Matrix chain multiplication

$A_{2\times5}$ $B_{5\times10}$ $C_{10\times3}$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}_{2\times3} \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix}_{3\times2}$$

$$AB = \begin{bmatrix} a_1\times b_1 + a_2\times b_3 + a_3\times b_5 & a_1\times b_2 + a_2\times b_4 + a_3\times b_6 \\ a_4\times b_1 + a_5\times b_3 + a_6\times b_5 & a_4\times b_2 + a_5\times b_4 + a_6\times b_6 \end{bmatrix}_{2\times2}$$

$A_{p\times q}$ $B_{q\times r}$

$(AB)_{p\times r}$

No. of scalar multiplication

$= pqr$

$= 2\times3\times2 = 12$

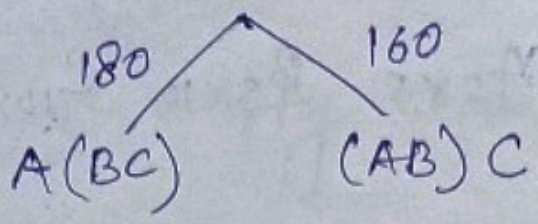
Optimization.

maximization minimization

No of scalar multiplication.

$A_{2\times5}$ $B_{5\times10}$ $C_{10\times3}$

$= 160$



$B_{5\times10}\ C_{10\times3} = (BC)_{5\times3} = 5\times10\times3 = 150$

$+ 180$

$A_{2\times5}\ (BC)_{5\times3} = (ABC)_{2\times3} = 2\times5\times3 = 30$

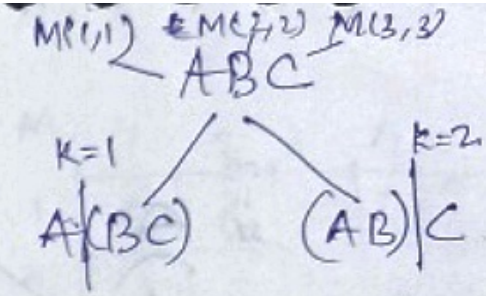
150

$A_{2\times5}\ B_{5\times10} = (AB)_{2\times10} = 2\times5\times10 = 100$

$+ 160$

$(AB)_{2\times10}\ C_{10\times3} = (ABC)_{2\times3} = 2\times10\times3 = 60$

100



A_1 A_2 A_3

$A_{2\times5}$ $B_{5\times10}$ $C_{10\times3}$

$P_0\times P_1$ $P_1\times P_2$ $P_2\times P_3$

$(BC)_{5\times3} = B_{5\times10}\, C_{10\times3} = 5\times10\times3 = 150$

$M[2,3] = M[2,2] + M[3,3] + 5\times10\times3 = 150$

$0 \quad 0 \qquad +=180$

$(ABC)_{2\times3} = A_{2\times5}(BC)_{5\times3} = 2\times5\times3 = 30$

$M[1,3] = M[1,1] + M[2,3] + P_0\times P_1\times P_3$

$i,j \qquad 0 + 150 + 30 = 180$

$$M[1,3] = \min_{k=1,2}\begin{cases} M[1,1] + M[2,3] + P_0P_1P_3 \\ M[1,2] + M[3,3] + P_0P_2P_3 \end{cases}$$

$i\ j$

$(AB)_{2\times10} = A_{2\times5}\, B_{5\times10} = 2\times5\times10 = 100$

$M[1,2] = M[1,1] + M[2,2] + 2\times5\times10 = 100$

$0 \quad 0 \qquad +=160$

$(ABC)_{2\times3} = (AB)_{2\times10}\, C_{10\times3} = 2\times10\times3 = 60$

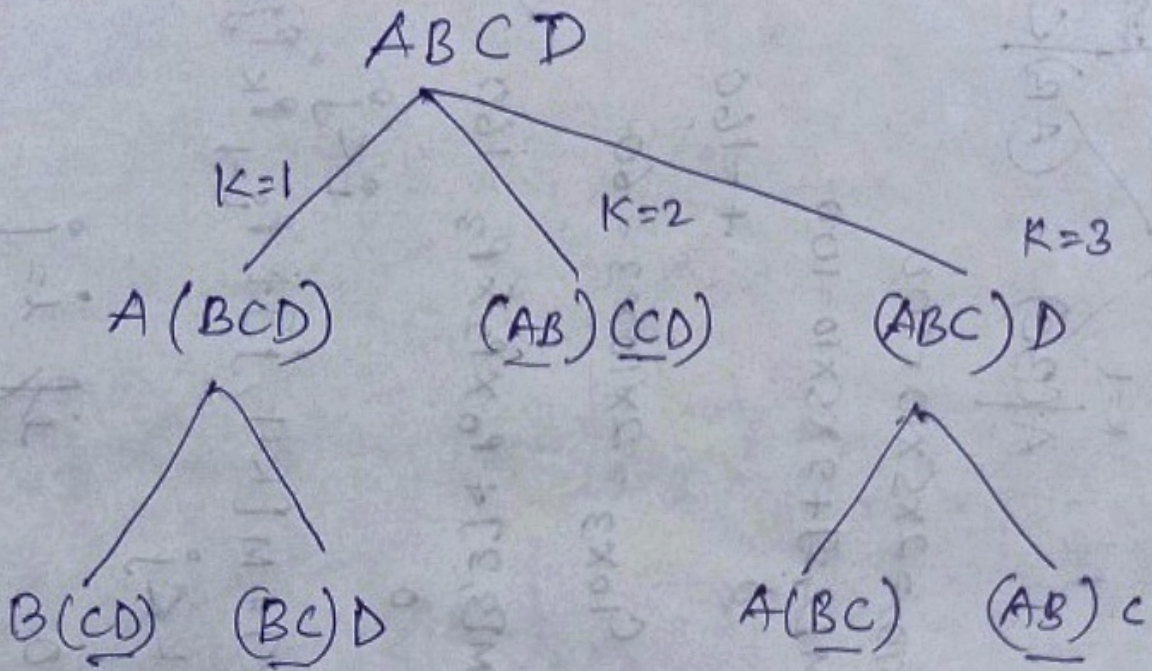
$M[1,3] = M[1,2] + M[3,3] + P_0\times P_2\times P_3 = 160$

$100 + 0$

$i<j$

$$= \min_{i\le k\le j}\{ M[i,k] + M[k+1,j] + P_{i-1}P_kP_j$$

$0 \quad \text{if } i=j$


ABCD
K=1
K=2
K=3
A(BCD)
(AB)(CD)
(ABC)D
B(CD)
(BC)D
A(BC)
(AB)C

Q.

$A_{2\times1}$ $B_{1\times3}$ $C_{3\times4}$ $D_{4\times5}$

$P_0\times P_1$ $P_1\times P_2$ $P_2\times P_3$ $P_3\times P_4$

- by dynamic programming (Bottom up)

M	1	2	3	4
1	0	6	20	42
2		0	12	32
3			0	60
4				0

$M[1,2] = M[1,1] + M[2,2] + 2\times1\times3$

$0 + 0 + 6 = 6$

$M[2,3] = M[2,2] + M[3,3] + 1\times3\times4 = 12$

$0 + 0 + 12$

$M[3,4] = M[3,3] + M[4,4] + 3\times4\times5 = 60$

$0 + 0 + 60$

K	1	2	3	4
1		1	1	1
2			2	3
3				3
4				

ABC < A(BC) = 20, (AB)C

BCD

$M[1,3] = \min\begin{cases} M[1,1] + M[2,3] + 2\times1\times4 = 0 + 12 + 8 = 20 \\ M[1,2] + M[3,3] + 2\times3\times4 = 6 + 0 + 24 = 30 \end{cases}$

32

$M[2,4] = \min\begin{cases} M[2,2] + M[3,4] + 1\times3\times5 = 0 + 60 + 15 = 75 \\ M[2,3] + M[4,4] + 1\times4\times5 = 12 + 0 + 20 = 32 \end{cases}$

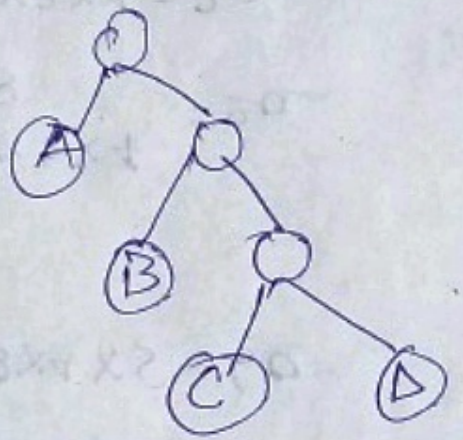
$M[i,j] = \min\begin{cases} M[i,k] + M[k+1,j] + P_{i-1}P_kP_j \\ i \le k < j \end{cases}$

0 if $i = j$

$$M[1,4] = \begin{cases} M[1,1] + M[2,4] + 2*1\times5 = 42 \\ M[1,2] + M[3,4] + 2\times3\times5 = 96 \\ M[1,3] + M[4,4] + 2\times4\times5 = 60 \end{cases}$$

$A_{2\times1}$ $B_{1\times3}$
$P_0\ P_1$ $P_1\ P_2$
$C_{3\times4}$ $D_{4\times5}$
$P_2\ P_3$ $P_3\ P_4$

A(B(C D))



A|BCD

AB|CD

ABC|D

$n = 4$

$1+2+3+4$

$\frac{n(n+1)}{2}$ $O(n)$

$= O(n^3)$