Solution of One-dimensional Heat equation by Bender-Schmidt method:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad (1)$$

where $c^2 = \frac{k}{s\rho}$ is the diffusivity of the substance (cm²/sec).

Schmidt method: Consider a rectangular mesh in the x-t plane with spacing h along x direction and k along time t-direction. Denoting a mesh pt $(x, t) = (ih, jk)$ as simply i, j, we have

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \frac{u_{i,j+1} - u_{i,j}}{k} \\ \& \quad \frac{\partial^2 u}{\partial x^2} &= \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \end{aligned}\right\}$$

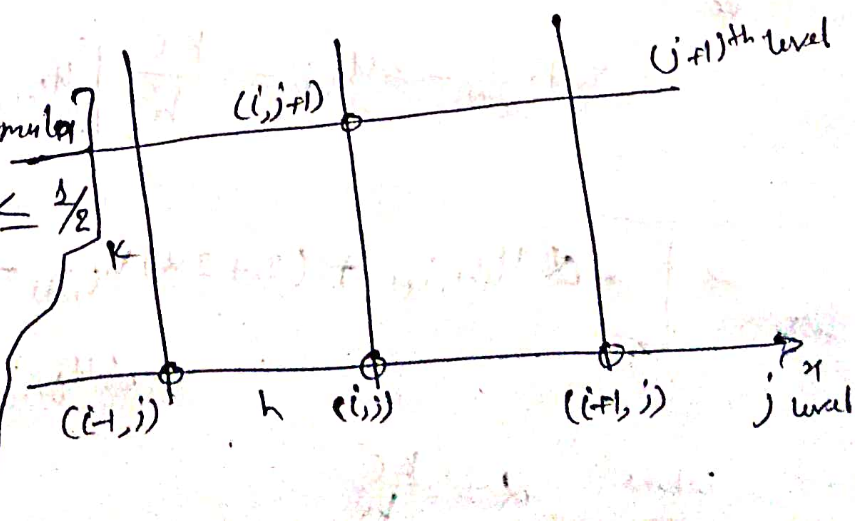
Substituting these in (1), we obtain

$$u_{i,j+1} - u_{i,j} = \frac{kc^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] \qquad (2)$$

or $$\boxed{u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha)u_{i,j} + \alpha u_{i+1,j}}$$ two level formula

where $\alpha = \frac{kc^2}{h^2}$ is the mesh ratio parameter.

[is called Schmidt explicit formula & it is valid only for $0 < \alpha \le 1/2$]



I) In particular when $\alpha = \frac{1}{2}$, (2) reduces to

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right) \qquad (3)$$

called Bendre–Schmidt recurrence relation, gives the value of u at the internal mesh pts with the help of boundary conditions.

→ Crank–Nicolson method:

In this method, we replace $\frac{\partial^2 u}{\partial x^2}$ by the average of its central-difference approximations on the j^{th} and $(j+1)^{th}$ time rows. So, (1) is reduced to

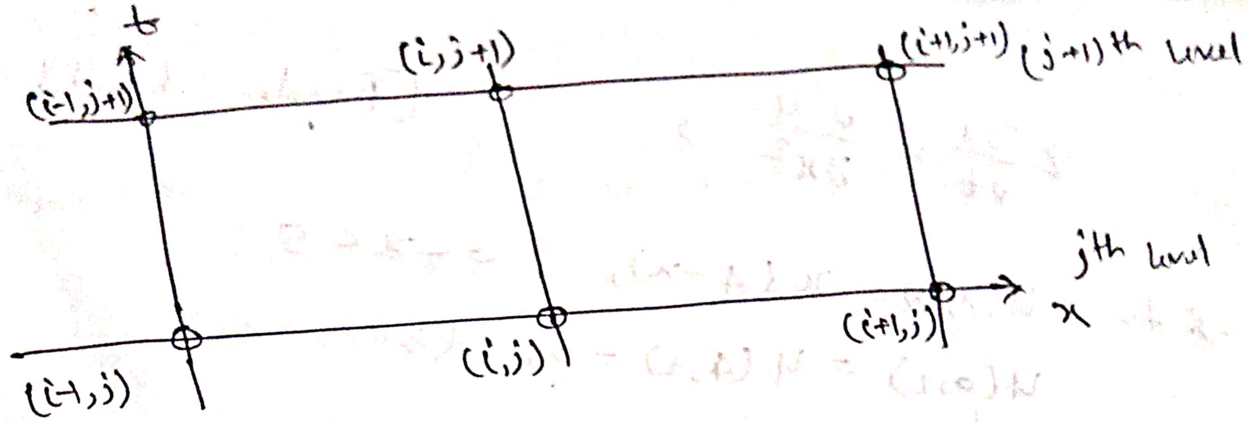
$$\frac{u_{i,j+1} - u_{i,j}}{k} = c^2 \frac{1}{2}\left[\left\{\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right\} + \left\{\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right\}\right]$$

or

$$2u_{i,j+1} - 2u_{i,j} = \frac{kc^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\right]$$

$$\Rightarrow \left[-\alpha u_{i-1,j+1} + (2+2\alpha)u_{i,j+1} - \alpha u_{i+1,j+1} = \alpha u_{i-1,j} + (2-2\alpha)u_{i,j} + \alpha u_{i+1,j}\right] \qquad (4)$$

where $\alpha = \frac{kc^2}{h^2}$



(4) contains three unknown values of u at the (j+1)th level while all the three values on the right are known values at the jth level. Thus (4) is a two-level implicit method and is known as Crank-Nicolson formula

[It is convergent for all values of α.]

→ If there are n-internal mesh pts on each row, then the relation (4) gives n-simultaneous eqns for the n-unknown values in terms of the known boundary values.

Exa
(29)
$$2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$ [Bender Schmidt].

s.t. $u(x,0) = x(4-x)$, $0 < x < 5$
$u(0,t) = u(4,t) = 0$, $t \geq 0$,

taking $h = 1$., Find the values of u upto $t = 5$.

(I) $\alpha = \frac{kc^2}{h^2} = k \times \frac{\frac{1}{2}}{1} = \frac{k}{2}$, $\alpha = 1/2 \Rightarrow k = 1$.

(30) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$ Crank – Nicolson.

s.t. $u(x,0) = 0$
$u(0,t) = 0$ & $u(1,t) = t$. for two time steps.

$h = 0.2$, $\alpha = \frac{kc^2}{h^2} = \frac{k \cdot 1}{0.04}$

(38) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

s.t. $u(x,0) = x^2(25 - x^2)$, $0 < x < 5$. Bender Schmidt.
taking $h = 1$, $k = 1/2$
$u(0,t) = u(5,t) = 0$, $t \geq 0$

(40) $$16\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \ t > 0$$

$u(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = 100t$

Compute u for one step in t-direction taking $h = 1/4$.

(29) **Solution**

From the given eq^n, we can write

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}.$$

Comparing it to $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, we get

$$c^2 = \frac{1}{2}$$

Given that $h=1$, and in case of Bender-Schmidt formula

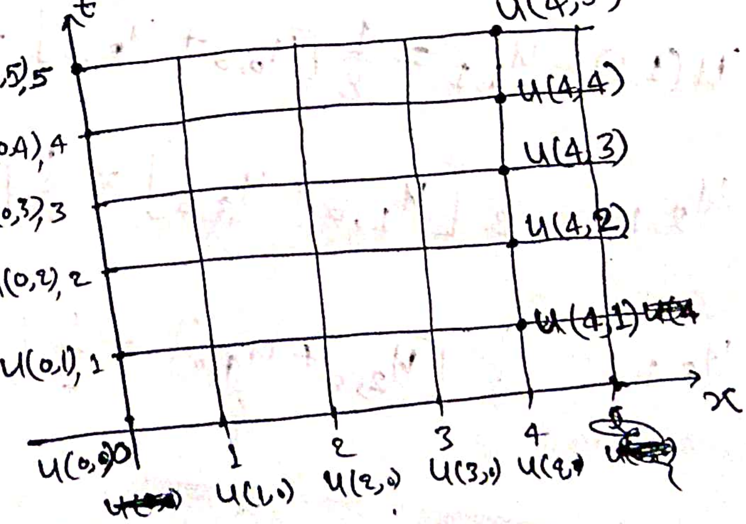
$\alpha = \frac{1}{2}$; where $\alpha = \frac{kc^2}{h^2} \Rightarrow \frac{1}{2} = \frac{k \times 1/2}{1} \Rightarrow k = 1.$

Given conditions are (Initial & boundary)

$$u(x,0) = x(4-x), \quad 0<x<5$$
$$u(0,t) = u(4,t) = 0, \quad t \geq 0$$

We have to calculate $u_{i,j}$: values of u at grid pts upto $t=5$.

Now, with $h=1$, $k=1$, we plot the network as follows:



As $u(x,0) = x(4-x)$, so that for $0<x<5$

$$u(1,0) = 1(4-1) = 3$$
$$u(2,0) = 2(4-2) = 4$$
$$u(3,0) = 3(4-3) = 3$$
$$u(4,0) = 0$$

~~u(5,0) = 5(4-5) = -5~~

$u(x,0) = x(4-x)$ for $0 < x < 5$.

We denote $u_{i,j} = u(ih, jk)$, $\because h = 1 = k$.

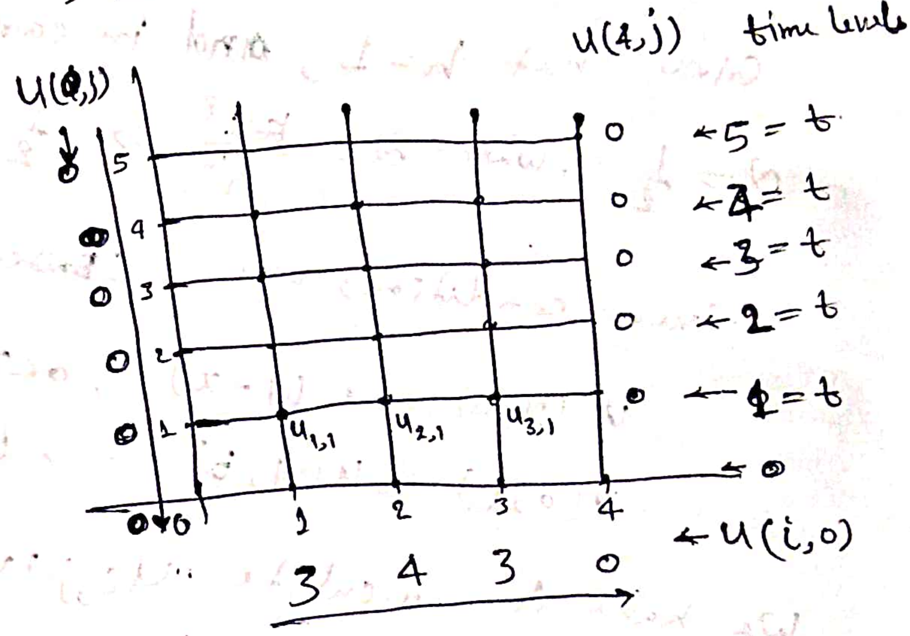
By conditions

$$u(0,t) = u(4,t) = 0 \quad , \forall\, t \geq 0$$

$$\Rightarrow u(0,0) = u(0,1) = u(0,2) = u(0,3) = u(0,4) = u(0,5) = 0$$

$$\& \; u(4,0) = u(4,1) = u(4,2) = u(4,3) = u(4,4) = u(4,5) = 0$$

From the given data, we have



By Bender–Schmidt formula

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right],$$

At time level $t = 1$,

$$u(1,1) \equiv u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{1}{2}[0+4] = 2$$

$$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{1}{2}[3+3] = 3$$

$$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{1}{2}[4+0] = 2$$

At time level $t=2$

$$u_{1,2} = \frac{1}{2}[u_{0,1} + u_{2,1}] = \frac{1}{2}[0+3] = 1.5$$

$$u_{2,2} = \frac{1}{2}[u_{1,1} + u_{3,1}] = \frac{1}{2}[2+2] = 2$$

$$u_{3,2} = \frac{1}{2}[u_{2,1} + u_{4,1}] = \frac{1}{2}[3+0] = 1.5$$

At time level $t=3$

$$u_{1,3} = \frac{1}{2}[u_{0,2} + u_{2,2}] = \frac{1}{2}[0+2] = 1$$

$$u_{2,3} = \frac{1}{2}[u_{1,2} + u_{3,2}] = \frac{1}{2}[1.5+1.5] = 1.5$$

$$u_{3,3} = \frac{1}{2}[u_{2,2} + u_{4,2}] = \frac{1}{2}[2+0] = 1$$

At time level $t=4$

$$u_{1,4} = \frac{1}{2}[u_{0,3} + u_{2,3}] = \frac{1}{2}[0+1.5] = 0.75$$

$$u_{2,4} = \frac{1}{2}[u_{1,3} + u_{3,3}] = \frac{1}{2}[1+1] = 1$$

$$u_{3,4} = \frac{1}{2}[u_{2,3} + u_{4,3}] = \frac{1}{2}[1.5+0] = 0.75$$

At time level $t=5$

$$u_{1,5} = \frac{1}{2}[u_{0,4} + u_{2,4}] = \frac{1}{2}[0+1] = 0.5$$

$$u_{2,5} = \frac{1}{2}[u_{1,4} + u_{3,4}] = \frac{1}{2}[0.75+0.75] = 0.75$$

$$u_{3,5} = \frac{1}{2}[u_{2,4} + u_{4,4}] = \frac{1}{2}[1+0] = 0.5$$

We see that network is symmetric about line $x = 2h = 2\times1 = 2$ i.e $x=2$. for respective values of heat at grid points.

2.30 $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$. ①

s.t $u(x,0)=0$ & $u(0,t)=0$, $u(1,t)=t$.

Evaluate all $u_{i,j}$ for two time steps.

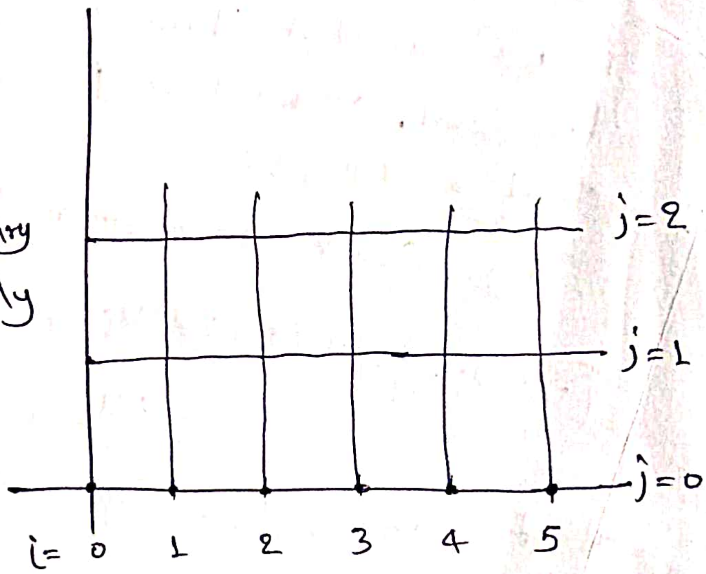
Comparing ① with eqn. $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, we get $c^2 = 1$.

Now, taking $h = 0.2$, then $\alpha = \dfrac{k c^2}{h^2} = \dfrac{k \times 1}{0.04}$

taking $\alpha = 1$, we get $1 = \dfrac{k}{0.04} \Rightarrow k = 0.04$.

Now, we have $h = 0.2$, $k = 0.04$ and $\alpha = 1$.

Denoting $u_{i,j} = u(ih, jk)$



Since values of u at $x=0$ and $x=1$ are given as boundary values, so we ~~could~~ take only grid points for $0<x<1$.

i) $u(x,0)=0$, given

So, $u_{0,0} = u(0\times h, 0\times k) = u(0,0) = 0$

$u_{1,0} = u(1\times h, 0\times k) = u(0.2, 0) = 0$

$u_{2,0} = u(2\times 0.2, 0\times 0.04) = u(0.4, 0) = 0$ [$\because h = 0.2$, $k = 0.04$]

$u_{3,0} = u(3\times 0.2, 0\times 0.04) = u(0.6, 0) = 0$ ["].

$u_{4,0} = u(4\times 0.2, 0\times 0.04) = u(0.8, 0) = 0$ ["].

$u_{5,0} = u(5\times 0.2, 0\times 0.04) = u(1, 0) = 0$ ["].

ii) $u(0,t) = 0$, given therefore

$u_{0,1} = u(0\times h, 1\times k) = u(0, 0.04) = 0.$

$u_{0,2} = u(0\times h, 2\times k) = u(0, 0.08) = 0$

iii) $u(1,t) = t$, so

$u_{5,0} = u(5\times 0.2, 0\times 0.04) = u(1,0) = 0$

$u_{5,1} = u(5\times 0.2, 1\times 0.04) = u(1, 0.04) = 0.04$

$u_{5,2} = u(5\times 0.2, 2\times 0.04) = u(1, 0.08) = 0.08.$

Now, we evaluate $u_{i,j}$ at time level ~~j=0~~ $j=1$ i.e $t=0.04$.

As we see that there are four interior mesh points at each time level, so we have four equations corresponding to each time level as follows:

By Crank - Nicolson method.

$$[-\alpha u_{i-1,j+1} + (2+2\alpha)u_{i,j+1} - \alpha u_{i+1,j+1} = \alpha u_{i-1,j} + (2-2\alpha)u_{i,j} + \alpha u_{i+1,j}] \quad -(2)$$

at time level $j=1$, i.e. $t=0.04$.

$$-u_{0,1} + (2+2\times 1)u_{1,1} - u_{2,1} = u_{0,0} + (2-2\times 1)u_{1,0} + 1\cdot u_{2,0}$$

or (i) $-u_{0,1} + 4u_{1,1} - u_{2,1} = 0 \quad [\because u_{0,0} = 0 = u_{1,0} = u_{2,0}].$

Similarily $-u_{1,1} + 4u_{2,1} - u_{3,1} = u_{1,0} + 0 + u_{2,0}$

or (ii) $-u_{1,1} + 4u_{2,1} - u_{3,1} = 0 \quad [\because u_{1,0} = 0 = u_{2,0}].$

and (iii) $-u_{2,1} + 4u_{3,1} - u_{4,1} = 0 \quad [\because u_{2,0} = 0 = u_{3,0} = u_{4,0}]$

$-u_{3,1} + 4u_{4,1} - u_{5,1} = u_{3,0} + 0\times u_{4,0} + u_{5,0}$

(iv) ⇒ $-u_{3,1} + 4u_{4,1} - 0.04 = 0 \quad [\because u_{3,0} = 0 = u_{5,0}]$ & $[u_{5,1} = 0.04]$

or

i) $4u_{1,1} - u_{2,1} = 0$

ii) $-u_{1,1} + 4u_{2,1} - u_{3,1} = 0$

iii) $-u_{2,1} + 4u_{3,1} - u_{4,1} = 0$

iv) $-u_{3,1} + 4u_{4,1} - 0.04 = 0$

$[\because u_{0,1} = 0]$.

From i), ii), iii) ~~4 iv)~~, we get

$u_{2,1} = 4u_{1,1}$

$u_{3,1} = -u_{1,1} + 4 \times 4u_{1,1} = 15u_{1,1}$

$u_{4,1} = -u_{2,1} + 4u_{3,1} = -4u_{1,1} + 4 \times 15u_{1,1} = 56u_{1,1}$

and ~~Putting~~ substituting $u_{3,1}$ & $u_{4,1}$ in (iv) we get

$-15u_{1,1} + 4 \times 56u_{1,1} = 0.04$

$\Rightarrow \quad 209u_{1,1} = 0.04$

$\Rightarrow \quad u_{1,1} = \frac{0.04}{209}$

$= 0.000191$

Therefore, $u_{2,1} = 0.000764$

$u_{3,1} = 15 \times u_{1,1} = 0.002865$

$u_{4,1} = 56 \times u_{1,1} = 0.010696$

Similarity, by using eqn (2), we can evaluate all $u_{i,j}$ at time level $j = 2$ i.e at $t = 0.08$.