Prob:- Show that the vector field $\vec{F} = 2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$

is conservative. Find its scalar potential and the work done in moving a particle from $(-1,2,1)$ to $(2,3,4)$.

Proof:- We have

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x(y^2+z^3) & 2x^2y & 3x^2z^2 \end{vmatrix}$$

$$= \hat{i}[0-0] + \hat{j}[6xz^2 - 6xz^2] + \hat{k}[4xy - 4xy]$$

$$= 0$$

Therefore, the vector field $\vec{F}$ is conservative. We have

$\vec{F} = \text{grad} f$. Hence

$$2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

i.e. $\frac{\partial f}{\partial x} = 2x(y^2+z^3)$, $\frac{\partial f}{\partial y} = 2x^2y$, $\frac{\partial f}{\partial z} = 3x^2z^2$.

Integrating the first equation, we get

$$f(x,y,z) = x^2(y^2+z^3) + g(y,z) \quad \text{--- (1)}$$

Substituting in the 2nd eqn

$$2x^2y = 2x^2y + g'(y,z) \Rightarrow g'(y,z) = 0$$

integrating, we get $g = h(z)$.

$$\therefore f(x,y,z) = x^2(y^2+z^3) + h(z) \quad \text{--- (2)}$$

Substituting in the 3rd eqn

$$3x^2z^2 = 3x^2z^2 + h'(z) \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$$

$$\therefore f(x,y,z) = x^2(y^2+z^3) + C$$

Therefore, the scalar potential is given by

$$f(x,y,z) = x^2(y^2+z^3) + C.$$

Hence, the work done by $\vec{F}$ in moving a particle from $P(-1,2,1)$ to $Q(2,3,4)$ is

$$W = \int_P^Q F \cdot d\vec{r} = \left[f(x,y,z)\right]_P^Q = \left[x^2(y^2+z^3)\right]_{(-1,2,1)}^{(2,3,4)}$$

$$= 287.$$

<u>Circulation</u> :- A line integral of a vector field $\vec{V}$ around a simple closed curve C is defined as the circulation of $\vec{V}$ around C.

$$\text{Circulation} = \oint_C \vec{V} \cdot d\vec{r} = \oint_C \vec{V} \cdot \frac{d\vec{r}}{ds}\, ds = \oint_C \vec{V} \cdot T\, ds$$

If Circulation > 0, then the fluid tends to rotate ~~do~~ C in the anti-clockwise dirc.

Circulation < 0, then - - - - - - - clockwise . . .

If Circulation $= 0$, then - - - - . direction $\perp r$ to T.

<u>Thm</u>:- $\int_C f\,dx + g\,dy + h\,dz$ is independent of path C iff

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x} \text{ and } \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

<u>Remark</u> ① If $\vec{F} = f\hat{i} + g\hat{j} + h\hat{k}$. Then

$$\int_C f\,dx + g\,dy + h\,dz = \int_C \vec{F} \cdot d\vec{r}$$

② If F is a conservative force field, the work done along any simple closed path is zero.

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Q6 :- Show that $\int_C (yz-1)\,dx + (z+xz+z^2)\,dy + (y+xy+2yz)\,dz$ is independent of the path of integration from (1,2,2) to (2,3,4). Evaluate the integral.

Solution :- We have

$f(x,y,z) = yz-1$, $g(x,y,z) = z+xz+z^2$

and $h(x,y,z) = y+xy+2yz$

Now, $\frac{\partial f}{\partial y} = z = \frac{\partial g}{\partial x}$, $\frac{\partial f}{\partial z} = y = \frac{\partial h}{\partial x}$ and $\frac{\partial g}{\partial z} =$

$\frac{\partial g}{\partial z} = 1+x+2z = \frac{\partial h}{\partial y}$.

Hence, the integral is independent of path.

Therefore, there exist a function $\phi(x,y,z)$ such that

$\frac{\partial \phi}{\partial x} = yz-1$, $\frac{\partial \phi}{\partial y} = z+xz+z^2$ and $\frac{\partial \phi}{\partial z} = y+xy+2yz$.

Integrating the first eqn w.r.to x, we get

$\phi(x,y,z) = xyz - x + h(y,z)$

Substituting in (2); we get

$z+xz+z^2 = xz + h'(y,z) \Rightarrow h'(y,z) = z+z^2$

$\Rightarrow \frac{\partial h(y,z)}{\partial y} = z+z^2 \Rightarrow h(y,z) = zy+z^2y+s(z)$

$\therefore \phi(x,y,z) = xyz - x + zy + yz^2 + s(z)$

Substituting in (3); we get

$y+xy+2yz = xy+y+2yz+\frac{\partial s}{\partial z} \Rightarrow \frac{\partial s}{\partial z} = 0 \Rightarrow s(z) = c$

Hence, $\phi(x,y,z) = xyz - x + yz + yz^2 + c$

The value of integral is

$$\int_C (yz-1)dx + (z+xz+z^2)\,dy + (y+xy+2yz)\,dz$$

$$= \Big[(xyz - z + yz + yz^2)\Big]_{(1,2,2)}^{(2,3,4)}$$

$$= 82-15 = 67.$$