JACOBIANS

If u & v are the functions of two independent variables x and y then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$ is called jacobian of u, v w.r.to x, y and written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$

Similiarly the jacobian of u, v, w with respect to x, y & z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \text{ or } J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Q1 If $x = r\cos\theta$, $y = r\sin\theta$

Evaluate $\frac{\partial(x,y)}{\partial(r,\theta)}$ & $\frac{\partial(r,\theta)}{\partial(x,y)}$

Sol We have

$x = r\cos\theta$, $y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$ $\quad \frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\quad \frac{\partial y}{\partial \theta} = r\cos\theta$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$
$$= r$$

also we have

$x^2 + y^2 = r^2$

& $\theta = \tan^{-1} y/x$

$\therefore \dfrac{\partial r}{\partial x} = \dfrac{x}{r} \qquad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times \left(-\frac{y}{x^2}\right) = -\frac{x^2}{x^2+y^2} \times \frac{y}{x^2} = -\frac{y}{r^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x}, & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \times \frac{1}{r} = 1$$

$$\therefore \boxed{\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1}$$

Q 2 If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ $\quad z = r\cos\theta$

show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

Sol We have

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$\dfrac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad \dfrac{\partial y}{\partial r} = \sin\theta\sin\phi \qquad \dfrac{\partial z}{\partial r} = \cos\theta$

$\dfrac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \qquad \dfrac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \qquad \dfrac{\partial z}{\partial \theta} = -r\sin\theta$

$\dfrac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \qquad \dfrac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \qquad \dfrac{\partial z}{\partial \phi} = 0$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r}, & \frac{\partial x}{\partial \theta}, & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}, & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r}. & \frac{\partial z}{\partial \theta}, & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi, & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi, & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & -0 \end{vmatrix}$$

$$= r^2\sin\theta\Big[\sin\theta\cos\phi\,(0+\sin\theta\cos\phi) - \cos\theta\cos\phi\,(0-\cos\theta\cos\phi) - \sin\phi\,(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi)\Big]$$

$$= r^2\sin\theta\Big[\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\phi(\sin^2\theta + \cos^2\theta)\Big]$$

$$= r^2\sin\theta\Big[\cos^2\phi(\sin^2\theta + \cos^2\theta) + \sin^2\phi\Big]$$

$$= r^2\sin\theta\Big[\cos^2\phi + \sin^2\phi\Big]$$

$$= r^2\sin\theta$$

Properties of Jacobians

Property first

1 If u, v are the functions of x & y

then $$\boxed{\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1}$$

or

$$\boxed{J \times J' = 1}$$

Q3 If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$

find $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol Since u, v, w are explicitly given so first we evaluate

$$J' = \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y}, & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x}, & \frac{\partial w}{\partial y}, & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} yz, & zx, & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} \qquad \begin{array}{l} C_1 \to C_1 - C_2 \\ C_3 \to C_3 - C_2 \end{array}$$

$$= \begin{vmatrix} yz - zx & zx & xy - zx \\ 2x - 2y & 2y & 2z - 2y \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -\cancel{z}(x-y) & zx & +x(y-z) \\ 2(x-y) & 2y & -2(y-z) \\ 0 & 1 & 0 \end{vmatrix}$$

$$= (x-y)(y-z) \begin{vmatrix} -z & zx & x \\ 2 & 2y & -2 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 2(x-y)(y-z) \begin{vmatrix} -z & zx & x \\ 1 & y & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

$C_1 \to C_1 + C_3$

$$= 2(x-y)(y-z) \begin{vmatrix} -z+x & zx & x \\ 0 & y & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 2(x-y)(y-z)(z-x) \begin{vmatrix} -1 & zx & x \\ 0 & y & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 2(x-y)(y-z)(z-x)\left[(-1)(0+1)\right]$$

$$= -2(x-y)(y-z)(z-x)$$

we know that

$$J \cdot J' = 1$$

$$\therefore J = \frac{1}{J'} = \frac{1}{-2(x-y)(y-z)(z-x)}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-1}{2(x-y)(y-z)(z-x)}$$

2- Second Property (Chain Rule)

If u, v are the functions of r, s and r, s are the functions of x, y

then $$\boxed{\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}}$$

Q4 $\frac{\partial(u,v)}{\partial(r,\theta)} = ?$ where $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$

Sol $u = x^2 - y^2$, $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial v}{\partial x} = 2y$

$\frac{\partial u}{\partial y} = -2y$ $\frac{\partial v}{\partial y} = 2x$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & x \end{vmatrix} = 4x^2 + 4y^2$$
$$= 4(x^2+y^2)$$
$$= 4r^2$$

$$\& \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

By chain Rule

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = 4r^2 \times r$$

$$\boxed{\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3}$$

Third Property If functions, u, v, w are the function of three independent variables x, y, z, are not independent then

$$\boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0}$$

Converse if it is given that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$ and, u, v, w are not independent of one another then they are connected by the relation

$$\boxed{f(u,v,w) = 0}$$

Q5 If $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$, and $w = x + y + z$ determine whether there is a functional relationship between u, v, w and if so, find it.

Sol We have $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$, $w = x + y + z$

$$\frac{\partial u}{\partial x} = y + z, \quad \frac{\partial v}{\partial x} = 2x, \quad \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = z + x, \quad \frac{\partial v}{\partial y} = 2y \quad \frac{\partial w}{\partial y} = 1$$

$$\frac{\partial u}{\partial z} = x + y, \quad \frac{\partial v}{\partial z} = 2z, \quad \frac{\partial w}{\partial z} = 1$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}, & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} -(x-y) & z+x & y-z \\ x-y & y & -(y-z) \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 2(x-y)(y-z)\begin{vmatrix} -1 & z+x & 1 \\ 1 & y & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

$R_1 \to R_1 + R_2$

$$= 2(x-y)(y-z)\begin{vmatrix} 0 & x+y+z & 0 \\ 1 & y & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 2(x-y)(y-z)(x+y+z)\begin{vmatrix} 0 & 1 & 0 \\ 1 & y & -1 \\ 0 & 1 & 0 \end{vmatrix} \qquad R_1 \equiv R_3$$

$= 0$ hence the functional relationship exists between u, v & w

Now $w^2 = (x+y+z)^2$

$= x^2+y^2+z^2+2(xy+yz+zx)$

$\boxed{w^2 = v + 2u}$ i.e $\boxed{2u + v - w^2 = 0}$

Jacobian of implicit functions

If the variables u, v, x, y are connected by implicit functions

$f_1(x, y, u, v) = 0$
$f_2(x, y, u, v) = 0$ where u, v are implicit functions of x & y

Then

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(f_1\ f_2)/\partial(x,y)}{\partial(f_1\ f_2)/\partial(u,v)}$$

In general the variables $x_1\ x_2\ x_3$ —— x_n are connected with $u_1\ u_2\ u_3$ - - - u_n implicitly as $f_1(x_1, x_2, x_3 - x_n, u_1, u_2 -- u_n) = 0$

$f_2(x_1\ x_2\ x_3 - — x_n, u_1\ u_2\ u_3 - - u_n) = 0$ - - —— ——

$f_n(x_1\ x_2\ x_3 \ldots u_1\ u_2\ u_3 — u_n) = 0$

Then

$$\frac{\partial(u_1 u_2 u_3 \text{ ---- } u_n)}{\partial(x_1 x_2 x_3 \text{ ---- } x_n)} = (-1)^n \cdot \frac{\dfrac{\partial(f_1 f_2 f_3 \text{ --- } f_n)}{\partial(x_1 x_2 x_3 \text{ --- } x_n)}}{\dfrac{\partial(f_1 f_2 f_3 \text{ --- } f_n)}{\partial(u_1 u_2 u_3 \text{ --- } u_n)}}$$

Q6 if, u, v, w are the roots of the equation $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$ in λ find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = ?$

Sol

$$\lambda^3 - x^3 - 3\lambda^2 x + 3\lambda x^2 + \lambda^3 - y^3 - 3\lambda^2 y + 3\lambda y^2 + \lambda^3 - z^3 - 3\lambda^2 z + 3\lambda z^2 = 0$$

i.e $3\lambda^3 - 3\lambda^2(x+y+z) + 3\lambda(x^2+y^2+z^2) - (x^3+y^3+z^3) = 0$

which is cubic equation in λ as $ax^3+bx^2+cx+d=0$

Sum of the roots $= -b/a$

i.e $u+v+w = \dfrac{(x+y+z)\cancel{(+3)}}{\cancel{3}}$ —— (I)

Product of the roots taken two at a time $= c/a$

$uv+vw+wu = \dfrac{3(x^2+y^2+z^2)}{3} = x^2+y^2+z^2$ —— (II)

& Product of the roots $= -d/a$

$uvw = (x^3+y^3+z^3)/3$ —— (III)

equations (I), (II) & (III) Can be written as

$$f_1 = u+v+w-x-y-z=0$$

$$f_2 = uv+vw+wu-x^2-y^2-z^2=0$$

$$f_3 = uvw - \tfrac{1}{3}(x^3+y^3+z^3)$$

Now

$$\frac{\partial(f_1 f_2 f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial y} & \frac{\partial f_3}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$$= (-1)(-2)(-1)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ x & -(x-y) & z-x \\ x^2 & -(x^2-y^2) & z^2-x^2 \end{vmatrix} = -2(x-y)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & -1 & 1 \\ x^2 & -(x+y) & z+x \end{vmatrix}$$

$$= -2(x-y)(z-x)[1\ (-z-\cancel{x}+\cancel{x}+y)$$

$$= -2(x-y)(y-z)(z-x)$$

&

$$\frac{\partial(f_1\ f_2\ f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix}$$

$C_2 \to C_2 - C_1 \qquad C_3 \to C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & -(w-u) \\ vw & w(u-v) & -v(w-u) \end{vmatrix}$$

$$= (u-v)(w-u)\begin{vmatrix} 1 & 0 & 0 \\ v+w & 1 & -1 \\ vw & w & -v \end{vmatrix}$$

$$= -(u-v)(v-w)(w-u)$$

$$\therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \cdot \frac{\partial(f_1\ f_2 f_3)\big/\partial(x, y, z)}{\partial(f_1\ f_2\ f_3)\big/\partial(u, v, w)}$$

$$= \frac{(\cancel{-1})(-2)(x-y)(y-z)(z-x)}{\cancel{(-1)}(u-v)(v-w)(w-u)}$$ Ans