Lecture 2

Geometric Series:- Geometric series are series the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

$$= \sum_{n=0}^{\infty} ar^n$$

The ratio 'r' can be positive or negative and $a \neq 0$.

n^{th} partial sum of the geometric series is

$$S_n = \frac{a(1-r^n)}{1-r} \quad (r \neq 1)$$

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ Converges to $\frac{a}{1-r}$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

If $|r| \geq 1$, the series diverges.

eg:- ① The series $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$

is a geometric series with $a = 5$ and $r = -1/4$.

It converges to $\frac{a}{1-r} = \frac{5}{1+1/4} = 4$.

② Find the sum of the "telescoping" series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Sol^n:- $S_K = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$\therefore S_K = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{K} - \frac{1}{K+1}\right)$

$S_K = 1 - \frac{1}{K+1} \rightarrow 1$ as $K \rightarrow \infty$

The series converges, and its sum is 1:

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

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nth term test for a Divergent series :-

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Ex. 7 The following are all examples of divergent series.

a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$

b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges $\because$ $\frac{n+1}{n} \to 1$ $\left[\lim_{n\to\infty} a_n \neq 0\right]$

c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges $\because$ $\lim_{n\to\infty} (-1)^{n+1}$ does not exist.

d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges $\because$ $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

Combining Series :- If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. $\sum k a_n = k \sum a_n = k.A$

Note :- ① Every nonzero constant multiple of a divergent series diverges.

② If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverge.

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Remark:- $\Sigma(a_n+b_n)$ can converge when Σa_n and Σb_n both diverge.

for eg. $\Sigma a_n = 1+1+1+\cdots$ diverges

and $\Sigma b_n = (-1)+(-1)+(-1)+\cdots$ converges to 0.

whereas $\Sigma(a_n+b_n) = 0+0+0+\cdots$

Ex. 9 Find the sum of the following series

a) $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \frac{4}{5}$ Ans.

b) $\sum_{n=0}^{\infty} \frac{4}{2^n} = 8$ Ans.

Problem:- 1) Using the nth term test, show that following series are divergent.

1) $\sum_{n=1}^{\infty} \frac{n}{n+10}$ 2) $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$

3) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ 4) $\sum_{n=1}^{\infty} \frac{e^n}{e^n+n}$

5) $\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$ 6) $\sum_{n=1}^{\infty} \cos n\pi$

Prob 2:- Which series converge and which diverge? If a series converges, find its sum.

1) $\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$ 2) $\sum_{n=0}^{\infty}(\sqrt{2})^n$ 3) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}$

4) $\sum_{n=0}^{\infty}\frac{\cos n\pi}{5^n}$ 5) $\sum_{n=1}^{\infty}\ln\left(\frac{1}{3^n}\right)$ 6) $\sum_{n=0}^{\infty}\frac{n!}{1000^n}$

7) $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ 8) $\sum_{n=0}^{\infty}\left(\frac{e}{\pi}\right)^n$

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integral Test:-

nr:- A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

Thm:- Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x $\forall$ $x \geq N$. Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

p-series:-

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p real const)

converges if $p > 1$, and
diverges if $p \leq 1$.

Problem:- ① Determine the convergence or divergence of the series

a) $\sum_{n=1}^{\infty} n e^{-n^2}$ ⓑ $\sum_{n=1}^{\infty} \frac{1}{2^{\ln(n)}}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Sol^n:- ⓐ We apply the integral test and find that

$$\int_1^{\infty} \frac{x}{e^{x^2}}\,dx = \frac{1}{2}\int_1^{\infty} \frac{du}{e^u} = \cancel{\frac{1}{2}[\ldots]}$$

Let $x^2 = u$
$2x\,dx = du$

$$= -\frac{1}{2}\lim_{b\to\infty}\left[e^{-u}\right]_1^b = -\frac{1}{2}\lim_{b\to\infty}\left[e^{-b} - e^{-1}\right]$$

$$= \frac{1}{2e} \quad \text{Ans}$$

Since the integral converges, the series also converges.

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b) Again applying the integral test

$$\int_1^{\infty} \frac{dx}{2^{\ln x}} = \int_0^{\infty} \frac{e^u\,du}{2^u} = \int_0^{\infty} \left(\frac{e}{2}\right)^u du$$

Put $\ln x = u$
$x = e^u$
$dx = e^u du$
$u : 0 \to \infty$

$$= \cancel{\frac{1}{u+1}\left(\frac{e}{2}\right)^{u+1}} \lim_{b\to\infty}\left[\frac{\left(\frac{e}{2}\right)^u}{\log(e/2)}\right]_0^b$$

$$= \frac{1}{\log(e/2)} \lim_{b\to\infty}\left[\left(\frac{e}{2}\right)^b - 1\right] = \infty \qquad \left(\because \frac{e}{2} > 1\right)$$

The improper integral diverges, so the series diverges also.

c) $$\int_1^{\infty} \frac{1}{x^2+1}dx = \lim_{b\to\infty}\left[\tan^{-1}x\right]_1^b$$

$$= \lim_{b\to\infty}\left[\tan^{-1}b - \tan^{-1}(1)\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Exercises Applying the integral test to determine series are converges or diverges..

(1) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (2) $\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$ (3) $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

(4) $\sum_{n=1}^{\infty} \frac{1}{n+4}$ (5) $\sum_{n=1}^{\infty} e^{-2n}$ (6) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

(7) $\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n}$ (8) $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$

(9) $\sum_{n=2}^{\infty} \frac{n-4}{n^2-2n+1}$

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