Fourier sine and Cosine series (Half range)

If a function $f(x)$ defined on an interval $[0, L]$ then the function $f(x)$ can be represented (expanded) in terms of infinite sum of cosine and sine angles. First is called Fourier cosine series and second is Fourier sine series.

F. Cosine series : $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad ; n = 0, 1, 2, \ldots$$

Formula of a_0 is merged with a_n but we calculate a_0 & a_n both separately as done in example.

F. Sine series : $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad ; n = 1, 2, \ldots$$

Ex: Find the Fourier sine and cosine series of the function $f(x) = k$ in the interval $0 < x < 5$.

Soln:- The F. sine series is $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ and

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Here, $f(x) = k$ & $L = 5$

$$\therefore b_n = \frac{2}{5}\int_0^5 k \sin\left(\frac{n\pi x}{5}\right) dx = \frac{2k}{5}\left[\frac{-\cos\frac{n\pi x}{5}}{\frac{n\pi}{5}}\right]_0^5$$

$$= \frac{2k}{n\pi}\left[-\cos n\pi + \cos 0\right] = \frac{2k}{n\pi}(1-\cos n\pi)$$

$\therefore$ The Fourier sine series is $\sum_{n=1}^{\infty} \frac{2k}{n\pi}(1-\cos n\pi)\sin\left(\frac{n\pi x}{5}\right)$

$$= \frac{2k}{\pi}(1-\cos\pi)\sin\left(\frac{\pi}{5}x\right) + \frac{2k}{2\pi}(1-\cos 2\pi)\sin\left(\frac{2\pi}{5}x\right) + \frac{2k}{3\pi}(1-\cos 3\pi)$$

$$\sin\left(\frac{3\pi}{5}x\right) + \frac{2k}{4\pi}(1-\cos 4\pi)\sin\left(\frac{4\pi}{5}x\right) + \frac{2k}{5\pi}(1-\cos 5\pi)\sin\left(\frac{5\pi}{5}x\right) + \dots$$

$$= \frac{2k}{\pi}\left\{2\sin\left(\frac{\pi}{5}x\right) + \frac{2}{3}\sin\left(\frac{3\pi}{5}x\right) + \frac{2}{5}\sin(\pi x) + \dots\right\}$$

$$= \frac{4k}{\pi}\left[\sin\frac{\pi}{5}x + \frac{1}{3}\sin\frac{3\pi}{5}x + \frac{1}{5}\sin\pi x + \dots\right\}$$

$$= \frac{4k}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)}\sin\left(\frac{2m-1}{5}\right)x$$

Cosine series is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$ &

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad ; n = 1, 2, \dots$$

Here, $f(x) = k$ & $L = 5$

$$\therefore a_n = \frac{2}{5}\int_0^5 k\cos\left(\frac{n\pi x}{5}\right)dx = \frac{2k}{5}\left[\frac{\sin\frac{n\pi x}{5}}{\frac{n\pi}{5}}\right]_0^5$$

$$= \frac{2k}{n\pi}\left[\sin n\pi - 0\right] = 0$$

& $a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{5}\int_0^5 k\,dx = \frac{2k}{5}(x)_0^5 = 2k$

$\therefore$ The F. cosine series is k.

Ex. Find the Fourier cosine and sine series of the function $f(x) = x$ in the interval $[0, \pi]$.

The Fourier cosine series is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$ &

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad ; \; n = 0, 1, 2, \ldots$$

Here, $f(x) = x$ & $L = \pi$.

$$\therefore \; a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^2}{2}\right)$$

$$= \pi.$$

$$\& \; a_n = \frac{2}{\pi}\int_0^{\pi} x\cos\frac{n\pi x}{\pi}\,dx = \frac{2}{\pi}\left[x\,\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi\,\frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - 0 - \frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left(\cos n\pi - 1\right)$$

$\therefore$ The Fourier cosine series is

$$\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}(\cos n\pi - 1)\cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[-2\cos x + 0 - \frac{2}{3^2}\cos 3x + 0 - \frac{2}{5^2}\cos 5x + \ldots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}\cos(2m-1)x$$

The Fourier sine series is $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ &

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Here $f(x) = x$ & $L = \pi$

$$\therefore\ b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos n\pi + 0 - 0 - 0\right] = -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n$$

$\therefore$ The Fourier sine series is

$$\sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx.$$