I

Review of Integrals of Functions of Single Variable

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c \; ; \; n \neq -1$

$\int \frac{1}{x} dx = \log x + c \; ; \; \int e^x dx = e^x + c ; \; \int a^x dx = \frac{a^x}{\log a} + c$

$\int \sin x \, dx = -\cos x + c ; \; \int \cos x \, dx = \sin x + c$

$\int \tan x \, dx = \log \sec x + c ; \; \int \cot x \, dx = \log \sin x + c$

$\int \sec x \, dx = \log(\sec x + \tan x) + c = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

$\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x) + c = \log \tan \frac{x}{2} + c$

$\int \sec x \tan x \, dx = \sec x + c ; \; \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

(2) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

$-\int \frac{dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\frac{x}{a}\right) + c$

$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c ;$

$-\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \cot^{-1}\left(\frac{x}{a}\right) + c ;$

$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right) + c ;$

$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + c$

$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$

$-\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + c$

(3) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

$\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left[x+\sqrt{a^2+x^2}\right] + C$

$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left[x+\sqrt{x^2-a^2}\right] + C$

$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C;$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C.$

(4) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x)=f(x) \\ 0, & \text{if } f(-x)=-f(x) \end{cases}$$

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a/2} f(x)\,dx, & \text{if } f(a-x)=f(x) \\ 0, & \text{if } f(a-x)=-f(x). \end{cases}$$

Double Integrals :-

$$\iint_A f(x,y)\,dx\,dy = \int_a^b \left[\int_{y_1}^{y_2} f(x,y)\,dy\right] dx$$

$$\iint_A f(x,y)\,dx\,dy = \int_c^d \left[\int_{x_1}^{x_2} f(x,y)\,dx\right] dy$$

Que:. (1) Find $\int_0^1 \int_0^y xy\,e^{-x^2}\,dx\,dy$

Sol^n:. let $I = \int_0^1 y\left[\int_0^y x\,e^{-x^2}\,dx\right] dy$

Put $x^2 = t$
$2x\,dx = dt$
$x\,dx = \frac{dt}{2}$

$= \frac{1}{2}\int_0^1 y\left[\int_0^{y^2} e^{-t}\,dt\right] dy$

$= -\frac{1}{2}\int_0^1 y\left(e^{-y^2} - 1\right) dy$

$= -\frac{1}{2}\left[\int_0^1 y\,e^{-y^2}\,dy - \int_0^1 y\,dy\right]$

$= -\frac{1}{2}\left[\frac{-e^{-y^2}}{2} - \frac{y^2}{2}\right]_0^1 = \frac{1}{4}\left[\frac{1}{e} + 1 - 1\right]$

$= \frac{1}{4e}$.

(2) $\int_0^1 \int_0^{x^2} e^{y/x}\,dy\,dx = \int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx$

$= \int_0^1 x\left(e^{x} - 1\right) dx = \int_0^1 x\,e^{x}\,dx - \int_0^1 x\,dx$

$= \left[x e^{x} - e^{x}\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1 = e - e + 1 - \frac{1}{2}$

$= 1/2$.

(3) $\int_1^{\log 8}\int_0^{\log y} e^{x+y}\,dx\,dy$

$= \int_1^{\log 8} e^y dy\, [\int_0^{\log y} e^x dx]\, dy$

$= 8\log 8 - 16 + e.$

(4). $\int_0^{a}\int_{x/a}^{x} \frac{x\,dy\,dx}{x^2+y^2}$

(5) $\int_0^{\infty}\int_0^{\infty} e^{-x^2(1+y^2)}\, x\,dx\,dy$

$= \int_0^{\infty} dy \int_0^{\infty} x\, e^{-x^2(1+y^2)} dx \qquad = \frac{\pi}{4}$

(6) $\int_1^{2}\int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2\,dx\,dy$

(7) $I = \int_{-1}^{1}\int_0^{1-x} x^{1/3}\, y^{-1/2}\, (1-x-y)^{1/2}\,dy\,dx$

Put $1-x=c$

$y=ct$

$dy=c\,dt$

$= \int_{-1}^{1} x^{1/3}dx \int_0^{c} y^{-1/2}(c-y)^{1/2}dy$

$= \int_{-1}^{1} x^{1/3}dx \int_0^{1} c^{-1/2}\, t^{-1/2}\, (c-ct)^{1/2}\, c\,dt$

$= \int_{-1}^{1} c\, x^{1/3}dx \int_0^{1} t^{-1/2}(1-t)^{1/2}dt \;=\; \int_{-1}^{c} c\, x^{1/3}dx.\; \beta\left(\frac{1}{2},\frac{3}{2}\right)$

$= \frac{9\pi}{28}$

Double Integral over a Rectangle

If $R = \{(x,y) ! \; 0 \le x \le 2$ and $1 \le y \le 4\}$,
evaluate $\iint_R (6x^2 + 4xy^3)\, dA$.

Solution:- $\iint_R (6x^2+4xy^3)\,dA = \int_1^4 \int_0^2 (6x^2+4xy^3)\,dx\,dy$

$$= \int_1^4 \left[\int_0^2 (6x^2+4xy^3)\,dx\right] dy$$

$$= \int_1^4 \left[\left(6\cdot\frac{x^3}{3} + 4\,\frac{x^2}{2}\cdot y^3\right)\right]_{x=0}^{x=2} dy$$

$$= \int_1^4 (16+8y^3)\,dy = \left[\left(16y + 8\,\frac{y^4}{4}\right)\right]_1^4$$

$$= [16(4)+2(4)^4] - [16+2(1)^4]$$

$$= 558.$$