Lecture 13

Properties of Beta function

(i) $\beta(m,n) = \beta(n,m)$ Hint $x = 1-t$

(ii) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$ Hint $x = \sin^2\theta$

(iii) $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$ Put $x = \frac{t}{1+t}$

(iv) $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

Relation between Beta and Gamma function :-

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Proof:- $\Gamma m = \int_0^{\infty} x^{m-1} e^{-x} dx = 2\int_0^{\infty} u^{2m-1} e^{-u^2} du$; Put $x = u^2$, $dx = 2u\,du$

$\therefore \Gamma n = \int_0^{\infty} x^{n-1} e^{-x} dx = 2\int_0^{\infty} v^{2n-1} e^{-v^2} dv$

$$\Gamma m \, \Gamma n = 4\int_0^{\infty}\int_0^{\infty} u^{2m-1} v^{2n-1} e^{-(u^2+v^2)} du\, dv \quad \text{—— (1)}$$

Changing into ~~plo~~ polar coordinates, $u = r\cos\theta$, $v = r\sin\theta$

we get $du\,dv = r\,dr\,d\theta$; $\theta : 0 \to \pi/2$; $r : 0 \to \infty$

From (1);

$$\Gamma m \, \Gamma n = 4\int_0^{\pi/2}\int_0^{\infty} \cos^{2m-1}\theta \cdot \sin^{2n-1}\theta \cdot r^{2m+2n-1} e^{-r^2} dr\, d\theta$$

$$= 4\left[\int_0^{\infty} r^{2m+2n-1} e^{-r^2} dr\right]\left[\int_0^{\pi/2} \cos^{2m-1}\theta \cdot \sin^{2n-1}\theta \, d\theta\right]$$

$$= 2\beta(m,n)\int_0^{\infty} r^{2m+2n-1} e^{-r^2} dr$$

Put $r^2 = t$, $r = \sqrt{t}$, $dr = \frac{1}{2\sqrt{t}} dt$, $t : 0 \to \infty$

$$= \beta(m,n)\int_0^{\infty} t^{m+n-1} e^{-t} dt$$

$$\Gamma m \, \Gamma n = \beta(m,n) \, \Gamma m+n$$

$$\therefore \beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \quad \text{proved}$$

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Problem (11) Given that $\int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}$,

show that $\Gamma p\, \Gamma 1-p = \frac{\pi}{\sin p\pi}$.

Solution:-

$$I = \int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx = \int_0^{\infty} \left(\frac{y}{1-y}\right)^{p-1} \cdot \frac{1}{\left(1+\frac{y}{1-y}\right)} \frac{1}{(1-y)^2}\, dy$$

Put $\frac{x}{1+x} = y$

$\Rightarrow x = \frac{y}{1-y}$

$dx = \frac{1}{(1-y)^2}\, dy$

$y: 0 \to \infty$

$$= \int_0^{\infty} \frac{y^{p-1}}{(1-y)^{p-1}} \frac{(1-y)}{(1-y)^2}\, dy$$

$$= \int_0^{\infty} \frac{y^{p-1}}{(1-y)^p}\, dy = \int_0^{\infty} y^{p-1} (1-y)^{-p}\, dy$$

$$= \int_0^{\infty} y^{p-1} (1-y)^{(1-p)-1}\, dy = \beta(p, 1-p)$$

i.e. $\frac{\pi}{\sin p\pi} = \beta(p, 1-p) = \frac{\Gamma p\, \Gamma 1-p}{\Gamma p+1-p} = \frac{\Gamma p\, \Gamma 1-p}{\Gamma 1}$

$$\Rightarrow \boxed{\Gamma p\, \Gamma 1-p = \frac{\pi}{\sin p\pi}} \qquad \because \Gamma 1 = 0! = 1$$

Exercise :- Using Beta and Gamma function

(i) Evaluate :- $\int_{-1}^{1} (1-x^2)^n\, dx$ Ans $\frac{2^{2n+1} (n!)^2}{\lfloor 2n+1}$

(ii) Show that

$$\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta = \frac{(2m-2)(2m-4)\cdots 2}{(2m-1)(2m-3)\cdots 3}$$

$$\int_0^{\pi/2} \sin^{2m}\theta\, d\theta = \frac{(2m-1)(2m-3)\cdots 1}{2m(2m-2)\cdots 2} \frac{\pi}{2}.$$

Problem:- Express $\int_0^1 x^m (1-x^p)^n dx$ in terms of Beta function and hence evaluate the integral

$$\int_0^1 x^{3/2}(1-\sqrt{x})^{1/2} dx$$

Solⁿ:- $I = \int_0^1 x^m (1-x^p)^n dx$

Let $x^p = y$

$x = y^{1/p}$

$dx = \frac{1}{p} y^{1/p - 1} dy$

$y: 0 \to 1$

$$= \int_0^1 y^{m/p}(1-y)^n \frac{1}{p} y^{(1/p - 1)} dy$$

$$= \frac{1}{p}\int_0^1 y^{\left(\frac{m+1}{p}-1\right)}(1-y)^{n+1-1} dy$$

$$\boxed{I = \frac{1}{p}\beta\left(\frac{m+1}{p}, n+1\right)} \quad \text{— (1)} \quad \text{Ans}$$

Now, $\int_0^1 x^{3/2}(1-\sqrt{x})^{1/2} dx = \frac{1}{1/2}\beta\left(\frac{3/2+1}{1/2}, \frac{1}{2}+1\right)$

$m = 3/2$

$p = 1/2$

$n = 1/2$

$$= 2\beta\left(5, \frac{3}{2}\right) = 2\frac{\Gamma 5\, \Gamma 3/2}{\Gamma 13/2}$$

$$= \frac{512}{3465} \quad \text{Ans}$$

Exercise:- (1) Show that, $\Gamma 2n = \frac{2^{2n-1}}{\sqrt{\pi}} \Gamma n+\frac{1}{2} \cdot \Gamma n$

and $\Gamma \frac{1}{4}\, \Gamma \frac{3}{4} = \pi\sqrt{2}$

Q (2) Show that $\int_0^{\pi/2} \sqrt{\tan x}\, dx = \frac{\pi}{\sqrt{2}}$

Proof:- $I = \int_0^{\pi/2} \sqrt{\tan x}\, dx = \int_0^{\pi/2} \sin^{1/2}x \cos^{-1/2}x\, dx = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$

$$= \frac{1}{2}\frac{\Gamma 3/4\, \Gamma 1/4}{\Gamma 1} = \frac{1}{2}\pi\sqrt{2} = \frac{\pi}{\sqrt{2}} \quad \text{Ans}$$