Lecture 3

Absolute Convergence

Defⁿ:- A series $\sum a_n$ converges absolutely if the $\sum_n |a_n|$, converges.

Thm:- If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

eg:- ① $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

Now $\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (by p-series test)

Hence $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2}$ converges. it is called absolutely convergent

② $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Now $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Hence $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges. and it is called absolutely convergent.

The Ratio Test:- let $\sum a_n$ be any series and suppose that $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = p$. Then

a) the series converges absolutely if $p < 1$

b) diverges if $p > 1$ or p infinite

c) the test is inconclusive if $p = 1$.

Scanned by CamScanner

Ex. (2) Investigate the convergence of the following series:

a) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$ (c) $\sum_{n=1}^{\infty} \frac{4^n\, n!\, n!}{(2n)!}$

Solution:- (a) $a_n = \frac{2^n+5}{3^n}$ $\therefore a_{n+1} = \frac{2^{n+1}+5}{3^{n+1}}$

Now, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2^{n+1}+5)\cdot 3^n}{3^{n+1}\cdot(2^n+5)} = \frac{1}{3}\cdot\left(\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{n-1}}\right)$

$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}\times 2 = \frac{2}{3} < 1$

The series $\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$ converges absolutely.

b) $a_n = \frac{(2n)!}{n!\,n!}$ $\therefore a_{n+1} = \frac{(2(n+1))!}{(n+1)!\,(n+1)!}$

Now, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n+2)!\; n!\, n!}{(n+1)!\,(n+1)!\; 2n!} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$

$= \frac{2(2n+1)}{(n+1)}$

$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2(2+1/n)}{(1+1/n)} = \frac{2(2+0)}{1+0} = 4 > 1$

The series diverges because $p = 4 > 1$.

c) $a_n = \frac{4^n\, n!\, n!}{(2n)!}$ $\therefore a_{n+1} = \frac{4^{n+1}(n+1)!\,(n+1)!}{(2n+2)!}$

Now, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+1}\cdot(n+1)!\,(n+1)!\,(2n)!}{4^n\, n!\, n!\cdot(2n+2)!} = \frac{4(n+1)}{2(2n+1)}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{4(1+1/n)}{2(2+1/n)} = \frac{4}{4} = 1$

$\because p = 1$, we cannot decide from the ratio test.

(series diverge)

Scanned by CamScanner

The Root Test:- Let $\sum a_n$ be any series and suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = p$. Then the series

a) converges absolutely if $p<1$

b) diverges if $1<p\leq\infty$

c) the test is inconclusive if $p=1$.

Ex (4) Which of the following series converge, and which diverge?

a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ (c) $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Solⁿ:- We apply the root test.

a) $\sqrt[n]{\left|\frac{n^2}{2^n}\right|} = \frac{(n^2)^{1/n}}{(2^n)^{1/n}} = \frac{n^{2/n}}{2} = \frac{(n^{1/n})^2}{2}$

$\therefore \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^2}{2^n}\right|} = \lim_{n\to\infty} \frac{(n^{1/n})^2}{2} = \frac{1}{2} < 1$.

absolutely converges

b) $\sqrt[n]{\left|\frac{2^n}{n^3}\right|} = \frac{(2^n)^{1/n}}{(n^3)^{1/n}} = \frac{2}{(n^{1/n})^3}$

$\therefore \lim_{n\to\infty} \sqrt[n]{\left|\frac{2^n}{n^3}\right|} = \frac{2}{1} = 2 > 1$ diverges.

c) $\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \left(\frac{1}{1+n}\right)$

$\lim_{n\to\infty} \frac{1}{1+n} = \frac{1}{\infty} = 0 < 1$ converges

Scanned by CamScanner

Summary of Tests:-

1. The nth term test:- If it is not true that $a_n \to 0$, then the series diverges.

2. Geometric Series:- $\sum ar^n$ converges if $|r|<1$; otherwise it diverges.

3. p-series:- $\sum \frac{1}{n^p}$ converges if $p>1$; otherwise it diverges.

4. Series with nonnegative terms:- Try the integral test or Root test or Ratio test.

5. Series with some negative terms:- Does $\sum |a_n|$ converge by the Root Test or Ratio test? Absolute convergence $\Rightarrow$ convergence

Note: (1) A series that converges but does not converge absolutely "converges conditionally."

(2) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Scanned by CamScanner