Chapter 1

RELATIONS AND FUNCTIONS

1.1 Overview

1.1.1 *Relation*

A relation R from a non-empty set A to a non empty set B is a subset of the Cartesian product A × B. The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R. The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the codomain of the relation R. Note that range is always a subset of codomain.

1.1.2 *Types of Relations*

A relation R in a set A is subset of A × A. Thus empty set ϕ and A × A are two extreme relations.

(i) A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

(ii) A relation R in a set A is called universal relation, if each element of A is related to every element of A, i.e., $R = A \times A$.

(iii) A relation R in A is said to be reflexive if aRa for all $a \in$ A, R is symmetric if $a\text{R}b \Rightarrow b\text{R}a$, $\forall\ a, b \in$ A and it is said to be transitive if aRb and bR$c \Rightarrow a$Rc $\forall\ a, b, c \in$ A. Any relation which is reflexive, symmetric and transitive is called an equivalence relation.

☛ **Note:** An important property of an equivalence relation is that it divides the set into pairwise disjoint subsets called equivalent classes whose collection is called a partition of the set. Note that the union of all equivalence classes gives the whole set.

1.1.3 *Types of Functions*

(i) A function $f : X \rightarrow Y$ is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct, i.e.,

$$x_1, x_2 \in X,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

(ii) A function $f : X \rightarrow Y$ is said to be onto (or surjective), if every element of Y is the image of some element of X under f, i.e., for every $y \in$ Y there exists an element $x \in$ X such that $f(x) = y$.

(iii) A function $f : \mathrm{X} \rightarrow \mathrm{Y}$ is said to be one-one and onto (or bijective), if f is both one-one and onto.

1.1.4 *Composition of Functions*

(i) Let $f : \mathrm{A} \rightarrow \mathrm{B}$ and $g : \mathrm{B} \rightarrow \mathrm{C}$ be two functions. Then, the composition of f and g, denoted by $g \circ f$, is defined as the function $g \circ f : \mathrm{A} \rightarrow \mathrm{C}$ given by

$$g \circ f(x) = g(f(x)),\ \forall\ x \in \mathrm{A}.$$

(ii) If $f : \mathrm{A} \rightarrow \mathrm{B}$ and $g : \mathrm{B} \rightarrow \mathrm{C}$ are one-one, then $g \circ f : \mathrm{A} \rightarrow \mathrm{C}$ is also one-one

(iii) If $f : \mathrm{A} \rightarrow \mathrm{B}$ and $g : \mathrm{B} \rightarrow \mathrm{C}$ are onto, then $g \circ f : \mathrm{A} \rightarrow \mathrm{C}$ is also onto.

However, converse of above stated results (ii) and (iii) need not be true. Moreover, we have the following results in this direction.

(iv) Let $f : \mathrm{A} \rightarrow \mathrm{B}$ and $g : \mathrm{B} \rightarrow \mathrm{C}$ be the given functions such that $g \circ f$ is one-one. Then f is one-one.

(v) Let $f : \mathrm{A} \rightarrow \mathrm{B}$ and $g : \mathrm{B} \rightarrow \mathrm{C}$ be the given functions such that $g \circ f$ is onto. Then g is onto.

1.1.5 *Invertible Function*

(i) A function $f : \mathrm{X} \rightarrow \mathrm{Y}$ is defined to be invertible, if there exists a function $g : \mathrm{Y} \rightarrow \mathrm{X}$ such that $g \circ f = \mathrm{I}_{\mathrm{X}}$ and $f \circ g = \mathrm{I}_{\mathrm{Y}}$. The function g is called the inverse of f and is denoted by f^{-1}.

(ii) A function $f : \mathrm{X} \rightarrow \mathrm{Y}$ is invertible if and only if f is a bijective function.

(iii) If $f : \mathrm{X} \rightarrow \mathrm{Y}$, $g : \mathrm{Y} \rightarrow \mathrm{Z}$ and $h : \mathrm{Z} \rightarrow \mathrm{S}$ are functions, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(iv) Let $f : \mathrm{X} \rightarrow \mathrm{Y}$ and $g : \mathrm{Y} \rightarrow \mathrm{Z}$ be two invertible functions. Then $g \circ f$ is also invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

1.1.6 *Binary Operations*

(i) A binary operation $*$ on a set A is a function $* : \mathrm{A} \times \mathrm{A} \rightarrow \mathrm{A}$. We denote $*(a, b)$ by $a * b$.

(ii) A binary operation $*$ on the set X is called commutative, if $a * b = b * a$ for every $a, b \in \mathrm{X}$.

(iii) A binary operation $* : \mathrm{A} \times \mathrm{A} \rightarrow \mathrm{A}$ is said to be associative if $(a * b) * c = a * (b * c)$, for every $a, b, c \in \mathrm{A}$.

(iv) Given a binary operation $* : \mathrm{A} \times \mathrm{A} \rightarrow \mathrm{A}$, an element $e \in \mathrm{A}$, if it exists, is called identity for the operation $*$, if $a * e = a = e * a$, $\forall\ a \in \mathrm{A}$.

(v) Given a binary operation $*: A \times A \to A$, with the identity element e in A, an element $a \in$ A, is said to be invertible with respect to the operation $*$, if there exists an element b in A such that $a * b = e = b * a$ and b is called the inverse of a and is denoted by a^{-1}.

1.2 Solved Examples

Short Answer (S.A.)

Example 1 Let A = {0, 1, 2, 3} and define a relation R on A as follows:

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}.$$

Is R reflexive? symmetric? transitive?

Solution R is reflexive and symmetric, but not transitive since for $(1, 0) \in R$ and $(0, 3) \in R$ whereas $(1, 3) \notin R$.

Example 2 For the set A = {1, 2, 3}, define a relation R in the set A as follows:

$$R = \{(1, 1), (2, 2), (3, 3), (1, 3)\}.$$

Write the ordered pairs to be added to R to make it the smallest equivalence relation.

Solution (3, 1) is the single ordered pair which needs to be added to R to make it the smallest equivalence relation.

Example 3 Let R be the equivalence relation in the set **Z** of integers given by $R = \{(a, b) : 2 \text{ divides } a - b\}$. Write the equivalence class [0].

Solution $[0] = \{0, \pm 2, \pm 4, \pm 6, ...\}$

Example 4 Let the function $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 4x - 1, \forall x \in \mathbf{R}$. Then, show that f is one-one.

Solution For any two elements $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) = f(x_2)$, we have

$$4x_1 - 1 = 4x_2 - 1$$

$$\Rightarrow \quad 4x_1 = 4x_2, \text{ i.e., } x_1 = x_2$$

Hence f is one-one.

Example 5 If $f = \{(5, 2), (6, 3)\}$, $g = \{(2, 5), (3, 6)\}$, write $f \circ g$.

Solution $f \circ g = \{(2, 2), (3, 3)\}$

Example 6 Let $f: \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 4x - 3 \ \forall x \in \mathbf{R}$. Then write f^{-1}.

Solution Given that $f(x) = 4x - 3 = y$ (say), then

$$4x = y + 3$$

$$\Rightarrow \qquad x = \frac{y+3}{4}$$

Hence $\quad f^{-1}(y) = \dfrac{y+3}{4} \quad \Rightarrow \quad f^{-1}(x) = \dfrac{x+3}{4}$

Example 7 Is the binary operation $*$ defined on **Z** (set of integer) by $m * n = m - n + mn \ \ \forall\, m, n \in \mathbf{Z}$ commutative?

Solution No. Since for $1, 2 \in \mathbf{Z}$, $1 * 2 = 1 - 2 + 1.2 = 1$ while $2 * 1 = 2 - 1 + 2.1 = 3$ so that $1 * 2 \neq 2 * 1$.

Example 8 If $f = \{(5, 2), (6, 3)\}$ and $g = \{(2, 5), (3, 6)\}$, write the range of f and g.

Solution The range of $f = \{2, 3\}$ and the range of $g = \{5, 6\}$.

Example 9 If $A = \{1, 2, 3\}$ and f, g are relations corresponding to the subset of $A \times A$ indicated against them, which of f, g is a function? Why?

$$f = \{(1, 3), (2, 3), (3, 2)\}$$

$$g = \{(1, 2), (1, 3), (3, 1)\}$$

Solution f is a function since each element of A in the first place in the ordered pairs is related to only one element of A in the second place while g is not a function because 1 is related to more than one element of A, namely, 2 and 3.

Example 10 If $A = \{a, b, c, d\}$ and $f = \{(a, b), (b, d), (c, a), (d, c)\}$, show that f is one-one from A onto A. Find f^{-1}.

Solution f is one-one since each element of A is assigned to distinct element of the set A. Also, f is onto since $f(A) = A$. Moreover, $f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$.

Example 11 In the set **N** of natural numbers, define the binary operation $*$ by $m * n = g.c.d\,(m, n)$, $m, n \in \mathbf{N}$. Is the operation $*$ commutative and associative?

Solution The operation is clearly commutative since

$$m * n = g.c.d\,(m, n) = g.c.d\,(n, m) = n * m \quad \forall\, m, n \in \mathbf{N}.$$

It is also associative because for $l, m, n \in \mathbf{N}$, we have

$$\begin{aligned} l * (m * n) &= g.\,c.\,d\,(l, g.c.d\,(m, n)) \\ &= g.c.d.\,(g.\,c.\,d\,(l, m), n) \\ &= (l * m) * n. \end{aligned}$$

Long Answer (L.A.)

Example 12 In the set of natural numbers **N**, define a relation R as follows: $\forall\, n, m \in \mathbf{N}$, nRm if on division by 5 each of the integers n and m leaves the remainder less than 5, i.e. one of the numbers 0, 1, 2, 3 and 4. Show that R is equivalence relation. Also, obtain the pairwise disjoint subsets determined by R.

Solution R is reflexive since for each $a \in \mathbf{N}$, aRa. R is symmetric since if aRb, then bRa for $a, b \in \mathbf{N}$. Also, R is transitive since for $a, b, c \in \mathbf{N}$, if aRb and bRc, then aRc. Hence R is an equivalence relation in **N** which will partition the set **N** into the pairwise disjoint subsets. The equivalent classes are as mentioned below:

$$A_0 = \{5, 10, 15, 20 \ldots\}$$
$$A_1 = \{1, 6, 11, 16, 21 \ldots\}$$
$$A_2 = \{2, 7, 12, 17, 22, \ldots\}$$
$$A_3 = \{3, 8, 13, 18, 23, \ldots\}$$
$$A_4 = \{4, 9, 14, 19, 24, \ldots\}$$

It is evident that the above five sets are pairwise disjoint and

$$A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{i=0}^{4} A_i = \mathbf{N}.$$

Example 13 Show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \dfrac{x}{x^2+1}, \forall\, x \in \mathbf{R}$, is neither one-one nor onto.

Solution For $x_1, x_2 \in \mathbf{R}$, consider

$$f(x_1) = f(x_2)$$

$$\Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow x_1 x_2^2 + x_1 = x_2 x_1^2 + x_2$$

$$\Rightarrow x_1 x_2 (x_2 - x_1) = x_2 - x_1$$

$$\Rightarrow x_1 = x_2 \text{ or } x_1 x_2 = 1$$

We note that there are point, x_1 and x_2 with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, for instance, if we take $x_1 = 2$ and $x_2 = \frac{1}{2}$, then we have $f(x_1) = \frac{2}{5}$ and $f(x_2) = \frac{2}{5}$ but $2 \neq \frac{1}{2}$. Hence f is not one-one. Also, f is not onto for if so then for $1 \in \mathbf{R}\ \exists\, x \in \mathbf{R}$ such that $f(x) = 1$

which gives $\frac{x}{x^2+1}=1$. But there is no such x in the domain **R**, since the equation $x^2 - x + 1 = 0$ does not give any real value of x.

Example 14 Let $f, g : \mathbf{R} \to \mathbf{R}$ be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x \ \forall \ x \in \mathbf{R}$. Then, find $f \circ g$ and $g \circ f$.

Solution Here $f(x) = |x| + x$ which can be redefined as

$$f(x) = \begin{cases} 2x \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$$

Similarly, the function g defined by $g(x) = |x| - x$ may be redefined as

$$g(x) = \begin{cases} 0 \text{ if } x \geq 0 \\ -2x \text{ if } x < 0 \end{cases}$$

Therefore, $g \circ f$ gets defined as :

For $x \geq 0$, $(g \circ f)(x) = g(f(x) = g(2x) = 0$

and for $x < 0$, $(g \circ f)(x) = g(f(x) = g(0) = 0$.

Consequently, we have $(g \circ f)(x) = 0$, $\forall \ x \in \mathbf{R}$.

Similarly, $f \circ g$ gets defined as:

For $x \geq 0$, $(f \circ g)(x) = f(g(x) = f(0) = 0$,

and for $x < 0$, $(f \circ g)(x) = f(g(x)) = f(-2x) = -4x$.

i.e. $$(f \circ g)(x) = \begin{cases} 0, x > 0 \\ -4x, x < 0 \end{cases}$$

Example 15 Let **R** be the set of real numbers and $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 4x + 5$. Show that f is invertible and find f^{-1}.

Solution Here the function $f : \mathbf{R} \to \mathbf{R}$ is defined as $f(x) = 4x + 5 = y$ (say). Then

$$4x = y - 5 \quad \text{or} \quad x = \frac{y-5}{4}.$$

This leads to a function $g : \mathbf{R} \to \mathbf{R}$ defined as

$$g(y) = \frac{y-5}{4}.$$

Therefore, $(g \circ f)(x) = g(f(x)) = g(4x + 5)$

$$= \frac{4x+5-5}{4} = x$$

or $g \circ f = I_R$

Similarly $(f \circ g)(y) = f(g(y))$

$$= f\left(\frac{y-5}{4}\right)$$

$$= 4\left(\frac{y-5}{4}\right) + 5 = y$$

or $f \circ g = I_R$.

Hence f is invertible and $f^{-1} = g$ which is given by

$$f^{-1}(x) = \frac{x-5}{4}$$

Example 16 Let $*$ be a binary operation defined on $\mathbf{Q}$. Find which of the following binary operations are associative

(i) $a * b = a - b$ for $a, b \in \mathbf{Q}$.

(ii) $a * b = \frac{ab}{4}$ for $a, b \in \mathbf{Q}$.

(iii) $a * b = a - b + ab$ for $a, b \in \mathbf{Q}$.

(iv) $a * b = ab^2$ for $a, b \in \mathbf{Q}$.

Solution

(i) $*$ is not associative for if we take $a = 1$, $b = 2$ and $c = 3$, then
$(a * b) * c = (1 * 2) * 3 = (1 - 2) * 3 = -1 - 3 = -4$ and
$a * (b * c) = 1 * (2 * 3) = 1 * (2 - 3) = 1 - (-1) = 2.$

Thus $(a * b) * c \neq a * (b * c)$ and hence $*$ is not associative.

(ii) $*$ is associative since **Q** is associative with respect to multiplication.

(iii) $*$ is not associative for if we take $a = 2$, $b = 3$ and $c = 4$, then

$(a * b) * c = (2 * 3) * 4 = (2 – 3 + 6) * 4 = 5 * 4 = 5 – 4 + 20 = 21$, and

$a * (b * c) = 2 * (3 * 4) = 2 * (3 – 4 + 12) = 2 * 11 = 2 – 11 + 22 = 13$

Thus $(a * b) * c \neq a * (b * c)$ and hence $*$ is not associative.

(iv) $*$ is not associative for if we take $a = 1$, $b = 2$ and $c = 3$, then $(a * b) * c = (1 * 2) * 3 = 4 * 3 = 4 \times 9 = 36$ and $a * (b * c) = 1 * (2 * 3) = 1 * 18 = 1 \times 18^2 = 324$.

Thus $(a * b) * c \neq a * (b * c)$ and hence $*$ is not associative.

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 17 to 25.

Example 17 Let R be a relation on the set **N** of natural numbers defined by $n\text{R}m$ if n divides m. Then R is

(A) Reflexive and symmetric
(B) Transitive and symmetric
(C) Equivalence
(D) Reflexive, transitive but not symmetric

Solution The correct choice is (D).

Since n divides n, $\forall\ n \in$ **N**, R is reflexive. R is not symmetric since for 3, 6 $\in$ **N**, 3 R 6 $\neq$ 6 R 3. R is transitive since for n, m, r whenever n/m and $m/r \Rightarrow n/r$, i.e., n divides m and m divides r, then n will devide r.

Example 18 Let L denote the set of all straight lines in a plane. Let a relation R be defined by $l\text{R}m$ if and only if l is perpendicular to m $\forall\ l, m \in$ L. Then R is

(A) reflexive
(B) symmetric
(C) transitive
(D) none of these

Solution The correct choice is (B).

Example 19 Let **N** be the set of natural numbers and the function f : **N** $\rightarrow$ **N** be defined by $f(n) = 2n + 3\ \forall\ n \in$ **N**. Then f is

(A) surjective
(B) injective
(C) bijective
(D) none of these

Solution (B) is the correct option.

Example 20 Set A has 3 elements and the set B has 4 elements. Then the number of injective mappings that can be defined from A to B is

(A) 144 (B) 12

(C) 24 (D) 64

Solution The correct choice is (C). The total number of injective mappings from the set containing 3 elements into the set containing 4 elements is ${}^4P_3 = 4! = 24$.

Example 21 Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \sin x$ and $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = x^2$, then $f \, o \, g$ is

(A) $x^2 \sin x$ (B) $(\sin x)^2$

(C) $\sin x^2$ (D) $\dfrac{\sin x}{x^2}$

Solution (C) is the correct choice.

Example 22 Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x - 4$. Then $f^{-1}(x)$ is given by

(A) $\dfrac{x+4}{3}$ (B) $\dfrac{x}{3} - 4$

(C) $3x + 4$ (D) None of these

Solution (A) is the correct choice.

Example 23 Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2 + 1$. Then, pre-images of 17 and – 3, respectively, are

(A) $\phi, \{4, -4\}$ (B) $\{3, -3\}, \phi$

(C) $\{4, -4\}, \phi$ (D) $\{4, -4, \{2, -2\}$

Solution (C) is the correct choice since for $f^{-1}(17) = x \Rightarrow f(x) = 17$ or $x^2 + 1 = 17$ $\Rightarrow x = \pm 4$ or $f^{-1}(17) = \{4, -4\}$ and for $f^{-1}(-3) = x \Rightarrow f(x) = -3 \Rightarrow x^2 + 1 = -3 \Rightarrow x^2 = -4$ and hence $f^{-1}(-3) = \phi$.

Example 24 For real numbers x and y, define xRy if and only if $x - y + \sqrt{2}$ is an irrational number. Then the relation R is

(A) reflexive (B) symmetric

(C) transitive (D) none of these

Solution (A) is the correct choice.

Fill in the blanks in each of the Examples 25 to 30.

Example 25 Consider the set A = {1, 2, 3} and R be the smallest equivalence relation on A, then R = ________

Solution R = {(1, 1), (2, 2), (3, 3)}.

Example 26 The domain of the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sqrt{x^2 - 3x + 2}$ is ________.

Solution Here $x^2 - 3x + 2 \geq 0$

$\Rightarrow \quad (x - 1)(x - 2) \geq 0$

$\Rightarrow \quad x \leq 1$ or $x \geq 2$

Hence the domain of $f = (-\infty, 1] \cup [2, \infty)$

Example 27 Consider the set A containing n elements. Then, the total number of injective functions from A onto itself is ________.

Solution $n!$

Example 28 Let **Z** be the set of integers and R be the relation defined in **Z** such that $a\mathrm{R}b$ if $a - b$ is divisible by 3. Then R partitions the set **Z** into ________ pairwise disjoint subsets.

Solution Three.

Example 29 Let **R** be the set of real numbers and $*$ be the binary operation defined on **R** as $a * b = a + b - ab \;\; \forall \; a, b \in \mathbf{R}$. Then, the identity element with respect to the binary operation $*$ is _______.

Solution 0 is the identity element with respect to the binary operation $*$.

State **True** or **False** for the statements in each of the Examples 30 to 34.

Example 30 Consider the set A = {1, 2, 3} and the relation R = {(1, 2), (1, 3)}. R is a transitive relation.

Solution True.

Example 31 Let A be a finite set. Then, each injective function from A into itself is not surjective.

Solution False.

Example 32 For sets A, B and C, let $f : A \to B$, $g : B \to C$ be functions such that $g\,o\,f$ is injective. Then both f and g are injective functions.

Solution False.

Example 33 For sets A, B and C, let $f : A \to B$, $g : B \to C$ be functions such that $g\,o\,f$ is surjective. Then g is surjective

Solution True.

Example 34 Let **N** be the set of natural numbers. Then, the binary operation $*$ in **N** defined as $a * b = a + b$, $\forall\ a, b \in$ **N** has identity element.

Solution False.

1.3 EXERCISE

Short Answer (S.A.)

1. Let A = $\{a, b, c\}$ and the relation R be defined on A as follows:

$$R = \{(a, a), (b, c), (a, b)\}.$$

Then, write minimum number of ordered pairs to be added in **R** to make **R** reflexive and transitive.

2. Let D be the domain of the real valued function f defined by $f(x) = \sqrt{25 - x^2}$. Then, write D.

3. Let $f, g : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$, $\forall x \in \mathbf{R}$, respectively. Then, find $g\ o\ f$.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 2x - 3\ \forall x \in$ R. write f^{-1}.

5. If A = $\{a, b, c, d\}$ and the function $f = \{(a, b), (b, d), (c, a), (d, c)\}$, write f^{-1}.

6. If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, write $f(f(x))$.

7. Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? If g is described by $g(x) = \alpha x + \beta$, then what value should be assigned to α and β.

8. Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective.

   (i) $\{(x, y)$: x is a person, y is the mother of $x\}$.

   (ii) $\{(a, b)$: a is a person, b is an ancestor of $a\}$.

9. If the mappings f and g are given by

   $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$, write $f\ o\ g$.

10. Let **C** be the set of complex numbers. Prove that the mapping $f : \mathbf{C} \to \mathbf{R}$ given by $f(z) = |z|$, $\forall\ z \in \mathbf{C}$, is neither one-one nor onto.

11. Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \cos x$, $\forall\ x \in \mathbf{R}$. Show that f is neither one-one nor onto.

12. Let X = $\{1, 2, 3\}$ and Y = $\{4, 5\}$. Find whether the following subsets of X $\times$Y are functions from X to Y or not.

(i)	$f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$	(ii)	$g = \{(1, 4), (2, 4), (3, 4)\}$
(iii)	$h = \{(1,4), (2, 5), (3, 5)\}$	(iv)	$k = \{(1,4), (2, 5)\}$.

13. If functions $f : A \to B$ and $g : B \to A$ satisfy $g\ o\ f = I_A$, then show that f is one-one and g is onto.

14. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = \dfrac{1}{2 - \cos x}$ $\forall\, x \in \mathbf{R}$. Then, find the range of f.

15. Let n be a fixed positive integer. Define a relation R in $\mathbf{Z}$ as follows: $\forall\, a, b \in \mathbf{Z}$, aRb if and only if $a - b$ is divisible by n. Show that R is an equivalance relation.

Long Answer (L.A.)

16. If A = {1, 2, 3, 4 }, define relations on A which have properties of being:

(a) reflexive, transitive but not symmetric

(b) symmetric but neither reflexive nor transitive

(c) reflexive, symmetric and transitive.

17. Let R be relation defined on the set of natural number $\mathbf{N}$ as follows:

R = $\{(x, y): x \in \mathbf{N}, y \in \mathbf{N}, 2x + y = 41\}$. Find the domain and range of the relation R. Also verify whether R is reflexive, symmetric and transitive.

18. Given A = {2, 3, 4}, B = {2, 5, 6, 7}. Construct an example of each of the following:

(a) an injective mapping from A to B

(b) a mapping from A to B which is not injective

(c) a mapping from B to A.

19. Give an example of a map

(i) which is one-one but not onto

(ii) which is not one-one but onto

(iii) which is neither one-one nor onto.

20. Let A = $\mathbf{R}$ – {3}, B = $\mathbf{R}$ – {1}. Let f : A $\to$ B be defined by $f(x) = \dfrac{x-2}{x-3}$ $\forall\, x \in$ A. Then show that f is bijective.

21. Let A = [–1, 1]. Then, discuss whether the following functions defined on A are one-one, onto or bijective:

(i) $f(x) = \dfrac{x}{2}$ (ii) $g(x) = |x|$

(iii) $h(x) = x|x|$ (iv) $k(x) = x^2$.

22. Each of the following defines a relation on $\mathbf{N}$:

(i) x is greater than y, $x, y \in \mathbf{N}$

(ii) $x + y = 10$, $x, y \in \mathbf{N}$

(iii) $x\,y$ is square of an integer $x, y \in \mathbf{N}$

(iv) $x + 4y = 10 \;\; x, y \in \mathbf{N}$.

Determine which of the above relations are reflexive, symmetric and transitive.

23. Let A = {1, 2, 3, ... 9} and R be the relation in A ×A defined by (a, b) R (c, d) if $a + d = b + c$ for $(a, b), (c, d)$ in A ×A. Prove that R is an equivalence relation and also obtain the equivalent class [(2, 5)].

24. Using the definition, prove that the function $f : A \to B$ is invertible if and only if f is both one-one and onto.

25. Functions $f, g : \mathbf{R} \to \mathbf{R}$ are defined, respectively, by $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, find

(i) $f \circ g$ (ii) $g \circ f$ (iii) $f \circ f$ (iv) $g \circ g$

26. Let $*$ be the binary operation defined on $\mathbf{Q}$. Find which of the following binary operations are commutative

(i) $a * b = a - b \;\forall\; a, b \in \mathbf{Q}$

(ii) $a * b = a^2 + b^2 \;\forall\; a, b \in \mathbf{Q}$

(iii) $a * b = a + ab \;\forall\; a, b \in \mathbf{Q}$

(iv) $a * b = (a - b)^2 \;\forall\; a, b \in \mathbf{Q}$

27. Let $*$ be binary operation defined on $\mathbf{R}$ by $a * b = 1 + ab, \;\forall\; a, b \in \mathbf{R}$. Then the operation $*$ is

(i) commutative but not associative

(ii) associative but not commutative

(iii) neither commutative nor associative

(iv) both commutative and associative

Objective Type Questions

Choose the correct answer out of the given four options in each of the Exercises from 28 to 47 (M.C.Q.).

28. Let T be the set of all triangles in the Euclidean plane, and let a relation R on T be defined as aRb if a is congruent to $b \;\forall\; a, b \in$ T. Then R is

(A) reflexive but not transitive

(B) transitive but not symmetric

(C) equivalence

(D) none of these

29. Consider the non-empty set consisting of children in a family and a relation R defined as aRb if a is brother of b. Then R is

(A) symmetric but not transitive

(B) transitive but not symmetric

(C) neither symmetric nor transitive

(D) both symmetric and transitive

30. The maximum number of equivalence relations on the set A = {1, 2, 3} are

(A) 1 (B) 2

(C) 3 (D) 5

31. If a relation R on the set {1, 2, 3} be defined by R = {(1, 2)}, then R is

(A) reflexive (B) transitive

(C) symmetric (D) none of these

32. Let us define a relation R in **R** as aRb if $a \geq b$. Then R is

(A) an equivalence relation (B) reflexive, transitive but not symmetric

(C) symmetric, transitive but not reflexive (D) neither transitive nor reflexive but symmetric.

33. Let A = {1, 2, 3} and consider the relation

R = {1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1,3)}.

Then R is

(A) reflexive but not symmetric (B) reflexive but not transitive

(C) symmetric and transitive (D) neither symmetric, nor transitive

34. The identity element for the binary operation $*$ defined on Q ~ {0} as

$a * b = \dfrac{ab}{2} \ \forall\, a, b \in$ Q ~ {0} is

(A) 1 (B) 0

(C) 2 (D) none of these

35. If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is

(A) 720 (B) 120

(C) 0 (D) none of these

36. Let A = {1, 2, 3, ...n} and B = {a, b}. Then the number of surjections from A into B is

(A) ${}^nP_2$ (B) $2^n - 2$

(C) $2^n - 1$ (D) None of these

37. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \frac{1}{x}$ $\forall\, x \in \mathbf{R}$. Then f is

(A) one-one (B) onto
(C) bijective (D) f is not defined

38. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x^2 - 5$ and $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = \frac{x}{x^2+1}$.

Then $g \circ f$ is

(A) $\frac{3x^2-5}{9x^4-30x^2+26}$ (B) $\frac{3x^2-5}{9x^4-6x^2+26}$

(C) $\frac{3x^2}{x^4+2x^2-4}$ (D) $\frac{3x^2}{9x^4+30x^2-2}$

39. Which of the following functions from **Z** into **Z** are bijections?

(A) $f(x) = x^3$ (B) $f(x) = x + 2$
(C) $f(x) = 2x + 1$ (D) $f(x) = x^2 + 1$

40. Let $f : \mathbf{R} \to \mathbf{R}$ be the functions defined by $f(x) = x^3 + 5$. Then $f^{-1}(x)$ is

(A) $(x+5)^{\frac{1}{3}}$ (B) $(x-5)^{\frac{1}{3}}$

(C) $(5-x)^{\frac{1}{3}}$ (D) $5 - x$

41. Let $f : A \to B$ and $g : B \to C$ be the bijective functions. Then $(g \circ f)^{-1}$ is

(A) $f^{-1} \circ g^{-1}$ (B) $f \circ g$
(C) $g^{-1} \circ f^{-1}$ (D) $g \circ f$

42. Let $f : \mathbf{R} - \left\{\frac{3}{5}\right\} \to \mathbf{R}$ be defined by $f(x) = \frac{3x+2}{5x-3}$. Then

(A) $f^{-1}(x) = f(x)$ (B) $f^{-1}(x) = -f(x)$

(C) $(f \circ f)\, x = -x$ (D) $f^{-1}(x) = \frac{1}{19} f(x)$

43. Let $f : [0, 1] \to [0, 1]$ be defined by $f(x) = \begin{cases} x, \text{ if } x \text{ is rational} \\ 1-x, \text{ if } x \text{ is irrational} \end{cases}$

Then $(f\ o\ f)\ x$ is

(A) constant (B) $1 + x$

(C) x (D) none of these

44. Let $f : [2, \infty) \to \mathbf{R}$ be the function defined by $f(x) = x^2 - 4x + 5$, then the range of f is

(A) $\mathbf{R}$ (B) $[1, \infty)$

(C) $[4, \infty)$ (B) $[5, \infty)$

45. Let $f : \mathbf{N} \to \mathbf{R}$ be the function defined by $f(x) = \dfrac{2x-1}{2}$ and $g : \mathbf{Q} \to \mathbf{R}$ be another function defined by $g(x) = x + 2$. Then $(g\ o\ f)\ \dfrac{3}{2}$ is

(A) 1 (B) 1

(C) $\dfrac{7}{2}$ (B) none of these

46. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 2x : x > 3 \\ x^2 : 1 < x \le 3 \\ 3x : x \le 1 \end{cases}$$

Then $f(-1) + f(2) + f(4)$ is

(A) 9 (B) 14

(C) 5 (D) none of these

47. Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = \tan x$. Then $f^{-1}(1)$ is

(A) $\dfrac{\pi}{4}$ (B) $\{n\pi + \dfrac{\pi}{4} : n \in Z\}$

(C) does not exist (D) none of these

Fill in the blanks in each of the Exercises 48 to 52.

48. Let the relation R be defined in **N** by aRb if $2a + 3b = 30$. Then R = ______.

49. Let the relation R be defined on the set
A = {1, 2, 3, 4, 5} by R = $\{(a, b) : |a^2 - b^2| < 8$. Then R is given by ______.

50. Let $f = \{(1, 2), (3, 5), (4, 1)$ and $g = \{(2, 3), (5, 1), (1, 3)\}$. Then $g\ o\ f$ = ______ and $f\ o\ g$ = ______.

51. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \frac{x}{\sqrt{1+x^2}}$. Then $(f \, o \, f \, o \, f)(x)$ = ________

52. If $f(x) = (4 - (x-7)^3\}$, then $f^{-1}(x)$ = ________.

State **True** or **False** for the statements in each of the Exercises 53 to 63.

53. Let R = {(3, 1), (1, 3), (3, 3)} be a relation defined on the set A = {1, 2, 3}. Then R is symmetric, transitive but not reflexive.

54. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = \sin(3x+2)$ $\forall x \in$ R. Then f is invertible.

55. Every relation which is symmetric and transitive is also reflexive.

56. An integer m is said to be related to another integer n if m is a integral multiple of n. This relation in **Z** is reflexive, symmetric and transitive.

57. Let A = {0, 1} and **N** be the set of natural numbers. Then the mapping $f : \mathbf{N} \to$ A defined by $f(2n-1) = 0$, $f(2n) = 1$, $\forall n \in \mathbf{N}$, is onto.

58. The relation R on the set A = {1, 2, 3} defined as R = {{1, 1), (1, 2), (2, 1), (3, 3)} is reflexive, symmetric and transitive.

59. The composition of functions is commutative.

60. The composition of functions is associative.

61. Every function is invertible.

62. A binary operation on a set has always the identity element.

Chapter 2

INVERSE TRIGONOMETRIC FUNCTIONS

2.1 Overview

2.1.1 *Inverse function*

Inverse of a function 'f' exists, if the function is one-one and onto, i.e, bijective. Since trigonometric functions are many-one over their domains, we restrict their domains and co-domains in order to make them one-one and onto and then find their inverse. The domains and ranges (principal value branches) of inverse trigonometric functions are given below:

Functions	**Domain**	**Range (Principal value branches)**
$y = \sin^{-1}x$	[–1,1]	$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$
$y = \cos^{-1}x$	[–1,1]	$[0,\pi]$
$y = \text{cosec}^{-1}x$	**R**– (–1,1)	$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$
$y = \sec^{-1}x$	**R**– (–1,1)	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$
$y = \tan^{-1}x$	**R**	$\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$
$y = \cot^{-1}x$	**R**	$(0,\pi)$

Notes:

(i) The symbol $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. Infact $\sin^{-1}x$ is an angle, the value of whose sine is x, similarly for other trigonometric functions.

(ii) The smallest numerical value, either positive or negative, of θ is called the principal value of the function.

(iii) Whenever no branch of an inverse trigonometric function is mentioned, we mean the principal value branch. The value of the inverse trigonometic function which lies in the range of principal branch is its principal value.

2.1.2 *Graph of an inverse trigonometric function*

The graph of an inverse trigonometric function can be obtained from the graph of original function by interchanging x-axis and y-axis, i.e, if (a, b) is a point on the graph of trigonometric function, then (b, a) becomes the corresponding point on the graph of its inverse trigonometric function.

It can be shown that the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (i.e., reflection) along the line $y = x$.

2.1.3 *Properties of inverse trigonometric functions*

1. $\sin^{-1}(\sin x) = x \quad : \quad x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\cos^{-1}(\cos x) = x \quad : \quad x \in [0, \pi]$

$\tan^{-1}(\tan x) = x \quad : \quad x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\cot^{-1}(\cot x) = x \quad : \quad x \in (0, \pi)$

$\sec^{-1}(\sec x) = x \quad : \quad x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\text{cosec}^{-1}(\text{cosec } x) = x \quad : \quad x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

2. $\sin(\sin^{-1} x) = x \quad : \quad x \in [-1,1]$

$\cos(\cos^{-1} x) = x \quad : \quad x \in [-1,1]$

$\tan(\tan^{-1} x) = x \quad : \quad x \in \mathbf{R}$

$\cot(\cot^{-1} x) = x \quad : \quad x \in \mathbf{R}$

$\sec(\sec^{-1} x) = x \quad : \quad x \in \mathbf{R} - (-1,1)$

$\text{cosec}(\text{cosec}^{-1} x) = x \quad : \quad x \in \mathbf{R} - (-1,1)$

3. $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x \quad : \quad x \in \mathbf{R} - (-1,1)$

$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x \quad : \quad x \in \mathbf{R} - (-1,1)$

$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x$: $x > 0$

$= -\pi + \cot^{-1}x$: $x < 0$

4. $\sin^{-1}(-x) = -\sin^{-1}x$: $x \in [-1,1]$

$\cos^{-1}(-x) = \pi - \cos^{-1}x$: $x \in [-1,1]$

$\tan^{-1}(-x) = -\tan^{-1}x$: $x \in \mathbf{R}$

$\cot^{-1}(-x) = \pi - \cot^{-1}x$: $x \in \mathbf{R}$

$\sec^{-1}(-x) = \pi - \sec^{-1}x$: $x \in \mathbf{R} - (-1,1)$

$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$: $x \in \mathbf{R} - (-1,1)$

5. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$: $x \in [-1,1]$

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$: $x \in \mathbf{R}$

$\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$: $x \in \mathbf{R} - [-1,1]$

6. $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$: $xy < 1$

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$; $xy > -1$

7. $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$: $-1 \le x \le 1$

$2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}$: $x \ge 0$

$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$: $-1 < x < 1$

2.2 Solved Examples

Short Answer (S.A.)

Example 1 Find the principal value of $\cos^{-1}x$, for $x = \frac{\sqrt{3}}{2}$.

Solution If $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \theta$, then $\cos\theta = \frac{\sqrt{3}}{2}$.

Since we are considering principal branch, $\theta \in [0, \pi]$. Also, since $\frac{\sqrt{3}}{2} > 0$, θ being in the first quadrant, hence $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

Example 2 Evaluate $\tan^{-1}\left(\sin\left(\frac{-\pi}{2}\right)\right)$.

Solution $\tan^{-1}\left(\sin\left(\frac{-\pi}{2}\right)\right) = \tan^{-1}\left(-\sin\left(\frac{\pi}{2}\right)\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$.

Example 3 Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Solution $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left(\cos(2\pi+\frac{\pi}{6})\right) = \cos^{-1}\left(\cos\frac{\pi}{6}\right)$

$$= \frac{\pi}{6}.$$

Example 4 Find the value of $\tan^{-1}\left(\tan\frac{9\pi}{8}\right)$.

Solution $\tan^{-1}\left(\tan\frac{9\pi}{8}\right) = \tan^{-1}\tan\left(\pi + \frac{\pi}{8}\right)$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right) = \frac{\pi}{8}$$

Example 5 Evaluate $\tan(\tan^{-1}(-4))$.

Solution Since $\tan(\tan^{-1}x) = x,\ \forall\ x \in \mathbf{R}$, $\tan(\tan^{-1}(-4) = -4$.

Example 6 Evaluate: $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.

Solution $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \tan^{-1}\sqrt{3} - [\pi - \sec^{-1}2]$

$$= \frac{\pi}{3} - \pi + \cos^{-1}\left(\frac{1}{2}\right) = -\frac{2\pi}{3} + \frac{\pi}{3} = -\frac{\pi}{3}.$$

Example 7 Evaluate: $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$.

Solution $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right] = \sin^{-1}\left[\cos\left(\frac{\pi}{3}\right)\right] = \sin^{-1}\left[\frac{1}{2}\right] = \frac{\pi}{6}$.

Example 8 Prove that $\tan(\cot^{-1}x) = \cot(\tan^{-1}x)$. State with reason whether the equality is valid for all values of x.

Solution Let $\cot^{-1}x = \theta$. Then $\cot\theta = x$

or, $\tan\left(\frac{\pi}{2} - \theta\right) = x \Rightarrow \tan^{-1}x = \frac{\pi}{2} - \theta$

So $\tan(\cot^{-1}x) = \tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) = \cot\left(\frac{\pi}{2} - \cot^{-1}x\right) = \cot(\tan^{-1}x)$

The equality is valid for all values of x since $\tan^{-1}x$ and $\cot^{-1}x$ are true for $x \in \mathbf{R}$.

Example 9 Find the value of $\sec\left(\tan^{-1}\frac{y}{2}\right)$.

Solution Let $\tan^{-1}\frac{y}{2} = \theta$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So, $\tan\theta = \frac{y}{2}$,

which gives $\sec\theta = \frac{\sqrt{4+y^2}}{2}$.

Therefore, $\sec\left(\tan^{-1}\frac{y}{2}\right) = \sec\theta = \frac{\sqrt{4+y^2}}{2}$.

Example 10 Find value of $\tan(\cos^{-1}x)$ and hence evaluate $\tan\left(\cos^{-1}\frac{8}{17}\right)$.

Solution Let $\cos^{-1}x = \theta$, then $\cos\theta = x$, where $\theta \in [0,\pi]$

Therefore, $\tan(\cos^{-1}x) = \tan\theta = \dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \dfrac{\sqrt{1-x^2}}{x}$.

Hence $\tan\left(\cos^{-1}\dfrac{8}{17}\right) = \dfrac{\sqrt{1-\left(\dfrac{8}{17}\right)^2}}{\dfrac{8}{17}} = \dfrac{15}{8}$.

Example 11 Find the value of $\sin\left[2\cot^{-1}\left(\dfrac{-5}{12}\right)\right]$

Solution Let $\cot^{-1}\left(\dfrac{-5}{12}\right) = y$. Then $\cot y = \dfrac{-5}{12}$.

Now $\sin\left[2\cot^{-1}\left(\dfrac{-5}{12}\right)\right] = \sin 2y$

$$= 2\sin y\cos y \quad = 2\left(\frac{12}{13}\right)\left(\frac{-5}{13}\right) \quad \left[\text{since } \cot y < 0,\text{ so } y \in \left(\frac{\pi}{2}, \pi\right)\right]$$

$$= \frac{-120}{169}$$

Example 12 Evaluate $\cos\left[\sin^{-1}\dfrac{1}{4}+\sec^{-1}\dfrac{4}{3}\right]$

Solution $\cos\left[\sin^{-1}\dfrac{1}{4}+\sec^{-1}\dfrac{4}{3}\right] = \cos\left[\sin^{-1}\dfrac{1}{4}+\cos^{-1}\dfrac{3}{4}\right]$

$$= \cos\left(\sin^{-1}\frac{1}{4}\right)\cos\left(\cos^{-1}\frac{3}{4}\right) - \sin\left(\sin^{-1}\frac{1}{4}\right)\sin\left(\cos^{-1}\frac{3}{4}\right)$$

$$= \frac{3}{4}\sqrt{1-\left(\frac{1}{4}\right)^2} - \frac{1}{4}\sqrt{1-\left(\frac{3}{4}\right)^2}$$

$$= \frac{3}{4}\frac{\sqrt{15}}{4} - \frac{1}{4}\frac{\sqrt{7}}{4} = \frac{3\sqrt{15}-\sqrt{7}}{16}.$$

Long Answer (L.A.)

Example 13 Prove that $2\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$

Solution Let $\sin^{-1}\frac{3}{5} = \theta$, then $\sin\theta = \frac{3}{5}$, where $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Thus $\tan\theta = \frac{3}{4}$, which gives $\theta = \tan^{-1}\frac{3}{4}$.

Therefore, $2\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{17}{31}$

$$= 2\theta - \tan^{-1}\frac{17}{31} \quad = 2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31}$$

$$= \tan^{-1}\left(\frac{2.\frac{3}{4}}{1-\frac{9}{16}}\right) - \tan^{-1}\frac{17}{31} = \tan^{-1}\frac{24}{7} - \tan^{-1}\frac{17}{31}$$

$$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}.\frac{17}{31}}\right) = \frac{\pi}{4}$$

Example 14 Prove that

$$\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$$

Solution We have

$$\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$$

$$= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} \qquad \left(\text{since } \cot^{-1}x = \tan^{-1}\frac{1}{x}, \text{ if } x > 0\right)$$

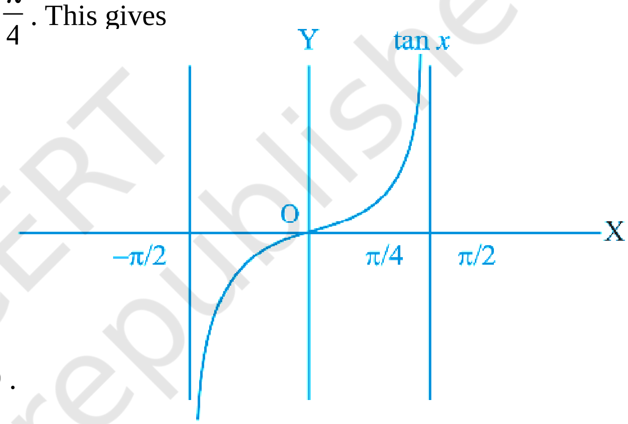
$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right) + \tan^{-1}\frac{1}{18} \qquad \left(\text{since } x\,.\,y = \frac{1}{7}.\frac{1}{8} < 1\right)$$

$$= \tan^{-1}\frac{3}{11} + \tan^{-1}\frac{1}{18} = \tan^{-1}\left(\frac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11}\times\frac{1}{18}}\right) \qquad \text{(since } xy < 1\text{)}$$

$$= \tan^{-1}\frac{65}{195} \qquad = \tan^{-1}\frac{1}{3} \qquad = \cot^{-1} 3$$

Example 15 Which is greater, tan 1 or $\tan^{-1} 1$?

Solution From Fig. 2.1, we note that tan x is an increasing function in the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, since $1 > \frac{\pi}{4} \Rightarrow \tan 1 > \tan\frac{\pi}{4}$. This gives



$$\tan 1 > 1$$

$$\Rightarrow \qquad \tan 1 > 1 > \frac{\pi}{4}$$

$$\Rightarrow \qquad \tan 1 > 1 > \tan^{-1}(1).$$

Example 16 Find the value of

$$\sin\left(2\tan^{-1}\frac{2}{3}\right) + \cos(\tan^{-1}\sqrt{3}).$$

Solution Let $\tan^{-1}\frac{2}{3} = x$ and $\tan^{-1}\sqrt{3} = y$ so that $\tan x = \frac{2}{3}$ and $\tan y = \sqrt{3}$.

Therefore, $\qquad \sin\left(2\tan^{-1}\frac{2}{3}\right) + \cos(\tan^{-1}\sqrt{3})$

$$= \sin(2x) + \cos y$$

$$= \frac{2\tan x}{1+\tan^2 x} + \frac{1}{\sqrt{1+\tan^2 y}} = \frac{2.\frac{2}{3}}{1+\frac{4}{9}} + \frac{1}{1+\sqrt{\left(\sqrt{3}\right)^2}}$$

$$= \frac{12}{13} + \frac{1}{2} = \frac{37}{26}.$$

Example 17 Solve for x

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x,\ x>0$$

Solution From given equation, we have $2\tan^{-1}\left(\frac{1-x}{1+x}\right)=\tan^{-1}x$

$$\Rightarrow \qquad 2\left[\tan^{-1}1-\tan^{-1}x\right]=\tan^{-1}x$$

$$\Rightarrow \qquad 2\left(\frac{\pi}{4}\right)=3\tan^{-1}x \Rightarrow \frac{\pi}{6}=\tan^{-1}x$$

$$\Rightarrow \qquad x=\frac{1}{\sqrt{3}}$$

Example 18 Find the values of x which satisfy the equation

$$\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x.$$

Solution From the given equation, we have

$$\sin(\sin^{-1}x+\sin^{-1}(1-x))=\sin(\cos^{-1}x)$$

$$\Rightarrow \sin(\sin^{-1}x)\cos(\sin^{-1}(1-x))+\cos(\sin^{-1}x)\sin(\sin^{-1}(1-x))=\sin(\cos^{-1}x)$$

$$\Rightarrow x\sqrt{1-(1-x)^2}+(1-x)\sqrt{1-x^2}=\sqrt{1-x^2}$$

$$\Rightarrow x\sqrt{2x-x^2}+\sqrt{1-x^2}(1-x-1)=0$$

$$\Rightarrow x\left(\sqrt{2x-x^2}-\sqrt{1-x^2}\right)=0$$

$$\Rightarrow x=0 \qquad \text{or} \qquad 2x-x^2=1-x^2$$

$$\Rightarrow x=0 \qquad \text{or} \qquad x=\frac{1}{2}.$$

Example 19 Solve the equation $\sin^{-1}6x+\sin^{-1}6\sqrt{3}\,x=-\frac{\pi}{2}$

Solution From the given equation, we have $\sin^{-1}6x=-\frac{\pi}{2}-\sin^{-1}6\sqrt{3}\,x$

$$\Rightarrow \qquad \sin(\sin^{-1} 6x) = \sin\left(-\frac{\pi}{2} - \sin^{-1} 6\sqrt{3}\,x\right)$$

$$\Rightarrow \qquad 6x = -\cos(\sin^{-1} 6\sqrt{3}\,x)$$

$$\Rightarrow \qquad 6x = -\sqrt{1-108x^2}\text{ . Squaring, we get}$$

$$36x^2 = 1 - 108x^2$$

$$\Rightarrow \qquad 144x^2 = 1 \qquad \Rightarrow x = \pm\frac{1}{12}$$

Note that $x = -\frac{1}{12}$ is the only root of the equation as $x = \frac{1}{12}$ does not satisfy it.

Example 20 Show that

$$2\tan^{-1}\left\{\tan\frac{\alpha}{2}.\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\} = \tan^{-1}\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}$$

Solution L.H.S. $= \tan^{-1}\dfrac{2\tan\frac{\alpha}{2}.\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{1-\tan^2\frac{\alpha}{2}\tan^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}$ $\qquad \left(\text{since } 2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}\right)$

$$= \tan^{-1}\frac{2\tan\frac{\alpha}{2}\,\dfrac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}}{1-\tan^2\frac{\alpha}{2}\left(\dfrac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}\right)^2}$$

$$= \tan^{-1}\frac{2\tan\frac{\alpha}{2}\,.\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan\frac{\beta}{2}\right)^2 - \tan^2\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)^2}$$

$$= \quad \tan^{-1}\frac{2\tan\frac{\alpha}{2}\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan^2\frac{\beta}{2}\right)\left(1-\tan^2\frac{\alpha}{2}\right)+2\tan\frac{\beta}{2}\left(1+\tan^2\frac{\alpha}{2}\right)}$$

$$= \quad \tan^{-1}\frac{\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\;\dfrac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}{\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\dfrac{2\tan\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}$$

$$= \quad \tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}\right) \quad = \text{R.H.S.}$$

Objective type questions

Choose the correct answer from the given four options in each of the Examples 21 to 41.

Example 21 Which of the following corresponds to the principal value branch of $\tan^{-1}$?

(A) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (B) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(C) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)-\{0\}$ (D) $(0, \pi)$

Solution (A) is the correct answer.

Example 22 The principal value branch of $\sec^{-1}$ is

(A) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ (B) $[0, \pi]-\left\{\frac{\pi}{2}\right\}$

(C) $(0, \pi)$ (D) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Solution (B) is the correct answer.

Example 23 One branch of $\cos^{-1}$ other than the principal value branch corresponds to

(A) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ (B) $[\pi, 2\pi] - \left\{\frac{3\pi}{2}\right\}$

(C) $(0, \pi)$ (D) $[2\pi, 3\pi]$

Solution (D) is the correct answer.

Example 24 The value of $\sin^{-1}\left(\cos\left(\frac{43\pi}{5}\right)\right)$ is

(A) $\frac{3\pi}{5}$ (B) $\frac{-7\pi}{5}$ (C) $\frac{\pi}{10}$ (D) $-\frac{\pi}{10}$

Solution (D) is the correct answer. $\sin^{-1}\left(\cos\frac{40\pi+3\pi}{5}\right) = \sin^{-1}\cos\left(8\pi+\frac{3\pi}{5}\right)$

$$= \sin^{-1}\left(\cos\frac{3\pi}{5}\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{2}-\frac{3\pi}{5}\right)\right)$$

$$= \sin^{-1}\left(\sin\left(-\frac{\pi}{10}\right)\right) = -\frac{\pi}{10}.$$

Example 25 The principal value of the expression $\cos^{-1}[\cos(-680°)]$ is

(A) $\frac{2\pi}{9}$ (B) $\frac{-2\pi}{9}$ (C) $\frac{34\pi}{9}$ (D) $\frac{\pi}{9}$

Solution (A) is the correct answer. $\cos^{-1}(\cos(680°)) = \cos^{-1}[\cos(720° - 40°)]$

$$= \cos^{-1}[\cos(-40°)] = \cos^{-1}[\cos(40°)] = 40° = \frac{2\pi}{9}.$$

Example 26 The value of $\cot(\sin^{-1}x)$ is

(A) $\frac{\sqrt{1+x^2}}{x}$ (B) $\frac{x}{\sqrt{1+x^2}}$

(C) $\frac{1}{x}$ (D) $\frac{\sqrt{1-x^2}}{x}$.

Solution (D) is the correct answer. Let $\sin^{-1} x = \theta$, then $\sin\theta = x$

$$\Rightarrow \qquad \operatorname{cosec}\theta = \frac{1}{x} \Rightarrow \operatorname{cosec}^2\theta = \frac{1}{x^2}$$

$$\Rightarrow \qquad 1 + \cot^2\theta = \frac{1}{x^2} \Rightarrow \cot\theta = \frac{\sqrt{1-x^2}}{x}.$$

Example 27 If $\tan^{-1}x = \frac{\pi}{10}$ for some $x \in \mathbf{R}$, then the value of $\cot^{-1}x$ is

(A) $\frac{\pi}{5}$ (B) $\frac{2\pi}{5}$ (C) $\frac{3\pi}{5}$ (D) $\frac{4\pi}{5}$

Solution (B) is the correct answer. We know $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$. Therefore

$$\cot^{-1}x = \frac{\pi}{2} - \frac{\pi}{10}$$

$$\Rightarrow \cot^{-1}x = \frac{\pi}{2} - \frac{\pi}{10} = \frac{2\pi}{5}.$$

Example 28 The domain of $\sin^{-1} 2x$ is

(A) $[0, 1]$ (B) $[-1, 1]$

(C) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (D) $[-2, 2]$

Solution (C) is the correct answer. Let $\sin^{-1}2x = \theta$ so that $2x = \sin\theta$.

Now $-1 \le \sin\theta \le 1$, i.e., $-1 \le 2x \le 1$ which gives $-\frac{1}{2} \le x \le \frac{1}{2}$.

Example 29 The principal value of $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ is

(A) $-\frac{2\pi}{3}$ (B) $-\frac{\pi}{3}$ (C) $\frac{4\pi}{3}$ (D) $\frac{5\pi}{3}$.

Solution (B) is the correct answer.

$$\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)=\sin^{-1}\left(-\sin\frac{\pi}{3}\right)=-\sin^{-1}\left(\sin\frac{\pi}{3}\right)=-\frac{\pi}{3}.$$

Example 30 The greatest and least values of $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$ are respectively

(A) $\frac{5\pi^2}{4}$ and $\frac{\pi^2}{8}$ (B) $\frac{\pi}{2}$ and $\frac{-\pi}{2}$

(C) $\frac{\pi^2}{4}$ and $\frac{-\pi^2}{4}$ (D) $\frac{\pi^2}{4}$ and 0.

Solution (A) is the correct answer. We have

$$(\sin^{-1}x)^2 + (\cos^{-1}x)^2 = (\sin^{-1}x + \cos^{-1}x)^2 - 2\sin^{-1}x\cos^{-1}x$$

$$= \frac{\pi^2}{4} - 2\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)$$

$$= \frac{\pi^2}{4} - \pi\sin^{-1}x + 2\left(\sin^{-1}x\right)^2$$

$$= 2\left[\left(\sin^{-1}x\right)^2 - \frac{\pi}{2}\sin^{-1}x + \frac{\pi^2}{8}\right]$$

$$= 2\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right].$$

Thus, the least value is $2\left(\frac{\pi^2}{16}\right)$ i.e. $\frac{\pi^2}{8}$ and the Greatest value is $2\left[\left(\frac{-\pi}{2} - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right]$,

i.e. $\frac{5\pi^2}{4}$.

Example 31 Let $\theta = \sin^{-1}(\sin(-600°))$, then value of θ is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{2\pi}{3}$ (D) $\frac{-2\pi}{3}$.

Solution (A) is the correct answer.

$$\sin^{-1}\sin\left(-600\times\frac{\pi}{180}\right)=\sin^{-1}\sin\left(\frac{-10\pi}{3}\right)$$

$$=\sin^{-1}\left[-\sin\left(4\pi-\frac{2\pi}{3}\right)\right]=\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

$$=\sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}.$$

Example 32 The domain of the function $y = \sin^{-1}(-x^2)$ is

(A) [0, 1] (B) (0, 1)

(C) [–1, 1] (D) ϕ

Solution (C) is the correct answer. $y = \sin^{-1}(-x^2) \Rightarrow \sin y = -x^2$

i.e. $-1 \le -x^2 \le 1$ (since $-1 \le \sin y \le 1$)

$\Rightarrow 1 \ge x^2 \ge -1$

$\Rightarrow 0 \le x^2 \le 1$

$\Rightarrow |x| \le 1$ *i.e.* $-1 \le x \le 1$

Example 33 The domain of $y = \cos^{-1}(x^2 - 4)$ is

(A) [3, 5] (B) $[0, \pi]$

(C) $\left[-\sqrt{5}, -\sqrt{3}\right] \cap \left[-\sqrt{5}, \sqrt{3}\right]$ (D) $\left[-\sqrt{5}, -\sqrt{3}\right] \cup \left[\sqrt{3}, \sqrt{5}\right]$

Solution (D) is the correct answer. $y = \cos^{-1}(x^2 - 4) \Rightarrow \cos y = x^2 - 4$

i.e. $-1 \le x^2 - 4 \le 1$ (since $-1 \le \cos y \le 1$)

$\Rightarrow 3 \le x^2 \le 5$

$\Rightarrow \sqrt{3} \le |x| \le \sqrt{5}$

$\Rightarrow x \in \left[-\sqrt{5}, -\sqrt{3}\right] \cup \left[\sqrt{3}, \sqrt{5}\right]$

Example 34 The domain of the function defined by $f(x) = \sin^{-1}x + \cos x$ is

(A) $[-1, 1]$ (B) $[-1, \pi + 1]$

(C) $(-\infty, \infty)$ (D) ϕ

Solution (A) is the correct answer. The domain of cos is **R** and the domain of $\sin^{-1}$ is $[-1, 1]$. Therefore, the domain of $\cos x + \sin^{-1}x$ is $\mathbf{R} \cap [-1,1]$, i.e., $[-1, 1]$.

Example 35 The value of $\sin(2\sin^{-1}(\cdot 6))$ is

(A) ·48 (B) ·96 (C) 1·2 (D) sin 1·2

Solution (B) is the correct answer. Let $\sin^{-1}(\cdot 6) = \theta$, i.e., $\sin\theta = \cdot 6$.

Now $\sin(2\theta) = 2\sin\theta\cos\theta = 2(\cdot 6)(\cdot 8) = \cdot 96$.

Example 36 If $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$, then value of $\cos^{-1} x + \cos^{-1} y$ is

(A) $\frac{\pi}{2}$ (B) π (C) 0 (D) $\frac{2\pi}{3}$

Solution (A) is the correct answer. Given that $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$.

Therefore, $\left(\frac{\pi}{2} - \cos^{-1} x\right) + \left(\frac{\pi}{2} - \cos^{-1} y\right) = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = \frac{\pi}{2}$.

Example 37 The value of $\tan\left(\cos^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4}\right)$ is

(A) $\frac{19}{8}$ (B) $\frac{8}{19}$ (C) $\frac{19}{12}$ (D) $\frac{3}{4}$

Solution (A) is the correct answer. $\tan\left(\cos^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4}\right) = \tan\left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{4}\right)$

$$= \tan \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{4}}{1-\frac{4}{3}\times\frac{1}{4}}\right) = \tan \tan^{-1}\left(\frac{19}{8}\right) = \frac{19}{8}.$$

Example 38 The value of the expression sin [cot^{-1} (cos (tan^{-1} 1))] is

(A) 0 (B) 1 (C) $\frac{1}{\sqrt{3}}$ (D) $\sqrt{\frac{2}{3}}$.

Solution (D) is the correct answer.

$$\sin [\cot^{-1} (\cos \frac{\pi}{4})] = \sin [\cot^{-1} \frac{1}{\sqrt{2}}] = \sin\left[\sin^{-1}\sqrt{\frac{2}{3}}\right] = \sqrt{\frac{2}{3}}$$

Example 39 The equation $\tan^{-1}x - \cot^{-1}x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ has

(A) no solution (B) unique solution
(C) infinite number of solutions (D) two solutions

Solution (B) is the correct answer. We have

$$\tan^{-1}x - \cot^{-1}x = \frac{\pi}{6} \text{ and } \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$

Adding them, we get $2\tan^{-1}x = \frac{2\pi}{3}$

$$\Rightarrow \tan^{-1}x = \frac{\pi}{3} \text{ i.e., } x = \sqrt{3}.$$

Example 40 If $\alpha \le 2\sin^{-1}x + \cos^{-1}x \le \beta$, then

(A) $\alpha = \frac{-\pi}{2}, \beta = \frac{\pi}{2}$ (B) $\alpha = 0, \beta = \pi$

(C) $\alpha = \frac{-\pi}{2}, \beta = \frac{3\pi}{2}$ (D) $\alpha = 0, \beta = 2\pi$

Solution (B) is the correct answer. We have $\frac{-\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$$\Rightarrow \quad \frac{-\pi}{2} + \frac{\pi}{2} \le \sin^{-1}x + \frac{\pi}{2} \le \frac{\pi}{2} + \frac{\pi}{2}$$

$$\Rightarrow \quad 0 \le \sin^{-1}x + (\sin^{-1}x + \cos^{-1}x) \le \pi$$

$$\Rightarrow \quad 0 \le 2\sin^{-1}x + \cos^{-1}x \le \pi$$

Example 41 The value of $\tan^2 (\sec^{-1}2) + \cot^2 (\text{cosec}^{-1}3)$ is

(A) 5 (B) 11 (C) 13 (D) 15

Solution (B) is the correct answer.

$\tan^2 (\sec^{-1}2) + \cot^2 (\text{cosec}^{-1}3) = \sec^2 (\sec^{-1}2) - 1 + \text{cosec}^2 (\text{cosec}^{-1}3) - 1$

$= 2^2 \times 1 + 3^2 - 2 = 11.$

2.3 EXERCISE

Short Answer (S.A.)

1. Find the value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

2. Evaluate $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$.

3. Prove that $\cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right) = 7$.

4. Find the value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\sin\left(\frac{-\pi}{2}\right)\right)$.

5. Find the value of $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$.

6. Show that $2\tan^{-1}(-3) = \frac{-\pi}{2} + \tan^{-1}\left(\frac{-4}{3}\right)$.

7. Find the real solutions of the equation

$$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}.$$

8. Find the value of the expression $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)$.

9. If 2 $\tan^{-1}$ (cos θ) = $\tan^{-1}$ (2 cosec θ), then show that θ = $\frac{\pi}{4}$,

where *n* is any integer.

10. Show that $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$.

11. Solve the following equation $\cos\left(\tan^{-1}x\right)=\sin\left(\cot^{-1}\frac{3}{4}\right)$.

Long Answer (L.A.)

12. Prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$

13. Find the simplified form of $\cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)$, where $x\in\left[\frac{-3\pi}{4},\frac{\pi}{4}\right]$.

14. Prove that $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{77}{85}$.

15. Show that $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$.

16. Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\sin^{-1}\frac{1}{\sqrt{5}}$.

17. Find the value of $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$.

18. Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$ and justify why the other value $\frac{4+\sqrt{7}}{3}$ is ignored?

19. If $a_1, a_2, a_3,...,a_n$ is an arithmetic progression with common difference d, then evaluate the following expression.

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1 a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2 a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3 a_4}\right)+...+\tan^{-1}\left(\frac{d}{1+a_{n-1} a_n}\right)\right].$$

Objective Type Questions

Choose the correct answers from the given four options in each of the Exercises from 20 to 37 (M.C.Q.).

20. Which of the following is the principal value branch of $\cos^{-1}x$?

(A) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (B) $(0, \pi)$

(C) $[0, \pi]$ (D) $(0, \pi) - \left\{\frac{\pi}{2}\right\}$

21. Which of the following is the principal value branch of $\text{cosec}^{-1}x$?

(A) $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ (B) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

(C) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (D) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right] - \{0\}$

22. If $3\tan^{-1} x + \cot^{-1} x = \pi$, then x equals

(A) 0 (B) 1 (C) –1 (D) $\frac{1}{2}$.

23. The value of $\sin^{-1}\left(\cos\left(\frac{33\pi}{5}\right)\right)$ is

(A) $\frac{3\pi}{5}$ (B) $\frac{-7\pi}{5}$ (C) $\frac{\pi}{10}$ (D) $\frac{-\pi}{10}$

24. The domain of the function $\cos^{-1}(2x - 1)$ is

(A) [0, 1] (B) [–1, 1]
(C) (–1, 1) (D) [0, π]

25. The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is

(A) [1, 2] (B) [–1, 1]
(C) [0, 1] (D) none of these

26. If $\cos\left(\sin^{-1}\frac{2}{5} + \cos^{-1}x\right) = 0$, then x is equal to

(A) $\frac{1}{5}$ (B) $\frac{2}{5}$ (C) 0 (D) 1

27. The value of $\sin(2\tan^{-1}(.75))$ is equal to

(A) ·75 (B) 1·5 (C) ·96 (D) sin 1·5

28. The value of $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{3\pi}{2}$ (C) $\frac{5\pi}{2}$ (D) $\frac{7\pi}{2}$

29. The value of the expression $2\sec^{-1}2 + \sin^{-1}\left(\frac{1}{2}\right)$ is

(A) $\frac{\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{7\pi}{6}$ (D) 1

30. If $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$, then $\cot^{-1}x + \cot^{-1}y$ equals

(A) $\frac{\pi}{5}$ (B) $\frac{2\pi}{5}$ (C) $\frac{3\pi}{5}$ (D) π

31. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $a, x \in]0, 1$, then the value of x is

(A) 0 (B) $\frac{a}{2}$ (C) a (D) $\frac{2a}{1-a^2}$

32. The value of cot $\left[\cos^{-1}\left(\frac{7}{25}\right)\right]$ is

(A) $\frac{25}{24}$ (B) $\frac{25}{7}$ (C) $\frac{24}{25}$ (D) $\frac{7}{24}$

33. The value of the expression tan $\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)$ is

(A) $2+\sqrt{5}$ (B) $\sqrt{5}-2$

(C) $\frac{\sqrt{5}+2}{2}$ (D) $5+\sqrt{2}$

$\left[\text{Hint}: \tan\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\ \right]$

34. If $|x| \leq 1$, then $2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is equal to

(A) $4\tan^{-1}x$ (B) 0 (C) $\frac{\pi}{2}$ (D) π

35. If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, then $\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$ equals

(A) 0 (B) 1 (C) 6 (D) 12

36. The number of real solutions of the equation $\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x)$ in $\left[\frac{\pi}{2}, \pi\right]$ is

(A) 0 (B) 1 (C) 2 (D) Infinite

37. If $\cos^{-1}x > \sin^{-1}x$, then

(A) $\frac{1}{\sqrt{2}} < x \leq 1$ (B) $0 \leq x < \frac{1}{\sqrt{2}}$

(C) $-1 \leq x < \frac{1}{\sqrt{2}}$ (D) $x > 0$

Fill in the blanks in each of the Exercises 38 to 48.

38. The principal value of $\cos^{-1}\left(-\dfrac{1}{2}\right)$ is__________.

39. The value of $\sin^{-1}\left(\sin\dfrac{3\pi}{5}\right)$ is__________.

40. If $\cos(\tan^{-1} x + \cot^{-1}\sqrt{3}) = 0$, then value of x is__________.

41. The set of values of $\sec^{-1}\left(\dfrac{1}{2}\right)$ is__________.

42. The principal value of $\tan^{-1}\sqrt{3}$ is__________.

43. The value of $\cos^{-1}\left(\cos\dfrac{14\pi}{3}\right)$ is__________.

44. The value of $\cos(\sin^{-1} x + \cos^{-1} x)$, $|x| \leq 1$ is______ .

45. The value of expression $\tan\left(\dfrac{\sin^{-1} x + \cos^{-1} x}{2}\right)$, when $x = \dfrac{\sqrt{3}}{2}$ is_________.

46. If $y = 2\tan^{-1} x + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ for all x, then____$< y <$____.

47. The result $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$ is true when value of xy is _____

48. The value of $\cot^{-1}(-x)$ for all $x \in \mathbf{R}$ in terms of $\cot^{-1}x$ is _______.

State **True** or **False** for the statement in each of the Exercises 49 to 55.

49. All trigonometric functions have inverse over their respective domains.

50. The value of the expression $(\cos^{-1} x)^2$ is equal to $\sec^2 x$.

51. The domain of trigonometric functions can be restricted to any one of their branch (not necessarily principal value) in order to obtain their inverse functions.

52. The least numerical value, either positive or negative of angle θ is called principal value of the inverse trigonometric function.

53. The graph of inverse trigonometric function can be obtained from the graph of their corresponding trigonometric function by interchanging x and y axes.

54. The minimum value of n for which $\tan^{-1}\frac{n}{\pi} > \frac{\pi}{4}, n \in \mathbf{N}$, is valid is 5.

55. The principal value of $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right]$ is $\frac{\pi}{3}$.

Chapter 3

Matrices

3.1 Overview

3.1.1 A matrix is an ordered rectangular array of numbers (or functions). For example,

$$A = \begin{bmatrix} x & 4 & 3 \\ 4 & 3 & x \\ 3 & x & 4 \end{bmatrix}$$

The numbers (or functions) are called the elements or the entries of the matrix.

The horizontal lines of elements are said to constitute rows of the matrix and the vertical lines of elements are said to constitute columns of the matrix.

3.1.2 *Order of a Matrix*

A matrix having m rows and n columns is called a matrix of order $m \times n$ or simply $m \times n$ matrix (read as an m by n matrix).

In the above example, we have A as a matrix of order 3×3 i.e., 3×3 matrix.

In general, an $m \times n$ matrix has the following rectangular array :

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mn} \end{bmatrix}_{m \times n} \quad 1 \le i \le m,\ 1 \le j \le n \quad i, j \in \mathbf{N}.$$

The element, a_{ij} is an element lying in the i^{th} row and j^{th} column and is known as the $(i, j)^{th}$ element of A. The number of elements in an $m \times n$ matrix will be equal to mn.

3.1.3 *Types of Matrices*

(i) A matrix is said to be a *row matrix* if it has only one row.

(ii) A matrix is said to be a *column matrix* if it has only one column.

(iii) A matrix in which the number of rows are equal to the number of columns, is said to be a square matrix. Thus, an $m \times n$ matrix is said to be a square matrix if $m = n$ and is known as a square matrix of order 'n'.

(iv) A square matrix B = $[b_{ij}]_{n\times n}$ is said to be a *diagonal matrix* if its all non diagonal elements are zero, that is a matrix B = $[b_{ij}]_{n\times n}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq$ j.

(v) A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal, that is, a square matrix B = $[b_{ij}]_{n\times n}$ is said to be a scalar matrix if

$b_{ij} = 0$, when $i \neq j$

$b_{ij} = k$, when $i = j$, for some constant k.

(vi) A square matrix in which elements in the diagonal are all 1 and rest are all zeroes is called an identity matrix.

In other words, the square matrix A = $[a_{ij}]_{n\times n}$ is an identity matrix, if $a_{ij} = 1$, when $i = j$ and $a_{ij} = 0$, when $i \neq j$.

(vii) A matrix is said to be *zero matrix* or *null matrix* if all its elements are zeroes. We denote zero matrix by O.

(ix) Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are said to be equal if

(a) they are of the same order, and

(b) each element of A is equal to the corresponding element of B, that is, $a_{ij} = b_{ij}$ for all i and j.

3.1.4 *Additon of Matrices*

Two matrices can be added if they are of the same order.

3.1.5 *Multiplication of Matrix by a Scalar*

If A = $[a_{ij}]_{m\times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by a scalar k, i.e. kA = $[ka_{ij}]_{m\times n}$

3.1.6 *Negative of a Matrix*

The negative of a matrix A is denoted by –A. We define –A = (–1)A.

3.1.7 *Multiplication of Matrices*

The multiplication of two matrices A and B is defined if the number of columns of A is equal to the number of rows of B.

Let A = $[a_{ij}]$ be an $m \times n$ matrix and B = $[b_{jk}]$ be an $n \times p$ matrix. Then the product of the matrices A and B is the matrix C of order $m \times p$. To get the $(i, k)^{\text{th}}$ element c_{ik} of the matrix C, we take the i^{th} row of A and k^{th} column of B, multiply them elementwise and take the sum of all these products i.e.,

$$c_{ik} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + a_{i3}\, b_{3k} + \ldots + a_{in}\, b_{nk}$$

The matrix C = $[c_{ik}]_{m\times p}$ is the product of A and B.

Notes:

1. If AB is defined, then BA need not be defined.
2. If A, B are, respectively $m \times n$, $k \times l$ matrices, then both AB and BA are defined if and only if $n = k$ and $l = m$.
3. If AB and BA are both defined, it is not necessary that AB = BA.
4. If the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.
5. For three matrices A, B and C of the same order, if A = B, then AC = BC, but converse is not true.
6. A. A = A^2, A. A. A = A^3, so on

3.1.8 *Transpose of a Matrix*

1. If A = $[a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A.

   Transpose of the matrix A is denoted by A′ or (A^T). In other words, if A = $[a_{ij}]_{m\times n}$, then $A^T = [a_{ji}]_{n\times m}$.

2. Properties of transpose of the matrices

For any matrices A and B of suitable orders, we have

(i) $(A^T)^T = A$,

(ii) $(kA)^T = kA^T$ (where k is any constant)

(iii) $(A + B)^T = A^T + B^T$

(iv) $(AB)^T = B^T\, A^T$

3.1.9 *Symmetric Matrix and Skew Symmetric Matrix*

(i) A square matrix A = $[a_{ij}]$ is said to be symmetric if $A^T = A$, that is, $a_{ij} = a_{ji}$ for all possible values of i and j.

(ii) A square matrix $A = [a_{ij}]$ is said to be skew symmetric matrix if $A^T = -A$, that is $a_{ji} = -a_{ij}$ for all possible values of i and j.

Note : Diagonal elements of a skew symmetric matrix are zero.

(iii) **Theorem 1:** For any square matrix A with real number entries, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.

(iv) **Theorem 2:** Any square matrix A can be expressed as the sum of a symmetric matrix and a skew symmetric matrix, that is

$$A = \frac{(A + A^T)}{2} + \frac{(A - A^T)}{2}$$

3.1.10 *Invertible Matrices*

(i) If A is a square matrix of order $m \times m$, and if there exists another square matrix B of the same order $m \times m$, such that $AB = BA = I_m$, then, A is said to be invertible matrix and B is called the inverse matrix of A and it is denoted by A^{-1}.

Note :

1. A rectangular matrix does not possess its inverse, since for the products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.

2. If B is the inverse of A, then A is also the inverse of B.

(ii) **Theorem 3** (Uniqueness of inverse) Inverse of a square matrix, if it exists, is unique.

(iii) **Theorem 4** : If A and B are invertible matrices of same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

3.1.11 *Inverse of a Matrix using Elementary Row or Column Operations*

To find A^{-1} using elementary row operations, write A = IA and apply a sequence of row operations on (A = IA) till we get, I = BA. The matrix B will be the inverse of A. Similarly, if we wish to find A^{-1} using column operations, then, write A = AI and apply a sequence of column operations on A = AI till we get, I = AB.

Note : In case, after applying one or more elementary row (or column) operations on A = IA (or A = AI), if we obtain all zeros in one or more rows of the matrix A on L.H.S., then A^{-1} does not exist.

3.2 Solved Examples

Short Answer (S.A.)

Example 1 Construct a matrix A = $[a_{ij}]_{2\times 2}$ whose elements a_{ij} are given by $a_{ij} = e^{2ix} \sin jx$.

Solution For $i = 1, j = 1,$ $a_{11} = e^{2x} \sin x$

For $i = 1, j = 2,$ $a_{12} = e^{2x} \sin 2x$

For $i = 2, j = 1,$ $a_{21} = e^{4x} \sin x$

For $i = 2, j = 2,$ $a_{22} = e^{4x} \sin 2x$

Thus $$A = \begin{bmatrix} e^{2x}\sin x & e^{2x}\sin 2x \\ e^{4x}\sin x & e^{4x}\sin 2x \end{bmatrix}$$

Example 2 If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 6 & 8 \\ 5 & 7 & 9 \end{bmatrix}$, then which of the sums A + B, B + C, C + D and B + D is defined?

Solution Only B + D is defined since matrices of the same order can only be added.

Example 3 Show that a matrix which is both symmetric and skew symmetric is a zero matrix.

Solution Let A = $[a_{ij}]$ be a matrix which is both symmetric and skew symmetric.

Since A is a skew symmetric matrix, so A′ = –A.

Thus for all i and j, we have $a_{ij} = -a_{ji}$. (1)

Again, since A is a symmetric matrix, so A′ = A.

Thus, for all i and j, we have

$$a_{ji} = a_{ij} \qquad (2)$$

Therefore, from (1) and (2), we get

$$a_{ij} = -a_{ij} \text{ for all } i \text{ and } j$$

or $2a_{ij} = 0$,

i.e., $a_{ij} = 0$ for all i and j. Hence A is a zero matrix.

Example 4 If $\begin{bmatrix} 2x & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 8 \end{bmatrix} = O$, find the value of x.

Solution We have

$$\begin{bmatrix} 2x & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 8 \end{bmatrix} = O \Rightarrow \begin{bmatrix} 2x-9 & 4x \end{bmatrix} \begin{bmatrix} x \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

or $$\begin{bmatrix} 2x^2 - 9x & +32x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \Rightarrow 2x^2 + 23x = 0$$

or $$x(2x+23) = 0 \qquad \Rightarrow \qquad x = 0,\ x = \frac{-23}{2}$$

Example 5 If A is 3 × 3 invertible matrix, then show that for any scalar k (non-zero), kA is invertible and $(k\text{A})^{-1} = \frac{1}{k}\text{A}^{-1}$

Solution We have

$$(k\text{A})\left(\frac{1}{k}\text{A}^{-1}\right) = \left(k.\frac{1}{k}\right)(\text{A}.\ \text{A}^{-1}) = 1\ (\text{I}) = \text{I}$$

Hence $(k\text{A})$ is inverse of $\left(\frac{1}{k}\text{A}^{-1}\right)$ or $(k\text{A})^{-1} = \frac{1}{k}\text{A}^{-1}$

Long Answer (L.A.)

Example 6 Express the matrix A as the sum of a symmetric and a skew symmetric matrix, where

$$\text{A} = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}.$$

Solution We have

$$\text{A} = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}, \qquad \text{then } \text{A}' = \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}$$

Hence $$\frac{A+A'}{2} = \frac{1}{2}\begin{bmatrix} 4 & 11 & -5 \\ 11 & 6 & 3 \\ -5 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 2 & \frac{11}{2} & \frac{-5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ \frac{-5}{2} & \frac{3}{2} & 4 \end{bmatrix}$$

and $$\frac{A-A'}{2} = \frac{1}{2}\begin{bmatrix} 0 & -3 & -7 \\ 3 & 0 & 7 \\ 7 & -7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-3}{2} & \frac{-7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & \frac{-7}{2} & 0 \end{bmatrix}$$

Therefore,

$$\frac{A+A'}{2} + \frac{A-A'}{2} = \begin{bmatrix} 2 & \frac{11}{2} & \frac{-5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ \frac{-5}{2} & \frac{3}{2} & 4 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-3}{2} & \frac{-7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & \frac{-7}{2} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix} = A.$$

Example 7 If $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$, then show that A satisfies the equation $A^3 - 4A^2 - 3A + 11I = O$.

Solution $$A^2 = A \times A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+2 & 3+0+4 & 2-3+6 \\ 2+0-1 & 6+0-2 & 4+0-3 \\ 1+4+3 & 3+0+6 & 2-2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$$

and $\quad A^3 = A^2 \times A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$$

Now $\quad A^3 - 4A^2 - 3A + 11(I)$

$$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11 & 37-28-9+0 & 26-20-6+0 \\ 10-4-6+0 & 5-16+0+11 & 1-4+3+0 \\ 35-32-3+0 & 42-36-6+0 & 34-36-9+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Example 8 Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$. Then show that $A^2 - 4A + 7I = O$.

Using this result calculate A^5 also.

Solution We have $A^2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$,

$$-4A = \begin{bmatrix} -8 & -12 \\ 4 & -8 \end{bmatrix} \text{ and } 7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}.$$

Therefore, $A^2 - 4A + 7I = \begin{bmatrix} 1-8+7 & 12-12+0 \\ -4+4+0 & 1-8+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

$\Rightarrow$ $A^2 = 4A - 7I$

Thus $A^3 = A.A^2 = A(4A - 7I) = 4(4A - 7I) - 7A$

$= 16A - 28I - 7A = 9A - 28I$

and so $A^5 = A^3A^2$

$= (9A - 28I)(4A - 7I)$

$= 36A^2 - 63A - 112A + 196I$

$= 36(4A - 7I) - 175A + 196I$

$= -31A - 56I$

$$= -31\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 56\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}$$

Objective Type Questions

Choose the correct answer from the given four options in Examples 9 to 12.

Example 9 If A and B are square matrices of the same order, then $(A + B)(A – B)$ is equal to

(A) $A^2 – B^2$ (B) $A^2 – BA – AB – B^2$

(C) $A^2 – B^2 + BA – AB$ (D) $A^2 – BA + B^2 + AB$

Solution (C) is correct answer. $(A + B)(A – B) = A(A – B) + B(A – B) = A^2 – AB + BA – B^2$

Example 10 If $A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix}$, then

(A) only AB is defined (B) only BA is defined

(C) AB and BA both are defined (D) AB and BA both are not defined.

Solution (C) is correct answer. Let $A = [a_{ij}]_{2\times3}$ $B = [b_{ij}]_{3\times2}$. Both AB and BA are defined.

Example 11 The matrix $A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}$ is a

(A) scalar matrix (B) diagonal matrix

(C) unit matrix (D) square matrix

Solution (D) is correct answer.

Example 12 If A and B are symmetric matrices of the same order, then $(AB' – BA')$ is a

(A) Skew symmetric matrix (B) Null matrix

(C) Symmetric matrix (D) None of these

Solution (A) is correct answer since

$$(AB' – BA')' = (AB')' – (BA')'$$

$$= (BA' - AB')$$

$$= -(AB' - BA')$$

Fill in the blanks in each of the Examples 13 to 15:

Example 13 If A and B are two skew symmetric matrices of same order, then AB is symmetric matrix if ________.

Solution AB = BA.

Example 14 If A and B are matrices of same order, then $(3A - 2B)'$ is equal to ________.

Solution $3A' - 2B'$.

Example 15 Addition of matrices is defined if order of the matrices is ________

Solution Same.

State whether the statements in each of the Examples 16 to 19 is true or false:

Example 16 If two matrices A and B are of the same order, then $2A + B = B + 2A$.

Solution True

Example 17 Matrix subtraction is associative

Solution False

Example 18 For the non singular matrix A, $(A')^{-1} = (A^{-1})'$.

Solution True

Example 19 $AB = AC \Rightarrow B = C$ for any three matrices of same order.

Solution False

3.3 EXERCISE

Short Answer (S.A.)

1. If a matrix has 28 elements, what are the possible orders it can have? What if it has 13 elements?

2. In the matrix $A = \begin{bmatrix} a & 1 & x \\ 2 & \sqrt{3} & x^2 - y \\ 0 & 5 & \frac{-2}{5} \end{bmatrix}$, write :

(i) The order of the matrix A

(ii) The number of elements

(iii) Write elements a_{23}, a_{31}, a_{12}

3. Construct $a_{2 \times 2}$ matrix where

(i) $a_{ij} = \dfrac{(i-2j)^2}{2}$

(ii) $a_{ij} = |-2i + 3j|$

4. Construct a 3 × 2 matrix whose elements are given by $a_{ij} = e^{i.x}\sin jx$

5. Find values of a and b if A = B, where

$$A = \begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix}, \qquad B = \begin{bmatrix} 2a+2 & b^2+2 \\ 8 & b^2-5b \end{bmatrix}$$

6. If possible, find the sum of the matrices A and B, where $A = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 3 \end{bmatrix}$,

and $B = \begin{bmatrix} x & y & z \\ a & b & 6 \end{bmatrix}$

7. If $X = \begin{bmatrix} 3 & 1 & -1 \\ 5 & -2 & -3 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 7 & 2 & 4 \end{bmatrix}$, find

(i) X +Y (ii) 2X – 3Y

(iii) A matrix Z such that X + Y + Z is a zero matrix.

8. Find non-zero values of x satisfying the matrix equation:

$$x\begin{bmatrix} 2x & 2 \\ 3 & x \end{bmatrix} + 2\begin{bmatrix} 8 & 5x \\ 4 & 4x \end{bmatrix} = 2\begin{bmatrix} (x^2+8) & 24 \\ (10) & 6x \end{bmatrix}.$$

9. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, show that $(A + B)(A - B) \neq A^2 - B^2$.

10. Find the value of x if

$$\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O.$$

11. Show that $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ satisfies the equation $A^2 - 3A - 7I = O$ and hence find A^{-1}.

12. Find the matrix A satisfying the matrix equation:

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

13. Find A, if $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} A = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$

14. If $A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, then verify $(BA)^2 \neq B^2A^2$

15. If possible, find BA and AB, where

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

16. Show by an example that for $A \neq O$, $B \neq O$, $AB = O$.

17. Given $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 9 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 1 & 3 \end{bmatrix}$. Is $(AB)' = B'A'$?

18. Solve for x and y:

$$x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}3\\5\end{bmatrix}+\begin{bmatrix}-8\\-11\end{bmatrix}=O.$$

19. If X and Y are 2×2 matrices, then solve the following matrix equations for X and Y

$$2X + 3Y = \begin{bmatrix}2 & 3\\4 & 0\end{bmatrix},\ 3X + 2Y = \begin{bmatrix}-2 & 2\\1 & -5\end{bmatrix}.$$

20. If $A = \begin{bmatrix}3 & 5\end{bmatrix}$, $B = \begin{bmatrix}7 & 3\end{bmatrix}$, then find a non-zero matrix C such that AC = BC.

21. Give an example of matrices A, B and C such that AB = AC, where A is non-zero matrix, but $B \neq C$.

22. If $A = \begin{bmatrix}1 & 2\\-2 & 1\end{bmatrix}$, $B = \begin{bmatrix}2 & 3\\3 & -4\end{bmatrix}$ and $C = \begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix}$, verify :

(i) (AB) C = A (BC) (ii) A (B + C) = AB + AC.

23. If $P = \begin{bmatrix}x & 0 & 0\\0 & y & 0\\0 & 0 & z\end{bmatrix}$ and $Q = \begin{bmatrix}a & 0 & 0\\0 & b & 0\\0 & 0 & c\end{bmatrix}$, prove that

$$PQ = \begin{bmatrix}xa & 0 & 0\\0 & yb & 0\\0 & 0 & zc\end{bmatrix} = QP.$$

24. If : $\begin{bmatrix}2 & 1 & 3\end{bmatrix}\begin{bmatrix}-1 & 0 & -1\\-1 & 1 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix} = A$, find A.

25. If $A = \begin{bmatrix}2 & 1\end{bmatrix}$, $B = \begin{bmatrix}5 & 3 & 4\\8 & 7 & 6\end{bmatrix}$ and $C = \begin{bmatrix}-1 & 2 & 1\\1 & 0 & 2\end{bmatrix}$, verify that A (B + C) = (AB + AC).

26. If $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$, then verify that $A^2 + A = A(A + I)$, where I is 3×3 unit matrix.

27. If $A = \begin{bmatrix} 0 & -1 & 2 \\ 4 & 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 1 & 3 \\ 2 & 6 \end{bmatrix}$, then verify that :

(i) $(A')' = A$

(ii) $(AB)' = B'A'$

(iii) $(kA)' = (kA')$.

28. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 6 & 4 \\ 7 & 3 \end{bmatrix}$, then verify that :

(i) $(2A + B)' = 2A' + B'$

(ii) $(A - B)' = A' - B'$.

29. Show that $A'A$ and AA' are both symmetric matrices for any matrix A.

30. Let A and B be square matrices of the order 3×3. Is $(AB)^2 = A^2 B^2$? Give reasons.

31. Show that if A and B are square matrices such that AB = BA, then

$$(A + B)^2 = A^2 + 2AB + B^2.$$

32. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$ and $a = 4$, $b = -2$.

Show that:

(a) $A + (B + C) = (A + B) + C$

(b) $A(BC) = (AB)C$

(c) $(a + b)B = aB + bB$

(d) $a\,(C–A) = aC – aA$

(e) $(A^T)^T = A$

(f) $(bA)^T = b\,A^T$

(g) $(AB)^T = B^T A^T$

(h) $(A – B)C = AC – BC$

(i) $(A – B)^T = A^T – B^T$

33. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then show that $A^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$.

34. If $A = \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $x^2 = –1$, then show that $(A + B)^2 = A^2 + B^2$.

35. Verify that $A^2 = I$ when $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$.

36. Prove by Mathematical Induction that $(A')^n = (A^n)'$, where $n \in \mathbf{N}$ for any square matrix A.

37. Find inverse, by elementary row operations (if possible), of the following matrices

(i) $\begin{bmatrix} 1 & 3 \\ -5 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$.

38. If $\begin{bmatrix} xy & 4 \\ z+6 & x+y \end{bmatrix} = \begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}$, then find values of x, y, z and w.

39. If $A = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 1 \\ 7 & 8 \end{bmatrix}$, find a matrix C such that 3A + 5B + 2C is a null matrix.

40. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, then find $A^2 - 5A - 14I$. Hence, obtain A^3.

41. Find the values of *a*, *b*, *c* and *d*, if

$$3\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 6 \\ -1 & 2d \end{bmatrix} + \begin{bmatrix} 4 & a+b \\ c+d & 3 \end{bmatrix}.$$

42. Find the matrix A such that

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}.$$

43. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$, find $A^2 + 2A + 7I$.

44. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, and $A^{-1} = A'$, find value of α.

45. If the matrix $\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is a skew symmetric matrix, find the values of *a*, *b* and *c*.

46. If $P(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$, then show that

$P(x) . P(y) = P(x + y) = P(y) . P(x)$.

47. If A is square matrix such that $A^2 = A$, show that $(I + A)^3 = 7A + I$.

48. If A, B are square matrices of same order and B is a skew-symmetric matrix, show that A′BA is skew symmetric.

Long Answer (L.A.)

49. If AB = BA for any two sqaure matrices, prove by mathematical induction that $(AB)^n = A^n B^n$.

50. Find x, y, z if $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies $A' = A^{-1}$.

51. If possible, using elementary row transformations, find the inverse of the following matrices

(i) $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

52. Express the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 4 & 1 & 2 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Objective Type Questions

Choose the correct answer from the given four options in each of the Exercises 53 to 67.

53. The matrix $P = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ is a

(A) square matrix (B) diagonal matrix

(C) unit matrix (D) none

54. Total number of possible matrices of order 3×3 with each entry 2 or 0 is

(A) 9 (B) 27 (C) 81 (D) 512

55. If $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$, then the value of $x + y$ is

(A) $x = 3, y = 1$ (B) $x = 2, y = 3$

(C) $x = 2, y = 4$ (D) $x = 3, y = 3$

56. If A = $\frac{1}{\pi}\begin{bmatrix} \sin^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$, B = $\frac{1}{\pi}\begin{bmatrix} -\cos^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$, then

A – B is equal to

(A) I (B) O (C) 2I (D) $\frac{1}{2}$I

57. If A and B are two matrices of the order 3 × m and 3 × n, respectively, and $m = n$, then the order of matrix (5A – 2B) is

(A) $m \times 3$ (B) 3 × 3 (C) $m \times n$ (D) $3 \times n$

58. If A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then A^2 is equal to

(A) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

59. If matrix A = $[a_{ij}]_{2 \times 2}$, where $a_{ij} = 1$ if $i \neq j$

$= 0$ if $i = j$

then A^2 is equal to

(A) I (B) A (C) 0 (D) None of these

60. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a

(A) identity matrix (B) symmetric matrix

(C) skew symmetric matrix (D) none of these

61. The matrix $\begin{bmatrix} 0 & -5 & 8 \\ 5 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix}$ is a

(A) diagonal matrix (B) symmetric matrix

(C) skew symmetric matrix (D) scalar matrix

62. If A is matrix of order $m \times n$ and B is a matrix such that AB′ and B′A are both defined, then order of matrix B is

(A) $m \times m$ (B) $n \times n$

(C) $n \times m$ (D) $m \times n$

63. If A and B are matrices of same order, then (AB′–BA′) is a

(A) skew symmetric matrix (B) null matrix

(C) symmetric matrix (D) unit matrix

64. If A is a square matrix such that $A^2 = I$, then $(A-I)^3 + (A + I)^3 - 7A$ is equal to

(A) A (B) I – A (C) I + A (D) 3A

65. For any two matrices A and B, we have

(A) AB = BA (B) AB ≠ BA

(C) AB = O (D) None of the above

66. On using elementary column operations $C_2 \to C_2 - 2C_1$ in the following matrix equation

$$\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \text{ we have :}$$

(A) $\begin{bmatrix} 1 & -5 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & -5 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -0 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & -5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & -5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$

67. On using elementary row operation $R_1 \to R_1 - 3R_2$ in the following matrix equation:

$\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, we have :

(A) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 4 & 2 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Fill in the blanks in each of the Exercises 68–81.

68. _________ matrix is both symmetric and skew symmetric matrix.

69. Sum of two skew symmetric matrices is always _________ matrix.

70. The negative of a matrix is obtained by multiplying it by _________.

71. The product of any matrix by the scalar _________ is the null matrix.

72. A matrix which is not a square matrix is called a _________ matrix.

73. Matrix multiplication is _________ over addition.

74. If A is a symmetric matrix, then A^3 is a _________ matrix.

75. If A is a skew symmetric matrix, then A^2 is a _________.

76. If A and B are square matrices of the same order, then

(i) $(AB)' =$ __________.

(ii) $(kA)' =$ __________. (k is any scalar)

(iii) $[k(A - B)]' =$ __________.

77. If A is skew symmetric, then kA is a __________. (k is any scalar)

78. If A and B are symmetric matrices, then

(i) AB – BA is a __________.

(ii) BA – 2AB is a __________.

79. If A is symmetric matrix, then B′AB is __________.

80. If A and B are symmetric matrices of same order, then AB is symmetric if and only if __________.

81. In applying one or more row operations while finding A^{-1} by elementary row operations, we obtain all zeros in one or more, then A^{-1} __________.

State Exercises 82 to 101 which of the following statements are **True** or **False**

82. A matrix denotes a number.

83. Matrices of any order can be added.

84. Two matrices are equal if they have same number of rows and same number of columns.

85. Matrices of different order can not be subtracted.

86. Matrix addition is associative as well as commutative.

87. Matrix multiplication is commutative.

88. A square matrix where every element is unity is called an identity matrix.

89. If A and B are two square matrices of the same order, then A + B = B + A.

90. If A and B are two matrices of the same order, then A – B = B – A.

91. If matrix AB = O, then A = O or B = O or both A and B are null matrices.

92. Transpose of a column matrix is a column matrix.

93. If A and B are two square matrices of the same order, then AB = BA.

94. If each of the three matrices of the same order are symmetric, then their sum is a symmetric matrix.

95. If A and B are any two matrices of the same order, then $(AB)' = A'B'$.

96. If $(AB)' = B'A'$, where A and B are not square matrices, then number of rows in A is equal to number of columns in B and number of columns in A is equal to number of rows in B.

97. If A, B and C are square matrices of same order, then AB = AC always implies that B = C.

98. AA' is always a symmetric matrix for any matrix A.

99. If $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then AB and BA are defined and equal.

100. If A is skew symmetric matrix, then A^2 is a symmetric matrix.

101. $(AB)^{-1} = A^{-1}. B^{-1}$, where A and B are invertible matrices satisfying commutative property with respect to multiplication.

Chapter 4

DETERMINANTS

4.1 Overview

To every square matrix A = $[a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the matrix A, written as det A, where a_{ij} is the $(i, j)th$ element of A.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A, denoted by |A| (or det A), is given by

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Remarks

(i) Only square matrices have determinants.

(ii) For a matrix A, $|A|$ is read as determinant of A and not, as modulus of A.

4.1.1 *Determinant of a matrix of order one*

Let A = $[a]$ be the matrix of order 1, then determinant of A is defined to be equal to a.

4.1.2 *Determinant of a matrix of order two*

Let $A = [a_{ij}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix of order 2. Then the determinant of A is defined as: det (A) = |A| = $ad - bc$.

4.1.3 *Determinant of a matrix of order three*

The determinant of a matrix of order three can be determined by expressing it in terms of second order determinants which is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) and each way gives the same value.

Consider the determinant of a square matrix A = $[a_{ij}]_{3\times3}$, i.e.,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding |A| along C_1, we get

$$|A| = a_{11}\ (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\ (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\ (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}\ a_{33} - a_{23}\ a_{32}) - a_{21}\ (a_{12}\ a_{33} - a_{13}\ a_{32}) + a_{31}\ (a_{12}\ a_{23} - a_{13}\ a_{22})$$

Remark In general, if A = kB, where A and B are square matrices of order n, then $|A| = k^n\ |B|$, n = 1, 2, 3.

4.1.4 *Properties of Determinants*

For any square matrix A, |A| satisfies the following properties.

(i) |A′| = |A|, where A′ = transpose of matrix A.

(ii) If we interchange any two rows (or columns), then sign of the determinant changes.

(iii) If any two rows or any two columns in a determinant are identical (or proportional), then the value of the determinant is zero.

(iv) Multiplying a determinant by k means multiplying the elements of only one row (or one column) by k.

(v) If we multiply each element of a row (or a column) of a determinant by constant k, then value of the determinant is multiplied by k.

(vi) If elements of a row (or a column) in a determinant can be expressed as the sum of two or more elements, then the given determinant can be expressed as the sum of two or more determinants.

(vii) If to each element of a row (or a column) of a determinant the equimultiples of corresponding elements of other rows (columns) are added, then value of determinant remains same.

Notes:

(i) If all the elements of a row (or column) are zeros, then the value of the determinant is zero.

(ii) If value of determinant 'Δ' becomes zero by substituting $x = \alpha$, then $x - \alpha$ is a factor of 'Δ'.

(iii) If all the elements of a determinant above or below the main diagonal consists of zeros, then the value of the determinant is equal to the product of diagonal elements.

4.1.5 *Area of a triangle*

Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

4.1.6 *Minors and co-factors*

(i) Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column, and it is denoted by M_{ij}.

(ii) Co-factor of an element a_{ij} is given by $A_{ij} = (-1)^{i+j} M_{ij}$.

(iii) Value of determinant of a matrix A is obtained by the sum of products of elements of a row (or a column) with corresponding co-factors. For example

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}.$$

(iv) If elements of a row (or column) are multiplied with co-factors of elements of any other row (or column), then their sum is zero. For example,

$$a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = 0.$$

4.1.7 *Adjoint and inverse of a matrix*

(i) The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix

$[a_{ij}]_{n\times n}$, where A_{ij} is the co-factor of the element a_{ij}. It is denoted by *adj* A.

If $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then $adj\ A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$, where A_{ij} is co-factor of a_{ij}.

(ii) A (*adj* A) = (*adj* A) A = |A| I, where A is square matrix of order n.

(iii) A square matrix A is said to be singular or non-singular according as |A| = 0 or |A| ≠ 0, respectively.

(iv) If A is a square matrix of order n, then $|adj\ A| = |A|^{n-1}$.

(v) If A and B are non-singular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.

(vi) The determinant of the product of matrices is equal to product of their respective determinants, that is, |AB| = |A| |B|.

(vii) If AB = BA = I, where A and B are square matrices, then B is called inverse of A and is written as $B = A^{-1}$. Also $B^{-1} = (A^{-1})^{-1} = A$.

(viii) A square matrix A is invertible if and only if A is non-singular matrix.

(ix) If A is an invertible matrix, then $A^{-1} = \dfrac{1}{|A|}(adj\ A)$

4.1.8 *System of linear equations*

(i) Consider the equations:

$$a_1x + b_1\,y + c_1\,z = d_1$$
$$a_2x + b_2\,y + c_2\,z = d_2$$
$$a_3x + b_3\,y + c_3\,z = d_3,$$

In matrix form, these equations can be written as A X = B, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

(ii) Unique solution of equation AX = B is given by $X = A^{-1}B$, where |A| ≠ 0.

(iii) A system of equations is consistent or inconsistent according as its solution exists or not.

(iv) For a square matrix A in matrix equation AX = B

(a) If $|A| \neq 0$, then there exists unique solution.

(b) If $|A| = 0$ and (*adj* A) B $\neq 0$, then there exists no solution.

(c) If $|A| = 0$ and (*adj* A) B $= 0$, then system may or may not be consistent.

4.2 Solved Examples

Short Answer (S.A.)

Example 1 If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$, then find x.

Solution We have $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$. This gives

$$2x^2 - 40 = 18 - 40 \quad \Rightarrow x^2 = 9 \quad \Rightarrow \quad x = \pm 3.$$

Example 2 If $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$, then prove that $\Delta + \Delta_1 = 0$.

Solution We have $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$

Interchanging rows and columns, we get

$$\Delta_1 = \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix} \qquad = \frac{1}{xyz}\begin{vmatrix} x & xyz & x^2 \\ y & xyz & y^2 \\ z & xyz & z^2 \end{vmatrix}$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} \qquad \text{Interchanging } C_1 \text{ and } C_2$$

$$= (-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -\Delta$$

$$\Rightarrow \quad \Delta_1 + \Delta = 0$$

Example 3 Without expanding, show that

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0.$$

Solution Applying $C_1 \rightarrow C_1 - C_2 - C_3$, we have

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta - \cot^2\theta - 1 & \cot^2\theta & 1 \\ \cot^2\theta - \operatorname{cosec}^2\theta + 1 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0$$

Example 4 Show that $\Delta = \begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x - p)\,(x^2 + px - 2q^2)$

Solution Applying $C_1 \rightarrow C_1 - C_2$, we have

$$\Delta = \begin{vmatrix} x-p & p & q \\ p-x & x & q \\ 0 & q & x \end{vmatrix} = (x-p)\begin{vmatrix} 1 & p & q \\ -1 & x & q \\ 0 & q & x \end{vmatrix}$$

$$= (x-p)\begin{vmatrix} 0 & p+x & 2q \\ -1 & x & q \\ 0 & q & x \end{vmatrix} \quad \text{Applying } R_1 \to R_1 + R_2$$

Expanding along C_1, we have

$$\Delta = (x-p)(px + x^2 - 2q^2) = (x-p)(x^2 + px - 2q^2)$$

Example 5 If $\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$, then show that Δ is equal to zero.

Solution Interchanging rows and columns, we get $\Delta = \begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$

Taking '–1' common from R_1, R_2 and R_3, we get

$$\Delta = (-1)^3 \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix} = -\Delta$$

$$\Rightarrow \quad 2\Delta = 0 \quad \text{or} \quad \Delta = 0$$

Example 6 Prove that $(A^{-1})' = (A')^{-1}$, where A is an invertible matrix.

Solution Since A is an invertible matrix, so it is non-singular.

We know that $|A| = |A'|$. But $|A| \neq 0$. So $|A'| \neq 0$ i.e. A' is invertible matrix.

Now we know that $AA^{-1} = A^{-1}A = I$.

Taking transpose on both sides, we get $(A^{-1})' A' = A' (A^{-1})' = (I)' = I$

Hence $(A^{-1})'$ is inverse of A', i.e., $(A')^{-1} = (A^{-1})'$

Long Answer (L.A.)

Example 7 If $x = -4$ is a root of $\Delta = \begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix} = 0$, then find the other two roots.

Solution Applying $R_1 \to (R_1 + R_2 + R_3)$, we get

$$\begin{vmatrix} x+4 & x+4 & x+4 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}.$$

Taking $(x + 4)$ common from R_1, we get

$$\Delta = (x+4)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = (x+4)\begin{vmatrix} 1 & 0 & 0 \\ 1 & x-1 & 0 \\ 3 & -1 & x-3 \end{vmatrix}.$$

Expanding along R_1,

$\Delta = (x + 4)\,[(x - 1)\,(x - 3) - 0]$. Thus, $\Delta = 0$ implies

$$x = -4, 1, 3$$

Example 8 In a triangle ABC, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0,$$

then prove that ΔABC is an isoceles triangle.

Solution Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ -\cos^2 A & -\cos^2 B & -\cos^2 C \end{vmatrix} \quad R_3 \to R_3 - R_2$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B - \sin A & \sin C - \sin B \\ -\cos^2 A & \cos^2 A - \cos^2 B & \cos^2 B - \cos^2 C \end{vmatrix} . \ (C_3 \to C_3 - C_2 \text{ and } C_2 \to C_2 - C_1)$$

Expanding along R_1, we get

$$\Delta = (\sin B - \sin A)(\sin^2 C - \sin^2 B) - (\sin C - \sin B)(\sin^2 B - \sin^2 A)$$

$$= (\sin B - \sin A)(\sin C - \sin B)(\sin C - \sin A) = 0$$

$\Rightarrow$ either $\sin B - \sin A = 0$ or $\sin C - \sin B$ or $\sin C - \sin A = 0$

$\Rightarrow$ $A = B$ or $B = C$ or $C = A$

i.e. triangle ABC is isoceles.

Example 9 Show that if the determinant $\Delta = \begin{vmatrix} 3 & -2 & \sin 3\theta \\ -7 & 8 & \cos 2\theta \\ -11 & 14 & 2 \end{vmatrix} = 0$, then $\sin\theta = 0$ or $\frac{1}{2}$.

Solution Applying $R_2 \to R_2 + 4R_1$ and $R_3 \to R_3 + 7R_1$, we get

$$\begin{vmatrix} 3 & -2 & \sin 3\theta \\ 5 & 0 & \cos 2\theta + 4\sin 3\theta \\ 10 & 0 & 2 + 7\sin 3\theta \end{vmatrix} = 0$$

or $\quad 2\,[5\,(2 + 7\sin 3\theta) - 10\,(\cos 2\theta + 4\sin 3\theta)] = 0$

or $\quad 2 + 7\sin 3\theta - 2\cos 2\theta - 8\sin 3\theta = 0$

or $\quad 2 - 2\cos 2\theta - \sin 3\theta = 0$

$\quad \sin\theta\,(4\sin^2\theta + 4\sin\theta - 3) = 0$

or $\qquad \sin\theta = 0$ or $(2\sin\theta - 1) = 0$ or $(2\sin\theta + 3) = 0$

or $\qquad \sin\theta = 0$ or $\sin\theta = \frac{1}{2}$ (Why ?).

Objective Type Questions

Choose the correct answer from the given four options in each of the Example 10 and 11.

Example 10 Let $\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$, then

(A) $\Delta_1 = -\Delta$ (B) $\Delta \neq \Delta_1$

(C) $\Delta - \Delta_1 = 0$ (D) None of these

Solution (C) is the correct answer since $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix} = \begin{vmatrix} A & x & yz \\ B & y & zx \\ C & z & xy \end{vmatrix}$

$$= \frac{1}{xyz}\begin{vmatrix} Ax & x^2 & xyz \\ By & y^2 & xyz \\ Cz & z^2 & xyz \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} = \Delta$$

Example 11 If $x, y \in \mathbf{R}$, then the determinant $\Delta = \begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ \cos(x+y) & -\sin(x+y) & 0 \end{vmatrix}$ lies in the interval

(A) $\left[-\sqrt{2}, \sqrt{2}\right]$ (B) [–1, 1]

(C) $\left[-\sqrt{2}, 1\right]$ (D) $\left[-1, -\sqrt{2},\right]$

Solution The correct choice is A. Indeed applying $R_3 \to R_3 - \cos y R_1 + \sin y R_2$, we get

$$\Delta = \begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ 0 & 0 & \sin y - \cos y \end{vmatrix}.$$

Expanding along R_3, we have

$$\Delta = (\sin y - \cos y)\ (\cos^2 x + \sin^2 x)$$

$$= (\sin y - \cos y) = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin y - \frac{1}{\sqrt{2}}\cos y\right]$$

$$= \sqrt{2}\left[\cos\frac{\pi}{4}\sin y - \sin\frac{\pi}{4}\cos y\right] \quad = \sqrt{2}\ \sin\left(y - \frac{\pi}{4}\right)$$

Hence $-\sqrt{2} \le \Delta \le \sqrt{2}$.

Fill in the blanks in each of the Examples 12 to 14.

Example 12 If A, B, C are the angles of a triangle, then

$$\Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix} = \ldots\ldots\ldots\ldots\ldots$$

Solution Answer is 0. Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$.

Example 13 The determinant $\Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$ is equal to ....

Solution Answer is 0.Taking $\sqrt{5}$ common from C_2 and C_3 and applying $C_1 \to C_3 - \sqrt{3}\ C_2$, we get the desired result.

Example 14 The value of the determinant

$$\Delta = \begin{vmatrix} \sin^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix} = \ldots\ldots\ldots$$

Solution $\Delta = 0$. Apply $C_1 \to C_1 + C_2 + C_3$.

State whether the statements in the Examples 15 to 18 is **True** or **False**.

Example 15 The determinant

$$\Delta = \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & \cos y \end{vmatrix}$$

is independent of x only.

Solution True. Apply $R_1 \to R_1 + \sin y R_2 + \cos y R_3$, and expand

Example 16 The value of

$$\begin{vmatrix} 1 & 1 & 1 \\ {}^nC_1 & {}^{n+2}C_1 & {}^{n+4}C_1 \\ {}^nC_2 & {}^{n+2}C_2 & {}^{n+4}C_2 \end{vmatrix} \text{ is 8.}$$

Solution True

Example 17 If $A = \begin{bmatrix} x & 5 & 2 \\ 2 & y & 3 \\ 1 & 1 & z \end{bmatrix}$, $xyz = 80$, $3x + 2y + 10z = 20$, then

$$A\ adj.\ A = \begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}.$$

Solution : False.

Example 18 If $A=\begin{bmatrix}0 & 1 & 3\\ 1 & 2 & x\\ 2 & 3 & 1\end{bmatrix}$, $A^{-1}=\begin{bmatrix}\frac{1}{2} & -4 & \frac{5}{2}\\ -\frac{1}{2} & 3 & -\frac{3}{2}\\ \frac{1}{2} & y & \frac{1}{2}\end{bmatrix}$

then $x = 1$, $y = -1$.

Solution True

4.3 EXERCISE

Short Answer (S.A.)

Using the properties of determinants in Exercises 1 to 6, evaluate:

1. $\begin{vmatrix}x^2-x+1 & x-1\\ x+1 & x+1\end{vmatrix}$

2. $\begin{vmatrix}a+x & y & z\\ x & a+y & z\\ x & y & a+z\end{vmatrix}$

3. $\begin{vmatrix}0 & xy^2 & xz^2\\ x^2y & 0 & yz^2\\ x^2z & zy^2 & 0\end{vmatrix}$

4. $\begin{vmatrix}3x & -x+y & -x+z\\ x-y & 3y & z-y\\ x-z & y-z & 3z\end{vmatrix}$

5. $\begin{vmatrix}x+4 & x & x\\ x & x+4 & x\\ x & x & x+4\end{vmatrix}$

6. $\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}$

Using the proprties of determinants in Exercises 7 to 9, prove that:

7. $\begin{vmatrix}y^2z^2 & yz & y+z\\ z^2x^2 & zx & z+x\\ x^2y^2 & xy & x+y\end{vmatrix} = 0$

8. $\begin{vmatrix}y+z & z & y\\ z & z+x & x\\ y & x & x+y\end{vmatrix} = 4xyz$

9. $$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

10. If A + B + C = 0, then prove that $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$

11. If the co-ordinates of the vertices of an equilateral triangle with sides of length 'a' are (x_1, y_1), (x_2, y_2), (x_3, y_3), then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}$.

12. Find the value of θ satisfying $\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$.

13. If $\begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix} = 0$, then find values of x.

14. If $a_1, a_2, a_3, \ldots, a_r$ are in G.P., then prove that the determinant

$$\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$$ is independent of r.

15. Show that the points $(a + 5, a - 4)$, $(a - 2, a + 3)$ and (a, a) do not lie on a straight line for any value of a.

16. Show that the ΔABC is an isosceles triangle if the determinant

$$\Delta=\begin{bmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{bmatrix}=0 .$$

17. Find A^{-1} if $A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and show that $A^{-1}=\frac{A^2-3I}{2}$.

Long Answer (L.A.)

18. If $A=\begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}.

Using A^{-1}, solve the system of linear equations $x-2y=10$, $2x-y-z=8$, $-2y+z=7$.

19. Using matrix method, solve the system of equations $3x+2y-2z=3$, $x+2y+3z=6$, $2x-y+z=2$.

20. Given $A=\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, find BA and use this to solve the system of equations $y+2z=7$, $x-y=3$, $2x+3y+4z=17$.

21. If $a+b+c\neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$, then prove that $a=b=c$.

22. Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ is divisible by $a+b+c$ and find the quotient.

23. If $x + y + z = 0$, prove that $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Objective Type Questions (M.C.Q.)

Choose the correct answer from given four options in each of the Exercises from 24 to 37.

24. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then value of x is

(A) 3 (B) ± 3

(C) ± 6 (D) 6

25. The value of determinant $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$

(A) $a^3 + b^3 + c^3$ (B) $3bc$

(C) $a^3 + b^3 + c^3 - 3abc$ (D) none of these

26. The area of a triangle with vertices (–3, 0), (3, 0) and (0, k) is 9 sq. units. The value of k will be

(A) 9 (B) 3

(C) – 9 (D) 6

27. The determinant $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$ equals

(A) $abc\,(b-c)\,(c-a)\,(a-b)$ (B) $(b-c)\,(c-a)\,(a-b)$

(C) $(a+b+c)\,(b-c)\,(c-a)\,(a-b)$ (D) None of these

28. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(A) 0 (B) 2

(C) 1 (D) 3

29. If A, B and C are angles of a triangle, then the determinant

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$ is equal to

(A) 0 (B) – 1

(C) 1 (D) None of these

30. Let $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$, then $\lim_{t \to 0} \frac{f(t)}{t^2}$ is equal to

(A) 0 (B) – 1

(C) 2 (D) 3

31. The maximum value of $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix}$ is (θ is real number)

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$

(C) $\sqrt{2}$ (D) $\frac{2\sqrt{3}}{4}$

32. If $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then

(A) $f(a) = 0$ (B) $f(b) = 0$

(C) $f(0) = 0$ (D) $f(1) = 0$

33. If $A = \begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}$, then A^{-1} exists if

(A) $\lambda = 2$ (B) $\lambda \neq 2$

(C) $\lambda \neq -2$ (D) None of these

34. If A and B are invertible matrices, then which of the following is not correct?

(A) $adj\ A = |A|.\ A^{-1}$ (B) $\det(A)^{-1} = [\det(A)]^{-1}$

(C) $(AB)^{-1} = B^{-1} A^{-1}$ (D) $(A + B)^{-1} = B^{-1} + A^{-1}$

35. If x, y, z are all different from zero and $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$, then value of $x^{-1} + y^{-1} + z^{-1}$ is

(A) $x\,y\,z$ (B) $x^{-1}\ y^{-1}\ z^{-1}$

(C) $-x\ -y\ -z$ (D) -1

36. The value of the determinant $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is

(A) $9x^2 (x + y)$ (B) $9y^2 (x + y)$

(C) $3y^2 (x + y)$ (D) $7x^2 (x + y)$

37. There are two values of a which makes determinant, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$, then sum of these number is

(A) 4 (B) 5

(C) – 4 (D) 9

Fill in the blanks

38. If A is a matrix of order 3 × 3, then |3A| = ________ .

39. If A is invertible matrix of order 3 × 3, then $|A^{-1}|$ ________ .

40. If $x, y, z \in$ R, then the value of determinant $\begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$ is equal to ________.

41. If $\cos 2\theta = 0$, then $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2 =$ _________.

42. If A is a matrix of order 3 × 3, then $(A^2)^{-1}$ = _________.

43. If A is a matrix of order 3 × 3, then number of minors in determinant of A are _________.

44. The sum of the products of elements of any row with the co-factors of corresponding elements is equal to __________.

45. If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then other two roots are ___________.

46. $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix}$ = ___________.

47. If $f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix}$ = A + Bx + Cx^2 + ..., then

A = ________.

State True or False for the statements of the following Exercises:

48. $\left(A^3\right)^{-1} = \left(A^{-1}\right)^3$, where A is a square matrix and $|A| \neq 0$.

49. $(aA)^{-1} = \frac{1}{a}A^{-1}$, where a is any real number and A is a square matrix.

50. $|A^{-1}| \neq |A|^{-1}$, where A is non-singular matrix.

51. If A and B are matrices of order 3 and $|A| = 5$, $|B| = 3$, then $|3AB| = 27 \times 5 \times 3 = 405$.

52. If the value of a third order determinant is 12, then the value of the determinant formed by replacing each element by its co-factor will be 144.

53. $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$, where a, b, c are in A.P.

54. $|adj.\ A| = |A|^2$, where A is a square matrix of order two.

55. The determinant $\begin{vmatrix} \sin A & \cos A & \sin A+\cos B \\ \sin B & \cos A & \sin B+\cos B \\ \sin C & \cos A & \sin C+\cos B \end{vmatrix}$ is equal to zero.

56. If the determinant $\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$ splits into exactly K determinants of order 3, each element of which contains only one term, then the value of K is 8.

57. Let $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$, then $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$.

58. The maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & (1+\sin\theta) & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ is $\frac{1}{2}$.

Chapter 5

CONTINUITY AND DIFFERENTIABILITY

5.1 Overview

5.1.1 *Continuity of a function at a point*

Let f be a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at $x = c$ exist and are equal to each other, i.e.,

$$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$$

then f is said to be continuous at $x = c$.

5.1.2 *Continuity in an interval*

(i) f is said to be continuous in an open interval (a, b) if it is continuous at every point in this interval.

(ii) f is said to be continuous in the closed interval $[a, b]$ if

- f is continuous in (a, b)
- $\lim_{x \to a^+} f(x) = f(a)$
- $\lim_{x \to b^-} f(x) = f(b)$

5.1.3 *Geometrical meaning of continuity*

(i) Function f will be continuous at $x = c$ if there is no break in the graph of the function at the point $\left(c, f(c)\right)$.

(ii) In an interval, function is said to be continuous if there is no break in the graph of the function in the entire interval.

5.1.4 *Discontinuity*

The function f will be discontinuous at $x = a$ in any of the following cases :

(i) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist but are not equal.

(ii) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist and are equal but not equal to $f(a)$.

(iii) $f(a)$ is not defined.

5.1.5 *Continuity of some of the common functions*

Function $f(x)$	**Interval in which f is continuous**
1. The constant function, i.e. $f(x) = c$	
2. The identity function, i.e. $f(x) = x$	$\mathbf{R}$
3. The polynomial function, i.e. $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$	
4. $\lvert x - a \rvert$	$(-\infty, \infty)$
5. x^{-n}, n is a positive integer	$(-\infty, \infty) - \{0\}$
6. $p(x) / q(x)$, where $p(x)$ and $q(x)$ are polynomials in x	$\mathbf{R} - \{x : q(x) = 0\}$
7. $\sin x$, $\cos x$	$\mathbf{R}$
8. $\tan x$, $\sec x$	$\mathbf{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\}$
9. $\cot x$, $\operatorname{cosec} x$	$\mathbf{R} - \{(n\pi : n \in \mathbf{Z}\}$

10. e^x	**R**
11. $\log x$	$(0, \infty)$
12. The inverse trigonometric functions, i.e., $\sin^{-1} x$, $\cos^{-1} x$ etc.	In their respective domains

5.1.6 *Continuity of composite functions*

Let f and g be real valued functions such that (fog) is defined at a. If g is continuous at a and f is continuous at $g(a)$, then (fog) is continuous at a.

5.1.7 *Differentiability*

The function defined by $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, wherever the limit exists, is defined to be the derivative of f at x. In other words, we say that a function f is differentiable at a point c in its domain if both $\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$, called left hand derivative, denoted by $Lf'(c)$, and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$, called right hand derivative, denoted by $Rf'(c)$, are finite and equal.

(i) The function $y = f(x)$ is said to be differentiable in an open interval (a, b) if it is differentiable at every point of (a, b)

(ii) The function $y = f(x)$ is said to be differentiable in the closed interval $[a, b]$ if $Rf'(a)$ and $Lf'(b)$ exist and $f'(x)$ exists for every point of (a, b).

(iii) Every differentiable function is continuous, but the converse is not true

5.1.8 *Algebra of derivatives*

If u, v are functions of x, then

(i) $\frac{d(u\pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

(ii) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(iii) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

5.1.9 Chain rule is a rule to differentiate composition of functions. Let $f = vou$. If $t = u(x)$ and both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist then $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$

5.1.10 Following are some of the standard derivatives (in appropriate domains)

1. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
4. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
5. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x|>1$
6. $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, |x|>1$

5.1.11 *Exponential and logarithmic functions*

(i) The exponential function with positive base $b > 1$ is the function $y = f(x) = b^x$. Its domain is **R**, the set of all real numbers and range is the set of all positive real numbers. Exponential function with base 10 is called the common exponential function and with base e is called the natural exponential function.

(ii) Let $b > 1$ be a real number. Then we say logarithm of a to base b is x if $b^x = a$, Logarithm of a to the base b is denoted by $\log_b a$. If the base $b = 10$, we say it is common logarithm and if $b = e$, then we say it is natural logarithms. $\log x$ denotes the logarithm function to base e. The domain of logarithm function is $\mathbf{R}^+$, the set of all positive real numbers and the range is the set of all real numbers.

(iii) The properties of logarithmic function to any base $b > 1$ are listed below:

1. $\log_b (xy) = \log_b x + \log_b y$

2. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

3. $\log_b x^n = n \log_b x$

4. $\log_b x = \dfrac{\log_c x}{\log_c b}$, where $c > 1$

5. $\log_b x = \dfrac{1}{\log_x b}$

6. $\log_b b = 1$ and $\log_b 1 = 0$

(iv) The derivative of e^x w.r.t., x is e^x , i.e. $\dfrac{d}{dx}(e^x) = e^x$. The derivative of $\log x$ w.r.t., x is $\dfrac{1}{x}$; i.e. $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$.

5.1.12 Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = (u(x))^{v(x)}$, where both f and u need to be positive functions for this technique to make sense.

5.1.13 Differentiation of a function with respect to another function

Let $u = f(x)$ and $v = g(x)$ be two functions of x, then to find derivative of $f(x)$ w.r.t. to $g(x)$, i.e., to find $\dfrac{du}{dv}$, we use the formula

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} .$$

5.1.14 *Second order derivative*

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$ is called the second order derivative of y w.r.t. x. It is denoted by y'' or y_2 , if $y = f(x)$.

5.1.15 *Rolle's Theorem*

Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b), such that $f(a) = f(b)$, where a and b are some real numbers. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Geometrically Rolle's theorem ensures that there is at least one point on the curve $y = f(x)$ at which tangent is parallel to x-axis (abscissa of the point lying in (a, b)).

5.1.16 *Mean Value Theorem (Lagrange)*

Let $f : [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$ and differentiable on (a, b). Then there exists at least one point c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Geometrically, Mean Value Theorem states that there exists at least one point c in (a, b) such that the tangent at the point $(c, f(c))$ is parallel to the secant joining the points $(a, f(a))$ and $(b, f(b))$.

5.2 Solved Examples

Short Answer (S.A.)

Example 1 Find the value of the constant k so that the function f defined below is continuous at $x = 0$, where $f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$.

Solution It is given that the function f is continuous at $x = 0$. Therefore, $\lim\limits_{x\to 0} f(x) = f(0)$

$$\Rightarrow \qquad \lim_{x\to 0}\frac{1-\cos 4x}{8x^2} = k$$

$$\Rightarrow \qquad \lim_{x\to 0}\frac{2\sin^2 2x}{8x^2} = k$$

$$\Rightarrow \qquad \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = k$$

$$\Rightarrow \qquad k = 1$$

Thus, f is continuous at $x = 0$ if $k = 1$.

Example 2 Discuss the continuity of the function $f(x) = \sin x \cdot \cos x$.

Solution Since $\sin x$ and $\cos x$ are continuous functions and product of two continuous function is a continuous function, therefore $f(x) = \sin x \cdot \cos x$ is a continuous function.

Example 3 If $f(x)=\begin{cases}\dfrac{x^3+x^2-16x+20}{(x-2)^2}, & x\neq 2\\ k, & x=2\end{cases}$ is continuous at $x = 2$, find the value of k.

Solution Given $f(2) = k$.

Now, $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=\lim_{x\to 2}\dfrac{x^3+x^2-16x+20}{(x-2)^2}$

$$=\lim_{x\to 2}\frac{(x+5)(x-2)^2}{(x-2)^2}=\lim_{x\to 2}(x+5)=7$$

As f is continuous at $x = 2$, we have

$$\lim_{x\to 2} f(x)=f(2)$$

$\Rightarrow$ $k = 7$.

Example 4 Show that the function f defined by

$$f(x)=\begin{cases}x\sin\dfrac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$$

is continuous at $x = 0$.

Solution Left hand limit at $x = 0$ is given by

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} x\sin\frac{1}{x} = 0 \qquad \left[\text{since, } -1<\sin\frac{1}{x}<1\right]$$

Similarly $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x\sin\dfrac{1}{x}=0$. Moreover $f(0) = 0$.

Thus $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$. Hence f is continuous at $x = 0$

Example 5 Given $f(x) = \dfrac{1}{x-1}$. Find the points of discontinuity of the composite function $y = f[f(x)]$.

Solution We know that $f(x) = \dfrac{1}{x-1}$ is discontinuous at $x = 1$

Now, for $x\neq 1$,

$$f(f(x)) = f\left(\frac{1}{x-1}\right) = \frac{1}{\dfrac{1}{x-1}-1} = \frac{x-1}{2-x},$$

which is discontinuous at $x = 2$.

Hence, the points of discontinuity are $x = 1$ and $x = 2$.

Example 6 Let $f(x) = x|x|$, for all $x \in \mathbf{R}$. Discuss the derivability of $f(x)$ at $x = 0$

Solution We may rewrite f as $f(x) = \begin{cases} x^2, \text{ if } x \geq 0 \\ -x^2, \text{if } x < 0 \end{cases}$

Now $\mathrm{L}f'(0) = \lim\limits_{h \to 0^-} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^-} \dfrac{-h^2 - 0}{h} = \lim\limits_{h \to 0^-} -h = 0$

Now $\mathrm{R}f'(0) = \lim\limits_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^+} \dfrac{h^2 - 0}{h} = \lim\limits_{h \to 0^-} h = 0$

Since the left hand derivative and right hand derivative both are equal, hence f is differentiable at $x = 0$.

Example 7 Differentiate $\sqrt{\tan\sqrt{x}}$ w.r.t. x

Solution Let $y = \sqrt{\tan\sqrt{x}}$. Using chain rule, we have

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan\sqrt{x}}} \cdot \frac{d}{dx}(\tan\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} \cdot \sec^2\sqrt{x}\,\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}}(\sec^2\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{(\sec^2\sqrt{x})}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}.$$

Example 8 If $y = \tan(x + y)$, find $\dfrac{dy}{dx}$.

Solution Given $y = \tan(x + y)$. differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx}=\sec^2(x+y)\frac{d}{dx}(x+y)$$

$$= \sec^2 (x + y) \left(1+\frac{dy}{dx}\right)$$

or $\quad [1 - \sec^2 (x + y] \dfrac{dy}{dx} = \sec^2 (x + y)$

Therefore, $\quad \dfrac{dy}{dx}=\dfrac{\sec^2(x+y)}{1-\sec^2(x+y)} = -\operatorname{cosec}^2 (x + y).$

Example 9 If $e^x + e^y = e^{x+y}$, prove that

$$\frac{dy}{dx}=-e^{y-x}.$$

Solution Given that $e^x + e^y = e^{x+y}$. Differentiating both sides w.r.t. x, we have

$$e^x + e^y\frac{dy}{dx} = e^{x+y}\left(1+\frac{dy}{dx}\right)$$

or $\quad (e^y - e^{x+y})\dfrac{dy}{dx} = e^{x+y} - e^x,$

which implies that $\dfrac{dy}{dx}=\dfrac{e^{x+y}-e^x}{e^y-e^{x+y}}=\dfrac{e^x+e^y-e^x}{e^y-e^x-e^y}=-e^{y-x}.$

Example 10 Find $\dfrac{dy}{dx}$, if $y = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), -\dfrac{1}{\sqrt{3}}<x<\dfrac{1}{\sqrt{3}}.$

Solution Put $x = \tan\theta$, where $\dfrac{-\pi}{6}<\theta<\dfrac{\pi}{6}$.

Therefore, $\quad y = \tan^{-1}\left(\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$

$= \tan^{-1}(\tan 3\theta)$

$= 3\theta \qquad$ (because $\dfrac{-\pi}{2}<3\theta<\dfrac{\pi}{2}$)

$= 3\tan^{-1}x$

Hence, $$\frac{dy}{dx} = \frac{3}{1+x^2}.$$

Example 11 If $y = \sin^{-1}\left\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right\}$ and $0 < x < 1$, then find $\frac{dy}{dx}$.

Solution We have $y = \sin^{-1}\left\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right\}$, where $0 < x < 1$.

Put $x = \sin A$ and $\sqrt{x} = \sin B$

Therefore, $$y = \sin^{-1}\left\{\sin A\sqrt{1-\sin^2 B} - \sin B\sqrt{1-\sin^2 A}\right\}$$

$$= \sin^{-1}\left\{\sin A\cos B - \sin B\cos A\right\}$$

$$= \sin^{-1}\left\{\sin(A - B)\right\} = A - B$$

Thus $$y = \sin^{-1} x - \sin^{-1}\sqrt{x}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

Example 12 If $x = a\sec^3\theta$ and $y = a\tan^3\theta$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.

Solution We have $x = a\sec^3\theta$ and $y = a\tan^3\theta$.

Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = 3a\sec^2\theta\frac{d}{d\theta}(\sec\theta) = 3a\sec^3\theta\tan\theta$$

and $$\frac{dy}{d\theta} = 3a\tan^2\theta\frac{d}{d\theta}(\tan\theta) = 3a\tan^2\theta\sec^2\theta.$$

Thus $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta.$$

Hence, $\left(\frac{dy}{dx}\right)_{at\,\theta=\frac{\pi}{3}} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Example 13 If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Solution We have $x^y = e^{x-y}$. Taking logarithm on both sides, we get

$$y \log x = x - y$$

$$\Rightarrow \quad y\,(1 + \log x) = x$$

$$\text{i.e.} \quad y = \frac{x}{1+\log x}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x).1 - x\left(\frac{1}{x}\right)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}.$$

Example 14 If $y = \tan x + \sec x$, prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$.

Solution We have $y = \tan x + \sec x$. Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \sec^2 x + \sec x \tan x$$

$$= \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \frac{1+\sin x}{\cos^2 x} = \frac{1+\sin x}{(1+\sin x)(1-\sin x)}.$$

thus $\frac{dy}{dx} = \frac{1}{1-\sin x}$.

Now, differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{-(-\cos x)}{(1-\sin x)^2} = \frac{\cos x}{(1-\sin x)^2}$$

Example 15 If $f(x) = |\cos x|$, find $f'\left(\frac{3\pi}{4}\right)$.

Solution When $\frac{\pi}{2} < x < \pi$, cos$x$ < 0 so that |cos x| = – cos x, i.e., $f(x) = -\cos x$

$\Rightarrow f'(x) = \sin x$.

Hence, $f'\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Example 16 If $f(x) = |\cos x - \sin x|$, find $f'\left(\frac{\pi}{6}\right)$.

Solution When $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$, so that $\cos x - \sin x > 0$, i.e.,

$f(x) = \cos x - \sin x$

$\Rightarrow f'(x) = -\sin x - \cos x$

Hence $f'\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} - \cos\frac{\pi}{6} = -\frac{1}{2}(1+\sqrt{3})$.

Example 17 Verify Rolle's theorem for the function, $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$.

Solution Consider $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$. Note that:

(i) The function f is continuous in $\left[0, \frac{\pi}{2}\right]$, as f is a sine function, which is always continuous.

(ii) $f'(x) = 2\cos 2x$, exists in $\left(0, \frac{\pi}{2}\right)$, hence f is derivable in $\left(0, \frac{\pi}{2}\right)$.

(iii) $f(0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}\right) = \sin\pi = 0 \Rightarrow f(0) = f\left(\frac{\pi}{2}\right)$.

Conditions of Rolle's theorem are satisfied. Hence there exists at least one $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$. Thus

$$2\cos 2c = 0 \quad \Rightarrow \quad 2c = \frac{\pi}{2} \quad \Rightarrow \quad c = \frac{\pi}{4}.$$

Example 18 Verify mean value theorem for the function $f(x) = (x-3)(x-6)(x-9)$ in [3, 5].

Solution (i) Function f is continuous in [3, 5] as product of polynomial functions is a polynomial, which is continuous.

(ii) $f'(x) = 3x^2 - 36x + 99$ exists in (3, 5) and hence derivable in (3, 5).

Thus conditions of mean value theorem are satisfied. Hence, there exists at least one $c \in (3, 5)$ such that

$$f'(c) = \frac{f(5) - f(3)}{5-3}$$

$$\Rightarrow 3c^2 - 36c + 99 = \frac{8-0}{2} = 4$$

$$\Rightarrow c = 6 \pm \sqrt{\frac{13}{3}}.$$

Hence $c = 6 - \sqrt{\frac{13}{3}}$ (since other value is not permissible).

Long Answer (L.A.)

Example 19 If $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, x \neq \dfrac{\pi}{4}$

find the value of $f\left(\dfrac{\pi}{4}\right)$ so that $f(x)$ becomes continuous at $x = \dfrac{\pi}{4}$.

Solution Given, $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, x \neq \dfrac{\pi}{4}$

Therefore, $$\lim_{x \to \frac{\pi}{4}} f(x) = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\left(\sqrt{2}\cos x - 1\right)\sin x}{\cos x - \sin x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\left(\sqrt{2}\cos x - 1\right)}{\left(\sqrt{2}\cos x + 1\right)} \cdot \frac{\left(\sqrt{2}\cos x + 1\right)}{\left(\cos x - \sin x\right)} \cdot \frac{\left(\cos x + \sin x\right)}{\left(\cos x + \sin x\right)} . \sin x$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{2\cos^2 x - 1}{\cos^2 x - \sin^2 x} \cdot \frac{\cos x + \sin x}{\sqrt{2}\cos x + 1} . (\sin x)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\cos 2x}{\cos 2x} \cdot \left(\frac{\cos x + \sin x}{\sqrt{2}\cos x + 1} \right) . (\sin x)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(\cos x + \sin x)}{\sqrt{2}\cos x + 1} \sin x$$

$$= \frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right)}{\sqrt{2} . \frac{1}{\sqrt{2}} + 1} = \frac{1}{2}$$

Thus, $\displaystyle \lim_{x \to \frac{\pi}{4}} f(x) = \frac{1}{2}$

If we define $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$, then $f(x)$ will become continuous at $x = \frac{\pi}{4}$. Hence for f to be continuous at $x = \frac{\pi}{4}$, $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$.

Example 20 Show that the function f given by $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

is discontinuous at $x = 0$.

Solution The left hand limit of f at $x = 0$ is given by

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \frac{0 - 1}{0 + 1} = -1$$

Similarly, $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$

$$= \lim_{x\to 0^+} \frac{1-\frac{1}{e^{\frac{1}{x}}}}{1+\frac{1}{e^{\frac{1}{x}}}} = \lim_{x\to 0^+} \frac{1-e^{\frac{-1}{x}}}{1+e^{\frac{-1}{x}}} = \frac{1-0}{1+0} = 1$$

Thus $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, therefore, $\lim_{x\to 0} f(x)$ does not exist. Hence f is discontinuous at $x = 0$.

Example 21 Let $$f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{if } x<0 \\ a, & \text{if } x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{if } x>0 \end{cases}$$

For what value of a, f is continuous at $x = 0$?

Solution Here $f(0) = a$ Left hand limit of f at 0 is

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1-\cos 4x}{x^2} = \lim_{x\to 0^-} \frac{2\sin^2 2x}{x^2}$$

$$= \lim_{2x\to 0^-} 8\left(\frac{\sin 2x}{2x}\right)^2 = 8\,(1)^2 = 8.$$

and right hand limit of f at 0 is

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}\,(\sqrt{16+\sqrt{x}}+4)}{(\sqrt{16+\sqrt{x}}+4)(\sqrt{16+\sqrt{x}}-4)}$$

$$= \lim_{x \to 0^+} \frac{\sqrt{x}\,(\sqrt{16+\sqrt{x}}+4)}{16+\sqrt{x}-16} = \lim_{x \to 0^+} \left(\sqrt{16+\sqrt{x}}+4\right) = 8$$

Thus, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = 8$. Hence f is continuous at $x = 0$ only if $a = 8$.

Example 22 Examine the differentiability of the function f defined by

$$f(x) = \begin{cases} 2x+3, & \text{if } -3 \le x < -2 \\ x+1, & \text{if } -2 \le x < 0 \\ x+2, & \text{if } 0 \le x \le 1 \end{cases}$$

Solution The only doubtful points for differentiability of $f(x)$ are $x = -2$ and $x = 0$.
Differentiability at $x = -2$.

Now $\mathrm{L}f'(-2) = \lim\limits_{h \to 0^-} \dfrac{f(-2+h) - f(-2)}{h}$

$$= \lim_{h \to 0^-} \frac{2(-2+h)+3-(-2+1)}{h} = \lim_{h \to 0^-} \frac{2h}{h} = \lim_{h \to 0^-} 2 = 2.$$

and $\mathrm{R}f'(-2) = \lim\limits_{h \to 0^+} \dfrac{f(-2+h) - f(-2)}{h}$

$$= \lim_{h \to 0^-} \frac{-2+h+1-(-2+1)}{h}$$

$$= \lim_{h \to 0^-} \frac{h-1-(-1)}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

Thus $\mathrm{R}f'(-2) \neq \mathrm{L}f'(-2)$. Therefore f is not differentiable at $x = -2$.
Similarly, for differentiability at $x = 0$, we have

$$\mathrm{L}(f'(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{0+h+1-(0+2)}{h}$$

$$= \lim_{h \to 0^-} \frac{h-1}{h} = \lim_{h \to 0^-} \left(1 - \frac{1}{h}\right)$$

which does not exist. Hence f is not differentiable at $x = 0$.

Example 23 Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$, where $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$.

Solution Let $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ and $v = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$.

We want to find $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

Now $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$. Put $x = \sin\theta$. $\left(\frac{\pi}{4} < \theta < \frac{\pi}{2}\right)$.

Then $u = \tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right) = \tan^{-1}(\cot\theta)$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \theta\right)\right\} = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sin^{-1} x$$

Hence $\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$.

Now $v = \cos^{-1}(2x\sqrt{1-x^2})$

$$= \frac{\pi}{2} - \sin^{-1}(2x\sqrt{1-x^2})$$

$$= \frac{\pi}{2} - \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta}) = \frac{\pi}{2} - \sin^{-1}(\sin 2\theta)$$

$$= \frac{\pi}{2} - \sin^{-1}\{\sin(\pi - 2\theta)\} \quad \left[\text{since } \frac{\pi}{2} < 2\theta < \pi\right]$$

$$= \frac{\pi}{2} - (\pi - 2\theta) = \frac{-\pi}{2} + 2\theta$$

$$\Rightarrow \qquad v = \frac{-\pi}{2} + 2\sin^{-1}x$$

$$\Rightarrow \qquad \frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}.$$

Hence
$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{-1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = \frac{-1}{2}.$$

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 24 to 35.

Example 24 The function $f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ k & \text{, if } x = 0 \end{cases}$

is continuous at $x = 0$, then the value of k is

(A) 3 (B) 2

(C) 1 (D) 1.5

Solution (B) is the Correct answer.

Example 25 The function $f(x) = [x]$, where $[x]$ denotes the greatest integer function, is continuous at

(A) 4 (B) – 2

(C) 1 (D) 1.5

Solution (D) is the correct answer. The greatest integer function$[x]$ is discontinuous at all integral values of x. Thus D is the correct answer.

Example 26 The number of points at which the function $f(x) = \frac{1}{x-[x]}$ is not continuous is

(A) 1 (B) 2

(C) 3 (D) none of these

Solution (D) is the correct answer. As $x - [x] = 0$, when x is an integer so $f(x)$ is discontinuous for all $x \in \mathbf{Z}$.

Example 27 The function given by $f(x) = \tan x$ is discontinuous on the set

(A) $\{n\pi : n \in \mathbf{Z}\}$ (B) $\{2n\pi : n \in \mathbf{Z}\}$

(C) $\left\{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\right\}$ (D) $\left\{\frac{n\pi}{2} : n \in \mathbf{Z}\right\}$

Solution C is the correct answer.

Example 28 Let $f(x) = |\cos x|$. Then,

(A) f is everywhere differentiable.

(B) f is everywhere continuous but not differentiable at $n = n\pi,\ n \in \mathbf{Z}$.

(C) f is everywhere continuous but not differentiable at $x = (2n + 1)\frac{\pi}{2}$, $n \in \mathbf{Z}$.

(D) none of these.

Solution C is the correct answer.

Example 29 The function $f(x) = |x| + |x - 1|$ is

(A) continuous at $x = 0$ as well as at $x = 1$.

(B) continuous at $x = 1$ but not at $x = 0$.

(C) discontinuous at $x = 0$ as well as at $x = 1$.

(D) continuous at $x = 0$ but not at $x = 1$.

Solution Correct answer is A.

Example 30 The value of k which makes the function defined by

$$f(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$, continuous at $x = 0$ is

(A) 8 (B) 1

(C) –1 (D) none of these

Solution (D) is the correct answer. Indeed $\lim_{x \to 0} \sin\frac{1}{x}$ does not exist.

Example 31 The set of points where the functions f given by $f(x) = |x - 3| \cos x$ is differentiable is

(A) **R** (B) **R** – {3}
(C) (0, ∞) (D) none of these

Solution B is the correct answer.

Example 32 Differential coefficient of sec $(\tan^{-1}x)$ w.r.t. x is

(A) $\dfrac{x}{\sqrt{1+x^2}}$ (B) $\dfrac{x}{1+x^2}$

(C) $x\sqrt{1+x^2}$ (D) $\dfrac{1}{\sqrt{1+x^2}}$

Solution (A) is the correct answer.

Example 33 If $u = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ and $v = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, then $\dfrac{du}{dv}$ is

(A) $\dfrac{1}{2}$ (B) x (C) $\dfrac{1-x^2}{1+x^2}$ (D) 1

Solution (D) is the correct answer.

Example 34 The value of c in Rolle's Theorem for the function $f(x) = e^x \sin x$, $x \in [0, \pi]$ is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{2}$ (D) $\dfrac{3\pi}{4}$

Solution (D) is the correct answer.

Example 35 The value of c in Mean value theorem for the function $f(x) = x(x-2)$, $x \in [1, 2]$ is

(A) $\dfrac{3}{2}$ (B) $\dfrac{2}{3}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{3}{2}$

Solution (A) is the correct answer.

Example 36 Match the following

COLUMN-I	**COLUMN-II**
(A) If a function $f(x)=\begin{cases} \dfrac{\sin 3x}{x}, & if\ x \neq 0 \\ \dfrac{k}{2}, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$, then k is equal to	(a) $\lvert x \rvert$

(B) Every continuous function is differentiable (b) True

(C) An example of a function which is continuous everywhere but not differentiable at exactly one point (c) 6

(D) The identity function i.e. $f(x) = x \ \forall x \in R$ is a continuous function (d) False

Solution A $\to$ *c*, B $\to$ *d*, C $\to$ *a*, D $\to$ *b*

Fill in the blanks in each of the Examples 37 to 41.

Example 37 The number of points at which the function $f(x) = \dfrac{1}{\log|x|}$ is discontinuous is ________.

Solution The given function is discontinuous at $x = 0, \pm 1$ and hence the number of points of discontinuity is 3.

Example 38 If $f(x) = \begin{cases} ax+1 \text{ if } x \geq 1 \\ x+2 \text{ if } x < 1 \end{cases}$ is continuous, then *a* should be equal to ________.

Solution $a = 2$

Example 39 The derivative of $\log_{10} x$ w.r.t. *x* is ________.

Solution $(\log_{10} e)\dfrac{1}{x}$.

Example 40 If $y = \sec^{-1}\left(\dfrac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, then $\dfrac{dy}{dx}$ is equal to ______.

Solution 0.

Example 41 The deriative of sin *x* w.r.t. cos *x* is ________.

Solution $-\cot x$

State whether the statements are True or False in each of the Exercises 42 to 46.

Example 42 For continuity, at $x = a$, each of $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ is equal to $f(a)$.

Solution True.

Example 43 $y = |x - 1|$ is a continuous function.

Solution True.

Example 44 A continuous function can have some points where limit does not exist.

Solution False.

Example 45 $|\sin x|$ is a differentiable function for every value of *x*.

Solution False.

Example 46 cos $|x|$ is differentiable everywhere.

Solution True.

5.3 EXERCISE

Short Answer (S.A.)

1. Examine the continuity of the function

$f(x) = x^3 + 2x^2 - 1$ at $x = 1$

Find which of the functions in Exercises 2 to 10 is continuous or discontinuous at the indicated points:

2. $f(x)=\begin{cases} 3x+5, \text{ if } x\geq 2 \\ x^2, \text{ if } x<2 \end{cases}$

at $x=2$

3. $f(x)=\begin{cases} \dfrac{1-\cos 2x}{x^2}, & \text{if } x\neq 0 \\ 5, & \text{if } x=0 \end{cases}$

at $x=0$

4. $f(x)=\begin{cases} \dfrac{2x^2-3x-2}{x-2}, & \text{if } x\neq 2 \\ 5, & \text{if } x=2 \end{cases}$

at $x=2$

5. $f(x)=\begin{cases} \dfrac{|x-4|}{2(x-4)}, & \text{if } x\neq 4 \\ 0, & \text{if } x=4 \end{cases}$

at $x = 4$

6. $f(x)=\begin{cases} |x|\cos\dfrac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$

at $x = 0$

7. $f(x)=\begin{cases} |x-a|\sin\dfrac{1}{x-a}, & \text{if } x\neq 0 \\ 0, & \text{if } x=a \end{cases}$

at $x = a$

8. $f(x)=\begin{cases} \dfrac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$

at $x = 0$

9. $f(x)=\begin{cases} \dfrac{x^2}{2}, & \text{if } 0\leq x\leq 1 \\ 2x^2-3x+\dfrac{3}{2}, & \text{if } 1<x\leq 2 \end{cases}$

at $x = 1$

10. $f(x)=|x|+|x-1|$ at $x = 1$

Find the value of k in each of the Exercises 11 to 14 so that the function f is continuous at the indicated point:

11. $f(x)=\begin{cases} 3x-8, & \text{if } x\le 5 \\ 2k, & \text{if } x>5 \end{cases}$ at $x=5$

12. $f(x)=\begin{cases} \dfrac{2^{x+2}-16}{4^x-16}, & \text{if } x\ne 2 \\ k, & \text{if } x=2 \end{cases}$ at $x=2$

13. $f(x)=\begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x\le 1 \end{cases}$ at $x=0$

14. $f(x)=\begin{cases} \dfrac{1-\cos kx}{x\sin x}, & \text{if } x\ne 0 \\ \dfrac{1}{2}, & \text{if } x=0 \end{cases}$ at $x=0$

15. Prove that the function f defined by

$$f(x)=\begin{cases} \dfrac{x}{|x|+2x^2}, & x\ne 0 \\ k, & x=0 \end{cases}$$

remains discontinuous at $x=0$, regardless the choice of k.

16. Find the values of a and b such that the function f defined by

$$f(x)=\begin{cases} \dfrac{x-4}{|x-4|}+a, & \text{if } x<4 \\ a+b, & \text{if } x=4 \\ \dfrac{x-4}{|x-4|}+b, & \text{if } x>4 \end{cases}$$

is a continuous function at $x=4$.

17. Given the function $f(x)=\dfrac{1}{x+2}$. Find the points of discontinuity of the composite function $y=f(f(x))$.

18. Find all points of discontinuity of the function $f(t)=\dfrac{1}{t^2+t-2}$, where $t=\dfrac{1}{x-1}$.

19. Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$.

Examine the differentiability of f, where f is defined by

20. $f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x, & \text{if } 2 \le x < 3 \end{cases}$

at $x = 2$.

21. $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

at $x = 0$.

22. $f(x) = \begin{cases} 1+x, & \text{if } x \le 2 \\ 5-x, & \text{if } x > 2 \end{cases}$

at $x = 2$.

23. Show that $f(x) = |x-5|$ is continuous but not differentiable at $x = 5$.

24. A function $f : \mathbf{R} \to \mathbf{R}$ satisfies the equation $f(x + y) = f(x) f(y)$ for all $x, y \in \mathbf{R}$, $f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Prove that $f'(x) = 2 f(x)$.

Differentiate each of the following w.r.t. x (Exercises 25 to 43) :

25. $2^{\cos^2 x}$ **26.** $\dfrac{8^x}{x^8}$ **27.** $\log\left(x+\sqrt{x^2+a}\right)$

28. $\log\left[\log\left(\log x^5\right)\right]$ **29.** $\sin\sqrt{x}+\cos^2\sqrt{x}$ **30.** $\sin^n(ax^2+bx+c)$

31. $\cos\left(\tan\sqrt{x+1}\right)$ **32.** $\sin x^2 + \sin^2 x + \sin^2(x^2)$ **33.** $\sin^{-1}\left(\dfrac{1}{\sqrt{x+1}}\right)$

34. $(\sin x)^{\cos x}$ **35.** $\sin^m x \cdot \cos^n x$ **36.** $(x + 1)^2 (x + 2)^3 (x + 3)^4$

37. $\cos^{-1}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right), \dfrac{-\pi}{4}<x<\dfrac{\pi}{4}$

38. $\tan^{-1}\left(\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right), -\dfrac{\pi}{4}<x<\dfrac{\pi}{4}$

39. $\tan^{-1}(\sec x+\tan x), -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$

40. $\tan^{-1}\left(\dfrac{a\cos x-b\sin x}{b\cos x+a\sin x}\right), -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$ and $\dfrac{a}{b}\tan x>-1$

41. $\sec^{-1}\left(\dfrac{1}{4x^3-3x}\right), 0<x<\dfrac{1}{\sqrt{2}}$

42. $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right), \dfrac{-1}{\sqrt{3}}<\dfrac{x}{a}<\dfrac{1}{\sqrt{3}}$

43. $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), -1<x<1, x\neq 0$

Find $\dfrac{dy}{dx}$ of each of the functions expressed in parametric form in Exercises from 44 to 48.

44. $x=t+\dfrac{1}{t},\ y=t-\dfrac{1}{t}$

45. $x=e^{\theta}\left(\theta+\dfrac{1}{\theta}\right),\ y=e^{-\theta}\left(\theta-\dfrac{1}{\theta}\right)$

46. $x=3\cos\theta-2\cos^3\theta,\ y=3\sin\theta-2\sin^3\theta.$

47. $\sin x=\dfrac{2t}{1+t^2},\ \tan y=\dfrac{2t}{1-t^2}.$

48. $x=\dfrac{1+\log t}{t^2},\ y=\dfrac{3+2\log t}{t}.$

49. If $x=e^{\cos 2t}$ and $y=e^{\sin 2t}$, prove that $\dfrac{dy}{dx}=\dfrac{-y\log x}{x\log y}$.

50. If $x=a\sin 2t\,(1+\cos 2t)$ and $y=b\cos 2t\,(1-\cos 2t)$, show that $\left(\dfrac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{4}}=\dfrac{b}{a}$.

51. If $x=3\sin t-\sin 3t,\ y=3\cos t-\cos 3t$, find $\dfrac{dy}{dx}$ at $t=\dfrac{\pi}{3}$.

52. Differentiate $\dfrac{x}{\sin x}$ w.r.t. $\sin x$.

53. Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1} x$ when $x \neq 0$.

Find $\dfrac{dy}{dx}$ when x and y are connected by the relation given in each of the Exercises 54 to 57.

54. $\sin (xy) + \dfrac{x}{y} = x^2 - y$

55. $\sec (x + y) = xy$

56. $\tan^{-1} (x^2 + y^2) = a$

57. $(x^2 + y^2)^2 = xy$

58. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then show that $\dfrac{dy}{dx} \cdot \dfrac{dx}{dy} = 1$.

59. If $x = e^{\frac{x}{y}}$, prove that $\dfrac{dy}{dx} = \dfrac{x-y}{x \log x}$.

60. If $y^x = e^{y-x}$, prove that $\dfrac{dy}{dx} = \dfrac{(1+\log y)^2}{\log y}$.

61. If $y = (\cos x)^{(\cos x)^{(\cos x)\cdots\infty}}$, show that $\dfrac{dy}{dx} = \dfrac{y^2 \tan x}{y \log \cos x - 1}$.

62. If $x \sin (a + y) + \sin a \cos (a + y) = 0$, prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

63. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a (x - y)$, prove that $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$.

64. If $y = \tan^{-1}x$, find $\dfrac{d^2y}{dx^2}$ in terms of y alone.

Verify the Rolle's theorem for each of the functions in Exercises 65 to 69.

65. $f(x) = x(x-1)^2$ in [0, 1].

66. $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$.

67. $f(x) = \log(x^2 + 2) - \log 3$ in [–1, 1].

68. $f(x) = x(x+3)e^{-x/2}$ in [–3, 0].

69. $f(x) = \sqrt{4-x^2}$ in [– 2, 2].

70. Discuss the applicability of Rolle's theorem on the function given by

$$f(x)=\begin{cases} x^2+1, & if \;\; 0\le x\le 1 \\ 3-x, & if \;\; 1\le x\le 2 \end{cases}.$$

71. Find the points on the curve $y = (\cos x - 1)$ in [0, 2π], where the tangent is parallel to x-axis.

72. Using Rolle's theorem, find the point on the curve $y = x(x-4)$, $x \in [0, 4]$, where the tangent is parallel to x-axis.

Verify mean value theorem for each of the functions given Exercises 73 to 76.

73. $f(x) = \dfrac{1}{4x-1}$ in [1, 4].

74. $f(x) = x^3 - 2x^2 - x + 3$ in [0, 1].

75. $f(x) = \sin x - \sin 2x$ in [0, π].

76. $f(x) = \sqrt{25-x^2}$ in [1, 5].

77. Find a point on the curve $y = (x-3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

78. Using mean value theorem, prove that there is a point on the curve $y = 2x^2 - 5x + 3$ between the points A(1, 0) and B (2, 1), where tangent is parallel to the chord AB. Also, find that point.

Long Answer (L.A.)

79. Find the values of p and q so that

$$f(x) = \begin{cases} x^2 + 3x + p, \text{ if } x \le 1 \\ qx + 2 \quad\quad, \text{ if } x > 1 \end{cases}$$

is differentiable at $x = 1$.

80. If $x^m . y^n = (x + y)^{m+n}$, prove that

(i) $\frac{dy}{dx} = \frac{y}{x}$ and (ii) $\frac{d^2y}{dx^2} = 0$.

81. If $x = \sin t$ and $y = \sin pt$, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

82. Find $\frac{dy}{dx}$, if $y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}}$.

Objective Type Questions

Choose the correct answers from the given four options in each of the Exercises 83 to 96.

83. If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$, then which of the following can be a discontinuous function

(A) $f(x) + g(x)$ (B) $f(x) - g(x)$

(C) $f(x) . g(x)$ (D) $\frac{g(x)}{f(x)}$

84. The function $f(x) = \frac{4-x^2}{4x-x^3}$ is

(A) discontinuous at only one point

(B) discontinuous at exactly two points

(C) discontinuous at exactly three points

(D) none of these

85. The set of points where the function f given by $f(x) = |2x-1| \sin x$ is differentiable is

(A) $\mathbf{R}$ (B) $\mathbf{R} - \left\{\frac{1}{2}\right\}$

(C) $(0, \infty)$ (D) none of these

86. The function $f(x) = \cot x$ is discontinuous on the set

(A) $\{x = n\pi : n \in \mathbf{Z}\}$ (B) $\{x = 2n\pi : n \in \mathbf{Z}\}$

(C) $\left\{x = (2n+1)\frac{\pi}{2};\ n \in \mathbf{Z}\right\}$ (iv) $\left\{x = \frac{n\pi}{2};\ n \in \mathbf{Z}\right\}$

87. The function $f(x) = e^{|x|}$ is

(A) continuous everywhere but not differentiable at $x = 0$

(B) continuous and differentiable everywhere

(C) not continuous at $x = 0$

(D) none of these.

88. If $f(x) = x^2 \sin\frac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is

(A) 0 (B) -1

(C) 1 (D) none of these

89. If $f(x) = \begin{cases} mx+1 & ,\text{if } x \le \frac{\pi}{2} \\ \sin x + n, & \text{if } x > \frac{\pi}{2} \end{cases}$, is continuous at $x = \frac{\pi}{2}$, then

(A) $m = 1, n = 0$ (B) $m = \frac{n\pi}{2} + 1$

(C) $n = \frac{m\pi}{2}$ (D) $m = n = \frac{\pi}{2}$

90. Let $f(x) = |\sin x|$. Then

(A) f is everywhere differentiable

(B) f is everywhere continuous but not differentiable at $x = n\pi,\ n \in \mathbf{Z}$.

(C) f is everywhere continuous but not differentiable at $x = (2n + 1)\frac{\pi}{2}$, $n \in \mathbf{Z}$.

(D) none of these

91. If $y = \log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(A) $\dfrac{4x^3}{1-x^4}$ (B) $\dfrac{-4x}{1-x^4}$

(C) $\dfrac{1}{4-x^4}$ (D) $\dfrac{-4x^3}{1-x^4}$

92. If $y = \sqrt{\sin x + y}$, then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{\cos x}{2y-1}$ (B) $\dfrac{\cos x}{1-2y}$

(C) $\dfrac{\sin x}{1-2y}$ (D) $\dfrac{\sin x}{2y-1}$

93. The derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t. $\cos^{-1}x$ is

(A) 2 (B) $\dfrac{-1}{2\sqrt{1-x^2}}$

(C) $\dfrac{2}{x}$ (D) $1 - x^2$

94. If $x = t^2$, $y = t^3$, then $\dfrac{d^2y}{dx^2}$ is

(A) $\dfrac{3}{2}$ (B) $\dfrac{3}{4t}$

(C) $\dfrac{3}{2t}$ (D) $\dfrac{3}{4}$

95. The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(A) 1 (B) – 1

(C) $\frac{3}{2}$ (D) $\frac{1}{3}$

96. For the function $f(x) = x + \frac{1}{x}, x \in [1, 3]$, the value of c for mean value theorem is

(A) 1 (B) $\sqrt{3}$

(C) 2 (D) none of these

Fill in the blanks in each of the Exercises 97 to 101:

97. An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is __________ .

98. Derivative of x^2 w.r.t. x^3 is _________.

99. If $f(x) = |\cos x|$, then $f'\left(\frac{\pi}{4}\right)$ = _______ .

100. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{3}\right)$ = _______.

101. For the curve $\sqrt{x} + \sqrt{y} = 1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$ is __________.

State **True** or **False** for the statements in each of the Exercises 102 to 106.

102. Rolle's theorem is applicable for the function $f(x) = |x - 1|$ in [0, 2].

103. If f is continuous on its domain D, then $|f|$ is also continuous on D.

104. The composition of two continuous function is a continuous function.

105. Trigonometric and inverse - trigonometric functions are differentiable in their respective domain.

106. If $f \cdot g$ is continuous at $x = a$, then f and g are separately continuous at $x = a$.

Chapter 6

APPLICATION OF DERIVATIVES

6.1 Overview

6.1.1 *Rate of change of quantities*

For the function $y = f(x)$, $\frac{d}{dx}(f(x))$ represents the rate of change of y with respect to x.

Thus if 's' represents the distance and 't' the time, then $\frac{ds}{dt}$ represents the rate of change of distance with respect to time.

6.1.2 *Tangents and normals*

A line touching a curve $y = f(x)$ at a point (x_1, y_1) is called the tangent to the curve at that point and its equation is given $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$.

The normal to the curve is the line perpendicular to the tangent at the point of contact, and its equation is given as:

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$$

The angle of intersection between two curves is the angle between the tangents to the curves at the point of intersection.

6.1.3 *Approximation*

Since $f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$, we can say that $f'(x)$ is approximately equal to $\frac{f(x + \Delta x) - f(x)}{\Delta x}$

$\Rightarrow$ approximate value of $f(x + \Delta x) = f(x) + \Delta x \,.f'(x)$.

6.1.4 *Increasing/decreasing functions*

A continuous function in an interval (a, b) is :

(i) strictly increasing if for all $x_1, x_2 \in (a, b)$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ or for all $x \in (a, b)$, $f'(x) > 0$

(ii) strictly decreasing if for all $x_1, x_2 \in (a, b)$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ or for all $x \in (a, b)$, $f'(x) < 0$

6.1.5 ***Theorem*** **:** Let f be a continuous function on $[a, b]$ and differentiable in (a, b) then

(i) f is increasing in $[a, b]$ if $f'(x) > 0$ for each $x \in (a, b)$

(ii) f is decreasing in $[a, b]$ if $f'(x) < 0$ for each $x \in (a, b)$

(iii) f is a constant function in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.

6.1.6 *Maxima and minima*

Local Maximum/Local Minimum for a real valued function f

A point c in the interior of the domain of f, is called

(i) local maxima, if there exists an $h > 0$, such that $f(c) > f(x)$, for all x in $(c - h, c + h)$.

The value $f(c)$ is called the local maximum value of f.

(ii) local minima if there exists an $h > 0$ such that $f(c) < f(x)$, for all x in $(c - h, c + h)$.

The value $f(c)$ is called the local minimum value of f.

A function f defined over $[a, b]$ is said to have maximum (or absolute maximum) at $x = c$, $c \in [a, b]$, if $f(x) \leq f(c)$ for all $x \in [a, b]$.

Similarly, a function $f(x)$ defined over $[a, b]$ is said to have a minimum [or absolute minimum] at $x = d$, if $f(x) \geq f(d)$ for all $x \in [a, b]$.

6.1.7 ***Critical point of f*** : A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a critical point of f.

Working rule for finding points of local maxima or local minima:

(a) **First derivative test:**

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, then c is a point of local maxima, and $f(c)$ is local maximum value.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, then c is a point of local minima, and $f(c)$ is local minimum value.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local minima nor a point of local maxima. Such a point is called a point of inflection.

(b) **Second Derivative test:** Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. In this case $f(c)$ is then the local maximum value.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. In this case $f(c)$ is the local minimum value.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$. In this case, we go back to first derivative test.

6.1.8 *Working rule for finding absolute maxima and or absolute minima :*

Step 1 : Find all the critical points of f in the given interval.

Step 2 : At all these points and at the end points of the interval, calculate the values of f.

Step 3 : Identify the maximum and minimum values of f out of the values calculated in step 2. The maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f.

6.2 Solved Examples

Short Answer Type (S.A.)

Example 1 For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then how fast is the slope of curve changing when $x = 3$?

Solution Slope of curve $= \dfrac{dy}{dx} = 5 - 6x^2$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{dy}{dx}\right) = -12x.\frac{dx}{dt}$$

$$= -12 \cdot (3) \cdot (2)$$

$$= -72 \text{ units/sec.}$$

Thus, slope of curve is decreasing at the rate of 72 units/sec when x is increasing at the rate of 2 units/sec.

Example 2 Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at the uniform rate of 2 cm^2/sec in the surface area, through a tiny hole at the vertex of the bottom. When the slant height of cone is 4 cm, find the rate of decrease of the slant height of water.

Solution If s represents the surface area, then

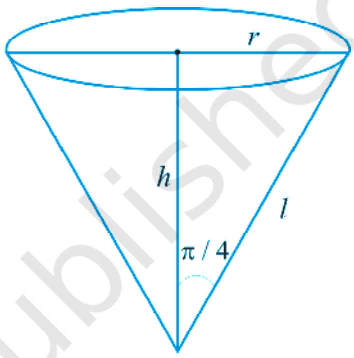


Fig. 6.1

$$\frac{ds}{dt} = 2\text{cm}^2 \text{ /sec}$$

$$s = \pi\, r.l = \pi l \,.\, \sin\frac{\pi}{4} \cdot l = \frac{\pi}{\sqrt{2}} l^2$$

Therefore, $\dfrac{ds}{dt} = \dfrac{2\pi}{\sqrt{2}} l . \dfrac{dl}{dt} = \sqrt{2}\pi l . \dfrac{dl}{dt}$

when l = 4 cm, $\dfrac{dl}{dt} = \dfrac{1}{\sqrt{2}\pi.4} . 2 = \dfrac{1}{2\sqrt{2}\pi} = \dfrac{\sqrt{2}}{4\pi}$ cm/s .

Example 3 Find the angle of intersection of the curves $y^2 = x$ and $x^2 = y$.

Solution Solving the given equations, we have $y^2 = x$ and $x^2 = y \Rightarrow x^4 = x$ or $x^4 - x = 0$

$\Rightarrow x (x^3 - 1) = 0 \Rightarrow x = 0, x = 1$

Therefore, $y = 0,\ y = 1$

i.e. points of intersection are (0, 0) and (1, 1)

Further $y^2 = x \Rightarrow 2y\dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2y}$

and $x^2 = y \Rightarrow \dfrac{dy}{dx} = 2x.$

At (0, 0), the slope of the tangent to the curve $y^2 = x$ is parallel to y-axis and the tangent to the curve $x^2 = y$ is parallel to x-axis.

$$\Rightarrow \text{angle of intersection} = \frac{\pi}{2}$$

At (1, 1), slope of the tangent to the curve $y_2 = x$ is equal to $\frac{1}{2}$ and that of $x^2 = y$ is 2.

$$\tan\theta = \left|\frac{2-\frac{1}{2}}{1+1}\right| = \frac{3}{4}. \qquad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

Example 4 Prove that the function $f(x) = \tan x - 4x$ is strictly decreasing on $\left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$.

Solution $f(x) = \tan x - 4x \Rightarrow f'(x) = \sec^2 x - 4$

When $\frac{-\pi}{3} < x < \frac{\pi}{3}$, $1 < \sec x < 2$

Therefore, $1 < \sec^2 x < 4 \Rightarrow -3 < (\sec^2 x - 4) < 0$

Thus for $\frac{-\pi}{3} < x < \frac{\pi}{3}$, $f'(x) < 0$

Hence f is strictly decreasing on $\left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$.

Example 5 Determine for which values of x, the function $y = x^4 - \frac{4x^3}{3}$ is increasing and for which values, it is decreasing.

Solution $y = x^4 - \frac{4x^3}{3}$ $\qquad \Rightarrow \frac{dy}{dx} = 4x^3 - 4x^2 = 4x^2 (x - 1)$

Now, $\frac{dy}{dx} = 0 \Rightarrow x = 0, x = 1.$

Since $f'(x) < 0 \ \forall x \in (-\infty, 0) \cup (0, 1)$ and f is continuous in $(-\infty, 0]$ and $[0, 1]$. Therefore f is decreasing in $(-\infty, 1]$ and f is increasing in $[1, \infty)$.

Note: Here f is strictly decreasing in $(-\infty, 0) \cup (0, 1)$ and is strictly increasing in $(1, \infty)$.

Example 6 Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima.

Solution $f(x) = 4x^3 - 18x^2 + 27x - 7$

$f'(x) = 12x^2 - 36x + 27 = 3(4x^2 - 12x + 9) = 3(2x - 3)^2$

$f'(x) = 0 \Rightarrow x = \frac{3}{2}$ (critical point)

Since $f'(x) > 0$ for all $x < \frac{3}{2}$ and for all $x > \frac{3}{2}$

Hence $x = \frac{3}{2}$ is a point of inflexion i.e., neither a point of maxima nor a point of minima.

$x = \frac{3}{2}$ is the only critical point, and f has neither maxima nor minima.

Example 7 Using differentials, find the approximate value of $\sqrt{0.082}$

Solution Let $f(x) = \sqrt{x}$

Using $f(x + \Delta x) \simeq f(x) + \Delta x \,.\, f'(x)$, taking $x = .09$ and $\Delta x = -0.008$,

we get $f(0.09 - 0.008) = f(0.09) + (-0.008) f'(0.09)$

$$\Rightarrow \sqrt{0.082} = \sqrt{0.09} - 0.008 \,.\, \left(\frac{1}{2\sqrt{0.09}}\right) = 0.3 - \frac{0.008}{0.6}$$

$= 0.3 - 0.0133 = 0.2867.$

Example 8 Find the condition for the curves $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; $xy = c^2$ to intersect orthogonally.

Solution Let the curves intersect at (x_1, y_1). Therefore,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$\Rightarrow$ slope of tangent at the point of intersection $(m_1) = \frac{b^2x_1}{a^2y_1}$

Again $xy = c^2 \Rightarrow x\frac{dy}{dx}+y=0 \Rightarrow \frac{dy}{dx}=\frac{-y}{x} \Rightarrow m_2 = \frac{-y_1}{x_1}$.

For orthoganality, $m_1 \times m_2 = -1 \Rightarrow \frac{b^2}{a^2}=1$ or $a^2 - b^2 = 0$.

Example 9 Find all the points of local maxima and local minima of the function $f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$.

Solution $f'(x) = -3x^3 - 24x^2 - 45x$

$= -3x(x^2 + 8x + 15) = -3x(x + 5)(x + 3)$

$f'(x) = 0 \Rightarrow x = -5,\ x = -3,\ x = 0$

$f''(x) = -9x^2 - 48x - 45$

$= -3(3x^2 + 16x + 15)$

$f''(0) = -45 < 0$. Therefore, $x = 0$ is point of local maxima

$f''(-3) = 18 > 0$. Therefore, $x = -3$ is point of local minima

$f''(-5) = -30 < 0$. Therefore $x = -5$ is point of local maxima.

Example 10 Show that the local maximum value of $x+\frac{1}{x}$ is less than local minimum value.

Solution Let $y = x+\frac{1}{x} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2}$,

$$\frac{dy}{dx} = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

$$\frac{d^2y}{dx^2} = + \frac{2}{x^3}\text{, therefore } \frac{d^2y}{dx^2}\text{(at } x = 1) > 0 \text{ and } \frac{d^2y}{dx^2}\text{(at } x = -1) < 0.$$

Hence local maximum value of y is at $x = -1$ and the local maximum value $= -2$.

Local minimum value of y is at $x = 1$ and local minimum value $= 2$.

Therefore, local maximum value (-2) is less than local minimum value 2.

Long Answer Type (L.A.)

Example 11 Water is dripping out at a steady rate of 1 cu cm/sec through a tiny hole at the vertex of the conical vessel, whose axis is vertical. When the slant height of water in the vessel is 4 cm, find the rate of decrease of slant height, where the vertical angle of the conical vessel is $\frac{\pi}{6}$.

Solution Given that $\frac{dv}{dt} = 1$ cm^3/s, where v is the volume of water in the conical vessel.

From the Fig.6.2, $l = 4$cm, $h = l\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}l$ and $r = l\sin\frac{\pi}{6} = \frac{l}{2}$.

Therefore, $v = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\,\frac{l^2}{4}\,\frac{\sqrt{3}}{2}l = \frac{\sqrt{3}\pi}{24}l^3$.

$$\frac{dv}{dt} = \frac{\sqrt{3}\pi}{8} l^2 \frac{dl}{dt}$$

Therefore, $1 = \dfrac{\sqrt{3}\pi}{8} 16 . \dfrac{dl}{dt}$

$$\Rightarrow \qquad \frac{dl}{dt} = \frac{1}{2\sqrt{3}\pi} \text{ cm/s.}$$

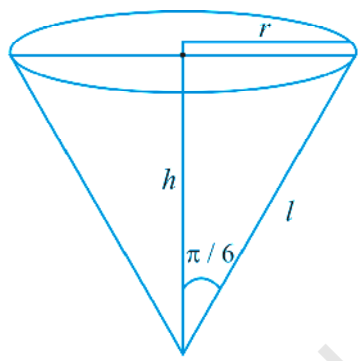


Fig. 6.2

Therefore, the rate of decrease of slant height = $\dfrac{1}{2\sqrt{3}\pi}$ cm/s.

Example 12 Find the equation of all the tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$, that are parallel to the line $x + 2y = 0$.

Solution Given that $y = \cos(x + y) \Rightarrow \dfrac{dy}{dx} = -\sin(x + y)\left[1 + \dfrac{dy}{dx}\right]$ ...(i)

or $$\frac{dy}{dx} = -\frac{\sin(x+y)}{1+\sin(x+y)}$$

Since tangent is parallel to $x + 2y = 0$, therefore slope of tangent $= -\dfrac{1}{2}$

Therefore, $-\dfrac{\sin(x+y)}{1+\sin(x+y)} = -\dfrac{1}{2} \Rightarrow \sin(x + y) = 1$ .... (ii)

Since $\cos(x + y) = y$ and $\sin(x + y) = 1 \Rightarrow \cos^2(x + y) + \sin^2(x + y) = y^2 + 1$

$\Rightarrow \quad 1 = y^2 + 1$ or $y = 0$.

Therefore, $\cos x = 0$.

Therefore, $x = (2n + 1)\dfrac{\pi}{2}$, $n = 0, \pm 1, \pm 2...$

Thus, $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, but $x = \frac{\pi}{2}$, $x = \frac{-3\pi}{2}$ satisfy equation (ii)

Hence, the points are $\left(\frac{\pi}{2}, 0\right), \left(\frac{-3\pi}{2}, 0\right)$.

Therefore, equation of tangent at $\left(\frac{\pi}{2}, 0\right)$ is $y = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)$ or $2x + 4y - \pi = 0$, and

equation of tangent at $\left(\frac{-3\pi}{2}, 0\right)$ is $y = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$ or $2x + 4y + 3\pi = 0$.

Example 13 Find the angle of intersection of the curves $y^2 = 4ax$ and $x^2 = 4by$.

Solution Given that $y^2 = 4ax$...(i) and $x^2 = 4by$... (ii). Solving (i) and (ii), we get

$$\left(\frac{x^2}{4b}\right)^2 = 4ax \Rightarrow x^4 = 64\, ab^2 x$$

or $\quad x(x^3 - 64\, ab^2) = 0 \Rightarrow x = 0,\; x = 4a^{\frac{1}{3}} b^{\frac{2}{3}}$

Therefore, the points of intersection are (0, 0) and $\left(4a^{\frac{1}{3}} b^{\frac{2}{3}}, 4a^{\frac{2}{3}} b^{\frac{1}{3}}\right)$.

Again, $y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$ and $x^2 = 4by \Rightarrow \frac{dy}{dx} = \frac{2x}{4b} = \frac{x}{2b}$

Therefore, at (0, 0) the tangent to the curve $y^2 = 4ax$ is parallel to y-axis and tangent to the curve $x^2 = 4by$ is parallel to x-axis.

$\Rightarrow$ Angle between curves $= \frac{\pi}{2}$

At $\left(4a^{\frac{1}{3}} b^{\frac{2}{3}}, 4a^{\frac{2}{3}} b^{\frac{1}{3}}\right)$, m_1 (slope of the tangent to the curve (i)) $= 2\left(\frac{a}{b}\right)^{\frac{1}{3}}$

$= \frac{2a}{4a^{\frac{2}{3}} b^{\frac{1}{3}}} = \frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{3}}$, m_2 (slope of the tangent to the curve (ii)) $= \frac{4a^{\frac{1}{3}} b^{\frac{2}{3}}}{2b} = 2\left(\frac{a}{b}\right)^{\frac{1}{3}}$

Therefore, $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\dfrac{2\left(\dfrac{a}{b}\right)^{\frac{1}{3}} - \dfrac{1}{2}\left(\dfrac{a}{b}\right)^{\frac{1}{3}}}{1 + 2\left(\dfrac{a}{b}\right)^{\frac{1}{3}} \dfrac{1}{2}\left(\dfrac{a}{b}\right)^{\frac{1}{3}}}\right| = \dfrac{3a^{\frac{1}{3}}.b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)}$

Hence, $\theta = \tan^{-1}\left(\dfrac{3a^{\frac{1}{3}}.b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)}\right)$

Example 14 Show that the equation of normal at any point on the curve $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$ is $4\,(y\cos^3\theta - x\sin^3\theta) = 3\sin 4\theta$.

Solution We have $x = 3\cos\theta - \cos^3\theta$

Therefore, $\dfrac{dx}{d\theta} = -3\sin\theta + 3\cos^2\theta\sin\theta = -3\sin\theta\,(1 - \cos^2\theta) = -3\sin^3\theta$.

$\dfrac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta = 3\cos\theta\;(1 - \sin^2\theta) = 3\cos^3\theta$

$\dfrac{dy}{dx} = -\dfrac{\cos^3\theta}{\sin^3\theta}$. Therefore, slope of normal $= +\dfrac{\sin^3\theta}{\cos^3\theta}$

Hence the equation of normal is

$$y - (3\sin\theta - \sin^3\theta) = \frac{\sin^3\theta}{\cos^3\theta}\,[x - (3\cos\theta - \cos^3\theta)]$$

$$\Rightarrow\; y\cos^3\theta - 3\sin\theta\cos^3\theta + \sin^3\theta\cos^3\theta = x\sin^3\theta - 3\sin^3\theta\cos\theta + \sin^3\theta\cos^3\theta$$

$$\Rightarrow\; y\cos^3\theta - x\sin^3\theta = 3\sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta)$$

$$= \frac{3}{2}\sin 2\theta \,.\, \cos 2\theta$$

$$= \frac{3}{4}\sin 4\theta$$

or $\quad 4\,(y\cos^3\theta - x\sin^3\theta) = 3\sin 4\theta.$

Example 15 Find the maximum and minimum values of

$$f(x) = \sec x + \log \cos^2 x,\ 0 < x < 2\pi$$

Solution $f(x) = \sec x + 2\log \cos x$

Therefore, $\quad f'(x) = \sec x \tan x - 2\tan x = \tan x\,(\sec x - 2)$

$$f'(x) = 0 \Rightarrow \tan x = 0 \text{ or } \sec x = 2 \text{ or } \cos x = \frac{1}{2}$$

Therefore, possible values of x are $x = 0$, or $x = \pi$ and

$$x = \frac{\pi}{3} \quad \text{or} \quad x = \frac{5\pi}{3}$$

Again, $\quad f''(x) = \sec^2 x\,(\sec x - 2) + \tan x\,(\sec x \tan x)$

$$= \sec^3 x + \sec x \tan^2 x - 2\sec^2 x$$

$$= \sec x\,(\sec^2 x + \tan^2 x - 2\sec x).$$ We note that

$f''(0) = 1\,(1 + 0 - 2) = -1 < 0$. Therefore, $x = 0$ is a point of maxima.

$f''(\pi) = -1\,(1 + 0 + 2) = -3 < 0$. Therefore, $x = \pi$ is a point of maxima.

$f''\left(\frac{\pi}{3}\right) = 2\,(4 + 3 - 4) = 6 > 0$. Therefore, $x = \frac{\pi}{3}$ is a point of minima.

$f''\left(\frac{5\pi}{3}\right) = 2\,(4 + 3 - 4) = 6 > 0$. Therefore, $x = \frac{5\pi}{3}$ is a point of minima.

Maximum Value of y at $x = 0$ is $\quad 1 + 0 = 1$

Maximum Value of y at $x = \pi$ is $\quad -1 + 0 = -1$

Minimum Value of y at $x = \dfrac{\pi}{3}$ is $\quad 2 + 2\log\dfrac{1}{2} = 2\,(1 - \log 2)$

Minimum Value of y at $x = \dfrac{5\pi}{3}$ is $\quad 2 + 2\log\dfrac{1}{2} = 2\,(1 - \log 2)$

Example 16 Find the area of greatest rectangle that can be inscribed in an ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Solution Let ABCD be the rectangle of maximum area with sides AB = $2x$ and BC = $2y$, where C (x, y) is a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ as shown in the Fig.6.3.

The area A of the rectangle is $4xy$ i.e. A = $4xy$ which gives $A^2 = 16x^2y^2 = s$ (say)

Therefore, $s = 16x^2\left(1-\dfrac{x^2}{a^2}\right).b^2 = \dfrac{16b^2}{a^2}\,(a^2x^2 - x^4)$

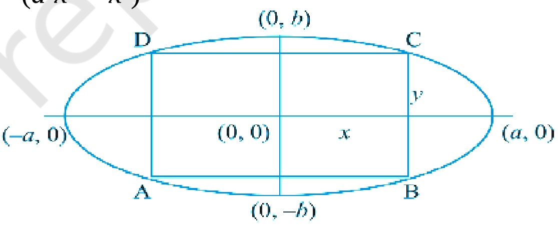


Fig. 6.3

$$\Rightarrow \quad \frac{ds}{dx} = \frac{16b^2}{a^2}\,.\,[2a^2x - 4x^3].$$

Again, $\dfrac{ds}{dx} = 0 \Rightarrow x = \dfrac{a}{\sqrt{2}}$ and $y = \dfrac{b}{\sqrt{2}}$

Now, $\dfrac{d^2s}{dx^2} = \dfrac{16b^2}{a^2}[2a^2 - 12x^2]$

At $x = \dfrac{a}{\sqrt{2}}$, $\dfrac{d^2s}{dx^2} = \dfrac{16b^2}{a^2}[2a^2 - 6a^2] = \dfrac{16b^2}{a^2}(-4a^2) < 0$

Thus at $x = \frac{a}{\sqrt{2}}, y = \frac{b}{\sqrt{2}}, s$ is maximum and hence the area A is maximum.

Maximum area $= 4.x.y = 4 . \frac{a}{\sqrt{2}} . \frac{b}{\sqrt{2}} = 2ab$ sq units.

Example 17 Find the difference between the greatest and least values of the function $f(x) = \sin 2x - x$, on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Solution $f(x) = \sin 2x - x$

$\Rightarrow \quad f'(x) = 2\cos 2x - 1$

Therefore, $f'(x) = 0 \Rightarrow \cos 2x = \frac{1}{2} \Rightarrow 2x$ is $\frac{-\pi}{3}$ or $\frac{\pi}{3} \Rightarrow x = -\frac{\pi}{6}$ or $\frac{\pi}{6}$

$$f\left(-\frac{\pi}{2}\right) = \sin(-\pi) + \frac{\pi}{2} = \frac{\pi}{2}$$

$$f\left(-\frac{\pi}{6}\right) = \sin\left(-\frac{2\pi}{6}\right) + \frac{\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}$$

$$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

$$f\left(\frac{\pi}{2}\right) = \sin(\pi) - \frac{\pi}{2} = -\frac{\pi}{2}$$

Clearly, $\frac{\pi}{2}$ is the greatest value and $-\frac{\pi}{2}$ is the least.

Therefore, difference $= \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Example 18 An isosceles triangle of vertical angle 2θ is inscribed in a circle of radius a. Show that the area of triangle is maximum when $\theta = \frac{\pi}{6}$.

Solution Let ABC be an isosceles triangle inscribed in the circle with radius a such that AB = AC.

AD = AO + OD = $a + a\cos 2\theta$ and BC = 2BD = $2a \sin 2\theta$ (see fig. 16.4)

Therefore, area of the triangle ABC i.e. $\Delta = \frac{1}{2} \text{BC} \cdot \text{AD}$

$$= \frac{1}{2} 2a \sin 2\theta \cdot (a + a\cos 2\theta)$$

$$= a^2 \sin 2\theta\,(1 + \cos 2\theta)$$

$$\Rightarrow \quad \Delta = a^2 \sin 2\theta + \frac{1}{2} a^2 \sin 4\theta$$

Therefore, $\frac{d\Delta}{d\theta} = 2a^2 \cos 2\theta + 2a^2 \cos 4\theta$

$$= 2a^2(\cos 2\theta + \cos 4\theta)$$

$$\frac{d\Delta}{d\theta} = 0 \Rightarrow \cos 2\theta = -\cos 4\theta = \cos(\pi - 4\theta)$$

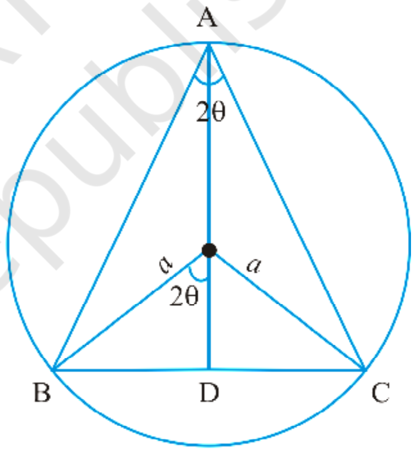


Fig. 6.4

Therefore, $2\theta = \pi - 4\theta \Rightarrow \theta = \frac{\pi}{6}$

$$\frac{d^2\Delta}{d\theta^2} = 2a^2(-2\sin 2\theta - 4\sin 4\theta) < 0 \text{ (at } \theta = \frac{\pi}{6}).$$

Therefore, Area of triangle is maximum when $\theta = \frac{\pi}{6}$.

Objective Type Questions

Choose the correct answer from the given four options in each of the following Examples 19 to 23.

Example 19 The abscissa of the point on the curve $3y = 6x - 5x^3$, the normal at which passes through origin is:

(A) 1 (B) $\frac{1}{3}$ (C) 2 (D) $\frac{1}{2}$

Solution Let (x_1, y_1) be the point on the given curve $3y = 6x - 5x^3$ at which the normal passes through the origin. Then we have $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2 - 5x_1^2$. Again the equation of the normal at (x_1, y_1) passing through the origin gives $2 - 5x_1^2 = \frac{-x_1}{y_1} = \frac{-3}{6 - 5x_1^2}$.

Since $x_1 = 1$ satisfies the equation, therefore, Correct answer is (A).

Example 20 The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 = 2$

(A) touch each other (B) cut at right angle

(C) cut at an angle $\frac{\pi}{3}$ (D) cut at an angle $\frac{\pi}{4}$

Solution From first equation of the curve, we have $3x^2 - 3y^2 - 6xy \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = (m_1)$ say and second equation of the curve gives

$6xy + 3x^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2} = (m_2)$ say

Since $m_1 . m_2 = -1$. Therefore, correct answer is (B).

Example 21 The tangent to the curve given by $x = e^t \cdot \cos t$, $y = e^t \cdot \sin t$ at $t = \frac{\pi}{4}$ makes with x-axis an angle:

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Solution $\frac{dx}{dt} = - e^t \cdot \sin t + e^t \cos t$, $\frac{dy}{dt} = e^t \cos t + e^t \sin t$

Therefore, $\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{\cos t + \sin t}{\cos t - \sin t} = \frac{\sqrt{2}}{0}$ and hence the correct answer is (D).

Example 22 The equation of the normal to the curve $y = \sin x$ at $(0, 0)$ is:

(A) $x = 0$ (B) $y = 0$ (C) $x + y = 0$ (D) $x - y = 0$

Solution $\frac{dy}{dx} = \cos x$. Therefore, slope of normal $= \left(\frac{-1}{\cos x}\right)_{x=0} = -1$. Hence the equation of normal is $y - 0 = -1(x - 0)$ or $x + y = 0$

Therefore, correct answer is (C).

Example 23 The point on the curve $y^2 = x$, where the tangent makes an angle of $\frac{\pi}{4}$ with x-axis is

(A) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (B) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (C) (4, 2) (D) (1, 1)

Solution $\frac{dy}{dx} = \frac{1}{2y} = \tan\frac{\pi}{4} = 1 \Rightarrow y = \frac{1}{2} \Rightarrow x = \frac{1}{4}$

Therefore, correct answer is B.

Fill in the blanks in each of the following Examples 24 to 29.

Example 24 The values of a for which $y = x^2 + ax + 25$ touches the axis of x are______.

Solution $\frac{dy}{dx} = 0 \Rightarrow 2x + a = 0$ i.e. $x = -\frac{a}{2}$,

Therefore, $\frac{a^2}{4} + a\left(-\frac{a}{2}\right) + 25 = 0$ $\Rightarrow$ $a = \pm 10$

Hence, the values of a are $\pm$ 10.

Example 25 If $f(x) = \frac{1}{4x^2 + 2x + 1}$, then its maximum value is _______.

Solution For f to be maximum, $4x^2 + 2x + 1$ should be minimum i.e.

$4x^2 + 2x + 1 = 4\left(x + \frac{1}{4}\right)^2 + \left(1 - \frac{1}{4}\right)$ giving the minimum value of $4x^2 + 2x + 1 = \frac{3}{4}$.

Hence maximum value of $f = \frac{4}{3}$.

Example 26 Let f have second deriative at c such that $f'(c) = 0$ and $f''(c) > 0$, then c is a point of ______.

Solution Local minima.

Example 27 Minimum value of f if $f(x) = \sin x$ in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is _____.

Solution –1

Example 28 The maximum value of $\sin x + \cos x$ is _____.

Solution $\sqrt{2}$.

Example 29 The rate of change of volume of a sphere with respect to its surface area, when the radius is 2 cm, is______.

Solution 1 cm^3/cm^2

$$v = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dr} = 4\pi r^2,\ s = 4\pi r^2 \Rightarrow \frac{ds}{dr} = 8\pi r \Rightarrow \frac{dv}{ds} = \frac{r}{2} = 1 \text{ at } r = 2.$$

6.3 EXERCISE

Short Answer (S.A.)

1. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is propotional to the surface. Prove that the radius is decreasing at a constant rate.

2. If the area of a circle increases at a uniform rate, then prove that perimeter varies inversely as the radius.

3. A kite is moving horizontally at a height of 151.5 meters. If the speed of kite is 10 m/s, how fast is the string being let out; when the kite is 250 m away from the boy who is flying the kite? The height of boy is 1.5 m.

4. Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, find the rate at which they are being seperated..

5. Find an angle θ, $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine.

6. Find the approximate value of $(1.999)^5$.

7. Find the approximate volume of metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm, respectively.

8. A man, 2m tall, walks at the rate of $1\frac{2}{3}$ m/s towards a street light which is $5\frac{1}{3}$ m above the ground. At what rate is the tip of his shadow moving? At what rate is the length of the shadow changing when he is $3\frac{1}{3}$ m from the base of the light?

9. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200(10 - t)^2$. How fast is the water running out at the end of 5 seconds? What is the average rate at which the water flows out during the first 5 seconds?

10. The volume of a cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.

11. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of second square with respect to the area of first square.

12. Find the condition that the curves $2x = y^2$ and $2xy = k$ intersect orthogonally.

13. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

14. Find the co-ordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes.

15. Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

16. Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1, 2).

17. Find the equation of the normal lines to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.

18. At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the y-axis?

19. Show that the line $\frac{x}{a} + \frac{y}{b} = 1$, touches the curve $y = b \cdot e^{\frac{-x}{a}}$ at the point where the curve intersects the axis of y.

20. Show that $f(x) = 2x + \cot^{-1}x + \log\left(\sqrt{1+x^2} - x\right)$ is increasing in **R**.

21. Show that for $a \geq 1$, $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ is decreasing in **R**.

22. Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

23. At what point, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum? Also find the maximum slope.

24. Prove that $f(x) = \sin x + \sqrt{3}\cos x$ has maximum value at $x = \frac{\pi}{6}$.

Long Answer (L.A.)

25. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

26. Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also find the corresponding local maximum and local minimum values.

27. A telephone company in a town has 500 subscribers on its list and collects fixed charges of Rs 300/- per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of Re 1/- one subscriber will discontinue the service. Find what increase will bring maximum profit?

28. If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$.

29. An open box with square base is to be made of a given quantity of card board of area c^2. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

30. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also find the maximum volume.

31. If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?

32. AB is a diameter of a circle and C is any point on the circle. Show that the area of Δ ABC is maximum, when it is isosceles.

33. A metal box with a square base and vertical sides is to contain 1024 cm^3. The material for the top and bottom costs Rs 5/cm^2 and the material for the sides costs Rs 2.50/cm^2 . Find the least cost of the box.

34. The sum of the surface areas of a rectangular parallelopiped with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of the sphere. Also find the minimum value of the sum of their volumes.

Objective Type Questions

Choose the correct answer from the given four options in each of the following questions 35 to 39:

35. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when side is 10 cm is:

(A) 10 cm^2/s (B) $\sqrt{3}$ cm^2/s (C) $10\sqrt{3}$ cm^2/s (D) $\frac{10}{3}$ cm^2/s

36. A ladder, 5 meter long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metres from the wall is:

(A) $\frac{1}{10}$ radian/sec (B) $\frac{1}{20}$ radian/sec (C) 20 radian/sec

(D) 10 radian/sec

37. The curve $y = x^{\frac{1}{5}}$ has at (0, 0)

(A) a vertical tangent (parallel to y-axis)

(B) a horizontal tangent (parallel to x-axis)

(C) an oblique tangent

(D) no tangent

38. The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is

(A) $3x - y = 8$ (B) $3x + y + 8 = 0$

(C) $x + 3y \pm 8 = 0$ (D) $x + 3y = 0$

39. If the curve $ay + x^2 = 7$ and $x^3 = y$, cut orthogonally at (1, 1), then the value of a is:

(A) 1 (B) 0 (C) – 6 (D) .6

40. If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the change in y

(A) .32 (B) .032 (C) 5.68 (D) 5.968

41. The equation of tangent to the curve $y(1 + x^2) = 2 - x$, where it crosses x-axis is:

(A) $x + 5y = 2$ (B) $x - 5y = 2$

(C) $5x - y = 2$ (D) $5x + y = 2$

42. The points at which the tangents to the curve $y = x^3 - 12x + 18$ are parallel to x-axis are:

(A) (2, –2), (–2, –34) (B) (2, 34), (–2, 0)

(C) (0, 34), (–2, 0) (D) (2, 2), (–2, 34)

43. The tangent to the curve $y = e^{2x}$ at the point (0, 1) meets x-axis at:

(A) (0, 1) (B) $\left(-\frac{1}{2}, 0\right)$ (C) (2, 0) (D) (0, 2)

44. The slope of tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point (2, –1) is:

(A) $\frac{22}{7}$ (B) $\frac{6}{7}$ (C) $\frac{-6}{7}$ (D) -6

45. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ intersect at an angle of

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{6}$

46. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is:

(A) $[-1, \infty)$ (B) $[-2, -1]$ (C) $(-\infty, -2]$ (D) $[-1, 1]$

47. Let the $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = 2x + \cos x$, then f :

(A) has a minimum at $x = \pi$ (B) has a maximum, at $x = 0$

(C) is a decreasing function (D) is an increasing function

48. $y = x(x - 3)^2$ decreases for the values of x given by :

(A) $1 < x < 3$ (B) $x < 0$ (C) $x > 0$ (D) $0 < x < \frac{3}{2}$

49. The function $f(x) = 4\sin^3x - 6\sin^2x + 12\sin x + 100$ is strictly

(A) increasing in $\left(\pi, \frac{3\pi}{2}\right)$ (B) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(C) decreasing in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (D) decreasing in $\left[0, \frac{\pi}{2}\right]$

50. Which of the following functions is decreasing on $\left(0, \frac{\pi}{2}\right)$

(A) $\sin 2x$ (B) $\tan x$ (C) $\cos x$ (D) $\cos 3x$

51. The function $f(x) = \tan x - x$

(A) always increases (B) always decreases

(C) never increases (D) sometimes increases and sometimes decreases.

52. If x is real, the minimum value of $x^2 - 8x + 17$ is

(A) –1 (B) 0 (C) 1 (D) 2

53. The smallest value of the polynomial $x^3 - 18x^2 + 96x$ in [0, 9] is

(A) 126 (B) 0 (C) 135 (D) 160

54. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has

(A) two points of local maximum (B) two points of local minimum

(C) one maxima and one minima (D) no maxima or minima

55. The maximum value of $\sin x \cdot \cos x$ is

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\sqrt{2}$ (D) $2\sqrt{2}$

56. At $x = \frac{5\pi}{6}$, $f(x) = 2 \sin 3x + 3 \cos 3x$ is:

(A) maximum (B) minimum

(C) zero (D) neither maximum nor minimum.

57. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is:

(A) 0 (B) 12 (C) 16 (D) 32

58. $f(x) = x^x$ has a stationary point at

(A) $x = e$ (B) $x = \frac{1}{e}$ (C) $x = 1$ (D) $x = \sqrt{e}$

59. The maximum value of $\left(\frac{1}{x}\right)^x$ is:

(A) e (B) e^e (C) $e^{\frac{1}{e}}$ (D) $\left(\frac{1}{e}\right)^{\frac{1}{e}}$

Fill in the blanks in each of the following Exercises 60 to 64:

60. The curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$ touch each other at the point_____.

61. The equation of normal to the curve $y = \tan x$ at (0, 0) is ________.

62. The values of a for which the function $f(x) = \sin x - ax + b$ increases on **R** are ______.

63. The function $f(x) = \dfrac{2x^2 - 1}{x^4}$, $x > 0$, decreases in the interval _______.

64. The least value of the function $f(x) = ax + \dfrac{b}{x}$ $(a > 0, b > 0, x > 0)$ is ______.

Chapter 7

INTEGRALS

7.1 Overview

7.1.1 Let $\frac{d}{dx}$ F $(x) = f(x)$. Then, we write $\int f(x)\,dx$ = F (x) + C. These integrals are called indefinite integrals or general integrals, C is called a constant of integration. All these integrals differ by a constant.

7.1.2 If two functions differ by a constant, they have the same derivative.

7.1.3 Geometrically, the statement $\int f(x)\,dx$ = F (x) + C = y (say) represents a family of curves. The different values of C correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. Further, the tangents to the curves at the points of intersection of a line $x = a$ with the curves are parallel.

7.1.4 ***Some properties of indefinite integrals***

(i) The process of differentiation and integration are inverse of each other, i.e., $\frac{d}{dx}\int f(x)\,dx = f(x)$ and $\int f'(x)\,dx = f(x) + \text{C}$, where C is any arbitrary constant.

(ii) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent. So if f and g are two functions such that $\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\int g(x)\,dx$, then $\int f(x)\,dx$ and $\int g(x)\,dx$ are equivalent.

(iii) The integral of the sum of two functions equals the sum of the integrals of the functions i.e., $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$.

(iv) A constant factor may be written either before or after the integral sign, i.e., $\int a\, f(x)\, dx = a\int f(x)\, dx$, where '$a$' is a constant.

(v) Properties (iii) and (iv) can be generalised to a finite number of functions $f_1, f_2, ..., f_n$ and the real numbers, $k_1, k_2, ..., k_n$ giving

$$\int (k_1 f_1(x) + k_2\, f_2(x) + ... + k_n f_n(x))\, dx = k_1 \int f_1(x)dx + k_2 \int f_2(x)dx + ... + k_n \int f_n(x)dx$$

7.1.5 *Methods of integration*

There are some methods or techniques for finding the integral where we can not directly select the antiderivative of function f by reducing them into standard forms. Some of these methods are based on

1. Integration by substitution
2. Integration using partial fractions
3. Integration by parts.

7.1.6 *Definite integral*

The definite integral is denoted by $\int_a^b f(x)\, dx$, where a is the lower limit of the integral and b is the upper limit of the integral. The definite integral is evaluated in the following two ways:

(i) The definite integral as the limit of the sum

(ii) $\int_a^b f(x)\, dx$ = F(b) – F(a), if F is an antiderivative of $f(x)$.

7.1.7 *The definite integral as the limit of the sum*

The definite integral $\int_a^b f(x)\, dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis and given by

$$\int_a^b f(x)\, dx = (b - a) \lim_{n\to\infty} \frac{1}{n}\left[f(a) + f(a+h) + ... f(a+(n-1)h)\right]$$

or

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right],$$

where $h = \dfrac{b-a}{n} \to 0$ as $n \to \infty$.

7.1.8 *Fundamental Theorem of Calculus*

(i) *Area function* : The function A (x) denotes the area function and is given by A (x) = $\int_a^x f(x)\,dx$.

(ii) *First Fundamental Theorem of integral Calculus*

Let f be a continuous function on the closed interval $[a, b]$ and let A (x) be the area function . Then A′ (x) = $f(x)$ for all $x \in [a, b]$.

(iii) *Second Fundamental Theorem of Integral Calculus*

Let f be continuous function defined on the closed interval $[a, b]$ and F be an antiderivative of f.

$$\int_a^b f(x)\,dx = [\mathrm{F}(x)]_a^b = \mathrm{F}(b) - \mathrm{F}(a).$$

7.1.9 *Some properties of Definite Integrals*

P_0 : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P_1 : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$, in particular, $\int_a^a f(x)\,dx = 0$

P_2 : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$$P_3 : \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$P_4 : \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$P_5 : \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$P_6 : \int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{if } f(2a-x) = f(x), \\ 0, \text{ if } f(2a-x) = -f(x). \end{cases}$$

P_7 : (i) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function i.e., $f(-x) = f(x)$

(ii) $\int_{-a}^{a} f(x)\,dx = 0$, if f is an odd function i.e., $f(-x) = -f(x)$

7.2 Solved Examples

Short Answer (S.A.)

Example 1 Integrate $\left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right)$ w.r.t. x

Solution $\int\left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right)dx$

$$= \int 2a(x)^{\frac{-1}{2}}\,dx - \int bx^{-2}\,dx + \int 3c\,x^{\frac{2}{3}}\,dx$$

$$= 4a\sqrt{x} + \frac{b}{x} + \frac{9cx^{\frac{5}{3}}}{5} + C\ .$$

Example 2 Evaluate $\int \frac{3ax}{b^2 + c^2x^2} dx$

Solution Let $v = b^2 + c^2x^2$, then $dv = 2c^2\, x dx$

Therefore, $\int \frac{3ax}{b^2 + c^2x^2} dx = \frac{3a}{2c^2} \int \frac{dv}{v}$

$$= \frac{3a}{2c^2} \log\left|b^2 + c^2x^2\right| + \text{C}.$$

Example 3 Verify the following using the concept of integration as an antiderivative.

$$\int \frac{x^3 dx}{x+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \log|x+1| + \text{C}$$

Solution $\frac{d}{dx}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \log|x+1| + \text{C}\right)$

$$= 1 - \frac{2x}{2} + \frac{3x^2}{3} - \frac{1}{x+1}$$

$$= 1 - x + x^2 - \frac{1}{x+1} = \frac{x^3}{x+1}.$$

Thus $\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \log|x+1| + \text{C}\right) = \int \frac{x^3}{x+1} dx$

Example 4 Evaluate $\int \sqrt{\frac{1+x}{1-x}}\, dx,\ x \neq 1.$

Solution Let $\text{I} = \int \sqrt{\frac{1+x}{1-x}}\, dx = \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{x\, dx}{\sqrt{1-x^2}} = \sin^{-1} x + \text{I}_1,$

where $\mathrm{I}_1 = \dfrac{x\,dx}{\sqrt{1-x^2}}$.

Put $1 - x^2 = t^2 \Rightarrow -2x\,dx = 2t\,dt$. Therefore

$$\mathrm{I}_1 = - \quad dt = -t + \mathrm{C} = -\sqrt{1-x^2} + \mathrm{C}$$

Hence $\qquad \mathrm{I} = \sin^{-1}x - \sqrt{1-x^2} + \mathrm{C}$.

Example 5 Evaluate $\displaystyle\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$, $\beta > \alpha$

Solution Put $x - \alpha = t^2$. Then $\beta - x = \beta - (t^2 + \alpha) = \beta - t^2 - \alpha = -t^2 - \alpha + \beta$ and $dx = 2t\,dt$. Now

$$\mathrm{I} = \int \frac{2t\,dt}{\sqrt{t^2(\beta-\alpha-t^2)}} = \int \frac{2\,dt}{\sqrt{(\beta-\alpha-t^2)}}$$

$$= 2 \quad \frac{dt}{\sqrt{k^2-t^2}}, \text{ where } k^2 = \beta - \alpha$$

$$= 2\sin^{-1}\frac{t}{k} + \mathrm{C} = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + \mathrm{C}.$$

Example 6 Evaluate $\displaystyle\int \tan^8 x \sec^4 x\,dx$

Solution $\mathrm{I} = \displaystyle\int \tan^8 x \sec^4 x\,dx$

$$= \int \tan^8 x\left(\sec^2 x\right)\sec^2 x\,dx$$

$$= \int \tan^8 x\left(\tan^2 x + 1\right)\sec^2 x\,dx$$

$$= \int \tan^{10} x \sec^2 x \, dx + \int \tan^8 x \sec^2 x \, dx$$

$$= \frac{\tan^{11} x}{11} + \frac{\tan^9 x}{9} + C .$$

Example 7 Find $\int \frac{x^3}{x^4 + 3x^2 + 2} dx$

Solution Put $x^2 = t$. Then $2x \, dx = dt$.

Now $$I = \int \frac{x^3 dx}{x^4 + 3x^2 + 2} = \frac{1}{2} \int \frac{t \, dt}{t^2 + 3t + 2}$$

Consider $$\frac{t}{t^2 + 3t + 2} = \frac{A}{t+1} + \frac{B}{t+2}$$

Comparing coefficient, we get A = –1, B = 2.

Then $$I = \frac{1}{2}\left[2\int \frac{dt}{t+2} - \int \frac{dt}{t+1}\right]$$

$$= \frac{1}{2}\left[2\log|t+2| - \log|t+1|\right]$$

$$= \log\left|\frac{x^2+2}{\sqrt{x^2+1}}\right| + C$$

Example 8 Find $\int \frac{dx}{2\sin^2 x + 5\cos^2 x}$

Solution Dividing numerator and denominator by $\cos^2 x$, we have

$$I = \int \frac{\sec^2 x \, dx}{2\tan^2 x + 5}$$

Put $\tan x = t$ so that $\sec^2 x\, dx = dt$. Then

$$\mathrm{I} = \int \frac{dt}{2t^2 + 5} = \frac{1}{2}\int \frac{dt}{t^2 + \left(\sqrt{\frac{5}{2}}\right)^2}$$

$$= \frac{1}{2}\frac{\sqrt{2}}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{2}t}{\sqrt{5}}\right) + \mathrm{C}$$

$$= \frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{\sqrt{2}\tan x}{\sqrt{5}}\right) + \mathrm{C}.$$

Example 9 Evaluate $\int_{-1}^{2}(7x-5)\,dx$ as a limit of sums.

Solution Here $a = -1$, $b = 2$, and $h = \frac{2+1}{n}$, i.e, $nh = 3$ and $f(x) = 7x - 5$.

Now, we have

$$\int_{-1}^{2}(7x-5)\,dx = \lim_{h\to 0} h\left[f(-1) + f(-1+h) + f(-1+2h) + ... + f(-1+(n-1)h)\right]$$

Note that

$f(-1) = -7 - 5 = -12$

$f(-1 + h) = -7 + 7h - 5 = -12 + 7h$

$f(-1 + (n-1)h) = 7(n-1)h - 12.$

Therefore,

$$\int_{-1}^{2}(7x-5)\,dx = \lim_{h\to 0} h\left[(-12) + (7h - 12) + (14h - 12) + ... + (7(n-1)h - 12)\right].$$

$$= \lim_{h\to 0} h\left[7h\left[1 + 2 + ... + (n-1)\right] - 12n\right]$$

$$= \lim_{h\to 0} h\left[7h\,\frac{(n-1)n}{2} - .12n\right] = \lim_{h\to 0}\left[\frac{7}{2}(nh)(nh-h) - 12nh\right]$$

$$= \frac{7}{2}(3)(3-0) - 12\times 3 = \frac{7\times 9}{2} - 36 = \frac{-9}{2}.$$

Example 10 Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\cot^7 x + \tan^7 x}\,dx$

Solution We have

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\cot^7 x + \tan^7 x}\,dx \qquad \text{...(1)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^7\left(\frac{\pi}{2}-x\right)}{\cot^7\left(\frac{\pi}{2}-x\right) + \tan^7\left(\frac{\pi}{2}-x\right)}\,dx \qquad \text{by } (\mathrm{P}_4)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cot^7(x)\,dx}{\cot^7 x\,dx + \tan^7 x} \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \left(\frac{\tan^7 x + \cot^7 x}{\tan^7 x + \cot^7 x}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} dx \text{ which gives } \mathrm{I} = \frac{\pi}{4}.$$

Example 11 Find $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx$

Solution We have

$$I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx \qquad ...(1)$$

$$= \int_2^8 \frac{\sqrt{10-(10-x)}}{\sqrt{10-x}+\sqrt{10-(10-x)}}dx \qquad \text{by } (P_3)$$

$$\Rightarrow \quad I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx \qquad (2)$$

Adding (1) and (2), we get

$$2I = \int_2^8 1\,dx = 8-2 = 6$$

Hence $\quad I = 3$

Example 12 Find $\int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x}\, dx$

Solution We have

$$I = \int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{(\sin x + \cos x)^2}\, dx$$

$$= \int_0^{\frac{\pi}{4}} (\sin x + \cos x)\, dx$$

$$= \left(-\cos x + \sin x\right)_0^{\frac{\pi}{4}}$$

$$I = 1.$$

Example 13 Find $\int x^2 \tan^{-1} x \, dx$.

Solution $I = \int x^2 \tan^{-1} x \, dx$

$$= \tan^{-1} x \int x^2 dx - \int \frac{1}{1+x^2} \cdot \frac{x^3}{3} dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \log \left| 1 + x^2 \right| + C.$$

Example 14 Find $\int \sqrt{10 - 4x + 4x^2} \, dx$

Solution We have

$$I = \int \sqrt{10 - 4x + 4x^2} \, dx = \int \sqrt{(2x-1)^2 + (3)^2} \, dx$$

Put $t = 2x - 1$, then $dt = 2dx$.

Therefore, $\quad I = \frac{1}{2} \int \sqrt{t^2 + (3)^2} \, dt$

$$= \frac{1}{2} t \frac{\sqrt{t^2 + 9}}{2} + \frac{9}{4} \log \left| t + \sqrt{t^2 + 9} \right| + C$$

$$= \frac{1}{4}(2x-1)\sqrt{(2x-1)^2 + 9} + \frac{9}{4} \log \left| (2x-1) + \sqrt{(2x-1)^2 + 9} \right| + C.$$

Long Answer (L.A.)

Example 15 Evaluate $\int \frac{x^2\,dx}{x^4+x^2-2}$.

Solution Let $x^2 = t$. Then

$$\frac{x^2}{x^4+x^2-2} = \frac{t}{t^2+t-2} = \frac{t}{(t+2)\,(t-1)} = \frac{\mathrm{A}}{t+2} + \frac{\mathrm{B}}{t-1}$$

So $\quad t = \mathrm{A}\,(t-1) + \mathrm{B}\,(t+2)$

Comparing coefficients, we get $\mathrm{A} = \frac{2}{3},\ \mathrm{B} = \frac{1}{3}$.

So $$\frac{x^2}{x^4+x^2-2} = \frac{2}{3}\,\frac{1}{x^2+2} + \frac{1}{3}\,\frac{1}{x^2-1}$$

Therefore,

$$\int \frac{x^2}{x^4+x^2-2}\,dx = \frac{2}{3}\int \frac{1}{x^2+2}\,dx + \frac{1}{3}\int \frac{dx}{x^2-1}$$

$$= \frac{2}{3}\,\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + \mathrm{C}$$

Example16 Evaluate $\int \frac{x^3+x}{x^4 \quad 9}\,dx$

Solution We have

$$\mathrm{I} = \int \frac{x^3+x}{x^4 \quad 9}\,dx \quad = \int \frac{x^3}{x^4 \quad 9}\,dx + \frac{x\,dx}{x^4 \quad 9} \quad = \mathrm{I}_1 + \mathrm{I}_2\,.$$

Now $\quad \mathrm{I}_1 = \int \frac{x^3}{x^4 - 9}$

Put $\quad t = x^4 - 9$ so that $4x^3\, dx = dt$. Therefore

$$\mathrm{I}_1 = \frac{1}{4}\int \frac{dt}{t} = \frac{1}{4}\log|t| + \mathrm{C}_1 = \frac{1}{4}\log\left|x^4 - 9\right| + \mathrm{C}_1$$

Again, $\quad \mathrm{I}_2 = \int \frac{x\,dx}{x^4 - 9}$.

Put $\quad x^2 = u$ so that $2x\, dx = du$. Then

$$\mathrm{I}_2 = \frac{1}{2}\int \frac{du}{u^2 - (3)^2} = \frac{1}{2\times 6}\log\left|\frac{u-3}{u+3}\right| + \mathrm{C}_2$$

$$= \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + \mathrm{C}_2.$$

Thus $\quad \mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2$

$$= \frac{1}{4}\log\left|x^4 - 9\right| + \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + \mathrm{C}.$$

Example 17 Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

Solution We have

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \text{(by P4)}$$

$$\Rightarrow \qquad I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x+\cos x}\,dx$$

Thus, we get $\quad 2I = \dfrac{1}{\sqrt{2}}\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{\cos\left(x-\frac{\pi}{4}\right)}$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \sec\left(x-\frac{\pi}{4}\right)dx = \frac{1}{\sqrt{2}}\left[\log\left(\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)-\log\sec\left(-\frac{\pi}{4}\right)+\tan\left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sqrt{2}+1\right)-\log\left(\sqrt{2}-1\right)\right] \quad = \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

$$= \frac{1}{\sqrt{2}}\log\left(\frac{(\sqrt{2}+1)^2}{1}\right) \quad = \frac{2}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

Hence $\qquad I = \dfrac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$.

Example 18 Find $\displaystyle\int_0^1 x\left(\tan^{-1}x\right)^2 dx$

Solution $\mathrm{I} = \int_0^1 x\left(\tan^{-1} x\right)^2 dx$.

Integrating by parts, we have

$$\mathrm{I} = \frac{x^2}{2}\left[\left(\tan^{-1} x\right)^2\right]_0^1 - \frac{1}{2}\int_0^1 x^2 . 2\frac{\tan^{-1} x}{1+x^2} dx$$

$$= \frac{\pi^2}{32} - \int_0^1 \frac{x^2}{1+x^2} . \tan^{-1} x \, dx$$

$$= \frac{\pi^2}{32} - \mathrm{I}_1 \text{ , where } \mathrm{I}_1 = \int_0^1 \frac{x^2}{1+x^2} \tan^{-1} x dx$$

Now $\mathrm{I}_1 = \int_0^1 \frac{x^2+1-1}{1+x^2} \tan^{-1} x \, dx$

$$= \int_0^1 \tan^{-1} x \, dx - \int_0^1 \frac{1}{1+x^2} \tan^{-1} x \, dx$$

$$= \mathrm{I}_2 - \frac{1}{2}\left(\left(\tan^{-1} x\right)^2\right)_0^1 = \mathrm{I}_2 - \frac{\pi^2}{32}$$

Here $\mathrm{I}_2 = \int_0^1 \tan^{-1} x \, dx = \left(x \tan^{-1} x\right)_0^1 - \int_0^1 \frac{x}{1+x^2} dx$

$$= \frac{\pi}{4} - \frac{1}{2}\left(\log\left|1+x^2\right|\right)_0^1 = \frac{\pi}{4} - \frac{1}{2}\log 2 .$$

Thus $\mathrm{I}_1 = \frac{\pi}{4} - \frac{1}{2}\log 2 - \frac{\pi^2}{32}$

Therefore, $\quad \mathrm{I} = \dfrac{\pi^2}{32} - \dfrac{\pi}{4} + \dfrac{1}{2}\log 2 + \dfrac{\pi^2}{32} = \dfrac{\pi^2}{16} - \dfrac{\pi}{4} + \dfrac{1}{2}\log 2$

$$= \frac{\pi^2 - 4\pi}{16} + \log\sqrt{2}\ .$$

Example 19 Evaluate $\int_{-1}^{2} f(x)\, dx$, where $f(x) = |x + 1| + |x| + |x - 1|$.

Solution We can redefine f as $f(x) = \begin{cases} 2 - x, & \text{if } -1 < x \le 0 \\ x + 2, & \text{if } 0 < x \le 1 \\ 3x, & \text{if } 1 < x \le 2 \end{cases}$

Therefore, $\quad \int_{-1}^{2} f(x)\,dx = \int_{-1}^{0} (2 - x)\,dx + \int_{0}^{1} (x + 2)\,dx + \int_{1}^{2} 3x\,dx \qquad$ (by P_2)

$$= \left(2x - \frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2} + 2x\right)_{0}^{1} + \left(\frac{3x^2}{2}\right)_{1}^{2}$$

$$= 0 - \left(-2 - \frac{1}{2}\right) + \left(\frac{1}{2} + 2\right) + 3\left(\frac{4}{2} - \frac{1}{2}\right) = \frac{5}{2} + \frac{5}{2} + \frac{9}{2} = \frac{19}{2}\ .$$

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples from 20 to 30.

Example 20 $\int e^x (\cos x - \sin x)\, dx$ is equal to

(A) $e^x \cos x + \mathrm{C}$ (B) $e^x \sin x + \mathrm{C}$

(C) $-e^x \cos x + \mathrm{C}$ (D) $-e^x \sin x + \mathrm{C}$

Solution (A) is the correct answer since $\int e^x \left[f(x) + f'(x) \right] dx = e^x f(x) + \text{C}$. Here $f(x) = \cos x$, $f'(x) = -\sin x$.

Example 21 $\int \frac{dx}{\sin^2 x \cos^2 x}$ is equal to

(A) $\tan x + \cot x + \text{C}$ (B) $(\tan x + \cot x)^2 + \text{C}$

(C) $\tan x - \cot x + \text{C}$ (D) $(\tan x - \cot x)^2 + \text{C}$

Solution (C) is the correct answer, since

$$\text{I} = \int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)\, dx}{\sin^2 x \cos^2 x}$$

$$= \int \sec^2 x\, dx + \int \text{cosec}^2 x\, dx = \tan x - \cot x + \text{C}$$

Example 22 If $\int \frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}} dx = ax + b \log |4e^x + 5e^{-x}| + \text{C}$, then

(A) $a = \frac{-1}{8}, \; b = \frac{7}{8}$ (B) $a = \frac{1}{8}, \; b = \frac{7}{8}$

(C) $a = \frac{-1}{8}, \; b = \frac{-7}{8}$ (D) $a = \frac{1}{8}, \; b = \frac{-7}{8}$

Solution (C) is the correct answer, since differentiating both sides, we have

$$\frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}} = a + b\, \frac{(4e^x - 5e^{-x})}{4e^x + 5e^{-x}},$$

giving $3e^x - 5e^{-x} = a(4e^x + 5e^{-x}) + b(4e^x - 5e^{-x})$. Comparing coefficients on both sides, we get $3 = 4a + 4b$ and $-5 = 5a - 5b$. This verifies $a = \frac{-1}{8}, \; b = \frac{7}{8}$.

Example 23 $\int_{a+c}^{b+c} f(x)\,dx$ is equal to

(A) $\int_a^b f(x-c)\,dx$ (B) $\int_a^b f(x+c)\,dx$

(C) $\int_a^b f(x)\,dx$ (D) $\int_{a-c}^{b-c} f(x)\,dx$

Solution (B) is the correct answer, since by putting $x = t + c$, we get

$$\text{I} = \int_a^b f(c+t)\,dt = \int_a^b f(x+c)\,dx.$$

Example 24 If f and g are continuous functions in [0, 1] satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = a$, then $\int_0^a f(x).g(x)\,dx$ is equal to

(A) $\frac{a}{2}$ (B) $\frac{a}{2}\int_0^a f(x)\,dx$

(C) $\int_0^a f(x)\,dx$ (D) $a\int_0^a f(x)\,dx$

Solution B is the correct answer. Since $\text{I} = \int_0^a f(x).g(x)\,dx$

$$= \int_0^a f(a-x)\,g(a-x)\,dx = \int_0^a f(x)\,(a-g(x))\,dx$$

$$= a\int_0^a f(x)\,dx - \int_0^a f(x).g(x)\,dx = a\int_0^a f(x)\,dx - \text{I}$$

or $\quad \mathrm{I} = \dfrac{a}{2}\displaystyle\int_0^a f(x)\,dx.$

Example 25 If $x = \displaystyle\int_0^y \frac{dt}{\sqrt{1+9t^2}}$ and $\dfrac{d^2y}{dx^2} = ay$, then a is equal to

(A) 3 (B) 6 (C) 9 (D) 1

Solution (C) is the correct answer, since $x = \displaystyle\int_0^y \frac{dt}{\sqrt{1+9t^2}} \Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{1+9y^2}}$

which gives $\dfrac{d^2y}{dx^2} = \dfrac{18y}{2\sqrt{1+9y^2}} \cdot \dfrac{dy}{dx} = 9y.$

Example 26 $\displaystyle\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx$ is equal to

(A) log 2 (B) 2 log 2 (C) $\dfrac{1}{2}\log 2$ (D) 4 log 2

Solution (B) is the correct answer, since $\mathrm{I} = \displaystyle\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx$

$$= \int_{-1}^{1} \frac{x^3}{x^2+2|x|+1} + \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1}\,dx \quad = \quad 0 + 2\int_0^1 \frac{|x|+1}{(|x|+1)^2}\,dx$$

[odd function + even function]

$$= 2\int_0^1 \frac{x+1}{(x+1)^2}\,dx = 2\int_0^1 \frac{1}{x+1}\,dx \qquad = 2\left|\log|x+1|\right|_0^1 = 2\log 2.$$

Example 27 If $\int_0^1 \frac{e^t}{1+t}dt = a$, then $\int_0^1 \frac{e^t}{(1+t)^2}dt$ is equal to

(A) $a - 1 + \frac{e}{2}$ (B) $a + 1 - \frac{e}{2}$ (C) $a - 1 - \frac{e}{2}$ (D) $a + 1 + \frac{e}{2}$

Solution (B) is the correct answer, since I = $\int_0^1 \frac{e^t}{1+t}\, dt$

$$= \left|\frac{1}{1+t}e^t\right|_0^1 + \int_0^1 \frac{e^t}{(1+t)^2}dt = a \text{ (given)}$$

Therefore, $\int_0^1 \frac{e^t}{(1+t)^2} = a - \frac{e}{2} + 1.$

Example 28 $\int_{-2}^{2} |x\cos \pi x|\, dx$ is equal to

(A) $\frac{8}{\pi}$ (B) $\frac{4}{\pi}$ (C) $\frac{2}{\pi}$ (D) $\frac{1}{\pi}$

Solution (A) is the correct answer, since I = $\int_{-2}^{2} |x\cos\pi x|\, dx = 2\int_0^2 |x\cos\pi x|\, dx$

$$= 2\left\{\int_0^{\frac{1}{2}} |x\cos\pi x|\, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} |x\cos\pi x|\, dx + \int_{\frac{3}{2}}^{2} |x\cos\pi x|\, dx\right\} = \frac{8}{\pi}.$$

Fill in the blanks in each of the Examples 29 to 32.

Example 29 $\int \frac{\sin^6 x}{\cos^8 x}\, dx =$ _______.

Solution $\dfrac{\tan^7 x}{7} + C$

Example 30 $\int_{-a}^{a} f(x)\,dx = 0$ if f is an _______ function.

Solution Odd.

Example 31 $\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$, if $f(2a - x) =$ _______.

Solution $f(x)$.

Example 32 $\int_{0}^{\frac{\pi}{2}} \dfrac{\sin^n x\,dx}{\sin^n x + \cos^n x} =$ _______.

Solution $\dfrac{\pi}{4}$.

7.3 EXERCISE

Short Answer (S.A.)

Verify the following :

1. $\int \dfrac{2x-1}{2x+3}\,dx = x - \log |(2x+3)^2| + C$

2. $\int \dfrac{2x+3}{x^2+3x}\,dx = \log |x^2 + 3x| + C$

Evaluate the following:

3. $\int \dfrac{(x^2+2)\,dx}{x+1}$

4. $\int \dfrac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}\,dx$

5. $\int \frac{(1+\cos x)}{x+\sin x}dx$

6. $\int \frac{dx}{1+\cos x}$

7. $\int \tan^2 x \sec^4 x\, dx$

8. $\int \frac{\sin x+\cos x}{\sqrt{1+\sin 2x}}dx$

9. $\int \sqrt{1+\sin x}\,dx$

10. $\int \frac{x}{\sqrt{x}+1}dx$ (**Hint** : Put $\sqrt{x} = z$)

11. $\int \sqrt{\frac{a+x}{a-x}}$

12. $\int \frac{x^{\frac{1}{2}}}{1+x^{\frac{3}{4}}}dx$ (**Hint** : Put $x = z^4$)

13. $\int \frac{\sqrt{1+x^2}}{x^4}dx$

14. $\int \frac{dx}{\sqrt{16-9x^2}}$

15. $\int \frac{dt}{\sqrt{3t-2t^2}}$

16. $\int \frac{3x-1}{\sqrt{x^2+9}}dx$

17. $\int \sqrt{5-2x+x^2}\,dx$

18. $\int \frac{x}{x^4-1}dx$

19. $\int \frac{x^2}{1-x^4}dx$ put $x^2 = t$

20. $\int \sqrt{2ax-x^2}\,dx$

21. $\int \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}}dx$

22. $\int \frac{(\cos 5x+\cos 4x)}{1-2\cos 3x}dx$

23. $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x \cos^2 x}dx$

24. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}\,dx$

25. $\displaystyle\int \frac{\cos x - \cos 2x}{1 - \cos x}\,dx$

26. $\displaystyle\int \frac{dx}{x\sqrt{x^4 - 1}}$ (**Hint :** Put $x^2 = \sec\theta$)

Evaluate the following as limit of sums:

27. $\displaystyle\int_0^2 (x^2 + 3)\,dx$

28. $\displaystyle\int_0^2 e^x\,dx$

Evaluate the following:

29. $\displaystyle\int_0^1 \frac{dx}{e^x + e^{-x}}$

30. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\tan x\,dx}{1 + m^2 \tan^2 x}$

31. $\displaystyle\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$

32. $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1 + x^2}}$

33. $\displaystyle\int_0^{\pi} x \sin x \cos^2 x\,dx$

34. $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{(1 + x^2)\sqrt{1 - x^2}}$

(**Hint:** let $x = \sin\theta$)

Long Answer (L.A.)

35. $\displaystyle\int \frac{x^2\,dx}{x^4 - x^2 - 12}$

36. $\displaystyle\int \frac{x^2\,dx}{(x^2 + a^2)(x^2 + b^2)}$

37. $\displaystyle\int_0^{\pi} \frac{x}{1 + \sin x}$

38. $\displaystyle\int \frac{2x - 1}{(x-1)(x+2)(x-3)}\,dx$

39. $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

40. $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

(**Hint**: Put $x = a\tan^2\theta$)

41. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{\frac{5}{2}}}$

42. $\int e^{-3x}\cos^3 x\,dx$

43. $\int \sqrt{\tan x}\,dx$ (**Hint:** Put $\tan x = t^2$)

44. $\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$

(**Hint:** Divide Numerator and Denominator by $\cos^4 x$)

45. $\int_0^1 x\log(1+2x)\,dx$

46. $\int_0^{\pi} x\log\sin x\,dx$

47. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\sin x + \cos x)dx$

Objective Type Questions

Choose the correct option from given four options in each of the Exercises from 48 to 63.

48. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta}dx$ is equal to

(A) $2(\sin x + x\cos\theta) + C$

(B) $2(\sin x - x\cos\theta) + C$

(C) $2(\sin x + 2x\cos\theta) + C$

(D) $2(\sin x - 2x\cos\theta) + C$

49. $\int \dfrac{dx}{\sin(x-a)\sin(x-b)}$ is equal to

(A) $\sin(b-a)\log\left|\dfrac{\sin(x-b)}{\sin(x-a)}\right| + C$

(B) $\operatorname{cosec}(b-a)\log\left|\dfrac{\sin(x-a)}{\sin(x-b)}\right| + C$

(C) $\operatorname{cosec}(b-a)\log\left|\dfrac{\sin(x-b)}{\sin(x-a)}\right| + C$

(D) $\sin(b-a)\log\left|\dfrac{\sin(x-a)}{\sin(x-b)}\right| + C$

50. $\int \tan^{-1}\sqrt{x}\, dx$ is equal to

(A) $(x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

(B) $x\tan^{-1}\sqrt{x} - \sqrt{x} + C$

(C) $\sqrt{x} - x\tan^{-1}\sqrt{x} + C$

(D) $\sqrt{x} - (x+1)\tan^{-1}\sqrt{x} + C$

51. $\int e^{x}\left(\dfrac{1-x}{1+x^2}\right)^2 dx$ is equal to

(A) $\dfrac{e^x}{1+x^2} + C$

(B) $\dfrac{-e^x}{1+x^2} + C$

(C) $\dfrac{e^x}{(1+x^2)^2} + C$

(D) $\dfrac{-e^x}{(1+x^2)^2} + C$

52. $\int \dfrac{x^9}{(4x^2+1)^6} dx$ is equal to

(A) $\dfrac{1}{5x}\left(4+\dfrac{1}{x^2}\right)^{-5} + C$

(B) $\dfrac{1}{5}\left(4+\dfrac{1}{x^2}\right)^{-5} + C$

(C) $\dfrac{1}{10x}(1+4)^{-5} + C$

(D) $\dfrac{1}{10}\left(\dfrac{1}{x^2}+4\right)^{-5} + C$

53. If $\int \frac{dx}{(x+2)(x^2+1)} = a \log |1 + x^2| + b \tan^{-1}x + \frac{1}{5} \log |x + 2| + C$, then

(A) $a = \frac{-1}{10}, b = \frac{-2}{5}$ (B) $a = \frac{1}{10}, b = -\frac{2}{5}$

(C) $a = \frac{-1}{10}, b = \frac{2}{5}$ (D) $a = \frac{1}{10}, b = \frac{2}{5}$

54. $\int \frac{x^3}{x+1}$ is equal to

(A) $x + \frac{x^2}{2} + \frac{x^3}{3} - \log|1-x| + C$ (B) $x + \frac{x^2}{2} - \frac{x^3}{3} - \log|1-x| + C$

(C) $x - \frac{x^2}{2} - \frac{x^3}{3} - \log|1+x| + C$ (D) $x - \frac{x^2}{2} + \frac{x^3}{3} - \log|1+x| + C$

55. $\int \frac{x+\sin x}{1+\cos x} dx$ is equal to

(A) $\log|1+\cos x| + C$ (B) $\log|x+\sin x| + C$

(C) $x - \tan\frac{x}{2} + C$ (D) $x.\tan\frac{x}{2} + C$

56. If $\int \frac{x^3\,dx}{\sqrt{1+x^2}} = a(1+x^2)^{\frac{3}{2}} + b\sqrt{1+x^2} + C$, then

(A) $a = \frac{1}{3}$, $b = 1$ (B) $a = \frac{-1}{3}$, $b = 1$

(C) $a = \frac{-1}{3}$, $b = -1$ (D) $a = \frac{1}{3}$, $b = -1$

57. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{1+\cos 2x}$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

58. $\int_0^{\frac{\pi}{2}} \sqrt{1-\sin 2x}\,dx$ is equal to

(A) $2\sqrt{2}$ (B) $2\left(\sqrt{2}+1\right)$ (C) 2 (D) $2\left(\sqrt{2}-1\right)$

59. $\int_0^{\frac{\pi}{2}} \cos x\, e^{\sin x} dx$ is equal to _______.

60. $\int \frac{x+3}{(x+4)^2} e^x dx$ = _______.

Fill in the blanks in each of the following Exercise 60 to 63.

61. If $\int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{8}$, then a = _______.

62. $\int \frac{\sin x}{3+4\cos^2 x} dx$ = _______.

63. The value of $\int_{-\pi}^{\pi} \sin^3 x \cos^2 x\, dx$ is _______.