{(1-x^2)}{(1-x^2)(1+x^2)}dx$$

$$= \frac{1}{2}\int \frac{1}{1-x^2}dx - \frac{1}{2}\int \frac{1}{1+x^2}dx = \frac{1}{2}\cdot\frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C_1 - \frac{1}{2}\tan^{-1}x + C_2$$

$$= \frac{1}{4}\log\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\tan^{-1}x + C \qquad [\because C = C_1 + C_2]$$

Q. 20 $\int \sqrt{2ax - x^2}\,dx$

Sol. Let $$I = \int \sqrt{2ax-x^2}\,dx = \int \sqrt{-(x^2-2ax)}\,dx$$

$$= \int \sqrt{-(x^2-2ax+a^2-a^2)}\,dx = \int \sqrt{-(x-a)^2 - a^2}\,dx$$

$$= \int \sqrt{a^2-(x-a)^2}\,dx$$

$$= \frac{x-a}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$$

$$= \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$$

Q. 21 $\int \frac{\sin^{-1} x}{(1-x^2)^{3/4}} dx$

Sol. Let $I = \int \frac{\sin^{-1} x}{(1-x^2)^{3/4}} dx = \int \frac{\sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} dx$

Put $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$

and $x = \sin t \Rightarrow 1 - x^2 = \cos^2 t$

$\Rightarrow$ $\cos t = \sqrt{1-x^2}$

$\therefore$
$$I = \int \frac{t}{\cos^2 t} dt = \int t \cdot \sec^2 t dt$$
$$= t \cdot \int \sec^2 t dt - \int \left(\frac{d}{dt} t \cdot \int \sec^2 t\, dt \right) dt$$
$$= t \cdot \tan t - \int 1 \cdot \tan t\, dt$$
$$= t \tan t + \log|\cos t| + C \qquad \left[\because \int \tan x dx = -\log|\cos x| + C\right]$$
$$= \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} + \log|\sqrt{1-x^2}| + C$$

Q. 22 $\int \frac{(\cos 5x + \cos 4x)}{1 - 2\cos 3x} dx$

Sol. Let
$$I = \int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx = \int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{1 - 2\left(2\cos^2 \frac{3x}{2} - 1\right)} dx$$
$$\left[\because \cos C + \cos D = 2\cos \frac{C+D}{2} . \cos \frac{C-D}{2} \text{ and } \cos 2x = 2\cos^2 x - 1\right]$$

$\therefore$
$$I = \int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{3 - 4\cos^2 \frac{3x}{2}} dx = -\int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{4\cos^2 \frac{3x}{2} - 3} dx$$
$$= -\int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{4\cos^3 \frac{3x}{2} - 3\cos \frac{3x}{2}} dx \qquad \left[\text{multiply and divide by } \cos \frac{3x}{2}\right]$$
$$= -\int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{\cos 3 \cdot \frac{3x}{2}} dx = -\int 2\cos \frac{3x}{2} \cdot \cos \frac{x}{2} dx$$
$$= -\int \left\{ \cos\left(\frac{3x}{2} + \frac{x}{2}\right) + \cos\left(\frac{3x}{2} - \frac{x}{2}\right) \right\} dx$$
$$= -\int (\cos 2x + \cos x) dx$$
$$= -\left[\frac{\sin 2x}{2} + \sin x\right] + C$$
$$= -\frac{1}{2}\sin 2x - \sin x + C$$

Q. 23 $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$

Thinking Process

Use $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ *and* $\sec^2 x = 1 + \tan^2 x$, $\text{cosec}^2 x = 1 + \cot^2 x$ *to solve the above problem.*

Sol. Let
$$I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cdot \cos^2 x} dx$$
$$= \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)}{\sin^2 x \cdot \cos^2 x} dx$$
$$= \int \frac{\sin^4 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^4 x}{\sin^2 x \cdot \cos^2 x} dx - \int \frac{\sin^2 x \cos^2 x}{\sin^2 x \cdot \cos^2 x} dx$$
$$= \int \tan^2 x \, dx + \int \cot^2 x \, dx - \int 1 dx$$
$$= \int (\sec^2 x - 1) \, dx + \int (\text{cosec}^2 x - 1) \, dx - \int 1 dx$$
$$= \int \sec^2 x dx + \int \text{cosec}^2 x dx - 3\int dx$$
$$I = \tan x - \cot x - 3x + C$$

Q. 24 $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$

Sol. Let
$$I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx = \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}$$

Put
$$x^{3/2} = t \Rightarrow \frac{3}{2} x^{1/2} dx = dt$$

$\therefore$
$$I = \frac{2}{3} \int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}} = \frac{2}{3} \sin^{-1} \frac{t}{a^{3/2}} + C$$
$$= \frac{2}{3} \sin^{-1} \frac{x^{3/2}}{a^{3/2}} + C = \frac{2}{3} \sin^{-1} \sqrt{\frac{x^3}{a^3}} + C$$

Q. 25 $\int \frac{\cos x - \cos 2x}{1 - \cos x} dx$

Thinking Process

Apply the formula, $\cos C - \cos D = 2 \sin \frac{C+D}{2} \cdot \sin \frac{D-C}{2}$ *and* $\cos x = 1 - 2\sin^2 \frac{x}{2}$ *to solve it.*

Sol. Let
$$I = \int \frac{\cos x - \cos 2x}{1 - \cos x} dx = \int \frac{2 \sin \frac{3x}{2} \cdot \sin \frac{x}{2}}{1 - 1 + 2\sin^2 \frac{x}{2}} dx$$
$$= 2 \int \frac{\sin \frac{3x}{2} \cdot \sin \frac{x}{2}}{2 \sin^2 \frac{x}{2}} dx = \int \frac{\sin \frac{3x}{2}}{\sin \frac{x}{2}} dx$$

$$= \int \frac{3\sin\frac{x}{2} - 4\sin^3\frac{x}{2}}{\sin\frac{x}{2}}dx \qquad [\because \sin 3x = 3\sin x - 4\sin^3 x]$$

$$= 3\int dx - 4\int \sin^2\frac{x}{2}dx = 3\int dx - 4\int \frac{1-\cos x}{2}dx$$

$$= 3\int dx - 2\int dx + 2\int \cos x\, dx$$

$$= \int dx + 2\int \cos x\, dx = x + 2\sin x + C = 2\sin x + x + C$$

Q. 26 $\int \frac{dx}{x\sqrt{x^4 - 1}}$

Sol. Let $$I = \int \frac{dx}{x\sqrt{x^4 - 1}}$$

Put $$x^2 = \sec\theta \Rightarrow \theta = \sec^{-1} x^2$$

$\Rightarrow$ $$2x\, dx = \sec\theta \cdot \tan\theta\, d\theta$$

$\therefore$ $$I = \frac{1}{2}\int \frac{\sec\theta \cdot \tan\theta}{\sec\theta \tan\theta}d\theta = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\sec^{-1}(x^2) + C$$

Q. 27 $\int_0^2 (x^2 + 3)dx$

Thinking Process

$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots + f\{a + (n-1)h\}]$, *where* $h = \frac{b-a}{n} \to 0$ *as* $n \to \infty$.

Sol. Let $$I = \int_0^2 (x^2 + 3)\, dx$$

Here, $$a = 0, b = 2 \text{ and } h = \frac{b-a}{n} = \frac{2-0}{n}$$

$\Rightarrow$ $$h = \frac{2}{n} \Rightarrow nh = 2 \Rightarrow f(x) = (x^2 + 3)$$

Now, $$\int_0^2 (x^2 + 3)\, dx = \lim_{h\to 0} h\,[f(0) + f(0+h) + f(0+2h) + \ldots + f\{0 + (n-1)h\}] \quad \ldots(i)$$

$\because$ $$f(0) = 3$$

$\Rightarrow$ $$f(0+h) = h^2 + 3,\ f(0+2h) = 4h^2 + 3 = 2^2h^2 + 3$$

$$f[0 + (n-1)h] = (n^2 - 2n + 1)h + 3 = (n-1)^2 h + 3$$

From Eq. (i),

$$\int_0^2 (x^2 + 3)dx = \lim_{h\to 0} h[3 + h^2 + 3 + 2^2h^2 + 3 + 3^2h^2 + 3 + \ldots + (n-1)^2h^2 + 3]$$

$$= \lim_{h\to 0} h[3n + h^2\{1^2 + 2^2 + \ldots + (n-1)^2\}]$$

$$= \lim_{h\to 0} h\left[3n + h^2\left(\frac{(n-1)(2n-2+1)(n-1+)}{6}\right)\right] \quad \left[\because \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{h\to 0} h\left[3n + h^2\left(\frac{(n^2-n)(2n-1)}{6}\right)\right]$$

$$= \lim_{h\to 0} h\left[3n + \frac{h^2}{6}(2n^3 - n^2 - 2n^2 + n)\right]$$

$$= \lim_{h\to 0}\left[3nh + \frac{2n^3h^3 - 3n^2h^2\cdot h + nh\cdot h^2}{6}\right]$$

$$= \lim_{h\to 0}\left[3\cdot 2 + \frac{2\cdot 8 - 3\cdot 2^2\cdot h + 2\cdot h^2}{6}\right] = \lim_{h\to 0}\left[6 + \frac{16 - 12h + 2h^2}{6}\right]$$

$$= 6 + \frac{16}{6} = 6 + \frac{8}{3} = \frac{26}{3}$$

Q. 28 $\int_0^2 e^x dx$

Sol. Let $I = \int_0^2 e^x\, dx$

Here, $a = 0$ and $b = 2$

$\therefore$ $h = \frac{b-a}{n}$

$\Rightarrow$ $nh = 2$ and $f(x) = e^x$

Now, $\int_0^2 e^x dx = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \dots + f\{0 + (n-1)h\}]$

$\therefore$ $I = \lim_{h\to 0} h[1 + e^h + e^{2h} + \dots + e^{(n-1)h}]$

$$= \lim_{h\to 0} h\left[\frac{1\cdot (e^h)^n - 1}{e^h - 1}\right] = \lim_{h\to 0} h\left(\frac{e^{nh} - 1}{e^h - 1}\right)$$

$$= \lim_{h\to 0} h\left(\frac{e^2 - 1}{e^h - 1}\right)$$

$$= e^2 \lim_{h\to 0}\frac{h}{e^h - 1} - \lim_{h\to 0}\frac{h}{e^h - 1} \qquad \left[\because \lim_{h\to 0}\frac{h}{e^h - 1} = 1\right]$$

$$= e^2 - 1 = e^2 - 1$$

Evaluate the following questions.

Q. 29 $\int_0^1 \frac{dx}{e^x + e^{-x}}$

Sol. Let $I = \int_0^1 \frac{dx}{e^x + e^{-x}} = \int_0^1 \frac{e^x}{1 + e^{2x}}dx$

Put $e^x = t$

$\Rightarrow$ $e^x\, dx = dt$

$\therefore$ $I = \int_1^e \frac{dt}{1 + t^2} = [\tan^{-1} t]_1^e$

$$= \tan^{-1} e - \tan^{-1} 1$$

$$= \tan^{-1} e - \frac{\pi}{4}$$

Q. 30 $\int_0^{\pi/2} \dfrac{\tan x}{1+m^2\tan^2 x}\,dx$

Sol. Let

$$I = \int_0^{\pi/2} \frac{\tan x\,dx}{1+m^2\tan^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\dfrac{\sin x}{\cos x}}{1+m^2\cdot\dfrac{\sin^2 x}{\cos^2 x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\dfrac{\sin x}{\cos x}}{\dfrac{\cos^2 x + m^2\sin^2 x}{\cos^2 x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin x\cos x\,dx}{1-\sin^2 x + m^2\sin^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin x\cos x}{1-\sin^2 x(1-m^2)}\,dx$$

Put $\sin^2 x = t$

$\Rightarrow$ $2\sin x\cos x\,dx = dt$

$\therefore$ $I = \dfrac{1}{2}\displaystyle\int_0^1 \frac{dt}{1-t(1-m^2)}$

$$= \frac{1}{2}\left[-\log|1-t(1-m^2)|\cdot\frac{1}{1-m^2}\right]_0^1$$

$$= \frac{1}{2}\left[-\log|1-1+m^2|\cdot\frac{1}{1+m^2} + \log|1|\cdot\frac{1}{1-m^2}\right]$$

$$= \frac{1}{2}\left[-\log|m^2|\cdot\frac{1}{1-m^2}\right] = \frac{2}{2}\cdot\frac{\log m}{(m^2-1)}$$

$$= \log\frac{m}{m^2-1}$$

Q. 31 $\int_1^2 \dfrac{dx}{\sqrt{(x-1)(2-x)}}$

Thinking Process

First of all convert the given function into $\dfrac{1}{\sqrt{a^2-x^2}}$ *form, then apply the formula i.e.,*

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C.$$

Sol. Let

$$I = \int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}} = \int_1^2 \frac{dx}{\sqrt{2x-x^2-2+x}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-(x^2-3x+2)}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-\left[x^2 - 2\cdot\frac{3}{2}x + \left(\frac{3}{2}\right)^2 + 2 - \frac{9}{4}\right]}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-\left\{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\}}}$$

$$= \int_1^2 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} = \left[\sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{1}{2}}\right)\right]_1^2$$

$$= [\sin^{-1}(2x - 3)]_1^2 = \sin^{-1} 1 - \sin^{-1}(-1)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} \qquad \left[\because \sin\frac{\pi}{2} = 1 \text{ and } \sin(-\theta) = -\sin\theta\right]$$

$$= \pi$$

Q. 32 $\int_0^1 \frac{x}{\sqrt{1 + x^2}}\, dx$

Sol. Let $I = \int_0^1 \frac{x}{\sqrt{1 + x^2}}\, dx$

Put $1 + x^2 = t^2$

$\Rightarrow$ $2x\, dx = 2t dt$

$\Rightarrow$ $x\, dx = t dt$

$\therefore$ $I = \int_1^{\sqrt{2}} \frac{t dt}{t}$

$= [t]_1^{\sqrt{2}} = \sqrt{2} - 1$

Q. 33 $\int_0^\pi x \sin x \cos^2 x\, dx$

Thinking Process

Here, use the property i.e., $\int_0^a f(x)dx = \int_0^a (a - x)dx$ and

$\sin(\pi - x) = \sin x, \cos(\pi - x) = \cos x$.

Sol. Let $I = \int_0^\pi x \sin x \cos^2 x\, dx$ ...(i)

and $I = \int_0^\pi (\pi - x)\sin(\pi - x)\cos^2(\pi - x)dx$

$\Rightarrow$ $I = \int_0^\pi (\pi - x)\sin x \cos^2 x\, dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^\pi \pi \sin x \cos^2 x\, dx$$

Put $\cos x = t$

$\Rightarrow$ $-\sin x\, dx = dt$

As $x \to 0$, then $t \to 1$
and $x \to \pi$, then $t \to -1$

$\therefore \quad I = -\pi\int_1^{-1} t^2\, dt \Rightarrow I = -\pi\left[\frac{t^3}{3}\right]_1^{-1}$

$\Rightarrow \quad 2I = -\frac{\pi}{3}[-1-1] \Rightarrow 2I = \frac{2\pi}{3}$

$\therefore \quad I = \frac{\pi}{3}$

Q. 34 $\displaystyle\int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Sol. Let $\displaystyle I = \int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put $x = \sin\theta$
$\Rightarrow dx = \cos\theta\, d\theta$
As $x \to 0$, then $\theta \to 0$
and $x \to \frac{1}{2}$, then $\theta \to \frac{\pi}{6}$

$$\therefore \quad I = \int_0^{\pi/6} \frac{\cos\theta}{(1+\sin^2\theta)\cos\theta}\, d\theta = \int_0^{\pi/6} \frac{1}{1+\sin^2\theta}\, d\theta$$

$$= \int_0^{\pi/6} \frac{1}{\cos^2\theta\,(\sec^2\theta + \tan^2\theta)}\, d\theta$$

$$= \int_0^{\pi/6} \frac{\sec^2\theta}{\sec^2\theta + \tan^2\theta}\, d\theta$$

$$= \int_0^{\pi/6} \frac{\sec^2\theta}{1+\tan^2\theta + \tan^2\theta}\, d\theta$$

$$= \int_0^{\pi/6} \frac{\sec^2\theta}{1+2\tan^2\theta}\, d\theta$$

Again, put $\tan\theta = t$
$\Rightarrow \sec^2\theta\, d\theta = dt$
As $\theta \to 0$, then $t \to 0$
and $\theta \to \frac{\pi}{6}$, then $t \to \frac{1}{\sqrt{3}}$

$$\therefore \quad I = \int_0^{1/\sqrt{3}} \frac{dt}{1+2t^2} = \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{dt}{\left(\frac{1}{\sqrt{2}}\right)^2 + t^2}$$

$$= \frac{1}{2}\cdot\frac{1}{1/\sqrt{2}}\left[\tan^{-1}\frac{t}{\frac{1}{\sqrt{2}}}\right]_0^{1/\sqrt{3}} = \frac{1}{\sqrt{2}}[\tan^{-1}(\sqrt{2}t)]_0^{1/\sqrt{3}}$$

$$= \frac{1}{\sqrt{2}}\left[\tan^{-1}\sqrt{\frac{2}{3}} - 0\right] = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

Long Answer Type Questions

Q. 35 $\displaystyle\int \frac{x^2}{x^4 - x^2 - 12}\, dx$

Thinking Process

Use $\dfrac{px+q}{(x-a)(x-b)} = \dfrac{A}{(x-a)} + \dfrac{B}{x-b}$, *where* $a \neq b$ *, then compare the coefficient of* x *to get the value of A and B.*

Sol. Let

$$I = \int \frac{x^2}{x^4 - x^2 - 12}\, dx$$

$$= \int \frac{x^2}{x^4 - 4x^2 + 3x^2 - 12} dx$$

$$= \int \frac{x^2 dx}{x^2(x^2 - 4) + 3(x^2 - 4)}$$

$$= \int \frac{x^2 dx}{(x^2 - 4)(x^2 + 3)}$$

Now, $\dfrac{x^2}{(x^2 - 4)(x^2 + 3)}$ [let $x^2 = t$]

$$\Rightarrow \quad \frac{t}{(t-4)(t+3)} = \frac{A}{t-4} + \frac{B}{t+3}$$

$$\Rightarrow \quad t = A(t+3) + B(t-4)$$

On comparing the coefficient of t on both sides, we get

$$A + B = 1 \quad \ldots\text{(i)}$$

$$\Rightarrow \quad 3A - 4B = 0 \quad \ldots\text{(ii)}$$

$$\Rightarrow \quad 3(1 - B) - 4B = 0$$

$$\Rightarrow \quad 3 - 3B - 4B = 0$$

$$\Rightarrow \quad 7B = 3$$

$$\Rightarrow \quad B = \frac{3}{7}$$

If $B = \dfrac{3}{7}$, then $A + \dfrac{3}{7} = 1$

$$\Rightarrow \quad A = 1 - \frac{3}{7} = \frac{4}{7}$$

$$\frac{x^2}{(x^2 - 4)(x^2 + 3)} = \frac{4}{7(x^2 - 4)} + \frac{3}{7(x^2 + 3)}$$

$$\therefore \quad I = \frac{4}{7}\int \frac{1}{x^2 - (2)^2} dx + \frac{3}{7}\int \frac{1}{x^2 + (\sqrt{3})^2} dx$$

$$= \frac{4}{7}\cdot\frac{1}{2\cdot 2}\log\left|\frac{x-2}{x+2}\right| + \frac{3}{7}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

$$= \frac{1}{7}\log\left|\frac{x-2}{x+2}\right| + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

Q. 36 $\int \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$

Sol. Let
$$I = \int \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$$

Now,
$$\frac{x^2}{(x^2+a^2)(x^2+b^2)} \qquad [\text{let } x^2 = t]$$
$$= \frac{t}{(t+a^2)(t+b^2)} = \frac{A}{(t+a^2)} + \frac{B}{(t+b^2)}$$
$$t = A(t+b^2) + B(t+a^2)$$

On comparing the coefficient of t, we get
$$A + B = 1 \qquad \ldots(\text{i})$$
$$b^2A + a^2B = 0 \qquad \ldots(\text{ii})$$
$$\Rightarrow \quad b^2(1-B) + a^2B = 0$$
$$\Rightarrow \quad b^2 - b^2B + a^2B = 0$$
$$\Rightarrow \quad b^2 + (a^2 - b^2)B = 0$$
$$\Rightarrow \quad B = \frac{-b^2}{a^2-b^2} = \frac{b^2}{b^2-a^2}$$

From Eq. (i),
$$A + \frac{b^2}{b^2-a^2} = 1$$
$$\Rightarrow \quad A = \frac{b^2-a^2-b^2}{b^2-a^2} = \frac{-a^2}{b^2-a^2}$$
$$\therefore \quad I = \int \frac{-a^2}{(b^2-a^2)(x^2+a^2)}dx + \int \frac{b^2}{b^2-a^2}\cdot\frac{1}{x^2+b^2}dx$$
$$= \frac{-a^2}{(b^2-a^2)}\int \frac{1}{x^2+a^2}dx + \frac{b^2}{b^2-a^2}\int \frac{1}{x^2+b^2}dx$$
$$= \frac{-a^2}{b^2-a^2}\cdot\frac{1}{a}\tan^{-1}\frac{x}{a} + \frac{b^2}{b^2-a^2}\cdot\frac{1}{b}\tan^{-1}\frac{x}{b}$$
$$= \frac{1}{b^2-a^2}\left[-a\tan^{-1}\frac{x}{a} + b\tan^{-1}\frac{x}{b}\right]$$
$$= \frac{1}{a^2-b^2}\left[a\tan^{-1}\frac{x}{a} - b\tan^{-1}\frac{x}{b}\right]$$

Q. 37 $\int_0^\pi \frac{x}{1+\sin x}$

Sol. Let
$$I = \int_0^\pi \frac{x}{1+\sin x}dx \qquad \ldots(\text{i})$$

and
$$I = \int_0^\pi \frac{\pi - x}{1+\sin(\pi - x)}dx = \int_0^\pi \frac{\pi - x}{1+\sin x}dx \qquad \ldots(\text{ii})$$

On adding Eqs. (i) and (ii), we get
$$2I = \pi\int_0^\pi \frac{1}{1+\sin x}dx$$
$$= \pi\int_0^\pi \frac{(1-\sin x)\,dx}{(1+\sin x)(1-\sin x)}$$

$$= \pi \int_0^{\pi} \frac{(1 - \sin x)\,dx}{\cos^2 x}$$

$$= \pi \int_0^{\pi} (\sec^2 x - \tan x \cdot \sec x)\,dx$$

$$= \pi \int_0^{\pi} \sec^2 x\,dx - \pi \int_0^{\pi} \sec x\, x \cdot \tan x\,dx$$

$$= \pi[\tan x]_0^{\pi} - \pi[\sec x]_0^{\pi}$$

$$= \pi[\tan x - \sec x]_0^{\pi}$$

$$= \pi[\tan \pi - \sec \pi - \tan 0 - \sec 0]$$

$$\Rightarrow \quad 2I = \pi[0 + 1 - 0 + 1]$$

$$2I = 2\pi$$

$$\therefore \quad I = \pi$$

Q. 38 $\int \frac{2x - 1}{(x - 1)(x + 2)(x - 3)}\,dx$

Thinking Process

Apply $\frac{px + q}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$, then get the values of A, B and C and use $\int \frac{1}{x}dx = \log|x| + C$.

Sol. Let $$I = \int \frac{(2x - 1)}{(x - 1)(x + 2)(x - 3)}dx$$

Now, $$\frac{2x - 1}{(x - 1)(x + 2)(x - 3)} = \frac{A}{(x - 1)} + \frac{B}{(x + 2)} + \frac{C}{(x - 3)}$$

$$\Rightarrow \quad 2x - 1 = A(x + 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x + 2)$$

Put $x = 3$, then

$$6 - 1 = C(3 - 1)(3 + 2)$$

$$\Rightarrow \quad 5 = 10C \quad \Rightarrow \quad C = \frac{1}{2}$$

Again, put $x = 1$, then

$$2 - 1 = A(1 + 2)(1 - 3)$$

$$\Rightarrow \quad 1 = -6A \quad \Rightarrow \quad A = -\frac{1}{6}$$

Now, put $x = -2$, then

$$-4 - 1 = B(-2 - 1)(-2 - 3)$$

$$\Rightarrow \quad -5 = 15B \quad \Rightarrow \quad B = -\frac{1}{3}$$

$$\therefore \quad I = -\frac{1}{6}\int \frac{1}{x - 1}dx - \frac{1}{3}\int \frac{1}{x + 2}dx + \frac{1}{2}\int \frac{1}{x - 3}dx$$

$$= -\frac{1}{6}\log|(x - 1)| - \frac{1}{3}\log|(x + 2)| + \frac{1}{2}\log|(x - 3)| + C$$

$$= -\log|(x - 1)|^{1/6} - \log|(x + 2)|^{1/3} + \log|(x - 3)|^{1/2} + C$$

$$= \log\left|\frac{\sqrt{x - 3}}{(x - 1)^{1/6}(x + 2)^{1/3}}\right| + C$$

Q. 39 $\int e^{\tan^{-1} x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

Sol. Let
$$I = \int e^{\tan^{-1} x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$$
$$= \int e^{\tan^{-1} x}\left(\frac{1+x^2}{1+x^2}+\frac{x}{1+x^2}\right)dx$$
$$= \int e^{\tan^{-1} x}dx + \int \frac{x\, e^{\tan^{-1} x}}{1+x^2}dx$$
$$I = I_1 + I_2 \quad \ldots\text{(i)}$$

Now,
$$I_2 = \int \frac{x\, e^{\tan^{-1} x}}{1+x^2}dx$$

Put $\tan^{-1} x = t \Rightarrow x = \tan t$

$\Rightarrow$ $\frac{1}{1+x^2}dx = dt$

$\therefore$
$$I = \int \underset{\text{I}}{\tan t}\cdot \underset{\text{II}}{e^t}\, dt$$
$$= \tan t \cdot e^t - \int \sec^2 t \cdot e^t dt + C$$
$$= \tan t \cdot e^t - \int (1+\tan^2 t)e^t dt + C \qquad [\because \sec^2\theta = 1+\tan^2\theta]$$
$$I_2 = \tan t \cdot e^t - \int (1+x^2)\frac{e^{\tan^{-1}x}}{1+x^2}dx + C$$
$$I_2 = \tan t \cdot e^t - \int e^{\tan^{-1} x}dx + C$$

$\therefore$
$$I = \int e^{\tan^{-1} x}dx + \tan t \cdot e^t - \int e^{\tan^{-1} x}dx + C$$
$$= \tan t \cdot e^t + C$$
$$= x\, e^{\tan^{-1} x} + C$$

Q. 40 $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$

Thinking Process

First of all put $x = \tan^2\theta$ and convert the given expression into two parts, then use the formulae for integration by part i.e., $\int \text{I}\cdot\text{II}\,dx = \text{I}\int \text{II}\,dx - \int\left(\frac{d}{dx}\text{I}\int \text{II}\,dx\right)dx$

Sol. Let
$$I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$$

Put
$$x = a\tan^2\theta$$

$\Rightarrow$
$$dx = 2a\tan\theta\sec^2\theta\, d\theta$$

$\therefore$
$$I = \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}(2a\tan\theta\cdot\sec^2\theta)\, d\theta$$
$$= 2a\int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\tan\theta\cdot\sec^2\theta\, d\theta$$
$$= 2a\int \sin^{-1}(\sin\theta)\tan\theta\cdot\sec^2\theta\, d\theta$$

$$= 2a\int \underset{\text{I}}{\theta}\cdot \underset{\text{II}}{\tan\theta\sec^2\theta}\, d\theta$$

$$= 2a\left[\theta\cdot\int \tan\theta\cdot\sec^2\theta\, d\theta - \int\left(\frac{d}{d\theta}\theta\cdot\int \tan\theta\cdot\sec^2\theta\, d\theta\right)d\theta\right]$$

$$\left[\begin{array}{ll} \text{Put} & \tan\theta = t \\ \Rightarrow & \sec\theta\cdot\tan\theta\cdot d\theta = dt \\ \Rightarrow \int \tan\theta\,\sec^2\theta\, d\theta & = \int t\, dt \end{array}\right]$$

$$= 2a\left[\theta\cdot\frac{\tan^2\theta}{2} - \int\frac{\tan^2\theta}{2}\, d\theta\right]$$

$$= a\theta\tan^2\theta - a\int(\sec^2\theta - 1)\, d\theta$$

$$= a\theta\cdot\tan^2\theta - a\tan\theta + a\theta + C$$

$$= a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + C$$

Q. 41 $\displaystyle\int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\, dx$

Sol. Let

$$I = \int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\, dx$$

$$= \int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^2\sqrt{1+\cos x}}\, dx$$

$$= \int_{\pi 3}^{\pi 2}\frac{1}{(1-\cos^2 x)}\, dx = \int_{\pi 3}^{\pi 2}\frac{1}{\sin^2 x}\, dx$$

$$= \int_{\pi 3}^{\pi 2}\operatorname{cosec}^2 x\, dx = [-\cot x]_{\pi/3}^{\pi/2}$$

$$= -\left[\cot\frac{\pi}{2} - \cot\frac{\pi}{3}\right] = -\left[0 - \frac{1}{\sqrt{3}}\right] = +\frac{1}{\sqrt{3}}$$

Alternate Method

Let

$$I = \int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\, dx = \int_{\pi/3}^{\pi/2}\frac{\left(2\cos^2\frac{x}{2}\right)^{1/2}}{\left(2\sin^2\frac{x}{2}\right)^{5/2}}\, dx$$

$$= \frac{\sqrt{2}}{4\sqrt{2}}\int_{\pi/3}^{\pi/2}\frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}\, dx \quad = \frac{1}{4}\int_{\pi/3}^{\pi/2}\frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}\, dx$$

Put $\sin\frac{x}{2} = t$

$\Rightarrow$ $\cos\frac{x}{2}\cdot\frac{1}{2}dx = dt$

$\Rightarrow$ $\cos\frac{x}{2}dx = 2dt$

As $\quad x \to \frac{\pi}{3}$, then $t \to \frac{1}{2}$

and $\ x \to \frac{\pi}{2}$, then $t \to \frac{1}{\sqrt{2}}$

$$\therefore \qquad I = \frac{2}{4}\int_{1/2}^{1/\sqrt{2}} \frac{dt}{t^5} = \frac{1}{2}\left[\frac{t^{-5+1}}{-5+1}\right]_{1/2}^{1/\sqrt{2}}$$

$$= -\frac{1}{8}\left[\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^4} - \frac{1}{\left(\frac{1}{2}\right)^4}\right]$$

$$= -\frac{1}{8}(4-16) = \frac{12}{8} = \frac{3}{2}$$

Note *If we integrate the trigonometric function in different ways [using different identities] then, we can get different answers.*

Q. 42 $\int e^{-3x} \cos^3 x\, dx$

Sol. Let $\quad I = \underset{\text{II}}{\int e^{-3x}} \underset{\text{I}}{\cos^3 x}\, dx$

$$= \cos^3 x \int e^{-3x}\, dx - \int\left(\frac{d}{dx}\cos^3 x \int e^{-3x}\, dx\right)dx$$

$$= \cos^3 x \cdot \frac{e^{-3x}}{-3} - \int (-3\cos^2 x)\sin x \cdot \frac{e^{-3x}}{-3}dx$$

$$= -\frac{1}{3}\cos^3 x e^{-3x} - \int \cos^2 x \sin x e^{-3x} dx$$

$$= -\frac{1}{3}\cos^3 x\, e^{-3x} - \int (1-\sin^2 x)\sin x\, e^{-3x} dx$$

$$= -\frac{1}{3}\cos^3 x e^{-3x} - \int \sin x\, e^{-3x} dx + \int \underset{\text{I}}{\sin^3 x}\, \underset{\text{II}}{e^{-3x}} dx$$

$$= -\frac{1}{3}\cos^3 x\, e^{-3x} - \int \sin x\, e^{-3x} dx + \sin^3 x \cdot \frac{e^{-3x}}{-3} - \int 3\sin^2 x \cos x \cdot \frac{e^{-3x}}{-3} dx$$

$$= -\frac{1}{3}\cos^3 x\, e^{-3x} - \int \sin x\, e^{-3x} dx - \frac{1}{3}\sin^3 x\, e^{-3x} + \int (1-\cos^2 x)\cos x\, e^{-3x} dx$$

$$I = -\frac{1}{3}\cos^3 x\, e^{-3x} - \int \underset{\text{I}}{\sin x}\, \underset{\text{II}}{e^{-3x}} - \frac{1}{3}\sin^3 x\, e^{-3x} + \int \cos x\, e^{-3x} dx - \int \cos^3 x\, e^{-3x} dx$$

$$2I = \frac{e^{-3x}}{3}[\cos^3 x + \sin^3 x] - \left[\sin x \cdot \frac{e^{-3x}}{-3} - \int \cos x \cdot \frac{e^{-3x}}{-3} dx\right] + \int \cos x\, e^{-3x} dx$$

$$2I = \frac{e^{-3x}}{-3}[\cos^3 x + \sin^3 x] + \frac{1}{3}\sin x \cdot e^{-3x} - \frac{1}{3}\int \cos x \cdot e^{-3x} dx + \int \cos x\, e^{-3x} dx$$

$$2I = \frac{e^{-3x}}{-3}[\cos^3 x + \sin^3 x] + \frac{1}{3}\sin x\, e^{-3x} + \frac{2}{3}\int \cos x\, e^{-3x} dx$$

Now, let $I_1 = \int \underset{\text{I}}{\cos x}\, \underset{\text{II}}{e^{-3x}} dx$

$$I_1 = \cos x \cdot \frac{e^{-3x}}{-3} - \int (-\sin x) \cdot \frac{e^{-3x}}{-3} dx$$

$$I_1 = \frac{-1}{3} \cos x \cdot e^{-3x} - \frac{1}{3} \int \sin x \cdot e^{-3x} dx$$

$$= -\frac{1}{3} \cos x \cdot e^{-3x} - \frac{1}{3}\left[\sin x \cdot \frac{e^{-3x}}{-3} - \int \cos x \cdot \frac{e^{-3x}}{-3} dx\right]$$

$$= -\frac{1}{3} \cos x \cdot e^{-3x} + \frac{1}{9} \sin x \cdot e^{-3x} - \frac{1}{9} \int \cos x \cdot e^{-3x} dx$$

$$I_1 + \frac{1}{9} I_1 = -\frac{1}{3} e^{-3x} \cdot \cos x + \frac{1}{9} \sin x \cdot e^{-3x}$$

$$\left(\frac{10}{9}\right) I_1 = -\frac{1}{3} e^{-3x} \cdot \cos x + \frac{1}{9} \sin x \cdot e^{-3x}$$

$$I_1 = \frac{-3}{10} e^{-3x} \cdot \cos x + \frac{1}{10} e^{-3x} \sin x$$

$$2I = -\frac{1}{3} e^{-3x} [\sin^3 x + \cos^3 x] + \frac{1}{3} \sin x \cdot e^{-3x} - \frac{3}{10} e^{-3x} \cdot \cos x + \frac{1}{10} e^{-3x} \cdot \sin x + C$$

$\therefore$
$$I = -\frac{1}{6} e^{-3x} [\sin^3 x + \cos^3 x] + \frac{13}{30} e^{-3x} \cdot \sin x - \frac{3}{10} e^{-3x} \cdot \cos x + C$$

$$\left[\begin{array}{l} \because \quad \sin 3x = 3\sin x - 4\sin^3 x \\ \text{and } \cos 3x = 4\cos^3 x - 3\cos x \end{array}\right]$$

$$= \frac{e^{-3x}}{24} [\sin 3x - \cos 3x] + \frac{3e^{-3x}}{40} [\sin x - 3\cos x] + C$$

Q. 43 $\int \sqrt{\tan x}\, dx$

Sol. Let $I = \int \sqrt{\tan x}\, dx$

Put $\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$

$\therefore$
$$I = \int t \cdot \frac{2t}{\sec^2 x} dt = 2\int \frac{t^2}{1 + t^4} dt$$

$$= \int \frac{(t^2 + 1) + (t^2 - 1)}{(1 + t^4)} dt$$

$$= \int \frac{t^2 + 1}{1 + t^4} dt + \int \frac{t^2 - 1}{1 + t^4} dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt + \int \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt$$

$$= \int \frac{1 - \left(-\frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 2} + \int \frac{1 + \left(-\frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 - 2} dt$$

Put $$u = t - \frac{1}{t} \Rightarrow du = \left(1 + \frac{1}{t^2}\right) dt$$

and $$v = t + \frac{1}{t} \Rightarrow dv = \left(1 - \frac{1}{t^2}\right) dt$$

$\therefore$ $$I = \int \frac{du}{u^2 + (\sqrt{2})^2} + \int \frac{dv}{v^2 - (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log \left| \frac{v - \sqrt{2}}{v + \sqrt{2}} \right| + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2 \tan x}} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1} \right| + C$$

Q. 44 $\displaystyle\int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$

Sol. Let $$I = \int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$$

Divide numerator and denominator by $\cos^4 x$, we get

$$I = \int_0^{\pi/2} \frac{\sec^4 x \, dx}{(a^2 + b^2 \tan^2 x)^2}$$

$$= \int_0^{\pi/2} \frac{(1 + \tan^2 x) \sec^2 x \, dx}{(a^2 + b^2 \tan^2 x)^2}$$

Put $\tan x = t$

$\Rightarrow$ $\sec^2 x \, dx = dt$

As $x \to 0$, then $t \to 0$

and $x \to \frac{\pi}{2}$, then $t \to \infty$ $$I = \int_0^{\infty} \frac{(1 + t^2)}{(a^2 + b^2 t^2)^2}$$

Now, $$\frac{1 + t^2}{(a^2 + b^2 t^2)^2} \qquad [\text{let } t^2 = u]$$

$$\frac{1 + u}{(a^2 + b^2 u)^2} = \frac{A}{(a^2 + b^2 u)} + \frac{B}{(a^2 + b^2 u)^2}$$

$\Rightarrow$ $$1 + u = A(a^2 + b^2 u) + B$$

On comparing the coefficient of x and constant term on both sides, we get

$$a^2 A + B = 1 \qquad \dots\text{(i)}$$

and $$b^2 A = 1 \qquad \dots\text{(ii)}$$

$\therefore$ $$A = \frac{1}{b^2}$$

Now, $$\frac{a^2}{b^2} + B = 1$$

$\Rightarrow$ $$B = 1 - \frac{a^2}{b^2} = \frac{b^2 - a^2}{b^2}$$

$\therefore$ $$I = \int_0^{\infty} \frac{(1 + t^2)}{(a^2 + b^2 t^2)^2}$$

$$= \frac{1}{b^2} \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} + \frac{b^2 - a^2}{b^2} \int_0^{\infty} \frac{dt}{(a^2 + b^2 t^2)^2}$$

$$= \frac{1}{b^2}\int_0^\infty \frac{dt}{b^2\left(\frac{a^2}{b^2}+t^2\right)} + \frac{b^2-a^2}{b^2}\int_0^\infty \frac{dt}{(a^2+b^2t^2)^2}$$

$$= \frac{1}{ab^3}\left[\tan^{-1}\left(\frac{tb}{a}\right)\right]_0^\infty + \frac{b^2-a^2}{b^2}\left(\frac{\pi}{4}\cdot\frac{1}{a^3b}\right)$$

$$= \frac{1}{ab^3}[\tan^{-1}\infty - \tan^{-1}0] + \frac{\pi}{4}\cdot\frac{b^2-a^2}{(a^3b^3)}$$

$$= \frac{\pi}{2ab^3} + \frac{\pi}{4}\cdot\frac{b^2-a^2}{(a^3b^3)}$$

$$= \pi\left(\frac{2a^2+b^2-a^2}{4a^3b^3}\right) = \frac{\pi}{4}\left(\frac{a^2+b^2}{a^3b^3}\right)$$

Q. 45 $\int_0^1 x\log(1+2x)\,dx$

Thinking Process

Use formula for integration by part i.e., $\int \mathrm{I}\cdot\mathrm{II}\,dx = \mathrm{I}\int \mathrm{II}\,dx - \int\left(\frac{d}{dx}\mathrm{I}\int \mathrm{I}\,dx\right)dx$ *and also use* $\int \frac{1}{x} = \log|x| + C.$

Sol. Let

$$I = \int_0^1 x\log(1+2x)dx$$

$$= \left[\log(1+2x)\frac{x^2}{2}\right]_0^1 - \int \frac{1}{1+2x}\cdot 2\cdot\frac{x^2}{2}dx$$

$$= \frac{1}{2}[x^2\log(1+2x)]_0^1 - \int\frac{x^2}{1+2x}dx$$

$$= \frac{1}{2}[1\log 3 - 0] - \left[\int_0^1\left(\frac{x}{2} - \frac{\frac{x}{2}}{1+2x}\right)dx\right]$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\int_0^1 x\,dx + \frac{1}{2}\int_0^1\frac{x}{1+2x}dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\int_0^1\frac{\frac{1}{2}(2x+1-1)}{(2x+1)}dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{1}{2} - 0\right] + \frac{1}{4}\int_0^1 dx - \frac{1}{4}\int_0^1\frac{1}{1+2x}dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4}[x]_0^1 - \frac{1}{8}[\log|(1+2x)|]_0^1$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4} - \frac{1}{8}[\log 3 - \log 1]$$

$$= \frac{1}{2}\log 3 - \frac{1}{8}\log 3$$

$$= \frac{3}{8}\log 3$$

Q. 46 $\int_0^{\pi} x \log \sin x \, dx$

Thinking Process

First of all use property of definite integral i.e., $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, *then use* $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

Sol. Let
$$I = \int_0^{\pi} x \log \sin x \, dx \qquad \text{...(i)}$$
$$I = \int_0^{\pi} (\pi - x) \log \sin(\pi - x)\, dx$$
$$= \int_0^{\pi} (\pi - x) \log \sin x \, dx \qquad \text{...(ii)}$$
$$2I = \pi \int_0^{\pi} \log \sin x \, dx \qquad \text{...(iii)}$$
$$2I = 2\pi \int_0^{\pi/2} \log \sin x \, dx \qquad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx\right]$$
$$I = \pi \int_0^{\pi/2} \log \sin x \, dx \qquad \text{...(iv)}$$

Now,
$$I = \pi \int_0^{\pi/2} \log \sin(\pi/2 - x)\, dx \qquad \text{...(v)}$$

On adding Eqs. (iv) and (v), we get
$$2I = \pi \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$
$$2I = \pi \int_0^{\pi/2} \log \sin x \cos x \, dx$$
$$= \pi \int_0^{\pi/22} \log \frac{2 \sin x \cos x}{2} dx$$
$$2I = \pi \int_0^{\pi/2} (\log \sin 2x - \log 2)\, dx$$
$$2I = \pi \int_0^{\pi/2} \log \sin 2x \, dx - \pi \int_0^{\pi/2} \log 2 \, dx$$

Put $2x = t \Rightarrow dx = \frac{1}{2} dt$

As $x \to 0$, then $t \to 0$

and $x \to \frac{\pi}{2}$, then $t \to \pi$

$\therefore$
$$2I = \frac{\pi}{2} \int_0^{\pi} \log \sin t \, dt - \frac{\pi^2}{2} \log 2$$

$\Rightarrow$
$$2I = \frac{\pi}{2} \int_0^{\pi} \log \sin x \, dx - \frac{\pi^2}{2} \log 2$$

$\Rightarrow$
$$2I = I - \frac{\pi^2}{2} \log 2 \qquad \text{[from Eq. (iii)]}$$

$\therefore$
$$I = -\frac{\pi^2}{2} \log 2 = \frac{\pi^2}{2} \log \left(\frac{1}{2}\right)$$

Q. 47 $\int_{\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$

Sol. Let $$I = \int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx \quad \text{...(i)}$$

$$I = \int_{-\pi/4}^{\pi/4} \log\left\{\sin\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right) + \cos\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right)\right\} dx$$

$$= \int_{-\pi/4}^{\pi/4} \log\{\sin(-x) + \cos(-x)\}\, dx$$

and $$I = \int_{-\pi/4}^{\pi 4} \log(\cos x - \sin x)\, dx \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$2I = \int_{-\pi/4}^{\pi 4} \log\cos 2x\, dx$$

$$2I = \int_{0}^{\pi/4} \log\cos 2x\, dx \quad \text{...(iii)}$$

$$\left[\because \int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x), \text{ if } f(-x) = f(x)\right]$$

Put $$2x = t \Rightarrow dx = \frac{dt}{2}$$

As $x \to 0$, then $t \to 0$

and $x \to \frac{\pi}{4}$, then $t \to \frac{\pi}{2}$

$$2I = \frac{1}{2}\int_0^{\pi/2} \log\cos t\, dt \quad \text{...(iv)}$$

$$2I = \frac{1}{2}\int_0^{\pi/2} \log\cos\left(\frac{\pi}{2} - t\right) dt \quad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$\Rightarrow$ $$2I = \frac{1}{2}\int_0^{\pi/2} \log\sin t\, dx \quad \text{...(v)}$$

On adding Eqs. (iv) and (v), we get

$$4I = \frac{1}{2}\int_0^{\pi/2} \log\sin t \cos t\, dt$$

$\Rightarrow$ $$4I = \frac{1}{2}\int_0^{\pi/2} \log\frac{\sin 2t}{2}\, dt$$

$\Rightarrow$ $$4I = \frac{1}{2}\int_0^{\pi/2} \log\sin 2x\, dx - \frac{1}{2}\int_0^{\pi/2} \log 2\, dx$$

$\Rightarrow$ $$4I = \frac{1}{2}\int_0^{\pi/2} \log\sin\left(\frac{\pi}{2} - 2x\right) dx - \log 2 \cdot \frac{\pi}{4}$$

$\Rightarrow$ $$4I = \frac{1}{2}\int_0^{\pi/2} \log\cos 2x\, dx - \frac{\pi}{4}\log 2$$

$\Rightarrow$ $$4I = \int_0^{\pi/4} \log\cos 2x\, dx - \frac{\pi}{4}\log 2 \quad \left[\because \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx\right]$$

$\Rightarrow$ $$4I = 2I - \frac{\pi}{4}\log 2 \quad \text{[from Eq. (iii)]}$$

$\therefore$ $$I = -\frac{\pi}{8}\log 2 = \frac{\pi}{8}\log\left(\frac{1}{2}\right)$$

Objective Type Questions

Q. 48 $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx$ is equal to

(a) $2(\sin x + x\cos\theta) + C$
(b) $2(\sin x - x\cos\theta) + C$
(c) $2(\sin x + 2x\cos\theta) + C$
(d) $2(\sin x - 2x\cos\theta) + C$

Thinking Process

Use formula $\cos 2\theta = 2\cos^2\theta - 1$ *to get simplest form, then apply* $\int \cos x\, dx = \sin x + C$.

Sol. *(a)* Let

$$I = \int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta} dx$$

$$= \int \frac{(2\cos^2 x - 1 - 2\cos^2\theta + 1)}{\cos x - \cos\theta} dx$$

$$= 2\int \frac{(\cos x + \cos\theta)(\cos x - \cos\theta)}{(\cos x - \cos\theta)} dx$$

$$= 2\int (\cos x + \cos\theta)\, dx$$

$$= 2(\sin x + x\cos\theta) + C$$

Q. 49 $\frac{dx}{\sin(x-a)\sin(x-b)}$ is equal to

(a) $\sin(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$
(b) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$
(c) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$
(d) $\sin(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$

Sol. *(c)* Let

$$I = \int \frac{dx}{\sin(x-a)\sin(x-b)}$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a-x+b)}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin\{(x-a)-(x-b)\}}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)}\int [\cot(x-b) - \cot(x-a)]\, dx$$

$$= \frac{1}{\sin(b-a)}[\log|\sin(x-b)| - \log|\sin(x-a)|] + C$$

$$= \operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$$

Q. 50 $\int \tan^{-1}\sqrt{x}\,dx$ is equal to

(a) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$ (b) $x\tan^{-1}\sqrt{x}-\sqrt{x}+C$

(c) $\sqrt{x}-x\tan^{-1}\sqrt{x}+C$ (d) $\sqrt{x}-(x+1)\tan^{-1}\sqrt{x}+C$

Thinking Process

Use formula for integration by part i.e., $\int \mathrm{I}\cdot\mathrm{II}\,dx=\mathrm{I}\int\mathrm{II}\,dx-\int\left(\frac{d}{dx}\mathrm{I}\int\mathrm{II}\,dx\right)dx$

Sol. *(a)* Let

$$I=\int 1\cdot\tan^{-1}\sqrt{x}\,dx$$

$$=\tan^{-1}\sqrt{x}\cdot x-\frac{1}{2}\int\frac{1}{(1+x)}\cdot\frac{2}{\sqrt{x}}dx$$

$$=x\tan^{-1}\sqrt{x}-\frac{1}{2}\int\frac{2}{\sqrt{x}(1+x)}dx$$

Put $x=t^2\Rightarrow dx=2t\,dt$

$\therefore$

$$I=x\tan^{-1}\sqrt{x}-\int\frac{t}{t(1+t^2)}dt$$

$$=x\tan^{-1}\sqrt{x}-\int\frac{t^2}{1+t^2}dt$$

$$=x\tan^{-1}\sqrt{x}-\int\left(1-\frac{1}{1+t^2}\right)dt$$

$$=x\tan^{-1}\sqrt{x}-\sqrt{x}+\tan^{-1}t+C$$

$$=x\tan^{-1}\sqrt{x}-\sqrt{x}+\tan^{-1}\sqrt{x}+C$$

$$=(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$$

Q. 51 $\int\frac{x^9}{(4x^2+1)^6}dx$ is equal to

(a) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+C$ (b) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+C$

(c) $\frac{1}{10x}(1+4)^{-5}+C$ (d) $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5}+C$

Sol. *(d)* Let

$$I=\int\frac{x^9}{(4x^2+1)^6}dx=\int\frac{x^9}{x^{12}\left(4+\frac{1}{x^2}\right)^6}dx$$

$$=\int\frac{dx}{x^3\left(4+\frac{1}{x^2}\right)^6}$$

Put $4+\frac{1}{x^2}=t\Rightarrow\frac{-2}{x^3}dx=dt$

$\Rightarrow$ $\frac{1}{x^3}dx=-\frac{1}{2}dt$

$\therefore$

$$I=-\frac{1}{2}\int\frac{dt}{t^6}=-\frac{1}{2}\left[\frac{t^{-6+1}}{-6+1}\right]+C$$

$$=\frac{1}{10}\left[\frac{1}{t^5}\right]+C=\frac{1}{10}\left(4+\frac{1}{x^2}\right)^{-5}+C$$

Q. 52 If $\int \frac{dx}{(x+2)(x^2+1)} = a\log|1+x^2| + b\tan^{-1}x + \frac{1}{5}\log|x+2| + C$, then

(a) $a = \frac{-1}{10}, b = \frac{-2}{5}$
(b) $a = \frac{1}{10}, b = -\frac{2}{5}$
(c) $a = \frac{-1}{10}, b = \frac{2}{5}$
(d) $a = \frac{1}{10}, b = \frac{2}{5}$

Thinking Process

Use method of partial fraction i.e., $\frac{1}{(x-a)(x^2+bx+c)} = \frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$

to solve the above problem.

Sol. *(c)* Given that, $\int \frac{dx}{(x+2)(x^2+1)} = a\log|1+x^2| + b\tan^{-1}x + \frac{1}{5}\log|x+2| + C$

Now, $I = \int \frac{dx}{(x+2)(x^2+1)}$

$$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$\Rightarrow \quad 1 = A(x^2+1) + (Bx+C)(x+2)$

$\Rightarrow \quad 1 = Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$

$\Rightarrow \quad 1 = (A+B)x^2 + (2B+C)x + A + 2C$

$\Rightarrow \quad A + B = 0, A + 2C = 1, 2B + C = 0$

We have, $A = \frac{1}{5}, B = -\frac{1}{5}$ and $C = \frac{2}{5}$

$$\therefore \quad \int \frac{dx}{(x+2)(x^2+1)} = \frac{1}{5}\int \frac{1}{x+2}dx + \int \frac{-\frac{1}{5}x + \frac{2}{5}}{x^2+1}dx$$

$$= \frac{1}{5}\int \frac{1}{x+2}dx - \frac{1}{5}\int \frac{x}{1+x^2}dx + \frac{1}{5}\int \frac{2}{1+x^2}dx$$

$$= \frac{1}{5}\log|x+2| - \frac{1}{10}\log|1+x^2| + \frac{2}{5}\tan^{-1}x + C$$

$\therefore \quad b = \frac{2}{5}$ and $a = \frac{-1}{10}$

Q. 53 $\int \frac{x^3}{x+1}$ is equal to

(a) $x + \frac{x^2}{2} + \frac{x^3}{3} - \log|1-x| + C$
(b) $x + \frac{x^2}{2} - \frac{x^3}{3} - \log|1-x| + C$
(c) $x - \frac{x^2}{2} - \frac{x^3}{3} - \log|1+x| + C$
(d) $x - \frac{x^2}{2} + \frac{x^3}{3} - \log|1+x| + C$

Sol. *(d)* Let $I = \int \frac{x^3}{x+1}dx$

$$= \int \left((x^2 - x + 1) - \frac{1}{(x+1)}\right)dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$$

Q. 54 $\int \frac{x+\sin x}{1+\cos x}\,dx$ is equal to

(a) $\log|1+\cos x|+C$ (b) $\log|x+\sin x|+C$

(c) $x-\tan\frac{x}{2}+C$ (d) $x\cdot\tan\frac{x}{2}+C$

Sol. *(d)* Let

$$I=\int\frac{x+\sin x}{1+\cos x}\,dx$$
$$=\int\frac{x}{1+\cos x}\,dx+\int\frac{\sin x}{1+\cos x}\,dx$$
$$=\int\frac{x}{2\cos^2 x/2}\,dx+\int\frac{2\sin x/2\cos x/2}{2\cos^2 x/2}\,dx$$
$$=\frac{1}{2}\int x\sec^2 x/2\,dx+\int\tan x/2\,dx$$
$$=\frac{1}{2}\left[x\cdot\tan x/2\cdot 2-\int\tan\frac{x}{2}\cdot 2\,dx\right]+\int\tan\frac{x}{2}\,dx$$
$$=x\cdot\tan\frac{x}{2}+C$$

Q. 55 If $\frac{x^3dx}{\sqrt{1+x^2}}=a(1+x^2)^{3/2}+b\sqrt{1+x^2}+C$, then

(a) $a=\frac{1}{3}, b=1$ (b) $a=\frac{-1}{3}, b=1$

(c) $a=\frac{-1}{3}, b=-1$ (d) $a=\frac{1}{3}, b=-1$

Sol. *(d)* Let $I=\int\frac{x^3}{\sqrt{1+x^2}}\,dx=a(1+x^2)^{3/2}+b\sqrt{1+x^2}+C$

$\because$ $$I=\int\frac{x^3}{\sqrt{1+x^2}}\,dx=\int\frac{x^2\cdot x}{\sqrt{1+x^2}}\,dx$$

Put $1+x^2=t^2$

$\Rightarrow$ $2x\,dx=2t\,dt$

$\therefore$ $$I=\int\frac{t(t^2-1)}{t}\,dt=\frac{t^3}{3}-t+C$$
$$=\frac{1}{3}(1+x^2)^{3/2}-\sqrt{1+x^2}+C$$

$\therefore$ $a=\frac{1}{3}$ and $b=-1$

Q. 56 $\int_{-\pi/4}^{\pi/4}\frac{dx}{1+\cos 2x}$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

Sol. *(a)* Let

$$I=\int_{-\pi/4}^{\pi/4}\frac{dx}{1+\cos 2x}=\int_{-\pi/4}^{\pi/4}\frac{dx}{2\cos^2 x}$$
$$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}\sec^2 x\,dx=\int_0^{\pi/4}\sec^2 x\,dx=[\tan x]_0^{\pi/4}=1$$

Q. 57 $\int_0^{\pi/2} \sqrt{1-\sin 2x}\, dx$ is equal to

(a) $2\sqrt{2}$ (b) $2(\sqrt{2}+1)$

(c) 2 (d) $2(\sqrt{2}-1)$

Sol. *(d)* Let

$$I = \int_0^{\pi/2} \sqrt{1-\sin 2x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2}\, dx + \int_{\pi/4}^{\pi/2} \sqrt{(\sin x - \cos x)^2}\, dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= 2\sqrt{2} - 2 = 2(\sqrt{2}-1)$$

Q. 58 $\int_0^{\pi/4} \cos x e^{\sin x} dx$ is equal to

(a) e + 1 (b) e − 1 (c) e (d) − e

Sol. *(b)* Let

$$I = \int_0^{\pi/2} \cos x\, e^{\sin x} dx$$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

As $x \to 0$, then $t \to 0$

and $x \to \pi/2$, then $t \to 1$

$\therefore$

$$I = \int_0^1 e^t dt = [e^t]_0^1$$

$$= e^1 - e^0 = e - 1$$

Q. 59 $\int \frac{x+3}{(x+4)^2} e^x dx$ is equal to

(a) $e^x \left(\frac{1}{x+4}\right) + C$ (b) $e^{-x} \left(\frac{1}{x+4}\right) + C$

(c) $e^{-x} \left(\frac{1}{x-4}\right) + C$ (d) $e^{2x} \left(\frac{1}{x-4}\right) + C$

Sol. *(a)* Let

$$I = \int \frac{x+3}{(x+4)^2} e^x\, dx$$

$$= \int \frac{e^x}{(x+4)} - \int \frac{e^x}{(x+4)^2}\, dx$$

$$= \int e^x \left(\frac{1}{(x+4)} - \frac{1}{(x+4)^2}\right) dx$$

$$= e^x \left(\frac{1}{x+4}\right) + C \qquad \left[\because \int e^x \{f(x) + f'(x)\}\, dx = e^x f(x) + C\right]$$

Fillers

Q. 60 If $\int_0^a \frac{1}{1+4x^2}\,dx = \frac{\pi}{8}$, then $a =$ ....

Sol. Let $I = \int_0^a \frac{1}{1+4x^2}\,dx = \frac{\pi}{8}$

Now,
$$\int_0^a \frac{1}{4\left(\frac{1}{4}+x^2\right)}\,dx = \frac{2}{4}[\tan^{-1}2x]_0^a$$
$$= \frac{1}{2}\tan^{-1}2a - 0 = \pi/8$$
$$\frac{1}{2}\tan^{-1}2a = \frac{\pi}{8}$$
$$\Rightarrow \quad \tan^{-1}2a = \pi/4$$
$$\Rightarrow \quad 2a = 1$$
$$\therefore \quad a = \frac{1}{2}$$

Q. 61 $\int \frac{\sin x}{3+4\cos^2 x}\,dx =$ ....

Sol. Let $I = \int \frac{\sin x}{3+4\cos^2 x}\,dx$

Put $\cos x = t \Rightarrow -\sin x\,dx = dt$

$$\therefore \quad I = -\int \frac{dt}{3+4t^2} = -\frac{1}{4}\int \frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2 + t^2}$$
$$= -\frac{1}{4}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2t}{\sqrt{3}} + C$$
$$= -\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right) + C$$

Q. 62 The value of $\int_{-\pi}^{\pi} \sin^3 x\cos^2 x\,dx$ is ....

Sol. We have,
$$f(x) = \int_{-\pi}^{\pi}\sin^3 x\cos^2 x\,dx$$
$$f(-x) = \int_{-\pi}^{\pi}\sin^3(-2) - \cos^2(-x)\,dx$$
$$= -f(x)$$

Since, $f(x)$ is an odd function.
$$\therefore \quad \int_{-\pi}^{\pi}\sin^3 x\cos^2 x\,dx = 0$$

6

Application of Derivatives

Short Answer Type Questions

Q. 1 A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is propotional to the surface. Prove that the radius is decreasing at a constant rate.

Thinking Process

First, let V be the volume of the ball and S be the surface area of the ball and then by using $\frac{dV}{dt} \propto S$, we can prove the required result.

Sol. We have, rate of decrease of the volume of spherical ball of salt at any instant is $\propto$ surface.
Let the radius of the spherical ball of the salt be r.

$\therefore$ Volume of the ball $(V) = \frac{4}{3}\pi r^3$

and surface area $(S) = 4\pi r^2$

$\because$ $$\frac{dV}{dt} \propto S \Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \propto 4\pi r^2$$

$$\Rightarrow \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} \propto 4\pi r^2 \Rightarrow \frac{dr}{dt} \propto \frac{4\pi r^2}{4\pi r^2}$$

$$\Rightarrow \frac{dr}{dt} = k \cdot 1$$ [where, k is the proportionality constant]

$$\Rightarrow \frac{dr}{dt} = k$$

Hence, the radius of ball is decreasing at a constant rate.

Q. 2 If the area of a circle increases at a uniform rate, then prove that perimeter varies inversely as the radius.

Sol. Let the radius of circle $= r$ And area of the circle, $A = \pi r^2$

$$\therefore \frac{d}{dt}A = \frac{d}{dt}\pi r^2$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} \quad \text{...(i)}$$

Since, the area of a circle increases at a uniform rate, then

$$\frac{dA}{dt} = k \qquad \text{...(ii)}$$

where, k is a constant.

From Eqs. (i) and (ii), $$2\pi r \cdot \frac{dr}{dt} = k$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{k}{2\pi r} = \frac{k}{2\pi} \cdot \left(\frac{1}{r}\right) \qquad \text{...(iii)}$$

Let the perimeter, $$P = 2\pi r$$

$$\therefore \quad \frac{dP}{dt} = \frac{d}{dt} \cdot 2\pi r \quad \Rightarrow \quad \frac{dP}{dt} = 2\pi \cdot \frac{dr}{dt}$$

$$= 2\pi \cdot \frac{k}{2\pi} \cdot \frac{1}{r} = \frac{k}{r} \qquad \text{[using Eq. (iii)]}$$

$$\Rightarrow \quad \frac{dP}{dt} \propto \frac{1}{r}$$

Hence proved.

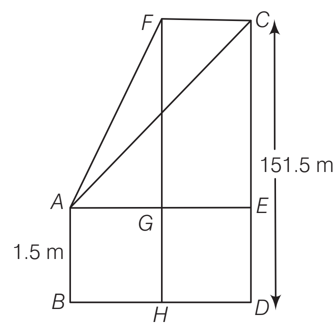
Q. 3 A kite is moving horizontally at a height of 151.5 m. If the speed of kite is 10 m/s, how fast is the string being let out, when the kite is 250 m away from the boy who is flying the kite, if the height of boy is 1.5 m?

Sol. We have , height $(h) = 151.5$ m, speed of kite $(v) = 10$ m/s

Let CD be the height of kite and AB be the height of boy.

Let $DB = x \text{ m} = EA$ and $AC = 250$ m

$$\therefore \quad \frac{dx}{dt} = 10 \text{ m/s}$$



From the figure, we see that

$$EC = 151.5 - 1.5 = 150 \text{ m}$$

and $AE = x$

Also, $AC = 250$ m

In right angled ΔCEA,

$$AE^2 + EC^2 = AC^2$$

$$\Rightarrow \quad x^2 + (150)^2 = y^2 \qquad \text{... (i)}$$

$$\Rightarrow \quad x^2 + (150)^2 = (250)^2$$

$$\Rightarrow \quad x^2 = (250)^2 - (150)^2$$

$$= (250 + 150)(250 - 150)$$

$$= 400 \times 100$$

$$\therefore \quad x = 20 \times 10 = 200$$

From Eq. (i), on differentiating w.r.t. t, we get

$$2x \cdot \frac{dx}{dt} + 0 = 2y \frac{dy}{dt}$$

$$\Rightarrow \quad 2y \frac{dy}{dt} = 2x \frac{dx}{dt}$$

$$\therefore \quad \frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt}$$

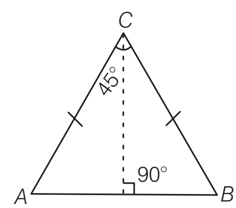
$$= \frac{200}{250} \cdot 10 = 8 \text{ m/s} \qquad \left[\because \frac{dx}{dt} = 10 \text{ m/s}\right]$$

So, the required rate at which the string is being let out is 8 m/s.

Q. 4 **Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, then find the rate at which they are being separated.**

Thinking Process

By drawing figure such that men start moving at a point C, A and B are separating points, then draw perpendicular from that point C to AB to get D. Now, get the value of $\angle ACD$ in terms of x and y, then by using $\frac{dy}{dt}$ get desired result. [let $AC = BC = x$ and $AB = y$]



Sol. Let two men start from the point C with velocity v each at the same time.

Also, $\angle BCA = 45°$

Since, A and B are moving with same velocity v, so they will cover same distance in same time.

Therefore, ΔABC is an isosceles triangle with $AC = BC$.

Now, draw $CD \perp AB$.

Let at any instant t, the distance between them is AB.

Let $AC = BC = x$ and $AB = y$

In ΔACD and ΔDCB,

$$\angle CAD = \angle CBD \quad [\because AC = BC]$$

$$\angle CDA = \angle CDB = 90°$$

$$\therefore \quad \angle ACD = \angle DCB$$

$$\text{or} \quad \angle ACD = \frac{1}{2} \times \angle ACB$$

$$\Rightarrow \quad \angle ACD = \frac{1}{2} \times 45°$$

$$\Rightarrow \quad \angle ACD = \frac{\pi}{8}$$

$$\therefore \quad \sin\frac{\pi}{8} = \frac{AD}{AC}$$

$$\Rightarrow \quad \sin\frac{\pi}{8} = \frac{y/2}{x} \quad [\because AD = y/2]$$

$$\Rightarrow \quad \frac{y}{2} = x\sin\frac{\pi}{8}$$

$$\Rightarrow \quad y = 2x \cdot \sin\frac{\pi}{8}$$

Now, differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = 2 \cdot \sin\frac{\pi}{8} \cdot \frac{dx}{dt}$$

$$= 2 \cdot \sin\frac{\pi}{8} . v \quad \left[\because v = \frac{dx}{dt}\right]$$

$$= 2v \cdot \frac{\sqrt{2-\sqrt{2}}}{2} \quad \left[\because \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}\right]$$

$$= \sqrt{2-\sqrt{2}}\, v \text{ unit/s}$$

which is the rate at which A and B are being separated.

Q. 5 Find an angle θ, where $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine.

Sol. Let θ increases twice as fast as its sine.

$$\Rightarrow \quad \theta = 2 \sin \theta$$

Now, on differentiating both sides w.r.t. t, we get

$$\frac{d\theta}{dt} = 2 \cdot \cos \theta \cdot \frac{d\theta}{dt} \Rightarrow 1 = 2\cos \theta$$

$$\Rightarrow \quad \frac{1}{2} = \cos \theta \Rightarrow \cos \theta = \cos \frac{\pi}{3}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

So, the required angle is $\frac{\pi}{3}$.

Q. 6 Find the approximate value of $(1.999)^5$.

Sol. Let $\quad x = 2$

and $\quad \Delta x = -0.001 \qquad [\because 2 - 0.001 = 1.999]$

Let $\quad y = x^5$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 5x^4$$

Now, $\quad \Delta y = \frac{dy}{dx} \cdot \Delta x = 5x^4 \times \Delta x$

$$= 5 \times 2^4 \times [-0.001]$$

$$= -80 \times 0.001 = -0.080$$

$$\therefore \quad (1.999)^5 = y + \Delta y$$

$$= 2^5 + (-0.080)$$

$$= 32 - 0.080 = 31.920$$

Q. 7 Find the approximate volume of metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm, respectively.

Sol. Let internal radius $= r$ and external radius $= R$

$\therefore$ Volume of hollow spherical shell, $V = \frac{4}{3}\pi (R^3 - r^3)$

$$\Rightarrow \quad V = \frac{4}{3}\pi [(3.0005)^3 - (3)^3] \qquad \text{... (i)}$$

Now, we shall use differentiation to get approximate value of $(3.0005)^3$.

Let $\quad (3.0005)^3 = y + \Delta y$

and $\quad x = 3, \Delta x = 0.0005$

Also, let $\quad y = x^3$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2$$

$$\therefore \quad \Delta y = \frac{dy}{dx} \times \Delta x = 3x^2 \times 0.0005$$

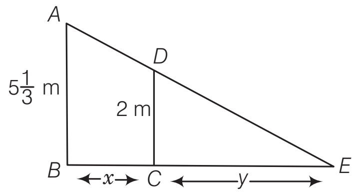
$$= 3 \times 3^2 \times 0.0005$$

$$= 27 \times 0.0005 = 0.0135$$

Also, $$(3.0005)^3 = y + \Delta y$$
$$= 3^3 + 0.0135 = 27.0135$$
$\therefore$ $$V = \frac{4}{3}\pi\,[27.0135 - 27.000] \quad \text{[using Eq. (i)]}$$
$$= \frac{4}{3}\pi\,[0.0135] = 4\pi \times (0.0045)$$
$$= 0.0180\pi \text{ cm}^3$$

Q. 8 **A man, 2 m tall, walks at the rate of $1\frac{2}{3}$ m/s towards a street light which is $5\frac{1}{3}$ m above the ground. At what rate is the tip of his shadow moving and at what rate is the length of the shadow changing when he is $3\frac{1}{3}$ m from the base of the light?**

Sol. Let AB be the street light post and CD be the height of man *i.e.,* $CD = 2$ m.



Let $BC = x$ m, $CE = y$ m and $\frac{dx}{dt} = \frac{-5}{3}$ m/s

From ΔABE and ΔDCE, we see that
$$\Delta ABE \sim \Delta DCE \quad \text{[by AAA similarity]}$$
$\therefore$ $$\frac{AB}{DC} = \frac{BE}{CE} \Rightarrow \frac{\frac{16}{3}}{2} = \frac{x + y}{y}$$
$\Rightarrow$ $$\frac{16}{6} = \frac{x + y}{y}$$
$\Rightarrow$ $$16y = 6x + 6y \Rightarrow 10y = 6x$$
$\Rightarrow$ $$y = \frac{3}{5}x$$

On differentiating both sides w.r.t. t, we get
$$\frac{dy}{dt} = \frac{3}{5}\cdot\frac{dx}{dt} = \frac{3}{5}\cdot\left(-1\frac{2}{3}\right)$$
[since, man is moving towards the light post]
$$= \frac{3}{5}\cdot\left(\frac{-5}{3}\right) = -1 \text{m/s}$$

Let $$z = x + y$$
Now, differentiating both sides w.r.t. t, we get
$$\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -\left(\frac{5}{3} + 1\right)$$
$$= -\frac{8}{3} = -2\frac{2}{3} \text{m/s}$$

Hence, the tip of shadow is moving at the rate of $2\frac{2}{3}$ m/s towards the light source and length of the shadow is decreasing at the rate of 1 m/s.

Q. 9 A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200\ (10 - t)^2$. How fast is the water running out at the end of 5 s and what is the average rate at which the water flows out during the first 5 s?

Sol. Let L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain, then

$$L = 200\ (10 - t)^2$$

$\therefore$ Rate at which the water is running out $= -\dfrac{dL}{dt}$

$$\frac{dL}{dt} = -200 \cdot 2\ (10 - t) \cdot (-1)$$
$$= 400\ (10 - t)$$

Rate at which the water is running out at the end of 5 s

$$= 400\ (10 - 5)$$
$$= 2000 \text{ L/s} = \text{Final rate}$$

Since, initial rate $= -\left(\dfrac{dL}{dt}\right)_{t=0} = 4000$ L/s

$\therefore$ Average rate during 5 s $= \dfrac{\text{Initial rate} + \text{Final rate}}{2}$

$$= \frac{4000 + 2000}{2}$$
$$= 3000 \text{ L/s}$$

Q. 10 The volume of a cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.

Sol. Let the side of a cube be x unit.

$\therefore$ Volume of cube $(V) = x^3$

On differentiating both side w.r.t. t, we get

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} = k \qquad \text{[constant]}$$

$$\Rightarrow \frac{dx}{dt} = \frac{k}{3x^2} \qquad \text{... (i)}$$

Also, surface area of cube, $S = 6x^2$

On differentiating w.r.t. t, we get

$$\frac{dS}{dt} = 12x \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dS}{dt} = 12\ x \cdot \frac{k}{3x^2} \qquad \text{[using Eq. (i)]}$$

$$\Rightarrow \frac{dS}{dt} = \frac{12k}{3x} = 4\left(\frac{k}{x}\right)$$

$$\Rightarrow \frac{dS}{dt} \propto \frac{1}{x}$$

Hence, the surface area of the cube varies inversely as the length of the side.

Q. 11 If x and y are the sides of two squares such that $y = x - x^2$, then find the rate of change of the area of second square with respect to the area of first square.

Thinking Process

First, let A_1 and A_2 be the areas of two squares and get their values in one variable and then by using dA_1/dt and dA_2/dt get the value of dA_2/dA_1

Sol. Since, x and y are the sides of two squares such that $y = x - x^2$.

$\therefore$ Area of the first square $(A_1) = x^2$

and area of the second square $(A_2) = y^2 = (x - x^2)^2$

$$\therefore \quad \frac{dA_2}{dt} = \frac{d}{dt}(x - x^2)^2 = 2(x - x^2)\left(\frac{dx}{dt} - 2x \cdot \frac{dx}{dt}\right)$$

$$= \frac{dx}{dt}(1 - 2x)\,2(x - x^2)$$

and
$$\frac{dA_1}{dt} = \frac{d}{dt}x^2 = 2x \cdot \frac{dx}{dt}$$

$$\therefore \quad \frac{dA_2}{dA_1} = \frac{dA_2/dt}{dA_1/dt} = \frac{\frac{dx}{dt} \cdot (1 - 2x)(2x - 2x^2)}{2x \cdot \frac{dx}{dt}}$$

$$= \frac{(1 - 2x)\,2x(1 - x)}{2x}$$

$$= (1 - 2x)(1 - x)$$

$$= 1 - x - 2x + 2x^2$$

$$= 2x^2 - 3x + 1$$

Q. 12 Find the condition that curves $2x = y^2$ and $2xy = k$ intersect orthogonally.

Thinking Process

First, get the intersection point of the curve and then get the slopes of both the curves at that point. Then, by using $m_1 \cdot m_2 = -1$, get the required condition.

Sol. Given, equation of curves are $\quad 2x = y^2 \quad$ ... (i)

and $\quad 2xy = k \quad$ ... (ii)

$\Rightarrow \quad y = \dfrac{k}{2x} \quad$ [from Eq. (ii)]

From Eq. (i), $\quad 2x = \left(\dfrac{k}{2x}\right)^2$

$\Rightarrow \quad 8x^3 = k^2$

$\Rightarrow \quad x^3 = \dfrac{1}{8}k^2$

$\Rightarrow \quad x = \dfrac{1}{2}k^{2/3}$

$\therefore \quad y = \dfrac{k}{2x} = \dfrac{k}{2 \cdot \frac{1}{2}k^{2/3}} = k^{1/3}$

Thus, we get point of intersection of curves which is $\left(\dfrac{1}{2}k^{2/3}, k^{1/3}\right)$.

From Eqs. (i) and (ii),

$$2 = 2y\frac{dy}{dx}$$

and $$2\left[x \cdot \frac{dy}{dx} + y \cdot 1\right] = 0$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{1}{y}$$

and $$\left(\frac{dy}{dx}\right) = \frac{-2y}{2x} = -\frac{y}{x}$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}k^{2/3},\, k^{1/3}\right)} = \frac{1}{k^{1/3}} \qquad \text{[say } m_1\text{]}$$

and $$\left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}k^{2/3},\, k^{1/3}\right)} = \frac{-k^{1/3}}{\frac{1}{2}k^{2/3}} = -2k^{-1/3} \qquad \text{[say } m_2\text{]}$$

Since, the curves intersect orthogonally.

i.e., $$m_1 \cdot m_2 = -1$$

$\Rightarrow$ $$\frac{1}{k^{1/3}} \cdot (-2k^{-1/3}) = -1$$

$\Rightarrow$ $$-2\,k^{-2/3} = -1$$

$\Rightarrow$ $$\frac{2}{k^{2/3}} = 1$$

$\Rightarrow$ $$k^{2/3} = 2$$

$\therefore$ $$k^2 = 8$$

which is the required condition.

Q. 13 Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

Thinking Process

First, find the intersection points of curves and then equate the slopes of both the curves at the obtained point.

Sol. Given equation of curves are

$$xy = 4 \qquad \text{...(i)}$$

and $$x^2 + y^2 = 8 \qquad \text{...(ii)}$$

$\Rightarrow$ $$x \cdot \frac{dy}{dx} + y = 0$$

and $$2x + 2y\frac{dy}{dx} = 0$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{-y}{x}$$

and $$\frac{dy}{dx} = \frac{-2x}{2y}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{-y}{x} = m_1 \qquad \text{[say]}$$

and $$\frac{dy}{dx} = \frac{-x}{y} = m_2 \qquad \text{[say]}$$

Since, both the curves should have same slope.

$\therefore$ $$\frac{-y}{x} = \frac{-x}{y} \Rightarrow -y^2 = -x^2$$

$\Rightarrow$ $$x^2 = y^2 \qquad \text{...(iii)}$$

Using the value of x^2 in Eq. (ii), we get

$$y^2 + y^2 = 8$$

$\Rightarrow$ $y^2 = 4 \Rightarrow y = \pm 2$

For $y = 2$, $x = \frac{4}{2} = 2$

and for $y = -2$, $x = \frac{4}{-2} = -2$

Thus, the required points of intersection are (2, 2) and (– 2, – 2).

For (2, 2), $m_1 = \frac{-y}{x} = \frac{-2}{2} = -1$

and $m_2 = \frac{-x}{y} = \frac{-2}{2} = -1$

$\because$ $m_1 = m_2$

For (–2,–2), $m_1 = \frac{-y}{x} = \frac{-(-2)}{-2} = -1$

and $m_2 = \frac{-x}{y} = \frac{-(-2)}{-2} = -1$

Thus, for both the intersection points, we see that slope of both the curves are same.
Hence, the curves touch each other.

Q. 14 Find the coordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes.

Sol. We have, $\sqrt{x} + \sqrt{y} = 4$ ...(i)

$\Rightarrow$ $x^{1/2} + y^{1/2} = 4$

$\Rightarrow$ $\frac{1}{2} \cdot \frac{1}{x^{1/2}} + \frac{1}{2} \cdot \frac{1}{y^{1/2}} \cdot \frac{dy}{dx} = 0$

$\therefore$ $\frac{dy}{dx} = -\frac{1}{2} \cdot x^{-1/2} \;\; 2 \cdot y^{1/2}$

$$= -\sqrt{\frac{y}{x}}$$

Since, tangent is equally inclined to the axes.

$\therefore$ $\frac{dy}{dx} = \pm 1$

$\Rightarrow$ $-\sqrt{\frac{y}{x}} = \pm 1$

$\Rightarrow$ $\frac{y}{x} = 1 \Rightarrow y = x$

From Eq. (i), $\sqrt{y} + \sqrt{y} = 4$

$\Rightarrow$ $2\sqrt{y} = 4$

$\Rightarrow$ $4y = 16$

$\therefore$ $y = 4$ and $x = 4$

When y=4, then $x = 4$

So, the required coordinates are (4, 4).

Q. 15 Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

Sol. We have,
$$y = 4 - x^2 \qquad \text{...(i)}$$
and
$$y = x^2 \qquad \text{... (ii)}$$
$$\Rightarrow \quad \frac{dy}{dx} = -2x$$
and
$$\frac{dy}{dx} = 2x$$
$$\Rightarrow \quad m_1 = -2x$$
and
$$m_2 = 2x$$
From Eqs. (i) and (ii),
$$x^2 = 4 - x^2$$
$$\Rightarrow \quad 2x^2 = 4$$
$$\Rightarrow \quad x^2 = 2$$
$$\Rightarrow \quad x = \pm\sqrt{2}$$
$$\therefore \quad y = x^2 = (\pm\sqrt{2})^2 = 2$$
So, the points of intersection are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$.

For point $(+\sqrt{2}, 2)$,
$$m_1 = -2x = -2\cdot\sqrt{2} = -2\sqrt{2}$$
and
$$m_2 = 2x = 2\sqrt{2}$$
and for point $(\sqrt{2}, 2)$,
$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-2\sqrt{2} - 2\sqrt{2}}{1 - 2\sqrt{2}\cdot 2\sqrt{2}}\right| = \left|\frac{-4\sqrt{2}}{-7}\right|$$
$$\therefore \quad \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

Q. 16 Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1, 2).

Sol. We have, $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$

Since, both the curves touch each other at (1, 2) *i.e.,* curves are passing through (1, 2).
$$\therefore \quad 2y\cdot\frac{dy}{dx} = 4$$
and
$$2x + 2y\frac{dy}{dx} = 6$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{4}{2y}$$
and
$$\frac{dy}{dx} = \frac{6 - 2x}{2y}$$
$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{4}{4} = 1$$
and
$$\left(\frac{dy}{dx}\right)_{(1,2)} = \frac{6 - 2\cdot 1}{2\cdot 2} = \frac{4}{4} = 1$$
$$\Rightarrow \quad m_1 = 1 \text{ and } m_2 = 1$$
Thus, we see that slope of both the curves are equal to each other *i.e.,* $m_1 = m_2 = 1$ at the point (1, 2).

Hence, both the curves touch each other.

Q. 17 **Find the equation of the normal lines to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.**

Sol. Given equation of the curve is

$$3x^2 - y^2 = 8 \qquad \text{...(i)}$$

On differentiating both sides w.r.t. x, we get

$$6x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y}$$

$$\Rightarrow \quad m_1 = \frac{3x}{y} \qquad \text{[say]}$$

and slope of normal $(m_2) = \dfrac{-1}{m_1} = \dfrac{-y}{3x}$ ...(ii)

Since, slope of normal to the curve should be equal to the slope of line $x + 3y = 4$, which is parallel to curve.

For line, $y = \dfrac{4 - x}{3} = \dfrac{-x}{3} + \dfrac{4}{3}$

$\Rightarrow$ Slope of the line $(m_3) = \dfrac{-1}{3}$

$$\therefore \quad m_2 = m_3$$

$$\Rightarrow \quad \frac{-y}{3x} = -\frac{1}{3}$$

$$\Rightarrow \quad -3y = -3x$$

$$\Rightarrow \quad y = x \qquad \text{...(iii)}$$

On substituting the value of y in Eq. (i), we get

$$3x^2 - x^2 = 8$$

$$\Rightarrow \quad x^2 = 4$$

$$\Rightarrow \quad x = \pm 2$$

For $x = 2$, $y = 2$ [using Eq. (iii)]

and for $x = -2$, $y = -2$ [using Eq. (iii)]

Thus,the points at which normal to the curve are parallel to the line $x + 3y = 4$ are (2, 2) and (– 2, – 2).

Required equations of normal are

$$y - 2 = m_2(x - 2) \text{ and } y + 2 = m_2(x + 2)$$

$$\Rightarrow \quad y - 2 = \frac{-2}{6}(x - 2) \text{ and } y + 2 = \frac{-2}{6}(x + 2)$$

$$\Rightarrow \quad 3y - 6 = -x + 2 \quad \text{and} \quad 3y + 6 = -x - 2$$

$$\Rightarrow \quad 3y + x = +8 \quad \text{and} \quad 3y + x = -8$$

So, the required equations are $3y + x = \pm 8$.

Q. 18 **At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the Y-axis?**

Sol. Given, equation of curve which is

$$x^2 + y^2 - 2x - 4y + 1 = 0 \qquad \text{... (i)}$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(2y-4) = 2-2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2(1-x)}{2(y-2)}$$

Since, the tangents are parallel to the Y-axis *i.e.,* $\tan\theta = \tan 90° = \frac{dy}{dx}$.

$$\therefore \quad \frac{1-x}{y-2} = \frac{1}{0}$$

$$\Rightarrow \quad y-2=0$$

$$\Rightarrow \quad y=2$$

For $y=2$ from Eq. (i), we get

$$x^2+2^2-2x-4\times 2+1=0$$

$$\Rightarrow \quad x^2-2x-3=0$$

$$\Rightarrow \quad x^2-3x+x-3=0$$

$$\Rightarrow \quad x(x-3)+1(x-3)=0$$

$$\Rightarrow \quad (x+1)(x-3)=0$$

$$\therefore \quad x=-1,\ x=3$$

So, the required points are $(-1, 2)$ and $(3, 2)$.

Q. 19 Show that the line $\frac{x}{a}+\frac{y}{b}=1$, touches the curve $y=b\cdot e^{-x/a}$ at the point, where the curve intersects the axis of Y.

Sol. We have the equation of line given by $\frac{x}{a}+\frac{y}{b}=1$, which touches the curve $y=b\cdot e^{-x/a}$ at the point, where the curve intersects the axis of Y *i.e.,* $x=0$.

$$\therefore \quad y = b\cdot e^{-0/a} = b \qquad [\because e^0 = 1]$$

So, the point of intersection of the curve with Y-axis is $(0, b)$.

Now, slope of the given line at $(0, b)$ is given by

$$\frac{1}{a}\cdot 1+\frac{1}{b}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-1}{a}\cdot b$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{1}{a}\cdot b=\frac{-b}{a}=m_1 \qquad \text{[say]}$$

Also, the slope of the curve at $(0, b)$ is

$$\frac{dy}{dx}=b\cdot e^{-x/a}\cdot\frac{-1}{a}$$

$$\frac{dy}{dx}=\frac{-b}{a}e^{-x/a}$$

$$\left(\frac{dy}{dx}\right)_{(0,\,b)}=\frac{-b}{a}e^{-0}=\frac{-b}{a}=m_2 \qquad \text{[say]}$$

Since, $\quad m_1=m_2=\frac{-b}{a}$

Hence, the line touches the curve at the point, where the curve intersects the axis of Y.

Q. 20 Show that $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$ is increasing in R.

Thinking Process

If $f'(x) \geq 0$, then we can say that $f(x)$ is increasing function. Use this condition to show the desired result.

Sol. We have, $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$

$$\therefore \quad f'(x) = 2 + \left(\frac{-1}{1+x^2}\right) + \frac{1}{(\sqrt{1+x^2} - x)}\left(\frac{1}{2\sqrt{1+x^2}} \cdot 2x - 1\right)$$

$$= 2 - \frac{1}{1+x^2} + \frac{1}{(\sqrt{1+x^2} - x)} \cdot \frac{(x - \sqrt{1+x^2})}{\sqrt{1+x^2}}$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{2 + 2x^2 - 1 - \sqrt{1+x^2}}{1+x^2} = \frac{1 + 2x^2 - \sqrt{1+x^2}}{1+x^2}$$

For increasing function, $f'(x) \geq 0$

$$\Rightarrow \quad \frac{1 + 2x^2 - \sqrt{1+x^2}}{1+x^2} \geq 0$$

$$\Rightarrow \quad 1 + 2x^2 \geq \sqrt{1+x^2}$$

$$\Rightarrow \quad (1 + 2x^2)^2 \geq 1 + x^2$$

$$\Rightarrow \quad 1 + 4x^4 + 4x^2 \geq 1 + x^2$$

$$\Rightarrow \quad 4x^4 + 3x^2 \geq 0$$

$$\Rightarrow \quad x^2(4x^2 + 3) \geq 0$$

which is true for any real value of x.

Hence, $f(x)$ is increasing in R.

Q. 21 Show that for $a \geq 1$, $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b$ is decreasing in R.

Thinking Process

If $f'(x) \leq 0$, then we can say that $f(x)$ is a decreasing function. So, use this condition to show the result.

Sol. We have, $a \geq 1$, $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$

$$\therefore \quad f'(x) = \sqrt{3}\cos x - (-\sin x) - 2a$$

$$= \sqrt{3}\cos x + \sin x - 2a$$

$$= 2\left[\frac{\sqrt{3}}{2} \cdot \cos x + \frac{1}{2} \cdot \sin x\right] - 2a$$

$$= 2\left[\cos\frac{\pi}{6} \cdot \cos x + \sin\frac{\pi}{6} \cdot \sin x\right] - 2a$$

$$= 2\left(\cos\frac{\pi}{6} - x\right) - 2a$$

$[\because \cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B]$

$$= 2\left[\left(\cos\frac{\pi}{6} - x\right) - a\right]$$

We know that, $\cos x \in [-1, 1]$

and $a \geq 1$

So, $2\left[\cos\left(\frac{\pi}{6} - x\right) - a\right] \leq 0$

$\therefore$ $f'(x) \leq 0$

Hence, $f(x)$ is a decreasing function in R.

Q. 22 Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Sol. We have, $f(x) = \tan^{-1}(\sin x + \cos x)$

$$\therefore \quad f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \cdot (\cos x - \sin x)$$

$$= \frac{1}{1 + \sin^2 x + \cos^2 x + 2\sin x \cdot \cos x}(\cos x - \sin x)$$

$$= \frac{1}{(2 + \sin 2x)}(\cos x - \sin x)$$

$[\because \sin 2x = 2\sin x \cos x \text{ and } \sin^2 x + \cos^2 x = 1]$

For $f'(x) \geq 0$,

$$\frac{1}{(2 + \sin 2x)} \cdot (\cos x - \sin x) \geq 0$$

$$\Rightarrow \quad \cos x - \sin x \geq 0 \quad \left[\because (2 + \sin 2x) \geq 0 \text{ in} \left(0, \frac{\pi}{4}\right)\right]$$

$$\Rightarrow \quad \cos x \geq \sin x$$

which is true, if $x \in \left(0, \frac{\pi}{4}\right)$.

Hence, $f(x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Q. 23 At what point, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum? Also, find the maximum slope.

Sol. We have, $y = -x^3 + 3x^2 + 9x - 27$

$$\therefore \quad \frac{dy}{dx} = -3x^2 + 6x + 9 = \text{Slope of tangent to the curve}$$

Now, $\frac{d^2y}{dx^2} = -6x + 6$

For $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$,

$$-6x + 6 = 0$$

$$\Rightarrow \quad x = \frac{-6}{-6} = 1$$

$$\therefore \quad \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = -6 < 0$$

So, the slope of tangent to the curve is maximum, when $x = 1$.

For $x = 1$, $\left(\frac{dy}{dx}\right)_{(x=1)} = -3 \cdot 1^2 + 6 \cdot 1 + 9 = 12$,

which is maximum slope.

Also, for $x = 1,\ y = -1^3 + 3\cdot1^2 + 9\cdot1 - 27$

$$= -1 + 3 + 9 - 27$$
$$= -16$$

So, the required point is $(1, -16)$.

Q. 24 Prove that $f(x) = \sin x + \sqrt{3}\cos x$ has maximum value at $x = \frac{\pi}{6}$.

Sol. We have, $f(x) = \sin x + \sqrt{3}\cos x$

$\therefore$ $f'(x) = \cos x + \sqrt{3}(-\sin x)$

$= \cos x - \sqrt{3}\sin x$

For $f'(x) = 0$, $\cos x = \sqrt{3}\sin x$

$\Rightarrow$ $\tan x = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\Rightarrow$ $x = \frac{\pi}{6}$

Again, differentiating $f'(x)$, we get

$$f''(x) = -\sin x - \sqrt{3}\cos x$$

At $x = \frac{\pi}{6}$, $f''(x) = -\sin\frac{\pi}{6} - \sqrt{3}\cos\frac{\pi}{6}$

$$= -\frac{1}{2} - \sqrt{3}\cdot\frac{\sqrt{3}}{2}$$
$$= -\frac{1}{2} - \frac{3}{2} = -2 < 0$$

Hence, at $x = \frac{\pi}{6}$, $f(x)$ has maximum value at $\frac{\pi}{6}$ is the point of local maxima.

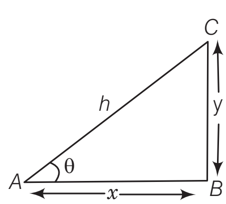
Long Answer Type Questions

Q. 25 If the sum of lengths of the hypotenuse and a side of a right angled triangle is given, then show that the area of triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

Sol. Let ABC be a triangle with $AC = h$, $AB = x$ and $BC = y$.

Also, $\angle CAB = \theta$

Let $h + x = k$ ...(i)



$\therefore$ $\cos\theta = \frac{x}{h}$

$\Rightarrow$ $x = h\cos\theta$

$\Rightarrow$ $h + h\cos\theta = k$ [using Eq. (i)]

$$\Rightarrow \quad h(1+\cos\theta)=k$$

$$\Rightarrow \quad h=\frac{k}{(1+\cos\theta)} \quad \ldots(ii)$$

Also, $$\text{area of } \Delta ABC = \frac{1}{2}(AB\cdot BC)$$

$$A=\frac{1}{2}\cdot x\cdot y$$

$$=\frac{1}{2}h\cos\theta\cdot h\sin\theta \qquad \left[\because \quad \sin\theta=\frac{y}{h}\right]$$

$$=\frac{1}{2}h^2\sin\theta\cdot\cos\theta$$

$$=\frac{2h^2}{4}\sin\theta\cdot\cos\theta$$

$$=\frac{1}{4}h^2\sin 2\theta \quad \ldots(iii)$$

Since, $$h=\frac{k}{1+\cos\theta}$$

$$\therefore \quad A=\frac{1}{4}\left(\frac{k}{1+\cos\theta}\right)^2\cdot\sin 2\theta$$

$$\Rightarrow \quad A=\frac{k^2}{4}\cdot\frac{\sin 2\theta}{(1+\cos\theta)^2} \quad \ldots(iv)$$

$$\therefore \quad \frac{dA}{d\theta}=\frac{k^2}{4}\left[\frac{(1+\cos\theta)^2\cdot\cos 2\theta\cdot 2-\sin 2\theta\cdot 2(1+\cos\theta)\cdot(0-\sin\theta)}{(1+\cos\theta)^4}\right]$$

$$=\frac{k^2}{4}\left\{\frac{2(1+\cos\theta)[(1+\cos\theta)\cdot\cos 2\theta+\sin 2\theta(\sin\theta)}{(1+\cos\theta)^4}\right\}$$

$$=\frac{k^2}{4}\cdot\frac{2}{(1+\cos\theta)^3}[(1+\cos\theta)\cdot\cos 2\theta+2\sin^2\theta\cdot\cos\theta]$$

$$=\frac{k^2}{2(1+\cos\theta)^3}[(1+\cos\theta)(1-2\sin^2\theta)+2\sin^2\theta\cdot\cos\theta]$$

$$=\frac{k^2}{2(1+\cos\theta)^3}[1+\cos\theta-2\sin^2\theta-2\sin^2\theta\cdot\cos\theta+2\sin^2\theta\cdot\cos\theta]$$

$$=\frac{k^2}{2(1+\cos\theta)^3}[(1+\cos\theta)-2\sin^2\theta]$$

$$=\frac{k^2}{2(1+\cos\theta)^3}[1+\cos\theta-2+2\cos^2\theta]$$

$$=\frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta+\cos\theta-1) \quad \ldots(v)$$

For $\frac{dA}{d\theta}=0$,

$$\frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta+\cos\theta-1)=0$$

$$\Rightarrow \quad 2\cos^2\theta+\cos\theta-1=0$$

$$\Rightarrow \quad 2\cos^2\theta+2\cos\theta-\cos\theta-1=0$$

$$\Rightarrow \quad 2\cos\theta(\cos\theta+1)-1(\cos\theta+1)=0$$

$$\Rightarrow \quad (2\cos\theta-1)(\cos\theta+1)=0$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \text{ or } \cos\theta = -1$$

$$\Rightarrow \quad \theta = \frac{\pi}{3} \qquad \text{[possible]}$$

$$\text{or} \quad \theta = 2n\pi \pm \pi \qquad \text{[not possible]}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

Again, differentiating w.r.t. θ in Eq. (v), we get

$$\frac{d}{d\theta}\left(\frac{dA}{d\theta}\right) = \frac{d}{d\theta}\left[\frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta + \cos\theta - 1)\right]$$

$$\therefore \quad \frac{d^2A}{d\theta^2} = \frac{d}{d\theta}\left[\frac{k^2(2\cos\theta - 1)(1+\cos\theta)}{2(1+\cos\theta)^3}\right] = \frac{d}{d\theta}\left[\frac{k^2}{2}\cdot\frac{(2\cos\theta-1)}{(1+\cos\theta)^2}\right]$$

$$= \frac{k^2}{2}\left[\frac{(1+\cos\theta)^2\cdot(-2\sin\theta) - 2(1+\cos\theta)\cdot(-\sin\theta)(2\cos\theta-1)}{(1+\cos\theta)^4}\right]$$

$$= \frac{k^2}{2}\left[\frac{(1+\cos\theta)\cdot[1+\cos\theta](-2\sin\theta) + 2\sin\theta\,(2\cos\theta-1)}{(1+\cos\theta)^4}\right]$$

$$= \frac{k^2}{2}\left[\frac{-2\sin\theta - 2\sin\theta\cdot\cos\theta + 4\sin\theta\cdot\cos\theta - 2\sin\theta}{(1+\cos\theta)^3}\right]$$

$$= \frac{k^2}{2}\left[\frac{-4\sin\theta - \sin 2\theta + 2\sin 2\theta}{(1+\cos\theta)^3}\right] = \frac{k^2}{2}\left[\frac{\sin 2\theta - 4\sin\theta}{(1+\cos\theta)^3}\right]$$

$$\therefore \quad \left(\frac{d^2A}{d\theta^2}\right)_{\text{at } \theta = \frac{\pi}{3}} = \frac{k^2}{2}\left[\frac{\sin\frac{2\pi}{3} - 4\sin\frac{\pi}{3}}{\left(1+\cos\frac{\pi}{3}\right)^3}\right] = \frac{k^2}{2}\left[\frac{\frac{\sqrt{3}}{2} - \frac{4\sqrt{3}}{2}}{\left(1+\frac{1}{2}\right)^3}\right]$$

$$= \frac{k^2}{2}\left[\frac{-3\sqrt{3}\cdot 8}{2\cdot 27}\right] = -k^2\left(\frac{2\sqrt{3}}{9}\right)$$

which is less than zero.

Hence, area of the right angled triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

Q. 26 **Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also, find the corresponding local maximum and local minimum values.**

Sol. Given that, $f(x) = x^5 - 5x^4 + 5x^3 - 1$

On differentiating w.r.t. x, we get

$$f'(x) = 5x^4 - 20x^3 + 15x^2$$

For maxima or minima, $f'(x) = 0$

$$\Rightarrow \quad 5x^4 - 20x^3 + 15x^2 = 0$$

$$\Rightarrow \quad 5x^2(x^2 - 4x + 3) = 0$$

$$\Rightarrow \quad 5x^2(x^2 - 3x - x + 3) = 0$$

$$\Rightarrow \quad 5x^2[x(x-3) - 1(x-3)] = 0$$

$$\Rightarrow \quad 5x^2[(x-1)(x-3)] = 0$$

$$\therefore \quad x = 0, 1, 3$$

Sign scheme for $\frac{dy}{dx} = 5x^2(x-1)(x-3)$

$$\begin{array}{ccccccc} -\infty\ + & & + & & - & & +\ \ +\infty \\ \longleftarrow & 0 & \text{---} & 1 & \text{---} & 3 & \longrightarrow \end{array}$$

So, y has maximum value at $x = 1$ and minimum value at $x = 3$.
At $x = 0$, y has neither maximum nor minimum value.

$\therefore$ Maximum value of $y = 1 - 5 + 5 - 1 = 0$

and minimum value $= (3)^5 - 5(3)^4 + 5(3)^3 - 1$

$$= 243 - 81 \times 5 - 27 \times 5 - 1 = -298$$

Q. 27 A telephone company in a town has 500 subscribers on its list and collects fixed charges of ₹ 300 per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of ₹ 1 per one subscriber will discontinue the service. Find what increase will bring maximum profit?

Sol. Consider that company increases the annual subscription by ₹ x.
So, x subscribes will discontinue the service.
$\therefore$ Total revenue of company after the increment is given by

$$R(x) = (500 - x)(300 + x)$$
$$= 15 \times 10^4 + 500x - 300x - x^2$$
$$= -x^2 + 200x + 150000$$

On differentiating both sides w.r.t. x, we get

$$R'(x) = -2x + 200$$

Now, $R'(x) = 0$

$\Rightarrow$ $2x = 200 \Rightarrow x = 100$

$\therefore$ $R''(x) = -2 < 0$

So, $R(x)$ is maximum when $x = 100$.
Hence, the company should increase the subscription fee by ₹ 100, so that it has maximum profit.

Q. 28 If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$.

Sol. Given, line is $x\cos\alpha + y\sin\alpha = p$ ... (i)

and curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow$ $b^2x^2 + a^2y^2 = a^2b^2$ ...(ii)

Now, differentiating Eq. (ii) w.r.t. x, we get

$$b^2 \cdot 2x + a^2 \cdot 2y \cdot \frac{dy}{dx} = 0$$

$\Rightarrow$ $\frac{dy}{dx} = \frac{-2b^2x}{2a^2y} = \frac{-xb^2}{ya^2}$ ... (iii)

From Eq. (i), $y\sin\alpha = p - x\cos\alpha$

$\Rightarrow$ $y = -x\cot\alpha + \frac{p}{\sin\alpha}$

Thus, slope of the line is $(-\cot\alpha)$.

So, the given equation of line will be tangent to the Eq. (ii), if $\left(-\frac{x}{y}\cdot\frac{b^2}{a^2}\right) = (-\cot\alpha)$

$$\Rightarrow \quad \frac{x}{a^2\cos\alpha} = \frac{y}{b^2\sin\alpha} = k \qquad \text{[say]}$$

$$\Rightarrow \quad x = ka^2\cos\alpha$$

and $\quad y = b^2 k\sin\alpha$

So, the line $x\cos\alpha + y\sin\alpha = p$ will touch the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ at point $(ka^2\cos\alpha, kb^2\sin\alpha)$.

From Eq. (i), $\quad ka^2\cos^2\alpha + kb^2\sin^2\alpha = p$

$$\Rightarrow \quad a^2\cos^2\alpha + b^2\sin^2\alpha = \frac{p}{k}$$

$$\Rightarrow \quad (a^2\cos^2\alpha + b^2\sin^2\alpha)^2 = \frac{p^2}{k^2} \qquad \text{...(iv)}$$

From Eq. (ii), $\quad b^2k^2a^4\cos^2\alpha + a^2k^2b^4\sin^2\alpha = a^2b^2$

$$\Rightarrow \quad k^2(a^2\cos^2\alpha + b^2\sin^2\alpha) = 1$$

$$\Rightarrow \quad (a^2\cos^2\alpha + b^2\sin^2\alpha) = \frac{1}{k^2} \qquad \text{... (v)}$$

On dividing Eq. (iv) by Eq. (v), we get

$$a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$$

Hence proved.

Alternate Method

We know that, if a line $y = mx + c$ touches ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

the required condition is $\quad c^2 = a^2m^2 + b^2$

Here, given equation of the line is

$$x\cos\alpha + y\sin\alpha = p$$

$$\Rightarrow \quad y = \frac{p - x\cos\alpha}{\sin\alpha}$$

$$= -x\cot\alpha + \frac{p}{\sin\alpha}$$

$$\Rightarrow \quad c = \frac{p}{\sin\alpha}$$

and $\quad m = -\cot\alpha$

$$\therefore \quad \left(\frac{p}{\sin\alpha}\right)^2 = a^2(-\cot\alpha)^2 + b^2$$

$$\Rightarrow \quad \frac{p^2}{\sin^2\alpha} = a^2\frac{\cos^2\alpha}{\sin^2\alpha} + b^2$$

$$\Rightarrow \quad p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$$

Hence proved.

Q. 29 If an open box with square base is to be made of a given quantity of card board of area c^2, then show that the maximum volume of the box is $\dfrac{c^3}{6\sqrt{3}}$ cu units.

Thinking Process

First, let the sides of box in x and y then find $\dfrac{dV}{dx}$ in terms c and x. Also, for $\dfrac{dV}{dx}=0$ get the value of x and if $\dfrac{d^2V}{dx^2}<0$ at the value of x, then by putting that value of x in the equation of V, get the desired result.

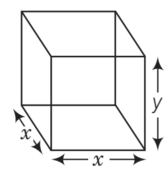
Sol. Let the length of side of the square base of open box be x units and its height be y units.

$\therefore$ Area of the metal used $= x^2 + 4xy$

$\Rightarrow \quad x^2 + 4xy = c^2$ [given]

$\Rightarrow \quad y = \dfrac{c^2 - x^2}{4x}$ ...(i)

Now, volume of the box $(V) = x^2 y$



$$\Rightarrow V = x^2 \cdot \left(\frac{c^2 - x^2}{4x}\right)$$
$$= \frac{1}{4} x (c^2 - x^2)$$
$$= \frac{1}{4} (c^2 x - x^3)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \frac{1}{4}(c^2 - 3x^2) \quad \text{... (ii)}$$

Now, $\dfrac{dV}{dx} = 0 \Rightarrow c^2 = 3x^2$

$\Rightarrow \quad x^2 = \dfrac{c^2}{3}$

$\Rightarrow \quad x = \dfrac{c}{\sqrt{3}}$ [using positive sign]

Again, differentiating Eq. (ii) w.r.t. x, we get

$$\frac{d^2V}{dx^2} = \frac{1}{4}(-6x) = \frac{-3}{2}x < 0$$

$$\therefore \left(\frac{d^2v}{dx^2}\right)_{\text{at } x = \frac{c}{\sqrt{3}}} = -\frac{3}{2} \cdot \left(\frac{c}{\sqrt{3}}\right) < 0$$

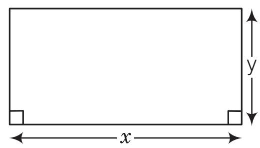
Thus, we see that volume (V) is maximum at $x = \dfrac{c}{\sqrt{3}}$.

$\therefore$ Maximum volume of the box, $(V)_{x = \frac{c}{\sqrt{3}}} = \dfrac{1}{4}\left(c^2 . \dfrac{c}{\sqrt{3}} - \dfrac{c^3}{3\sqrt{3}}\right)$

$$= \frac{1}{4} \cdot \frac{(3c^3 - c^3)}{3\sqrt{3}} = \frac{1}{4} \cdot \frac{2c^3}{3\sqrt{3}}$$
$$= \frac{c^3}{6\sqrt{3}} \text{ cu units}$$

Q. 30 Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also, find the maximum volume.

Sol. Let breadth and length of the rectangle be x and y, respectively.



$\because$ Perimeter of the rectangle = 36 cm

$\Rightarrow \quad 2x + 2y = 36$

$\Rightarrow \quad x + y = 18$

$\Rightarrow \quad y = 18 - x \qquad \text{... (i)}$

Let the rectangle is being revolved about its length y.

Then, volume (V) of resultant cylinder = $\pi x^2 \cdot y$

$\Rightarrow \quad V = \pi x^2 \cdot (18 - x) \qquad [\because V = \pi r^2 h]$ [using Eq. (i)]

$= 18\pi x^2 - \pi x^3 = \pi [18x^2 - x^3]$

On differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \pi (36x - 3x^2)$$

Now, $\dfrac{dV}{dx} = 0$

$\Rightarrow \quad 36x = 3x^2$

$\Rightarrow \quad 3x^2 - 36x = 0$

$\Rightarrow \quad 3(x^2 - 12x) = 0$

$\Rightarrow \quad 3x(x - 12) = 0$

$\Rightarrow \quad x = 0, x = 12$

$\therefore \quad x = 12 \qquad [\because, x \neq 0]$

Again, differentiating w.r.t. x, we get

$$\frac{d^2V}{dx^2} = \pi (36 - 6x)$$

$$\Rightarrow \quad \left(\frac{d^2V}{dx^2}\right)_{x=12} = \pi(36 - 6 \times 12) = -36\pi < 0$$

At $x = 12$, volume of the resultant cylinder is the maximum.

So, the dimensions of rectangle are 12 cm and 6 cm, respectively. [using Eq. (i)]

$\therefore$ Maximum volume of resultant cylinder,

$$(V)_{x=12} = \pi [18 \cdot (12)^2 - (12)^3]$$

$$= \pi [12^2 (18 - 12)]$$

$$= \pi \times 144 \times 6$$

$$= 864\,\pi \text{ cm}^3$$

Q. 31 I the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?

Sol. Let length of one edge of cube be x units and radius of sphere be r units.

$\therefore$ Surface area of cube $= 6x^2$

and surface area of sphere $= 4\pi r^2$

Also, $6x^2 + 4\pi r^2 = k$ [constant, given]

$\Rightarrow$ $6x^2 = k - 4\pi r^2$

$\Rightarrow$ $x^2 = \dfrac{k - 4\pi r^2}{6}$

$\Rightarrow$ $x = \left[\dfrac{k - 4\pi r^2}{6}\right]^{1/2}$ ... (i)

Now, volume of cube $= x^3$

and volume of sphere $= \dfrac{4}{3}\pi r^3$

Let sum of volume of the cube and volume of the sphere be given by

$$S = x^3 + \frac{4}{3}\pi r^3 = \left[\frac{k - 4\pi r^2}{6}\right]^{3/2} + \frac{4}{3}\pi r^3$$

On differentiating both sides w.r.t. r, we get

$$\frac{dS}{dr} = \frac{3}{2}\left[\frac{k - 4\pi r^2}{6}\right]^{1/2} \cdot \left(\frac{-8\pi r}{6}\right) + \frac{12}{3}\pi r^2$$

$$= -2\pi r\left[\frac{k - 4\pi r^2}{6}\right]^{1/2} + 4\pi r^2 \quad \text{... (ii)}$$

$$= -2\pi r\left[\left\{\frac{k - 4\pi r^2}{6}\right\}^{1/2} - 2r\right]$$

Now, $\dfrac{dS}{dr} = 0$

$\Rightarrow$ $r = 0$ or $2r = \left(\dfrac{k - 4\pi r^2}{6}\right)^{1/2}$

$\Rightarrow$ $4r^2 = \dfrac{k - 4\pi r^2}{6} \Rightarrow 24r^2 = k - 4\pi r^2$

$\Rightarrow$ $24r^2 + 4\pi r^2 = k \Rightarrow r^2\,[24 + 4\pi] = k$

$\therefore$ $r = 0$ or $r = \sqrt{\dfrac{k}{24 + 4\pi}} = \dfrac{1}{2}\sqrt{\dfrac{k}{6 + \pi}}$

We know that, $r \neq 0$

$\therefore$ $r = \dfrac{1}{2}\sqrt{\dfrac{k}{6 + \pi}}$

Again, differentiating w.r.t. r in Eq. (ii), we get

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[-2\pi r\left\{\left(\frac{k - 4\pi r^2}{6}\right)^{1/2} + 4\pi r^2\right\}\right]$$

$$= -2\pi\left[r\cdot\frac{1}{2}\left(\frac{k-4\pi r^2}{6}\right)^{-1/2}\cdot\left(\frac{-8\pi r}{6}\right)+\left(\frac{k-4\pi r^2}{6}\right)^{1/2}\cdot 1\right]+4\pi\cdot 2r$$

$$= -2\pi\left[r\cdot\frac{1}{2\sqrt{\frac{k-4\pi r^2}{6}}}\cdot\left(\frac{-8\pi r}{6}\right)+\sqrt{\frac{k-4\pi r^2}{6}}\right]+8\pi r$$

$$= -2\pi\left[\frac{-8\pi r^2+12\left(k-\frac{4\pi r^2}{6}\right)}{12\sqrt{\frac{k-4\pi r^2}{6}}}\right]+8\pi r$$

$$= -2\pi\left[\frac{-48\pi r^2+72k-48\pi r^2}{72\sqrt{\frac{k-4\pi r^2}{6}}}\right]+8\pi r = -2\pi\left[\frac{-96\pi r^2+72k}{72\sqrt{\frac{k-4\pi r^2}{6}}}\right]+8\pi r > 0$$

For $r=\frac{1}{2}\sqrt{\frac{k}{6+\pi}}$, then the sum of their volume is minimum.

For $r=\frac{1}{2}\sqrt{\frac{k}{6+\pi}}$,
$$x=\left[\frac{k-4\pi\cdot\frac{1}{4}\frac{k}{(6+\pi)}}{6}\right]^{1/2}$$

$$=\left[\frac{(6+\pi)k-\pi k}{6(6+\pi)}\right]^{1/2}=\left[\frac{k}{6+\pi}\right]^{1/2}=2r$$

Since, the sum of their volume is minimum when $x=2r$.
Hence, the ratio of an edge of cube to the diameter of the sphere is 1:1.

Q. 32 If AB is a diameter of a circle and C is any point on the circle, then show that the area of ΔABC is maximum, when it is isosceles.

Sol. We have, $AB = 2r$

and $\angle ACB = 90°$ [since, angle in the semi-circle is always 90°]

Let $AC = x$ and $BC = y$

$\therefore$ $(2r)^2 = x^2+y^2$

$\Rightarrow$ $y^2 = 4r^2-x^2$

$\Rightarrow$ $y=\sqrt{4r^2-x^2}$ ... (i)

Now, area of ΔABC, $A=\frac{1}{2}\times x\times y$

$$=\frac{1}{2}\times x\times(4r^2-x^2)^{1/2}$$ [using Eq. (i)]

Now, differentiating both sides w.r.t. x, we get

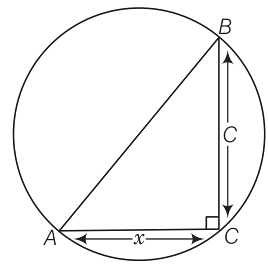
$$\frac{dA}{dx}=\frac{1}{2}\left[x\cdot\frac{1}{2}(4r^2-x^2)^{-1/2}\cdot(0-2x)+(4r^2-x^2)^{1/2}\cdot 1\right]$$

$$=\frac{1}{2}\left[\frac{-2x^2}{2\sqrt{4r^2-x^2}}+(4r^2-x^2)^{1/2}\right]$$

$$= \frac{1}{2}\left[\frac{-x^2}{\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2}\right]$$

$$= \frac{1}{2}\left[\frac{-x^2 + 4r^2 - x^2}{\sqrt{4r^2 - x^2}}\right] = \frac{1}{2}\left[\frac{-2x^2 + 4r^2}{\sqrt{4r^2 - x^2}}\right]$$

$$\Rightarrow \quad \frac{dA}{dx} = \left[\frac{(-x^2 + 2r^2)}{\sqrt{4r^2 - x^2}}\right]$$



Now, $\frac{dA}{dx} = 0$

$\Rightarrow \quad -x^2 + 2r^2 = 0$

$\Rightarrow \quad r^2 = \frac{1}{2}x^2$

$\Rightarrow \quad r = \frac{1}{\sqrt{2}}x$

$\therefore \quad x = r\sqrt{2}$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}\cdot(-2x) + (2r^2 - x^2)\cdot\frac{1}{2}(4r^2 - x^2)^{-1/2}(-2x)}{(\sqrt{4r^2 - x^2})^2}$$

$$= \frac{-2x\left[\sqrt{4r^2 - x^2} + (2r^2 - x^2)\cdot\frac{1}{2\sqrt{4r^2 - x^2}}\right]}{(\sqrt{4r^2 - x^2})^2}$$

$$= \frac{-4x\cdot\left(\sqrt{4r^2 - x^2}\right)^2 + (2r^2 - x^2)(-2x)}{2\cdot(4r^2 - x^2)^{3/2}}$$

$$= \frac{-4x(4r^2 - x^2) + (2r^2 - x^2)\cdot(-2x)}{2\cdot(4r^2 - x^2)^{3/2}}$$

$$= \frac{-16xr^2 + 4x^3 + (2r^2 - x^2)(-2x)}{2\cdot(4r^2 - x^2)^{3/2}}$$

$$\left(\frac{d^2A}{dx^2}\right)_{x = r\sqrt{2}} = \frac{-16\cdot r\sqrt{2}\cdot r^2 + 4\cdot(r\sqrt{2})^3 + [2r^2 - (r\sqrt{2})^2]\cdot(-2\cdot r\sqrt{2})}{2\cdot(4r^2 - 2r^2)^{3/2}} \qquad [\because x = r\sqrt{2}]$$

$$= \frac{-16\sqrt{2}\cdot r^3 + 8\sqrt{2}r^3}{2(2r^2)^{3/2}} = \frac{8\sqrt{2}\,r^2\,[r - 2r]}{4r^3}$$

$$= \frac{-8\sqrt{2}\,r^3}{4r^3} = -2\sqrt{2} < 0$$

For $x = r\sqrt{2}$, the area of triangle is maximum.

For $x = r\sqrt{2}$, $\quad y = \sqrt{4r^2 - (r\sqrt{2})^2} = \sqrt{2r^2} = r\sqrt{2}$

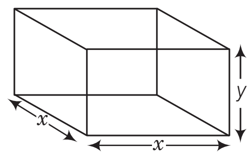
Since, $\quad x = r\sqrt{2} = y$

Hence, the triangle is isosceles.

Q. 33 A metal box with a square base and vertical sides is to contain $1024\ \text{cm}^3$. If the material for the top and bottom costs ₹ 5per cm^2 and the material for the sides costs ₹ 2.50 per cm^2. Then, find the least cost of the box.

Sol. Since, volume of the box = 1024 cm^3

Let length of the side of square base be x cm and height of the box be y cm.



$\therefore$ Volume of the box $(V) = x^2 \cdot y = 1024$

Since, $$x^2 y = 1024 \Rightarrow y = \frac{1024}{x^2}$$

Let C denotes the cost of the box.

$$\therefore \quad C = 2x^2 \times 5 + 4xy \times 2.50$$
$$= 10x^2 + 10xy = 10x(x + y)$$
$$= 10x\left(x + \frac{1024}{x^2}\right)$$
$$= \frac{10x}{x^2}(x^3 + 1024)$$
$$\Rightarrow \quad C = 10x^2 + \frac{10240}{x} \quad \text{... (i)}$$

On differentiating both sides w.r.t. x, we get
$$\frac{dC}{dx} = 20x + 10240(-x)^{-2}$$
$$= 20x - \frac{10240}{x^2} \quad \text{...(ii)}$$

Now, $$\frac{dC}{dx} = 0$$
$$\Rightarrow \quad 20x = \frac{10240}{x^2}$$
$$\Rightarrow \quad 20x^3 = 10240$$
$$\Rightarrow \quad x^3 = 512 = 8^3 \Rightarrow x = 8$$

Again, differentiating Eq. (ii) w.r.t. x, we get
$$\frac{d^2C}{dx^2} = 20 - 10240(-2)\cdot\frac{1}{x^3}$$
$$= 20 + \frac{20480}{x^3} > 0$$
$$\therefore \quad \left(\frac{d^2C}{dx^2}\right)_{x=8} = 20 + \frac{20480}{512} = 60 > 0$$

For $x = 8$, cost is minimum and the corresponding least cost of the box,
$$C(8) = 10 \cdot 8^2 + \frac{10240}{8}$$
$$= 640 + 1280 = 1920$$
$\therefore$ Least cost = ₹ 1920

Q. 34 The sum of surface areas of a rectangular parallelopiped with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of the sphere. Also, find the minimum value of the sum of their volumes.

Sol. We have given that, the sum of the surface areas of a rectangular parallelopiped with sides x, $2x$ and $\frac{x}{3}$ and a sphere is constant.

Let S be the sum of both the surface area.

$$\therefore \quad S = 2\left(x \cdot 2x + 2x \cdot \frac{x}{3} + \frac{x}{3} \cdot x\right) + 4\pi r^2 = k$$

$$k = 2\left[2x^2 + \frac{2x^2}{3} + \frac{x^2}{3}\right] + 4\pi r^2$$

$$= 2\,[3x^2] + 4\pi r^2 = 6x^2 + 4\pi r^2$$

$$\Rightarrow \quad 4\pi r^2 = k - 6x^2$$

$$\Rightarrow \quad r^2 = \frac{k - 6x^2}{4\pi}$$

$$\Rightarrow \quad r = \sqrt{\frac{k - 6x^2}{4\pi}} \qquad \text{... (i)}$$

Let V denotes the volume of both the parallelopiped and the sphere.

Then, $$V = 2x \cdot x \cdot \frac{x}{3} + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi\left(\frac{k - 6x^2}{4\pi}\right)^{3/2}$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \frac{1}{8\pi^{3/2}}(k - 6x^2)^{3/2}$$

$$= \frac{2}{3}x^3 + \frac{1}{6\sqrt{\pi}}(k - 6x^2)^{3/2} \qquad \text{...(ii)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \frac{2}{3} \cdot 3x^2 + \frac{1}{6\sqrt{\pi}} \cdot \frac{3}{2}(k - 6x^2)^{1/2} \cdot (-12x)$$

$$= 2x^2 - \frac{12x}{4\sqrt{\pi}}\sqrt{k - 6x^2}$$

$$= 2x^2 - \frac{3x}{\sqrt{\pi}}(k - 6x^2)^{1/2} \qquad \text{...(iii)}$$

$$\because \quad \frac{dV}{dx} = 0$$

$$\Rightarrow \quad 2x^2 = \frac{3x}{\sqrt{\pi}}(k - 6x^2)^{1/2}$$

$$\Rightarrow \quad 4x^4 = \frac{9x^2}{\pi}(k - 6x^2)$$

$$\Rightarrow \quad 4\pi x^4 = 9kx^2 - 54x^4$$

$$\Rightarrow \quad 4\pi x^4 + 54x^4 = 9kx^2$$

$$\Rightarrow \quad x^4[4\pi + 54] = 9 \cdot k \cdot x^2$$

$$\Rightarrow \quad x^2 = \frac{9k}{4\pi + 54}$$

$$\Rightarrow \quad x = 3 \cdot \sqrt{\frac{k}{4\pi + 54}} \qquad \text{...(iv)}$$

Again, differentiating Eq. (iii) w.r.t. x, we get

$$\frac{d^2V}{dx^2} = 4x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{1}{2}(k - 6x^2)^{-1/2} \cdot (-12x) + (k - 6x^2)^{1/2} \cdot 1\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}[-6x^2 \cdot (k - 6x^2)^{-1/2} + (k - 6x^2)^{1/2}]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2 + k - 6x^2}{\sqrt{k - 6x^2}}\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{k - 12x^2}{\sqrt{k - 6x^2}}\right]$$

Now, $$\left(\frac{d^2V}{dx^2}\right)_{x = 3\cdot\sqrt{\frac{k}{4\pi + 54}}} = 4\cdot 3\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{k - 12\cdot 9 \cdot \frac{k}{4\pi + 54}}{\sqrt{k - \frac{6\cdot 9 \cdot k}{4\pi + 54}}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{k - \frac{108k}{4\pi + 54}}{\sqrt{k - \frac{54\,k}{4\pi + 54}}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{4k\pi + 54k - 108\,k / 4\pi + 54}{\sqrt{4k\pi + 54k - 54\,k / 4\pi + 54}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{4k\pi - 54k}{\sqrt{4k\pi}\sqrt{4\pi + 54}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{6}{\sqrt{\pi}}\left[\frac{k(2\pi - 27)}{\sqrt{k}\sqrt{16\pi^2 + 216\pi}}\right]$$

$$\left[\text{since, } (2\pi - 27) < 0 \Rightarrow \frac{d^2V}{dx^2} > 0; k > 0\right]$$

For $x = 3\sqrt{\frac{k}{4\pi + 54}}$, the sum of volumes is minimum.

For $x = 3\sqrt{\frac{k}{4\pi + 54}}$, then $r = \sqrt{\frac{k - 6x^2}{4\pi}}$ [using Eq. (i)]

$$= \frac{1}{2\sqrt{\pi}}\sqrt{k - 6\cdot\frac{9k}{4\pi + 54}}$$

$$= \frac{1}{2\sqrt{\pi}}\cdot\sqrt{\frac{4k\pi + 54\,k - 54\,k}{4\pi + 54}}$$

$$= \frac{1}{2\sqrt{\pi}}\sqrt{\frac{4k\pi}{4\pi + 54}} = \frac{\sqrt{k}}{\sqrt{4\pi + 54}} = \frac{1}{3}x$$

$\Rightarrow$ $x = 3r$ **Hence proved.**

$\therefore$ Minimum sum of volume,

$$V_{\left(x = 3\cdot\sqrt{\frac{k}{4\pi + 54}}\right)} = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi\cdot\left(\frac{1}{3}x\right)^3$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi\cdot\frac{x^3}{27} = \frac{2}{3}x^3\left(1 + \frac{2\pi}{27}\right)$$

Objective Type Questions

Q. 35 If the sides of an equilateral triangle are increasing at the rate of 2 cm/s then the rate at which the area increases, when side is 10 cm, is

(a) $10\ \text{cm}^2/\text{s}$ (b) $\sqrt{3}\ \text{cm}^2/\text{s}$

(c) $10\sqrt{3}\ \text{cm}^2/\text{s}$ (d) $\dfrac{10}{3}\ \text{cm}^2/\text{s}$

Sol. *(c)* Let the side of an equilateral triangle be x cm.

$\therefore$ Area of equilateral triangle, $A = \dfrac{\sqrt{3}}{4}x^2$ ...(i)

Also, $\dfrac{dx}{dt} = 2\text{cm/s}$

On differentiating Eq. (i) w.r.t. t, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}\cdot 2x\cdot\frac{dx}{dt}$$

$$= \frac{\sqrt{3}}{4}\cdot 2\cdot 10\cdot 2 \quad \left[\because x = 10 \text{ and } \frac{dx}{dt} = 2\right]$$

$$= 10\sqrt{3}\,\text{cm}^2/\text{s}$$

Q. 36 A ladder, 5 m long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/s, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 m from the wall is

(a) $\dfrac{1}{10}$ rad/s (b) $\dfrac{1}{20}$ rad/s

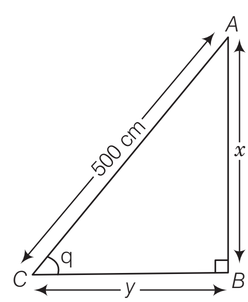
(c) 20 rad/s (d) 10 rad/s

Sol. *(b)* Let the angle between floor and the ladder be θ.

Let $AB = x$ cm and $BC = y$ cm

$\therefore$ $\sin\theta = \dfrac{x}{500}$ and $\cos\theta = \dfrac{y}{500}$

$\Rightarrow$ $x = 500\sin\theta$ and $y = 500\cos\theta$



Also, $\dfrac{dx}{dt} = 10$ cm/s

$\Rightarrow$ $500\cdot\cos\theta\cdot\dfrac{d\theta}{dt} = 10$ cm/s

$\Rightarrow$ $\dfrac{d\theta}{dt} = \dfrac{10}{500\cos\theta} = \dfrac{1}{50\cos\theta}$

For $y = 2$ m $= 200$ cm,

$$\frac{d\theta}{dt} = \frac{1}{50\cdot\dfrac{y}{500}} = \frac{10}{y}$$

$$= \frac{10}{200} = \frac{1}{20}\text{ rad/s}$$

Q. 37 The curve $y = x^{1/5}$ has at (0, 0)

(a) a vertical tangent (parallel to Y-axis)
(b) a horizontal tangent (parallel to X-axis)
(c) an oblique tangent
(d) no tangent

Sol. *(a)* We have, $y = x^{1/5}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{5} x^{\frac{1}{5} - 1} = \dfrac{1}{5} x^{-4/5}$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(0,0)} = \dfrac{1}{5} \times (0)^{-4/5} = \infty$

So, the curve $y = x^{1/5}$ has a vertical tangent at (0, 0), which is parallel to Y-axis.

Q. 38 The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is

(a) $3x - y = 8$
(b) $3x + y + 8 = 0$
(c) $x + 3y \pm 8 = 0$
(d) $x + 3y = 0$

Sol. *(c)* We have, the equation of the curve is $3x^2 - y^2 = 8$ ...(i)

Also, the given equation of the line is $x + 3y = 8$.

$\Rightarrow \quad 3y = 8 - x$

$\Rightarrow \quad y = -\dfrac{x}{3} + \dfrac{8}{3}$

Thus, slope of the line is $-\dfrac{1}{3}$ which should be equal to slope of the equation of normal to the curve.

On differentiating Eq. (i) w.r.t. x, we get

$$6x - 2y\frac{dy}{dx} = 0$$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{6x}{2y} = \dfrac{3x}{y} =$ Slope of the curve

Now, slope of normal to the curve $= -\dfrac{1}{\left(\dfrac{dy}{dx}\right)}$

$$= -\frac{1}{\left(\dfrac{3x}{y}\right)} = -\frac{y}{3x}$$

$\therefore \quad -\left(\dfrac{y}{3x}\right) = -\dfrac{1}{3}$

$\Rightarrow \quad -3y = -3x$

$\Rightarrow \quad y = x$

On substituting the value of the given equation of the curve, we get

$$3x^2 - x^2 = 8$$

$\Rightarrow \quad x^2 = \dfrac{8}{2}$

$\Rightarrow \quad x = \pm 2$

For $x = 2$, $3(2)^2 - y^2 = 8$

$\Rightarrow$ $y^2 = 4$

$\Rightarrow$ $y = \pm 2$

and for $x = -2$, $3(-2)^2 - y^2 = 8$

$\Rightarrow$ $y = \pm 2$

So, the points at which normals are parallel to the given line are $(\pm 2, \pm 2)$.

Hence, the equation of normal at $(\pm 2, \pm 2)$ is

$$y - (\pm 2) = -\frac{1}{3}[x - (\pm 2)]$$

$$\Rightarrow \quad 3[y - (\pm 2)] = -[x - (\pm 2)]$$

$$\Rightarrow \quad x + 3y \pm 8 = 0$$

Q. 39 If the curve $ay + x^2 = 7$ and $x^3 = y$, cut orthogonally at (1, 1), then the value of a is

(a) 1 (b) 0 (c) -6 (d) 6

Sol. *(d)* We have, $ay + x^2 = 7$ and $x^3 = y$

On differentiating w.r.t. x in both equations, we get

$$a \cdot \frac{dy}{dx} + 2x = 0 \quad \text{and } 3x^2 = \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a} \text{ and } \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1, 1)} = \frac{-2}{a} = m_1$$

$$\text{and} \quad \left(\frac{dy}{dx}\right)_{(1, 1)} = 3 \cdot 1 = 3 = m_2$$

Since, the curves cut orthogonally at (1, 1).

$$\therefore \quad m_1 \cdot m_2 = -1$$

$$\Rightarrow \quad \left(\frac{-2}{a}\right) \cdot 3 = -1$$

$$\therefore \quad a = 6$$

Q. 40 If $y = x^4 - 10$ and x changes from 2 to 1.99, then what is the change in y?

(a) 0.32 (b) 0.032 (c) 5.68 (d) 5.968

Sol. *(a)* We have, $y = x^4 - 10 \Rightarrow \dfrac{dy}{dx} = 4x^3$

and $\Delta x = 2.00 - 1.99 = 0.01$

$$\therefore \quad \Delta y = \frac{dy}{dx} \times \Delta x$$

$$= 4x^3 \times \Delta x$$

$$= 4 \times 2^3 \times 0.01$$

$$= 32 \times 0.01 = 0.32$$

So, the approximate change in y is 0.32.

Q. 41 **The equation of tangent to the curve $y(1+x^2)=2-x$, where it crosses X-axis, is**

(a) $x+5y=2$ (b) $x-5y=2$

(c) $5x-y=2$ (d) $5x+y=2$

Sol. *(a)* We have, equation of the curve $y(1+x^2)=2-x$ ...(i)

$\therefore \quad y\cdot(0+2x)+(1+x^2)\cdot\dfrac{dy}{dx}=0-1$ [on differentiating w.r.t. x]

$\Rightarrow \quad 2xy+(1+x^2)\dfrac{dy}{dx}=-1$

$\Rightarrow \quad \dfrac{dy}{dx}=\dfrac{-1-2xy}{1+x^2}$ ...(ii)

Since, the given curve passes through X-axis *i.e.,* $y=0$.

$\therefore \quad 0(1+x^2)=2-x$ [using Eq. (i)]

$\Rightarrow \quad x=2$

So, the curve passes through the point (2, 0).

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(2,\,0)}=\dfrac{-1-2\times 0}{1+2^2}=-\dfrac{1}{5}=$ Slope of the curve

$\therefore$ Slope of tangent to the curve $=-\dfrac{1}{5}$

$\therefore$ Equation of tangent of the curve passing through (2, 0) is

$$y-0=-\frac{1}{5}(x-2)$$

$\Rightarrow \quad y+\dfrac{x}{5}=+\dfrac{2}{5}$

$\Rightarrow \quad 5y+x=2$

Q. 42 **The points at which the tangents to the curve $y=x^3-12x+18$ are parallel to X-axis are**

(a) $(2,-2),(-2,-34)$ (b) $(2,34),(-2,0)$

(c) $(0,34),(-2,0)$ (d) $(2,2),(-2,34)$

Sol. *(d)* The given equation of curve is

$$y=x^3-12x+18$$

$\therefore \quad \dfrac{dy}{dx}=3x^2-12$ [on differentiating w.r.t. x]

So, the slope of line parallel to the X-axis.

$\therefore \quad \left(\dfrac{dy}{dx}\right)=0$

$\Rightarrow \quad 3x^2-12=0$

$\Rightarrow \quad x^2=\dfrac{12}{3}=4$

$\therefore \quad x=\pm 2$

For $x=2$, $\quad y=2^3-12\times 2+18=2$

and for $x=-2$, $y=(-2)^3-12(-2)+18=34$

So, the points are $(2, 2)$ and $(-2, 34)$.

Q. 43 The tangent to the curve $y = e^{2x}$ at the point (0, 1) meets X-axis at

(a) (0, 1) (b) $\left(-\frac{1}{2}, 0\right)$ (c) (2, 0) (d) (0, 2)

Sol. *(b)* The equation of curve is $y = e^{2x}$

Since, it passes through the point (0, 1).

$$\therefore \quad \frac{dy}{dx} = e^{2x} \cdot 2 = 2 \cdot e^{2x}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,\,1)} = 2 \cdot e^{2 \cdot 0} = 2 = \text{Slope of tangent to the curve}$$

$\therefore$ Equation of tangent is $y - 1 = 2(x - 0)$

$$\Rightarrow \quad y = 2x + 1$$

Since, tangent to curve $y = e^{2x}$ at the point (0, 1) meets X-axis *i.e.,* $y = 0$.

$$\therefore \quad 0 = 2x + 1 \Rightarrow \quad x = -\frac{1}{2}$$

So, the required point is $\left(\frac{-1}{2}, 0\right)$.

Q. 44 The slope of tangent to the curve $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$ at the point (2, − 1) is

(a) $\frac{22}{7}$ (b) $\frac{6}{7}$ (c) $-\frac{6}{7}$ (d) -6

Sol. *(b)* Equation of curve is given by

$$x = t^2 + 3t - 8 \quad \text{and} \quad y = 2t^2 - 2t - 5.$$

$$\therefore \quad \frac{dx}{dt} = 2t + 3 \quad \text{and} \quad \frac{dy}{dt} = 4t - 2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t - 2}{2t + 3} \qquad \text{...(i)}$$

Since, the curve passes through the point (2, − 1).

$$\therefore \quad 2 = t^2 + 3t - 8$$

$$\text{and} \quad -1 = 2t^2 - 2t - 5$$

$$\Rightarrow \quad t^2 + 3t - 10 = 0$$

$$\text{and} \quad 2t^2 - 2t - 4 = 0$$

$$\Rightarrow \quad t^2 + 5t - 2t - 10 = 0$$

$$\text{and} \quad 2t^2 + 2t - 4t - 4 = 0$$

$$\Rightarrow \quad t(t + 5) - 2(t + 5) = 0$$

$$\text{and} \quad 2t(t + 1) - 4(t + 1) = 0$$

$$\Rightarrow \quad (t - 2)(t + 5) = 0$$

$$\text{and} \quad (2t - 4)(t + 1) = 0$$

$$\Rightarrow \quad t = 2, -5 \text{ and } t = -1, 2$$

$$\Rightarrow \quad t = 2$$

$\therefore$ Slope of tangent,

$$\left(\frac{dy}{dx}\right)_{\text{at } t = 2} = \frac{4 \times 2 - 2}{2 \times 2 + 3} = \frac{6}{7} \qquad \text{[using Eq. (i)]}$$

Q. 45 Two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ intersect at an angle of

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

Sol. ***(c)*** Equation of two curves are given by

$$x^3 - 3xy^2 + 2 = 0$$

and $$3x^2y - y^3 - 2 = 0 \quad \text{[on differentiating w.r.t. } x\text{]}$$

$$\Rightarrow \quad 3x^2 - 3\left[x \cdot 2y\frac{dy}{dx} + y^2 \cdot 1\right] + 0 = 0$$

and $$3\left[x^2\frac{dy}{dx} + y \cdot 2x\right] - 3y^2\frac{dy}{dx} - 0 = 0$$

$$\Rightarrow \quad 3x \cdot 2y\frac{dy}{dx} + 3y^2 = 3x^2$$

and $$3y^2\frac{dy}{dx} = 3x^2\frac{dy}{dx} + 6xy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2 - 3y^2}{6xy}$$

and $$\frac{dy}{dx} = \frac{6xy}{3y^2 - 3x^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = \frac{3(x^2 - y^2)}{6xy}$$

and $$\left(\frac{dy}{dx}\right) = \frac{-6xy}{3(x^2 - y^2)}$$

$$\Rightarrow \quad m_1 = \frac{(x^2 - y^2)}{2xy}$$

and $$m_2 = \frac{-2xy}{x^2 - y^2}$$

$$\therefore \quad m_1m_2 = \frac{x^2 - y^2}{2xy} \cdot \frac{-(2xy)}{x^2 - y^2} = -1$$

Hence, both the curves are intersecting at right angle *i.e.,* making $\frac{\pi}{2}$ with each other.

Q. 46 The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is

(a) $[-1, \infty)$ (b) $[-2, -1]$ (c) $(-\infty, -2]$ (d) $[-1, 1]$

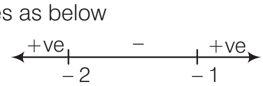
Sol. ***(b)*** We have, $$f(x) = 2x^3 + 9x^2 + 12x - 1$$

$$\therefore \quad f'(x) = 6x^2 + 18x + 12$$

$$= 6(x^2 + 3x + 2) = 6(x + 2)(x + 1)$$

So, $f'(x) \le 0$, for decreasing.

On drawing number lines as below



We see that $f'(x)$ is decreasing in $[-2, -1]$.

Q. 47 If $f : R \to R$ be defined by $f(x) = 2x + \cos x$, then f

(a) has a minimum at $x = \pi$ (b) has a maximum at $x = 0$

(c) is a decreasing function (d) is an increasing function

Sol. *(d)* We have, $f(x) = 2x + \cos x$

$\therefore$ $f'(x) = 2 + (-\sin x) = 2 - \sin x$

Since, $f'(x) > 0, \forall\ x$

Hence, $f(x)$ is an increasing function.

Q. 48 If $y = x(x-3)^2$ decreases for the values of x given by

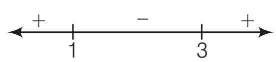
(a) $1 < x < 3$ (b) $x < 0$ (c) $x > 0$ (d) $0 < x < \frac{3}{2}$

Sol. *(a)* We have, $y = x(x-3)^2$

$$\therefore \quad \frac{dy}{dx} = x \cdot 2(x-3) \cdot 1 + (x-3)^2 \cdot 1$$

$$= 2x^2 - 6x + x^2 + 9 - 6x = 3x^2 - 12x + 9$$

$$= 3(x^2 - 3x - x + 3) = 3(x-3)(x-1)$$



So, $y = x(x-3)^2$ decreases for (1, 3).

[since, $y' < 0$ for all $x \in (1, 3)$, hence y is decreasing on (1, 3)]

Q. 49 The function $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$ is strictly

(a) increasing in $\left(\pi, \frac{3\pi}{2}\right)$ (b) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(c) decreasing in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (d) decreasing in $\left[0, \frac{\pi}{2}\right]$

Sol. *(b)* We have, $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$

$$\therefore \quad f'(x) = 12\sin^2 x \cdot \cos x - 12\sin x \cdot \cos x + 12\cos x$$

$$= 12[\sin^2 x \cdot \cos x - \sin x \cdot \cos x + \cos x]$$

$$= 12\cos x[\sin^2 x - \sin x + 1]$$

$$\Rightarrow \quad f'(x) = 12\cos x[\sin^2 x + (1 - \sin x)] \quad \text{...(i)}$$

$\because$ $1 - \sin x \ge 0$ and $\sin^2 x \ge 0$

$\therefore$ $\sin^2 x + 1 - \sin x \ge 0$

Hence, $f'(x) > 0$, when $\cos x > 0$ *i.e.*, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

So, $f(x)$ is increasing when $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $f'(x) < 0$, when $\cos x < 0$ *i.e.*, $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence, $f(x)$ is decreasing when $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Since, $\left(\frac{\pi}{2}, \pi\right) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence, $f(x)$ is decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

Q. 50 Which of the following functions is decreasing on $\left(0, \frac{\pi}{2}\right)$?

(a) $\sin 2x$ (b) $\tan x$ (c) $\cos x$ (d) $\cos 3x$

Sol. *(c)* In the interval $\left(0, \frac{\pi}{2}\right)$, $f(x) = \cos x$

$\Rightarrow$ $f'(x) = -\sin x$

which gives $f'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

Hence, $f(x) = \cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$.

Q. 51 The function $f(x) = \tan x - x$

(a) always increases

(b) always decreases

(c) never increases

(d) sometimes increases and sometimes decreases

Sol. *(a)* We have, $f(x) = \tan x - x$

$\therefore$ $f'(x) = \sec^2 x - 1$

$\Rightarrow$ $f'(x) > 0, \forall x \in R$

So, $f(x)$ always increases.

Q. 52 If x is real, then the minimum value of $x^2 - 8x + 17$ is

(a) -1 (b) 0 (c) 1 (d) 2

Sol. *(c)* Let $f(x) = x^2 - 8x + 17$

$\therefore$ $f'(x) = 2x - 8$

So, $f'(x) = 0$, gives $x = 4$

Now, $f''(x) = 2 > 0, \forall x$

So, $x = 4$ is the point of local minima.

$\therefore$ Minimum value of $f(x)$ at $x = 4$,

$$f(4) = 4 \times 4 - 8 \times 4 + 17 = 1$$

Q. 53 The smallest value of polynomial $x^3 - 18x^2 + 96x$ in $[0, 9]$ is

(a) 126 (b) 0 (c) 135 (d) 160

Sol. *(b)* We have, $f(x) = x^3 - 18x^2 + 96x$

$\therefore$ $f'(x) = 3x^2 - 36x + 96$

So, $f'(x) = 0$

Gives, $3x^2 - 36x + 96 = 0$

$\Rightarrow$ $3(x^2 - 12x + 32) = 0$

$\Rightarrow$ $(x - 8)(x - 4) = 0$

$\Rightarrow$ $x = 8, 4 \in [0, 9]$

We shall now evaluate the value of f at these points and at the end points of the interval $[0, 9]$ *i.e.,* at $x = 4$ and $x = 8$ and at $x = 0$ and at $x = 9$.

$$\therefore \qquad f(4) = 4^3 - 18\cdot 4^2 + 96\cdot 4$$
$$= 64 - 288 + 384 = 160$$
$$f(8) = 8^3 - 18\cdot 8^2 + 96\cdot 8 = 128$$
$$f(9) = 9^3 - 18\cdot 9^2 + 96\cdot 9$$
$$= 729 - 1458 + 864 = 135$$

and
$$f(0) = 0^3 - 18\cdot 0^2 + 96\cdot 0 = 0$$

Thus, we conclude that absolute minimum value of f on $[0, 9]$ is 0 occurring at $x = 0$.

Q. 54 The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has

(a) two points of local maximum

(b) two points of local minimum

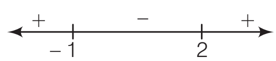
(c) one maxima and one minima

(d) no maxima or minima

Sol. *(c)* We have
$$f(x) = 2x^3 - 3x^2 - 12x + 4$$
$$\therefore \qquad f'(x) = 6x^2 - 6x - 12$$

Now,
$$f'(x) = 0 \Rightarrow 6(x^2 - x - 2) = 0$$
$$\Rightarrow \qquad 6(x+1)(x-2) = 0$$
$$\Rightarrow \qquad x = -1 \text{ and } x = +2$$

On number line for $f'(x)$, we get



Hence $x = -1$ is point of local maxima and $x = 2$ is point of local minima.

So, $f(x)$ has one maxima and one minima.

Q. 55 The maximum value of $\sin x \cdot \cos x$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\sqrt{2}$ (d) $2\sqrt{2}$

Sol. *(b)* We have,
$$f(x) = \sin x \cdot \cos x = \frac{1}{2}\sin 2x$$
$$\therefore \qquad f'(x) = \frac{1}{2}\cdot \cos 2x \cdot 2 = \cos 2x$$

Now,
$$f'(x) = 0 \Rightarrow \cos 2x = 0$$
$$\Rightarrow \qquad \cos 2x = \cos\frac{\pi}{2} \Rightarrow x = \frac{\pi}{4}$$

Also
$$f''(x) = \frac{d}{dx}\cos 2x = -\sin 2x \cdot 2 = -2\sin 2x$$
$$\therefore \qquad [f''(x)]_{\text{at } x = \pi/4} = -2\cdot \sin 2\cdot\frac{\pi}{4} = -2\sin\frac{\pi}{2} = -2 < 0$$

At $\frac{\pi}{4}$, $f(x)$ is maximum and $\frac{\pi}{4}$ is point of maxima.

$$\therefore \qquad f\left(\frac{\pi}{4}\right) = \frac{1}{2}\cdot \sin 2\cdot\frac{\pi}{4} = \frac{1}{2}$$

Q. 56 At $x = \frac{5\pi}{6}$, $f(x) = 2\sin 3x + 3\cos 3x$ is

(a) maximum (b) minimum
(c) zero (d) neither maximum nor minimum

Sol. *(d)* We have,

$$f(x) = 2\sin 3x + 3\cos 3x$$

$\therefore$ $$f'(x) = 2\cdot\cos 3x\cdot 3 + 3(-\sin 3x)\cdot 3$$

$\Rightarrow$ $$f'(x) = 6\cos 3x - 9\sin 3x \quad \text{...(i)}$$

Now, $$f''(x) = -18\sin 3x - 27\cos 3x$$
$$= -9(2\sin 3x + 3\cos 3x)$$

$\therefore$ $$f'\left(\frac{5\pi}{6}\right) = 6\cos\left(3\cdot\frac{5\pi}{6}\right) - 9\sin\left(3\cdot\frac{5\pi}{6}\right)$$
$$= 6\cos\frac{5\pi}{2} - 9\sin\frac{5\pi}{2}$$
$$= 6\cos\left(2\pi + \frac{\pi}{2}\right) - 9\sin\left(2\pi + \frac{\pi}{2}\right)$$
$$= 0 - 9 \neq 0$$

So, $x = \frac{5\pi}{6}$ cannot be point of maxima or minima.

Hence, $f(x)$ at $x = \frac{5\pi}{6}$ is neither maximum nor minimum.

Q. 57 The maximum slope of curve $y = -x^3 + 3x^2 + 9x - 27$ is

(a) 0 (b) 12 (c) 16 (d) 32

Sol. *(b)* We have,

$$y = -x^3 + 3x^2 + 9x - 27$$

$\therefore$ $$\frac{dy}{dx} = -3x^2 + 6x + 9 = \text{Slope of the curve}$$

and $$\frac{d^2y}{dx^2} = -6x + 6 = -6(x-1)$$

$\therefore$ $$\frac{d^2y}{dx^2} = 0$$

$\Rightarrow$ $$-6(x-1) = 0 \Rightarrow x = 1 > 0$$

Now, $$\frac{d^3y}{dx^3} = -6 < 0$$

So, the maximum slope of given curve is at $x = 1$.

$\therefore$ $$\left(\frac{dy}{dx}\right)_{(x=1)} = -3\cdot 1^2 + 6\cdot 1 + 9 = 12$$

Q. 58 The functin $f(x) = x^x$ has a stationary point at

(a) $x = e$ (b) $x = \frac{1}{e}$ (c) $x = 1$ (d) $x = \sqrt{e}$

Sol. *(b)* We have,

$$f(x) = x^x$$

Let $$y = x^x$$

and $$\log y = x\log x$$

$\therefore$ $$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x\cdot 1$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + \log x) \cdot x^x$$

$$\therefore \quad \frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1 + \log x) \cdot x^x = 0$$

$$\Rightarrow \quad \log x = -1$$

$$\Rightarrow \quad \log x = \log e^{-1}$$

$$\Rightarrow \quad x = e^{-1}$$

$$\Rightarrow \quad x = \frac{1}{e}$$

Hence, $f(x)$ has a stationary point at $x = \frac{1}{e}$.

Q. 59 The maximum value of $\left(\frac{1}{x}\right)^x$ is

(a) e (b) e^e (c) $e^{1/e}$ (d) $\left(\frac{1}{e}\right)^{1/e}$

Sol. *(c)* Let $\quad y = \left(\frac{1}{x}\right)^x$

$$\Rightarrow \quad \log y = x \cdot \log \frac{1}{x}$$

$$\therefore \quad \frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) + \log \frac{1}{x} \cdot 1$$

$$= -1 + \log \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = \left(\log \frac{1}{x} - 1\right) \cdot \left(\frac{1}{x}\right)^x$$

Now, $\quad \frac{dy}{dx} = 0$

$$\Rightarrow \quad \log \frac{1}{x} = 1 = \log e$$

$$\Rightarrow \quad \frac{1}{x} = e$$

$$\therefore \quad x = \frac{1}{e}$$

Hence, the maximum value of $f\left(\frac{1}{e}\right) = (e)^{1/e}$.

Fillers

Q. 60 The curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$ touch each other at the point ....

Sol. The curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$ touch each other at the point (3, 34).

Given, equation of curves are $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$

$$\therefore \quad \frac{dy}{dx} = 8x + 2$$

$$\text{and} \quad \frac{dy}{dx} = 3x^2 - 1$$

So, the slope of both curves should be same

$$\therefore \quad 8x + 2 = 3x^2 - 1$$

$$\Rightarrow \quad 3x^2 - 8x - 3 = 0$$

$$\Rightarrow \quad 3x^2 - 9x + x - 3 = 0$$

$$\Rightarrow \quad 3x(x - 3) + 1(x - 3) = 0$$

$$\Rightarrow \quad (3x + 1)(x - 3) = 0$$

$$\therefore \quad x = -\frac{1}{3} \text{ and } x = 3,$$

For $x = -\frac{1}{3}$,

$$y = 4\cdot\left(-\frac{1}{3}\right)^2 + 2\cdot\left(\frac{-1}{3}\right) - 8$$

$$= \frac{4}{9} - \frac{2}{3} - 8 = \frac{4 - 6 - 72}{9}$$

$$= -\frac{74}{9}$$

and for $x = 3$, $y = 4\cdot(3)^2 + 2\cdot(3) - 8$

$$= 36 + 6 - 8 = 34$$

So, the required points are (3, 34) and $\left(-\frac{1}{3}, \frac{-74}{9}\right)$.

Q. 61 The equation of normal to the curve $y = \tan x$ at (0, 0) is ....

Sol. The equation of normal to the curve $y = \tan x$ at (0, 0) is $x + y = 0$.

$$\because \quad y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,0)} = \sec^2 0 = 1 \text{ and } -\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{1}{1}$$

$\therefore$ Equation of normal to the curve $y = \tan x$ at (0, 0) is

$$y - 0 = -1(x - 0)$$

$$\Rightarrow \quad y + x = 0$$

Q. 62 The values of a for which the function $f(x) = \sin x - ax + b$ increases on R are ....

Sol. The values of a for which the function $f(x) = \sin x - ax + b$ increases on R are $(-\infty, -1)$.

$\because \quad f'(x) = \cos x - a$

and $\quad f'(x) > 0 \Rightarrow \cos x > a$

Since, $\quad \cos x \in [-1, 1]$

$\Rightarrow \quad a < -1 \Rightarrow a \in (-\infty, -1)$

Q. 63 The function $f(x) = \dfrac{2x^2 - 1}{x^4}$, (where, $x > 0$) decreases in the interval ....

Sol. The function $f(x) = \dfrac{2x^2 - 1}{x^4}$, where $x > 0$, decreases in the interval $(1, \infty)$.

$$\because \quad f'(x) = \frac{x^4 \cdot 4x - (2x^2 - 1) \cdot 4x^3}{x^8} = \frac{4x^5 - 8x^5 + 4x^3}{x^8}$$

$$= \frac{-4x^5 + 4x^3}{x^8} = \frac{4x^3(-x^2 + 1)}{x^8}$$

Also, $\quad f'(x) < 0$

$$\Rightarrow \quad \frac{4x^3(1 - x^2)}{x^8} < 0 \Rightarrow x^2 > 1$$

$\Rightarrow \quad x > \pm 1$

$\therefore \quad x \in (1, \infty)$

Q. 64 The least value of function $f(x) = ax + \dfrac{b}{x}$ (where, $a > 0$, $b > 0$, $x > 0$) is ....

Sol. The least value of function $f(x) = ax + \dfrac{b}{x}$ (where, $a > 0, b > 0, x > 0$) is $2\sqrt{ab}$.

$$\because \quad f'(x) = a - \frac{b}{x^2} \text{ and } f'(x) = 0$$

$$\Rightarrow \quad a = \frac{b}{x^2}$$

$$\Rightarrow \quad x^2 = \frac{b}{a} \Rightarrow x = \pm \sqrt{\frac{b}{a}}$$

Now, $\quad f''(x) = -b \cdot \dfrac{(-2)}{x^3} = + \dfrac{2b}{x^3}$

At $x = \sqrt{\dfrac{b}{a}}$, $\quad f''(x) = + \dfrac{2b}{\left(\dfrac{b}{a}\right)^{3/2}} = \dfrac{+2b \cdot a^{3/2}}{b^{3/2}}$

$$= +2b^{-1/2} \cdot a^{3/2} = +2\sqrt{\frac{a^3}{b}} > 0 \qquad [\because a, b > 0]$$

$\therefore$ Least value of $f(x)$, $\quad f\left(\sqrt{\dfrac{b}{a}}\right) = a \cdot \sqrt{\dfrac{b}{a}} + \dfrac{b}{\sqrt{\dfrac{b}{a}}}$

$$= a \cdot a^{-1/2} \cdot b^{1/2} + b \cdot b^{-1/2} \cdot a^{1/2}$$

$$= \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$$

5

Continuity and Differentiability

Short Answer Type Questions

Q. 1 Examine the continuity of the function $f(x) = x^3 + 2x^2 - 1$ at $x = 1$.

Thinking Process

We know that, function f will be continuous at $x = a$, if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.

Sol. We have, $f(x) = x^3 + 2x^2 - 1$ at $x = 1$.

$$\therefore \quad \lim_{x \to 1^+} f(x) = \lim_{h \to 0} (1 + h)^3 + 2(1 + h)^2 - 1 = 2$$

and $$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} (1 - h)^3 + 2(1 - h)^2 - 1 = 2$$

$$\therefore \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$$

and $$f(1) = 1 + 2 - 1 = 2$$

So, $f(x)$ is continuous at $x = 1$.

Note *Every polynomial function is continuous at any real point.*

Q. 2 $f(x) = \begin{cases} 3x + 5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$ at $x = 2$.

Sol. We have, $$f(x) = \begin{cases} 3x + 5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases} \text{ at } x = 2.$$

At $x = 2$, $$\text{LHL} = \lim_{x \to 2^-} (x)^2$$

$$= \lim_{h \to 0} (2 - h)^2 = \lim_{h \to 0} (4 + h^2 - 4h) = 4$$

and $$\text{RHL} = \lim_{x \to 2^+} (3x + 5)$$

$$= \lim_{h \to 0} [3(2 + h) + 5] = 11$$

Since, $\text{LHL} \neq \text{RHL}$ at $x = 2$

So, $f(x)$ is discontinuous at $x = 2$.

Q. 3 $f(x)=\begin{cases}\dfrac{1-\cos 2x}{x^2}, & \text{if } x\neq 0\\ 5, & \text{if } x=0\end{cases}$ at $x=0$.

Sol. We have, $$f(x)=\begin{cases}\dfrac{1-\cos 2x}{x^2}, & \text{if } x\neq 0\\ 5, & \text{if } x=0\end{cases} \text{ at } x=0.$$

At $x=0$,
$$\text{LHL}=\lim_{x\to 0^-}\frac{1-\cos 2x}{x^2}$$
$$=\lim_{h\to 0}\frac{1-\cos 2(0-h)}{(0-h)^2}$$
$$=\lim_{h\to 0}\frac{1-\cos 2h}{h^2} \qquad [\because \cos(-\theta)=\cos\theta]$$
$$=\lim_{h\to 0}\frac{1-1+2\sin^2 h}{h^2} \qquad [\because \cos 2\theta=1-2\sin^2\theta]$$
$$=\lim_{h\to 0}\frac{2(\sin h)^2}{(h)^2} \qquad \left[\because \lim_{h\to 0}\frac{\sin h}{h}=1\right]$$
$$=2$$
$$\text{RHL}=\lim_{x\to 0^+}\frac{1-\cos 2x}{x^2}$$
$$=\lim_{h\to 0}\frac{1-\cos 2(0+h)}{(0+h)^2}$$
$$=\lim_{h\to 0}\frac{2\sin^2 h}{h^2}=2 \qquad \left[\because \lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

and $\quad f(0)=5$

Since, $\quad \text{LHL}=\text{RHL}\neq f(0)$

Hence, $f(x)$ is not continuous at $x=0$.

Q. 4 $f(x)=\begin{cases}\dfrac{2x^2-3x-2}{x-2}, & \text{if } x\neq 2\\ 5, & \text{if } x=2\end{cases}$ at $x=2$.

Sol. We have, $f(x)=\begin{cases}\dfrac{2x^2-3x-2}{x-2}, & \text{if } x\neq 2\\ 5, & \text{if } x=2\end{cases}$ at $x=2$.

At $x=2$,
$$\text{LHL}=\lim_{x\to 2^-}\frac{2x^2-3x-2}{x-2}$$
$$=\lim_{h\to 0}\frac{2(2-h)^2-3(2-h)-2}{(2-h)-2}$$
$$=\lim_{h\to 0}\frac{8+2h^2-8h-6+3h-2}{-h}$$
$$=\lim_{h\to 0}\frac{2h^2-5h}{-h}=\lim_{h\to 0}\frac{h(2h-5)}{-h}=5$$
$$\text{RHL}=\lim_{x\to 2^+}\frac{2x^2-3x-2}{x-2}$$

$$= \lim_{h \to 0} \frac{2(2+h)^2 - 3(2+h) - 2}{(2+h) - 2}$$

$$= \lim_{h \to 0} \frac{8 + 2h^2 + 8h - 6 - 3h - 2}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 + 5h}{h} = \lim_{h \to 0} \frac{h(2h+5)}{h} = 5$$

and $\quad f(2) = 5$

$\therefore \quad \text{LHL} = \text{RHL} = f(2)$

So, $f(x)$ is continuous at $x = 2$.

Q. 5 $f(x) = \begin{cases} \dfrac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$ **at $x = 4$.**

Sol. We have, $\quad f(x) = \begin{cases} \dfrac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$ at $x = 4$.

At $x = 4$, $\quad \text{LHL} = \lim_{x \to 4^-} \dfrac{|x-4|}{2(x-4)}$

$$= \lim_{h \to 0} \frac{|4-h-4|}{2[(4-h)-4]} = \lim_{h \to 0} \frac{|0-h|}{(8-2h-8)}$$

$$= \lim_{h \to 0} \frac{h}{-2h} = \frac{-1}{2} \quad \text{and} \quad f(4) = 0 \neq \text{LHL}$$

So, $f(x)$ is discontinuous at $x = 4$.

Q. 6 $f(x) = \begin{cases} |x| \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ **at $x = 0$.**

Sol. We have, $\quad f(x) = \begin{cases} |x| \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

At $x = 0$, $\quad \text{LHL} = \lim_{x \to 0^-} |x| \cos \dfrac{1}{x} = \lim_{h \to 0} |0-h| \cos \dfrac{1}{0-h}$

$$= \lim_{h \to 0} h \cos \left(\frac{-1}{h}\right)$$

$$= 0 \times [\text{an oscillating number between } -1 \text{ and } 1] = 0$$

$$\text{RHL} = \lim_{x \to 0^+} |x| \cos \frac{1}{x}$$

$$= \lim_{h \to 0} |0+h| \cos \frac{1}{(0+h)}$$

$$= \lim_{h \to 0} h \cos \frac{1}{h}$$

$$= 0 \times [\text{an oscillating number between } -1 \text{ and } 1] = 0$$

and $\quad f(0) = 0$

Since, $\quad \text{LHL} = \text{RHL} = f(0)$

So, $f(x)$ is continuous at $x = 0$.

Q. 7 $f(x)=\begin{cases}|x-a|\sin\dfrac{1}{x-a}, & \text{if } x\neq 0\\ 0, & \text{if } x=a\end{cases}$ at $x=a$.

Sol. We have, $f(x)=\begin{cases}|x-a|\sin\dfrac{1}{x-a}, & \text{if } x\neq 0\\ 0, & \text{if } x=a\end{cases}$ at $x=a$

At $x=a$,
$$\text{LHL}=\lim_{x\to a^-}|x-a|\sin\frac{1}{x-a}$$
$$=\lim_{h\to 0}|a-h-a|\sin\left(\frac{1}{a-h-a}\right)$$
$$=\lim_{h\to 0}-h\sin\left(\frac{1}{h}\right) \qquad [\because \sin(-\theta)=-\sin\theta]$$
$$=0\times[\text{an oscillating number between } -1 \text{ and } 1]=0$$
$$\text{RHL}=\lim_{x\to a^+}|x-a|\sin\left(\frac{1}{x-a}\right)$$
$$=\lim_{h\to 0}|a+h-a|\sin\left(\frac{1}{a+h-a}\right)=\lim_{h\to 0}h\sin\frac{1}{h}$$
$$=0\times[\text{an oscillating number between } -1 \text{ and } 1]=0$$
and $f(a)=0$

$\therefore$ LHL = RHL = $f(a)$

So, $f(x)$ is continuous at $x=a$.

Q. 8 $f(x)=\begin{cases}\dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ at $x=0$.

Sol. We have, $f(x)=\begin{cases}\dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ at $x=0$

At $x=0$,
$$\text{LHL}=\lim_{x\to 0^-}\frac{e^{1/x}}{1+e^{1/x}}=\lim_{h\to 0}\frac{e^{1/0-h}}{1+e^{1/0-h}}$$
$$=\lim_{h\to 0}\frac{e^{-1/h}}{1+e^{-1/h}}=\lim_{h\to 0}\frac{1}{e^{1/h}(1+e^{-1/h})}$$
$$=\lim_{h\to 0}\frac{1}{e^{1/h}+1}=\frac{1}{e^{\infty}+1}=\frac{1}{\infty+1} \qquad [\because e^{\infty}=\infty]$$
$$=\frac{1}{\frac{1}{0}}=0$$
$$\text{RHL}=\lim_{x\to 0^+}\frac{e^{1/x}}{1+e^{1/x}}$$
$$=\lim_{h\to 0}\frac{e^{1/0+h}}{1+e^{1/0+h}}=\lim_{h\to 0}\frac{e^{1/h}}{1+e^{1/h}}$$

$$= \lim_{h \to 0} \frac{1}{e^{-1/h} + 1} = \frac{1}{e^{-\infty} + 1}$$

$$= \frac{1}{0 + 1} = 1 \qquad [\because e^{-\infty} = 0]$$

Hence, LHL $\neq$ RHL at $x = 0$.

So, $f(x)$ is discontinuous at $x = 0$.

Q. 9 $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ at $x = 1$.

Sol. We have, $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ at $x = 1$

At $x = 1$,

$$\text{HL} = \lim_{x \to 1^-} \frac{x^2}{2} = \lim_{h \to 0} \frac{(1-h)^2}{2}$$

$$= \lim_{h \to 0} \frac{1 + h^2 - 2h}{2} = \frac{1}{2}$$

$$\text{RHL} = \lim_{x \to 1^+} \left(2x^2 - 3x + \frac{3}{2}\right)$$

$$= \lim_{h \to 0} \left[2(1+h)^2 - 3(1+h) + \frac{3}{2}\right]$$

$$= \lim_{h \to 0} \left(2 + 2h^2 + 4h - 3 - 3h + \frac{3}{2}\right) = -1 + \frac{3}{2} = \frac{1}{2}$$

and $f(1) = \dfrac{1^2}{2} = \dfrac{1}{2}$

$\therefore$ LHL = RHL = $f(1)$

Hence, $f(x)$ is continuous at $x = 1$.

Q. 10 $f(x) = |x| + |x - 1|$ at $x = 1$.

Sol. We have, $f(x) = |x| + |x - 1|$ at $x = 1$

At $x = 1$,

$$\text{LHL} = \lim_{x \to 1^-} [|x| + |x - 1|]$$

$$= \lim_{h \to 0} [|1 - h| + |1 - h - 1|] = 1 + 0 = 1$$

and

$$\text{RHL} = \lim_{x \to 1^-} [|x| + |x - 1|]$$

$$= \lim_{h \to 0} [|1 + h| + |1 + h - 1|] = 1 + 0 = 1$$

and $f(1) = |1| + |0| = 1$

$\therefore$ LHL = RHL = $f(1)$

Hence, $f(x)$ is continuous at $x = 1$.

Note *Every modulus function is a continuous function at any real point.*

Q. 11 $f(x)=\begin{cases}3x-8, \text{ if } x\le 5\\ 2k, \quad \text{ if } x>5\end{cases}$ at $x=5$.

Sol. We have, $f(x)=\begin{cases}3x-8, \text{ if } x\le 5\\ 2k, \quad \text{ if } x>5\end{cases}$ at $x=5$

Since, $f(x)$ is continuous at $x=5$.

$\therefore$ LHL = RHL = $f(5)$

Now, LHL = $\lim_{x\to 5^-}(3x-8)=\lim_{h\to 0}[3(5-h)-8]$

$=\lim_{h\to 0}[15-3h-8]=7$

RHL = $\lim_{x\to 5^+}2k=\lim_{h\to 0}2k=2k=7$ [$\because$ LHL = RHL]

and $f(5)=3\times 5-8=7$

$\therefore$ $2k=7 \Rightarrow k=\frac{7}{2}$

Q. 12 $f(x)=\begin{cases}\dfrac{2^{x+2}-16}{4^x-16}, & \text{if } x\ne 2\\ k, & \text{if } x=2\end{cases}$ at $x=2$.

Sol. We have, $f(x)=\begin{cases}\dfrac{2^{x+2}-16}{4^x-16}, & \text{if } x\ne 2\\ k, & \text{if } x=2\end{cases}$ at $x=2$

Since, $f(x)$ is continuous at $x=2$.

$\therefore$ LHL = RHL = $f(2)$

At $x=2$, $\lim_{x\to 2}\dfrac{2^x\cdot 2^2-2^4}{4^x-4^2}=\lim_{x\to 2}\dfrac{4\cdot(2^x-4)}{(2^x)^2-(4)^2}$

$=\lim_{x\to 2}\dfrac{4\cdot(2^x-4)}{(2^x-4)(2^x+4)}$ [$\because a^2-b^2=(a+b)(a-b)$]

$=\lim_{x\to 2}\dfrac{4}{2^x+4}=\dfrac{4}{8}=\dfrac{1}{2}$

But $f(2)=k$

$\therefore$ $k=\frac{1}{2}$

Q.13 $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x\le 1\end{cases}$ at $x=0$.

Sol. We have, $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x\le 1\end{cases}$ at $x=0$.

$\therefore$ $$\text{LHL} = \lim_{x \to 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}$$

$$= \lim_{x \to 0^-} \left(\frac{\sqrt{1+kx} - \sqrt{1-kx}}{x} \right) \cdot \left(\frac{\sqrt{1+kx} + \sqrt{1-kx}}{\sqrt{1+kx} + \sqrt{1-kx}} \right)$$

$$= \lim_{x \to 0^-} \frac{1+kx-1+kx}{x[\sqrt{1+kx} + \sqrt{1-kx}]}$$

$$= \lim_{x \to 0^-} \frac{2kx}{x\sqrt{1+kx} + \sqrt{1-kx}}$$

$$= \lim_{h \to 0} \frac{2k}{\sqrt{1+k(0-h)} + \sqrt{1-k(0-h)}}$$

$$= \lim_{h \to 0} \frac{2k}{\sqrt{1-kh} + \sqrt{1+kh}} \qquad = \frac{2k}{2} = k$$

and $$f(0) = \frac{2 \times 0 + 1}{0 - 1} = -1$$

$\Rightarrow$ $k = -1$ $\qquad$ [$\because$ LHL = RHL = $f(0)$]

Q. 14 $f(x) = \begin{cases} \dfrac{1 - \cos kx}{x \sin x}, & \text{if } x \neq 0 \\ \dfrac{1}{2}, & \text{if } x = 0 \end{cases}$ at $x = 0$.

Sol. We have, $f(x) = \begin{cases} \dfrac{1 - \cos kx}{x \sin x}, & \text{if } x \neq 0 \\ \dfrac{1}{2}, & \text{if } x = 0 \end{cases}$ at $x = 0$

At $x = 0$, $$\text{LHL} = \lim_{x \to 0^-} \frac{1 - \cos kx}{x \sin x} = \lim_{h \to 0} \frac{1 - \cos k(0-h)}{(0-h)\sin(0-h)}$$

$$= \lim_{h \to 0} \frac{1 - \cos(-kh)}{-h \sin(-h)}$$

$$= \lim_{h \to 0} \frac{1 - \cos kh}{h \sin h} \qquad [\because \cos(-\theta) = \cos\theta, \sin(-\theta) = -\sin\theta]$$

$$= \lim_{h \to 0} \frac{1 - 1 + 2\sin^2 \frac{kh}{2}}{h \sin h} \qquad \left[\because \cos\theta = 1 - 2\sin^2\frac{\theta}{2}\right]$$

$$= \lim_{h \to 0} \frac{2\sin^2 \frac{kh}{2}}{h \sin h}$$

$$= \lim_{h \to 0} \frac{2\sin\frac{kh}{2}}{\frac{kh}{2}} \cdot \frac{\sin\frac{kh}{2}}{\frac{kh}{2}} \cdot \frac{1}{\frac{\sin h}{h}} \cdot \frac{k^2 h/4}{h}$$

$$= \frac{2k^2}{4} = \frac{k^2}{2} \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

Also, $$f(0) = \frac{1}{2} \Rightarrow \frac{k^2}{2} = \frac{1}{2} \Rightarrow k = \pm 1$$ p

Q. 15 Prove that the function f defined by $f(x) = \begin{cases} \dfrac{x}{|x| + 2x^2}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$

remains discontinuous at $x = 0$, regardless the choice of k.

Sol. We have, $f(x) = \begin{cases} \dfrac{x}{|x| + 2x^2}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$

At $x = 0$, $\text{LHL} = \lim_{x \to 0^-} \dfrac{x}{|x| + 2x^2} = \lim_{h \to 0} \dfrac{(0 - h)}{|0 - h| + 2(0 - h)^2}$

$= \lim_{h \to 0} \dfrac{-h}{h + 2h^2} = \lim_{h \to 0} \dfrac{-h}{h(1 + 2h)} = -1$

$\text{RHL} = \lim_{x \to 0^+} \dfrac{x}{|x| + 2x^2} = \lim_{h \to 0} \dfrac{0 + h}{|0 + h| + 2(0 + h)^2}$

$= \lim_{h \to 0} \dfrac{h}{h + 2h^2} = \lim_{h \to 0} \dfrac{h}{h(1 + 2h)} = 1$

and $f(0) = k$

Since, LHL $\neq$ RHL for any value of k.

Hence, $f(x)$ is discontinuous at $x = 0$ regardless the choice of k.

Q. 16 Find the values of a and b such that the function f defined by

$$f(x) = \begin{cases} \dfrac{x - 4}{|x - 4|} + a, & \text{if } x < 4 \\ a + b, & \text{if } x = 4 \\ \dfrac{x - 4}{|x - 4|} + b, & \text{if } x > 4 \end{cases}$$

is a continuous function at $x = 4$.

Sol. We have, $f(x) = \begin{cases} \dfrac{x - 4}{|x - 4|} + a, & \text{if } x < 4 \\ a + b, & \text{if } x = 4 \\ \dfrac{x - 4}{|x - 4|} + b, & \text{if } x > 4 \end{cases}$

At $x = 4$, $\text{LHL} = \lim_{x \to 4^-} \dfrac{x - 4}{|x - 4|} + a$

$= \lim_{h \to 0} \dfrac{4 - h - 4}{|4 - h - 4|} + a = \lim_{h \to 0} \dfrac{-h}{h} + a$

$= -1 + a$

$\text{RHL} = \lim_{x \to 4^+} \dfrac{x - 4}{|x - 4|} + b$

$= \lim_{h \to 0} \dfrac{4 + h - 4}{|4 + h - 4|} + b = \lim_{h \to 0} \dfrac{h}{h} + b = 1 + b$

$f(4) = a + b \Rightarrow -1 + a = 1 + b = a + b$

$\Rightarrow$ $-1 + a = a + b$ and $1 + b = a + b$

$\therefore$ $b = -1$ and $a = 1$

Q. 17 If the function $f(x)=\dfrac{1}{x+2}$, then find the points of discontinuity of the composite function $y=f\{f(x)\}$.

Sol. We have,
$$f(x)=\frac{1}{x+2}$$
$$\therefore \quad y=f\{f(x)\}$$
$$=f\left(\frac{1}{x+2}\right)=\frac{1}{\dfrac{1}{x+2}+2}$$
$$=\frac{1}{1+2x+4}\cdot(x+2)=\frac{(x+2)}{(2x+5)}$$

So, the function y will not be continuous at those points, where it is not defined as it is a rational function.

Therefore, $y=\dfrac{x+2}{(2x+5)}$ is not defined, when $2x+5=0$
$$\therefore \quad x=\frac{-5}{2}$$
Hence, y is discontinuous at $x=\dfrac{-5}{2}$.

Q. 18 Find all points of discontinuity of the function $f(t)=\dfrac{1}{t^2+t-2}$, where $t=\dfrac{1}{x-1}$.

Sol. We have,
$$f(t)=\frac{1}{t^2+t-2} \text{ and } t=\frac{1}{x-1}$$
$$\therefore \quad f(t)=\frac{1}{\left(\dfrac{1}{x^2+1-2x}\right)+\left(\dfrac{1}{x-1}\right)-\dfrac{2}{1}}$$
$$=\frac{1}{\left(\dfrac{1+x-1+[-2(x-1)^2]}{(x^2+1-2x)}\right)}$$
$$=\frac{x^2+1-2x}{x-2x^2-2+4x}$$
$$=\frac{x^2+1-2x}{-2x^2+5x-2}$$
$$=\frac{(x-1)^2}{-(2x^2-5x+2)}$$
$$=\frac{(x-1)^2}{(2x-1)(2-x)}$$

So, $f(t)$ is discontinuous at $2x-1=0 \Rightarrow x=1/2$

and $2-x=0 \Rightarrow x=2$.

Q. 19 Show that the function $f(x)=|\sin x+\cos x|$ is continuous at $x=\pi$.

Sol. We have, $f(x)=|\sin x+\cos x|$ at $x=\pi$

Let $g(x)=\sin x+\cos x$

and $h(x)=|x|$

$$\therefore \quad hog(x)=h[g(x)] = h(\sin x+\cos x) = |\sin x+\cos x|$$

Since, $g(x)=\sin x+\cos x$ is a continuous function as it is forming with addition of two continuous functions $\sin x$ and $\cos x$.

Also, $h(x)=|x|$ is also a continuous function. Since, we know that composite functions of two continuous functions is also a continuous function.

Hence, $f(x)=|\sin x+\cos x|$ is a continuous function everywhere.

So, $f(x)$ is continuous at $x=\pi$.

Q. 20 Examine the differentiability of f, where f is defined by

$$f(x)=\begin{cases} x[x], & \text{if } 0\le x<2 \\ (x-1)x, & \text{if } 2\le x<3 \end{cases} \text{ at } x=2.$$

Thinking Process

We know that, a function f is differentiable at a point a in its domain, if both $Lf'(a)$ and $Rf'(a)$ are finite and equal, where $Lf'(c)=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$ and $Rf'(c)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

Sol. We have, $f(x)=\begin{cases} x[x], & \text{if } 0\le x<2 \\ (x-1)x & \text{if } 2\le x<3 \end{cases}$ at $x=2$.

At $x=2$,

$$Lf'(2)=\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$=\lim_{h\to 0}\frac{(2-h)[2-h]-(2-1)2}{-h}$$

$\{\because [a-h]=[a-1], \text{where } a \text{ is any positive number}\}$

$$=\lim_{h\to 0}\frac{(2-h)(1)-2}{-h}$$

$$=\lim_{h\to 0}\frac{2-h-2}{-h}=\lim_{h\to 0}\frac{-h}{-h}=1$$

$$Rf'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{(2+h-1)(2+h)-(2-1)\cdot 2}{h}$$

$$=\lim_{h\to 0}\frac{(1+h)(2+h)-2}{h}$$

$$=\lim_{h\to 0}\frac{2+h+2h+h^2-2}{h}$$

$$=\lim_{h\to 0}\frac{h^2+3h}{h}=\lim_{h\to 0}\frac{h(h+3)}{h}=3$$

$$\therefore \quad Lf'(2)\ne Rf'(2)$$

So, $f(x)$ is not differentiable at $x=2$.

Q. 21 $f(x)=\begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x=0$.

Sol. We have, $f(x)=\begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x=0$

For differentiability at $x=0$,

$$Lf'(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{x^2\sin\frac{1}{x}-0}{x-0}$$

$$=\lim_{h\to 0}\frac{(0-h)^2\sin\left(\frac{1}{0-h}\right)}{0-h}=\lim_{h\to 0}\frac{h^2\sin\left(\frac{-1}{h}\right)}{-h}$$

$$=\lim_{h\to 0}+h\sin\left(\frac{1}{h}\right) \qquad [\because \sin(-\theta)=-\sin\theta]$$

$$=0\times[\text{an oscillating number between } -1 \text{and} 1]=0$$

$$Rf'(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x^2\sin\frac{1}{x}-0}{x-0}$$

$$=\lim_{h\to 0}\frac{(0+h)^2\sin\left(\frac{1}{0+h}\right)}{0+h}=\lim_{h\to 0}\frac{h^2\sin(1/h)}{h}$$

$$=\lim_{h\to 0}h\sin(1/h)$$

$$=0\times[\text{an oscillating number between } -1 \text{ and } 1]=0$$

$\because$ $Lf'(0)=Rf'(0)$

So, $f(x)$ is differentiable at $x=0$.

Q. 22 $f(x)=\begin{cases} 1+x, & \text{if } x \le 2 \\ 5-x, & \text{if } x > 2 \end{cases}$ at $x=2$.

Sol. We have, $f(x)=\begin{cases} 1+x, \text{ if } x \le 2 \\ 5-x, \text{ if } x > 2 \end{cases}$ at $x=2$.

For differentiability at $x=2$,

$$Lf'(2)=\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^-}\frac{(1+x)-(1+2)}{x-2}$$

$$=\lim_{h\to 0}\frac{(1+2-h)-3}{2-h-2}=\lim_{h\to 0}\frac{-h}{-h}=1$$

$$Rf'(2)=\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^+}\frac{(5-x)-3}{x-2}$$

$$=\lim_{h\to 0}\frac{5-(2+h)-3}{2+h-2}$$

$$=\lim_{h\to 0}\frac{5-2-h-3}{h}=\lim_{h\to 0}\frac{-h}{+h}$$

$$=-1$$

$\because$ $Lf'(2)\neq Rf'(2)$

So, $f(x)$ is not differentiable at $x=2$.

Q. 23 Show that $f(x)=|x-5|$ is continuous but not differentiable at $x=5$.

Sol. We have, $$f(x)=|x-5|$$

$\therefore$ $$f(x)=\begin{cases}-(x-5), & \text{if } x<5\\ x-5, & \text{if } x\ge 5\end{cases}$$

For continuity at $x=5$,

$$\text{LHL}=\lim_{x\to 5^-}(-x+5)$$

$$=\lim_{h\to 0}[-(5-h)+5]=\lim_{h\to 0}h=0$$

$$\text{RHL}=\lim_{x\to 5^+}(x-5)$$

$$=\lim_{h\to 0}(5+h-5)=\lim_{h\to 0}h=0$$

$\therefore$ $$f(5)=5-5=0$$

$\Rightarrow$ $$\text{LHL}=\text{RHL}=f(5)$$

Hence, $f(x)$ is continuous at $x=5$.

Now, $$Lf'(5)=\lim_{x\to 5^-}\frac{f(x)-f(5)}{x-5}$$

$$=\lim_{x\to 5^-}\frac{-x+5-0}{x-5}=-1$$

$$Rf'(5)=\lim_{x\to 5^+}\frac{f(x)-f(5)}{x-5}$$

$$=\lim_{x\to 5^+}\frac{x-5-0}{x-5}=1$$

$\therefore$ $$Lf'(5)\ne Rf'(5)$$

So, $f(x)=|x-5|$ is not differentiable at $x=5$.

Q. 24 A function $f:R\to R$ satisfies the equation $f(x+y)=f(x)\cdot f(y)$ for all $x, y\in R$, $f(x)\ne 0$. Suppose that the function is differentiable at $x=0$ and $f'(0)=2$, then prove that $f'(x)=2f(x)$.

Sol. Let $f:R\to R$ satisfies the equation $f(x+y)=f(x)\cdot f(y)$, $\forall\, x, y\in R$, $f(x)\ne 0$.
Let $f(x)$ is differentiable at $x=0$ and $f'(0)=2$.

$\Rightarrow$ $$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$$

$\Rightarrow$ $$2=\lim_{x\to 0}\frac{f(x)-f(0)}{x}$$

$\Rightarrow$ $$2=\lim_{h\to 0}\frac{f(0+h)-f(0)}{0+h}$$

$\Rightarrow$ $$2=\lim_{h\to 0}\frac{f(0)\cdot f(h)-f(0)}{h}$$

$\Rightarrow$ $$2=\lim_{h\to 0}\frac{f(0)\,[f(h)-1]}{h} \qquad [\because f(0)=f(h)]\ \ldots(\text{i})$$

Also, $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h} \qquad [\because f(x+y)=f(x)\cdot f(y)]$$

$$=\lim_{h\to 0}\frac{f(x)\,[f(h)-1]}{h}=2f(x) \qquad [\text{using Eq. (i)}]$$

$\therefore$ $$f'(x)=2f(x)$$

Q. 25 $2^{\cos^2 x}$

Sol. Let $$y = 2^{\cos^2 x}$$

$\therefore$ $$\log y = \log 2^{\cos^2 x} = \cos^2 x \cdot \log 2$$

On differentiating w.r.t. x, we get

$$\frac{d}{dy} \log y. \frac{dy}{dx} = \frac{d}{dx} \log 2 \cdot \cos^2 x$$

$\Rightarrow$ $$\frac{1}{y} \cdot \frac{dy}{dx} = \log 2 \frac{d}{dx} (\cos x)^2$$

$\Rightarrow$ $$\frac{1}{y} \cdot \frac{dy}{dx} = \log 2 \cdot [2 \cos x] \cdot \frac{d}{dx} \cos x$$

$$= \log 2 \cdot 2 \cos x \cdot (-\sin x)$$

$$= \log 2 \cdot [-(\sin 2x)]$$

$\therefore$ $$\frac{dy}{dx} = -y \cdot \log 2 (\sin 2x)$$

$$= -2^{\cos^2 x} \cdot \log 2 (\sin 2x)$$

Q. 26 $\dfrac{8^x}{x^8}$

Sol. Let $$y = \frac{8^x}{x^8} \Rightarrow \log y = \log \frac{8^x}{x^8}$$

$\Rightarrow$ $$\frac{d}{dy} \log y \cdot \frac{dy}{dx} = \frac{d}{dx} [\log 8^x - \log x^8]$$

$\Rightarrow$ $$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx} [x \cdot \log 8 - 8 \cdot \log x]$$

On differentiating w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log 8 \cdot 1 - 8 \cdot \frac{1}{x}$$

$\Rightarrow$ $$\frac{1}{y} \cdot \frac{dy}{dx} = \log 8 - \frac{8}{x}$$

$\therefore$ $$\frac{dy}{dx} = y \left(\log 8 - \frac{8}{x} \right) = \frac{8^x}{x^8} \left(\log 8 - \frac{8}{x} \right)$$

Q. 27 $\log (x + \sqrt{x^2 + a})$

Sol. Let $$y = \log (x + \sqrt{x^2 + a})$$

$\therefore$ $$\frac{dy}{dx} = \frac{d}{dx} \log (x + \sqrt{x^2 + a})$$

$$= \frac{1}{(x + \sqrt{x^2 + a})} . \frac{d}{dx} [x + \sqrt{x^2 + a}]$$

$$= \frac{1}{(x + \sqrt{x^2 + a})} \left[1 + \frac{1}{2} (x^2 + a)^{-1/2} \cdot 2x \right]$$

$$= \frac{1}{(x + \sqrt{x^2 + a})} \cdot \left(1 + \frac{x}{\sqrt{x^2 + a}} \right)$$

$$= \frac{(\sqrt{x^2 + a} + x)}{(x + \sqrt{x^2 + a})(\sqrt{x^2 + a})} = \frac{1}{(\sqrt{x^2 + a})}$$

Q. 28 $\log[\log\ (\log x^5)]$

Sol. Let $y = \log[\log(\log x^5)]$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{d}{dx}[\log\ (\log\ \log x^5)]$

$= \dfrac{1}{\log\log x^5} \cdot \dfrac{d}{dx}(\log \cdot \log x^5)$

$= \dfrac{1}{\log\log x^5} \cdot \left(\dfrac{1}{\log x^5}\right) . \dfrac{d}{dx}\log x^5$

$= \dfrac{1}{\log\log x^5} \cdot \dfrac{1}{\log x^5} \cdot \dfrac{d}{dx}(5\log x) = \dfrac{5}{x \cdot \log(\log x^5) \cdot \log(x^5)}$

Q. 29 $\sin\sqrt{x} + \cos^2\sqrt{x}$

Sol. Let $y = \sin\sqrt{x} + (\cos\sqrt{x})^2$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{d}{dx}\sin(x^{1/2}) + \dfrac{d}{dx}[\cos(x^{1/2})]^2$

$= \cos x^{1/2} \cdot \dfrac{d}{dx}x^{1/2} + 2\cos(x^{1/2})\ \dfrac{d}{dx}[\cos(x^{1/2})]$

$= \cos(x^{1/2})\ \dfrac{1}{2}x^{-1/2} + 2 \cdot \cos(x^{1/2}) \cdot \left[-\sin(x^{1/2}) . \dfrac{d}{dx}x^{1/2}\right]$

$= \cos\sqrt{x} \cdot \dfrac{1}{2\sqrt{x}}[-2\cos(x^{1/2})] \cdot \sin x^{1/2} \cdot \dfrac{1}{2\sqrt{x}}$

$= \dfrac{1}{2\sqrt{x}}[\cos(\sqrt{x}) - \sin(2\sqrt{x})]$

Q. 30 $\sin^n(ax^2 + bx + c)$

Sol. Let $y = \sin^n(ax^2 + bx + c)$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{d}{dx}[\sin(ax^2 + bx + c)]^n$

$= n \cdot [\sin(ax^2 + bx + c)]^{n-1} \cdot \dfrac{d}{dx}\sin(ax^2 + bx + c)$

$= n \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c) . \dfrac{d}{dx}(ax^2 + bx + c)$

$= n \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c) \cdot (2ax + b)$

$= n \cdot (2ax + b) \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c)$

Q. 31 $\cos(\tan\sqrt{x+1})$

Sol. Let $y = \cos(\tan\sqrt{x+1})$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{d}{dx}\cos(\tan\sqrt{x+1}) = -\sin(\tan\sqrt{x+1}) \cdot \dfrac{d}{dx}(\tan\sqrt{x+1})$

$= -\sin(\tan\sqrt{x+1}) \cdot \sec^2\sqrt{x+1} \cdot \dfrac{d}{dx}(x+1)^{1/2}$ $\left[\because \dfrac{d}{dx}(\tan x) = \sec^2 x\right]$

$= -\sin(\tan\sqrt{x+1}) \cdot (\sec\sqrt{x+1})^2 \cdot \dfrac{1}{2}(x+1)^{-1/2} \cdot \dfrac{d}{dx}(x+1)$

$= \dfrac{-1}{2\sqrt{x+1}} \cdot \sin(\tan\sqrt{x+1}) \cdot \sec^2(\sqrt{x+1})$

Q. 32 $\sin x^2 + \sin^2 x + \sin^2 (x^2)$

Sol. Let $y = \sin x^2 + \sin^2 x + \sin^2 (x^2)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{d}{dx}\sin (x^2) + \dfrac{d}{dx}(\sin x)^2 + \dfrac{d}{dx}(\sin x^2)^2$

$= \cos (x^2)\dfrac{d}{dx}(x^2) + 2\sin x \cdot \dfrac{d}{dx}\sin x + 2\sin x^2 \cdot \dfrac{d}{dx}\sin x^2$

$= \cos x^2 2x + 2 \cdot \sin x \cdot \cos x + 2\sin x^2 \cos x^2 \cdot \dfrac{d}{dx}x^2$

$= 2x\cos (x)^2 + 2 \cdot \sin x \cdot \cos x + 2\sin x^2 \cdot \cos x^2 \cdot 2x$

$= 2x\cos (x)^2 + \sin 2x + \sin 2\,(x)^2 \cdot 2\,x$

$= 2x\cos (x^2) + 2x \cdot \sin 2\,(x^2) + \sin 2\,x$

Q. 33 $\sin^{-1}\dfrac{1}{\sqrt{x+1}}$

Sol. Let $y = \sin^{-1}\dfrac{1}{\sqrt{x+1}}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{d}{dx}\sin^{-1}\dfrac{1}{\sqrt{x+1}}$

$= \dfrac{1}{\sqrt{1-\left(\dfrac{1}{\sqrt{x+1}}\right)^2}} \cdot \dfrac{d}{dx}\dfrac{1}{(x+1)^{1/2}} \qquad \left[\because \dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1-x^2}}\right]$

$= \dfrac{1}{\sqrt{\dfrac{x+1-1}{x+1}}} \cdot \dfrac{d}{dx} \cdot (x+1)^{-1/2}$

$= \sqrt{\dfrac{x+1}{x}} \cdot \dfrac{-1}{2}(x+1)^{-\frac{1}{2}-1} \cdot \dfrac{d}{dx}(x+1)$

$= \dfrac{(x+1)^{1/2}}{x^{1/2}} \cdot \left(-\dfrac{1}{2}\right)(x+1)^{-3/2} = \dfrac{-1}{2\sqrt{x}} \cdot \left(\dfrac{1}{x+1}\right)$

Q. 34 $(\sin x)^{\cos x}$

Sol. Let $y = (\sin x)^{\cos x}$

$\Rightarrow \quad \log y = \log(\sin x)^{\cos x} = \cos x \log \sin x$

$\therefore \quad \dfrac{d}{dy}\log y \cdot \dfrac{dy}{dx} = \dfrac{d}{dx}(\cos x \cdot \log \sin x)$

$\Rightarrow \quad \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \cos x \cdot \dfrac{d}{dx}\log \sin x + \log \sin x \cdot \dfrac{d}{dx}\cos x$

$= \cos x \cdot \dfrac{1}{\sin x} \cdot \dfrac{d}{dx}\sin x + \log \sin x \cdot (-\sin x)$

$= \cot x \cdot \cos x - \log (\sin x) \cdot \sin x \qquad \left[\because \cot x = \dfrac{\cos x}{\sin x}\right]$

$\therefore \quad \dfrac{dy}{dx} = y\left[\dfrac{\cos^2 x}{\sin x} - \sin x \cdot \log (\sin x)\right]$

$= \sin x^{\cos x}\left[\dfrac{\cos^2 x}{\sin x} - \sin x \cdot \log (\sin x)\right]$

Q. 35 $\sin^m x \cdot \cos^n x$

Sol. Let $y = \sin^m x \cdot \cos^n x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[(\sin x)^m \cdot (\cos x)^n]$$

$$= (\sin x)^m \cdot \frac{d}{dx}(\cos x)^n + (\cos x)^n \cdot \frac{d}{dx}(\sin x)^m$$

$$= (\sin x)^m \cdot n(\cos x)^{n-1} \cdot \frac{d}{dx}\cos x + (\cos x)^n\ m(\sin x)^{m-1} \cdot \frac{d}{dx}\sin x$$

$$= (\sin x)^m \cdot n(\cos x)^{n-1}\ (-\sin x) + (\cos x)^n \cdot m(\sin x)^{m-1}\cos x$$

$$= -n\sin^m x \cdot \cos^{n-1} x \cdot (\sin x) + m\cos^n x \cdot \sin^{m-1} x \cdot \cos x$$

$$= -n \cdot \sin^m x \cdot \sin x \cdot \cos^n x \cdot \frac{1}{\cos x} + m \cdot \sin^m x. \frac{1}{\sin x} \cdot \cos^n x \cdot \cos x$$

$$= -n \cdot \sin^m x \cdot \cos^n x \cdot \tan x + m\sin^m x \cdot \cos^n x \cdot \cot x$$

$$= \sin^m x \cdot \cos^n x\,[-n\tan x + m\cot x]$$

Q. 36 $(x+1)^2(x+2)^3(x+3)^4$

Sol. Let $y = (x+1)^2\,(x+2)^3(x+3)^4$

$$\therefore \quad \log y = \log\{(x+1)^2 \cdot (x+2)^3(x+3)^4\}$$

$$= \log(x+1)^2 + \log(x+2)^3 + \log(x+3)^4$$

and $$\frac{d}{dy}\log y \cdot \frac{dy}{dx} = \frac{d}{dx}[2\log(x+1)] + \frac{d}{dx}[3\log(x+2)] + \frac{d}{dx}[4\log(x+3)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{(x+1)} \cdot \frac{d}{dx}(x+1) + 3 \cdot \frac{1}{(x+2)} \cdot \frac{d}{dx}(x+2)$$

$$+ 4 \cdot \frac{1}{(x+3)} \cdot \frac{d}{dx}(x+3) \qquad \left[\because \frac{d}{dx}(\log x) = \frac{1}{x}\right]$$

$$= \left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)}\right]$$

$$= (x+1)^2 \cdot (x+2)^3 \cdot (x+3)^4\left[\frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)}\right]$$

$$= (x+1)^2 \cdot (x+2)^3 \cdot (x+3)^4$$
$$\left[\frac{2(x+2)(x+3) + 3(x+1)(x+3) + 4(x+1)(x+2)}{(x+1)(x+2)(x+3)}\right]$$

$$= \frac{(x+1)^2\,(x+2)^3\,(x+3)^4}{(x+1)(x+2)(x+3)}$$
$$[2(x^2+5x+6) + 3(x^2+4x+3) + 4(x^2+3x+2)]$$

$$= (x+1)(x+2)^2\,(x+3)^3$$
$$[2x^2 + 10x + 12 + 3x^2 + 12x + 9 + 4x^2 + 12x + 8]$$

$$= (x+1)(x+2)^2\,(x+3)^3\,[9x^2 + 34x + 29]$$

Q. 37 $\cos^{-1}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right), -\dfrac{\pi}{4}<x<\dfrac{\pi}{4}$

Sol. Let $$y=\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$\therefore \quad \frac{dy}{dx}=\frac{d}{dx}\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$=\frac{-1}{\sqrt{1-\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)^2}}\cdot\frac{d}{dx}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$\left[\because \frac{d}{dx}(\cos x)=-\frac{1}{\sqrt{1-x^2}}\right]$$

$$=\frac{-1}{\sqrt{4-\frac{(\sin^2 x+\cos^2 x+2\sin x\cdot\cos x)}{2}}}\cdot\frac{1}{\sqrt{2}}(\cos x-\sin x)$$

$$=\frac{-1\cdot\sqrt{2}}{\sqrt{1-\sin 2x}}\cdot\frac{1}{\sqrt{2}}(\cos x-\sin x)$$

$$[\because 1-\sin 2x=(\cos x-\sin x)^2=\cos^2 x+\sin^2 x-2\sin x\cos x]$$

$$=\frac{-1(\cos x-\sin x)}{(\cos x-\sin x)}=-1$$

Q. 38 $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}, -\dfrac{\pi}{4}<x<\dfrac{\pi}{4}$

Sol. Let $$y=\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$\therefore \quad \frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$=\frac{1}{1+\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)^2}}\cdot\frac{d}{dx}\left[\frac{1-\cos x}{1+\cos x}\right]^{1/2} \qquad \left[\because \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}\right]$$

$$=\frac{1}{1+\frac{1-\cos x}{1+\cos x}}\cdot\frac{1}{2}\left[\frac{1-\cos x}{1+\cos x}\right]^{-1/2}\cdot\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$=\frac{1}{\frac{1+\cos x+1-\cos x}{1+\cos x}}\cdot\frac{1}{2}\left[\frac{(1-\cos x)}{(1+\cos x)}\cdot\frac{(1-\cos x)}{(1-\cos x)}\right]^{-1/2}$$

$$\cdot\frac{(1+\cos x)\cdot\sin x+(1-\cos x)\cdot\sin x}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)}{2}\cdot\frac{1}{2}\left[\frac{(1-\cos x)^2}{(1-\cos^2 x)}\right]^{-1/2}\left[\frac{\sin x\,(1+\cos x+1-\cos x)}{(1+\cos x)^2}\right]$$

$$=\frac{(1+\cos x)}{2}\cdot\frac{1}{2}\left[\frac{(1-\cos x)^2}{(1-\cos^2 x)}\right]^{-1/2}\left[\frac{\sin x\,(1+\cos x+1-\cos x)}{(1+\cos x)^2}\right]$$

$$= \frac{(1+\cos x)}{2} \cdot \frac{1}{2}\left[\frac{(1-\cos x)^2}{\sin x}\right]^{-1/2} \cdot \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)}{2} \cdot \frac{1}{2} \cdot \frac{\sin x}{(1-\cos x)} \cdot \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{2\sin^2 x}{4(1+\cos x)(1-\cos x)} = \frac{1}{2} \cdot \frac{\sin^2 x}{(1-\cos^2 x)}$$

$$= \frac{1}{2} \cdot \frac{\sin^2 x}{\sin^2 x} = \frac{1}{2}$$

Alternate Method

Let $\quad y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

$$= \tan^{-1}\left(\sqrt{\frac{1-1+2\sin^2\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1}}\right) \qquad \left[\because \cos x = 1-2\sin^2\frac{x}{2} = 2\cos^2\frac{x}{2}-1\right]$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}$$

Q. 39 $\tan^{-1}(\sec x + \tan x), \frac{-\pi}{2} < x < \frac{\pi}{2}$

Sol. Let $\quad y = \tan^{-1}(\sec x + \tan x)$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\tan^{-1}(\sec x + \tan x)$

$$= \frac{1}{1+(\sec x + \tan x)^2} \cdot \frac{d}{dx}(\sec x + \tan x) \qquad \left[\because \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}\right]$$

$$= \frac{1}{1+\sec^2 x + \tan^2 x + 2\sec x \cdot \tan x} \cdot [\sec x \cdot \tan x + \sec^2 x]$$

$$= \frac{1}{(\sec^2 x + \sec^2 x + 2\sec x \cdot \tan x)} \cdot \sec x \cdot (\sec x + \tan x)$$

$$= \frac{1}{2\sec x(\tan x + \sec x)} \cdot \sec x(\sec x + \tan x) = \frac{1}{2}$$

Q. 40 $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right), \frac{-\pi}{2} < x < \frac{\pi}{2}$ and $\frac{a}{b}\tan x > -1.$

Sol. Let $\quad y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$

$$= \tan^{-1}\left[\frac{\frac{a\cos x}{b\cos x} - \frac{b\sin x}{b\cos x}}{\frac{b\cos x}{b\cos x} + \frac{a\sin x}{b\cos x}}\right] = \tan^{-1}\left[\frac{\frac{a}{b} - \tan x}{1+\frac{a}{b}\tan x}\right]$$

$$= \tan^{-1}\frac{a}{b} - \tan^{-1}\tan x \qquad \left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\frac{a}{b} - x$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{a}{b}\right) - \frac{d}{dx}(x)$$

$$= 0 - 1 \qquad \left[\because \frac{d}{dx}\left(\frac{a}{b}\right) = 0\right]$$

$$= -1$$

Q. 41 $\sec^{-1}\left(\dfrac{1}{4x^3 - 3x}\right), 0 < x < \dfrac{1}{\sqrt{2}}$

Sol. Let $$y = \sec^{-1}\left(\frac{1}{4x^3 - 3x}\right) \qquad \text{...(i)}$$

On putting $x = \cos\theta$ in Eq. (i), we get

$$y = \sec^{-1}\frac{1}{4\cos^3\theta - 3\cos\theta}$$

$$= \sec^{-1}\frac{1}{\cos 3\theta}$$

$$= \sec^{-1}(\sec 3\theta) = 3\theta$$

$$= 3\cos^{-1}x \qquad [\because \theta = \cos^{-1}x]$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(3\cos^{-1}x)$$

$$= 3\cdot\frac{-1}{\sqrt{1 - x^2}}$$

Q. 42 $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right), \dfrac{-1}{\sqrt{3}} < \dfrac{x}{a} < \dfrac{1}{\sqrt{3}}$

Sol. Let $$y = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$$

Put $$x = a\tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$$

$$\therefore \quad y = \tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right] \qquad \left[\because \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right]$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta$$

$$= 3\tan^{-1}\frac{x}{a} \qquad \left[\because \theta = \tan^{-1}\frac{x}{a}\right]$$

$$\therefore \quad \frac{dy}{dx} = 3\cdot\frac{d}{dx}\tan^{-1}\frac{x}{a} = 3\cdot\left[\frac{1}{1 + \frac{x^2}{a^2}}\right]\cdot\frac{d}{dx}\cdot\left(\frac{x}{a}\right)$$

$$= 3\cdot\frac{a^2}{a^2 + x^2}\cdot\frac{1}{a} = \frac{3a}{a^2 + x^2}$$

Q. 43 $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right], -1 < x < 1, x \neq 0$

Sol. Let $\quad y = \tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$

Put $\quad x^2 = \cos 2\theta$

$$\therefore \quad y = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-1+2\sin^2\theta}}{\sqrt{1+2\cos^2\theta-1}-\sqrt{1-1+2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right) = \tan^{-1}\left[\frac{\sqrt{2}(\cos\theta+\sin\theta)}{\sqrt{2}(\cos\theta-\sin\theta)}\right]$$

$$= \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \tan^{-1}\left(\frac{\frac{\cos\theta+\sin\theta}{\cos\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}+\theta\right) \qquad \left[\because \tan(a+b) = \frac{\tan a+\tan b}{1-\tan a\cdot\tan b}\right]$$

$$= \frac{\pi}{4}+\theta = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 \qquad \left[\because 2\theta = \cos^{-1}x^2 \Rightarrow \theta = \frac{1}{2}\cos^{-1}x^2\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}\right)+\frac{d}{dx}\left(\frac{1}{2}\cos^{-1}x^2\right)$$

$$= 0+\frac{1}{2}\cdot\frac{-1}{\sqrt{1-x^4}}\cdot\frac{d}{dx}x^2 = \frac{1}{2}\cdot\frac{-2x}{\sqrt{1-x^4}} = \frac{-x}{\sqrt{1-x^4}}$$

Find $\frac{dy}{dx}$ of each of the functions expressed in parametric form.

Q. 44 $x = t+\frac{1}{t},\ y = t-\frac{1}{t}$

Sol. $\because \quad x = t+\frac{1}{t}$ and $y = t-\frac{1}{t}$

$$\therefore \quad \frac{dx}{dt} = \frac{d}{dt}\left(t+\frac{1}{t}\right) \quad \text{and} \quad \frac{dy}{dt} = \frac{d}{dt}\left(t-\frac{1}{t}\right)$$

$$\Rightarrow \quad \frac{dx}{dt} = 1+(-1)t^{-2} \quad \text{and} \quad \frac{dy}{dt} = 1-(-1)t^{-2}$$

$$\Rightarrow \quad \frac{dx}{dt} = 1-\frac{1}{t^2} \quad \text{and} \quad \frac{dy}{dt} = 1+\frac{1}{t^2}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{t^2-1}{t^2} \quad \text{and} \quad \frac{dy}{dt} = \frac{t^2+1}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t^2+1/t^2}{t^2-1/t^2} = \frac{t^2+1}{t^2-1}$$

Q. 45 $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right), y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Sol. $\because$ $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$ and $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

$\therefore$
$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left[e^{\theta}\cdot\left(\theta + \frac{1}{\theta}\right)\right]$$
$$= e^{\theta}\cdot\frac{d}{d\theta}\left(\theta + \frac{1}{\theta}\right) + \left(\theta + \frac{1}{\theta}\right)\cdot\frac{d}{d\theta}e^{\theta}$$
$$= e^{\theta}\left(1 - \frac{1}{\theta^2}\right) + \left(\theta + \frac{1}{\theta}\right)e^{\theta}$$
$$= e^{\theta}\left(1 - \frac{1}{\theta^2} + \theta + \frac{1}{\theta}\right)$$
$$= e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right) \quad \text{...(i)}$$

and
$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[e^{-\theta}\cdot\left(\theta - \frac{1}{\theta}\right)\right]$$
$$= e^{-\theta}\cdot\frac{d}{d\theta}\left(\theta - \frac{1}{\theta}\right) + \frac{d}{d\theta}e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$$
$$= e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) + \left(\theta - \frac{1}{\theta}\right)e^{-\theta}\cdot\frac{d}{d\theta}(-\theta)$$
$$= e^{-\theta}\left[\frac{\theta^2 + 1}{\theta^2} - \frac{\theta^2 - 1}{\theta}\right] = e^{-\theta}\left[\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right] \quad \text{...(ii)}$$

$\therefore$
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{e^{-\theta}\left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right)}{e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right)}$$
$$= e^{-2\theta}\left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^3 + \theta^2 + \theta - 1}\right)$$

Q. 46 $x = 3\cos\theta - 2\cos^3\theta,\ y = 3\sin\theta - 2\sin^3\theta$

Sol. $\because$ $x = 3\cos\theta - 2\cos^3\theta$ and $y = 3\sin\theta - 2\sin^3\theta$

$\therefore$
$$\frac{dx}{d\theta} = \frac{d}{d\theta}(3\cos\theta) - \frac{d}{d\theta}(2\cos^3\theta)$$
$$= 3\cdot(-\sin\theta) - 2\cdot 3\cos^2\theta\cdot\frac{d}{d\theta}\cdot\cos\theta$$
$$= -3\sin\theta + 6\cos^2\theta\sin\theta$$

and
$$\frac{dy}{d\theta} = 3\cos A - 2\cdot 3\sin^2\theta\cdot\frac{d}{d\theta}\cdot\sin\theta$$
$$= 3\cos\theta - 6\sin^2\theta\cdot\cos\theta$$

Now,
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta - 6\sin^2\theta\cos\theta}{-3\sin\theta + 6\cos^2\theta\sin\theta}$$
$$= \frac{3\cos\theta(1 - 2\sin^2\theta)}{3\sin\theta(-1 + 2\cos^2\theta)} = \cot\theta\cdot\frac{\cos 2\theta}{\cos 2\theta} = \cot\theta$$

Q. 47 $\sin x = \dfrac{2t}{1+t^2}, \tan y = \dfrac{2t}{1-t^2}$

Sol. $\because$ $$\sin x = \frac{2t}{1+t^2} \quad \text{...(i)}$$

and $$\tan y = \frac{2t}{1-t^2} \quad \text{...(ii)}$$

$$\therefore \quad \frac{d}{dx}\sin x \cdot \frac{dx}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$$\Rightarrow \quad \cos x \frac{dx}{dt} = \frac{(1+t^2)\cdot\frac{d}{dt}(2t) - (2t)\cdot\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{2(1+t^2) - 2t\cdot 2t}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\cos x}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\sqrt{1-\sin^2 x}} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{(1+t^2)}{(1-t^2)} = \frac{2}{1+t^2} \quad \text{...(iii)}$$

Also, $$\frac{d}{dy}\tan y\cdot\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$$

$$\sec^2 y\frac{dy}{dt} = \frac{(1-t^2)\frac{d}{dt}\cdot(2t) - 2t\cdot\frac{d}{dt}(1-t^2)}{(1-t^2)^2}$$

$$\frac{dy}{dt} = \frac{2-2t^2+4t^2}{(1-t^2)^2}\cdot\frac{1}{\sec^2 y}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{1}{(1+\tan^2 y)} = \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{1}{1+\frac{4t^2}{(1-t^2)^2}}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{(1-t^2)^2}{(1+t^2)^2} = \frac{2}{1+t^2} \quad \text{...(iv)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2/1+t^2}{2/1+t^2} = 1 \quad \text{[from Eqs. (iii) and (iv)]}$$

Q. 48 $x = \dfrac{1+\log t}{t^2}, y = \dfrac{3+2\log t}{t}$

Sol. $\because$ $$x = \frac{1+\log t}{t^2} \text{ and } y = \frac{3+2\log t}{t}$$

$$\therefore \quad \frac{dx}{dt} = \frac{t^2.\frac{d}{dt}(1+\log t) - (1+\log t).\frac{d}{dt}t^2}{(t^2)^2}$$

$$= \frac{t^2 \cdot \frac{1}{t} - (1 + \log t) \cdot 2t}{t^4} = \frac{t - (1 + \log t) \cdot 2t}{t^4}$$

$$= \frac{t}{t^4} [1 - 2(1 + \log t) = \frac{-1 - 2\log t}{t^3} \quad \text{... (i)}$$

and $$\frac{dy}{dt} = \frac{t \cdot \frac{d}{dt}(3 + 2\log t) - (3 + 2\log t) \cdot \frac{d}{dt} t}{t^2}$$

$$= \frac{t \cdot 2 \cdot \frac{1}{t} - (3 + 2\log t) \cdot 1}{t^2}$$

$$= \frac{2 - 3 - 2\log t}{t^2} = \frac{-1 - 2\log t}{t^2} \quad \text{...(ii)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1 - 2\log t / t^2}{-1 - 2\log t / t^3} = t$$

Q. 49 If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, then prove that $\dfrac{dy}{dx} = -\dfrac{y \log x}{x \log y}$.

Sol. $\because$ $$x = e^{\cos 2t} \text{ and } y = e^{\sin 2t}$$

$\therefore$ $$\frac{dx}{dt} = \frac{d}{dt} e^{\cos 2t} = e^{\cos 2t} \cdot \frac{d}{dt} \cos 2t$$

$$= e^{\cos 2t} \cdot (-\sin 2t) \cdot \frac{d}{dt}(2t)$$

$$\frac{dx}{dt} = -2e^{\cos 2t} \cdot \sin 2t \quad \text{...(i)}$$

and $$\frac{dy}{dt} = \frac{d}{dt} e^{\sin 2t} = e^{\sin 2t} \cdot \frac{d}{dt} \sin 2t$$

$$= e^{\sin 2t} \cos 2t \cdot \frac{d}{dt} 2t$$

$$= 2e^{\sin 2t} \cdot \cos 2t \quad \text{...(ii)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{\sin 2t} \cdot \cos 2t}{-2e^{\cos 2t} \cdot \sin 2t}$$

$$= \frac{e^{\sin 2t} \cdot \cos 2t}{e^{\cos 2t} \cdot \sin 2t} \quad \text{...(iii)}$$

We know that, $$\log x = \cos 2t \cdot \log e = \cos 2t \quad \text{...(iv)}$$

and $$\log y = \sin 2t \cdot \log e = \sin 2t \quad \text{...(v)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{-y \log x}{x \log y}$$

[using Eqs. (iv) and (v) in Eq. (iii) and $x = e^{\cos 2t}$, $y = e^{\sin 2t}$]

Hence proved.

Q. 50 If $x = a \sin 2t\,(1 + \cos 2t)$ and $y = b \cos 2t\,(1 - \cos 2t)$, then show that $\left(\dfrac{dy}{dx}\right)_{t = \pi/4} = \dfrac{b}{a}$.

Sol. $\because$ $$x = a \sin 2t\,(1 + \cos 2t) \text{ and } y = b \cos 2t\,(1 - \cos 2t)$$

$\therefore$ $$\frac{dx}{dt} = a\left[\sin 2t \cdot \frac{d}{dt}(1 + \cos 2t) + (1 + \cos 2t) \cdot \frac{d}{dt} \sin 2t\right]$$

$$= a\left[\sin 2t\cdot(-\sin 2t)\cdot\frac{d}{dt}2t + (1+\cos 2t)\cdot\cos 2t\cdot\frac{d}{dt}2t\right]$$

$$= -2a\sin^2 2t + 2a\cos 2t\,(1+\cos 2t)$$

$\Rightarrow$ $$\frac{dx}{dt} = -2a\,[\sin^2 2t - \cos 2t\,(1+\cos 2t)] \quad \text{...(i)}$$

and $$\frac{dy}{dt} = b\left[\cos 2t\cdot\frac{d}{dt}(1-\cos 2t) + (1-\cos 2t)\cdot\frac{d}{dt}\cos 2t\right]$$

$$= b\left[\cos 2t\cdot(\sin 2t)\frac{d}{dt}2t + (1-\cos 2t)(-\sin 2t)\cdot\frac{d}{dt}2t\right]$$

$$= b\,[2\sin 2t\cdot\cos 2t + 2\,(1-\cos 2t)(-\sin 2t)]$$

$$= 2b\,[\sin 2t\cdot\cos 2t - (1-\cos 2t)\sin 2t] \quad \text{...(ii)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2b\,[-\sin 2t\cdot\cos 2t + (1-\cos 2t)\sin 2t]}{-2a\,[\sin^2 2t - \cos 2t\,(1+\cos 2t)]}$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{t=\pi/4} = \frac{b}{a}\,\frac{\left[-\sin\frac{\pi}{2}\cos\frac{\pi}{2} + \left(1-\cos\frac{\pi}{2}\right)\sin\frac{\pi}{2}\right]}{\left[\sin^2\frac{\pi}{2} - \cos\frac{\pi}{2}\left(1+\cos\frac{\pi}{2}\right)\right]}$$

$$= \frac{b}{a}\cdot\frac{(0+1)}{(1-0)} \qquad \left[\because \sin\frac{\pi}{2} = 1 \text{ and } \cos\frac{\pi}{2} = 0\right]$$

$$= \frac{b}{a}$$

Hence proved.

Q. 51 If $x = 3\sin t - \sin 3t,\ y = 3\cos t - \cos 3t$, then find $\dfrac{dy}{dx}$ at $t = \dfrac{\pi}{3}$.

Sol. $\because$ $$x = 3\sin t - \sin 3t \text{ and } y = 3\cos t - \cos 3t$$

$\therefore$ $$\frac{dx}{dt} = 3\cdot\frac{d}{dt}\sin t - \frac{d}{dt}\sin 3t$$

$$= 3\cos t - \cos 3t\cdot\frac{d}{dt}3t = 3\cos t - 3\cos 3t \quad \text{...(i)}$$

and $$\frac{dy}{dt} = 3\cdot\frac{d}{dt}\cos t - \frac{d}{dt}\cos 3t$$

$$= -3\sin t + \sin 3t\cdot\frac{d}{dt}3t$$

$$\frac{dy}{dt} = 3\sin 3t - 3t\sin t \quad \text{...(ii)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\,(\sin 3t - \sin t)}{3\,(\cos t - \cos 3t)}$$

Now, $$\left(\frac{dy}{dx}\right)_{t=\pi/3} = \frac{\sin\frac{3\pi}{3} - \sin\frac{\pi}{3}}{\left(\cos\frac{\pi}{3} - \cos 3\,\frac{\pi}{3}\right)} = \frac{0 - \sqrt{3}/2}{\frac{1}{2} - (-1)}$$

$$= \frac{-\sqrt{3}/2}{3/2} = \frac{-\sqrt{3}}{3} = \frac{-1}{\sqrt{3}}$$

Q. 52 Differentiate $\dfrac{x}{\sin x}$ w.r.t. $\sin x$.

Sol. Let $u = \dfrac{x}{\sin x}$ and $v = \sin x$

$$\therefore \quad \frac{du}{dx} = \frac{\sin x \cdot \frac{d}{dx} x - x \cdot \frac{d}{dx} \sin x}{(\sin x)^2}$$

$$= \frac{\sin x - x \cos x}{\sin^2 x} \quad \text{...(i)}$$

and $$\frac{dv}{dx} = \frac{d}{dx} \sin x = \cos x \quad \text{...(ii)}$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\sin x - x \cos x / \sin^2 x}{\cos x}$$

$$= \frac{\sin x - x \cos x}{\sin^2 x \cos x} = \frac{\dfrac{\sin x - x \cos x}{\cos x}}{\dfrac{\sin^2 x \cos x}{\cos x}}$$

[dividing by $\cos x$ in both numerator and denominator]

$$= \frac{\tan x - x}{\sin^2 x}$$

Q. 53 Differentiate $\tan^{-1} \dfrac{\sqrt{1 + x^2} - 1}{x}$ w.r.t. $\tan^{-1} x$, when $x \neq 0$.

Sol. Let $u = \tan^{-1} \dfrac{\sqrt{1 + x^2} - 1}{x}$ and $v = \tan^{-1} x$

$$\therefore \quad x = \tan \theta$$

$$\Rightarrow \quad u = \tan^{-1} \frac{\sqrt{1 + \tan^2 \theta} - 1}{\tan \theta}$$

$$= \tan^{-1} \frac{(\sec \theta - 1) \cos \theta}{\sin \theta}$$

$$= \tan^{-1} \left(\frac{1 - \cos \theta}{\sin \theta} \right)$$

$$= \tan^{-1} \left[\frac{1 - 1 + 2 \sin^2 \theta/2}{2 \sin \theta/2 \cdot \cos \theta/2} \right] \quad [\because \cos \theta = 1 - 2 \sin^2 \theta]$$

$$= \tan^{-1} \left[\tan \frac{\theta}{2} \right]$$

$$= \frac{\theta}{2} = \frac{1}{2} \tan^{-1} x$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{2} \frac{d}{dx} \tan^{-1} x = \frac{1}{2} \cdot \frac{1}{1 + x^2} \quad \text{...(i)}$$

and $$\frac{dv}{dx} = \frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2} \quad \text{...(ii)}$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$= \frac{1/2\,(1 + x^2)}{1/(1 + x^2)} = \frac{(1 + x^2)}{2(1 + x^2)} = \frac{1}{2}$$

Find $\frac{dy}{dx}$ *when x and y are connected by the relation given.*

Q. 54 $\sin(xy) + \frac{x}{y} = x^2 - y$

Sol. We have, $\sin(xy) + \frac{x}{y} = x^2 - y$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\sin xy) + \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}x^2 - \frac{d}{dx}y$$

$$\Rightarrow \quad \cos xy \cdot \frac{d}{dx}(xy) + \frac{y\frac{d}{dx}x - x \cdot \frac{d}{dx}y}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \quad \cos xy \cdot \left[x \cdot \frac{d}{dx}y + y \cdot \frac{d}{dx} \cdot x\right] + \frac{y - x\frac{dy}{dx}}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \quad x\cos xy \cdot \frac{dy}{dx} + y\cos xy + \frac{y}{y^2} - \frac{x}{y^2}\frac{dy}{dx} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[x\cos xy - \frac{x}{y^2} + 1\right] = 2x - y\cos xy - \frac{y}{y^2}$$

$$\therefore \quad \frac{dy}{dx} = \left[\frac{2xy - y^2\cos xy - 1}{y}\right]\left[\frac{y^2}{xy^2\cos xy - x + y^2}\right]$$

$$= \frac{(2xy - y^2\cos xy - 1)\,y}{(xy^2\cos xy - x + y^2)}$$

Q. 55 $\sec(x + y) = xy$

Sol. We have, $\sec(x + y) = xy$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\sec(x + y) = \frac{d}{dx}(xy)$$

$$\Rightarrow \quad \sec(x + y) \cdot \tan(x + y) \cdot \frac{d}{dx}(x + y) = x \cdot \frac{d}{dx}y + y \cdot \frac{d}{dx}x$$

$$\Rightarrow \quad \sec(x + y) \cdot \tan(x + y) \cdot \left(1 + \frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad \sec(x + y)\tan(x + y) + \sec(x + y) \cdot \tan(x + y) \cdot \frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad \frac{dy}{dx}[\sec(x + y) \cdot \tan(x + y) - x] = y - \sec(x + y) \cdot \tan(x + y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{y - \sec(x + y) \cdot \tan(x + y)}{\sec(x + y) \cdot \tan(x + y) - x}$$

Q. 56 $\tan^{-1}(x^2+y^2)=a$

Sol. We have, $\tan^{-1}(x^2+y^2)=a$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\tan^{-1}(x^2+y^2)=\frac{d}{dx}(a)$$

$$\Rightarrow \quad \frac{1}{1+(x^2+y^2)^2}\cdot\frac{d}{dx}(x^2+y^2)=0$$

$$\Rightarrow \quad 2x+\frac{d}{dy}y^2\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \quad 2y\cdot\frac{dy}{dx}=-2x$$

$$\therefore \quad \frac{dy}{dx}=\frac{-2x}{2y}=\frac{-x}{y}$$

Q. 57 $(x^2+y^2)^2=xy$

Sol. We have, $(x^2+y^2)^2=xy$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(x^2+y^2)^2=\frac{d}{dx}(xy)$$

$$\Rightarrow \quad 2(x^2+y^2)\cdot\frac{d}{dx}(x^2+y^2)=x\cdot\frac{d}{dx}y+y\cdot\frac{d}{dx}x$$

$$\Rightarrow \quad 2(x^2+y^2)\cdot\left(2x+2y\frac{dy}{dx}\right)=x\frac{dy}{dx}+y$$

$$\Rightarrow \quad 2x^2\cdot 2x+2x^2\cdot 2y\frac{dy}{dx}+2y^2\cdot 2x+2y^2\cdot 2y\frac{dy}{dx}=x\frac{dy}{dx}+y$$

$$\Rightarrow \quad \frac{dy}{dx}[4x^2y+4y^3-x]=y-4x^3-4xy^2$$

$$\therefore \quad \frac{dy}{dx}=\frac{(y-4x^3-4xy^2)}{(4x^2y+4y^3-x)}$$

Q. 58 If $ax^2+2hxy+by^2+2gx+2fy+c=0$, then show that $\frac{dy}{dx}\cdot\frac{dx}{dy}=1$.

Sol. We have, $ax^2+2hxy+by^2+2gx+2fy+c=0$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(ax^2)+\frac{d}{dx}(2hxy)+\frac{d}{dx}(by^2)+\frac{d}{dx}(2gx)+\frac{d}{dx}(2fy)+\frac{d}{dx}(c)=0$$

$$\Rightarrow \quad 2ax+2h\left(x\cdot\frac{dy}{dx}+y\cdot 1\right)+b\cdot 2y\frac{dy}{dx}+2g+2f\frac{dy}{dx}+0=0$$

$$\Rightarrow \quad \frac{dy}{dx}[2hx+2by+2f]=-2ax-2hy-2g$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-2(ax+hy+g)}{2(hx+by+f)}$$

$$=\frac{-(ax+hy+g)}{(hx+by+f)} \quad \text{...(ii)}$$

Now, differentiating Eq. (i) w.r.t. y, we get

$$\frac{d}{dy}(ax^2)+\frac{d}{dy}(2hxy)+\frac{d}{dy}(by^2)+\frac{d}{dy}(2gx)+\frac{d}{dy}(2fy)+\frac{d}{dy}(c)=0$$

$$\Rightarrow \quad a\cdot 2x\cdot\frac{dx}{dy}+2h\cdot\left(x\cdot\frac{d}{dy}y+y\cdot\frac{d}{dy}x\right)+b\cdot 2y+2g\cdot\frac{dx}{dy}+2f+0=0$$

$$\Rightarrow \quad \frac{dx}{dy}[2ax+2hy+2g]=-2hx-2by-2f$$

$$\Rightarrow \quad \frac{dx}{dy}=\frac{-2(hx+by+f)}{2(ax+hy+g)}=\frac{-(hx+by+f)}{(ax+hy+g)} \quad \text{... (iii)}$$

$$\therefore \quad \frac{dy}{dx}\cdot\frac{dx}{dy}=\frac{-(ax+hy+g)}{(hx+by+f)}\cdot\frac{-(hx+by+f)}{(ax+hy+g)} \quad \text{[using Eqs. (ii) and (iii)]}$$

$$=1=\text{RHS}$$

Hence proved.

Q. 59 If $x=e^{x/y}$, then prove that $\dfrac{dy}{dx}=\dfrac{x-y}{x\log x}$.

Sol. We have, $$x=e^{x/y}$$

$$\therefore \quad \frac{d}{dx}x=\frac{d}{dx}e^{x/y}$$

$$\Rightarrow \quad 1=e^{x/y}\cdot\frac{d}{dx}(x/y)$$

$$\Rightarrow \quad 1=e^{x/y}\cdot\left[\frac{y\cdot 1-x\cdot dy/dx}{y^2}\right]$$

$$\Rightarrow \quad y^2=y\cdot e^{x/y}-x\cdot\frac{dy}{dx}\cdot e^{x/y}$$

$$\Rightarrow \quad x\cdot\frac{dy}{dx}\cdot e^{x/y}=ye^{x/y}-y^2$$

$$\therefore \quad \frac{dy}{dx}=\frac{y(e^{x/y}-y)}{x\cdot e^{x/y}}$$

$$=\frac{(e^{x/y}-y)}{e^{x/y}\cdot\frac{x}{y}} \quad \left[\because x=e^{x/y}\Rightarrow \log x=\frac{x}{y}\right]$$

$$=\frac{x-y}{x\cdot\log x}$$

Hence proved.

Q. 60 If $y^x=e^{y-x}$, then prove that $\dfrac{dy}{dx}=\dfrac{(1+\log y)^2}{\log y}$.

Sol. We have, $$y^x=e^{y-x}$$

$$\Rightarrow \quad \log y^x=\log e^{y-x}$$

$$\Rightarrow \quad x\log y=y-x\cdot\log_e=(y-x) \quad [\because \log_e=1]$$

$$\Rightarrow \quad \log y=\frac{(y-x)}{x} \quad \text{...(i)}$$

Now, differentiating w.r.t. x, we get

$$\frac{d}{dy}\log y\cdot\frac{dy}{dx}=\frac{d}{dx}\frac{(y-x)}{x}$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx}=\frac{x\cdot\frac{d}{dx}(y-x)-(y-x)\cdot\frac{d}{dx}\cdot x}{x^2}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx}=\frac{x\left(\frac{dy}{dx}-1\right)-(y-x)}{x^2}$$

$$\Rightarrow \quad \frac{x^2}{y}\cdot\frac{dy}{dx}=x\frac{dy}{dx}-x-y+x$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{x^2}{y}-x\right)=-y$$

$$\therefore \quad \frac{dy}{dx}=\frac{-y^2}{x^2-xy}=\frac{-y^2}{x(x-y)}$$

$$=\frac{y^2}{x(y-x)}\cdot\frac{x}{x}=\frac{y^2}{x^2}\cdot\frac{1}{\frac{(y-x)}{x}}$$

$$=\frac{(1+\log y)^2}{\log y} \quad \left[\because \log y=\frac{y-x}{x} \quad \log y=\frac{y}{x}-1\Rightarrow 1+\log y=\frac{y}{x}\right]$$

Hence proved.

Q. 61 If $y=(\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$, then show that $\dfrac{dy}{dx}=\dfrac{y^2\tan x}{y\log\cos x-1}$.

Sol. We have,

$$y=(\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$$

$$\Rightarrow \quad y=(\cos x)^y$$

$$\therefore \quad \log y=\log(\cos x)^y$$

$$\Rightarrow \quad \log y=y\log\cos x$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=y\cdot\frac{d}{dx}\log\cos x+\log\cos x\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx}=\frac{y}{\cos x}\cdot\frac{d}{dx}\cos x+\log\cos x\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1}{y}-\log\cos x\right]=\frac{-y\sin x}{\cos x}=-y\tan x$$

$$\therefore \quad \frac{dy}{dx}=\frac{-y^2\tan x}{(1-y\log\cos x)}$$

$$=\frac{y^2\tan x}{y\log\cos x-1}$$

Hence proved.

Q. 62 If $x\sin(a+y)+\sin a\cdot\cos(a+y)=0$, then prove that

$$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}.$$

Sol. We have,

$$x\sin(a+y)+\sin a\cdot\cos(a+y)=0$$

$$\Rightarrow \quad x\sin(a+y)=-\sin a\cdot\cos(a+y)$$

$$\Rightarrow \quad x=\frac{-\sin a\cdot\cos(a+y)}{\sin(a+y)}$$

$$\Rightarrow \quad x = -\sin a \cdot \cot(a+y)$$

$$\therefore \quad \frac{dx}{dy} = -\sin a \cdot [-\operatorname{cosec}^2(a+y)] \cdot \frac{d}{dy}(a+y)$$

$$= \sin a \cdot \frac{1}{\sin^2(a+y)} \cdot 1$$

$$= \frac{\sin^2(a+y)}{\sin a}$$

Hence proved.

Q. 63 If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, then prove that $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$.

Sol. We have,

$$\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$$

On putting $x=\sin\alpha$ and $y=\sin\beta$, we get

$$\sqrt{1-\sin^2\alpha}+\sqrt{1-\sin^2\beta}=a(\sin\alpha-\sin\beta)$$

$$\Rightarrow \quad \cos\alpha+\cos\beta=a(\sin\alpha-\sin\beta)$$

$$\Rightarrow \quad 2\cos\frac{\alpha+\beta}{2}.\cos\frac{\alpha-\beta}{2}=a\left(2\cos\frac{\alpha+\beta}{2}.\sin\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad \cos\frac{\alpha-\beta}{2}=a\sin\frac{\alpha-\beta}{2}$$

$$\Rightarrow \quad \cot\frac{\alpha-\beta}{2}=a$$

$$\Rightarrow \quad \frac{\alpha-\beta}{2}=\cot^{-1}a$$

$$\Rightarrow \quad \alpha-\beta=2\cot^{-1}a$$

$$\Rightarrow \quad \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a \qquad [\because x=\sin\alpha \text{ and } y=\sin\beta]$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0$$

$$\therefore \quad \frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}=\sqrt{\frac{1-y^2}{1-x^2}}$$

Hence proved.

Q. 64 If $y=\tan^{-1}x$, then find $\frac{d^2y}{dx^2}$ in terms of y alone.

Sol. We have,

$$y=\tan^{-1}x \qquad \text{[on differentiating w.r.t. } x]$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{1+x^2} \qquad \text{[again differentiating w.r.t. } x]$$

Now,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(1+x^2)^{-1}$$

$$=-1(1+x^2)^{-2}.\frac{d}{dx}(1+x^2)$$

$$=-\frac{1}{(1+x^2)^2}\cdot 2x$$

$$=\frac{-2\tan y}{(1+\tan^2 y)^2} \qquad [\because y=\tan^{-1}x \Rightarrow \tan y=x]$$

$$= \frac{-2\tan y}{(\sec^2 y)^2}$$

$$= -2\frac{\sin y}{\cos y}\cdot\cos^2 y\cdot\cos^2 y$$

$$= -\sin 2y\cdot\cos^2 y \qquad [\because \sin 2x = 2\sin x\cos x]$$

Verify the Rolle's theorem for each of the functions in following questions.

Q. 65 $f(x) = x(x-1)^2$ in $[0,1]$

Thinking Process

We know that, Rolle's theorem states that, if f be a real valued function, defined in the closed interval [a, b], such that (i) f is continuous on [a, b]. (ii) f is differentiable on]a, b[.(iii) f(a) = f(b).

Then, there exists a real number c in the open interval] a, b [, such that $f'(c) = 0$'. Here, we shall verify the Rolle's theorem for the given function.

Sol. We have, $f(x) = x(x-1)^2$ in [0, 1].

(i) Since, $f(x) = x(x-1)^2$ is a polynomial function.

So, it is continuous in [0, 1].

(ii) Now,

$$f'(x) = x\cdot\frac{d}{dx}(x-1)^2 + (x-1)^2\frac{d}{dx}x$$

$$= x\cdot 2(x-1)\cdot 1 + (x-1)^2$$

$$= 2x^2 - 2x + x^2 + 1 - 2x$$

$$= 3x^2 - 4x + 1, \text{ which exists in } (0, 1).$$

So, $f(x)$ is differentiable in (0, 1).

(iii) Now, $f(0) = 0$ and $f(1) = 0 \Rightarrow f(0) = f(1)$

f satisfies the above conditions of Rolle's theorem.

Hence, by Rolle's theorem $\exists\, c \in (0, 1)$ such that

$$f'(c) = 0$$

$$\Rightarrow 3c^2 - 4c + 1 = 0$$

$$\Rightarrow 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow 3c(c-1) - 1(c-1) = 0$$

$$\Rightarrow (3c-1)(c-1) = 0$$

$$\Rightarrow c = \frac{1}{3}, 1 \Rightarrow \frac{1}{3} \in (0, 1)$$

Thus, we see that there exists a real number c in the open interval (0, 1).

Hence, Rolle's theorem has been verified.

Q. 66 $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$

Sol. We have, $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$ ...(i)

(i) $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$

[since, $\sin^4 x$ and $\cos^4 x$ are continuous functions and we know that, if g and h be continuous functions, then $(g + h)$ is a continuous function.]

(ii)
$$f'(x) = 4(\sin x)^3 \cdot \cos x + 4(\cos x)^3 \cdot (-\sin x)$$
$$= 4\sin^3 x \cdot \cos x - 4\sin x \cdot \cos^3 x$$
$$= 4\sin x \cos x (\sin^2 x - \cos^2 x) \text{ which exists in } \left(0, \frac{\pi}{2}\right) \quad \text{...(ii)}$$

Hence, $f(x)$ is differentiable in $\left(0, \frac{\pi}{2}\right)$.

(iii) Also, $f(0) = 0 + 1 = 1$ and $f\left(\frac{\pi}{2}\right) = 1 + 0 = 1$

$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$

Conditions of Rolle's theorem are satisfied.

Hence, there exists atleast one $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

$\therefore \quad 4\sin c \cos c (\sin^2 c - \cos^2 c) = 0$

$\Rightarrow \quad 4\sin c \cos c (-\cos 2c) = 0$

$\Rightarrow \quad -2\sin 2c \cdot \cos 2c = 0$

$\Rightarrow \quad -\sin 4c = 0$

$\Rightarrow \quad \sin 4c = 0$

$\Rightarrow \quad 4c = \pi$

$\Rightarrow \quad c = \frac{\pi}{4}$

and $\quad \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

Hence, Rolle's theorem has been verified.

Q. 67 $f(x) = \log(x^2 + 2) - \log 3$ in $[-1, 1]$

Sol. We have, $\quad f(x) = \log(x^2 + 2) - \log 3.$

(i) Logarithmic functions are continuous in their domain.
Hence, $f(x) = \log(x^2 + 2) - \log 3$ is continuous in $[-1, 1]$.

(ii) $f'(x) = \frac{1}{x^2 + 2} \cdot 2x - 0$

$= \frac{2x}{x^2 + 2}$, which exists in $(-1, 1)$.

Hence, $f(x)$ is differentiable in $(-1, 1)$.

(iii) $f(-1) = \log[(-1)^2 + 2] - \log 3 = \log 3 - \log 3 = 0$ and

$f(1) = \log(1^2 + 2) - \log 3 = \log 3 - \log 3 = 0$

$\Rightarrow \quad f(-1) = f(1)$

Conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that

$$f'(c) = 0.$$

$$\Rightarrow \quad \frac{2c}{c^2 + 2} = 0$$

$$\Rightarrow \quad c = 0 \in (-1, 1)$$

Hence, Rolle's theorem has been verified.

Q. 68 $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$

Sol. We have, $\quad f(x) = x(x+3)e^{-x/2}$

(i) $f(x)$ is a continuous function. [since, it is a combination of polynomial functions $x(x+3)$ and an exponential function $e^{-x/2}$ which are continuous functions]

So, $f(x) = x(x+3)e^{-x/2}$ is continuous in $[-3, 0]$.

(ii) $\therefore$

$$f'(x) = (x^2 + 3x)\cdot\frac{d}{dx}e^{-x/2} + e^{-x/2}\cdot\frac{d}{dx}(x^2 + 3x)$$

$$= (x^2 + 3x)\cdot e^{-x/2}\cdot\left(-\frac{1}{2}\right) + e^{-x/2}\cdot(2x + 3)$$

$$= e^{-x/2}\left[2x + 3 - \frac{1}{2}\cdot(x^2 + 3x)\right]$$

$$= e^{-x/2}\left[\frac{4x + 6 - x^2 - 3x}{2}\right]$$

$$= e^{-x/2}\cdot\frac{1}{2}[-x^2 + x + 6]$$

$$= \frac{-1}{2}e^{-x/2}[x^2 - x - 6]$$

$$= \frac{-1}{2}e^{-x/2}[x^2 - 3x + 2x - 6]$$

$$= \frac{-1}{2}e^{-x/2}[(x+2)(x-3)] \text{ which exists in } (-3, 0).$$

Hence, $f(x)$ is differentiable in $(-3, 0)$.

(iii) $\therefore \quad f(-3) = -3(-3+3)e^{-3/2} = 0$

and $\quad f(0) = 0(0+3)e^{-0/2} = 0$

$\Rightarrow \quad f(-3) = f(0)$

Since, conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that $f'(c) = 0$

$$\Rightarrow \quad -\frac{1}{2}e^{-c/2}(c+2)(c-3) = 0$$

$$\Rightarrow \quad c = -2, 3, \text{ where } -2 \in (-3, 0)$$

Therefore, Rolle's theorem has been verified.

Q. 69 $f(x)=\sqrt{4-x^2}$ in $[-2, 2]$

Sol. We have, $f(x)=\sqrt{4-x^2}=(4-x^2)^{1/2}$

(i) $f(x)=\sqrt{4-x^2}$ is a continuous function.

[since every polynomial function is a continuous function]

Hence, $f(x)$ is continuous in $[-2, 2]$.

(ii) $f'(x)=\frac{1}{2}(4-x^2)^{-1/2}\cdot(-2x)$

$=-x.\frac{1}{\sqrt{4-x^2}}$, which exists everywhere except at $x=\pm 2$.

Hence, $f(x)$ is differentiable in $(-2, 2)$.

(iii) $f(-2)=\sqrt{(4-4)}=0$ and $f(2)=\sqrt{(4-4)}=0$

$\Rightarrow \quad f(-2)=f(2)$

conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that $f'(c)=0$.

$\Rightarrow \quad -c\frac{1}{\sqrt{4-c^2}}=0$

$\Rightarrow \quad c=0\in(-2, 2)$

Hence, Rolle's theorem has been verified.

Q. 70 Discuss the applicability of Rolle's theorem on the function given by

$$f(x)=\begin{cases}x^2+1, & \text{if } 0\le x\le 1\\ 3-x, & \text{if } 1\le x\le 2\end{cases}$$

Sol. We have, $\quad f(x)=\begin{cases}x^2+1, \text{if } 0\le x\le 1\\ 3-x, \ \text{if } 1\le x\le 2\end{cases}$

We know that, polynomial function is everywhere continuous and differentiability.

So, $f(x)$ is continuous and differentiable at all points except possibly at $x=1$.

Now, check the differentiability at $x=1$,

At $x=1$,

$$\text{LDH}=\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to 1}\frac{(x^2+1)-(1+1)}{x-1} \qquad [\because f(x)=x^2+1, \forall\ 0\le x\le 1]$$

$$=\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x+1)(x-1)}{x-1}$$

$$=2$$

and $\quad \text{RDH}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{(3-x)\,f(1+1)}{(x-1)}$

$$=\lim_{x\to 1}\frac{3-x-2}{x-1}=\lim_{x\to 1}\frac{-(x-1)}{x-1}=-1$$

$\therefore \quad \text{LHD}\ne\text{RHD}$

So, $f(x)$ is not differentiable at $x=1$.

Hence, polle's theorem is not applicable on the interval [0, 2].

Q. 71 Find the points on the curve $y = (\cos x - 1)$ in $[0, 2\pi]$, where the tangent is parallel to X-axis.

Thinking Process

We know that, if f be a real valued function defined in the closed interval $[a, b]$ such that it follows all the three conditions of Rolle's theorem, then $f'(c) = 0$ shows that the tangent to the curve at $x = c$ has a slope 0, i.e., it is parallel to the X-axis. So, by getting the value of c' we can get the required point.

Sol. The equation of the curve is $y = \cos x - 1$.
Now, we have to find a point on the curve in $[0, 2\pi]$,
where the tangent is parallel to X-axis *i.e.,* the tangent to the curve at $x = c$ has a slope o, where $c \in]0, 2\pi[$.
Let us apply Rolle's theorem to get the point.

(i) $y = \cos x - 1$ is a continuous function in $[0, 2\pi]$.
[since it is a combination of cosine function and a constant function]

(ii) $y' = -\sin x$, which exists in $(0, 2\pi)$.
Hence, y is differentiable in $(0, 2\pi)$.

(iii) $y(0) = \cos 0 - 1 = 0$ and $y(2\pi) = \cos 2\pi - 1 = 0$,

$\therefore \quad y(0) = y(2\pi)$

Since, conditions of Rolle's theorem are satisfied.
Hence, there exists a real number c such that

$$f'(c) = 0$$
$$\Rightarrow \quad -\sin c = 0$$
$$\Rightarrow \quad c = \pi \text{ or } 0, \text{ where } \pi \in (0, 2\pi)$$
$$\Rightarrow \quad x = \pi$$
$$\therefore \quad y = \cos \pi - 1 = -2$$

Hence, the required point on the curve, where the tangent drawn is parallel to the X-axis is $(\pi, -2)$.

Q. 72 Using Rolle's theorem, find the point on the curve $y = x(x - 4)$, $x \in [0, 4]$, where the tangent is parallel to X-axis.

Sol. We have, $y = x(x - 4)$, $x \in [0, 4]$

(i) y is a continuous function since $x(x - 4)$ is a polynomial function.
Hence, $y = x(x - 4)$ is continuous in $[0, 4]$.

(ii) $y' = (x - 4)\cdot 1 + x \cdot 1 = 2x - 4$ which exists in $(0, 4)$.
Hence, y is differentiable in $(0, 4)$.

(iii) $y(0) = 0(0 - 4) = 0$

and $\quad y(4) = 4(4 - 4) = 0$

$\Rightarrow \quad y(0) = y(4)$

Sicne, conditions of Rolle's theorem are satisfied.
Hence, there exists a point c such that

$$f'(c) = 0 \text{ in } (0, 4) \qquad [\because f'(x) = y']$$
$$\Rightarrow \quad 2c - 4 = 0$$
$$\Rightarrow \quad c = 2$$
$$\Rightarrow \quad x = 2;\ y = 2(2 - 4) = -4$$

Thus, $(2, -4)$ is the point on the curve at which the tangent drawn is parallel to X-axis.

Verify mean value theorem for each of the functions.

Q. 73 $f(x)=\dfrac{1}{4x-1}$ in $[1, 4]$

Thinking Process

We know that, mean value theorem states that, if f be a real function such that
(i) $f(x)$ *is continuous on [a,b]*
(ii) $f(x)$ *is differentiable on]a,b[*

Then, there exists a real number $c \in]a,b[$ *such that* $f'(c)=\dfrac{f(b)-f(a)}{b-a}$, *thus we can verify it for given function.*

Sol. We have, $f(x)=\dfrac{1}{4x-1}$ in [1, 4]

(i) $f(x)$ is continuous in [1, 4].
Also, at $x=\dfrac{1}{4}$, $f(x)$ is discontinuous.
Hence, $f(x)$ is continuous in [1, 4].

(ii) $f'(x)=-\dfrac{4}{(4x-1)^2}$, which exists in (1, 4).

Since, conditions of mean value theorem are satisfied.
Hence, there exists a real number $c \in]1, 4[$ such that

$$f'(c)=\frac{f(4)-f(1)}{4-1}$$

$$\Rightarrow \quad \frac{-4}{(4c-1)^2}=\frac{\frac{1}{16-1}-\frac{1}{4-1}}{4-1}=\frac{\frac{1}{15}-\frac{1}{3}}{3}$$

$$\Rightarrow \quad \frac{-4}{(4c-1)^2}=\frac{1-5}{45}=\frac{-4}{45}$$

$$\Rightarrow \quad (4c-1)^2=45$$

$$\Rightarrow \quad 4c-1=\pm 3\sqrt{5}$$

$$\Rightarrow \quad c=\frac{3\sqrt{5}+1}{4}\in(1, 4) \qquad \text{[neglecting (– ve) value]}$$

Hence, mean value theorem has been verified.

Q. 74 $f(x)=x^3-2x^2-x+3$ in $[0, 1]$

Sol. We have, $f(x)=x^3-2x^2-x+3$ in [0, 1]

(i) Since, $f(x)$ is a polynomial function.
Hence, $f(x)$ is continuous in [0, 1].

(ii) $f'(x)=3x^2-4x-1$, which exists in (0,1).
Hence, $f(x)$ is differentiable in (0,1).
Since, conditions of mean value theorem are satisfied.
Therefore, by mean value theorem $\exists\, c \in (0,1)$, such that

$$f'(c)=\frac{f(1)-f(0)}{1-0}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = \frac{[1-2-1+3]-[0+3]}{1-0}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = \frac{-2}{1}$$

$$\Rightarrow \quad 3c^2 - 4c + 1 = 0$$

$$\Rightarrow \quad 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow \quad 3c(c-1) - 1(c-1) = 0$$

$$\Rightarrow \quad (3c-1)(c-1) = 0$$

$$\Rightarrow \quad c = 1/3,\ 1, \text{ where } \frac{1}{3} \in (0, 1)$$

Hence, the mean value theorem has been verified.

Q. 75 $f(x) = \sin x - \sin 2x$ in $[0, \pi]$

Sol. We have, $f(x) = \sin x - \sin 2x$ in $[0,\pi]$

(i) Since, we know that sine functions are continuous functions hence $f(x) = \sin x - \sin 2x$ is a continuous function in $[0,\pi]$.

(ii) $f'(x) = \cos x - \cos 2x \cdot 2 = \cos x - 2\cos 2x$, which exists in $(0, \pi)$.

So, $f(x)$ is differentiable in $(0, \pi)$. Conditions of mean value theorem are satisfied.

Hence, $\exists\, c \in (0, \pi)$ such that, $f'(c) = \dfrac{f(\pi) - f(0)}{\pi - 0}$

$$\Rightarrow \quad \cos c - 2\cos 2c = \frac{\sin\pi - \sin 2\pi - \sin 0 + \sin 2\cdot 0}{\pi - 0}$$

$$\Rightarrow \quad 2\cos 2c - \cos c = \frac{0}{\pi}$$

$$\Rightarrow \quad 2\cdot(2\cos^2 c - 1) - \cos c = 0$$

$$\Rightarrow \quad 4\cos^2 c - 2 - \cos c = 0$$

$$\Rightarrow \quad 4\cos^2 c - \cos c - 2 = 0$$

$$\Rightarrow \quad \cos c = \frac{1 \pm \sqrt{1+32}}{8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\therefore \quad c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$$

Also, $\quad \cos^{-1}\left(\dfrac{1 \pm \sqrt{33}}{8}\right) \in (0, \pi)$

Hence, mean value theorem has been verified.

Q. 76 $f(x) = \sqrt{25 - x^2}$ in $[1, 5]$

Sol. We have, $f(x) = \sqrt{25 - x^2}$ in $[1, 5]$

(i) Since, $f(x) = (25 - x^2)^{1/2}$, where $25 - x^2 \geq 0$

$$\Rightarrow \quad x^2 \leq \pm 5 \Rightarrow -5 \leq x \leq 5$$

Hence, $f(x)$ is continuous in $[1, 5]$.

(ii) $f'(x) = \dfrac{1}{2}(25 - x^2)^{-1/2} \cdot -2x = \dfrac{-x}{\sqrt{25 - x^2}}$, which exists in $(1, 5)$.

Hence, $f'(x)$ is differentiable in $(1, 5)$.

Since, conditions of mean value theorem are satisfied.

By mean value theorem $\exists\, c \in (1, 5)$ such that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} \Rightarrow \frac{-c}{\sqrt{25 - c^2}} = \frac{0 - \sqrt{24}}{4}$$

$$\Rightarrow \quad \frac{c^2}{25 - c^2} = \frac{24}{16}$$

$$\Rightarrow \quad 16c^2 = 600 - 24c^2$$

$$\Rightarrow \quad c^2 = \frac{600}{40} = 15$$

$$\therefore \quad c = \pm\sqrt{15}$$

Also, $c = \sqrt{15} \in (1, 5)$

Hence, the mean value theorem has been verified.

Q. 77 Find a point on the curve $y = (x - 3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

Thinking Process

We know that, if $y = f(x)$ be a function defined on $[a, b]$ which follows mean value theorem, then there exists atleast one point c in (a, b) such that the tangent at the point $[c, f(c)]$ is parallel to the secant joining the points $[a, f(a)]$ and $[b, f(b)]$. So, we shall use this concept.

Sol. We have, $y = (x - 3)^2$, which is continuous in $x_1 = 3$ and $x_2 = 4$ *i.e.,* $[3, 4]$.

Also, $y' = 2(x - 3)\cdot 1 = 2(x - 3)$ which exists in $(3, 4)$.

Hence, by mean value theorem there exists a point on the curve at which tangent drawn is parallel to the chord joining the points (3,0) and (4,1).

Thus, $$f'(c) = \frac{f(4) - f(3)}{4 - 3}$$

$$\Rightarrow \quad 2(c - 3) = \frac{(4 - 3)^2 - (3 - 3)^2}{4 - 3}$$

$$\Rightarrow \quad 2c - 6 = \frac{1 - 0}{1} \Rightarrow c = \frac{7}{2}$$

For $x = \frac{7}{2}$, $$y = \left(\frac{7}{2} - 3\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

So, $\left(\frac{7}{2}, \frac{1}{4}\right)$ is the point on the curve at which tangent drawn is parallel to the chord joining the points (3, 0) and (4, 1).

Q. 78 Using mean value theorem, prove that there is a point on the curve $y = 2x^2 - 5x + 3$ between the points $A(1, 0)$ and $B(2, 1)$, where tangent is parallel to the chord AB. Also, find that point.

Sol. We have, $y = 2x^2 - 5x + 3$, which is continuous in $[1, 2]$ as it is a polynomial function.

Also, $y' = 4x - 5$, which exists in $(1, 2)$.

By mean value theorem, $\exists\, c \in (1, 2)$ at which drawn tangent is parallel to the chord AB, where A and B are (1, 0) and (2,1), respectively.

$$\therefore \quad f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$\Rightarrow \quad 4c-5=\dfrac{(8-10+3)-(2-5+3)}{1}$

$\Rightarrow \quad 4c-5=1$

$\therefore \quad c=\dfrac{6}{4}=\dfrac{3}{2}\in(1,2)$

For $x=\dfrac{3}{2}$, $\quad y=2\left(\dfrac{3}{2}\right)^2-5\left(\dfrac{3}{2}\right)+3$

$=2\times\dfrac{9}{4}-\dfrac{15}{2}+3=\dfrac{9-15+6}{2}=0$

Hence, $\left(\dfrac{3}{2},0\right)$ is the point on the curve $y=2x^2-5x+3$ between the points A (1, 0) and B (2, 1), where tangent is parallel to the chord AB.

Long Answer Type Questions

Q. 79 Find the values of p and q, so that $f(x)=\begin{cases} x^2+3x+p, & \text{if } x\le 1 \\ qx+2, & \text{if } x>1\end{cases}$ is differentiable at $x=1$.

Sol. We have, $f(x)=\begin{cases} x^2+3x+p, & \text{if } x\le 1 \\ qx+2, & \text{if } x>1\end{cases}$ is differentiable at $x=1$.

$\therefore \quad Lf'(1)=\lim\limits_{x\to1^-}\dfrac{f(x)-f(1)}{x-1}$

$=\lim\limits_{x\to1^-}\dfrac{(x^2+3x+p)-(1+3+p)}{x-1}$

$=\lim\limits_{h\to0}\dfrac{[(1-h)^2+3(1-h)+p]-[1+3+p]}{(1-h)-1}$

$=\lim\limits_{h\to0}\dfrac{[1+h^2-2h+3-3h+p]-[4+p]}{-h}$

$=\lim\limits_{h\to0}\dfrac{[h^2-5h+p+4-4-p]}{-h}=\lim\limits_{h\to0}\dfrac{h[h-5]}{-h}$

$=\lim\limits_{h\to0}-[h-5]=5$

$Rf'(1)=\lim\limits_{x\to1^+}\dfrac{f(x)-f(1)}{x-1}=\lim\limits_{x\to1^+}\dfrac{(qx+2)-(1+3+p)}{x-1}$

$=\lim\limits_{h\to0}\dfrac{[q(1+h)+2]-(4+p)}{1+h-1}$

$=\lim\limits_{h\to0}\dfrac{[q+qh+2-4-p]}{h}=\lim\limits_{h\to0}\dfrac{qh+(q-2-p)}{h}$

$\Rightarrow \quad q-2-p=0\Rightarrow p-q=-2$ ...(i)

$\Rightarrow \quad \lim\limits_{h\to0}\dfrac{qh+0}{h}=q$ [for existing the limit]

If $Lf'(1)=Rf'(1)$, then $5=q$

$\Rightarrow \quad p-5=-2\Rightarrow p=3$

$\therefore \quad p=3$ and $q=5$

Q. 80 If $x^m \cdot y^n = (x+y)^{m+n}$, prove that

(i) $\dfrac{dy}{dx} = \dfrac{y}{x}$ and (ii) $\dfrac{d^2y}{dx^2} = 0$

Sol. We have, $$x^m \cdot y^n = (x+y)^{m+n} \quad \ldots(i)$$

(i) Differentiating Eq. (i) w.r.t. x, we get

$$\frac{d}{dx}(x^m \cdot y^n) = \frac{d}{dx}(x+y)^{m+n}$$

$$\Rightarrow \quad x^m \cdot \frac{d}{dy} y^n \cdot \frac{dy}{dx} + y^n \cdot \frac{d}{dx} x^m = (m+n)(x+y)^{m+n-1} \frac{d}{dx}(x+y)$$

$$\Rightarrow \quad x^m \cdot ny^{n-1} \frac{dy}{dx} + y^n \cdot mx^{m-1} = (m+n)(x+y)^{m+n-1}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}[x^m \cdot ny^{n-1} - (m+n)\cdot(x+y)^{m+n-1}] = (m+n)(x+y)^{m+n-1} - y^n mx^{m-1}$$

$$\Rightarrow \quad \frac{dy}{dx}[nx^m y^{n-1} - (m+n)(x+y)^{m+n-1}] = (m+n)\cdot(x+y)^{m+n-1} - \frac{y^{n-1}\cdot y \cdot mx^m}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\dfrac{(m+n)(x+y)^{m+n}}{(x+y)} - \dfrac{y^{n-1}\cdot y\cdot mx^m}{x}}{\dfrac{n\,x^m y^n}{y} - (m+n)(x+y)^{m+n}\dfrac{1}{(x+y)}}$$

$$= \frac{\dfrac{x(m+n)(x+y)^{m+n} - (x+y)\cdot y^{n-1}\, y\cdot mx^m}{(x+y)\cdot x}}{\dfrac{(x+y)\,n\,x^m y^n - y(m+n)(x+y)^{m+n}}{(x+y)\cdot y}}$$

$$= \frac{\dfrac{x(m+n)\cdot x^m \cdot y^n - m(x+y)\,y^n\,x^m}{(x+y)\cdot x}}{\dfrac{(x+y)\,nx^m \cdot y^n - y(m+n)\cdot x^m \cdot y^n}{(x+y)\cdot y}} \quad [\because (x+y)^{m+n} = x^m \cdot y^n]$$

$$= \frac{x^m\, y^n[mx + nx - mx - my]\cdot(x+y)\,y}{x^m\, y^n[nx + ny - my - ny]\cdot(x+y)\cdot x}$$

$$= \frac{y}{x} \quad \ldots(ii)$$

Hence proved.

(ii) Further, differentiating Eq. (ii) *i.e.*, $\dfrac{dy}{dx} = \dfrac{y}{x}$ on both the sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{x\cdot\dfrac{dy}{dx} - y\cdot 1}{x^2}$$

$$= \frac{x\cdot\dfrac{y}{x} - y}{x^2} \quad \left[\because \frac{dy}{dx} = \frac{y}{x}\right]$$

$$= 0$$

Hence proved.

Q. 81 If $x = \sin t$ and $y = \sin pt$, then prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0.$$

Sol. We have, $x = \sin t$ and $y = \sin pt$

$\therefore \quad \frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = \cos pt \cdot p$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{p \cdot \cos pt}{\cos t}$ ...(i)

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{\cos t \cdot \frac{d}{dt}(p \cdot \cos pt)\frac{dt}{dx} - p\cos pt \cdot \frac{d}{dt}\cos t \cdot \frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{[\cos t \cdot p \cdot (-\sin pt) \cdot p - p\cos pt \cdot (-\sin t)]\frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{[-p^2 \sin pt \cdot \cos t + p\sin t \cdot \cos pt] \cdot \frac{1}{\cos t}}{\cos^2 t}$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-p^2 \sin pt \cdot \cos t + p\cos pt \cdot \sin t}{\cos^3 t}$ ...(ii)

Since, we have to prove

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

$\therefore \quad \text{LHS} = (1 - \sin^2 t)\frac{[-p^2 \sin pt \cdot \cos t + p\cos pt \cdot \sin t]}{\cos^3 t}$

$$- \sin t \cdot \frac{p\cos pt}{\cos t} + p^2 \sin pt$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix} (1 - \sin^2 t)(-p^2 \sin pt \cdot \cos t + p\cos pt \cdot \sin t) \\ - p\cos pt \cdot \sin t \cdot \cos^2 t + p^2 \sin pt \cdot \cos^3 t \end{bmatrix}$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix} -p^2 \sin pt \cdot \cos^3 t + p\cos pt \cdot \sin t \cdot \cos^2 t \\ - p\cos pt \cdot \sin t \cdot \cos^2 t + p^2 \sin pt \cdot \cos^3 t \end{bmatrix} \quad [\because 1 - \sin^2 t = \cos^2 t]$$

$$= \frac{1}{\cos^3 t} \cdot 0$$

$$= 0$$

Hence proved.

Q. 82 Find the value of $\frac{dy}{dx}$, if $y = x^{\tan x} + \sqrt{\frac{x^2 + 1}{2}}$.

Sol. We have, $\quad y = x^{\tan x} + \sqrt{\frac{x^2 + 1}{2}}$ ...(i)

Taking $\quad u = x^{\tan x}$ and $v = \sqrt{\frac{x^2 + 1}{2}}$,

$\log u = \tan x \log x$ ...(ii)

and $\quad v^2 = \frac{x^2 + 1}{2}$ ...(iii)

On, differentiating Eq. (ii) w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\tan x\cdot\frac{1}{x}+\log x\cdot\sec^2 x$$

$$\Rightarrow \quad \frac{du}{dx}=u\left[\frac{\tan x}{x}+\log x\cdot\sec^2 x\right]$$

$$=x^{\tan x}\left[\frac{\tan x}{x}+\log x\cdot\sec^2 x\right] \quad \ldots\text{(iv)}$$

Also, differentiating Eq. (iii) w.r.t. x, we get

$$2v\cdot\frac{dv}{dx}=\frac{1}{2}(2x)\Rightarrow\frac{dv}{dx}=\frac{1}{4v}\cdot(2x)$$

$$\Rightarrow \quad \frac{dv}{dx}=\frac{1}{4\cdot\sqrt{\frac{x^2+1}{2}}}\cdot 2x=\frac{x\cdot\sqrt{2}}{2\sqrt{x^2+1}}$$

$$\Rightarrow \quad \frac{dv}{dx}=\frac{x}{\sqrt{2(x^2+1)}} \quad \ldots\text{(v)}$$

Now, $\quad y=u+v$

$$\therefore \quad \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

$$=x^{\tan x}\left[\frac{\tan x}{x}+\log x\cdot\sec^2 x\right]+\frac{x}{\sqrt{2(x^2+1)}}$$

Objective Type Questions

Q. 83 If $f(x)=2x$ and $g(x)=\dfrac{x^2}{2}+1$, then which of the following can be a discontinuous function?

(a) $f(x)+g(x)$ (b) $f(x)-g(x)$

(c) $f(x)\cdot g(x)$ (d) $\dfrac{g(x)}{f(x)}$

Sol. *(d)* We know that, if f and g be continuous functions, then

(a) $f+g$ is continuous **(b)** $f-g$ is continuous.

(c) fg is continuous **(d)** $\dfrac{f}{g}$ is continuous at these points, where $g(x)\neq 0$.

Here, $$\frac{g(x)}{f(x)}=\frac{\frac{x^2}{2}+1}{2x}=\frac{x^2+2}{4x}$$

which is discontinuous at $x=0$.

Q. 84 The function $f(x) = \dfrac{4 - x^2}{4x - x^3}$ is

(a) discontinuous at only one point
(b) discontinuous at exactly two points
(c) discontinuous at exactly three points
(d) None of the above

Sol. *(c)* We have,
$$f(x) = \frac{4 - x^2}{4x - x^3} = \frac{(4 - x^2)}{x\,(4 - x^2)}$$
$$= \frac{(4 - x^2)}{x\,(2^2 - x^2)} = \frac{4 - x^2}{x\,(2 + x)\,(2 - x)}$$

Clearly, $f(x)$ is discontinuous at exactly three points $x = 0$, $x = -2$ and $x = 2$.

Q. 85 The set of points where the function f given by $f(x) = |2x - 1| \sin x$ is differentiable is

(a) R (b) $R - \left(\frac{1}{2}\right)$
(c) $(0, \infty)$ (d) None of these

Sol. *(b)* We have, $f(x) = |2x - 1| \sin x$

At $x = \frac{1}{2}$, $f(x)$ is not differentiable.

Hence, $f(x)$ is differentiable in $R - \left(\frac{1}{2}\right)$.

$\because$
$$Rf'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h}$$
$$= \lim_{h \to 0} \frac{\left|2\left(\frac{1}{2} + h\right) - 1\right| \sin\left(\frac{1}{2} + h\right) - 0}{h}$$
$$= \lim_{h \to 0} \frac{|2h| \cdot \sin\left(\frac{1 + 2h}{2}\right)}{h} = 2 \cdot \sin\frac{1}{2}$$

and
$$Lf'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2} - h\right) - f\left(\frac{1}{2}\right)}{-h}$$
$$= \lim_{h \to 0} \frac{\left|2\left(\frac{1}{2} - h\right)^{-1}\right| - \sin\left(\frac{1}{2} - h\right) - 0}{-h}$$
$$= \lim_{h \to 0} \frac{|0 - 2h| - \sin\left(\frac{1}{2} - h\right)}{-h} = -2\sin\left(\frac{1}{2}\right)$$

$\because$
$$Rf'\left(\frac{1}{2}\right) \neq Lf'\left(\frac{1}{2}\right)$$

So, $f(x)$ is not differentiable at $x = \frac{1}{2}$.

Q. 86 The function $f(x) = \cot x$ is discontinuous on the set

(a) $\{x = n\pi : n \in Z\}$ (b) $\{x = 2n\pi : n \in Z\}$

(c) $\left\{ x = (2n+1)\frac{\pi}{2}; n \in Z \right\}$ (d) $\left\{ x = \frac{n\pi}{2}; n \in Z \right\}$

Sol. *(a)* We know that, $f(x) = \cot x$ is continuous in $R - \{n\pi : n \in Z\}$.

Since, $f(x) = \cot x = \frac{\cos x}{\sin x}$ [since, $\sin x = 0$ at $n\pi$, $n \in Z$]

Hence, $f(x) = \cot x$ is discontinuous on the set $\{x = n\pi : n \in Z\}$.

Q. 87 The function $f(x) = e^{|x|}$ is

(a) continuous everywhere but not differentiable at $x = 0$

(b) continuous and differentiable everywhere

(c) not continuous at $x = 0$

(d) None of the above

Sol. *(a)* Let $u(x) = |x|$ and $v(x) = e^x$

$$\therefore \quad f(x) = vou(x) = v[u(x)]$$
$$= v|x| = e^{|x|}$$

Since, $u(x)$ and $v(x)$ are both continuous functions.

So, $f(x)$ is also continuous function but $u(x) = |x|$ is not differentiable at $x = 0$, whereas $v(x) = e^x$ is differentiable at everywhere.

Hence, $f(x)$ is continuous everywhere but not differentiable at $x = 0$.

Q. 88 If $f(x) = x^2 \sin\frac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is

(a) 0 (b) -1

(c) 1 (d) None of these

Sol. *(a)* $\because$ $f(x) = x^2 \sin\left(\frac{1}{x}\right)$, where $x \neq 0$

Hence, value of the function f at $x = 0$, so that it is continuous at $x = 0$ is 0.

Q. 89 If $f(x) = \begin{cases} mx + 1, & \text{if } x \leq \frac{\pi}{2} \\ \sin x + n, & \text{if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then

(a) $m = 1, n = 0$ (b) $m = \frac{n\pi}{2} + 1$

(c) $n = \frac{m\pi}{2}$ (d) $m = n = \frac{\pi}{2}$

Sol. *(c)* We have, $f(x) = \begin{cases} mx + 1, & \text{if } x \leq \frac{\pi}{2} \\ (\sin x + n), & \text{if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$

$\therefore$ LHL $= \lim_{x \to \frac{\pi^-}{2}} (mx + 1) = \lim_{h \to 0} \left[m\left(\frac{\pi}{2} - h\right) + 1\right] = \frac{m\pi}{2} + 1$

and RHL $= \lim_{x \to \frac{\pi^+}{2}} (\sin x + n) = \lim_{h \to 0} \left[\sin\left(\frac{\pi}{2} + h\right) + n\right]$

$= \lim_{h \to 0} \cos h + n = 1 + n$

$\therefore$ LHL = RHL $\left[\text{to be continuous at } x = \frac{\pi}{2}\right]$

$\Rightarrow m \cdot \frac{\pi}{2} + 1 = n + 1$

$\therefore n = m \cdot \frac{\pi}{2}$

Q. 90 If $f(x) = |\sin x|$, then

(a) f is everywhere differentiable
(b) f is everywhere continuous but not differentiable at $x = n\pi$, $n \in Z$
(c) f is everywhere continuous but not differentiable at $x = (2n + 1)\frac{\pi}{2}$, $n \in Z$
(d) None of the above

Sol. *(b)* We have, $f(x) = |\sin x|$

Let $f(x) = vou\ (x) = v\ [u(x)]$ [where, $u(x) = \sin x$ and $v(x) = |x|$]

$= v(\sin x) = |\sin x|$

where, $u(x)$ and $v(x)$ are both continuous.

Hence, $f(x) = vo\,u(x)$ is also a continuous function but $v(x)$ is not differentiable at $x = 0$.

So, $f(x)$ is not differentiable where $\sin x = 0 \Rightarrow x = n\pi$, $n \in Z$

Hence, $f(x)$ is continuous everywhere but not differentiable at $x = n\pi$, $n \in Z$.

Q. 91 If $y = \log\left(\frac{1 - x^2}{1 + x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{4x^3}{1 - x^4}$ (b) $\frac{-4x}{1 - x^4}$ (c) $\frac{1}{4 - x^4}$ (d) $\frac{-4x^3}{1 - x^4}$

Sol. *(b)* We have, $y = \log\left(\frac{1 - x^2}{1 + x^2}\right)$

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{1 - x^2}{1 + x^2}} \cdot \frac{d}{dx}\left(\frac{1 - x^2}{1 + x^2}\right)$$

$$= \frac{(1 + x^2)}{(1 - x^2)} \cdot \frac{(1 + x^2)\cdot(-2x) - (1 - x^2)\cdot 2x}{(1 + x^2)^2}$$

$$= \frac{-2x[1 + x^2 + 1 - x^2]}{(1 - x^2)\cdot(1 + x^2)} = \frac{-4x}{1 - x^4}$$

Q. 92 If $y = \sqrt{\sin x + y}$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{\cos x}{2y-1}$ (b) $\dfrac{\cos x}{1-2y}$ (c) $\dfrac{\sin x}{1-2y}$ (d) $\dfrac{\sin x}{2y-1}$

Sol. *(a)* $\because$ $y = (\sin x + y)^{1/2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(\sin x + y)^{-1/2} \cdot \frac{d}{dx}(\sin x + y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{(\sin x + y)^{1/2}} \cdot \left(\cos x + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}\left(\cos x + \frac{dy}{dx}\right) \qquad [\because (\sin x + y)^{1/2} = y]$$

$$\Rightarrow \quad \frac{dy}{dx}\left(1 - \frac{1}{2y}\right) = \frac{\cos x}{2y}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x}{2y} \cdot \frac{2y}{2y-1} = \frac{\cos x}{2y-1}$$

Q. 93 The derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t. $\cos^{-1} x$ is

(a) 2 (b) $\dfrac{-1}{2\sqrt{1-x^2}}$

(b) $\dfrac{2}{x}$ (d) $1 - x^2$

Sol. *(a)* Let $u = \cos^{-1}(2x^2 - 1)$ and $v = \cos^{-1} x$

$$\therefore \quad \frac{dv}{dx} = \frac{+\,-1}{\sqrt{1-(2x^2-1)^2}} \cdot 4x = \frac{-4x}{\sqrt{1-(4x^4+1-4x^2)}}$$

$$= \frac{-4x}{\sqrt{-4x^4+4x^2}} = \frac{-4x}{\sqrt{4x^2(1-x^2)}}$$

$$= \frac{-2}{\sqrt{1-x^2}}$$

and
$$\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{dx}{dv} = \frac{du/dx}{dv/dx} = \frac{-2/\sqrt{1-x^2}}{-1/\sqrt{1-x^2}} = 2$$

Q. 94 If $x = t^2$ and $y = t^3$, then $\dfrac{d^2y}{dx^2}$ is equal to

(a) $\dfrac{3}{2}$ (b) $\dfrac{3}{4t}$ (c) $\dfrac{3}{2t}$ (d) $\dfrac{3}{2t}$

Sol. *(b)* We have, $x = t^2$ and $y = t^3$

$$\therefore \quad \frac{dx}{dt} = 2t \text{ and } \frac{dy}{dt} = 3t^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t$$

On further differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{3}{2} \cdot \frac{d}{dt} t \cdot \frac{dt}{dx}$$

$$= \frac{3}{2} \cdot \frac{1}{2t} \qquad \left[\because \frac{dt}{dx} = \frac{1}{2t}\right]$$

$$= \frac{3}{4t}$$

Q. 95 The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(a) 1 (b) -1 (c) $\frac{3}{2}$ (d) $\frac{1}{3}$

Sol. *(a)* $\because \quad f'(c) = 0 \qquad [\because f'(x) = 3x^2 - 3]$

$\Rightarrow \quad 3c^2 - 3 = 0$

$\Rightarrow \quad c^2 = \frac{3}{3} = 1$

$\Rightarrow \quad c = \pm 1$, where $1 \in (0, \sqrt{3})$

$\therefore \quad c = 1$

Q. 96 For the function $f(x) = x + \frac{1}{x}$, $x \in [1, 3]$, the value of c for mean value theorem is

(a) 1 (b) $\sqrt{3}$

(c) 2 (d) None of these

Sol. *(b)* $\because \quad f'(c) = \frac{f(b) - f(a)}{b - a}$

$$\Rightarrow \quad 1 - \frac{1}{c^2} = \frac{\left[3 + \frac{1}{3}\right] - \left[1 + \frac{1}{1}\right]}{3 - 1} \qquad \left[\begin{array}{l}\because f'(x) = 1 - \frac{1}{x^2} \\ \text{and } b = 3, a = 1\end{array}\right]$$

$$\Rightarrow \quad \frac{c^2 - 1}{c^2} = \frac{\frac{10}{3} - 2}{2}$$

$$\Rightarrow \quad \frac{c^2 - 1}{c^2} = \frac{4}{3 \times 2} = \frac{2}{3}$$

$\Rightarrow \quad 3(c^2 - 1) = 2c^2$

$\Rightarrow \quad 3c^2 - 2c^2 = 3$

$\Rightarrow \quad c^2 = 3 \Rightarrow c = \pm \sqrt{3}$

$\because \quad c = \sqrt{3} \in (1, 3)$

Fillers

Q. 97 An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is ....

Sol. $|x| + |x - 1|$ is continuous everywhere but fails to be differentiable exactly at two points $x = 0$ and $x = 1$.

So, there can be more such examples of functions.

Q. 98 Derivative of x^2 w.r.t. x^3 is ....

Sol. Derivative of x^2 w.r.t. x^3 is $\frac{2}{3x}$.

Let $u = x^2$ and $v = x^3$

$\therefore$ $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 3x^2$

$\Rightarrow$ $\frac{du}{dv} = \frac{2x}{3x^2} = \frac{2}{3x}$

Q. 99 If $f(x) = |\cos x|$, then $f'\left(\frac{\pi}{4}\right)$ is equal to ....

Sol. If $f(x) = |\cos x|$, then $f'\left(\frac{\pi}{4}\right)$

$\because$ $0 < x < \frac{\pi}{2}, \cos x > 0.$

$f(x) = +\cos x$

$\therefore$ $f'(x) = (-\sin x)$

$\Rightarrow$ $f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$ $\left[\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$

Q. 100 If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{3}\right)$ is equal to ....

Sol. $\because$ $f(x) = |\cos x - \sin x|,$

$\therefore$ $f'\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}+1}{2}$

We know that, $\frac{\pi}{4} < x < \frac{\pi}{2}, \sin x > \cos x$

$\therefore \cos x - \sin x \le 0$ *i.e.*, $f(x) = -(\cos x - \sin x)$

$f'(x) = -[-\sin x - \cos x]$

$\therefore$ $f'\left(\frac{\pi}{3}\right) = -\left(\frac{-\sqrt{3}}{2} - \frac{1}{2}\right) = \left(\frac{\sqrt{3}+1}{2}\right)$

Q. 101 For the curve $\sqrt{x}+\sqrt{y}=1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4},\frac{1}{4}\right)$ is ....

Sol. For the curve $\sqrt{x}+\sqrt{y}=1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4},\frac{1}{4}\right)$ is -1.

We have, $\sqrt{x}+\sqrt{y}=1$

$$\Rightarrow \quad \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)}=\frac{-\frac{1}{2}}{\frac{1}{2}}=-1$$

True/False

Q. 102 Rolle's theorem is applicable for the function $f(x)=|x-1|$ in $[0, 2]$.

Sol. ***False***

Hence, $f(x)=|x-1|$ in $[0, 2]$ is not differentiable at $x=1\in(0,2)$.

Q. 103 If f is continuous on its domain D, then $|f|$ is also continuous on D.

Sol. ***True***

Q. 104 The composition of two continuous function is a continuous function.

Sol. ***True***

Q. 105 Trigonometric and inverse trigonometric functions are differentiable in their respective domain.

Sol. ***True***

Q. 106 If $f \cdot g$ is continuous at $x=a$, then f and g are separately continuous at $x=a$.

Sol. ***False***

Let $f(x)=\sin x$ and $g(x)=\cot x$

$$\therefore \quad f(x)\cdot g(x)=\sin x\cdot\frac{\cos x}{\sin x}=\cos x$$

which is continuous at $x=0$ but $\cot x$ is not continuous at $x=0$.

4

Determinants

Short Answer Type Questions

Q. 1 $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Sol. We have, $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = \begin{vmatrix} x^2 - 2x + 2 & x - 1 \\ 0 & x + 1 \end{vmatrix}$ $[\because C_1 \to C_1 - C_2]$

$$= (x^2 - 2x + 2) \cdot (x + 1) - (x - 1) \cdot 0$$
$$= x^3 - 2x^2 + 2x + x^2 - 2x + 2$$
$$= x^3 - x^2 + 2$$

Q. 2 $\begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix}$

Sol. We have, $\begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix} = \begin{vmatrix} a & -a & 0 \\ 0 & a & -a \\ x & y & a + z \end{vmatrix}$ $\left[\begin{array}{l} \because \ R_1 \to R_1 - R_2 \\ \text{and } R_2 \to R_2 - R_3 \end{array}\right]$

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & a & -a \\ x & x + y & a + z \end{vmatrix}$$ $[\because C_2 \to C_2 + C_1]$

$$= a\,(a^2 + az + ax + ay)$$
$$= a^2\,(a + z + x + y)$$

Q. 3 $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

Sol. We have, $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix} = x^2y^2z^2 \begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix}$

[taking x^2, y^2 and z^2 common from C_1, C_2 and C_3, respectively]

$$= x^2y^2z^2 \begin{vmatrix} 0 & 0 & x \\ y & -y & y \\ z & z & 0 \end{vmatrix} \qquad [\because C_2 \to C_2 - C_3]$$

$$= x^2y^2z^2\,[x\,(yz + yz)]$$

$$= x^2y^2z^2 \cdot 2xyz = 2x^3y^3z^3$$

Q. 4 $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

Sol. We have, $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

Applying, $C_1 \to C_1 + C_2 + C_3$,

$$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

$$= (x+y+z) \begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

[taking $(x+y+z)$ common from column C_1]

$$= (x+y+z) \begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & x-y \\ 0 & x-z & 2z+x \end{vmatrix}$$

[$\because$ $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

Now, expanding along first column, we get

$$(x+y+z) \cdot 1\,[(2y+x)(2z+x) - (x-y)(x-z)]$$

$$= (x+y+z)(4yz + 2yx + 2xz + x^2 - x^2 + xz + yx - yz)$$

$$= (x+y+z)(3yz + 3yx + 3xz)$$

$$= 3\,(x+y+z)(yz + yx + xz)$$

Q. 5 $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$

Sol. We have, $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = \begin{vmatrix} 2x+4 & 2x+4 & 2x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$ $[\because R_1 \to R_1 + R_2]$

$$= \begin{vmatrix} 2x & 2x & 2x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} + \begin{vmatrix} 4 & 4 & 0 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

[here, given determinant is expressed in sum of two determinants]

$$= 2x \begin{vmatrix} 1 & 1 & 1 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} + 4 \begin{vmatrix} 1 & 1 & 0 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

[taking $2x$ common from first row of first determinant and 4 from first row of second determinant]

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$ in first and applying $C_1 \to C_1 - C_2$ in second, we get

$$= 2x \begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & x \\ -4 & -4 & x+4 \end{vmatrix} + 4 \begin{vmatrix} 0 & 1 & 0 \\ -4 & x+4 & x \\ 0 & x & x+4 \end{vmatrix}$$

Expanding both the along first column, we get

$$2x\,[-4(-4)] + 4\,[4(x+4-0)]$$
$$= 2x \times 16 + 16(x+4)$$
$$= 32x + 16x + 64$$
$$= 16(3x+4)$$

Q. 6 $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Sol. Wehave, $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad [\because R_1 \to R_1 + R_2 + R_3]$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

[taking $(a+b+c)$common from the first row]

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & -(a+b+c) & 2b \\ (a+b+c) & (a+b+c) & (c-a-b) \end{vmatrix}$$

$[\because C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3]$

Expanding along R_1,

$$= (a+b+c)\,[1\{0+(a+b+c^2\}]$$
$$= (a+b+c)\,[(a+b+c)^2]$$
$$= (a+b+c)^3$$

Q. 7 $\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = 0$

Sol. We have to prove,

$$\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = 0$$

$$\therefore \quad \text{LHS} = \begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} xy^2z^2 & xyz & xy+xz \\ x^2yz^2 & xyz & yz+xy \\ x^2y^2z & xyz & xz+yz \end{vmatrix}$$

[$\because R_1 \to xR_1, R_2 \to yR_2, R_3 \to zR_3$]

$$= \frac{1}{xyz}(xyz)^2 \begin{vmatrix} yz & 1 & xy+xz \\ xz & 1 & yz+xy \\ xy & 1 & xz+yz \end{vmatrix}$$

[taking (xyz) common from C_1 and C_2]

$$= xyz \begin{vmatrix} yz & 1 & xy+yz+zx \\ xz & 1 & xy+yz+zx \\ xy & 1 & xy+yz+zx \end{vmatrix} \quad [C_3 \to C_3 + C_1]$$

$$= xyz\,(xy+yz+zx) \begin{vmatrix} yz & 1 & 1 \\ xz & 1 & 1 \\ xy & 1 & 1 \end{vmatrix}$$

[taking $(xy+yz+zx)$ common from C_3]

$= 0$ [since, C_2 and C_3 are identicals]

$=$ RHS **Hence proved.**

Q. 8 $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$

Thinking Process

First in LHS use $C_1 \to C_1 + C_2 + C_3$ and then by using $C_1 \to C_1 - C_2$ and $R_1 \to R_1 - R_3$, we can get two zeroes in column 1 and then by simplification we will get the desired result.

Sol. We have to prove,

$$\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$$

$$\therefore \quad \text{LHS} = \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} y+z+z+y & z & y \\ z+z+x+x & z+x & x \\ y+x+x+y & x & x+y \end{vmatrix} \qquad [\because C_1 \to C_1 + C_2 + C_3]$$

$$= 2\begin{vmatrix} (y+z) & z & y \\ (z+x) & z+x & x \\ (x+y) & x & x+y \end{vmatrix} \qquad [\text{taking 2 common from } C_1]$$

$$= 2\begin{vmatrix} y & z & y \\ 0 & z+x & x \\ y & x & x+y \end{vmatrix} \qquad [\because C_1 \to C_1 - C_2]$$

$$= 2\begin{vmatrix} 0 & z-x & -x \\ 0 & z+x & x \\ y & x & x+y \end{vmatrix} \qquad [\because R_1 \to R_1 - R_3]$$

$$= 2\ [y(xz - x^2 + xz + x^2)]$$

$$= 4xyz = \text{RHS}$$

Hence proved.

Q. 9 $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$

Thinking Process

Here, by using $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ in LHS, we can easily get the desired result.

Sol. We have to prove,

$$= \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

$$\therefore \quad \text{LHS} = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+2a-2a-1 & 2a+1-a-2 & 0 \\ 2a+1-3 & a+2-3 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$[\because R_1 \to R_1 - R_2 \text{ and } R_2 \to R_2 - R_3]$$

$$= \begin{vmatrix} (a-1)(a+1) & (a-1) & 0 \\ 2(a-1) & (a-1) & 0 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^2 \begin{vmatrix} (a+1) & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$[\text{taking } (a-1) \text{ common from } R_1 \text{ and } R_2 \text{ each}]$$

$$= (a-1)^2[1(a+1) - 2] = (a-1)^3$$

$$= \text{RHS}$$

Hence proved.

Q. 10 If $A+B+C=0$, then prove that $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0.$

Thinking Process

We have, given $A+B+C=0$, so on solving the determinant by expansion, we can use $\cos(A+B)=\cos(-C)$ and similarly after simplification this expansion we will get the desired result.

Sol. We have to prove, $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$

$$\therefore \quad \text{LHS} = \begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}$$

$$= 1(1-\cos^2 A) - \cos C(\cos C - \cos A \cdot \cos B) + \cos B(\cos C \cdot \cos A - \cos B)$$

$$= \sin^2 A - \cos^2 C + \cos A \cdot \cos B \cdot \cos C + \cos A \cdot \cos B \cdot \cos C - \cos^2 B$$

$$= \sin^2 A - \cos^2 B + 2\cos A \cdot \cos B \cdot \cos C - \cos^2 C$$

$$= -\cos(A+B) \cdot \cos(A-B) + 2\cos A \cdot \cos B \cdot \cos C - \cos^2 C$$

$$[\because \cos^2 B - \sin^2 A = \cos(A+B) \cdot \cos(A-B)]$$

$$= -\cos(-C) \cdot \cos(A-B) + \cos C(2\cos A \cdot \cos B - \cos C) \quad [\because \cos(-\theta) = \cos\theta]$$

$$= -\cos C(\cos A \cdot \cos B + \sin A \cdot \sin B - 2\cos A \cdot \cos B + \cos C)$$

$$= \cos C(\cos A \cdot \cos B - \sin A \cdot \sin B - \cos C)$$

$$= \cos C[\cos(A+B) - \cos C]$$

$$= \cos C(\cos C - \cos C) = 0 = \text{RHS}$$

Hence proved.

Q. 11 If the coordinates of the vertices of an equilateral triangle with sides of length 'a' are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}.$$

Sol. Since, we know that area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3), is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta^2 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 \quad \text{...(i)}$$

We know that, area of an equilateral triangle with side a,

$$\Delta = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)a^2 = \frac{\sqrt{3}}{4}a^2$$

$$\Rightarrow \quad \Delta^2 = \frac{3}{16}a^4 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $\quad \dfrac{3}{16} a^4 = \dfrac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$

$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \dfrac{3}{4} a^4$ **Hence proved**.

Q. 12 Find the value of θ satisfying $\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$

Sol. We have, $\quad \begin{vmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{vmatrix} = 0$

$\Rightarrow \quad \begin{vmatrix} 0 & 1 & \sin 3\theta \\ -7 & 3 & \cos 2\theta \\ 14 & -7 & -2 \end{vmatrix} = 0 \quad [\because C_1 \to C_1 - C_2]$

$\Rightarrow \quad 7\begin{vmatrix} 0 & 1 & \sin 3\theta \\ -1 & 3 & \cos 2\theta \\ 2 & -7 & -2 \end{vmatrix} = 0 \quad$ [taking 7 common from C_1]

$\Rightarrow \quad 7\,[0 - 1(2 - 2\cos 2\theta) + \sin 3\theta\,(7 - 6)] = 0 \quad$ [expanding along R_1]

$\Rightarrow \quad 7\,[-2\,(1 - \cos 2\theta) + \sin 3\theta] = 0$

$\Rightarrow \quad -14 + 14\cos 2\theta + 7\sin 3\theta = 0$

$\Rightarrow \quad 14\cos 2\theta + 7\sin 3\theta = 14$

$\Rightarrow \quad 14\,(1 - 2\sin^2\theta) + 7\,(3\sin\theta - 4\sin^3\theta) = 14$

$\Rightarrow \quad -28\sin^2\theta + 14 + 21\sin\theta - 28\sin^3\theta = 14$

$\Rightarrow \quad -28\sin^2\theta - 28\sin^3\theta + 21\sin\theta = 0$

$\Rightarrow \quad 28\sin^3\theta + 28\sin^2\theta - 21\sin\theta = 0$

$\Rightarrow \quad 4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow \quad \sin\theta\,(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\Rightarrow \quad$ Either $\sin\theta = 0$,

$\Rightarrow \quad \theta = n\pi \quad$ or $\quad 4\sin^2\theta + 4\sin\theta - 3 = 0$

$\therefore \quad \sin\theta = \dfrac{-4 \pm \sqrt{16 + 48}}{8} = \dfrac{-4 \pm \sqrt{64}}{8}$

$= \dfrac{-4 \pm 8}{8} = \dfrac{4}{8}, \dfrac{-12}{8}$

$\sin\theta = \dfrac{1}{2}, \dfrac{-3}{2}$

If $\quad \sin\theta = \dfrac{1}{2} = \sin\dfrac{\pi}{6}$, then

$\theta = n\pi + (-1)^n \dfrac{\pi}{6}$

Hence, $\quad \sin\theta = \dfrac{-3}{2} \quad$ [not possible because $-1 \le \sin\theta \le 1$]

Q. 13 If $\begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix} = 0$, then find the value of x.

Sol. Given,
$$\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 12+x & 12+x & 12+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0 \qquad [\because R_1 \to R_1 + R_2 + R_3]$$

$$\Rightarrow (12+x)\begin{vmatrix} 1 & 1 & 1 \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0 \qquad [\text{taking } (12+x) \text{ common from } R_1]$$

$$\Rightarrow (12+x)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 8 & 4+x \\ 2x & 8 & 4-x \end{vmatrix} = 0 \qquad [\because C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 + C_3]$$

$$\Rightarrow (12+x)[1 \cdot (-16x)] = 0$$
$$\Rightarrow (12+x)(-16x) = 0$$
$$\therefore x = -12, 0$$

Q. 14 If $a_1, a_2, a_3, \ldots, a_r$ are in GP, then prove that the determinant $\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$ is independent of r.

Thinking Process

We know that, nth term of a GP has value ar^{n-1}, where a = first term and r = common ratio. So, by using this result, we can prove the given determinant as independent of r.

Sol. We know that, $a_{r+1} = AR^{(r+1)-1} = AR^r$

where $r = r$ th term of a GP, A = First term of a GP and R = Common ratio of GP

We have,
$$\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$$
$$= \begin{vmatrix} AR^r & AR^{r+4} & AR^{r+8} \\ AR^{r+6} & AR^{r+10} & AR^{r+14} \\ AR^{r+10} & AR^{r+16} & AR^{r+20} \end{vmatrix}$$
$$= AR^r \cdot AR^{r+6} \cdot AR^{r+10} \begin{vmatrix} 1 & AR^4 & AR^8 \\ 1 & AR^4 & AR^8 \\ 1 & AR^6 & AR^{10} \end{vmatrix}$$

[taking AR^r, AR^{r+6} and AR^{r+10} common from R_1, R_2 and R_3, respectively]

$= 0$ [since, R_1 and R_2 are identicals]

Q. 15 Show that the points $(a+5, a-4)$, $(a-2, a+3)$ and (a, a) do not lie on a straight line for any value of a.

Thinking Process

We know that, if three points lie in a straight line, then area formed by these points will be equal to zero. So, by showing area formed by these points other than zero, we can prove the result.

Sol. Given, the points are $(a+5, a-4)$, $(a-2, a+3)$ and (a, a).

$$\therefore \quad \Delta = \frac{1}{2}\begin{vmatrix} a+5 & a-4 & 1 \\ a-2 & a+3 & 1 \\ a & a & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 5 & -4 & 0 \\ -2 & 3 & 0 \\ a & a & 1 \end{vmatrix} \quad [\because R_1 \to R_1 - R_3 \text{ and } R_2 \to R_2 - R_3]$$

$$= \frac{1}{2}[1(15-8)]$$

$$\Rightarrow \quad = \frac{7}{2} \neq 0$$

Hence, given points form a triangle *i.e.,* points do not lie in a straight line.

Q. 16 Show that ΔABC is an isosceles triangle, if the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0.$$

Sol. We have, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos C & \cos B - \cos C & 1+\cos C \\ \cos^2 A + \cos A - \cos^2 C - \cos C & \cos^2 B + \cos B - \cos^2 C - \cos C & \cos^2 C + \cos C \end{vmatrix} = 0$$

$[\therefore \quad C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 - C_3]$

$$\Rightarrow (\cos A - \cos C) \cdot (\cos B - \cos C) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1+\cos C \\ \cos A + \cos C + 1 & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix} = 0$$

[taking $(\cos A - \cos C)$ common from C_1 and $(\cos B - \cos C)$ common from C_2]

$$\Rightarrow \quad (\cos A - \cos C) \cdot (\cos B - \cos C)[(\cos B + \cos C + 1) - (\cos A + \cos C + 1)] = 0$$

$$\Rightarrow \quad (\cos A - \cos C) \cdot (\cos B - \cos C)\ (\cos B + \cos C + 1 - \cos A - \cos C - 1) = 0$$

$$\Rightarrow \quad (\cos A - \cos C) \cdot (\cos B - \cos C)(\cos B - \cos A) = 0$$

i.e., $\quad \cos A = \cos C$ or $\cos B = \cos C$ or $\cos B = \cos A$

$\Rightarrow \quad A = C$ or $B = C$ or $B = A$

Hence, ABC is an isosceles triangle.

Q. 17 Find A^{-1}, if $A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ and show that $A^{-1} = \dfrac{A^2 - 3I}{2}$.

Sol. We have, $A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$\therefore \quad A_{11} = -1, A_{12} = 1, A_{13} = 1, A_{21} = 1, A_{22} = -1, A_{23} = 1, A_{31} = 1, A_{32} = 1$ and $A_{33} = -1$

$$\therefore \quad \text{adj } A = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}^T = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

and $\quad |A| = -1(-1) + 1 \cdot 1 = 2$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{...(i)}$$

and
$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{...(ii)}$$

$$\therefore \quad \frac{A^2 - 3I}{2} = \frac{1}{2}\left\{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix}\right\} = \frac{1}{2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= A^{-1}$$ [using Eq. (i)]

Hence proved.

Long Answer Type Questions

Q. 18 If $A = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix}$, then find the value of A^{-1}.

Using A^{-1}, solve the system of linear equations $x - 2y = 10$, $2x - y - z = 8$ and $-2y + z = 7$.

Sol. We have,
$$A = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \quad \text{...(i)}$$

$\therefore \quad |A| = 1(-3) - 2(-2) + 0 = 1 \neq 0$

Now, $A_{11} = -3, A_{12} = 2, A_{13} = 2, A_{21} = -2, A_{22} = 1, A_{23} = 1, A_{31} = -4, A_{32} = 2$ and $A_{33} = 3$

$$\therefore \quad \text{adj } (A) = \begin{vmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{vmatrix}^T = \begin{vmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\therefore \qquad A^{-1} = \frac{\text{adj } A}{|A|}$$

$$= \frac{1}{1}\begin{vmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\Rightarrow \qquad A^{-1} = \begin{vmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix} \qquad \text{...(ii)}$$

Also, we have the system of linear equations as

$$x - 2y = 10,$$
$$2x - y - z = 8$$
and
$$-2y + z = 7$$

In the form of $CX = D$,

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

where,
$$C = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

We know that, $\qquad (A^T)^{-1} = (A^{-1})^T$

$$\therefore \qquad C^T = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix} = A \qquad \text{[using Eq. (i)]}$$

$$\therefore \qquad X = C^{-1} D$$

$$\Rightarrow \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -30 + 16 + 14 \\ -20 + 8 + 7 \\ -40 + 16 + 21 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$$

$$\therefore \qquad x = 0, y = -5 \text{ and } z = -3$$

Q. 19 Using matrix method, solve the system of equations $3x + 2y - 2z = 3$, $x + 2y + 3z = 6$ and $2x - y + z = 2$.

Thinking Process

We know that, for given system of equations in the matrix form, we get $AX = B \Rightarrow X = A^{-1}B$, where $A^{-1} = \frac{\text{adj}(A)}{|A|}$ and then by getting inverse of A and determinant of A, we can get the desired result.

Sol. Given system of equations is

$$3x + 2y - 2z = 3,$$
$$x + 2y + 3z = 6$$
and
$$2x - y + z = 2$$

In the form of $AX = B$,

$$\begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

For A^{-1}, $\quad |A| = |3(5) - 2(1-6) + (-2)(-5)|$

$$= |15 + 10 + 10| = |35| \neq 0$$

$\therefore \quad A_{11} = 5, A_{12} = 5, A_{13} = -5, \quad A_{21} = 0, A_{22} = 7, A_{23} = 7, \quad A_{31} = 10, A_{32} = -11$ and $A_{33} = 4$

$$\therefore \qquad \text{adj } A = \begin{vmatrix} 5 & 5 & -5 \\ 0 & 7 & 7 \\ 10 & -11 & 4 \end{vmatrix}^T = \begin{vmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{vmatrix}$$

Now, $\quad A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{35}\begin{vmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{vmatrix}$

For $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} 15 + 20 \\ 15 + 42 - 22 \\ -15 + 42 + 8 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 35 \\ 35 \\ 35 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore \qquad x = 1, y = 1$ and $z = 1$

Q. 20 If $A = \begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix}$ and $B = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$, then find BA and use this to solve the system of equations $y + 2z = 7$, $x - y = 3$ and $2x + 3y + 4z = 17$.

Sol. We have, $\quad A = \begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix}$ and $B = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$

$$\therefore \qquad BA = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}\begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 6I$$

$$\therefore \qquad B^{-1} = \frac{A}{6} = \frac{1}{6}A = \frac{1}{6}\begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix} \qquad \text{...(i)}$$

Also, $\quad x - y = 3, 2x + 3y + 4z = 17$ and $y + 2z = 7$

$$\Rightarrow \qquad \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} \qquad \text{[using Eq. (i)]}$$

$$= \frac{1}{6} \begin{bmatrix} 6 + 34 - 28 \\ -12 + 34 - 28 \\ 6 - 17 + 35 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$\therefore \quad x = 2, y = -1$ and $z = 4$

Q. 21 If $a + b + c \neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then prove that $a = b = c$.

Sol. Let

$$A = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} \qquad [\because R_1 \to R_1 + R_2 + R_3]$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ b-a & c-a & a \\ c-b & a-b & b \end{vmatrix} \qquad [\because C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 - C_3]$$

Expanding along R_1,

$$= (a+b+c)\,[1\,(b-a)(a-b) - (c-a)(c-b)]$$

$$= (a+b+c)\,(ba - b^2 - a^2 + ab - c^2 + cb + ac - ab)$$

$$= \frac{-1}{2}(a+b+c) \times (-2)\ (-a^2 - b^2 - c^2 + ab + bc + ca)$$

$$= \frac{-1}{2}(a+b+c)[a^2 + b^2 + c^2 - 2ab - 2bc - 2ca + a^2 + b^2 + c^2]$$

$$= -\frac{1}{2}(a+b+c)[a^2 + b^2 - 2ab + b^2 + c^2 - 2bc + c^2 + a^2 - 2ac]$$

$$= \frac{-1}{2}(a+b+c)\,[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

Also, $A = 0$

$$= \frac{-1}{2}(a+b+c)\,[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 0 \qquad [\because a + b + c \neq 0, \text{ given}]$$

$$\Rightarrow \quad a - b = b - c = c - a = 0$$

$$a = b = c$$

Hence proved.

Q. 22 Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ is divisible by $(a+b+c)$ and find the quotient.

Sol. Let
$$\Delta = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} bc-a^2-ca+b^2 & ca-b^2-ab+c^2 & ab-c^2 \\ ca-b^2-ab+c^2 & ab-c^2-bc+a^2 & bc-a^2 \\ ab-c^2-bc+a^2 & bc-a^2-ca+b^2 & ca-b^2 \end{vmatrix}$$

$[\because C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3]$

$$= \begin{vmatrix} (b-a)(a+b+c) & (c-b)(a+b+c) & ab-c^2 \\ (c-b)(a+b+c) & (a-c)(a+b+c) & bc-a^2 \\ (a-c)(a+b+c) & (b-a)(a+b+c) & ca-b^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b-a & c-b & ab-c^2 \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

[taking $(a+b+c)$ common from C_1 and C_2 each]

$$= (a+b+c)^2 \begin{vmatrix} 0 & 0 & ab+bc+ca-(a^2+b^2+c^2) \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

$[\because R_1 \to R_1 + R_2 + R_3]$

Now, expanding along R_1,

$$= (a+b+c)^2\, [ab+bc+ca-(a^2+b^2+c^2)][(c-b)(b-a)-(a-c)^2]$$

$$= (a+b+c)^2\, (ab+bc+ca-a^2-b^2-c^2)$$
$$(cb-ac-b^2+ab-a^2-c^2+2ac)$$

$$= (a+b+c)^2\, (a^2+b^2+c^2-ab-bc-ca)$$
$$(a^2+b^2+c^2-ac-ab-bc)$$

$$= \frac{1}{2}(a+b+c)\,[(a+b+c)(a^2+b^2+c^2-ab-bc-ca)]$$
$$[(a-b)^2+(b-c)^2+(c-a)^2]$$

$$= \frac{1}{2}(a+b+c)(a^3+b^3+c^3-3abc)\,[(a-b)^2+(b-c)^2+(c-a)^2]$$

Hence, given determinant is divisible by $(a+b+c)$ and quotient is
$$(a^3+b^3+c^3-3abc)[(a-b)^2+(b-c)^2+(c-a)^2].$$

Q. 23 If $x + y + z = 0$, then prove that $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

Thinking Process

We have, given $x + y + z = 0 \Rightarrow x^3 + y^3 + z^3 = 3xyz$. So, by using this in solving the given determinant from both the sides, we can equate the obtained result from both the sides to desired result.

Sol. Since, $x + y + z = 0$, also we have to prove

$$\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$\therefore$

$$\text{LHS} = \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix}$$

$$= xa\,(za \cdot ya - xb \cdot xc) - yb\,(yc \cdot ya - xb \cdot zb) + zc\,(yc \cdot xc - za \cdot zb)$$

$$= xa\,(a^2yz - x^2bc) - yb\,(y^2ac - b^2xz) + zc\,(c^2xy - z^2ab)$$

$$= xyza^3 - x^3abc - y^3abc + b^3xyz + c^3xyz - z^3abc$$

$$= xyz\,(a^3 + b^3 + c^3) - abc\,(x^3 + y^3 + z^3)$$

$$= xyz\,(a^3 + b^3 + c^3) - abc\,(3xyz)$$

$[\because x + y + z = 0 \Rightarrow x^3 + y^3 + z^3 - 3xyz]$

$$= xyz\,(a^3 + b^3 + c^3 - 3abc) \qquad \text{...(i)}$$

Now,

$$\text{RHS} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = xyz \begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix} \qquad [\because C_1 \to C_1 + C_2 + C_3]$$

$$= xyz\,(a + b + c) \begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix} \qquad [\text{taking } (a + b + c) \text{ common from } C_1]$$

$$= xyz\,(a + b + c) \begin{vmatrix} 0 & b-c & c-a \\ 0 & a-c & b-a \\ 1 & c & a \end{vmatrix}$$

$[\because R_1 \to R_1 - R_3 \text{ and } R_2 \to R_2 - R_3]$

Expanding along C_1,

$$= xyz\,(a + b + c)\,[1\,(b - c)(b - a) - (a - c)(c - a)]$$

$$= xyz\,(a + b + c)\,(b^2 - ab - bc + ac + a^2 + c^2 - 2ac)$$

$$= xyz\,(a + b + c)\,(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$= xyz\,(a^3 + b^3 + c^3 - 3abc) \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$\text{LHS} = \text{RHS}$$

$$\Rightarrow \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Hence proved.

Objective Type Questions

Q. 24 If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then the value of x is

(a) 3 (b) $\pm$ 3 (c) $\pm$ 6 (d) 6

Sol. *(c)* $\because$ $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

$\Rightarrow$ $2x^2 - 40 = 18 + 14$

$\Rightarrow$ $2x^2 = 32 + 40$

$\Rightarrow$ $x^2 = \dfrac{72}{2} = 36$

$\therefore$ $x = \pm 6$

Q. 25 The value of $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$ is

(a) $a^3 + b^3 + c^3$ (b) $3bc$

(c) $a^3 + b^3 + c^3 - 3abc$ (d) None of these

Sol. *(d)* We have,

$$\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix} = \begin{vmatrix} a+c & b+c+a & a \\ b+c & c+a+b & b \\ c+b & a+b+c & c \end{vmatrix} \quad [\because C_1 \to C_1 + C_2 \text{ and } C_2 \to C_2 + C_3]$$

$$= (a+b+c)\begin{vmatrix} a+c & 1 & a \\ b+c & 1 & b \\ c+b & 1 & c \end{vmatrix} \quad [\text{taking } (a+b+c) \text{ common from } C_2]$$

$$= (a+b+c)\begin{vmatrix} a-b & 0 & a-c \\ 0 & 0 & b-c \\ c+b & 1 & c \end{vmatrix} \quad [\because R_2 \to R_2 - R_3 \text{ and } R_1 \to R_1 - R_3]$$

$$= (a+b+c)[-(b-c)\cdot(a-b)] \quad [\text{expanding along } R_2]$$

$$= (a+b+c)(c-b)(a-b)$$

Q. 26 If the area of a triangle with vertices (–3, 0), (3, 0) and (0, k) is 9 sq units. Then, the value of k will be

(a) 9 (b) 3 (c) –9 (d) 6

Sol. *(b)* We know that, area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$\therefore$ $$\Delta = \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}$$

Expanding along R_1,

$$9 = \frac{1}{2}[-3(-k) - 0 + 1(3k)]$$

$$\Rightarrow \quad 18 = 3k + 3k = 6k$$

$$\therefore \quad k = \frac{18}{6} = 3$$

Q. 27 The determinant

$$\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix} \text{ equals to}$$

(a) $abc(b-c)(c-a)(a-b)$
(b) $(b-c)(c-a)(a-b)$
(c) $(a+b+c)(b-c)(c-a)(a-b)$
(d) None of these

Sol. *(d)* We have,

$$\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix} = \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$$

$$= (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

[on taking $(b-a)$ common from C_1 and C_3 each]

$$= (b-a)^2 \begin{vmatrix} b-c & b-c & c \\ a-b & a-b & b \\ c-a & c-a & a \end{vmatrix} \qquad [\because C_1 \to C_1 - C_3]$$

$$= 0$$

[since, two columns C_1 and C_2 are identical, so the value of determinant is zero]

Q. 28 The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(a) 0
(b) 2
(c) 1
(d) 3

Sol. *(c)* We have,

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$,

$$\begin{vmatrix} 2\cos x + \sin x & \cos x & \cos x \\ 2\cos x + \sin x & \sin x & \cos x \\ 2\cos x + \sin x & \cos x & \sin x \end{vmatrix} = 0$$

On taking $(2\cos x + \sin x)$ common from C_1, we get

$$\Rightarrow \quad (2\cos x + \sin x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$$\Rightarrow \quad (2\cos x + \sin x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & (\sin x - \cos x) \end{vmatrix} = 0$$

$[\because R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

Expanding along C_1,

$$(2\cos x + \sin x)\,[1\cdot(\sin x - \cos x)^2] = 0$$

$$\Rightarrow \quad (2\cos x + \sin x)(\sin x - \cos x)^2 = 0$$

Either $\quad 2\cos x = -\sin x$

$$\Rightarrow \quad \cos x = -\frac{1}{2}\sin x$$

$$\Rightarrow \quad \tan x = -2 \qquad \text{...(i)}$$

But here for $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$, we get $-1 \le \tan x \le 1$ so, no solution possible

and for $\quad (\sin x - \cos x)^2 = 0, \sin x = \cos x$

$$\Rightarrow \quad \tan x = 1 = \tan\frac{\pi}{4}$$

$$\therefore \quad x = \frac{\pi}{4}$$

So, only one distinct real root exist.

Q. 29 If A, B and C are angles of a triangle, then the determinant

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} \text{ is equal to}$$

(a) 0 (b) -1 (c) 1 (d) None of these

Sol. ***(a)*** We have, $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$

Applying $C_1 \to aC_1 + bC_2 + cC_3$,

$$\begin{vmatrix} -a + b\cos C + c\cos B & \cos C & \cos B \\ a\cos C - b + c\cos A & -1 & \cos A \\ a\cos B + b\cos A - c & \cos A & -1 \end{vmatrix}$$

Also, by projection rule in a triangle, we know that

$$a = b\cos C + c\cos B,\ b = c\cos A + a\cos C \text{ and } c = a\cos B + b\cos A$$

Using above equation in column first, we get

$$\begin{vmatrix} -a + a & \cos C & \cos B \\ b - b & -1 & \cos A \\ c - c & \cos A & -1 \end{vmatrix} = \begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = 0$$

[since, determinant having all elements of any column or row gives value of determinant as zero]

Q. 30 If $f(t)=\begin{bmatrix}\cos t & t & 1\\ 2\sin t & t & 2t\\ \sin t & t & t\end{bmatrix}$, then $\lim\limits_{t\to 0}\dfrac{f(t)}{t^2}$ is equal to

(a) 0 (b) −1 (c) 2 (d) 3

Sol. *(a)* We have,

$$f(t)=\begin{vmatrix}\cos t & t & 1\\ 2\sin t & t & 2t\\ \sin t & t & t\end{vmatrix}$$

Expanding along C_1,

$$=\cos t\,(t^2-2t^2)-2\sin t\,(t^2-t)+\sin t\,(2t^2-t)$$
$$=-t^2\cos t-(t^2-t)2\sin t+(2t^2-t)\sin t$$
$$=-t^2\cos t-t^2\cdot 2\sin t+t\cdot 2\sin t+2t^2\sin t$$
$$=-t^2\cos t+2t\sin t$$

$\therefore$
$$\lim_{t\to 0}\frac{f(t)}{t^2}=\lim_{t\to 0}\frac{(-t^2\cos t)}{t^2}+\lim_{t\to 0}\frac{2t\sin t}{t^2}$$
$$=-\lim_{t\to 0}\cos t+2\cdot\lim_{t\to 0}\frac{\sin t}{t}$$
$$=-1+1 \qquad \left[\because \lim_{t\to 0}\frac{\sin t}{t}=1 \text{ and } \cos 0=1\right]$$
$$=0$$

Q. 31 The maximum value of

$$\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & 1+\sin\theta & 1\\ 1+\cos\theta & 1 & 1\end{vmatrix}$$ is (where, θ is real number)

(a) $\dfrac{1}{2}$ (b) $\dfrac{\sqrt{3}}{2}$ (c) $\sqrt{2}$ (d) $\dfrac{2\sqrt{3}}{4}$

Sol. *(a)* Since,

$$\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & 1+\sin\theta & 1\\ 1+\cos\theta & 1 & 1\end{vmatrix}$$
$$=\begin{vmatrix}0 & 0 & 1\\ 0 & \sin\theta & 1\\ \cos\theta & 0 & 1\end{vmatrix} \qquad [\because C_1\to C_1-C_3 \text{ and } C_2\to C_2-C_3]$$
$$=1(\sin\theta\cdot\cos\theta)$$
$$=\frac{1}{2}\cdot 2\sin\theta\cos\theta=\frac{1}{2}\sin 2\theta$$

Since, the maximum value of $\sin 2\theta$ is 1. So, for maximum value of θ should be 45°.

$\therefore$
$$\Delta=\frac{1}{2}\sin 2\cdot 45^\circ$$
$$=\frac{1}{2}\sin 90^\circ=\frac{1}{2}\,.\,1=\frac{1}{2}$$

Q. 32 If $f(x)=\begin{vmatrix}0 & x-a & x-b\\ x+a & 0 & x-c\\ x+b & x+c & 0\end{vmatrix}$, then

(a) $f(a)=0$ (b) $f(b)=0$ (c) $f(0)=0$ (d) $f(1)=0$

Sol. *(c)* We have,

$$f(x)=\begin{vmatrix}0 & x-a & x-b\\ x+a & 0 & x-c\\ x+b & x+c & 0\end{vmatrix}$$

$$\Rightarrow \quad f(a)=\begin{vmatrix}0 & 0 & a-b\\ 2a & 0 & a-c\\ a+b & a+c & 0\end{vmatrix}$$

$$=[(a-b)\{2a\cdot(a+c)\}]\neq 0$$

$$\therefore \quad f(b)=\begin{vmatrix}0 & b-a & 0\\ b+a & 0 & b-c\\ 2b & b+c & 0\end{vmatrix}$$

$$=-(b-a)[2b(b-c)]$$

$$=-2b(b-a)(b-c)\neq 0$$

$$\therefore \quad f(0)=\begin{vmatrix}0 & -a & -b\\ a & 0 & -c\\ b & c & 0\end{vmatrix}$$

$$=a(bc)-b(ac)$$

$$=abc-abc=0$$

Q. 33 If $A=\begin{vmatrix}2 & \lambda & -3\\ 0 & 2 & 5\\ 1 & 1 & 3\end{vmatrix}$, then A^{-1} exists, if

(a) $\lambda=2$ (b) $\lambda\neq 2$

(c) $\lambda\neq -2$ (d) None of these

Sol. *(d)* We have,

$$A=\begin{vmatrix}2 & \lambda & -3\\ 0 & 2 & 5\\ 1 & 1 & 3\end{vmatrix}$$

Expanding along R_1,

$$|A|=2(6-5)-\lambda(-5)-3(-2)=2+5\lambda+6$$

We know that, A^{-1} exists, if A is non-singular matrix *i.e.,* $|A|\neq 0$.

$$\therefore \quad 2+5\lambda+6\neq 0$$

$$\Rightarrow \quad 5\lambda\neq -8$$

$$\therefore \quad \lambda\neq \frac{-8}{5}$$

So, A^{-1} exists if and only if $\lambda\neq \dfrac{-8}{5}$.

Q. 34 If A and B are invertible matrices, then which of the following is not correct?

(a) $\text{adj } A = |A| \cdot A^{-1}$
(b) $\det (A)^{-1} = [\det (A)]^{-1}$
(c) $(AB)^{-1} = B^{-1} A^{-1}$
(d) $(A + B)^{-1} = B^{-1} + A^{-1}$

Sol. *(d)* Since, A and B are invertible matrices. So, we can say that

$$(AB)^{-1} = B^{-1} A^{-1} \quad \text{...(i)}$$

Also, $$A^{-1} = \frac{1}{|A|} (\text{adj } A)$$

$\Rightarrow$ $$\text{adj } A = |A| \cdot A^{-1} \quad \text{...(ii)}$$

Also, $$\det (A)^{-1} = [\det (A)]^{-1}$$

$\Rightarrow$ $$\det (A)^{-1} = \frac{1}{[\det (A)]}$$

$\Rightarrow$ $$\det (A) \cdot \det (A)^{-1} = 1 \quad \text{...(iii)}$$

which is true.

Again, $$(A + B)^{-1} = \frac{1}{|(A + B)|} \text{adj } (A + B)$$

$\Rightarrow$ $$(A + B)^{-1} \neq B^{-1} + A^{-1} \quad \text{...(iv)}$$

So, only option (d) is incorrect.

Q. 35 If x, y and z are all different from zero and $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$, then the value of $x^{-1} + y^{-1} + z^{-1}$ is

(a) xyz
(b) $x^{-1}y^{-1}z^{-1}$
(c) $-x-y-z$
(d) -1

Sol. *(d)* We have, $$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$,

$\Rightarrow$ $$\begin{vmatrix} x & 0 & 1 \\ 0 & y & 1 \\ -z & -z & 1+z \end{vmatrix} = 0$$

Expanding along R_1,

$$x [y (1 + z) + z] - 0 + 1 (yz) = 0$$

$\Rightarrow$ $$x (y + yz + z) + yz = 0$$

$\Rightarrow$ $$xy + xyz + xz + yz = 0$$

$\Rightarrow$ $$\frac{xy}{xyz} + \frac{xyz}{xyz} + \frac{xz}{xyz} + \frac{yz}{xyz} = 0 \quad \text{[on dividing } (xyz) \text{ from both sides]}$$

$\Rightarrow$ $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1 = 0$$

$\Rightarrow$ $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -1$$

$\therefore$ $$x^{-1} + y^{-1} + z^{-1} = -1$$

Q. 36 The value of $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is

(a) $9x^2(x+y)$ (b) $9y^2(x+y)$
(c) $3y^2(x+y)$ (d) $7x^2(x+y)$

Sol. *(b)* We have, $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$$= \begin{vmatrix} 3(x+y) & x+y & y \\ 3(x+y) & x & y \\ 3(x+y) & x+2y & -2y \end{vmatrix} \quad [\because C_1 \to C_1 + C_2 + C_3 \text{ and } C_3 \to C_3 - C_2]$$

$$= 3(x+y)\begin{vmatrix} 1 & (x+y) & y \\ 1 & x & y \\ 1 & (x+2y) & -2y \end{vmatrix} \quad [\text{taking } 3(x+y) \text{ common from first column}]$$

$$= 3(x+y)\begin{vmatrix} 0 & y & 0 \\ 1 & x & y \\ 1 & (x+2y) & -2y \end{vmatrix} \quad [\because R_1 \to R_1 - R_2]$$

Expanding along R_1,

$$= 3(x+y)[-y(-2y-y)]$$
$$= 3y^2 \cdot 3(x+y) = 9y^2(x+y)$$

Q. 37 If there are two values of a which makes determinant,

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86, \text{ then the sum of these number is}$$

(a) 4 (b) 5 (c) – 4 (d) 9

Sol. *(c)* We have,

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$$

$\Rightarrow \quad 1(2a^2 + 4) - 2(-4a - 20) + 0 = 86$ [expanding along first column]

$\Rightarrow \quad 2a^2 + 4 + 8a + 40 = 86$

$\Rightarrow \quad 2a^2 + 8a + 44 - 86 = 0$

$\Rightarrow \quad a^2 + 4a - 21 = 0$

$\Rightarrow \quad a^2 + 7a - 3a - 21 = 0$

$\Rightarrow \quad (a+7)(a-3) = 0$

$a = -7$ and 3

$\therefore \quad$ Required sum $= -7 + 3 = -4$

Fillers

Q. 38 If A is a matrix of order 3×3, then $|3A|$ is equal to ....

Sol. If A is a matrix of order 3×3, then $|3A| = 3 \times 3 \times 3 |A| = 27 |A|$

Q. 39 If A is invertible matrix of order 3×3, then $|A^{-1}|$ is equal to ....

Sol. If A is invertible matrix of order 3×3, then $|A^{-1}| = \dfrac{1}{|A|}$. [since, $|A| \cdot |A^{-1}| = 1$]

Q. 40 If $x, y, z \in R$, then the value of $\begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$ is

Sol. We have,

$$\begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (2 \cdot 2^x)(2 \cdot 2^{-x}) & (2^x - 2^{-x})^2 & 1 \\ (2 \cdot 3^x)(2 \cdot 3^{-x}) & (3^x - 3^{-x})^2 & 1 \\ (2 \cdot 4^x)(2 \cdot 4^{-x}) & (4^x - 4^{-x})^2 & 1 \end{vmatrix} \quad \begin{matrix} [\because (a+b)^2 - (a-b)^2 = 4ab] \\ [\because C_1 \to C_1 - C_2] \end{matrix}$$

$$= \begin{vmatrix} 4 & (2^x - 2^{-x})^2 & 1 \\ 4 & (3^x - 3^{-x})^2 & 1 \\ 4 & (4^x - 4^{-x})^2 & 1 \end{vmatrix} = 0 \quad \text{[since, } C_1 \text{ and } C_3 \text{ are proportional to each other]}$$

Q. 41 If $\cos 2\theta = 0$, then $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$ is equal to ....

Sol. Since, $\cos 2\theta = 0$

$$\Rightarrow \quad \cos 2\theta = \cos\frac{\pi}{2} \Rightarrow 2\theta = \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

$$\therefore \quad \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \begin{vmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix}^2$$

Expanding along R_1,

$$= \left[-\frac{1}{\sqrt{2}}\left(\frac{1}{2}\right) + \frac{1}{\sqrt{2}}\left(-\frac{1}{2}\right)\right]^2 = \left[\frac{-2}{2\sqrt{2}}\right]^2 = \left(\frac{-1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

Q. 42 **If A is a matrix of order 3×3, then $(A^2)^{-1}$ is equal to ....**

Sol. If A is a matrix of order 3×3, then $(A^2)^{-1} = (A^{-1})^2$.

Q. 43 **If A is a matrix of order 3×3, then the number of minors in determinant of A are ....**

Sol. If A is a matrix of order 3×3, then the number of minors in determinant of A are 9. [since, in a 3×3 matrix, there are 9 elements]

Q. 44 **The sum of products of elements of any row with the cofactors of corresponding elements is equal to ....**

Sol. The sum of products of elements of any row with the cofactors of corresponding elements is equal to value of the determinant.

$$\text{Let } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_1,

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

= Sum of products of elements of R_1 with their corresponding cofactors

Q. 45 **If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then other two roots are ....**

Sol. Since, $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

Expanding along R_1,

$$x(x^2 - 12) - 3(2x - 14) + 7(12 - 7x) = 0$$

$$\Rightarrow \quad x^3 - 12x - 6x + 42 + 84 - 49x = 0$$

$$\Rightarrow \quad x^3 - 67x + 126 = 0 \quad \text{...(i)}$$

Here, $126 \times 1 = 9 \times 2 \times 7$

For $x = 2$, $2^3 - 67 \times 2 + 126 = 134 - 134 = 0$

Hence, $x = 2$ is a root.

For $x = 7$, $7^3 - 67 \times 7 + 126 = 469 - 469 = 0$

Hence, $x = 7$ is also a root.

Q. 46 $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix}$ is equal to ....

Sol. We have, $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix} = \begin{vmatrix} z-x & xyz & x-z \\ z-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix}$ $[\because C_1 \to C_1 - C_3]$

$$= (z-x)\begin{vmatrix} 1 & xyz & x-z \\ 1 & 0 & y-z \\ 1 & z-y & 0 \end{vmatrix}$$

[taking $(z-x)$ common from column 1]

Expanding along R_1,

$= (z-x)[1\cdot\{-(y-z)(z-y)\} - xyz(z-y) + (x-z)(z-y)]$
$= (z-x)(z-y)\ (-y+z-xyz+x-z)$
$= (z-x)(z-y)(x-y-xyz)$
$= (z-x)(y-z)(y-x+xyz)$

Q. 47 If $f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix}$

$= A + Bx + Cx^2 + \ldots$, then A is equal to ....

Sol. Since,

$$f(x) = (1+x)^{17}(1+x)^{23}(1+x)^{41}\begin{vmatrix} 1 & (1+x)^2 & (1+x)^6 \\ 1 & (1+x)^6 & (1+x)^{11} \\ 1 & (1+x)^2 & (1+x)^6 \end{vmatrix} = 0$$

[since, R_1 and R_3 are identical]

$\therefore \quad A = 0$

True/False

Q. 48 $(A^3)^{-1} = (A^{-1})^3$, where A is a square matrix and $|A| \neq 0$.

Sol. ***True***

Since, $(A^n)^{-1} = (A^{-1})^n$, where $n \in N$.

Q. 49 $(aA)^{-1} = \frac{1}{a} A^{-1}$, where a is any real number and A is a square matrix.

Sol. ***False***

Since, we know that, if A is a non-singular square matrix, then for any scalar a (non-zero), aA is invertible such that

$$(aA)\left(\frac{1}{a}A^{-1}\right) = \left(a \cdot \frac{1}{a}\right)(A \cdot A^{-1})$$
$$= I$$

i.e., (aA) is inverse of $\left(\frac{1}{a}A^{-1}\right)$ or $(aA)^{-1} = \frac{1}{a}A^{-1}$, where a is any non-zero scalar.

In the above statement a is any real number. So, we can conclude that above statement is false.

Q. 50 $|A^{-1}| \neq |A|^{-1}$, where A is a non-singular matrix.

Sol. ***False***

$|A^{-1}| = |A|^{-1}$, where A is a non-singular matrix.

Q. 51 If A and B are matrices of order 3 and $|A| = 5$, $|B| = 3$, then $|3AB| = 27 \times 5 \times 3 = 405$.

Sol. ***True***

We know that, $\quad |AB| = |A| \cdot |B|$

$\therefore \quad |3AB| = 27\,|AB|$

$= 27\,|A| \cdot |B|$

$= 27 \times 5 \times 3 = 405$

Q. 52 If the value of a third order determinant is 12, then the value of the determinant formed by replacing each element by its cofactor will be 144.

Sol. ***True***

Let A is the determinant.

$\therefore \quad |A| = 12$

Also, we know that, if A is a square matrix of order n, then $|\text{adj } A| = |A|^{n-1}$

For $n = 3$, $|\text{adj } A| = |A|^{3-1} = |A|^2$

$= (12)^2 = 144$

Q. 53 $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$, where a, b and c are in AP.

Sol. ***True***

Since, a, b and c are in AP, then $2b = a + c$

$$\therefore \quad \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2x+4 & 2x+6 & 2x+a+c \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0 \quad [\because R_1 \to R_1 + R_3]$$

$$\Rightarrow \quad \begin{vmatrix} 2(x+2) & 2(x+3) & 2(x+b) \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0 \quad [\because 2b = a + c]$$

$\Rightarrow \quad 0 = 0$ [since, R_1 and R_2 are in proportional to each other]

Hence, statement is true.

Q. 54 $|\text{adj } A| = |A|^2$, where A is a square matrix of order two.

Sol. ***False***

If A is a square matrix of order n, then

$$|\text{adj } A| = |A|^{n-1}$$

$$\Rightarrow \quad |\text{adj } A| = |A|^{2-1} = |A| \quad [\because n = 2]$$

Q. 55 The determinant $\begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix}$ is equal to zero.

Sol. ***True***

Since, $\begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix} = \begin{vmatrix} \sin A & \cos A & \sin A \\ \sin B & \cos A & \sin B \\ \sin C & \cos A & \sin C \end{vmatrix} + \begin{vmatrix} \sin A & \cos A & \cos B \\ \sin B & \cos A & \cos B \\ \sin C & \cos A & \cos B \end{vmatrix}$

$$= 0 + \begin{vmatrix} \sin A & \cos A & \cos B \\ \sin B & \cos A & \cos B \\ \sin C & \cos A & \cos B \end{vmatrix}$$

[since, in first determinant C_1 and C_3 are identicals]

$$= \cos A \cdot \cos B \begin{vmatrix} \sin A & 1 & 1 \\ \sin B & 1 & 1 \\ \sin C & 1 & 1 \end{vmatrix}$$

[taking $\cos A$ common from C_2 and $\cos B$ common from C_3]

$= 0$ [since, C_2 and C_3 are identicals]

Q. 56 If the determinant $\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$ splits into exactly k determinants of order 3, each element of which contains only one term, then the value of k is 8.

Sol. ***True***

Since,
$$\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$$

$$= \begin{vmatrix} x & p & l \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix} + \begin{vmatrix} a & u & f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix} \qquad \text{[splitting first row]}$$

$$= \begin{vmatrix} x & p & l \\ y & q & m \\ z+c & r+m & n+h \end{vmatrix} + \begin{vmatrix} x & p & l \\ b & v & g \\ z+c & r+w & n+h \end{vmatrix}$$

$$+ \begin{vmatrix} a & u & f \\ y & q & m \\ z+c & r+w & n+h \end{vmatrix} + \begin{vmatrix} a & u & f \\ b & v & g \\ z+c & r+w & n+h \end{vmatrix} \qquad \text{[splitting second row]}$$

Similarly, we can split these 4 determinants in 8 determinants by splitting each one in two determinants further. So, given statement is true.

Q. 57 If $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$, then $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$.

Sol. ***True***

We have,
$$\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$$

and we have to prove,
$$\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$$

$$\Delta_1 = \begin{vmatrix} 2p+2x+2a & a+x & a+p \\ 2q+2y+2b & b+y & b+q \\ 2r+2z+2c & c+z & c+r \end{vmatrix} \qquad [\because C_1 \to C_1 + C_2 + C_3]$$

$$= 2\begin{vmatrix} p & x-p & a+p \\ q & y-q & b+q \\ r & z-r & c+r \end{vmatrix}$$

[taking 2 common from C_1 and then $C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$]

$$= 2\left[\begin{vmatrix} p & x & a+p \\ q & y & b+q \\ r & z & c+r \end{vmatrix} - \begin{vmatrix} p & p & a+p \\ q & q & b+q \\ r & r & c+r \end{vmatrix}\right]$$

$$= 2\begin{vmatrix} p & x & a+p \\ q & y & b+q \\ r & z & c+r \end{vmatrix} - 0$$

[since, two columns C_1 and C_2 are identicals]

$$= 2\begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix} + 2\begin{vmatrix} p & x & p \\ q & y & q \\ r & z & r \end{vmatrix}$$

$$= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} + 0$$

[since, C_1 and C_3 are identical in second determinant and in first determinant, $C_1 \leftrightarrow C_2$ and then $C_1 \leftrightarrow C_3$]

$= 2 \times 16$ $\quad [\because \Delta = 16]$

$= 32$

Hence proved.

Q. 58 The maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ is $\frac{1}{2}$.

Sol. ***True***

Since, $\begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \cos\theta \end{vmatrix}$ $\quad [\because R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

On expanding along third row, we get the value of the determinant

$$= \cos\theta \cdot \sin\theta = \frac{1}{2}\sin 2\theta = \frac{1}{2}$$

[when θ is $45°$ which gives maximum value]

3

Matrices

Short Answer Type Questions

Q. 1 If a matrix has 28 elements, what are the possible orders it can have? What if it has 13 elements?

Sol. We know that, if a matrix is of order $m \times n$, it has mn elements, where m and n are natural numbers.

We have, $m \times n = 28$

$\Rightarrow \quad (m, n) = \{(1, 28), (2, 14), (4, 7), (7, 4), (14, 2), (28, 1)\}$

So, the possible orders are $1 \times 28, 2 \times 14, 4 \times 7, 7 \times 4, 14 \times 2, 28 \times 1$.

Also, if it has 13 elements, then $m \times n = 13$

$\Rightarrow \quad (m, n) = \{(1,13), (13,1)\}$

Hence, the possible orders are $1 \times 13, 13 \times 1$.

Q. 2 In the matrix $A = \begin{bmatrix} a & 1 & x \\ 2 & \sqrt{3} & x^2 - y \\ 0 & 5 & \frac{-2}{5} \end{bmatrix}$, write

(i) the order of the matrix A.

(ii) the number of elements.

(iii) elements a_{23}, a_{31} and a_{12}.

Sol. We have,
$$A = \begin{bmatrix} a & 1 & x \\ 2 & \sqrt{3} & x^2 - y \\ 0 & 5 & \frac{-2}{5} \end{bmatrix}$$

(i) the order of matrix $A = 3 \times 3$

(ii) the number of elements $= 3 \times 3 = 9$

[since, the number of elements in an $m \times n$ matrix will be equal to $m \times n = mn$]

(iii) $a_{23} = x^2 - y$, $a_{31} = 0$, $a_{12} = 1$

[since, we know that a_{ij}, is a representation of element lying in the ith row and jth column]

Q. 3 Construct $a_{2\times 2}$ matrix, where

(i) $a_{ij} = \dfrac{(i-2j)^2}{2}$ (ii) $a_{ij} = |-2i+3j|$

Sol. We know that, the notation, namely $A = [a_{ij}]_{m\times n}$ indicates that A is a matrix of order $m\times n$, also $1 \le i \le m, 1 \le j \le n;\ i, j \in N$.

(i) Here, $A = [a_{ij}]_{2\times 2}$

$$\Rightarrow \quad A = \frac{(i-2j)^2}{2}, 1 \le i \le 2 ; 1 \le j \le 2 \quad \text{...(i)}$$

$$\therefore \quad a_{11} = \frac{(1-2)^2}{2} = \frac{1}{2}$$

$$a_{12} = \frac{(1-2\times 2)^2}{2} = \frac{9}{2}$$

$$a_{21} = \frac{(2-2\times 1)^2}{2} = 0$$

$$a_{22} = \frac{(2-2\times 2)^2}{2} = 2$$

Thus, $$A = \begin{bmatrix} \frac{1}{2} & \frac{9}{2} \\ 0 & 2 \end{bmatrix}_{2\times 2}$$

(ii) Here, $A = [a_{ij}]_{2\times 2} = |-2i+3j|,\ 1 \le i \le 2;\ 1 \le j \le 2$

$$\therefore \quad a_{11} = |-2\times 1 + 3\times 1| = 1$$

$$a_{12} = |-2\times 1 + 3\times 2| = 4 \quad [\because |-1| = 1]$$

$$a_{21} = |-2\times 2 + 3\times 1| = 1$$

$$a_{22} = |-2\times 2 + 3\times 2| = 2$$

$$\therefore \quad A = \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix}_{2\times 2}$$

Q. 4 Construct a 3×2 matrix whose elements are given by $a_{ij} = e^{i\cdot x} = \sin jx$.

Sol. Since, $A = [a_{ij}]_{m\times n}$ $1 \le i \le m$ and $1 \le j \le n, i, j \in N$

$$\therefore \quad A = [e^{i\cdot x} \sin jx]_{3\times 2};\ 1 \le i \le 3;\ 1 \le j \le 2$$

$$\Rightarrow \quad a_{11} = e^{1\cdot x} \cdot \sin 1\cdot x = e^x \sin x$$

$$a_{12} = e^{1\cdot x} \cdot \sin 2\cdot x = e^x \sin 2x$$

$$a_{21} = e^{2\cdot x} \cdot \sin 1\cdot x = e^{2x} \sin x$$

$$a_{22} = e^{2\cdot x} \cdot \sin 2\cdot x = e^{2x} \sin 2x$$

$$a_{31} = e^{3\cdot x} \cdot \sin 1\cdot x = e^{3x} \sin x$$

$$a_{32} = e^{3\cdot x} \cdot \sin 2\cdot x = e^{3x} \sin 2x$$

$$\therefore \quad A = \begin{bmatrix} e^x \sin x & e^x \sin 2x \\ e^{2x} \sin x & e^{2x} \sin 2x \\ e^{3x} \sin x & e^{3x} \sin 2x \end{bmatrix}_{3\times 2}$$

Q. 5 Find the values of a and b, if $A = B$, where

$$A = \begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2a+2 & b^2+2 \\ 8 & b^2-5b \end{bmatrix}$$

Thinking Process

By using equality of two matrices, we know that each element of A is equal to corresponding element of B.

Sol. We have, $A = \begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix}_{2\times 2}$ and $B = \begin{bmatrix} 2a+2 & b^2+2 \\ 8 & b^2-5b \end{bmatrix}_{2\times 2}$

Also, $A = B$

By equality of matrices we know that each element of A is equal to the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j.

$\therefore \quad a_{11} = b_{11} \Rightarrow a+4 = 2a+2 \Rightarrow a = 2$

$a_{12} = b_{12} \Rightarrow 3b = b^2+2 \Rightarrow b^2 = 3b-2$

and $\quad a_{22} = b_{22} \Rightarrow -6 = b^2-5b$

$\Rightarrow \quad -6 = 3b-2-5b \quad [\because b^2 = 3b-2]$

$\Rightarrow \quad 2b = 4 \Rightarrow b = 2$

$\therefore \quad a = 2$ and $b = 2$

Q. 6 If possible, find the sum of the matrices A and B, where $A = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix}$.

Thinking Process

We know that, two matrices are added, if they have same order.

Sol. We have, $A = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 3 \end{bmatrix}_{2\times 2}$ and $B = \begin{bmatrix} x & y & z \\ a & b & 6 \end{bmatrix}_{2\times 3}$

Here, A and B are of different orders. Also, we know that the addition of two matrices A and B is possible only if order of both the matrices A and B should be same.

Hence, the sum of matrices A and B is not possible.

Q. 7 If $X = \begin{bmatrix} 3 & 1 & -1 \\ 5 & -2 & -3 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 7 & 2 & 4 \end{bmatrix}$, then find

(i) $X + Y$.

(ii) $2X - 3Y$.

(iii) a matrix Z such that $X + Y + Z$ is a zero matrix.

Sol. We have, $X = \begin{bmatrix} 3 & 1 & -1 \\ 5 & -2 & -3 \end{bmatrix}_{2\times 3}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 7 & 2 & 4 \end{bmatrix}_{2\times 3}$

(i) $X + Y = \begin{bmatrix} 3+2 & 1+1 & -1-1 \\ 5+7 & -2+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & -2 \\ 12 & 0 & 1 \end{bmatrix}$

(ii) $\because\ 2X = 2\begin{bmatrix} 3 & 1 & -1 \\ 5 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 2 & -2 \\ 10 & -4 & -6 \end{bmatrix}$

and $\quad 3Y = 3\begin{bmatrix} 2 & 1 & -1 \\ 7 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 3 & -3 \\ 21 & 6 & 12 \end{bmatrix}$

$\therefore \quad 2X - 3Y = \begin{bmatrix} 6-6 & 2-3 & -2+3 \\ 10-21 & -4-6 & -6-12 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -11 & -10 & -18 \end{bmatrix}$

(iii) $X + Y = \begin{bmatrix} 3+2 & 1+1 & -1-1 \\ 5+7 & -2+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & -2 \\ 12 & 0 & +1 \end{bmatrix}$

Also, $\quad X + Y + Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

We see that Z is the additive inverse of $(X+Y)$ or negative of $(X+Y)$.

$\therefore \quad Z = \begin{bmatrix} -5 & -2 & 2 \\ -12 & 0 & -1 \end{bmatrix} \qquad [\because Z = -(X+Y)]$

Q. 8 Find non-zero values of x satisfying the matrix equation

$$x\begin{bmatrix} 2x & 2 \\ 3 & x \end{bmatrix} + 2\begin{bmatrix} 8 & 5x \\ 4 & 4x \end{bmatrix} = 2\begin{bmatrix} (x^2+8) & 24 \\ (10) & 6x \end{bmatrix}.$$

Sol. Given that,

$$x\begin{bmatrix} 2x & 2 \\ 3 & x \end{bmatrix} + 2\begin{bmatrix} 8 & 5x \\ 4 & 4x \end{bmatrix} = 2\begin{bmatrix} (x^2+8) & 24 \\ 10 & 6x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x^2 & 2x \\ 3x & x^2 \end{bmatrix} + \begin{bmatrix} 16 & 10x \\ 8 & 8x \end{bmatrix} = \begin{bmatrix} 2x^2+16 & 48 \\ 20 & 12x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x^2+16 & 2x+10x \\ 3x+8 & x^2+8x \end{bmatrix} = \begin{bmatrix} 2x^2+16 & 48 \\ 20 & 12x \end{bmatrix}$$

$$\Rightarrow 2x + 10x = 48$$

$$\Rightarrow 12x = 48$$

$$\therefore x = \frac{48}{12} = 4$$

Q. 9 If $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then show that

$$(A+B)\ (A-B) \neq A^2 - B^2.$$

Sol. We have, $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\therefore \quad (A+B) = \begin{bmatrix} 0+0 & 1-1 \\ 1+1 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}_{2\times 2}$

and $\quad (A-B) = \begin{bmatrix} 0-0 & 1+1 \\ 1-1 & 1-0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

Since, $(A+B)\cdot(A-B)$ is defined, if the number of columns of $(A+B)$ is equal to the number of rows of $(A-B)$, so here multiplication of matrices $(A+B)\cdot(A-B)$ is possible.

Now, $\quad (A+B)_{2\times 2}\cdot(A-B)_{2\times 2} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 4+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \qquad \ldots(i)$

Also, $$A^2 = A \cdot A$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0+1 & 0+1 \\ 0+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and $B^2 = B \cdot B$ $$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0-1 & 0+0 \\ 0+0 & -1+0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\therefore$ $$A^2 - B^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{...(ii)}$$

Thus, we see that
$$(A+B) \cdot (A-B) \neq A^2 - B^2 \quad \text{[using Eqs. (i) and (ii)]}$$
$\Rightarrow$ $$\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$ **Hence proved.**

Q. 10 Find the value of x, if $[1\ x\ 1] \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0$.

Sol. We have, $$[1\ x\ 1]_{1 \times 3} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}_{3 \times 1} = 0$$

$$\Rightarrow [1+2x+15 \quad 3+5x+3 \quad 2+x+2]_{1 \times 3} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}_{3 \times 1} = 0$$

$$\Rightarrow [16+2x \quad 5x+6 \quad x+4]_{1 \times 3} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}_{3 \times 1} = 0$$

$$\Rightarrow [16+2x+(5x+6) \cdot 2+(x+4) \cdot x]_{1 \times 1} = 0$$
$$\Rightarrow [16+2x+10x+12+x^2+4x] = 0$$
$$\Rightarrow [x^2+16x+28] = 0$$
$$\Rightarrow [x^2+2x+14x+28] = 0$$
$$\Rightarrow (x+2)(x+14) = 0$$
$$\therefore x = -2, -14$$

Q. 11 Show that $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ satisfies the equation $A^2 - 3A - 7I = O$ and hence find the value of A^{-1}.

Sol. We have, $$A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$$

$\therefore$ $$A^2 = A \cdot A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix}$$

and $$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$\therefore$ $$A^2 - 3A - 7I = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 22-15-7 & 9-9-0 \\ -3+3-0 & 1+6-7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0$$ **Hence proved.**

Since, $$A^2 - 3A - 7I = 0$$

$\Rightarrow$ $$A^{-1}[(A^2) - 3A - 7I] = A^{-1}0$$

$\Rightarrow$ $$A^{-1}A \cdot A - 3A^{-1}A - 7A^{-1}I = 0 \quad [\because A^{-1}0 = 0]$$

$\Rightarrow$ $$IA - 3I - 7A^{-1} = 0 \quad [\because A^{-1}A = I]$$

$\Rightarrow$ $$A - 3I - 7A^{-1} = 0 \quad [\because A^{-1}I = A^{-1}]$$

$\Rightarrow$ $$-7A^{-1} = -A + 3I$$

$$= \begin{bmatrix} -5 & -3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{-1}{7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

Q. 12 Find the matrix A satisfying the matrix equation

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thinking Process

We know that, if two matrices A and B of order $m \times n$ and $p \times q$ respectively are multiplied, then necessity condition to multiplication of $A \cdot B$ is $n = p$. So, by taking a matrix of correct order we can get the desired elements of the required matrix.

Sol. We have, $$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}_{2\times 2} A \cdot \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

Let $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$$

$\therefore$ $$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $$\begin{bmatrix} 2a+c & 2b+d \\ 3a+2c & 3b+2d \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $$\begin{bmatrix} -6a-3c+10b+5d & 4a+2c-6b-3d \\ -9a-6c+15b+10d & 6a+4c-9b-6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $$-6a - 3c + 10b + 5d = 1 \quad \ldots(\text{i})$$

$$\Rightarrow \quad 4a + 2c - 6b - 3d = 0 \quad \ldots(ii)$$
$$\Rightarrow \quad -9a - 6c + 15b + 10d = 0 \quad \ldots(iii)$$
$$\Rightarrow \quad 6a + 4c - 9b - 6d = 1 \quad \ldots(iv)$$

On adding Eqs. (i) and (iv), we get

$$c + b - d = 2 \Rightarrow d = c + b - 2 \quad \ldots(v)$$

On adding Eqs. (ii) and (iii), we get

$$-5a - 4c + 9b + 7d = 0 \quad \ldots(vi)$$

On adding Eqs. (vi) and (iv), we get

$$a + 0 + 0 + d = 1 \Rightarrow d = 1 - a \quad \ldots(vii)$$

From Eqs. (v) and (vii),

$$c + b - 2 = 1 - a \Rightarrow a + b + c = 3 \quad \ldots(viii)$$
$$\Rightarrow \quad a = 3 - b - c$$

Now, using the values of a and d in Eq. (iii), we get

$$-9(3 - b - c) - 6c + 15b + 10(-2 + b + c) = 0$$
$$\Rightarrow \quad -27 + 9b + 9c - 6c + 15b - 20 + 10b + 10c = 0$$
$$\Rightarrow \quad 34b + 13c = 47 \quad \ldots(ix)$$

Now, using the values of a and d in Eq. (ii), we get

$$4(3 - b - c) + 2c - 6b - 3(b + c - 2) = 0$$
$$\Rightarrow \quad 12 - 4b - 4c + 2c - 6b - 3b - 3c + 6 = 0$$
$$\Rightarrow \quad -13b - 5c = -18 \quad \ldots(x)$$

On multiplying Eq. (ix) by 5 and Eq. (x) by 13, then adding, we get

$$\begin{array}{r} -169b - 65c = -234 \\ \underline{170\,b + 65c = 235} \\ b = 1 \end{array}$$

$$\Rightarrow \quad -13 \times 1 - 5c = -18 \quad \text{[from Eq. (x)]}$$
$$\Rightarrow \quad -5c = -18 + 13 = -5 \Rightarrow c = 1$$
$$\therefore \quad a = 3 - 1 - 1 = 1 \text{ and } d = 1 - 1 = 0$$
$$\therefore \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Q. 13 Find A, if $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} A = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$.

Sol. We have,
$$\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}_{3\times1} A = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}_{3\times3}$$

Let
$$A = [x\; y\; z]$$

$$\therefore \quad \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}_{3\times1} [x\; y\; z]_{1\times3} = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}_{3\times3}$$

$$\Rightarrow \quad \begin{bmatrix} 4x & 4y & 4z \\ x & y & z \\ 3x & 3y & 3z \end{bmatrix} = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$$

$$\Rightarrow \quad 4x = -4 \Rightarrow x = -1,\ 4y = 8$$
$$\Rightarrow \quad y = 2 \text{ and } 4z = 4$$
$$\Rightarrow \quad z = 1$$
$$\therefore \quad A = [-1\ 2\ 1]$$

Q. 14 If $A\begin{bmatrix} 3 & -4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, then verify $(BA)^2 \neq B^2 A^2$.

Sol. We have, $A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}_{3\times2}$ and $B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2\times3}$

$$\therefore \quad BA = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2\times3} \begin{bmatrix} 3 & -4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}_{3\times2}$$

$$= \begin{bmatrix} 6+1+4 & -8+1+0 \\ 3+2+8 & -4+2+0 \end{bmatrix} = \begin{bmatrix} 11 & -7 \\ 13 & -2 \end{bmatrix}$$

and $$(BA)\cdot(BA) = \begin{bmatrix} 11 & -7 \\ 13 & -2 \end{bmatrix} \begin{bmatrix} 11 & -7 \\ 13 & -2 \end{bmatrix}$$

$$\Rightarrow \quad (BA)^2 = \begin{bmatrix} 121-91 & -77+14 \\ 143-26 & -91+4 \end{bmatrix} = \begin{bmatrix} 30 & -63 \\ 117 & -87 \end{bmatrix} \quad \text{...(i)}$$

Also, $$B^2 = B\cdot B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2\times3} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2\times3}$$

So, B^2 is not possible, since the B is not a square matrix.
Hence, $(BA)^2 \neq B^2 A^2$.

Q. 15 If possible, find the value of BA and AB, where

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

Sol. We have, $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}_{3\times2}$

So, AB and BA both are possible.
[since, in both $A\cdot B$ and $B\cdot A$, the number of columns of first is equal to the number of rows of second.]

$$\therefore \quad AB = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2\times3} \cdot \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}_{3\times2}$$

$$= \begin{bmatrix} 8+2+2 & 2+3+4 \\ 4+4+4 & 1+6+8 \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ 12 & 15 \end{bmatrix}$$

and $$BA = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}_{3 \times 2} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2 \times 3}$$

$$= \begin{bmatrix} 4 \times 2 + 1 & 4 + 2 & 8 + 4 \\ 4 + 3 & 2 + 6 & 4 + 12 \\ 2 + 2 & 1 + 4 & 2 + 8 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 12 \\ 7 & 8 & 16 \\ 4 & 5 & 10 \end{bmatrix}$$

Q. 16 Show by an example that for $A \neq 0$, $B \neq 0$ and $AB = 0$.

Sol. Let $$A = \begin{bmatrix} 0 & -4 \\ 0 & 2 \end{bmatrix} \neq 0 \text{ and } B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} \neq 0$$

$\therefore$ $$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$ **Hence proved.**

Q. 17 Given, $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 9 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 1 & 3 \end{bmatrix}$. is $(AB)' = B'\ A'$?

Sol. We have, $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 9 & 6 \end{bmatrix}_{2 \times 3}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 1 & 3 \end{bmatrix}_{3 \times 2}$

$\therefore$ $$AB = \begin{bmatrix} 2 + 8 + 0 & 8 + 32 + 0 \\ 3 + 18 + 6 & 12 + 72 + 18 \end{bmatrix} = \begin{bmatrix} 10 & 40 \\ 27 & 102 \end{bmatrix}$$

and $$(AB)' = \begin{bmatrix} 10 & 27 \\ 40 & 102 \end{bmatrix} \quad \text{...(i)}$$

Also, $$B' = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 3 \end{bmatrix}_{2 \times 3} \text{ and } A' = \begin{bmatrix} 2 & 3 \\ 4 & 9 \\ 0 & 6 \end{bmatrix}_{3 \times 2}$$

$\therefore$ $$B'A' = \begin{bmatrix} 2 + 8 + 0 & 3 + 18 + 6 \\ 8 + 32 + 0 & 12 + 72 + 18 \end{bmatrix} = \begin{bmatrix} 10 & 27 \\ 40 & 102 \end{bmatrix} \quad \text{...(ii)}$$

Thus, we see that, $(AB)' = B'A'$ [using Eqs. (i) and (ii)]

Q. 18 Solve for x and y, $x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} -8 \\ -11 \end{bmatrix} = 0.$

Sol. We have, $$x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} -8 \\ -11 \end{bmatrix} = 0$$

$\Rightarrow$ $$\begin{bmatrix} 2x \\ x \end{bmatrix} + \begin{bmatrix} 3 \cdot y \\ 5 \cdot y \end{bmatrix} + \begin{bmatrix} -8 \\ -11 \end{bmatrix} = 0$$

$\Rightarrow$ $$\begin{bmatrix} 2x & 3y & -8 \\ x & 5y & -11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ $$2x + 3y - 8 = 0$$

$\Rightarrow$ $$4x + 6y = 16 \quad \text{...(i)}$$

and $$x + 5y - 11 = 0$$

$\Rightarrow$ $$4x + 20y = 44 \quad \text{...(ii)}$$

On subtracting Eq. (i) from Eq. (ii), we get

$$14y = 28 \Rightarrow y = 2$$

$\therefore$ $$2x + 3 \times 2 - 8 = 0$$

$\Rightarrow$ $$2x = 2 \Rightarrow x = 1$$

$\therefore$ $$x = 1 \text{ and } y = 2$$

Q. 19 If X and Y are 2×2 matrices, then solve the following matrix equations for X and Y

$$2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}, 3X + 2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}.$$

Sol. We have,

$$2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{...(i)}$$

and $$3X + 2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix} \quad \text{...(ii)}$$

On subtracting Eq. (i) from Eq. (ii), we get

$\therefore$ $$(3X + 2Y) - (2X + 3Y) = \begin{bmatrix} -2-2 & 2-3 \\ 1-4 & -5-0 \end{bmatrix}$$

$$(X - Y) = \begin{bmatrix} -4 & -1 \\ -3 & -5 \end{bmatrix} \quad \text{...(iii)}$$

On adding Eqs. (i) and (ii), we get

$$(5X + 5Y) = \begin{bmatrix} 0 & 5 \\ 5 & -5 \end{bmatrix}$$

$\Rightarrow$ $$(X + Y) = \frac{1}{5}\begin{bmatrix} 0 & 5 \\ 5 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{...(iv)}$$

On adding Eqs. (iii) and (iv), we get

$$(X - Y) + (X + Y) = \begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$$

$\Rightarrow$ $$2X = 2\begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}$$

$\therefore$ $$X = \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}$$

From Eq. (iv),

$$\begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix} + Y = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$\therefore$ $$Y = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \text{ and } X = \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}$$

Q. 20 **If $A = [3\ \ 5]$ and $B = [7\ \ 3]$, then find a non-zero matrix C such that $AC = BC$.**

Sol. We have, $A = [3\ \ 5]_{1\times 2}$ and $B = [7\ \ 3]_{1\times 2}$

Let $C = \begin{bmatrix} x \\ y \end{bmatrix}_{2\times 1}$ is a non-zero matrix of order 2×1.

$$\therefore \qquad AC = [3\ \ 5]\begin{bmatrix} x \\ y \end{bmatrix} = [3x + 5y]$$

$$\text{and} \qquad BC = [7\ \ 3]\begin{bmatrix} x \\ y \end{bmatrix} = [7x + 3y]$$

For $AC = BC$,

$$[3x + 5y] = [7x + 3y]$$

On using equality of matrix, we get

$$3x + 5y = 7x + 3y$$

$$\Rightarrow \qquad 4x = 2y$$

$$\Rightarrow \qquad x = \frac{1}{2}y$$

$$\Rightarrow \qquad y = 2x$$

$$\therefore \qquad C = \begin{bmatrix} x \\ 2x \end{bmatrix}$$

We see that on taking C of order $2 \times 1, 2 \times 2, 2 \times 3, \ldots$, we get

$$C = \begin{bmatrix} x \\ 2x \end{bmatrix}, \begin{bmatrix} x & x \\ 2x & 2x \end{bmatrix}, \begin{bmatrix} x & x & x \\ 2x & 2x & 2x \end{bmatrix} \ldots$$

In general,

$$C = \begin{bmatrix} k \\ 2k \end{bmatrix}, \begin{bmatrix} k & k \\ 2k & 2k \end{bmatrix} \text{ etc...}$$

where, k is any real number.

Q. 21 **Give an example of matrices A, B and C, such that $AB = AC$, where A is non-zero matrix but $B \neq C$.**

Sol. Let $$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix} \qquad [\because B \neq C]$$

$$\therefore \qquad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \qquad \ldots(\text{i})$$

$$\text{and} \qquad AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \qquad \ldots(\text{ii})$$

Thus, we see that $AB = AC$ [using Eqs. (i) and (ii)]

where, A is non-zero matrix but $B \neq C$.

Q. 22 If $A=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}$, $B=\begin{bmatrix}2 & 3\\ 3 & -4\end{bmatrix}$ and $C=\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}$, verify

(i) $(AB)\,C = A(BC)$.

(ii) $A\,(B+C) = AB + AC$.

Sol. We have, $A=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}$, $B=\begin{bmatrix}2 & 3\\ 3 & -4\end{bmatrix}$ and $C=\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}$

(i) $(AB)=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\ 3 & -4\end{bmatrix}=\begin{bmatrix}2+6 & 3-8\\ -4+3 & -6-4\end{bmatrix}=\begin{bmatrix}8 & -5\\ -1 & -10\end{bmatrix}$

and $(AB)C=\begin{bmatrix}8 & -5\\ -1 & -10\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}$

$$=\begin{bmatrix}8+5 & 0\\ -1+10 & 0\end{bmatrix}=\begin{bmatrix}13 & 0\\ 9 & 0\end{bmatrix} \quad \text{...(i)}$$

Again, $(BC)=\begin{bmatrix}2 & 3\\ 3 & -4\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}$

$$=\begin{bmatrix}2-3 & 0\\ 3+4 & 0\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 7 & 0\end{bmatrix}$$

and $A\,(BC)=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 7 & 0\end{bmatrix}$

$$=\begin{bmatrix}-1+14 & 0\\ +2+7 & 0\end{bmatrix}=\begin{bmatrix}13 & 0\\ 9 & 0\end{bmatrix} \quad \text{...(ii)}$$

$\therefore$ $(AB)\,C = A(BC)$ [using Eqs. (i) and (ii)]

(ii) $(B+C)=\begin{bmatrix}2 & 3\\ 3 & -4\end{bmatrix}+\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}=\begin{bmatrix}3 & 3\\ 2 & -4\end{bmatrix}$

and $A\cdot(B+C)=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}3 & 3\\ 2 & -4\end{bmatrix}$

$$=\begin{bmatrix}3+4 & 3-8\\ -6+2 & -6-4\end{bmatrix}=\begin{bmatrix}7 & -5\\ -4 & -10\end{bmatrix} \quad \text{...(iii)}$$

Also, $AB=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}\cdot\begin{bmatrix}2 & 3\\ 3 & -4\end{bmatrix}$

$$=\begin{bmatrix}2+6 & 3-8\\ -4+3 & -6-4\end{bmatrix}=\begin{bmatrix}8 & -5\\ -1 & -10\end{bmatrix}$$

and $AC=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}=\begin{bmatrix}1-2 & 0\\ -2-1 & 0\end{bmatrix}=\begin{bmatrix}-1 & 0\\ -3 & 0\end{bmatrix}$

$\therefore$ $AB+AC=\begin{bmatrix}8 & -5\\ -1 & -10\end{bmatrix}+\begin{bmatrix}-1 & 0\\ -3 & 0\end{bmatrix}$

$\Rightarrow$ $AB+AC=\begin{bmatrix}7 & -5\\ -4 & -10\end{bmatrix}$ ...(iv)

From Eqs. (iii) and (iv),

$$A\,(B+C)=AB+AC$$

Q. 23 If $P=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ and $Q=\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then prove that

$$PQ=\begin{bmatrix} xa & 0 & 0 \\ 0 & yb & 0 \\ 0 & 0 & zc \end{bmatrix}=QP.$$

Sol. $PQ=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}=\begin{bmatrix} xa & 0 & 0 \\ 0 & yb & 0 \\ 0 & 0 & zc \end{bmatrix}$ ...(i)

and $QP=\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}=\begin{bmatrix} ax & 0 & 0 \\ 0 & by & 0 \\ 0 & 0 & zc \end{bmatrix}$ ...(ii)

Thus, we see that, $PQ=QP$ [using Eqs. (i) and (ii)]

Hence proved.

Q. 24 If $[2 \ 1 \ 3]\begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}=A$, then find the value of A.

Sol. We have, $[2 \ 1 \ 3]\begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}=A$

$\therefore \quad [2 \ 1 \ 3]\begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}=[-2-1+0 \quad 0+1+3 \quad -2+0+3]$

$=[-3 \quad 4 \quad 1]$

Now, $[-3 \ 4 \ 1]\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}=A$

$\therefore \quad A=[-3 \ 4 \ 1]\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$=[-3+0-1]=[-4]$

Q. 25 If $A=[2 \ 1]$, $B=\begin{bmatrix} 5 & 3 & 4 \\ 8 & 7 & 6 \end{bmatrix}$ and $C=\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, then verify that $A(B+C)=(AB+AC)$.

Sol. We have to verify that, $A(B+C)=AB+AC$

We have, $A=[2 \ 1]$, $B=\begin{bmatrix} 5 & 3 & 4 \\ 8 & 7 & 6 \end{bmatrix}$ and $C=\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

$$\therefore \quad A(B+C) = [2 \quad 1]\begin{bmatrix} 5-1 & 3+2 & 4+1 \\ 8+1 & 7+0 & 6+2 \end{bmatrix}$$

$$= [2 \quad 1]\begin{bmatrix} 4 & 5 & 5 \\ 9 & 7 & 8 \end{bmatrix}$$

$$= [8+9 \quad 10+7 \quad 10+8]$$

$$= [17 \quad 17 \quad 18] \quad \text{...(i)}$$

Also, $$AB = [2 \quad 1]\begin{bmatrix} 5 & 3 & 4 \\ 8 & 7 & 6 \end{bmatrix}$$

$$= [10+8 \quad 6+7 \quad 8+6] = [18 \quad 13 \quad 14]$$

and $$AC = [2 \quad 1]\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= [-2+1 \quad 4+0 \quad 2+2] = [-1 \quad 4 \quad 4]$$

$$\therefore \quad AB + AC = [18 \quad 13 \quad 14] + [-1 \quad 4 \quad 4]$$

$$= [17 \quad 17 \quad 18] \quad \text{...(ii)}$$

$$\therefore \quad A(B+C) = (AB + AC)$$ [using Eqs. (i) and (ii)]

Hence proved.

Q. 26 If $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$, then verify that $A^2 + A = (A + I)$, where I is 3×3 unit matrix.

Sol. We have, $$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 4 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix}$$

$$\therefore \quad A^2 + A = \begin{bmatrix} 1 & -1 & -2 \\ 4 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & -3 \\ 6 & 5 & 7 \\ 2 & 3 & 5 \end{bmatrix} \quad \text{...(i)}$$

Now, $$A + I = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

and $$A(A + I) = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -3 \\ 6 & 5 & 7 \\ 2 & 3 & 5 \end{bmatrix} \quad \text{...(ii)}$$

Thus, we see that $A^2 + A = A(A + I)$ [using Eqs. (i) and (ii)]

Q. 27 If $A = \begin{bmatrix} 0 & -1 & 2 \\ 4 & 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 1 & 3 \\ 2 & 6 \end{bmatrix}$, then verify that

(i) $(A')' = A$
(ii) $(AB)' = B'A'$
(iii) $(kA)' = (kA')$.

Sol. We have, $A = \begin{bmatrix} 0 & -1 & 2 \\ 4 & 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 1 & 3 \\ 2 & 6 \end{bmatrix}$

(i) We have to verify that, $A' = A$

$$\therefore \quad A' = \begin{bmatrix} 0 & 4 \\ -1 & 3 \\ 2 & -4 \end{bmatrix}$$

and $$A' = \begin{bmatrix} 0 & -1 & 2 \\ 4 & 3 & -4 \end{bmatrix} = A$$ **Hence proved.**

(ii) We have to verify that, $AB' = B'A'$

$$\therefore \quad AB = \begin{bmatrix} 3 & 9 \\ 11 & -15 \end{bmatrix}$$

$$\Rightarrow \quad (AB)' = \begin{bmatrix} 3 & 11 \\ 9 & -15 \end{bmatrix}$$

and $$B'A' = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -1 & 3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 11 \\ 9 & -15 \end{bmatrix}$$

$$= (AB)'$$ **Hence proved.**

(iii) We have to verify that, $(kA)' = (kA')$

Now, $$(kA) = \begin{bmatrix} 0 & -k & 2k \\ 4k & 3k & -4k \end{bmatrix}$$

and $$(kA)' = \begin{bmatrix} 0 & 4k \\ -k & 3k \\ 2k & -4k \end{bmatrix}$$

Also, $$kA' = \begin{bmatrix} 0 & 4k \\ -k & 3k \\ 2k & -4k \end{bmatrix}$$

$$= (kA)'$$ **Hence proved.**

Q. 28 If $A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{bmatrix}$, then verify that

(i) $(2A + B)' = 2A\mathrm{A} + B'$.

(ii) $(A - B)' = A' - B'$.

Sol. We have,
$$A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{bmatrix}$$

(i) $\therefore$
$$(2A + B) = \begin{bmatrix} 2 & 4 \\ 8 & 2 \\ 10 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 14 & 6 \\ 17 & 15 \end{bmatrix}$$

and
$$(2A + B)' = \begin{bmatrix} 3 & 14 & 17 \\ 6 & 6 & 15 \end{bmatrix}$$

Also,
$$2A' + B' = 2\begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 7 \\ 2 & 4 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 14 & 17 \\ 6 & 6 & 15 \end{bmatrix} = (2A + B)'$$

Hence proved.

(ii)
$$(A - B) = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & -3 \\ -2 & 3 \end{bmatrix}$$

and
$$(A - B)' = \begin{bmatrix} 0 & -2 & -2 \\ 0 & -3 & 3 \end{bmatrix}$$

Also,
$$A' - B' = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 6 & 7 \\ 2 & 4 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -2 & -2 \\ 0 & -3 & 3 \end{bmatrix}$$
$$= (A - B)'$$

Hence proved.

Q. 29 Show that $A'A$ and AA' are both symmetric matrices for any matrix A.

Thinking Process

We know that, for a matrix A to be symmetric matrix, $A' = A$. Also by using the result $(AB)' = BA'$, we can prove that $A'A$ and AA' are both symmetric matrices for any matrix A.

Sol. Let $P = A'A$

$\therefore$ $P' = (AA')'$

$= A'(A')'$ $\quad [\because (AB')' = B'A']$

$= A'A = P$

So, $A'A$ is symmetric matrix for any matrix A.

Similarly, let $Q = AA'$

$\therefore$ $Q' = (AA')' = (A')'(A)'$

$= A(A')' = Q$

So, AA' is symmetric matrix for any matrix A.

Q. 30 Let A and B be square matrices of the order 3×3. Is $(AB)^2 = A^2B^2$? Give reasons.

Sol. Since, A and B are square matrices of order 3×3.

$$\therefore \quad AB^2 = AB \cdot AB$$
$$= ABAB$$
$$= AABB \qquad [\because AB = BA]$$
$$= A^2B^2$$

So, $AB^2 = A^2B^2$ is true when $AB = BA$.

Q. 31 Show that, if A and B are square matrices such that $AB = BA$, then $(A+B)^2 = A^2 + 2AB + B^2$.

Sol. Since, A and B are square matrices such that $AB = BA$.

$$\therefore \quad (A+B)^2 = (A+B) \cdot (A+B)$$
$$= A^2 + AB + BA + B^2$$
$$= A^2 + AB + AB + B^2 \qquad [\because AB = BA]$$
$$= A^2 + 2AB + B^2$$

Hence proved.

Q. 32 If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}, a = 4,$ and $b = -2,$ then show that

(i) $A + (B + C) = (A + B) + C$

(ii) $A\,(BC) = (AB)\,C$

(iii) $(a + b)B = aB + bB$

(iv) $a\,(C - A) = aC - aA$

(v) $(A^T)^T = A$

(vi) $(bA)^T = b\,A^T$

(vii) $(AB)^T = B^T A^T$

(viii) $(A - B)C = AC - BC$

(ix) $(A - B)^T = A^T - B^T$

Sol. We have,

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} \text{ and } a = 4,\ b = -2$$

(i) $A + (B + C) = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix}$

and $\quad (A + B) + C = \begin{bmatrix} 5 & 2 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix} = A + (B + C)$$

Hence proved.

(ii) $(BC) = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 7 & -10 \end{bmatrix}$

and $A\ (BC) = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 7 & -10 \end{bmatrix}$

$= \begin{bmatrix} 8+14 & 0-20 \\ -8+21 & 0-30 \end{bmatrix} = \begin{bmatrix} 22 & -20 \\ 13 & -30 \end{bmatrix}$

Also, $(AB) = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ -1 & 15 \end{bmatrix}$

$(AB)\ C = \begin{bmatrix} 6 & 10 \\ -1 & 15 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 22 & -20 \\ 13 & -30 \end{bmatrix} = A(BC)$ **Hence proved.**

(iii) $(a+b)\ B = (4-2) \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}$ $[\because a = 4, b = -2]$

$= \begin{bmatrix} 8 & 0 \\ 2 & 10 \end{bmatrix}$

and $aB + bB = 4B - 2B$

$= \begin{bmatrix} 16 & 0 \\ 4 & 20 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 2 & 10 \end{bmatrix}$

$= \begin{bmatrix} 8 & 0 \\ 2 & 10 \end{bmatrix}$

$= (a+b)B$ **Hence proved.**

(iv) $(C - A) = \begin{bmatrix} 2-1 & 0-2 \\ 1+1 & -2-3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -5 \end{bmatrix}$

and $a(C - A) = \begin{bmatrix} 4 & -8 \\ 8 & -20 \end{bmatrix}$ $[\because a = 4]$

Also, $aC - aA = \begin{bmatrix} 8 & 0 \\ 4 & -8 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ -4 & 12 \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ 8 & -20 \end{bmatrix}$

$= a(C - A)$ **Hence proved.**

(v) $A^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Now, $(A^T)^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}^T$

$= A$ **Hence proved.**

(vi) $(bA)^T = \begin{bmatrix} -2 & -4 \\ 2 & -6 \end{bmatrix}^T$ $[\because b = -2]$

$= \begin{bmatrix} -2 & 2 \\ -4 & -6 \end{bmatrix}$

and $A^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$\therefore$ $bA^T = \begin{bmatrix} -2 & 2 \\ -4 & -6 \end{bmatrix} = (bA)^T$ **Hence proved.**

(vii) $AB = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 4+2 & 0+10 \\ -4+3 & 0+15 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ -1 & 15 \end{bmatrix}$

$\therefore \quad (AB)^T = \begin{bmatrix} 6 & -1 \\ 10 & 15 \end{bmatrix}$

Now, $\quad B^T A^T = \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 10 & 15 \end{bmatrix}$

$= (AB)^T$ **Hence proved.**

(viii) $(A-B) = \begin{bmatrix} 1-4 & 2-0 \\ -1-1 & 3-5 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & -2 \end{bmatrix}$

$(A-B)\ C = \begin{bmatrix} -3 & 2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -6 & 4 \end{bmatrix}$ ...(i)

Now, $\quad AC = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 1 & -6 \end{bmatrix}$ ...(ii)

and $\quad BC = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 7 & -10 \end{bmatrix}$ ...(iii)

$\therefore \quad AC - BC = \begin{bmatrix} 4-8 & -4-0 \\ 1-7 & -6+10 \end{bmatrix}$ [using Eqs. (ii) and (iii)]

$= \begin{bmatrix} -4 & -4 \\ -6 & 4 \end{bmatrix}$

$= (A-B)C$ [using Eq. (i)] **Hence proved.**

(ix) $(A-B)^T = \begin{bmatrix} 1-4 & 2-0 \\ -1-1 & 3-5 \end{bmatrix}^T$

$= \begin{bmatrix} -3 & 2 \\ -2 & -2 \end{bmatrix}^T = \begin{bmatrix} -3 & -2 \\ 2 & -2 \end{bmatrix}$

$A^T - B^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} -3 & -2 \\ 2 & -2 \end{bmatrix} = (A-B)^T$ **Hence proved.**

Q. 33 If $A = \begin{bmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{bmatrix}$, then show that $A^2 = \begin{bmatrix} \cos 2q & \sin 2q \\ -\sin 2q & \cos 2q \end{bmatrix}$.

Sol. We have, $A = \begin{bmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{bmatrix}$

$\therefore \quad A^2 = A \cdot A = \begin{bmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{bmatrix} \cdot \begin{bmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{bmatrix}$

$= \begin{bmatrix} \cos^2 q - \sin^2 q & \cos q \cdot \sin q + \sin q \cos q \\ -\sin q \cos q - \cos q \sin q & -\sin^2 q + \cos^2 q \end{bmatrix}$

$= \begin{bmatrix} \cos 2q & 2 \sin q \cos q \\ -2 \sin q \cos q & \cos 2q \end{bmatrix}$ $[\because \cos^2 \theta - \sin^2 \theta = \cos 2\theta]$

$= \begin{bmatrix} \cos 2q & \sin 2q \\ -\sin 2q & \cos 2q \end{bmatrix}$ $[\because \sin 2\theta = 2 \sin \theta \cdot \cos \theta]$ **Hence proved.**

Q. 34 If $A=\begin{bmatrix}0 & -x\\ x & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $x^2=-1$, then show that $(A+B)^2=A^2+B^2$.

Sol. We have, $A=\begin{bmatrix}0 & -x\\ x & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $x^2=-1$

$\therefore$ $(A+B)=\begin{bmatrix}0 & -x+1\\ x+1 & 0\end{bmatrix}$

and $(A+B)^2=\begin{bmatrix}0 & -x+1\\ x+1 & 0\end{bmatrix}\begin{bmatrix}0 & -x+1\\ x+1 & 0\end{bmatrix}$

$=\begin{bmatrix}1-x^2 & 0\\ 0 & 1-x^2\end{bmatrix}$ ...(i)

Also, $A^2=A\cdot A=\begin{bmatrix}0 & -x\\ x & 0\end{bmatrix}\begin{bmatrix}0 & -x\\ x & 0\end{bmatrix}=\begin{bmatrix}-x^2 & 0\\ 0 & -x^2\end{bmatrix}$

and $B^2=B\cdot B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Now, $A^2+B^2=\begin{bmatrix}-x^2+1 & 0\\ 0 & -x^2+1\end{bmatrix}=\begin{bmatrix}1-x^2 & 0\\ 0 & 1-x^2\end{bmatrix}$ [using Eq. (i)]

$=(A+B)^2$ **Hence proved.**

Q. 35 Verify that $A^2=I$, when $A=\begin{bmatrix}0 & 1 & -1\\ 4 & -3 & 4\\ 3 & -3 & 4\end{bmatrix}$.

Sol. We have, $A=\begin{bmatrix}0 & 1 & -1\\ 4 & -3 & 4\\ 3 & -3 & 4\end{bmatrix}$

$\therefore$ $A^2=\begin{bmatrix}0 & 1 & -1\\ 4 & -3 & 4\\ 3 & -3 & 4\end{bmatrix}\cdot\begin{bmatrix}0 & 1 & -1\\ 4 & -3 & 4\\ 3 & -3 & 4\end{bmatrix}$ $[\because A^2=A\cdot A]$

$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$ **Hence proved.**

Q. 36 Prove by mathematical induction that $(A')^n=(A^n)'$ where $n\in N$ for any square matrix A.

Sol. Let $P(n):\ (A')^n=(A^n)'$

$\therefore$ $P(1):\ (A')^1=(A)'$

$\Rightarrow$ $A'=A'\Rightarrow P(1)$ is true.

Now, $P(k):(A')^k=(A^k)'$,

where $k\in N$

and $P(k+1):(A')^{k+1}=(A^{k+1})'$

where $P(k+1)$ is true whenever $P(k)$ is true.

$\therefore$
$$P(k+1):(A')^k.(A')^1=[A^{k+1}]'$$
$$(A^k)'.(A)'=[A^{k+1}]'$$
$$(A\cdot A^k)'=[A^{k+1}]' \qquad [\because (A')^k=(A^k)' \text{ and } (AB)=B'A']$$
$$(A^{k+1})'=[A^{k+1}]'$$

Hence proved.

Q. 37 Find inverse, by elementary row operations (if possible), of the following matrices.

(i) $\begin{bmatrix}1 & 3\\ -5 & 7\end{bmatrix}$ (ii) $\begin{bmatrix}1 & -3\\ -2 & 6\end{bmatrix}$

Thinking Process

To find the inverse of a matrix A, we know that $A=IA$ is used for elementary row operations. So, with the help of this method we can get the desired result.

Sol. **(i)** Let $A=\begin{bmatrix}1 & 3\\ -5 & 7\end{bmatrix}$

In order to use elementary row operations we may write $A=IA$.

$\therefore$
$$\begin{bmatrix}1 & 3\\ -5 & 7\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}A$$

$\Rightarrow$
$$\begin{bmatrix}1 & 3\\ 0 & 22\end{bmatrix}=\begin{bmatrix}1 & 0\\ 5 & 1\end{bmatrix}A \qquad [\because R_2\to R_2+5R_1]$$

$\Rightarrow$
$$\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 5/22 & 1/22\end{bmatrix}A \qquad \left[\because R_2\to\frac{1}{22}R_2\right]$$

$\Rightarrow$
$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}7/22 & -3/22\\ 5/22 & 1/22\end{bmatrix}A \qquad [\because R_1\to R_1-3R_2]$$

$\Rightarrow$
$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\frac{1}{22}\begin{bmatrix}7 & -3\\ 5 & 1\end{bmatrix}A$$

$\Rightarrow I=BA$, where B is the inverse of A.

$\therefore$
$$B=\frac{1}{22}\begin{bmatrix}7 & -3\\ 5 & -1\end{bmatrix}$$

(ii) Let $A=\begin{bmatrix}1 & -3\\ -2 & 6\end{bmatrix}$

In order to use elementary row operations, we write $A=IA$

$\Rightarrow$
$$\begin{bmatrix}1 & -3\\ -2 & 6\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}A$$

$\Rightarrow$
$$\begin{bmatrix}1 & -3\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}A \qquad [\because R_2\to R_2+2R_1]$$

Since, we obtain all zeroes in a row of the matrix A on LHS, so A^{-1} does not exist.

Q. 38 If $\begin{bmatrix} xy & 4 \\ z+6 & x+y \end{bmatrix} = \begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}$, then find the values of x, y, z and w.

Sol. We have, $$\begin{bmatrix} xy & 4 \\ z+6 & x+y \end{bmatrix} = \begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}$$

By equality of matrix, $x + y = 6$ and $xy = 8$

$\Rightarrow$ $x = 6 - y$ and $(6 - y) \cdot y = 8$

$\Rightarrow$ $y^2 - 6y + 8 = 0$

$\Rightarrow$ $y^2 - 4y - 2y + 8 = 0$

$\Rightarrow$ $(y - 2)(y - 4) = 0$

$\Rightarrow$ $y = 2$ or $y = 4$

$\therefore$ $x = 6 - 2 = 4$

or $x = 6 - 4 = 2$ $\quad [\because x = 6 - y]$

Also, $z + 6 = 0$

$\Rightarrow$ $z = -6$ and $w = 4$

$\therefore$ $x = 2, y = 4$ or $x = 4, y = 2, z = -6$ and $w = 4$

Q. 39 If $A = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 1 \\ 7 & 8 \end{bmatrix}$, then find a matrix C such that $3A + 5B + 2C$ is a null matrix.

Sol. We have, $A = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 1 \\ 7 & 8 \end{bmatrix}$

Let $C = \begin{bmatrix} a & b \\ C & d \end{bmatrix}$

$\therefore$ $3A + 5B + 2C = 0$

$\Rightarrow$ $\begin{bmatrix} 3 & 15 \\ 21 & 36 \end{bmatrix} + \begin{bmatrix} 45 & 5 \\ 35 & 40 \end{bmatrix} + \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 48 + 2a & 20 + 2b \\ 56 + 2c & 76 + 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $2a + 48 = 0 \Rightarrow a = -24$

Also, $20 + 2b = 0 \Rightarrow b = -10$

$56 + 2c = 0 \Rightarrow c = -28$

and $76 + 2d = 0 \Rightarrow d = -38$

$\therefore$ $C = \begin{bmatrix} -24 & -10 \\ -28 & -38 \end{bmatrix}$

Q. 40 If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, then find $A^2 - 5A - 14I$. Hence, obtain A^3.

Sol. We have, $$A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} \quad \ldots(i)$$

$\therefore$ $$A^2 = A \cdot A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} \quad \ldots(ii)$$

$\therefore \quad A^2 - 5A - 14I = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - \begin{bmatrix} 15 & -25 \\ -20 & 10 \end{bmatrix} - \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Now, $\quad A^2 - 5A - 14\,I = 0$

$\Rightarrow \quad A \cdot A^2 - 5\,A \cdot A - 14\,A\,I = 0$

$\Rightarrow \quad A^3 - 5A^2 - 14A = 0 \qquad [\because A\,I = A]$

$\Rightarrow \quad A^3 = 5A^2 = 14A$

$= 5\begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} + 14\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$ [using Eqs. (i) and (ii)]

$= \begin{bmatrix} 145 & -125 \\ -100 & 120 \end{bmatrix} + \begin{bmatrix} 42 & -70 \\ -56 & 28 \end{bmatrix}$

$= \begin{bmatrix} 187 & -195 \\ -156 & 148 \end{bmatrix}$

Q. 41 Find the values of a, b, c and d, if

$$3\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 6 \\ -1 & 2\ d \end{bmatrix} + \begin{bmatrix} 4 & a+b \\ c+d & 3 \end{bmatrix} z.$$

Sol. We have,

$$3\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 6 \\ -1 & 2d \end{bmatrix} + \begin{bmatrix} 4 & a+b \\ c+d & 3 \end{bmatrix}$$

$\Rightarrow \quad \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix} = \begin{bmatrix} a+4 & 6+a+b \\ c+d-1 & 3+2\,d \end{bmatrix}$

$\Rightarrow \quad 3a = a + 4 \Rightarrow a = 2;$

$3b = 6 + a + b$

$\Rightarrow \quad 3b - b = 8 \Rightarrow b = 4;$

$3d = 3 + 2d \Rightarrow d = 3$

and $\Rightarrow \quad 3c = c + d - 1$

$\Rightarrow \quad 2c = 3 - 1\ c = 1$

$\therefore \quad a = 2,\ b = 4,\ c = 1 \text{ and } d = 3$

Q. 42 Find the matrix A such that $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$.

Sol. We have, $\quad \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}_{3\times 2} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}_{3\times 3}$

From the given equation, it is clear that order of A should be 2×3.

Let $\quad A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2a-d & 2b-e & 2c-f \\ a+0d & b+0\cdot e & c+0\cdot f \\ -3a+4d & -3b+4e & -3c+4f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2a-d & 2b-e & 2c-f \\ a & b & c \\ -3a+4d & -3b+4e & -3c+4f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

By equality of matrices, we get

$$a = 1, b = -2, c = -5$$

and $\quad 2a - d = -1 \Rightarrow d = 2a + 1 = 3;$

$\Rightarrow \quad 2b - e = -8 \Rightarrow e = 2(-2) + 8 = 4$

$$2c - f = -10 \Rightarrow f = 2c + 10 = 0$$

$$\therefore \quad A = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$$

Q. 43 If $A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$, then find $A^2 + 2A + 7I$.

Sol. We have, $A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$

$$\therefore \quad A^2 = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \qquad [\because A^2 = A \cdot A]$$

$$= \begin{bmatrix} 1+8 & 2+2 \\ 4+4 & 8+1 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 8 & 9 \end{bmatrix}$$

$$\therefore \quad A^2 + 2A + 7I = \begin{bmatrix} 9 & 4 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 16 & 18 \end{bmatrix}$$

Q. 44 If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ and $A^{-1} = A'$, then find the value of α.

Sol. We have,

$$A = \begin{bmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{bmatrix} \text{ and } A' = \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix}$$

Also, $\quad A^{-1} = A'$

$\Rightarrow \quad AA^{-1} = AA'$

$$\Rightarrow \quad I = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 \\ 0 & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

By using equality of matrices, we get

$$\cos^2\alpha + \sin^2\alpha = 1$$

which is true for all real values of α.

Q. 45 If matrix $\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is a skew-symmetric matrix, then find the values of a, b and c.

Thinking Process

We know that, a matrix A is skew-symmetric matrix, if A′ = – A, so by using this we can get the values of a, b and c.

Sol. Let $A = \begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$

Since, A is skew-symmetric matrix.

$$\therefore \quad A' = -A$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 2 & c \\ a & b & 1 \\ 3 & -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 2 & c \\ a & b & 1 \\ 3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & -3 \\ -2 & -b & +1 \\ -c & -1 & 0 \end{bmatrix}$$

By equality of matrices, we get

$$a = -2,\ c = -3 \text{ and } b = -b \Rightarrow b = 0$$

$$\therefore \quad a = -2,\ b = 0 \text{ and } c = -3$$

Q. 46 If $P(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$, then show that $P(x) \cdot P(y) = P(x + y) = P(y) \cdot P(x)$.

Sol. We have,

$$P(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

$$\therefore \quad P(y) = \begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix}$$

Now, $$P(x) \cdot P(y) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix}$$

$$= \begin{bmatrix} \cos x \cdot \cos y - \sin x \cdot \sin y & \cos x \cdot \sin y + \sin x \cdot \cos y \\ -\sin x \cdot \cos y - \cos x \cdot \sin y & -\sin x \cdot \sin y + \cos x \cdot \cos y \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x + y) & \sin(x + y) \\ -\sin(x + y) & \cos(x + y) \end{bmatrix} \quad \text{...(i)}$$

$$\begin{bmatrix} \because \cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y \\ \text{and } \sin(x + y) = \sin x \cdot \cos y + \cos x \cdot \sin y \end{bmatrix}$$

and $$P(x + y) = \begin{bmatrix} \cos(x + y) & \sin(x + y) \\ -\sin(x + y) & \cos(x + y) \end{bmatrix} \quad \text{...(ii)}$$

Also, $$P(y)\cdot P(x)=\begin{bmatrix}\cos y & \sin y\\ -\sin y & \cos y\end{bmatrix}\begin{bmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{bmatrix}$$

$$=\begin{bmatrix}\cos y\cdot\cos x-\sin y\cdot\sin x & \cos y\cdot\sin x+\sin y\cdot\cos x\\ -\sin y\cdot\cos x-\sin x\cdot\cos y & -\sin y\cdot\sin x+\cos y\cdot\cos x\end{bmatrix}$$

$$=\begin{bmatrix}\cos(x+y) & \sin(x+y)\\ -\sin(x+y) & \cos(x+y)\end{bmatrix} \quad \text{...(iii)}$$

Thus, we see from the Eqs. (i), (ii) and (iii) that,

$$P(x)\cdot P(y)=P(x+y)=P(y)\cdot P(x)$$ **Hence proved.**

Q. 47 If A is square matrix such that $A^2=A$, then show that $(I+A)^3=7A+I$.

Sol. Since, $A^2=A$ and $(I+A)\cdot(I+A)=I^2+IA+AI+A^2$

$$=I^2+2AI+A^2$$
$$=I+2A+A=I+3A$$

and $$(I+A)\cdot(I+A)(I+A)=(I+A)(I+3A)$$
$$=I^2+3AI+AI+3A^2$$
$$=I+4AI+3A$$
$$=I+7A=7A+I$$ **Hence proved.**

Q. 48 If A, B are square matrices of same order and B is a skew-symmetric matrix, then show that $A'BA$ is skew-symmetric.

Sol. Since, A and B are square matrices of same order and B is a skew-symmetric matrix *i.e.,* $B'=-B$.

Now, we have to prove that $A'BA$ is a skew-symmetric matrix.

$\therefore$ $$A'BA'=A'BA'=BA'A' \quad [\because AB'=B'A']$$
$$=A'B'A=A'-BA=-A'BA$$

Hence, $A'BA$ is a skew-symmetric matrix.

Long Answer Type Questions

Q. 49 If $AB=BA$ for any two square matrices, then prove by mathematical induction that $(AB)^n=A^nB^n$.

Sol. Let $$P(n):(AB)^n=A^nB^n$$

$\therefore$ $$P(1):(AB)^1=A^1B^1\Rightarrow AB=AB$$

So, $P(1)$ is true.

Now, $$P(k):(AB)^k=A^kB^k,\ k\in N$$

So, $P(K)$ is true, whenever $P(k+1)$ is true.

$\therefore$ $$P(K+1:AB)^{k+1}=A^{k+1}B^{k+1} \quad \text{...(i)}$$

$\Rightarrow$ $$AB^k\cdot AB^1 \quad [\because AB=BA]$$

$\Rightarrow$ $$A^kB^k\cdot BA\Rightarrow A^kB^{k+1}A$$

$\Rightarrow$ $$A^k\cdot A\cdot B^{k+1}\Rightarrow A^{k+1}B^{k+1}$$

$\Rightarrow$ $$(A\cdot B)^{k+1}=A^{k+1}B^{k+1}$$

So, $P(k+1)$ is true for all $n\in N$, whenever $P(k)$ is true.

By mathematical induction $(AB)=A^nB^n$ is true for all $n\in N$.

Q. 50 Find x, y and z, if $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies $A' = A^{-1}$.

Sol. We have, $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ and $A' = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$

By using elementary row transformations, we get

$$A = IA$$

$$\Rightarrow \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ 0 & -2y & 2z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \qquad [\because R_3 \to R_3 - R_2]$$

$$\Rightarrow \begin{bmatrix} 0 & 2y & z \\ x & 3y & 0 \\ 0 & 0 & 3z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} A \qquad \left[\begin{matrix} \because R_3 \to R_3 + R_1 \\ \text{and } R_2 \to R_2 + R_1 \end{matrix}\right]$$

$$\Rightarrow \begin{bmatrix} -x & -y & z \\ x & 3y & 0 \\ 0 & 0 & z \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \end{bmatrix} A \qquad \left[\begin{matrix} \because R_1 \to R_1 - R_2 \\ \text{and } R_3 \to \frac{1}{3} R_3 \end{matrix}\right]$$

$$\Rightarrow \begin{bmatrix} -x & -y & 0 \\ x & 3y & 0 \\ 0 & 0 & z \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} & \frac{-2}{3} & \frac{-1}{3} \\ 1 & 1 & 0 \\ \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \end{bmatrix} A \qquad [\because R_1 \to R_1 - R_3]$$

$$\Rightarrow \begin{bmatrix} -x & -y & 0 \\ 0 & 2y & 0 \\ 0 & 0 & z \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} & \frac{-2}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \end{bmatrix} A \qquad [\because R_2 \to R_2 + R_1]$$

$$\Rightarrow \begin{bmatrix} -x & 0 & 0 \\ 0 & 2y & 0 \\ 0 & 0 & z \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-1}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \end{bmatrix} A \qquad \left[\because R_1 \to R_1 + \frac{1}{2} R_2\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2x} & \frac{1}{2x} \\ \frac{1}{3y} & \frac{1}{6y} & \frac{-1}{6y} \\ \frac{1}{3z} & \frac{-1}{3z} & \frac{1}{3z} \end{bmatrix} A \qquad \left[\begin{matrix} \because R_1 \to \frac{-1}{x} R_1, \\ R_2 \to \frac{1}{2y} R_2 \\ \text{and } R_3 \to \frac{1}{z} R_3 \end{matrix}\right]$$

$\therefore \quad A^{-1} = \begin{bmatrix} 0 & \frac{1}{2x} & \frac{1}{2x} \\ \frac{1}{3y} & \frac{1}{6y} & \frac{-1}{6y} \\ \frac{1}{3z} & \frac{-1}{3z} & \frac{1}{3z} \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$

$\Rightarrow \quad \frac{1}{2x} = x \Rightarrow \; = \pm \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \frac{1}{6y} = y \Rightarrow y = \pm \frac{1}{\sqrt{6}}$

and $\quad \frac{1}{3z} = z \Rightarrow z = \pm \frac{1}{\sqrt{3}}$

Alternate Method

We have,

$$A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \text{ and } A' = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$$

Also, $\quad A' = A^{-1}$

$\Rightarrow \quad AA' = AA^{-1}$ $\quad [\because AA^{-1} = I]$

$\Rightarrow \quad AA' = I$

$$\Rightarrow \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4y^2 + z^2 & 2y^2 - z^2 & -2y^2 + z^2 \\ 2y^2 - z^2 & x^2 + y^2 + z^2 & x^2 - y^2 - z^2 \\ -2y^2 + z^2 & x^2 - y^2 - z^2 & x^2 + y^2 + z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad 2y^2 - z^2 = 0 \Rightarrow 2y^2 = z^2$

$\Rightarrow \quad 4y^2 + z^2 = 1$

$\Rightarrow \quad 2 \cdot z^2 + z^2 = 1$

$\quad z = \pm \frac{1}{\sqrt{3}}$

$\therefore \quad y^2 = \frac{z^2}{2} \Rightarrow y = \pm \frac{1}{\sqrt{6}}$

Also, $\quad x^2 + y^2 + z^2 = 1$

$\Rightarrow \quad x^2 = 1 - y^2 - z^2 = 1 - \frac{1}{6} - \frac{1}{3}$

$\quad = 1 - \frac{3}{6} = \frac{1}{2}$

$\Rightarrow \quad x = \pm \frac{1}{\sqrt{2}}$

$\therefore \quad x = \pm, \frac{1}{\sqrt{2}}, y = \pm \frac{1}{\sqrt{6}}$

and $\quad z = \pm \frac{1}{\sqrt{3}}$

Q. 51 If possible, using elementary row transformations, find the inverse of the following matrices.

(i) $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Sol. For getting the inverse of the given matrix A by row elementary operations we may write the given matrix as

$$A = IA$$

(i) $\because \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ -3 & 2 & 4 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad [\because R_2 \to R_2 + R_1]$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ -3 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \qquad [\because R_3 \to R_3 - R_2]$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 7 \\ -3 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \qquad [\because R_1 \to R_1 + R_2]$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 7 \\ 0 & -1 & -17 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -5 & -2 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \qquad [\because R_2 \to R_2 - 3R_1]$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & -10 \\ 0 & -1 & -17 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 0 \\ -5 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} A \qquad \begin{bmatrix} \because R_1 \to R_1 + R_2 \\ \text{and } R_3 \to -1 \cdot R_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ 1 & 1 & -1 \end{bmatrix} A \qquad \begin{bmatrix} \because R_1 \to R_1 + 10R_3 \\ \text{and } R_2 \to R_2 + 17R_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix} A \qquad \begin{bmatrix} \because R_1 \to -1R_1 \\ \text{and } R_2 \to -1R_2 \end{bmatrix}$$

So, the inverse of A is $\begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}$.

(ii) $\therefore$

$$\begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \left[\begin{array}{l} \because R_2 \to R_2 + R_3 \\ \text{and } R_1 \to R_1 - 2R_3 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} A \qquad [\because R_2 \to R_2 + R_1]$$

Since, second row of the matrix A on LHS is containing all zeroes, so we can say that inverse of matrix A does not exist.

(iii) $\therefore$

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad [\because R_2 \to R_2 - R_1]$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \qquad \left[\begin{array}{l} \because R_2 \to R_2 - R_1 \\ \text{and } R_3 \to R_3 + R_1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & \frac{5}{2} \\ 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-5}{2} & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A \qquad \left[\begin{array}{l} \because R_3 \to R_3 + R_1 \\ \text{and } R_2 \to R_2 - \frac{1}{2}R_1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & \frac{5}{2} \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-5}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad [\because R_3 \to R_3 - 2R_1]$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-5}{2} & 1 & 0 \\ \frac{5}{2} & -1 & 1 \end{bmatrix} A \qquad [\because R_3 \to R_3 - R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-5}{2} & 1 & 0 \\ 5 & -2 & 2 \end{bmatrix} A \qquad \left[\begin{array}{l} \because R_1 \to \frac{1}{2}R_1 \\ \text{and} R_3 \to 2R_3 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A \qquad \left[\begin{array}{l} \because R_1 \to R_1 + \frac{1}{2}R_3 \\ \text{and } R_2 \to R_2 - \frac{5}{2}R_3 \end{array}\right]$$

Hence, $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ is the inverse of given matrix A.

Q. 52 Express the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 4 & 1 & 2 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Thinking Process

We know that, any square matrix A can be expressed as the sum of a symmetric matrix and skew-symmetric matrix, i.e., $A = \dfrac{A + A'}{2} + \dfrac{A - A'}{2}$, *where* $A + A'$ *and* $A - A'$ *are a symmetric matrix and a skew-symmetric matrix, respectively.*

Sol. We have,
$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 4 & 1 & 2 \end{bmatrix}$$

$\therefore$
$$A' = \begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Now,
$$\frac{A + A'}{2} = \frac{1}{2}\begin{bmatrix} 4 & 4 & 5 \\ 4 & -2 & 3 \\ 5 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & \frac{5}{2} \\ 2 & -1 & \frac{3}{2} \\ \frac{5}{2} & \frac{3}{2} & 2 \end{bmatrix}$$

and
$$\frac{A - A'}{2} = \frac{1}{2}\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \frac{-3}{2} \\ -1 & 0 & \frac{1}{2} \\ \frac{3}{2} & \frac{-1}{2} & 0 \end{bmatrix}$$

$\therefore$
$$\frac{A + A'}{2} + \frac{A - A'}{2} = \begin{bmatrix} 2 & 2 & \frac{5}{2} \\ 2 & -1 & \frac{3}{2} \\ \frac{5}{2} & \frac{3}{2} & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & \frac{-3}{2} \\ -1 & 0 & \frac{1}{2} \\ \frac{3}{2} & \frac{-1}{2} & 0 \end{bmatrix}$$

which is the required expression.

Objective Type Questions

Q. 53 The matrix $P = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ is a

(a) square matrix (b) diagonal matrix
(c) unit matrix (d) None of these

Sol. *(a)* We know that, in a square matrix number of rows are equal to the number of columns, so the matrix $P = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ is a square matrix.

Q. 54 Total number of possible matrices of order 3×3 with each entry 2 or 0 is

(a) 9 (b) 27
(c) 81 (d) 512

Sol. *(d)* Total number of possible matrices of order 3×3 with each entry 2 or 0 is 2^9 *i.e.,* 512.

Q. 55 $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$, then the value of $x + y$ is

(a) $x = 3, y = 1$ (b) $x = 2, y = 3$
(c) $x = 2, y = 4$ (d) $x = 3, y = 3$

Sol. *(b)* We have, $4x = x + 6 \Rightarrow x = 2$

and $4x = 7y - 13 \Rightarrow 8 = 7y - 13$

$\Rightarrow$ $7y = 21 \Rightarrow y = 3$

$\therefore$ $x + y = 2 + 3 = 5$

Q. 56 If $A = \dfrac{1}{\pi}\begin{bmatrix} \sin^{-1}(x\pi) & \tan^{-1}\left(\dfrac{x}{\pi}\right) \\ \sin^{-1}\left(\dfrac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$ and $B = \dfrac{1}{\pi}\begin{bmatrix} -\cos^{-1}(x\pi) & \tan^{-1}\left(\dfrac{x}{\pi}\right) \\ \sin^{-1}\left(\dfrac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$,

then $A - B$ is equal to

(a) I (b) 0 (c) $2I$ (d) $\dfrac{1}{2}I$

Sol. *(d)* We have, $A = \begin{bmatrix} \dfrac{1}{\pi}\sin^{-1} x\pi & \dfrac{1}{\pi}\tan^{-1}\dfrac{x}{\pi} \\ \dfrac{1}{\pi}\sin^{-1}\dfrac{x}{\pi} & \dfrac{1}{\pi}\cot^{-1}\pi x \end{bmatrix}$

and $B = \begin{bmatrix} \dfrac{-1}{\pi}\cos^{-1} x\pi & \dfrac{1}{\pi}\tan^{-1}\dfrac{x}{\pi} \\ \dfrac{1}{\pi}\sin^{-1}\dfrac{x}{\pi} & \dfrac{-1}{\pi}\tan^{-1}\pi x \end{bmatrix}$

$$\therefore \quad A-B=\begin{bmatrix} \frac{1}{\pi}(\sin^{-1}x\pi+\cos^{-1}x\pi) & \frac{1}{\pi}\left(\tan^{-1}\frac{x}{\pi}-\tan^{-1}\frac{x}{\pi}\right) \\ \frac{1}{\pi}\left(\sin^{-1}\frac{x}{\pi}-\sin^{-1}\frac{x}{\pi}\right) & \frac{1}{\pi}\cot^{-1}\pi x+\tan^{-1}\pi x \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{\pi}\cdot\frac{\pi}{2} & 0 \\ 0 & \frac{1}{\pi}\cdot\frac{\pi}{2} \end{bmatrix} \qquad \begin{bmatrix} \because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2} \\ \text{and} \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2} \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\frac{1}{2}I$$

Q. 57 If A and B are two matrices of the order $3\times m$ and $3\times n$, respectively and $m=n$, then order of matrix $(5A-2B)$ is

(a) $m\times 3$
(b) 3×3
(c) $m\times n$
(d) $3\times n$

Sol. *(d)* $A_{3\times m}$ and $B_{3\times n}$ are two matrices. If $m=n$, then A and B have same orders as $3\times n$ each, so the order of $(5A-2B)$ should be same as $3\times n$.

Q. 58 If $A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then A^2 is equal to

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
(b) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$
(c) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$
(d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Sol. *(d)* $\because A^2=A\cdot A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q. 59 If matrix $A=[a_{ij}]_{2\times 2}$, where $a_{ij}=1$, if $i\neq j=0$ and if $i=j$, then A^2 is equal to

(a) I
(b) A
(c) 0
(d) None of these

Sol. *(a)* We have, $A=[a_{ij}]_{2\times 2}$, where $a_{ij}=1$, if $i\neq j=0$ and if $i=j$

$$\therefore \quad A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $$A^2=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

Q. 60 The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a

(a) identity matrix
(b) symmetric matrix
(c) skew-symmetric matrix
(d) None of these

Sol. *(b)* Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\therefore \quad A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = A$$

So, the given matrix is a symmetric matrix.

[since, in a square matrix A, if $A' = A$, then A is called symmetric matrix]

Q. 61 The matrix $\begin{bmatrix} 0 & -5 & 8 \\ 5 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix}$ is a

(a) diagonal matrix (b) symmetric matrix

(c) skew-symmetric matrix (d) scalar matrix

Sol. *(c)* We know that, in a square matrix, if $b_{ij} = 0$, when $i \neq j$, then it is said to be a diagonal matrix. Here, $b_{12}, b_{13}, \ldots \neq 0$, so the given matrix is not a diagonal matrix.

Now,

$$B = \begin{bmatrix} 0 & -5 & 8 \\ 5 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix}$$

$$\therefore \quad B' = \begin{bmatrix} 0 & 5 & -8 \\ -5 & 0 & -12 \\ 8 & 12 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -5 & 8 \\ 5 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix} = -B$$

So, the given matrix is a skew-symmetric matrix, since we know that in a square matrix B, if $B' = -B$, then it is called skew-symmetric matrix.

Q. 62 If A is matrix of order $m \times n$ and B is a matrix such that AB' and $B'A$ are both defined, then order of matrix B is

(a) $m \times m$ (b) $n \times n$ (c) $n \times m$ (d) $m \times n$

Sol. *(d)* Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$

$\therefore \quad B' = [b_{ji}]_{q \times p}$

Now, AB' is defined, so $n = q$

and $B'A$ is also defined, so $p = m$

$\therefore$ Order of $B' = [b_{ji}]_{n \times m}$

and order of $B = [b_{ij}]_{m \times n}$

Q. 63 If A and B are matrices of same order, then $(AB' - BA')$ is a

(a) skew-symmetric matrix (b) null matrix

(c) symmetric matrix (d) unit matrix

Sol. *(a)* We have matrices A and B of same order.

Let $P = (AB' - BA')$

Then,

$$\begin{aligned} P' &= (AB' - BA')' = (AB')' - (BA')' \\ &= (B')'(A)' - (A')'B' = BA' - AB' \\ &= -(AB' - BA') = -P \end{aligned}$$

Hence, $(AB' - BA')$ is a skew-symmetric matrix.

Q. 64 If A is a square matrix such that $A^2 = I$, then $(A - I)^3 + (A + I)^3 - 7A$ is equal to

(a) A (b) $I - A$ (c) $I + A$ (d) $3A$

Sol. *(a)* We have, $A^2 = I$

$\therefore (A - I)^3 + (A + I)^3 - 7A = [(A - I) + (A + I)\{(A - I)^2 + (A + I)^2 - (A - I)(A + I)\}] - 7A$

$[\because a^3 + b^3 = (a + b)(a^2 + b^2 - ab)]$

$= [(2A)\{A^2 + I^2 - 2AI + A^2 + I^2 + AI - (A^2 - I^2)\}] - 7A$

$= 2A[I + I^2 + I + I^2 - A^2 + I^2] - 7A$ $[\because A^2 = AI]$

$= 2A[5I - I] - 7A$

$= 8AI - 7AI$ $[\because A = AI]$

$= AI = A$

Q. 65 For any two matrices A and B, we have

(a) $AB = BA$ (b) $AB \neq BA$ (c) $AB = O$ (d) None of these

Sol. *(d)* For any two matrices A and B, we may have $AB = BA = I$, $AB \neq BA$ and $AB = O$ but it is not always true.

Q. 66 On using elementary column operations $C_2 \to C_2 - 2C_1$ in the following matrix equation $\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, we have

(a) $\begin{bmatrix} 1 & -5 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -5 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$

Sol. *(d)* Given that, $\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

On using $C_2 \to C_2 - 2C_1$, $\begin{bmatrix} 1 & -5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$

Since, on using elementary column operation on $X = AB$, we apply these operations simultaneously on X and on the second matrix B of the product AB on RHS.

Q. 67 On using elementary row operation $R_1 \to R_1 - 3R_2$ in the following matrix equation $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, we have

(a) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 2 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Sol. *(a)* We have, $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Using elementary row operation $R_1 \rightarrow R_1 - 3R_2$,

$$\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Since, on using elementary row operation on $X = AB$, we apply these operation simultaneously on X and on the first matrix A of the product AB on RHS.

Fillers

Q. 68 .... matrix is both symmetric and skew-symmetric matrix.

Sol. Null matrix is both symmetric and skew-symmetric matrix.

Q. 69 Sum of two skew-symmetric matrices is always .... matrix.

Sol. Let A is a given matrix, then $(-A)$ is a skew-symmetric matrix.
Similarly, for a given matrix $-B$ is *a* skew-symmetric matrix.
Hence, $-A - B = -(A + B) \Rightarrow$ sum of two skew-symmetric matrices is always skew-symmetric matrix.

Q. 70 The negative of a matrix is obtained by multiplying it by ....

Sol. Let A is a given matrix.

$\therefore \qquad -A = -1[A]$

So, the negative of a matrix is obtained by multiplying it by -1.

Q. 71 The product of any matrix by the scalar .... is the null matrix.

Sol. The product of any matrix by the scalar 0 is the null matrix. *i.e.,* $0 \cdot A = 0$.

[where, A is any matrix]

Q. 72 A matrix which is not a square matrix is called a .... matrix.

Sol. A matrix which is not a square matrix is called a rectangular matrix. For example a rectangular matrix is $A = [a_{ij}]_{m \times n}$, where $m \neq n$.

Q. 73 Matrix multiplication is ...... over addition.

Sol. Matrix multiplication is distributive over addition.
e.g., For three matrices A, B and C,

(i) $A(B + C) = AB + AC$

(ii) $(A + B)C = AC + BC$

Q. 74 If A is a symmetric matrix, then A^3 is a .... matrix.

Sol. If A is a symmetric matrix, then A^3 is a symmetric matrix.

$\because$ $A' = A$

$\therefore$ $(A^3)' = A'^3$

$= A^3$ $\quad [\because (A')^n = (A^n)']$

Q. 75 If A is a skew-symmetric matrix, then A^2 is a ....

Sol. If A is a skew-symmetric matrix, then A^2 is a symmetric matrix.

$\because$ $A' = -A$

$\therefore$ $(A^2)' = (A')^2$

$= (-A)^2$ $\quad [\because A' = -A]$

$= A^2$

So, A^2 is a symmetric matrix.

Q. 76 If A and B are square matrices of the same order, then

(i) $(AB)' =$ ....

(ii) $(kA)' =$ .... (where, k is any scalar)

(iii) $[k(A - B)]' =$ ....

Sol. **(i)** $(AB)' = B'A'$

(ii) $(kA)' = k\,A'$

(iii) $[k(A - B)]' = k(A' - B')$

Q. 77 If A is a skew-symmetric, then kA is a .... (where, k is any scalar).

Sol. If A is a skew-symmetric, then kA is a skew-symmetric matrix (where, k is any scalar).

$[\because A' = -A \Rightarrow (kA)' = k(A)' = -(kA)]$

Q. 78 If A and B are symmetric matrices, then

(i) $AB - BA$ is a ....

(ii) $BA - 2AB$ is a ....

Sol. **(i)** $AB - BA$ is a skew-symmetric matrix.

Since, $[AB - BA]' = (AB)' - (BA)'$

$= B'A' - A'B'$ $\quad [\because (AB)' = B'A']$

$= BA - AB$ $\quad [\because A' = A$ and $B' = B]$

$= -[AB - BA]$

So, $[AB - BA]$ is a skew-symmetric matrix.

(ii) $[BA - 2AB]$ is a neither symmetric nor skew-symmetric matrix.

$\therefore$ $(BA - 2AB)' = (BA)' - 2(AB)'$

$= A'B' - 2B'A'$

$= AB - 2BA$

$= -(2BA - AB)$

So, $[BA - 2AB]$ is neither symmetric nor skew-symmetric matrix.

Q. 79 If A is symmetric matrix, then $B'AB$ is ....

Sol. If A is a symmetric matrix, then $B'AB$ is a symmetric metrix.

$$\because \quad [B'AB]' = [B'(AB)]'$$
$$= (AB)'(B')' \qquad [\because (AB)' = B'A']$$
$$= B'A'B$$
$$= [B'A'B] \qquad [\because A' = A]$$

So, $B'AB$ is a symmetric matrix.

Q. 80 If A and B are symmetric matrices of same order, then AB is symmetric if and only if......... .

Sol. If A and B are symmetric matrices of same order, then AB is symmetric if and only if $AB = BA$.

$$\therefore \quad (AB)'$$
$$= B'A' = BA$$
$$= AB \qquad [\because AB = BA]$$

Q. 81 In applying one or more row operations while finding A^{-1} by elementary row operations, we obtain all zeroes in one or more, then A^{-1} ....

Sol. In applying one or more row operations while finding A^{-1} by elementary row operations, we obtain all zeroes in one or more, then A^{-1} does not exist.

True/False

Q. 82 A matrix denotes a number.

Sol. *False*

A matrix is an ordered rectangular array of numbers or functions.

Q. 83 Matrices of any order can be added.

Sol. *False*

Two matrices are added, if they are of the same order.

Q. 84 Two matrices are equal, if they have same number of rows and same number of columns.

Sol. *False*

If two matrices have same number of rows and same number of columns, then they are said to be square matrix and if two square matrices have same elements in both the matrices, only then they are called equal.

Q. 85 Matrices of different order cannot be subtracted.

Sol. *True*

Two matrices of same order can be subtracted

Q. 86 Matrix addition is associative as well as commutative.

Sol. ***True***

Matrix addition is associative as well as commutative *i.e.,*
$(A + B) + C = A + (B + C)$ and $A + B = B + A$, where A, B and C are matrices of same order.

Q. 87 Matrix multiplication is commutative.

Sol. ***False***

Since, $AB \neq BA$ is possible when AB and BA are both defined.

Q. 88 A square matrix where every element is unity is called an identity matrix.

Sol. ***False***

Since, in an identity matrix, the diagonal elements are all one and rest are all zero.

Q. 89 If A and B are two square matrices of the same order, then $A + B = B + A$.

Sol. ***True***

Since, matrix addition is commutative *i.e.,* $A + B = B + A$, where A and B are two square matrices.

Q. 90 If A and B are two matrices of the same order, then $A - B = B - A$.

Sol. ***False***

Since, the addition of two matrices of same order are commutative.

$$\therefore \quad A + (-B) = A - B = -[B - A] \neq B - A$$

Q. 91 If matrix $AB = 0$, then $A = 0$ or $B = 0$ or both A and B are null matrices.

Sol. ***False***

Since, for two non-zero matrices A and B of same order, it can be possible that $A \cdot B = 0 =$ null matrix

Q. 92 Transpose of a column matrix is a column matrix.

Sol. ***False***

Transpose of a column matrix is a row matrix.

Q. 93 If A and B are two square matrices of the same order, then $AB = BA$.

Sol. ***False***

For two square matrices of same order it is not always true that $AB = BA$.

Q. 94 If each of the three matrices of the same order are symmetric, then their sum is a symmetric matrix.

Sol. ***True***

Let A, B and C are three matrices of same order

$$\therefore \quad A' = A,\ B' = B \text{ and } C' = C$$

$$\therefore \quad (A + B + C)' = A' + B' + C'$$

$$= (A + B + C)$$

Q. 95 If A and B are any two matrices of the same order, then $(AB)' = A'B'$.

Sol. ***False***

$\because \quad (AB)' = B'A'$

Q. 96 If $(AB)' = B'\ A'$, where A and B are not square matrices, then number of rows in A is equal to number of columns in B and number of columns in A is equal to number of rows in B.

Sol. ***True***

Let A is of order $m \times n$ and B is of order $p \times q$.

Since, $\quad (AB)' = B'A'$

$\therefore \quad A_{(m \times n)} B_{(p \times q)}$ is defined $\Rightarrow n = p \quad$ ...(i)

and AB is of order $m \times q$.

$\Rightarrow \quad (AB)'$ is of order $q \times m \quad$ ...(ii)

Also, B' is of order $q \times p$ and A' is of order $n \times m$

$\therefore \quad B'A'$ is defined $\Rightarrow p = n$

and $\quad B'A'$ is of order $q \times m$. $\quad$ ...(iii)

Also, equality of matrices $(AB)' = B'A'$, we get the given statement as true.

e.g., If A is of order (3×1) and B is of order (1×3), we get

Order of $(AB)'$ = Order of $(B'A') = 3 \times 3$

Q. 97 If A, B and C are square matrices of same order, then $AB = AC$ always implies that $B = C$.

Sol. ***False***

If $AB = AC = 0$, then it can be possible that B and C are two non-zero matrices such that $B \neq C$.

$\therefore \quad A \cdot B = 0 = A \cdot C$

Let $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$

and $\quad C = \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}$

$\therefore \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

and $\quad AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \quad AB = AC \quad$ but $B \neq C$

Q. 98 AA' is always a symmetric matrix for any matrix A.

Sol. ***True***

$\because \ [AA']' = (A')'\ A' = [AA']$

Q. 99 If $A = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 4 & 2 \end{vmatrix}$ and $B = \begin{vmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{vmatrix}$, then AB and BA are defined and equal.

Sol. ***False***

Since, AB is defined.

$$\therefore \quad AB = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 20 \\ 22 & 25 \end{bmatrix}$$

Also, BA is defined.

$$\therefore \quad BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 18 & 4 \\ 13 & 32 & 6 \\ 5 & 10 & 0 \end{bmatrix}$$

$$\therefore \quad AB \neq BA$$

Q. 100 If A is skew-symmetric matrix, then A^2 is a symmetric matrix.

Sol. ***True***

$$\because \quad [A^2]' = [A']^2$$

$$= [-A]^2 \quad [\because A' = -A]$$

$$= A^2$$

Hence, A^2 is symmetric matrix.

Q. 101 $(AB)^{-1} = A^{-1} \cdot B^{-1}$, where A and B are invertible matrices satisfying commutative property with respect to multiplication.

Sol. ***True***

We know that, if A and B are invertible matrices of the same order, then

$$(AB)^{-1} = (BA)^{-1} \quad [\because AB = BA]$$

Here, $\quad (AB)^{-1} = (AB)^{-1}$

$$\Rightarrow \quad B^{-1}A^{-1} = A^{-1}B^{-1}$$

[since, A and B are satisfying commutative property with respect to multiplications].

2

Inverse Trigonometric Functions

Short Answer Type Questions

Q. 1 Find the value of $\tan^{-1}\left(\tan\dfrac{5\pi}{6}\right)+\cos^{-1}\left(\cos\dfrac{13\pi}{6}\right)$.

Thinking Process

Use the property, $\tan^{-1}\tan x = x,\ x \in \left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ *and* $\cos^{-1}(\cos x) = x,\ x\in[0,\pi]$ *to get the answer.*

Sol. We know that, $\tan^{-1}\tan x = x;\ x \in \left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ and $\cos^{-1}\cos x = x;\ x \in [0,\pi]$

$$\therefore \quad \tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$$

$$= \tan^{-1}\left[\tan\left(\pi-\frac{\pi}{6}\right)\right]+\cos^{-1}\left[\cos\left(\pi+\frac{7\pi}{6}\right)\right]$$

$$= \tan^{-1}\left(-\tan\frac{\pi}{6}\right)+\cos^{-1}\left(-\cos\frac{7\pi}{6}\right) \qquad [\because \cos(\pi+\theta) = -\cos\theta]$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\left[\cos^{-1}\cos\left(\frac{7\pi}{6}\right)\right]$$

$$\{\because \tan^{-1}(-x) = -\tan^{-1}x;\ x\in R \text{ and } \cos^{-1}(-x) = \pi-\cos^{-1}x;\ x\in[-1,1]\}$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\cos^{-1}\left[\cos\left(\pi+\frac{\pi}{6}\right)\right]$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\left[\cos^{-1}\left(-\cos\frac{\pi}{6}\right)\right] \qquad [\because \cos(\pi+\theta) = -\cos\theta]$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\pi+\cos^{-1}\left(\cos\frac{\pi}{6}\right) \qquad [\because \cos^{-1}(-x) = \pi-\cos^{-1}x]$$

$$= -\frac{\pi}{6}+0+\frac{\pi}{6} = 0$$

Note *Remember that,* $\tan^{-1}\left(\tan\dfrac{5\pi}{6}\right)\neq\dfrac{5\pi}{6}$ *and* $\cos^{-1}\left(\cos\dfrac{13\pi}{6}\right)\neq\dfrac{13\pi}{6}$

Since, $\dfrac{5\pi}{6}\notin\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ *and* $\dfrac{13\pi}{6}\notin[0,\pi]$

Q. 2 Evaluate $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$.

Sol. We have, $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]=\cos\left[\cos^{-1}\left(\cos\frac{5\pi}{6}\right)+\frac{\pi}{6}\right]$ $\left[\because \cos\frac{5\pi}{6}=\frac{-\sqrt{3}}{2}\right]$

$$=\cos\left(\frac{5\pi}{6}+\frac{\pi}{6}\right) \qquad \{\because \cos^{-1}\cos x = x;\ x\in[0,\pi]\}$$

$$=\cos\left(\frac{6\pi}{6}\right)$$

$$=\cos(\pi)=-1$$

Q. 3 Prove that $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=7$.

Sol. We have to prove, $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=7$

$\Rightarrow$ $\left(\frac{\pi}{4}-2\cot^{-1}3\right)=\cot^{-1}7$

$\Rightarrow$ $(2\cot^{-1}3)=\frac{\pi}{4}-\cot^{-1}7$

$\Rightarrow$ $2\tan^{-1}\frac{1}{3}=\frac{\pi}{4}-\tan^{-1}\frac{1}{7}$

$\Rightarrow$ $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\frac{2/3}{1-(1/3)^2}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\frac{2/3}{8/9}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\frac{(21+4)/28}{(28-3)/28}=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\frac{25}{25}=\frac{\pi}{4}$

$\Rightarrow$ $1=\tan\frac{\pi}{4}$

$\Rightarrow$ $1=1$

$\Rightarrow$ $\text{LHS}=\text{RHS}$ **Hence proved.**

Q. 4 Find the value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$.

Sol. We have, $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$

$$= \tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cot^{-1}\left(\cot\frac{\pi}{3}\right)+\tan^{-1}(-1)$$

$$= \tan^{-1}\left[\tan\left(\pi-\frac{\pi}{6}\right)\right]+\cot^{-1}\left[\cot\left(\frac{\pi}{3}\right)\right]+\tan^{-1}\left[\tan\left(\pi-\frac{\pi}{4}\right)\right]$$

$$= \tan^{-1}\left(-\tan\frac{\pi}{6}\right)+\cot^{-1}\left(\cot\frac{\pi}{3}\right)+\tan^{-1}\left(-\tan\frac{\pi}{4}\right)$$

$$\left[\begin{array}{l}\because\quad \tan^{-1}(\tan x)=x,\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\\ \quad\ \cot^{-1}(\cot x)=x, x\in(0,\pi)\\ \text{and}\quad \tan^{-1}(-x)=-\tan^{-1}x\end{array}\right]$$

$$= -\frac{\pi}{6}+\frac{\pi}{3}-\frac{\pi}{4}=\frac{-2\pi+4\pi-3\pi}{12}$$

$$= \frac{-5\pi+4\pi}{12}=-\frac{\pi}{12}$$

Q. 5 Find the value of $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$.

Sol. We have, $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)=\tan^{-1}\tan\left(\pi-\frac{\pi}{3}\right)$

$$= \tan^{-1}\left(-\tan\frac{\pi}{3}\right) \qquad [\because \tan^{-1}(-x)=-\tan^{-1}x]$$

$$= -\tan^{-1}\tan\frac{\pi}{3}=-\frac{\pi}{3} \qquad \left[\because \tan^{-1}(\tan x)=x,\ x\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$$

Note *Remember that,* $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)\neq\frac{2\pi}{3}$

Since, $\tan^{-1}(\tan x)=x$, *if* $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *and* $\frac{2\pi}{3}\notin\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

Q. 6 Show that $2\tan^{-1}(-3)=\frac{-\pi}{2}+\tan^{-1}\left(\frac{-4}{3}\right)$.

Sol. LHS $=2\tan^{-1}(-3)=-2\tan^{-1}3 \qquad [\because \tan^{-1}(-x)=-\tan^{-1}x,\ x\in R]$

$$= -\left[\cos^{-1}\frac{1-3^2}{1+3^2}\right] \qquad \left[\because 2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}, x\geq 0\right]$$

$$= -\left[\cos^{-1}\left(\frac{-8}{10}\right)\right]=-\left[\cos^{-1}\left(\frac{-4}{5}\right)\right]$$

$$= -\left[\pi-\cos^{-1}\left(\frac{4}{5}\right)\right] \qquad \{\because \cos^{-1}(-x)=\pi-\cos^{-1}x,\ x\in[-1,1]\}$$

$$= -\pi+\cos^{-1}\left(\frac{4}{5}\right) \quad \left[\text{let }\cos^{-1}\left(\frac{4}{5}\right)=\theta\Rightarrow\cos\theta=\frac{4}{5}\Rightarrow\tan\theta=\frac{3}{4}\Rightarrow\theta=\tan^{-1}\frac{3}{4}\right]$$

$$= -\pi + \tan^{-1}\left(\frac{3}{4}\right) = -\pi + \left[\frac{\pi}{2} - \cot^{-1}\left(\frac{3}{4}\right)\right]$$

$$= -\frac{\pi}{2} - \cot^{-1}\frac{3}{4} = -\frac{\pi}{2} - \tan^{-1}\frac{4}{3}$$

$$= -\frac{\pi}{2} + \tan^{-1}\left(\frac{-4}{3}\right) \qquad [\because \tan^{-1}(-x) = -\tan^{-1}x]$$

$= \text{RHS}$ **Hence proved.**

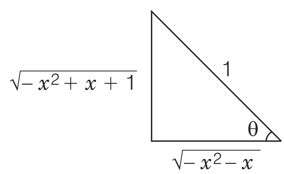
Q. 7 Find the real solution of

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}.$$

Thinking Process

Convert the $\sin^{-1}\sqrt{x^2+x+1}$ *into inverse of tangent function and then use the property* $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$.

Sol. We have, $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ ...(i)



Let $\sin^{-1}\sqrt{x^2+x+1} = \theta$

$$\Rightarrow \quad \sin\theta = \sqrt{\frac{x^2+x+1}{1}}$$

$$\Rightarrow \quad \tan\theta = \frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}} \qquad \left[\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

$$\because \quad \theta = \tan^{-1}\frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}}$$

$$= \sin^{-1}\sqrt{x^2+x+1}$$

On putting the value of θ in Eq. (i), we get

$$\tan^{-1}\sqrt{x(x+1)} + \tan^{-1}\frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}} = \frac{\pi}{2}$$

We know that, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

$$\therefore \quad \tan^{-1}\left[\frac{\sqrt{x(x+1)} + \sqrt{\frac{x^2+x+1}{-x^2-x}}}{1 - \sqrt{x(x+1)}\cdot\sqrt{\frac{x^2+x+1}{-x^2-x}}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\sqrt{x^2+x} + \sqrt{\frac{x^2+x+1}{-1(x^2+x)}}}{1 - \sqrt{(x^2+x)\cdot\frac{(x^2+x+1)}{-1(x^2+x)}}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{x^2+x+\sqrt{-(x^2+x+1)}}{[1-\sqrt{-(x^2+x+1]}\sqrt{(x^2+x)}} = \tan\frac{\pi}{2} = \frac{1}{0}$$

$\Rightarrow \quad [1 - \sqrt{-(x^2 + x + 1)}]\sqrt{(x^2 + x)} = 0$

$\Rightarrow \quad -(x^2 + x + 1) = 1 \quad \text{or} \quad x^2 + x = 0$

$\Rightarrow \quad -x^2 - x - 1 = 1 \quad \text{or} \quad x(x + 1) = 0$

$\Rightarrow \quad x^2 + x + 2 = 0 \quad \text{or} \quad x(x + 1) = 0$

$\therefore \quad x = \dfrac{-1 \pm \sqrt{1 - 4 \times 2}}{2}$

$\Rightarrow \quad x = 0 \quad \text{or} \quad x = -1$

For real solution, we have $x = 0, -1$.

Q. 8 Find the value of $\sin\left(2\tan^{-1}\dfrac{1}{3}\right) + \cos(\tan^{-1} 2\sqrt{2})$.

Sol. We have, $\sin\left(2\tan^{-1}\dfrac{1}{3}\right) + \cos(\tan^{-1} 2\sqrt{2})$

$$= \sin\left[\sin^{-1}\left\{\frac{2 \times \frac{1}{3}}{1 + \left(\frac{1}{3}\right)^2}\right\}\right] + \cos\left(\cos^{-1}\frac{1}{3}\right) \qquad \left[\because \tan^{-1} x = \cos^{-1}\frac{1}{\sqrt{1 + x^2}}\right]$$

$$\left[\because 2\tan^{-1} x = \sin^{-1}\frac{2x}{1 + x^2}, -1 \le x \le 1 \text{ and } \tan^{-1}(2\sqrt{2}) = \cos^{-1}\frac{1}{3}\right]$$

$$= \sin\left[\sin^{-1}\left(\frac{\frac{2}{3}}{1 + \frac{1}{9}}\right)\right] + \frac{1}{3} \qquad \{\because \cos(\cos^{-1} x) = x;\ x \in [-1, 1]\}$$

$$= \sin\left[\sin^{-1}\left(\frac{2 \times 9}{3 \times 10}\right)\right] + \frac{1}{3} = \sin\left[\sin^{-1}\left(\frac{3}{5}\right)\right] + \frac{1}{3} \qquad [\because \sin(\sin^{-1} x) = x]$$

$$= \frac{3}{5} + \frac{1}{3} = \frac{9 + 5}{15} = \frac{14}{15}$$

Q. 9 If $2\tan^{-1}(\cos\theta) = \tan^{-1}(2\operatorname{cosec}\theta)$, then show that $\theta = \dfrac{\pi}{4}$, where n is any integer.

Thinking Process

Use the property, $2\tan^{-1} x = \tan^{-1}\left(\dfrac{2x}{1 - x^2}\right)$ *to prove the desired result.*

Sol. We have, $2\tan^{-1}(\cos\theta) = \tan^{-1}(2\operatorname{cosec}\theta)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos\theta}{1 - \cos^2\theta}\right) = \tan^{-1}(2\operatorname{cosec}\theta)$$

$$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)\right]$$

$$\Rightarrow \quad \left(\frac{2\cos\theta}{\sin^2\theta}\right) = (2\operatorname{cosec}\theta)$$

$$\Rightarrow \quad (\cot\theta \cdot 2\operatorname{cosec}\theta) = (2\operatorname{cosec}\theta) \Rightarrow \cot\theta = 1$$

$$\Rightarrow \quad \cot\theta = \cot\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$$

Q. 10 Show that $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$.

Thinking Process

Use the property $2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}$ *and* $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$, *to prove LHS = RHS.*

Sol. We have, $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{1-\left(\frac{1}{7}\right)^2}{1+\left(\frac{1}{7}\right)^2}\right)\right]=\sin\left[2\cdot 2\tan^{-1}\frac{1}{3}\right] \quad \left[\because 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{\frac{48}{49}}{\frac{50}{49}}\right)\right]=\sin\left[2\cdot\left(\tan^{-1}\frac{\frac{2}{3}}{1-\left(\frac{1}{3}\right)^2}\right)\right] \quad \left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{48\times 49}{50\times 49}\right)\right]=\sin\left[2\tan^{-1}\left(\frac{18}{24}\right)\right]$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{24}{25}\right)\right]=\sin\left(2\tan^{-1}\frac{3}{4}\right)$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{24}{25}\right)\right]=\sin\left(\sin^{-1}\frac{2\times\frac{3}{4}}{1+\frac{9}{16}}\right) \quad \left[\because 2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}\right]$$

$$\Rightarrow \frac{24}{25}=\sin\left(\sin^{-1}\frac{3/2}{25/16}\right)$$

$$\Rightarrow \frac{24}{25}=\frac{48}{50} \Rightarrow \frac{24}{25}=\frac{24}{25}$$

$\therefore$ LHS = RHS **Hence proved.**

Q. 11 Solve the equation $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$.

Sol. We have, $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$

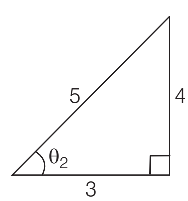
$$\Rightarrow \cos\left(\cos^{-1}\frac{1}{\sqrt{x^2+1}}\right)=\sin\left(\sin^{-1}\frac{4}{5}\right)$$

Let $\tan^{-1}x=\theta_1 \Rightarrow \tan\theta_1=\frac{x}{1}$

$\Rightarrow \cos\theta_1=\frac{1}{\sqrt{x^2+1}} \Rightarrow \theta_1=\cos^{-1}\frac{1}{\sqrt{x^2+1}}$

and $\cot^{-1}\frac{3}{4}=\theta_2 \Rightarrow \cot\theta_2=\frac{3}{4}$

$\Rightarrow \sin\theta_2=\frac{4}{5} \Rightarrow \theta_2=\sin^{-1}\frac{4}{5}$



$\Rightarrow \quad \dfrac{1}{\sqrt{x^2+1}} = \dfrac{4}{5}$

$\{\because \cos(\cos^{-1} x) = x,\ x \in [-1, 1] \text{ and } \sin(\sin^{-1} x) = x,\ x \in [-1, 1]\}$

On squaring both sides, we get

$$16(x^2+1) = 25$$

$\Rightarrow \quad 16x^2 = 9$

$\Rightarrow \quad x^2 = \left(\dfrac{3}{4}\right)^2$

$\therefore \quad x = \pm\dfrac{3}{4} = \dfrac{-3}{4}, \dfrac{3}{4}$

Long Answer Type Questions

Q. 12 Prove that $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1} x^2$.

Sol. We have,

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$

$\therefore \quad \text{LHS} = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) \quad \ldots(\text{i})$

$[\text{let } x^2 = \cos 2\theta = (\cos^2\theta - \sin^2\theta) = 1 - 2\sin^2\theta = 2\cos^2\theta - 1]$

$\Rightarrow \quad \cos^{-1} x^2 = 2\theta \Rightarrow \theta = \dfrac{1}{2}\cos^{-1} x^2$

$\therefore \quad \sqrt{1+x^2} = \sqrt{1+\cos 2\theta}$

$$= \sqrt{1+2\cos^2\theta - 1} = \sqrt{2}\cos\theta$$

and $\quad \sqrt{1-x^2} = \sqrt{1-\cos 2\theta}$

$$= \sqrt{1-1+2\sin^2\theta} = \sqrt{2}\sin\theta$$

$\therefore \quad \text{LHS} = \tan^{-1}\left(\dfrac{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta}\right)$

$$= \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\cdot\tan\theta}\right)$$

$= \tan^{-1}\left[\tan\left(\dfrac{\pi}{4}+\theta\right)\right] \quad \left[\because \tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \cdot \tan y}\right]$

$$= \frac{\pi}{4} + \theta = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$

$= \text{RHS}$ **Hence proved.**

Q. 13 Find the simplified form of

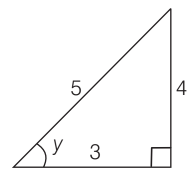
$$\cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right), \text{ where } x\in\left[\frac{-3\pi}{4},\frac{\pi}{4}\right].$$

Sol. We have, $\cos^{-1}\left[\frac{3}{5}\cos x+\frac{4}{5}\sin x\right], x\in\left[\frac{-3\pi}{4},\frac{\pi}{4}\right]$

Let $\cos y=\frac{3}{5}$

$\Rightarrow$ $\sin y=\frac{4}{5}$

$\Rightarrow$ $y=\cos^{-1}\frac{3}{5}=\sin^{-1}\frac{4}{5}=\tan^{-1}\left(\frac{4}{3}\right)$



$\therefore$ $\cos^{-1}[\cos y\cdot\cos x+\sin y\cdot\sin x]$

$=\cos^{-1}[\cos(y-x)]$ $\quad [\because \cos(A-B)=\cos A\cdot\cos B+\sin A\cdot\sin B]$

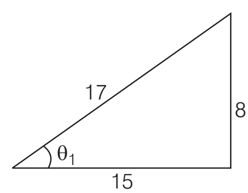
$=y-x=\tan^{-1}\frac{4}{3}-x$ $\quad \left[\because y=\tan^{-1}\frac{4}{3}\right]$

Q. 14 Prove that $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{77}{85}$.

Sol. We have, $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{77}{85}$

$\therefore$ $\text{LHS}=\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}$

$=\tan^{-1}\frac{8}{15}+\tan^{-1}\frac{3}{4}$



Let $\sin^{-1}\frac{8}{17}=\theta_1 \Rightarrow \sin\theta_1=\frac{8}{17}$

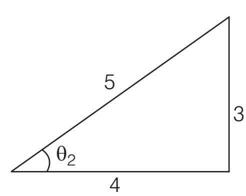
$\Rightarrow$ $\tan\theta_1=\frac{8}{15} \Rightarrow \theta_1=\tan^{-1}\frac{8}{15}$

and $\sin^{-1}\frac{3}{5}=\theta_2 \Rightarrow \sin\theta_2=\frac{3}{5}$

$\Rightarrow$ $\tan\theta_2=\frac{3}{4} \Rightarrow \theta_2=\tan^{-1}\frac{3}{4}$

$=\tan^{-1}\left[\dfrac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}\right]$ $\quad \left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$=\tan^{-1}\left[\dfrac{\frac{32+45}{60}}{\frac{60-24}{60}}\right]=\tan^{-1}\left(\frac{77}{36}\right)$



Let $\theta_3=\tan^{-1}\frac{77}{36} \Rightarrow \tan\theta_3=\frac{77}{36}$

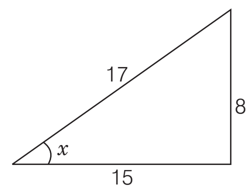
$\Rightarrow$ $\sin\theta_3=\frac{77}{\sqrt{5929+1296}}=\frac{77}{85}$

$\therefore$ $\theta_3=\sin^{-1}\frac{77}{85}$

$=\sin^{-1}\frac{77}{85}=\text{RHS}$ **Hence proved.**

Alternate Method

To prove, $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{77}{85}$

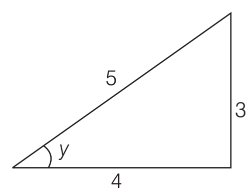


Let $\sin^{-1}\frac{8}{17}=x$

$\Rightarrow \sin x=\frac{8}{17}$

$\Rightarrow \cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\left(\frac{8}{17}\right)^2}$

$=\sqrt{\frac{289-64}{289}}=\sqrt{\frac{225}{289}}=\frac{15}{17}$



Let $\sin^{-1}\frac{3}{5}=y$

$\Rightarrow \sin y=\frac{3}{5} \Rightarrow \sin^2 y=\frac{9}{25}$

$\therefore \cos^2 y=1-\frac{9}{25}$

$\Rightarrow \cos^2 y=\left(\frac{4}{5}\right)^2 \Rightarrow \cos y=\frac{4}{5}$

Now, $\sin(x+y)=\sin x\cdot\cos y+\cos x\cdot\sin y$

$=\frac{8}{17}\cdot\frac{4}{5}+\frac{15}{17}\cdot\frac{3}{5}$

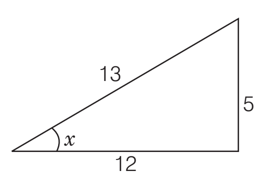
$=\frac{32}{85}+\frac{45}{85}=\frac{77}{85}$

$\Rightarrow (x+y)=\sin^{-1}\left(\frac{77}{85}\right)$

$\Rightarrow \sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{77}{85}$

Q. 15 Show that $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$.

Sol. We have, $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$ ...(i)



Let $\sin^{-1}\frac{5}{13}=x$

$\Rightarrow \sin x=\frac{5}{13}$

and $\cos^2 x=1-\sin^2 x$

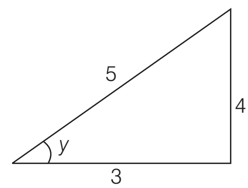
$=1-\frac{25}{169}=\frac{144}{169}$

$\Rightarrow \cos x=\sqrt{\frac{144}{169}}=\frac{12}{13}$

$\therefore \tan x=\frac{\sin x}{\cos x}=\frac{5/13}{12/13}=\frac{5}{12}$ ...(ii)

$\Rightarrow \tan x=5/12$ ...(iii)

Again, let $\cos^{-1}\frac{3}{5} = y \Rightarrow \cos y = \frac{3}{5}$



$\therefore$ $$\sin y = \sqrt{1-\cos^2 y}$$

$$= \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}}$$

$$\sin y = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$\Rightarrow$ $$\tan y = \frac{\sin y}{\cos y} = \frac{4/5}{3/5} = \frac{4}{3} \quad \text{...(iii)}$$

We know that,

$$\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x \cdot \tan y}$$

$\Rightarrow$ $$\tan(x+y) = \frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\cdot\frac{4}{3}} \Rightarrow \tan(x+y) = \frac{\frac{15+48}{36}}{\frac{36-20}{36}}$$

$\Rightarrow$ $$\tan(x+y) = \frac{63/36}{16/36}$$

$\Rightarrow$ $$\tan(x+y) = \frac{63}{16}$$

$\Rightarrow$ $$x+y = \tan^{-1}\frac{63}{16}$$

$\Rightarrow$ $$\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3} = \tan^{-1}\frac{63}{16}$$ **Hence proved.**

Q. 16 Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}}$.

Sol. We have, $$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}} \quad \text{...(i)}$$

Let $$\tan^{-1}\frac{1}{4} = x$$

$\Rightarrow$ $$\tan x = \frac{1}{4}$$

$\Rightarrow$ $$\tan^2 x = \frac{1}{16}$$

$\Rightarrow$ $$\sec^2 x - 1 = \frac{1}{16}$$

$\Rightarrow$ $$\sec^2 x = 1+\frac{1}{16} = \frac{17}{16}$$

$\Rightarrow$ $$\frac{1}{\cos^2 x} = \frac{17}{16}$$

$\Rightarrow$ $$\cos^2 x = \frac{16}{17}$$

$\Rightarrow$ $$\cos x = \frac{4}{\sqrt{17}}$$

$\Rightarrow$ $$\sin^2 x = 1-\cos^2 x = 1-\frac{16}{17} = \frac{1}{17}$$

$\Rightarrow$ $$\sin x = \frac{1}{\sqrt{17}} \quad \text{...(ii)}$$

Again, let $\tan^{-1}\frac{2}{9} = y$

$\Rightarrow \quad \tan y = \frac{2}{9} \Rightarrow \tan^2 y = \frac{4}{81}$

$\Rightarrow \quad \sec^2 y - 1 = \frac{4}{81}$

$\Rightarrow \quad \sec^2 y = \frac{4}{81} + 1 = \frac{85}{81}$

$\Rightarrow \quad \cos^2 y = \frac{81}{85} \Rightarrow \cos y = \frac{9}{\sqrt{85}}$

$\Rightarrow \quad \sin^2 y = 1 - \cos^2 y = 1 - \frac{81}{85} = \frac{4}{85}$

$\Rightarrow \quad \sin y = \frac{2}{\sqrt{85}}$ ...(iii)

We know that, $\sin(x + y) = \sin x \cdot \cos y + \cos x \cdot \sin y$

$$= \frac{1}{\sqrt{17}} \cdot \frac{9}{\sqrt{85}} + \frac{4}{\sqrt{17}} \cdot \frac{2}{\sqrt{85}}$$

$$= \frac{17}{\sqrt{17} \cdot \sqrt{85}} = \frac{\sqrt{17}}{\sqrt{17} \cdot \sqrt{5}} = \frac{1}{\sqrt{5}}$$

$\Rightarrow \quad (x + y) = \sin^{-1}\frac{1}{\sqrt{5}}$

$\Rightarrow \quad \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}}$ **Hence proved.**

Q. 17 Find the value of $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$.

Thinking Process

Use the properties $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ to get the desired value.

Sol. We have, $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

$$= 2 \cdot 2\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

$$= 2 \cdot \left[\tan^{-1}\frac{\frac{2}{5}}{1-\left(\frac{1}{5}\right)^2}\right] - \tan^{-1}\frac{1}{239} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= 2 \cdot \left[\tan^{-1}\left(\frac{\frac{2}{5}}{1-\frac{1}{25}}\right)\right] - \tan^{-1}\frac{1}{239}$$

$$= 2 \cdot \left[\tan^{-1}\left(\frac{2/5}{24/25}\right)\right] - \tan^{-1}\frac{1}{239}$$

$$= 2\tan^{-1}\frac{5}{12} - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\frac{2\cdot\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2} - \tan^{-1}\frac{1}{239} \qquad \left[\because\ 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left(\frac{\frac{5}{6}}{1-\frac{25}{144}}\right) - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\left(\frac{144\times 5}{119\times 6}\right) - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\left(\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}}\right) \qquad \left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{120\times 239-119}{119\times 239+120}\right)$$

$$= \tan^{-1}\left[\frac{28680-119}{28441+120}\right] = \tan^{-1}\frac{28561}{28561}$$

$$= \tan^{-1}(1) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$$

Q. 18 Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$ and justify why the other value $\frac{4+\sqrt{7}}{3}$ is ignored?

Sol. We have, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

$\therefore$ $\text{LHS} = \tan\left[\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)\right]$

Let $\frac{1}{2}\sin^{-1}\frac{3}{4} = \theta \Rightarrow \sin^{-1}\frac{3}{4} = 2\theta$

$\Rightarrow$ $\sin 2\theta = \frac{3}{4} \Rightarrow \frac{2\tan\theta}{1+\tan^2\theta} = \frac{3}{4}$

$\Rightarrow$ $3 + 3\tan^2\theta = 8\tan\theta$

$\Rightarrow$ $3\tan^2\theta - 8\tan\theta + 3 = 0$

Let $\tan\theta = y$

$\therefore$ $3y^2 - 8y + 3 = 0$

$\Rightarrow$ $$y = \frac{+8 \pm \sqrt{64-4\times 3\times 3}}{2\times 3} = \frac{8\pm\sqrt{28}}{6}$$

$$= \frac{2[4\pm\sqrt{7}]}{2\cdot 3}$$

$\Rightarrow$ $$\tan\theta = \frac{4\pm\sqrt{7}}{3}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left[\frac{4 \pm \sqrt{7}}{3}\right]$$

$$\left\{\text{but } \frac{4+\sqrt{7}}{3} > \frac{1}{2}.\frac{\pi}{2}, \text{ since max}\left[\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)\right] = 1\right\}$$

$$\therefore \qquad \text{LHS} = \tan\tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right) = \frac{4-\sqrt{7}}{3} = \text{RHS}$$

Note *Since,* $$-\frac{\pi}{2} \le \sin^{-1}\frac{3}{4} \le \pi/2$$

$$\Rightarrow \qquad \frac{-\pi}{4} \le \frac{1}{2}\sin^{-1}\frac{3}{4} \le \pi/4$$

$$\therefore \qquad \tan\left(\frac{-\pi}{4}\right) \le \tan\frac{1}{2}\left(\sin^{-1}\frac{3}{4}\right) \le \tan\frac{\pi}{4}$$

$$\Rightarrow \qquad -1 \le \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \le 1$$

Q. 19 If $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression with common difference d, then evaluate the following expression.

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \tan^{-1}\left(\frac{d}{1+a_3a_4}\right) + \ldots + \tan^{-1}\left(\frac{d}{1+a_{n-1}\,a_n}\right)\right]$$

Sol. We have, $a_1 = a,\ a_2 = a+d,\ a_3 = a+2d$

and $d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots = a_n - a_{n-1}$

Given that, $$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \tan^{-1}\left(\frac{d}{1+a_3a_4}\right) + \ldots + \tan^{-1}\left(\frac{d}{1+a_{n-1}\cdot a_n}\right)\right]$$

$$= \tan\left[\tan^{-1}\frac{a_2-a_1}{1+a_2\cdot a_1} + \tan^{-1}\frac{a_3-a_2}{1+a_3\cdot a_2} + \ldots + \tan^{-1}\frac{a_n-a_{n-1}}{1+a_n\cdot a_{n-1}}\right]$$

$$= \tan\left[(\tan^{-1}a_2 - \tan^{-1}a_1) + (\tan^{-1}a_3 - \tan^{-1}a_2) + \ldots + (\tan^{-1}a_n - \tan^{-1}a_{n-1})\right]$$

$$= \tan[\tan^{-1}a_n - \tan^{-1}a_1]$$

$$= \tan\left[\tan^{-1}\frac{a_n - a_1}{1+a_n\cdot a_1}\right] \qquad \left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

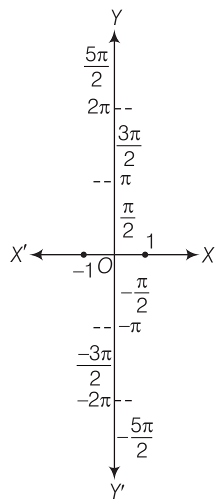
$$= \frac{a_n - a_1}{1+a_n\cdot a_1} \qquad [\because \tan(\tan^{-1}x) = x]$$

Objective Type Questions

Q. 20 Which of the following is the principal value branch of $\cos^{-1} x$?

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) $(0, \pi)$ (c) $[0, \pi]$ (d) $(0, \pi)-\left\{\frac{\pi}{2}\right\}$

Sol. *(c)* We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$.

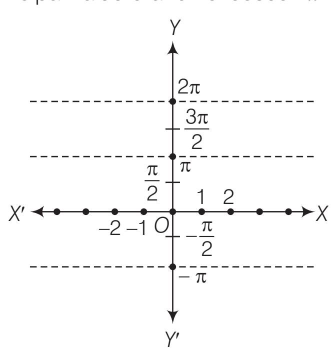


$\therefore$ $y=\cos^{-1} x$

Q. 21 Which of the following is the principal value branch of $\operatorname{cosec}^{-1} x$?

(a) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ (b) $[0, \pi]-\left\{\frac{\pi}{2}\right\}$ (c) $\left[\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]-[0]$

Sol. *(d)* We know that, the principal value branch of $\operatorname{cosec}^{-1} x$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]-0$.



$\therefore$ $y=\operatorname{cosec}^{-1} x$

Q. 22 If $3\tan^{-1}x + \cot^{-1}x = \pi$, then x equals to

(a) 0 (b) 1 (c) -1 (d) $\frac{1}{2}$

Sol. ***(b)*** Given that, $3\tan^{-1}x + \cot^{-1}x = \pi$ ...(i)

$\Rightarrow \quad 2\tan^{-1}x + \tan^{-1}x + \cot^{-1}x = \pi$

$\Rightarrow \quad 2\tan^{-1}x = \pi - \frac{\pi}{2}$ $\quad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$

$\Rightarrow \quad 2\tan^{-1}x = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{2}$ $\quad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}, \forall x \in (-1, 1)\right]$

$\Rightarrow \quad \frac{2x}{1-x^2} = \tan\frac{\pi}{2}$

$\Rightarrow \quad \frac{2x}{1-x^2} = \frac{1}{0} \Rightarrow 1 - x^2 = 0$

$\Rightarrow \quad x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow x = 1$

Hence, only $x = 1$ satisfies the given equation.

Note *Here, putting $x = -1$ in the given equation, we get*

$$3\tan^{-1}(-1) + \cot^{-1}(-1) = \pi$$

$$\Rightarrow \quad 3\tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] + \cot^{-1}\left[\cot\left(\frac{-\pi}{4}\right)\right] = \pi$$

$$\Rightarrow \quad 3\tan^{-1}\left(-\tan\frac{\pi}{4}\right) + \cot^{-1}\left(-\cot\frac{\pi}{4}\right) = \pi$$

$$\Rightarrow \quad -3\tan^{-1}\left(\tan\frac{\pi}{4}\right) + \pi - \cot^{-1}\left(\cot\frac{\pi}{4}\right) = \pi$$

$$\Rightarrow \quad -3\cdot\frac{\pi}{4} + \pi - \frac{\pi}{4} = \pi$$

$$\Rightarrow \quad -\pi + \pi = \pi \Rightarrow 0 \neq \pi$$

Hence, $x = -1$ does not satisfy the given equation.

Q. 23 The value of $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]$ is

(a) $\frac{3\pi}{5}$ (b) $\frac{-7\pi}{5}$ (c) $\frac{\pi}{10}$ (d) $\frac{-\pi}{10}$

Sol. ***(d)*** We have,

$$\sin^{-1}\left(\cos\frac{33\pi}{5}\right) = \sin^{-1}\left[\cos\left(6\pi + \frac{3\pi}{5}\right)\right] = \sin^{-1}\left[\cos\left(\frac{3\pi}{5}\right)\right] \quad [\because \cos(2n\pi + \theta) = \cos\theta]$$

$$= \sin^{-1}\left[\cos\left(\frac{\pi}{2} + \frac{\pi}{10}\right)\right] = \sin^{-1}\left(-\sin\frac{\pi}{10}\right)$$

$$= -\sin^{-1}\left(\sin\frac{\pi}{10}\right) \quad [\because \sin^{-1}(-x) = -\sin^{-1}x]$$

$$= -\frac{\pi}{10} \quad \left[\because \sin^{-1}(\sin x) = x, x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right]$$

Q. 24 The domain of the function $\cos^{-1}(2x-1)$ is

(a) [0, 1] (b) [−1, 1] (c) (−1, 1) (d) [0, π]

Sol. *(a)* We have, $f(x) = \cos^{-1}(2x-1)$

$$\therefore \quad -1 \le 2x-1 \le 1$$
$$\Rightarrow \quad 0 \le 2x \le 2$$
$$\Rightarrow \quad 0 \le x \le 1$$
$$\therefore \quad x \in [0, 1]$$

Q. 25 The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is

(a) [1, 2] (b) [−1, 1] (c) [0, 1] (d) None of these

Sol. *(a)* $\because \quad f(x) = \sin^{-1}\sqrt{x-1}$

$$\Rightarrow \quad 0 \le x-1 \le 1 \qquad [\because \sqrt{x-1} \ge 0 \text{ and } -1 \le \sqrt{x-1} \le 1]$$
$$\Rightarrow \quad 1 \le x \le 2$$
$$\therefore \quad x \in [1, 2]$$

Q. 26 If $\cos\left(\sin^{-1}\frac{2}{5} + \cos^{-1}x\right) = 0$, then x is equal to

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) 0 (d) 1

Sol. *(b)* We have, $\cos\left(\sin^{-1}\frac{2}{5} + \cos^{-1}x\right) = 0$

$$\Rightarrow \quad \sin^{-1}\frac{2}{5} + \cos^{-1}x = \cos^{-1}0$$
$$\Rightarrow \quad \sin^{-1}\frac{2}{5} + \cos^{-1}x = \cos^{-1}\cos\frac{\pi}{2}$$
$$\Rightarrow \quad \sin^{-1}\frac{2}{5} + \cos^{-1}x = \frac{\pi}{2}$$
$$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}\frac{2}{5}$$
$$\Rightarrow \quad \cos^{-1}x = \cos^{-1}\frac{2}{5} \qquad \left[\because \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}\right]$$
$$\therefore \quad x = \frac{2}{5}$$

Q. 27 The value of $\sin[2\tan^{-1}(0.75)]$ is

(a) 0.75 (b) 1.5 (c) 0.96 (d) sin 1.5

Sol. *(c)* We have, $\sin[2\tan^{-1}(0.75)] = \sin\left(2\tan^{-1}\frac{3}{4}\right) \qquad \left[\because 0.75 = \frac{75}{100} = \frac{3}{4}\right]$

$$= \sin\left(\sin^{-1}\frac{2\cdot\frac{3}{4}}{1+\frac{9}{16}}\right) = \sin\left[\sin^{-1}\frac{3/2}{25/16}\right]$$
$$= \sin\left[\sin^{-1}\left(\frac{48}{50}\right)\right] = \sin\left[\sin^{-1}\left(\frac{24}{25}\right)\right] = \frac{24}{25} = 0.96$$

Q. 28 The value of $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{5\pi}{2}$ (d) $\frac{7\pi}{2}$

Sol. *(a)* We have, $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$

$$= \cos^{-1}\cos\left(2\pi - \frac{\pi}{2}\right) \qquad \left[\because \cos\left(2\pi - \frac{\pi}{2}\right) = \cos\frac{\pi}{2}\right]$$

$$= \cos^{-1}\cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \qquad \{\because \cos^{-1}(\cos x) = x,\ x \in [0, \pi]\}$$

Note *Remember that,* $\cos^{-1}\left(\cos\frac{3\pi}{2}\right) \neq \frac{3\pi}{2}$

$\because$ $\frac{3\pi}{2} \notin (0, \pi)$

Q. 29 The value of $2\sec^{-1} 2 + \sin^{-1}\left(\frac{1}{2}\right)$ is

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{7\pi}{6}$ (d) 1

Sol. *(b)* We have, $2\sec^{-1}2 + \sin^{-1}\frac{1}{2} = 2\sec^{-1}\sec\frac{\pi}{3} + \sin^{-1}\sin\frac{\pi}{6}$

$$= 2\cdot\frac{\pi}{3} + \frac{\pi}{6} \qquad [\because \sec^{-1}(\sec x) = x \text{ and } \sin^{-1}(\sin x) = x]$$

$$= \frac{4\pi + \pi}{6} = \frac{5\pi}{6}$$

Q. 30 If $\tan^{-1} x + \tan^{-1} y = \frac{4\pi}{5}$, then $\cot^{-1} x + \cot^{-1} y$ equals to

(a) $\frac{\pi}{5}$ (b) $\frac{2\pi}{5}$ (c) $\frac{3\pi}{5}$ (d) π

Sol. *(a)* We have, $\tan^{-1} x + \tan^{-1} y = \frac{4\pi}{5}$

$$\Rightarrow \quad \frac{\pi}{2} - \cot^{-1} x + \frac{\pi}{2} - \cot^{-1} y = \frac{4\pi}{5}$$

$$\Rightarrow \quad -(\cot^{-1} x + \cot^{-1} y) = \frac{4\pi}{5} - \pi \qquad \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$$

$$\Rightarrow \quad \cot^{-1} x + \cot^{-1} y = -\left(-\frac{\pi}{5}\right)$$

$$\Rightarrow \quad \cot^{-1} x + \cot^{-1} y = \frac{\pi}{5}$$

Q. 31 If $\sin^{-1}\left(\dfrac{2a}{1+a^2}\right)+\cos^{-1}\left(\dfrac{1-a^2}{1+a^2}\right)=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, where $a, x \in]0, 1[$, then the value of x is

(a) 0 (b) $\dfrac{a}{2}$ (c) a (d) $\dfrac{2a}{1-a^2}$

Sol. *(d)* We have, $\sin^{-1}\left(\dfrac{2a}{1+a^2}\right)+\cos^{-1}\left(\dfrac{1-a^2}{1+a^2}\right)=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

Let $a=\tan\theta \Rightarrow \theta=\tan^{-1}a$

$$\therefore \quad \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\tan^{-1}\frac{2x}{1-x^2}$$

$$\Rightarrow \quad \sin^{-1}\sin 2\theta+\cos^{-1}\cos 2\theta=\tan^{-1}\frac{2x}{1-x^2}$$

$$\Rightarrow \quad 2\theta+2\theta=\tan^{-1}\frac{2x}{1-x^2}$$

$$\Rightarrow \quad 4\tan^{-1}a=\tan^{-1}\frac{2x}{1-x^2}$$

$$\Rightarrow \quad 2\cdot 2\tan^{-1}a=\tan^{-1}\frac{2x}{1-x^2}$$

$$\Rightarrow \quad 2\cdot\tan^{-1}\frac{2a}{1-a^2}=\tan^{-1}\frac{2x}{1-x^2} \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow \quad \tan^{-1}\frac{2\cdot\left(\dfrac{2a}{1-a^2}\right)}{1-\left(\dfrac{2a}{1-a^2}\right)^2}=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\therefore \quad x=\frac{2a}{1-a^2}$$

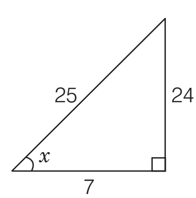
Q. 32 The value of $\cot\left[\cos^{-1}\left(\dfrac{7}{25}\right)\right]$ is

(a) $\dfrac{25}{24}$ (b) $\dfrac{25}{7}$ (c) $\dfrac{24}{25}$ (d) $\dfrac{7}{24}$

Sol. *(d)* We have, $\cot\left[\cos^{-1}\left(\dfrac{7}{25}\right)\right]$

Let $\cos^{-1}\dfrac{7}{25}=x$

$$\Rightarrow \quad \cos x=\frac{7}{25}$$



$$\therefore \quad \sin x=\sqrt{1-\cos^2 x}=\sqrt{1-\left(\frac{7}{25}\right)^2}$$

$$=\sqrt{\frac{625-49}{625}}=\frac{24}{25}$$

$$\therefore \qquad \cot x = \frac{\cos x}{\sin x} = \frac{\frac{7}{25}}{\frac{24}{25}} = \frac{7}{24} \qquad \text{...(i)}$$

$$\Rightarrow \qquad x = \cot^{-1}\left(\frac{7}{24}\right) = \cos^{-1}\left(\frac{7}{25}\right)$$

$$\therefore \qquad \cot\left(\cos^{-1}\frac{7}{25}\right) = \cot\left(\cot^{-1}\frac{7}{24}\right) = \frac{7}{24} \qquad \left[\because \cot^{-1}\frac{7}{24} = \cos^{-1}\frac{7}{25}\right]$$

Q. 33 The value of $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)$ is

(a) $2+\sqrt{5}$ (b) $\sqrt{5}-2$ (c) $\frac{\sqrt{5}+2}{2}$ (d) $5+\sqrt{2}$

Sol. *(b)* We have, $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)$

Let $\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}} = \theta$

$$\Rightarrow \qquad \cos^{-1}\frac{2}{\sqrt{5}} = 2\theta \Rightarrow \cos 2\theta = \frac{2}{\sqrt{5}}$$

$$\therefore \qquad (1-2\sin^2\theta) = \frac{2}{\sqrt{5}}$$

$$\Rightarrow \qquad 2\sin^2\theta = 1-\frac{2}{\sqrt{5}}$$

$$\Rightarrow \qquad \sin^2\theta = \frac{1}{2}-\frac{1}{\sqrt{5}}$$

$$\Rightarrow \qquad \sin\theta = \sqrt{\frac{1}{2}-\frac{1}{\sqrt{5}}}$$

$$\therefore \qquad \cos^2\theta = 1-\sin^2\theta$$

$$= 1-\frac{1}{2}+\frac{1}{\sqrt{5}} = \frac{1}{2}+\frac{1}{\sqrt{5}}$$

$$\Rightarrow \qquad \cos\theta = \sqrt{\frac{1}{2}+\frac{1}{\sqrt{5}}}$$

$$\therefore \qquad \tan\theta = \sqrt{\frac{\frac{1}{2}-\frac{1}{\sqrt{5}}}{\frac{1}{2}+\frac{1}{\sqrt{5}}}} = \sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}} \qquad \left[\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

$$\Rightarrow \qquad \theta = \tan^{-1}\sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}} = \frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}$$

$$\therefore \qquad \tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right) = \tan\tan^{-1}\sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}}$$

$$= \sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}\cdot\frac{\sqrt{5}-2}{\sqrt{5}-2}}$$

$$= \sqrt{\frac{(\sqrt{5}-2)^2}{5-4}} = \sqrt{5}-2$$

Q. 34 If $|x| \le 1$, then $2\tan^{-1} x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is equal to

(a) $4\tan^{-1} x$ (b) 0 (c) $\frac{\pi}{2}$ (d) π

Sol. *(a)* We have, $2\tan^{-1} x + \sin^{-1}\frac{2x}{1+x^2}$

Let $x = \tan\theta$

$\therefore$ $2\tan^{-1}\tan\theta + \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$ $[\because \tan^{-1}(\tan x) = x]$

$= 2\theta + \sin^{-1}\sin 2\theta$ $\left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$

$= 2\theta + 2\theta$ $[\because \sin^{-1}(\sin x) = x]$

$= 4\theta$ $[\because \theta = \tan^{-1} x]$

$= 4\tan^{-1} x$

Q. 35 If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, then $\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$ equals to

(a) 0 (b) 1 (c) 6 (d) 12

Sol. *(c)* We have, $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$

We know that, $0 \le \cos^{-1} x \le \pi$

$\Rightarrow$ $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$

If and only if, $\cos^{-1}\alpha = \cos^{-1}\beta = \cos^{-1}\gamma = \pi$

$\Rightarrow$ $\cos\pi = \alpha = \beta = \gamma$

$\Rightarrow$ $-1 = \alpha = \beta = \gamma$

$\Rightarrow$ $\alpha = \beta = \gamma = -1$

$\therefore$ $\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$

$= -1(-1-1) - 1(-1-1) - 1(-1-1)$

$= 2 + 2 + 2 = 6$

Q. 36 The number of real solutions of the equation

$$\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x) \text{ in } \left[\frac{\pi}{2}, \pi\right] \text{ is}$$

(a) 0 (b) 1 (c) 2 (d) ∞

Sol. *(a)* We have, $\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x), \left[\frac{\pi}{2}, \pi\right]$

$\Rightarrow$ $\sqrt{1+2\cos^2 x - 1} = \sqrt{2}\cos^{-1}(\cos x)$

$\Rightarrow$ $\sqrt{2}\cos x = \sqrt{2}\cos^{-1}(\cos x)$

$\Rightarrow$ $\cos x = \cos^{-1}(\cos x)$

$\Rightarrow$ $\cos x = x$ $[\because \cos^{-1}(\cos x) = x]$

which is not true for any real value of x.

Hence, there is no solution possible for the given equation.

Q. 37 If $\cos^{-1} x > \sin^{-1} x$, then

(a) $\frac{1}{\sqrt{2}} < x \le 1$ (b) $0 \le x < \frac{1}{\sqrt{2}}$ (c) $-1 \le x < \frac{1}{\sqrt{2}}$ (d) $x > 0$

Sol. *(c)* We have, $\cos^{-1} x > \sin^{-1} x$, where $x \in [-1, 1]$

$\Rightarrow \quad x < \cos(\sin^{-1} x)$

$\Rightarrow \quad x < \cos[\cos^{-1}\sqrt{1-x^2}] \qquad \left[\text{let } \sin^{-1} x = \theta \Rightarrow \sin\theta = \frac{x}{1}\right]$

$\left[\because \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2} \Rightarrow \theta = \cos^{-1}\sqrt{1-x^2}\right]$

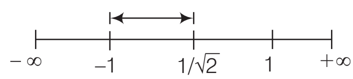
$\Rightarrow \quad x < \sqrt{1-x^2}$

$\Rightarrow \quad x^2 < 1 - x^2 \Rightarrow 2x^2 < 1$

$\Rightarrow \quad x^2 < \frac{1}{2} \Rightarrow x < \pm\left(\frac{1}{\sqrt{2}}\right) \qquad \text{...(i)}$

Also, $-1 \le x \le 1 \qquad \text{...(ii)}$

$\therefore \quad -1 \le x \le \frac{1}{\sqrt{2}}$



Alternate Method

$$\frac{\pi}{2} - \sin^{-1} x > \sin^{-1} x$$

$$\frac{\pi}{2} > 2\sin^{-1} x \Rightarrow \frac{\pi}{4} > \sin^{-1} x$$

$$\frac{1}{\sqrt{2}} > x \Rightarrow \frac{1}{\sqrt{2}} < x \le 1$$

We know that, $\sin^{-1} x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Fillers

Q. 38 The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is ....

Sol. $\because \quad 0 \le \cos^{-1} x \le \pi$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$$

Q. 39 The value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ is ....

Sol. $\because \quad -\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$\therefore \quad \sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\sin\left(\pi - \frac{2\pi}{5}\right) = \sin^{-1}\left(\sin\frac{2\pi}{5}\right) = \frac{2\pi}{5}$

Q. 40 If $\cos(\tan^{-1} x + \cot^{-1}\sqrt{3}) = 0$, then the value of x is ....

Sol. We have, $\cos(\tan^{-1} x + \cot^{-1}\sqrt{3}) = 0$

$\Rightarrow \tan^{-1} x + \cot^{-1}\sqrt{3} = \cos^{-1} 0$

$\Rightarrow \tan^{-1} x + \cot^{-1}\sqrt{3} = \cos^{-1}\cos\frac{\pi}{2}$

$\Rightarrow \tan^{-1} x + \cot^{-1}\sqrt{3} = \frac{\pi}{2}$

$\Rightarrow \tan^{-1} x = \frac{\pi}{2} - \cot^{-1}\sqrt{3}$

$\Rightarrow \tan^{-1} x = \tan^{-1}\sqrt{3}$ $\qquad \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$

$\therefore x = \sqrt{3}$

Q. 41 The set of values of $\sec^{-1}\frac{1}{2}$ is ....

Sol. Since, domain of $\sec^{-1} x$ is $R - (-1, 1)$.

$\Rightarrow (-\infty, -1] \cup [1, \infty)$

So, there is no set of values exist for $\sec^{-1}\frac{1}{2}$.

So, ϕ is the answer.

Q. 42 The principal value of $\tan^{-1}\sqrt{3}$ is ....

Sol. $\because \tan^{-1}\sqrt{3} = \tan^{-1}\tan\left(\frac{\pi}{3}\right)$

$= \left(\frac{\pi}{3}\right)$ $\qquad \left[\because \tan^{-1}(\tan x) = x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$

Q. 43 The value of $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$ is ....

Sol. We have, $\cos^{-1}\left(\cos\frac{14\pi}{3}\right) = \cos^{-1}\cos\left(4\pi + \frac{2\pi}{3}\right)$

$= \cos^{-1}\cos\frac{2\pi}{3}$ $\qquad [\because \cos(2n\pi + \theta) = \cos\theta]$

$= \frac{2\pi}{3}$ $\qquad \{\because \cos^{-1}(\cos x) = x, x \in [0, \pi]\}$

Note *Remember that,* $\cos^{-1}\left(\cos\frac{14\pi}{3}\right) \neq \frac{14\pi}{3}$

Since, $\frac{14\pi}{3} \notin [0, \pi]$

Q. 44 The value of $\cos(\sin^{-1} x + \cos^{-1} x)$, where $|x| \le 1$, is ....

Sol. $\cos(\sin^{-1} x + \cos^{-1} x)$

$= \cos\frac{\pi}{2} = 0$ $\qquad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$

Q. 45 The value of $\tan\left(\frac{\sin^{-1} x + \cos^{-1} x}{2}\right)$, when $x = \frac{\sqrt{3}}{2}$, is ....

Sol. $\because$ $$\tan\left(\frac{\sin^{-1} x + \cos^{-1} x}{2}\right) = \tan\left(\frac{\pi/2}{2}\right) \quad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$$

$$= \tan\frac{\pi}{4} = 1$$

Q. 46 If $y = 2\tan^{-1} x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then .... $< y <$ ....

Sol. We have, $$y = 2\tan^{-1} x + \sin^{-1}\frac{2x}{1+x^2}$$

$\therefore$ $$y = 2\tan^{-1}\tan\theta + \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} \quad [\text{let } x = \tan\theta]$$

$\Rightarrow$ $$y = 2\theta + \sin^{-1}\sin 2\theta \quad \left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$\Rightarrow$ $$y = 2\theta + 2\theta = 4\theta \quad [\because \theta = \tan^{-1} x]$$

$\Rightarrow$ $$y = 4\tan^{-1} x$$

$\because$ $$-\pi/2 < \tan^{-1} x < \pi/2$$

$\therefore$ $$-\frac{4\pi}{2} < 4\tan^{-1} x < 4\pi/2$$

$\Rightarrow$ $$-2\pi < 4\tan^{-1} x < 2\pi$$

$\Rightarrow$ $$-2\pi < y < 2\pi \quad [\because y = 4\tan^{-1} x]$$

Q. 47 The result $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ is true when the value of xy is ....

Sol. We know that, $$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

where, $$xy > -1$$

Q. 48 The value of $\cot^{-1}(-x)$ $x \in R$ in terms of $\cot^{-1} x$ is ....

Sol. We know that,

$$\cot^{-1}(-x) = \pi - \cot^{-1} x, \; x \in R$$

True/False

Q. 49 All trigonometric functions have inverse over their respective domains.

Sol. *False*

We know that, all trigonometric functions have inverse over their restricted domains.

Q. 50 The value of the expression $(\cos^{-1} x)^2$ is equal to $\sec^2 x$.

Sol. ***False***

$$\because \qquad [\cos^{-1} x]^2 = \left[\sec^{-1}\frac{1}{x}\right]^2 \neq \sec^2 x$$

Q. 51 The domain of trigonometric functions can be restricted to any one of their branch (not necessarily principal value) in order to obtain their inverse functions.

Sol. ***True***

We know that, the domain of trigonometric functions are restricted in their domain to obtain their inverse functions.

Q. 52 The least numerical value, either positive or negative of angle θ is called principal value of the inverse trigonometric function.

Sol. ***True***

We know that, the smallest numerical value, either positive or negative of θ is called the principal value of the function.

Q. 53 The graph of inverse trigonometric function can be obtained from the graph of their corresponding function by interchanging X and Y-axes.

Sol. ***True***

We know that, the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (*i.e.,* reflection) along the line $y = x$.

Q. 54 The minimum value of n for which $\tan^{-1}\frac{n}{\pi} > \frac{\pi}{4}$, $n \in N$, is valid is 5.

Sol. ***False***

$$\because \qquad \tan^{-1}\frac{n}{\pi} > \frac{\pi}{4} \Rightarrow \frac{n}{\pi} > \tan\frac{\pi}{4}$$

$$\Rightarrow \qquad \frac{n}{\pi} > 1 \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$\Rightarrow \qquad n > \pi$$

So, the minimum value of n is 4. $[\because n \in N \text{ and } \pi = 3.14\ldots]$

Q. 55 The principal value of $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right]$ is $\frac{\pi}{3}$.

Sol. ***True***

Given that,
$$\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right] = \sin^{-1}\left[\cos \sin^{-1}\left(\sin\frac{\pi}{6}\right)\right]$$

$$= \sin^{-1}\left[\cos\frac{\pi}{6}\right] \qquad [\because \sin^{-1}(\sin x) = x]$$

$$= \sin^{-1}\frac{\sqrt{3}}{2}$$

$$= \sin^{-1}\sin\frac{\pi}{3} = \frac{\pi}{3}$$

1

Relations and Functions

Short Answer Type Questions

Q. 1 Let $A = \{a, b, c\}$ and the relation R be defined on A as follows

$$R = \{(a, a), (b, c), (a, b)\}$$

Then, write minimum number of ordered pairs to be added in R to make R reflexive and transitive.

Sol. Given relation, $R = \{(a, a), (b, c), (a, b)\}$.

To make R is reflexive we must add (b, b) and (c, c) to R. Also, to make R is transitive we must add (a, c) to R.

So, minimum number of ordered pair is to be added are (b, b), (c, c), (a, c).

Q. 2 Let D be the domain of the real valued function f defined by $f(x) = \sqrt{25 - x^2}$. Then, write D.

Sol. Given function is, $f(x) = \sqrt{25 - x^2}$

For real valued of $f(x)$ $\quad 25 - x^2 \geq 0$

$$x^2 \leq 25$$

$$-5 \leq x \leq +5$$

$\therefore \quad D = [-5, 5]$

Q. 3 If $f, g : R \to R$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$, $\forall\ x \in R$, respectively. Then, find gof.

Thinking Process

If $f, g : R \to R$ be two functions, then $gof(x) = g\{f(x)\}\ \forall x \in R$.

Sol. Given that, $\quad f(x) = 2x + 1$ and $g(x) = x^2 - 2$, $\forall x \in R$

$\therefore \quad gof = g\{f(x)\}$

$$= g(2x + 1) = (2x + 1)^2 - 2$$

$$= 4x^2 + 4x + 1 - 2$$

$$= 4x^2 + 4x - 1$$

Q. 4 Let $f: R \to R$ be the function defined by $f(x) = 2x - 3, \forall\, x \in R$. Write f^{-1}.

Sol. Given that, $f(x) = 2x - 3, \forall\, x \in R$

Now, let

$$y = 2x - 3$$
$$2x = y + 3$$
$$x = \frac{y+3}{2}$$

$\therefore$ $$f^{-1}(x) = \frac{x+3}{2}$$

Q. 5 If $A = \{a, b, c, d\}$ and the function $f = \{(a, b), (b, d), (c, a), (d, c)\}$, write f^{-1}.

Sol. Given that, $A = \{a, b, c, d\}$

and $f = \{(a, b), (b, d), (c, a), (d, c)\}$

$$f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$$

Q. 6 If $f: R \to R$ is defined by $f(x) = x^2 - 3x + 2$, write $f\{f(x)\}$.

Thinking Process

To solve this problem use the formula i.e., $(a + b + c)^2 = (a^2 + b^2 + c^2 + 2ab + 2bc + 2ca)$

Sol. Given that, $f(x) = x^2 - 3x + 2$

$\therefore$
$$\begin{aligned} f\{f(x)\} &= f(x^2 - 3x + 2) \\ &= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2 \\ &= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2 \\ &= x^4 + 10x^2 - 6x^3 - 3x \\ f\{f(x)\} &= x^4 - 6x^3 + 10x^2 - 3x \end{aligned}$$

Q. 7 Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? If g is described by $g(x) = \alpha x + \beta$, then what value should be assigned to α and β?

Sol. Given that, $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$.

Here, each element of domain has unique image. So, g is a function.

Now given that, $g(x) = \alpha x + \beta$

$$g(1) = \alpha + \beta$$
$$\alpha + \beta = 1 \quad \ldots\text{(i)}$$
$$g(2) = 2\alpha + \beta$$
$$2\alpha + \beta = 3 \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii),

$$2(1 - \beta) + \beta = 3$$
$$\Rightarrow \quad 2 - 2\beta + \beta = 3$$
$$\Rightarrow \quad 2 - \beta = 3$$
$$\beta = -1$$

If $\beta = -1$, then $\alpha = 2$

$$\alpha = 2,\ \beta = -1$$

Q. 8 **Are the following set of ordered pairs functions? If so examine whether the mapping is injective or surjective.**

(i) $\{(x, y) : x$ is a person, y is the mother of $x\}$.

(ii) $\{(a, b) : a$ is a person, b is an ancestor of $a\}$.

Sol. **(i)** Given set of ordered pair is $\{(x, y) : x$ is a person, y is the mother of $x\}$.

It represent a function. Here, the image of distinct elements of x under f are not distinct, so it is not a injective but it is a surjective.

(ii) Set of ordered pairs $= \{(a, b) : a$ is a person, b is an ancestor of $a\}$

Here, each element of domain does not have a unique image. So, it does not represent function.

Q. 9 **If the mappings f and g are given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$, write fog.**

Sol. Given that, $f = \{(1, 2), (3, 5), (4, 1)\}$

and $g = \{(2, 3), (5, 1), (1, 3)\}$

Now,

$$fog(2) = f\{g(2)\} = f(3) = 5$$

$$fog(5) = f\{g(5)\} = f(1) = 2$$

$$fog(1) = f\{g(1)\} = f(3) = 5$$

$$fog = \{(2, 5), (5, 2), (1, 5)\}$$

Q. 10 **Let C be the set of complex numbers. Prove that the mapping $f : C \to R$ given by $f(z) = |z|, \forall z \in C$, is neither one-one nor onto.**

Sol. The mapping $f : C \to R$

Given,

$$f(z) = |z|, \forall z \in C$$

$$f(1) = |1| = 1$$

$$f(-1) = |-1| = 1$$

$$f(1) = f(-1)$$

But $1 \neq -1$

So, $f(z)$ is not one-one. Also, $f(z)$ is not onto as there is no pre-image for any negative element of R under the mapping $f(z)$.

Q. 11 **Let the function $f : R \to R$ be defined by $f(x) = \cos x, \forall x \in R$. Show that f is neither one-one nor onto.**

Sol. Given function, $f(x) = \cos x, \forall x \in R$

Now,

$$f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$\Rightarrow$

$$f\left(\frac{-\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$\Rightarrow$

$$f\left(\frac{\pi}{2}\right) = f\left(\frac{-\pi}{2}\right)$$

But

$$\frac{\pi}{2} \neq \frac{-\pi}{2}$$

So, $f(x)$ is not one-one.

Now, $f(x) = \cos x, \forall x \in R$ is not onto as there is no pre-image for any real number. Which does not belonging to the intervals $[-1, 1]$, the range of $\cos x$.

Q. 12 Let $X = \{1, 2, 3\}$ and $Y = \{4, 5\}$. Find whether the following subsets of $X \times Y$ are functions from X to Y or not.

(i) $f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$ (ii) $g = \{(1, 4), (2, 4), (3, 4)\}$

(iii) $h = \{(1, 4), (2, 5), (3, 5)\}$ (iv) $k = \{(1, 4), (2, 5)\}$

Sol. Given that, $X = \{1, 2, 3\}$ and $Y = \{4, 5\}$

$$X \times Y = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$$

(i) $f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$

f is not a function because f has not unique image.

(ii) $g = \{(1, 4), (2, 4), (3, 4)\}$

Since, g is a function as each element of the domain has unique image.

(iii) $h = \{(1, 4), (2, 5), (3, 5)\}$

It is clear that h is a function.

(iv) $k = \{(1, 4), (2, 5)\}$

k is not a function as 3 has not any image under the mapping.

Q. 13 If functions $f : A \to B$ and $g : B \to A$ satisfy $gof = I_A$, then show that f is one-one and g is onto.

Sol. Given that,

$$f : A \to B \text{ and } g : B \to A \text{ satisfy } gof = I_A$$

$\because$ $gof = I_A$

$\Rightarrow$ $gof\{f(x_1)\} = gof\{f(x_2)\}$

$\Rightarrow$ $g(x_1) = g(x_2)$ $[\because gof = I_A]$

$\therefore$ $x_1 = x_2$

Hence, f is one-one and g is onto.

Q. 14 Let $f : R \to R$ be the function defined by $f(x) = \dfrac{1}{2 - \cos x}$, $\forall\ x \in R$. Then, find the range of f.

Thinking Process

Range of $f = \{y \in Y : y = f(x) : \text{for some in } x\}$ and use range of $\cos x$ is $[-1, 1]$

Sol. Given function, $f(x) = \dfrac{1}{2 - \cos x}$, $\forall\ x \in R$

Let $y = \dfrac{1}{2 - \cos x}$

$\Rightarrow$ $2y - y\cos x = 1$

$\Rightarrow$ $y\cos x = 2y - 1$

$\Rightarrow$ $\cos x = \dfrac{2y - 1}{y} = 2 - \dfrac{1}{y}$ $\Rightarrow$ $\cos x = 2 - \dfrac{1}{y}$

$\Rightarrow$ $-1 \le \cos x \le 1$ $\Rightarrow$ $-1 \le 2 - \dfrac{1}{y} \le 1$

$\Rightarrow$ $-3 \le -\dfrac{1}{y} \le -1$ $\Rightarrow$ $1 \le \dfrac{1}{y} \le 3$

$\Rightarrow$ $\dfrac{1}{3} \le \dfrac{1}{y} \le 1$

So, y range is $\left[\dfrac{1}{3}, 1\right]$.

Q. 15 Let n be a fixed positive integer. Define a relation R in Z as follows $\forall\ a, b \in Z$, aRb if and only if $a - b$ is divisible by n. Show that R is an equivalence relation.

Sol. Given that, $\forall\ a, b \in Z$, aRb if and only if $a - b$ is divisible by n.

Now,

I. **Reflexive**

$aRa \Rightarrow (a - a)$ is divisible by n, which is true for any integer a as 'O' is divisible by n.

Hence, R is reflexive.

II. **Symmetric**

$$aRb$$
$\Rightarrow$ $a - b$ is divisible by n.
$\Rightarrow$ $-b + a$ is divisible by n.
$\Rightarrow$ $-(b - a)$ is divisible by n.
$\Rightarrow$ $(b - a)$ is divisible by n.
$\Rightarrow$ bRa

Hence, R is symmetric.

III. **Transitive**

Let aRb and bRc

$\Rightarrow$ $(a - b)$ is divisible by n and $(b - c)$ is divisible by n
$\Rightarrow$ $(a - b) + (b - c)$ is divisibly by n
$\Rightarrow$ $(a - c)$ is divisible by n
$\Rightarrow$ aRc

Hence, R is transitive.

So, R is an equivalence relation.

Long Answer Type Questions

Q. 16 If $A = \{1, 2, 3, 4\}$, define relations on A which have properties of being

(i) reflexive, transitive but not symmetric.
(ii) symmetric but neither reflexive nor transitive.
(iii) reflexive, symmetric and transitive.

Sol. Given that, $A = \{1, 2, 3, 4\}$

(i) Let $R_1 = \{(1, 1), (1, 2), (2, 3), (2, 2), (1, 3), (3, 3)\}$

R_1 is reflexive, since, (1, 1) (2, 2) (3, 3) lie in R_1.

Now, $(1, 2) \in R_1, (2, 3) \in R_1 \Rightarrow (1, 3) \in R_1$

Hence, R_1 is also transitive but $(1, 2) \in R_1 \Rightarrow (2, 1) \notin R_1$.

So, it is not symmetric.

(ii) Let $R_2 = \{(1, 2), (2, 1)\}$

Now, $(1, 2) \in R_2, (2, 1) \in R_2$

So, it is symmetric.

(iii) Let $R_3 = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3), (1, 3), (3, 1), (2, 3)\}$

Hence, R_3 is reflexive, symmetric and transitive.

Q. 17 **Let R be relation defined on the set of natural number N as follows, $R = \{(x, y): x \in N, y \in N, 2x + y = 41\}$. Find the domain and range of the relation R. Also verify whether R is reflexive, symmetric and transitive.**

Sol. Given that, $R = \{(x, y): x \in N, y \in N, 2x + y = 41\}$.

$$\text{Domain} = \{1, 2, 3, \ldots, 20\}$$
$$\text{Range} = \{1, 3, 5, 7, \ldots, 39\}$$
$$R = \{(1, 39), (2, 37), (3, 35), \ldots, (19, 3), (20, 1)\}$$

R is not reflexive as $(2, 2) \notin R$

$$2 \times 2 + 2 \neq 41$$

So, R is not symmetric.

As $(1, 39) \in R$ but $(39, 1) \notin R$

So, R is not transitive.

As $(11, 19) \in R, (19, 3) \in R$

But $(11, 3) \notin R$

Hence, R is neither reflexive, nor symmetric and nor transitive.

Q. 18 **Given, $A = \{2, 3, 4\}$, $B = \{2, 5, 6, 7\}$. Construct an example of each of the following**

(i) an injective mapping from A to B.

(ii) a mapping from A to B which is not injective.

(iii) a mapping from B to A.

Sol. Given that, $A = \{2, 3, 4\}, B = \{2, 5, 6, 7\}$

(i) Let $f : A \to B$ denote a mapping

$$f = \{(x, y): y = x + 3\}$$

i.e., $f = \{(2, 5), (3, -6), (4, 7)\}$, which is an injective mapping.

(ii) Let $g : A \to B$ denote a mapping such that $g = \{(2, 2), (3, 5), (4, 5)\}$, which is not an injective mapping.

(iii) Let $h : B \to A$ denote a mapping such that $h = \{(2, 2), (5, 3), (6, 4), (7, 4)\}$, which is a mapping from B to A.

Q. 19 **Give an example of a map**

(i) which is one-one but not onto.

(ii) which is not one-one but onto.

(iii) which is neither one-one nor onto.

Sol. **(i)** Let $f : N \to N$, be a mapping defined by $f(x) = 2x$ which is one-one.

For $f(x_1) = f(x_2)$

$\Rightarrow$ $2x_1 = 2x_2$

$$x_1 = x_2$$

Further f is not onto, as for $1 \in N$, there does not exist any x in N such that $f(x) = 2x + 1$.

(ii) Let $f : N \to N$, given by $f(1) = f(2) = 1$ and $f(x) = x - 1$ for every $x > 2$ is onto but not one-one. f is not one-one as $f(1) = f(2) = 1$. But f is onto.

(iii) The mapping $f : R \to R$ defined as $f(x) = x^2$, is neither one-one nor onto.

Q. 20 Let $A = R - \{3\}$, $B = R - \{1\}$. If $f : A \to B$ be defined by $f(x) = \dfrac{x-2}{x-3}$, $\forall\ x \in A$. Then, show that f is bijective.

Thinking Process

A function $f : x \to y$ is said to be bijective, if f is both one-one and onto.

Sol. Given that, $A = R - \{3\},\ B = R - \{1\}$.

$f : A \to B$ is defined by $f(x) = \dfrac{x-2}{x-3}, \forall\ x \in A$

For injectivity

Let $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1 - 2}{x_1 - 3} = \dfrac{x_2 - 2}{x_2 - 3}$

$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$

$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$

$\Rightarrow -3x_1 - 2x_2 = -3x_2 - 2x_1$

$\Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$

So, $f(x)$ is an injective function.

For surjectivity

Let $y = \dfrac{x-2}{x-3} \Rightarrow x - 2 = xy - 3y$

$\Rightarrow x(1-y) = 2 - 3y \Rightarrow x = \dfrac{2-3y}{1-y}$

$\Rightarrow x = \dfrac{3y-2}{y-1} \in A, \forall\ y \in B$ [codomain]

So, $f(x)$ is surjective function.

Hence, $f(x)$ is a bijective function.

Q. 21 Let $A = [-1, 1]$, then, discuss whether the following functions defined on A are one-one onto or bijective.

(i) $f(x) = \dfrac{x}{2}$ (ii) $g(x) = |x|$

(iii) $h(x) = x|x|$ (iv) $k(x) = x^2$

Sol. Given that, $A = [-1, 1]$

(i) $f(x) = \dfrac{x}{2}$

Let $f(x_1) = f(x_2)$

$\Rightarrow \dfrac{x_1}{2} = \dfrac{x_2}{2} \Rightarrow x_1 = x_2$

So, $f(x)$ is one-one.

Now, let $y = \dfrac{x}{2}$

$\Rightarrow x = 2y \notin A, \forall\ y \in A$

As for $y = 1 \in A, x = 2 \notin A$

So, $f(x)$ is not onto.

Also, $f(x)$ is not bijective as it is not onto.

(ii) $g(x) = |x|$

Let $\quad g(x_1) = g(x_2)$

$\Rightarrow \quad |x_1| = |x_2| \Rightarrow x_1 = \pm x_2$

So, $g(x)$ is not one-one.

Now, $\quad y = |x| \Rightarrow x = \pm y \notin A, \forall y \in A$

So, $g(x)$ is not onto, also, $g(x)$ is not bijective.

(iii) $h(x) = x|x|$

Let $\quad h(x_1) = h(x_2)$

$\Rightarrow \quad x_1|x_1| = x_2|x_2| \Rightarrow x_1 = x_2$

So, $h(x)$ is one-one.

Now, let $\quad y = x|x|$

$\Rightarrow \quad y = x^2 \in A, \forall x \in A$

So, $h(x)$ is onto also, $h(x)$ is a bijective.

(iv) $k(x) = x^2$

Let $\quad k(x_1) = k(x_2)$

$\Rightarrow \quad x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$

Thus, $k(x)$ is not one-one.

Now, let $\quad y = x^2$

$\Rightarrow \quad x = \sqrt{y} \notin A, \forall y \in A$

As for $y = -1$, $x = \sqrt{-1} \notin A$

Hence, $k(x)$ is neither one-one nor onto.

Q. 22 Each of the following defines a relation of N

(i) x is greater than y, $x, y \in N$.

(ii) $x + y = 10$, $x, y \in N$.

(iii) xy is square of an integer $x, y \in N$.

(iv) $x + 4y = 10$, $x, y \in N$

Determine which of the above relations are reflexive, symmetric and transitive.

Sol. (i) x is greater than y, $x, y \in N$

$(x, x) \in R$

For $xRx \quad x > x$ is not true for any $x \in N$.

Therefore, $\quad R$ is not reflexive.

Let $\quad (x, y) \in R \Rightarrow xRy$

$x > y$

but $y > x$ is not true for any $x, y \in N$

Thus, R is not symmetric.

Let $\quad xRy$ and yRz

$x > y$ and $y > z \Rightarrow x > z$

$\Rightarrow \quad xRz$

So, R is transitive.

(ii) $x + y = 10, x, y \in N$

$$R = \{(x, y); x + y = 10, x, y \in N\}$$

$$R = \{(1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)\} \ (1, 1) \notin R$$

So, R is not reflexive.

$$(x, y) \in R \Rightarrow (y, x) \in R$$

Therefore, R is symmetric.

$$(1, 9) \in R, (9, 1) \in R \Rightarrow (1, 1) \notin R$$

Hence, R is not transitive.

(iii) Given xy, is square of an integer $x, y \in N$.

$\Rightarrow$ $R = \{(x, y) : xy \text{ is a square of an integer } x, y \in N\}$

$$(x, x) \in R, \forall x \in N$$

As x^2 is square of an integer for any $x \in N$.

Hence, R is reflexive.

If $(x, y) \in R \Rightarrow (y, x) \in R$

Therefore, R is symmetric.

If $(x, y) \in R, (y, z) \in R$

So, xy is square of an integer and yz is square of an integer.

Let $xy = m^2$ and $yz = n^2$ for some $m, n \in Z$

$$x = \frac{m^2}{y} \text{ and } z = \frac{x^2}{y}$$

$$xz = \frac{m^2n^2}{y^2}, \text{ which is square of an integer.}$$

So, R is transitive.

(iv) $x + 4y = 10, x, y \in N$

$$R = \{(x, y) : x + 4y = 10, x, y \in N\}$$

$$R = \{(2, 2), (6, 1)\}$$

$$(1, 1), (3, 3), \ldots, \notin R$$

Thus, R is not reflexive.

$$(6, 1) \in R \text{ but } (1, 6) \notin R$$

Hence, R is not symmetric.

$$(x, y) \in R \Rightarrow x + 4y = 10 \text{ but } (y, z) \in R$$

$$y + 4z = 10 \Rightarrow (x, z) \in R$$

So, R is transitive.

Q. 23 Let $A = \{1, 2, 3, \ldots, 9\}$ and R be the relation in $A \times A$ defined by $(a, b) R (c, d)$ if $a + d = b + c$ for $(a, b), (c, d)$ in $A \times A$. Prove that R is an equivalence relation and also obtain the equivalent class $[(2, 5)]$.

Sol. Given that, $A = \{1, 2, 3, \ldots, 9\}$ and $(a, b) R (c, d)$ if $a + d = b + c$ for $(a, b) \in A \times A$ and $(c, d) \in A \times A$.

Let $(a, b) R (a, b)$

$\Rightarrow$ $a + b = b + a, \forall a, b \in A$

which is true for any $a, b \in A$.

Hence, R is reflexive.

Let $(a, b) R (c, d)$ $\quad a + d = b + c$

$$c + b = d + a \Rightarrow (c, d) R (a, b)$$

So, R is symmetric.

Let $\quad (a, b)\, R\, (c, d)$ and $(c, d)\, R\, (e, f)$

$$a + d = b + c \text{ and } c + f = d + e$$
$$a + d = b + c \text{ and } d + e = c + f$$
$$(a + d) - (d + e) = (b + c) - (c + f)$$
$$(a - e) = b - f$$
$$a + f = b + e$$
$$(a, b)\, R\, (e, f)$$

So, R is transitive.

Hence, R is an equivalence relation.

Now, equivalence class containing $[(2, 5)]$ is $\{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$.

Q. 24 **Using the definition, prove that the function $f : A \to B$ is invertible if and only if f is both one-one and onto.**

Sol. A function $f : X \to Y$ is defined to be invertible, if there exist a function $g = Y \to X$ such that $gof = I_X$ and $fog = I_Y$. The function is called the inverse of f and is denoted by f^{-1}.

A function $f = X \to Y$ is invertible iff f is a bijective function.

Q. 25 **Functions $f, g : R \to R$ are defined, respectively, by $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, find**

(i) fog (ii) gof (iii) fof (iv) gog

Sol. Given that, $f(x) = x^2 + 3x + 1, g(x) = 2x - 3$

(i)
$$\begin{aligned} fog &= f\{g(x)\} = f(2x - 3) \\ &= (2x - 3)^2 + 3(2x - 3) + 1 \\ &= 4x^2 + 9 - 12x + 6x - 9 + 1 = 4x^2 - 6x + 1 \end{aligned}$$

(ii)
$$\begin{aligned} gof &= g\{f(x)\} = g(x^2 + 3x + 1) \\ &= 2(x^2 + 3x + 1) - 3 \\ &= 2x^2 + 6x + 2 - 3 = 2x^2 + 6x - 1 \end{aligned}$$

(iii)
$$\begin{aligned} fof &= f\{f(x)\} = f(x^2 + 3x + 1) \\ &= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1 \\ &= x^4 + 9x^2 + 1 + 6x^3 + 6x + 2x^2 + 3x^2 + 9x + 3 + 1 \\ &= x^4 + 6x^3 + 14x^2 + 15x + 5 \end{aligned}$$

(iv)
$$\begin{aligned} gog &= g\{g(x)\} = g(2x - 3) \\ &= 2(2x - 3) - 3 \\ &= 4x - 6 - 3 = 4x - 9 \end{aligned}$$

Q. 26 **Let $*$ be the binary operation defined on Q. Find which of the following binary operations are commutative**

(i) $a * b = a - b, \forall\, a, b \in Q$ (ii) $a * b = a^2 + b^2, \forall\, a, b \in Q$

(iii) $a * b = a + ab, \forall\, a, b \in Q$ (iv) $a * b = (a - b)^2, \forall\, a, b \in Q$

Sol. Given that $*$ be the binary operation defined on Q.

(i) $a * b = a - b, \forall\, a, b \in Q$ and $b * a = b - a$

So, $\quad a * b \neq b * a \quad [\because b - a \neq a - b]$

Hence, $*$ is not commutative.

(ii) $a * b = a^2 + b^2$

$b * a = b^2 + a^2$

So, $*$ is commutative. [since, '+' is on rational is commutative]

(iii) $a * b = a + ab$

$b * a = b + ab$

Clearly, $a + ab \neq b + ab$

So, $*$ is not commutative.

(iv) $a * b = (a - b)^2, \forall a, b \in Q$

$b * a = (b - a)^2$

$\because \quad (a - b)^2 = (b - a)^2$

Hence, $*$ is commutative.

Q. 27 If $*$ be binary operation defined on R by $a * b = 1 + ab, \forall a, b \in R$. Then, the operation $*$ is

(i) commutative but not associative.

(ii) associative but not commutative.

(iii) neither commutative nor associative.

(iv) both commutative and associative.

Sol. (i) Given that, $a * b = 1 + ab, \forall a, b \in R$

$a * b = ab + 1 = b * a$

So, $*$ is a commutative binary operation.

Also, $a * (b * c) = a * (1 + bc) = 1 + a(1 + bc)$

$a * (b * c) = 1 + a + abc \quad \ldots(i)$

$(a * b) * c = (1 + ab) * c$

$= 1 + (1 + ab)c = 1 + c + abc \quad \ldots(ii)$

From Eqs. (i) and (ii),

$$a * (b * c) \neq (a * b) * c$$

So, $*$ is not associative

Hence, $*$ is commutative but not associative.

Objective Type Questions

Q. 28 Let T be the set of all triangles in the Euclidean plane and let a relation R on T be defined as aRb, if a is congruent to b, $\forall a, b \in T$. Then, R is

(a) reflexive but not transitive (b) transitive but not symmetric

(c) equivalence (d) None of these

Sol. *(c)* Consider that aRb, if a is congruent to b, $\forall a, b \in T$.

Then, $aRa \Rightarrow a \cong a$,

which is true for all $a \in T$

So, R is reflexive, $\quad \ldots(i)$

Let $\quad aRb \Rightarrow a \cong b$

$\Rightarrow \quad b \cong a \Rightarrow b \cong a$

$\Rightarrow \quad bRa$

So, R is symmetric. ...(ii)

Let aRb and bRc

$\Rightarrow \quad a \cong b$ and $b \cong c$

$\Rightarrow \quad a \cong c \Rightarrow aRc$

So, R is transitive. ...(iii)

Hence, R is equivalence relation.

Q. 29 Consider the non-empty set consisting of children in a family and a relation R defined as aRb, if a is brother of b. Then, R is

(a) symmetric but not transitive
(b) transitive but not symmetric
(c) neither symmetric nor transitive
(d) both symmetric and transitive

Sol. *(b)* Given, $\quad aRb \Rightarrow a$ is brother of b

$\therefore \quad aRa \Rightarrow a$ is brother of a, which is not true.

So, R is not reflexive.

$aRb \Rightarrow a$ is brother of b.

This does not mean b is also a brother of a and b can be a sister of a.

Hence, R is not symmetric.

$aRb \Rightarrow a$ is brother of b

and $\quad bRc \Rightarrow b$ is a brother of c.

So, a is brother of c.

Hence, R is transitive.

Q. 30 The maximum number of equivalence relations on the set $A = \{1, 2, 3\}$ are

(a) 1 (b) 2 (c) 3 (d) 5

Sol. *(d)* Given that, $A = \{1, 2, 3\}$

Now, number of equivalence relations as follows

$$R_1 = \{(1, 1), (2, 2), (3, 3)\}$$
$$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$
$$R_3 = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$$
$$R_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$$
$$R_5 = \{(1, 2, 3) \Leftrightarrow A \times A = A^2\}$$

$\therefore$ Maximum number of equivalence relation on the set $A = \{1, 2, 3\} = 5$

Q. 31 If a relation R on the set $\{1, 2, 3\}$ be defined by $R = \{(1, 2)\}$, then R is

(a) reflexive (b) transitive (c) symmetric (d) None of these

Sol. *(b)* R on the set $\{1, 2, 3\}$ be defined by $R = \{(1, 2)\}$

It is clear that R is transitive.

Q. 32 Let us define a relation R in R as aRb if $a \geq b$. Then, R is

(a) an equivalence relation
(b) reflexive, transitive but not symmetric
(c) symmetric, transitive but not reflexive
(d) neither transitive nor reflexive but symmetric

Sol. *(b)* Given that, aRb if $a \geq b$

$\Rightarrow \quad aRa \Rightarrow a \geq a$ which is true.

Let aRb, $a \geq b$, then $b \geq a$ which is not true R is not symmetric.

But aRb and $b\ R\ c$

$\Rightarrow \quad a \geq b$ and $b \geq c$

$\Rightarrow \quad a \geq c$

Hence, R is transitive.

Q. 33 If $A = \{1, 2, 3\}$ and consider the relation

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$$

Then, R is

(a) reflexive but not symmetric
(b) reflexive but not transitive
(c) symmetric and transitive
(d) neither symmetric nor transitive

Sol. *(a)* Given that, $A = \{1, 2, 3\}$

and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$

$\because \quad (1, 1), (2, 2), (3, 3) \in R$

Hence, R is reflexive.

$$(1, 2) \in R \text{ but } (2, 1) \notin R$$

Hence, R is not symmetric.

$$(1, 2) \in R \text{ and } (2, 3) \in R$$

$\Rightarrow \quad (1, 3) \in R$

Hence, R is transitive.

Q. 34 The identity element for the binary operation $*$ defined on $Q - \{0\}$ as $a * b = \dfrac{ab}{2}, \forall\ a, b \in Q - \{0\}$ is

(a) 1
(b) 0
(c) 2
(d) None of these

Thinking Process

For given binary operation $* : A \times A \to A$, an element $e \in A$, if it exists, is called identity for the operation $*$, if $a * e = a = e * a, \forall\ a \in A$.

Sol. *(c)* Given that, $a * b = \dfrac{ab}{2}, \forall\ a, b \in Q - \{0\}$.

Let e be the identity element for $*$.

$\therefore \quad a * e = \dfrac{ae}{2}$

$\Rightarrow \quad a = \dfrac{ae}{2} \Rightarrow e = 2$

Q. 35 If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is

(a) 720 (b) 120 (c) 0 (d) None of these

Sol. *(c)* We know that, if A and B are two non-empty finite set containing m and n elements respectively, then the number of one-one and onto mapping from A to B is

$$n!, \text{ if } m = n$$
$$0, \text{ if } m \neq n$$

Given that, $m = 5$ and $n = 6$

$\therefore$ $m \neq n$

Number of mapping = 0

Q. 36 If $A = \{1, 2, 3, \ldots, n\}$ and $B = \{a, b\}$. Then, the number of surjections from A into B is

(a) ${}^nP_2$ (b) $2^n - 2$ (c) $2^n - 1$ (d) None of these

Sol. *(d)* Given that, $A = \{1, 2, 3, \ldots, n\}$ and $B = \{a, b\}$.

We know that, if A and B are two non-empty finite sets containing m and n elements respectively, then the number of surjection from A into B is

$${}^nC_m \times m!, \text{ if } n \geq m$$
$$0, \text{ if } n < m$$

Here, $m = 2$

$\therefore$ Number of surjection from A into B is ${}^nC_2 \times 2! = \dfrac{n!}{2!(n-2)!} \times 2!$

$$= \frac{n(n-1)(n-2)!}{2 \times 1(n-2)} \times 2! = n^2 - n$$

Q. 37 If $f : R \to R$ be defined by $f(x) = \dfrac{1}{x}$, $\forall\, x \in R$. Then, f is

(a) one-one (b) onto (c) bijective (d) f is not defined

Thinking Process

In the given function at $x = 0$, $f(x) = \infty$. So, the function is not define.

Sol. *(d)* Given that, $f(x) = \dfrac{1}{x}$, $\forall\, x \in R$

For $x = 0$,

$f(x)$ is not defined.

Hence, $f(x)$ is a not define function.

Q. 38 If $f : R \to R$ be defined by $f(x) = 3x^2 - 5$ and $g : R \to R$ by $g(x) = \dfrac{x}{x^2 + 1}$. Then, gof is

(a) $\dfrac{3x^2 - 5}{9x^4 - 30x^2 + 26}$ (b) $\dfrac{3x^2 - 5}{9x^4 - 6x^2 + 26}$

(c) $\dfrac{3x^2}{x^4 + 2x^2 - 4}$ (d) $\dfrac{3x^2}{9x^4 + 30x^2 - 2}$

Sol. *(a)* Given that, $f(x) = 3x^2 - 5$ and $g(x) = \dfrac{x}{x^2 + 1}$

$$gof = g\{f(x)\} = g(3x^2 - 5)$$
$$= \frac{3x^2 - 5}{(3x^2 - 5)^2 + 1} = \frac{3x^2 - 5}{9x^4 - 30x^2 + 25 + 1}$$
$$= \frac{3x^2 - 5}{9x^4 - 30x^2 + 26}$$

Q. 39 Which of the following functions from Z into Z are bijections?

(a) $f(x) = x^3$ (b) $f(x) = x + 2$ (c) $f(x) = 2x + 1$ (d) $f(x) = x^2 + 1$

Sol. *(b)* Here, $f(x) = x + 2 \Rightarrow f(x_1) = f(x_2)$

$$x_1 + 2 = x_2 + 2 \Rightarrow x_1 = x_2$$

Let $y = x + 2$

$$x = y - 2 \in Z, \forall\ y \in x$$

Hence, $f(x)$ is one-one and onto.

Q. 40 If $f : R \to R$ be the functions defined by $f(x) = x^3 + 5$, then $f^{-1}(x)$ is

(a) $(x + 5)^{\frac{1}{3}}$ (b) $(x - 5)^{\frac{1}{3}}$ (c) $(5 - x)^{\frac{1}{3}}$ (d) $5 - x$

Sol. *(b)* Given that, $f(x) = x^3 + 5$

Let $y = x^3 + 5 \Rightarrow x^3 = y - 5$

$$x = (y - 5)^{\frac{1}{3}} \Rightarrow f(x)^{-1} = (x - 5)^{\frac{1}{3}}$$

Q. 41 If $f : A \to B$ and $g : B \to C$ be the bijective functions, then $(gof)^{-1}$ is

(a) $f^{-1}og^{-1}$ (b) fog (c) $g^{-1}of^{-1}$ (d) gof

Sol. *(a)* Given that, $f : A \to B$ and $g : B \to C$ be the bijective functions.

$$(gof)^{-1} = f^{-1}og^{-1}$$

Q. 42 If $f : R - \left\{\dfrac{3}{5}\right\} \to R$ be defined by $f(x) = \dfrac{3x + 2}{5x - 3}$, then

(a) $f^{-1}(x) = f(x)$ (b) $f^{-1}(x) = -f(x)$ (c) $(fof)x = -x$ (d) $f^{-1}(x) = \dfrac{1}{19}f(x)$

Sol. *(a)* Given that, $f(x) = \dfrac{3x + 2}{5x - 3}$

Let $y = \dfrac{3x + 2}{5x - 3}$

$$3x + 2 = 5xy - 3y \Rightarrow x(3 - 5y) = -3y - 2$$
$$x = \frac{3y + 2}{5y - 3} \Rightarrow f^{-1}(x) = \frac{3x + 2}{5x - 3}$$

$\therefore$ $f^{-1}(x) = f(x)$

Q. 43 If $f:[0, 1] \to [0, 1]$ be defined by $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} \end{cases}$

then $(fof)x$ is

(a) constant (b) $1 + x$ (c) x (d) None of these

Sol. *(c)* Given that, $f: [0, 1] \to [0, 1]$ be defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} \end{cases}$$

$\therefore$ $(fof)x = f(f(x)) = x$

Q. 44 If $f : [2, \infty) \to R$ be the function defined by $f(x) = x^2 - 4x + 5$, then the range of f is

(a) R (b) $[1, \infty)$ (c) $[4, \infty)$ (d) $[5, \infty)$

Thinking Process

Range of $f = \{y \in Y : y = f(x)$ for some in $X\}$

Sol. *(b)* Given that, $f(x) = x^2 - 4x + 5$

Let $y = x^2 - 4x + 5$

$\Rightarrow$ $y = x^2 - 4x + 4 + 1 = (x - 2)^2 + 1$

$\Rightarrow$ $(x - 2)^2 = y - 1 \Rightarrow x - 2 = \sqrt{y - 1}$

$\Rightarrow$ $x = 2 + \sqrt{y - 1}$

$\therefore$ $y - 1 \geq 0, \; y \geq 1$

Range $= [1, \infty)$

Q. 45 If $f : N \to R$ be the function defined by $f(x) = \dfrac{2x - 1}{2}$ and $g : Q \to R$ be another function defined by $g(x) = x + 2$. Then, $(gof)\dfrac{3}{2}$ is

(a) 1 (b) 1 (c) $\dfrac{7}{2}$ (d) None of these

Sol. *(d)* Given that, $f(x) = \dfrac{2x - 1}{2}$ and $g(x) = x + 2$

$$(gof)\frac{3}{2} = g\left[f\left(\frac{3}{2}\right)\right] = g\left(\frac{2 \times \frac{3}{2} - 1}{2}\right)$$

$$= g(1) = 1 + 2 = 3$$

Q. 46 If $f: R \to R$ be defined by $f(x) = \begin{cases} 2x : x > 3 \\ x^2 : 1 < x \leq 3 \\ 3x : x \leq 1 \end{cases}$

Then, $f(-1) + f(2) + f(4)$ is

(a) 9 (b) 14 (c) 5 (d) None of these

Sol. *(a)* Given that,

$$f(x) = \begin{cases} 2x : x > 3 \\ x^2 : 1 < x \leq 3 \\ 3x : x \leq 1 \end{cases}$$

$$f(-1) + f(2) + f(4) = 3(-1) + (2)^2 + 2 \times 4$$
$$= -3 + 4 + 8 = 9$$

Q. 47 If $f: R \to R$ be given by $f(x) = \tan x$, then $f^{-1}(1)$ is

(a) $\frac{\pi}{4}$ (b) $\left\{n\pi + \frac{\pi}{4} : n \in Z\right\}$

(c) Does not exist (d) None of these

Sol. *(a)* Given that, $f(x) = \tan x$

Let $y = \tan x \Rightarrow x = \tan^{-1} y$

$\Rightarrow f^{-1}(x) = \tan^{-1} x \Rightarrow f^{-1}(1) = \tan^{-1} 1$

$\Rightarrow = \tan^{-1}\tan\frac{\pi}{4} = \frac{\pi}{4}$ $\left[\because \tan\frac{\pi}{4} = 1\right]$

Fillers

Q. 48 Let the relation R be defined in N by aRb, if $2a + 3b = 30$. Then, $R = \ldots\ldots$.

Sol. Given that,

$$2a + 3b = 30$$
$$3b = 30 - 2a$$
$$b = \frac{30 - 2a}{3}$$

For

$$a = 3, b = 8$$
$$a = 6, b = 6$$
$$a = 9, b = 4$$
$$a = 12, b = 2$$
$$R = \{(3, 8), (6, 6), (9, 4), (12, 2)\}$$

Q. 49 If the relation R be defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(a, b) : |a^2 - b^2| < 8\}$. Then, R is given by ……… .

Sol. Given, $A = \{1, 2, 3, 4, 5\}$,

$R = \{(a, b) : |a^2 - b^2| < 8\}$

$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 3), (3, 4), (4, 4), (5, 5)\}$

Q. 50 If $f=\{(1, 2), (3, 5), (4, 1)\}$ and $g=\{(2, 3), (5, 1), (1, 3)\}$, then $gof=$ .... and $fog=$ ....

Sol. Given that, $f=\{(1, 2), (3, 5), (4, 1)\}$ and $g=\{(2, 3), (5, 1), (1, 3)\}$

$$gof(1)=g\{f(1)\}=g(2)=3$$
$$gof(3)=g\{f(3)\}=g(5)=1$$
$$gof(4)=g\{f(4)\}=g(1)=3$$
$$gof=\{(1, 3), (3, 1), (4, 3)\}$$

Now,
$$fog(2)=f\{g(2)\}=f(3)=5$$
$$fog(5)=f\{g(5)\}=f(1)=2$$
$$fog(1)=f\{g(1)\}=f(3)=5$$
$$fog=\{(2, 5), (5, 2), (1, 5)\}$$

Q. 51 If $f: R\rightarrow R$ be defined by $f(x)=\dfrac{x}{\sqrt{1+x^2}}$, then $(fofof)(x)=$ ....

Sol. Given that, $f(x)=\dfrac{x}{\sqrt{1+x^2}}$

$$(fofof)(x)=f[f\{f(x)\}]$$
$$=f\left[f\left(\frac{x}{\sqrt{1+x^2}}\right)\right]=f\left(\frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{1+x^2}}}\right)$$
$$=f\left[\frac{x\sqrt{1+x^2}}{\sqrt{1+x^2}(\sqrt{2x^2}+1)}\right]=f\left(\frac{x}{\sqrt{1+2x^2}}\right)$$
$$=\frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\frac{x^2}{1+2x^2}}}=\frac{x\sqrt{1+2x^2}}{\sqrt{1+2x^2}\sqrt{1+3x^2}}$$
$$=\frac{x}{\sqrt{1+3x^2}}=\frac{x}{\sqrt{3x^2+1}}$$

Q. 52 If $f(x)=[4-(x-7)^3]$, then $f^{-1}(x)=$ ....

Sol. Given that, $f(x)=\{4-(x-7)^3\}$

Let $y=[4-(x-7)^3]$

$$(x-7)^3=4-y$$
$$(x-7)=(4-y)^{1/3}$$
$$\Rightarrow \quad x=7+(4-y)^{1/3}$$
$$f^{-1}(x)=7+(4-x)^{1/3}$$

True/False

Q. 53 Let $R = \{(3, 1), (1, 3), (3, 3)\}$ be a relation defined on the set $A = \{1, 2, 3\}$. Then, R is symmetric, transitive but not reflexive.

Sol. ***False***

Given that, $R = \{(3, 1), (1, 3), (3, 3)\}$ be defined on the set $A = \{1, 2, 3\}$

$$(1, 1) \notin R$$

So, R is not reflexive. $\quad (3, 1) \in R, (1, 3) \in R$

Hence, R is symmetric.

Since, $\quad (3, 1) \in R, (1, 3) \in R$

But $\quad (1, 1) \notin R$

Hence, R is not transitive.

Q. 54 If $f : R \to R$ be the function defined by $f(x) = \sin(3x + 2) \ \forall \ x \in R$. Then, f is invertible.

Sol. ***False***

Given that, $f(x) = \sin(3x + 2), \forall \ x \in R$ is not one-one function for all $x \in R$.

So, f is not invertible.

Q. 55 Every relation which is symmetric and transitive is also reflexive.

Sol. ***False***

Let R be a relation defined by

$$R = \{(1, 2), (2, 1), (1, 1), (2, 2)\} \text{ on the set } A = \{1, 2, 3\}$$

It is clear that $(3, 3) \notin R$. So, it is not reflexive.

Q. 56 An integer m is said to be related to another integer n, if m is a integral multiple of n. This relation in Z is reflexive, symmetric and transitive.

Sol. ***False***

The given relation is reflexive and transitive but not symmetric.

Q. 57 If $A = \{0, 1\}$ and N be the set of natural numbers. Then, the mapping $f : N \to A$ defined by $f(2n - 1) = 0, \ f(2n) = 1, \ \forall \ n \in N$, is onto.

Sol. ***True***

Given, $\quad A = \{0, 1\}$

$$f(2n - 1) = 0, f(2n) = 1, \forall \ n \in N$$

So, the mapping $f : N \to A$ is onto.

Q. 58 The relation R on the set $A = \{1, 2, 3\}$ defined as $R = \{(1, 1), (1, 2), (2, 1), (3, 3)\}$ is reflexive, symmetric and transitive.

Sol. ***False***

Given that, $\quad R = \{(1, 1), (1, 2), (2, 1), (3, 3)\}$

$$(2, 2) \notin R$$

So, R is not reflexive.

Q. 59 The composition of function is commutative.

Sol. ***False***

Let $f(x) = x^2$

and $g(x) = x + 1$

$$fog(x) = f\{g(x)\} = f(x+1)$$
$$= (x+1)^2 = x^2 + 2x + 1$$
$$gof(x) = g\{f(x)\} = g(x^2) = x^2 + 1$$

$\therefore$ $fog(x) \neq gof(x)$

Q. 60 The composition of function is associative.

Sol. ***True***

Let $f(x) = x,\ g(x) = x + 1$

and $h(x) = 2x - 1$

Then, $fo\{goh(x)\} = f\,[g\{h(x)\}]$

$$= f\{g(2x-1)\}$$
$$= f(2x - 1 + 1)$$
$$= f(2x) = 2x$$

$\therefore$ $(fog)\,oh(x) = (fog)\{h(x)\}$

$$= (fog)(2x - 1)$$
$$= f\{g(2x-1)\}$$
$$= f(2x - 1 + 1)$$
$$= f(2x) = 2x$$

Q. 61 Every function is invertible.

Sol. ***False***

Only bijective functions are invertible.

Q. 62 A binary operation on a set has always the identity element.

Sol. ***False***

'+' is a binary operation on the set N but it has no identity element.