Change of Variables

(1) $x = \phi(u,v)$, $y = \psi(u,v)$. Then

$$\iint_R f(x,y)\,dx\,dy = \iint_D f\{x(u,v), y(u,v)\} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

(2) <u>Polar coordinates</u> $x = r\cos\theta$, $y = r\sin\theta$

$$\iint_R f(x,y)\,dx\,dy = \iint_D f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$\left[\because dx\,dy = \frac{\partial(x,y)}{\partial(r,\theta)}\,dr\,d\theta\right]$$

(3) <u>Spherical Coordinates</u>

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint f(r,\theta,\phi)\, r^2\sin\theta\,dr\,d\theta\,d\phi.$$

<u>Que(1)</u> Evaluate the Integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$ by changing into polar coordinates?

<u>Sol^n:-</u> The region of integration is

$x: 0 \longrightarrow \sqrt{a^2-y^2}$ i.e. $x^2+y^2=a^2$

$y: 0 \longrightarrow a$

In polar coordinates, $r^2 = a^2 \Rightarrow r = a$ $\cancel{x=0}$

$x = 0 \Rightarrow r\cos\theta = 0 \Rightarrow \theta = \pi/2$ and $r = 0$.

$y = 0 \Rightarrow r\sin\theta = 0 \Rightarrow \theta = 0$

The limits of ~~r are 0 and a~~ $r: 0 \to a$, $\theta: 0 \to \pi/2$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy = \int_0^{\pi/2} \int_0^a r^2 \cdot r\,dr\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} [r^4]_0^a\,d\theta = \frac{a^4}{4}\int_0^{\pi/2} d\theta = \frac{a^4}{4}\cdot\frac{\pi}{2} = \frac{\pi a^4}{8}.$$

(2) Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing to polar coordinates.

Sol^n: Given limit $y: 0 \to \sqrt{2x-x^2}$

$x: 0 \to 2$

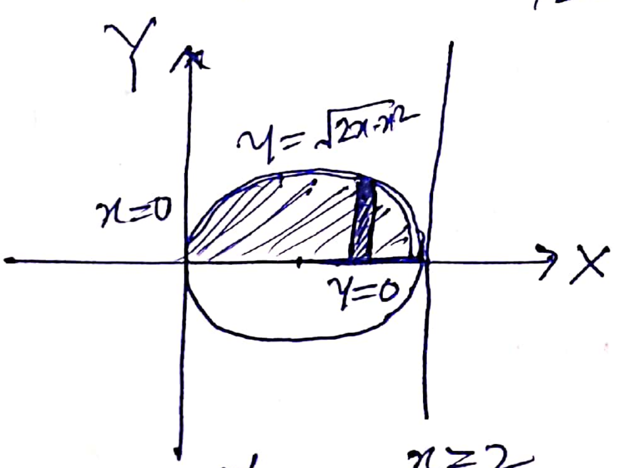
$$y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2+y^2 = 2x$$

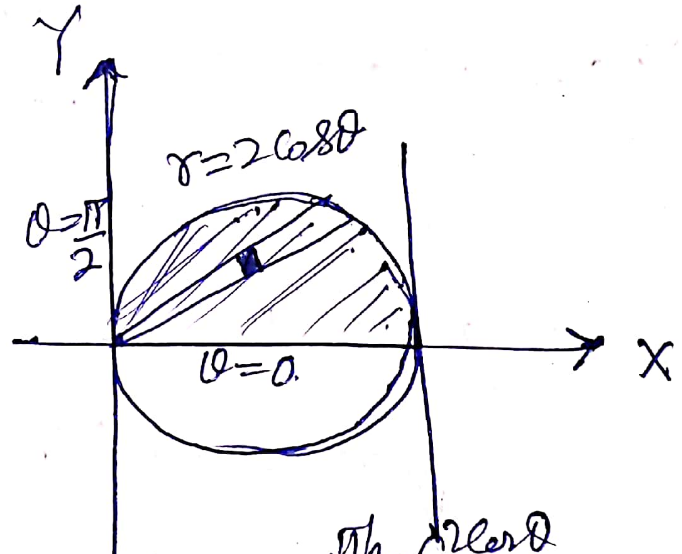
In polar cordinate $r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$

$\therefore$ For the region of integration, $x = r\cos\theta$, $y = r\sin\theta$

$r: 0 \to 2\cos\theta$

$\theta: 0 \to \pi/2$



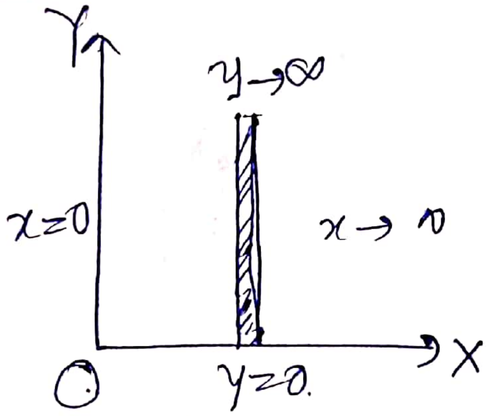


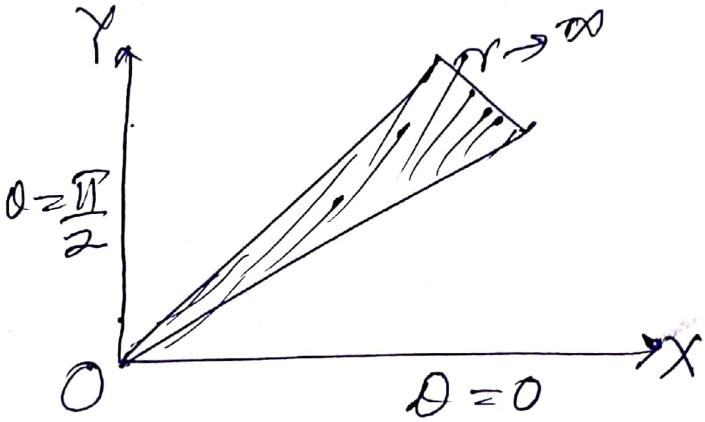
$$\therefore I = \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r\cos\theta}{r} \cdot r\,dr\,d\theta = \int_0^{\pi/2} \int_0^{2\cos\theta} \cos\theta\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi/2} 2\cos^3\theta\,d\theta = 2\cdot\frac{2}{3} = \frac{4}{3}$$

) Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar co-ordinates. Hence show that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

Sol^n:- $y: 0 \to \infty$, $x: 0 \to \infty$





New limit $r: 0 \to \infty$, then $\theta: 0 \to \frac{\pi}{2}$.

Hence, $I = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta$ Put $r^2 = t$, $2r\,dr = dt$

$$= \frac{1}{2}\int_0^{\pi/2}\int_0^\infty e^{-t}\,dt\,d\theta = -\frac{1}{2}\int_0^{\pi/2}(-1)\,d\theta = \frac{\pi}{4}.$$

Let $I = \int_0^\infty e^{-x^2}dx$ — (1)

Also, $I = \int_0^\infty e^{-y^2}dy$ —— (2) [By

Multiplying (1) and (2), we get

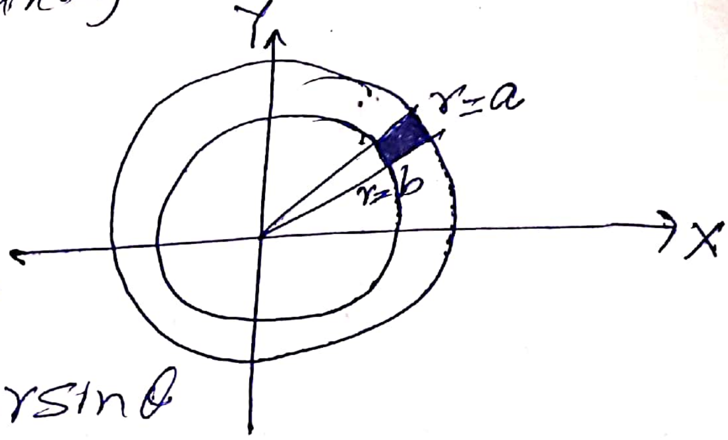
$$I^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4}$$

$$\therefore I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}.$$

(4) Evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region between circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$.

Sol^n:- Here, we have

$$I=\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy \qquad \text{—— (1)}$$



$x^2+y^2=a^2 \Rightarrow \quad r=a$

$x^2+y^2=b^2 \Rightarrow \quad r=b$

Let us convert (1) into polar coordinates.

By putting $x=r\cos\theta,\ y=r\sin\theta$

$r: b \to a$

$\theta: 0 \to \pi/2$ in first quadrant

$$I=4\int_0^{\pi/2}\int_b^a \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{r^2}\cdot r\,d\theta\,dr \qquad \text{—— (2)}$$

$$=4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\int_b^a r^3\,dr\,d\theta=(a^4-b^4)\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$=(a^4-b^4)\,\frac{\Gamma 3/2\,\Gamma 3/2}{2\Gamma 3}=(a^4-b^4)\,\frac{\frac{1}{2}\Gamma\frac{1}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{2\times 2!}$$

$$I=\frac{(a^4-b^4)}{16}\cdot\pi.$$

Change of Variables

① If $x = g(u,v)$, $y = h(u,v)$. Then

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

② If $x = r\cos\theta$, $y = r\sin\theta$

$$\iint_R f(x,y)\,dx\,dy = \iint_{S} f(r,\theta)\, r\,dr\,d\theta$$

(3) If $x = g(u,v,w)$, $y = h(u,v,w)$, $z = l(u,v,w)$

Then

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_S f(u,v,w) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.$$

Prob ①:- Evaluate the integral $\iint_R (x^2+2xy)\,dA$, where R is the region bounded by the lines $y = 2x+3$, $y = 2x+1$, $y = 5-x$ and $y = 2-x$.

Solution:- $y-2x = 3$, $y-2x = 1$

$x+y = 5$, $x+y = 2$

Taking $u = y-2x$, $v = x+y$

i.e. $u: 1 \to 3$, $v: 2 \to 5$

and $x = \frac{1}{3}(v-u)$, $y = \frac{1}{3}(2v+u)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} = -\frac{2}{9} - \frac{1}{3} = -\frac{1}{3}$$

$\therefore \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3}$

$$I = \iint_R (x+2xy)\,dA = \int_2 \int_1 \left[\frac{1}{3}(v-u)^2 + \frac{2}{3}(v-u)\frac{1}{3}(2v+u)\right] \times \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$I = \frac{1}{9\times 3}\int_2^5 \int_1^3 \left[(v-u)^2 + 2(v-u)(2v+u)\right] du\,dv$$

$$= \frac{196}{27} \quad \text{Ans.}$$

Prob ②:- Evaluate the double integral $\iint_R \frac{e^{x-y}}{x+y}\,dA$, where R is the rectangle bounded by the lines $y=x$, $y=x+5$, $y=2-x$ and $y=4-x$.

<u>Solution:-</u> $y-x=0$, $y-x=5$.

$x+y=2$, $x+y=4$

Taking $u=y-x$, $v=x+y$

Then $u: 0 \to 5$, $v: 2 \to 4$

and. $x=\frac{1}{2}(v-u)$; $y=\frac{1}{2}(u+v)$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\therefore \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$$

$$\therefore I = \iint_R \frac{e^{x-y}}{x+y}\,dA = \int_2^4 \int_0^5 \frac{e^{-u}}{v} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| . du\,dv$$

$$= \frac{1}{2}\int_2^4 \int_0^5 \frac{e^{-u}}{v}\,du\,dv = \frac{1}{2}\int_2^4 \frac{1}{v}(1-e^{-5})\,dv$$

$$= \frac{1}{2}(1-e^{-5})(\ln 4 - \ln 2) \quad \text{Ans}$$

$$\simeq 0.34424$$