Lecture 6

Fourier series of $f(x)$ in $(0, 2L)$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where $a_0 = \frac{2}{L}\int_0^{2L} f(x)\,dx$, $a_n = \frac{2}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$$b_n = \frac{2}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier series of $f(x)$ in $(0, 2\pi)$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

Half Range Expansion Procedure to expand a non-periodic function $f(x)$ defined in half of the $(0, 2L)$ say $(0, L)$ of length L, are known as half range Fourier series.

Fourier half range cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{2}{L}\int_0^{L} f(x)\,dx$ and $a_n = \frac{2}{L}\int_0^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

Fourier half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right);\ \text{where}$$

$$b_n = \frac{2}{L}\int_0^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Problem: ① Find the Fourier sine and cosine series of the function $f(x) = K$ in the interval $0 < x < 5$.

Soln:- $a_0 = \frac{2}{5}\int_0^5 f(x)\,dx = \frac{2}{5}\int_0^5 K\,dx = 2K$

$a_n = \frac{2}{5}\int_0^5 f(x)\cos\left(\frac{n\pi x}{5}\right)dx = \frac{2K}{5}\int_0^5 \cos\left(\frac{n\pi x}{5}\right)dx$

$= \frac{2K}{5}\left(\frac{5}{n\pi}\right)\left[\sin\left(\frac{n\pi x}{5}\right)\right]_0^5 = 0$

Hence Fourier cosine series is $\frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right)$

$= \frac{2K}{2} = K$

Fourier sine series $= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

$b_n = \frac{2}{5}\int_0^5 f(x)\sin\left(\frac{n\pi x}{5}\right)dx = \frac{2K}{5}\left[\cos\left(\frac{n\pi x}{5}\right)\right]_0^5$

$= -\frac{2K}{5}\left[\cos\left(\frac{5n\pi}{5}\right) - 1\right] \times \frac{5}{n\pi}$

$= \frac{2K}{n\pi}\left[1 - \cos\left(\frac{5n\pi}{5}\right)\right] = \frac{2K}{n\pi}\left[1 - (-1)^n\right]$

Hence sine series is

$= \frac{2K}{n\pi}\sum_{n=1}^{\infty}\left[1-(-1)^n\right]\sin\left(\frac{n\pi x}{5}\right)$ Ans

Prob ②:- Find the Fourier sine and cosine series of the function $f(x)$

$$f(x) = \begin{cases} x & ; \quad 0 < x < 2 \\ 2 & \quad 2 \le x < 4 \end{cases}$$

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Solⁿ:- $L=4$

Fourier cosine series is

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) \quad \text{——— (1)}$$

where $a_0 = \frac{2}{4}\int_0^4 f(x)\,dx = \frac{1}{2}\left[\int_0^2 x\,dx + \int_2^4 2\,dx\right]$

$$= \frac{1}{2}\left[\left[\frac{x^2}{2}\right]_0^2 + 2(4-2)\right] = \frac{1}{2}[2+4] = \frac{6}{2} = 3$$

$$a_n = \frac{2}{4}\int_0^4 f(x)\cos\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{1}{2}\left[\int_0^2 x\cos\left(\frac{n\pi x}{4}\right)dx + \int_2^4 2\cos\left(\frac{n\pi x}{4}\right)dx\right]$$

$$= \frac{8}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1\right] \quad \{\text{after calculation}\}$$

Hence, from (1); required fourier cosine series

$$= \frac{3}{2} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos\left(\frac{n\pi x}{4}\right) \quad \text{Ans}$$

Fourier sine series is $= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right)$

where $b_n = \frac{2}{4}\int_0^4 f(x)\sin\left(\frac{n\pi x}{4}\right)dx$

$$= \frac{1}{2}\left[\int_0^2 x\sin\left(\frac{n\pi x}{4}\right)dx + \int_2^4 2\cdot\sin\left(\frac{n\pi x}{4}\right)dx\right]$$

$$= \frac{4}{\pi}\left[\frac{2}{\pi n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n}\cos(n\pi)\right]$$

Hence, Fourier sine series is

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{2}{\pi n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n}\cos(n\pi)\right]\sin\left(\frac{n\pi x}{4}\right)$$

Ans

Ex:- (1) $f(x) = x$; $0 < x < 2$

(2) $f(x) = x + x^2$; $0 < x < \pi$