Composite functions of several variables

If $z = f(x,y)$ and x and y are both functions of the variable 't'. Then $z = f(x,y)$ is said to be Composite functions of variables x and y.

x, y are intermediate variable

t : independent variable.

Chain rule:- If $z = f(x(t), y(t))$, where $x(t)$ and $y(t)$ are differentiable and $f(x,y)$ is a differentiable function of x and y, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

OR,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Prob(1) By using the chain rule find $\frac{df}{dt}$ where $f(x,y) = x^2 e^y$, $x(t) = t^2 - 1$ and $y(t) = \sin t$.

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Solution:- $f_x = 2xe^y$, $f_y = x^2e^y$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = \cos t.$$

$\therefore$ By the chain rule,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= 2xe^y \times 2t + x^2e^y \times \cos t$$

$$\frac{df}{dt} = 4xte^y + x^2e^y\cos t$$

$$= 4(t^2-1)te^{\sin t} + (t^2-1)^2e^{\sin t}\cos t \quad \underline{\text{Ans}}$$

Prob (2):- Find the derivative $g'(t)$, where $g(t) = f(x(t), y(t))$, $f(x,y) = x^2y - \sin y$, $x(t) = \sqrt{t^2+1}$, $y(t) = e^t$.

Solution:- $f_x = 2xy$, $f_y = x^2 - \cos y$

$$\frac{dx}{dt} = \frac{2t}{2\sqrt{t^2+1}} = \frac{t}{\sqrt{t^2+1}}, \quad \frac{dy}{dt} = e^t$$

By chain rule

$$g'(t) = \frac{dg}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2xy)\frac{t}{\sqrt{t^2+1}} + (x^2 - \cos y)e^t$$

$$= 2e^t\sqrt{t^2+1}\cdot\frac{t}{\sqrt{t^2+1}} + (t^2+1-\cos e^t)e^t$$

$$= 2te^t + (t^2+1-\cos e^t)e^t$$

$$g'(t) = (2t + t^2 + 1 - \cos e^t)e^t \quad \underline{\text{Ans}}$$

Lecture 19

Chain Rule

(i) For single variable $w = f(x)$; $x = g(t)$

$$\therefore \frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

(ii) For two variable $w = f(x, y)$; $x = x(t)$, $y = y(t)$

Then,

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Branch Diagram The branch diagram in the margin provides a convenient way to remember the chain rule.

Ex. ① Use the chain rule to find the derivative of $w = xy$ w.r.to 't' along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$

Sol^n:- We apply the chain rule

$x = \cos t$
$\frac{dx}{dt} = -\sin t$
$y = \sin t$
$\frac{dy}{dt} = \cos t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$= y(-\sin t) + x(\cos t)$$

$$\frac{dw}{dt} = -\sin^2 t + \cos^2 t$$

at $t = \pi/2$ $\quad \frac{dw}{dt} = \cos 2(\pi/2) = \cos \pi = -1$

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Chain Rule for One independent variable

$$w = f(x,y,z) \quad ; \quad x,y,z \equiv f(t)$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

Chain Rule for two independent variable

$$w = f(x,y,z) \quad ; \quad x,y,z \equiv g(r,s)$$

Then,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}.$$

Ex (3) Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $w = x+2y+z^2$; $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$.

Ans $\frac{\partial w}{\partial r} = \frac{1}{s} + 12r$, $\frac{\partial w}{\partial s} = \frac{2}{s} - \frac{r}{s^2}$.

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Implicit Differentiation

(i) $F(x,y)=0$.

Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$

Ex ⑤ Find $\dfrac{dy}{dx}$ if $y^2-x^2-\sin xy=0$

Solⁿ:- Take $F(x,y)=y^2-x^2-\sin xy$.

Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{-2x-y\cos xy}{2y-x\cos xy}$

$= \dfrac{2x+y\cos xy}{2y-x\cos xy}$

(ii) $z=f(x,y)$

$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$

Ex. ⑥ Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

if $x^3+z^2+ye^{xz}+z\cos y=0$.

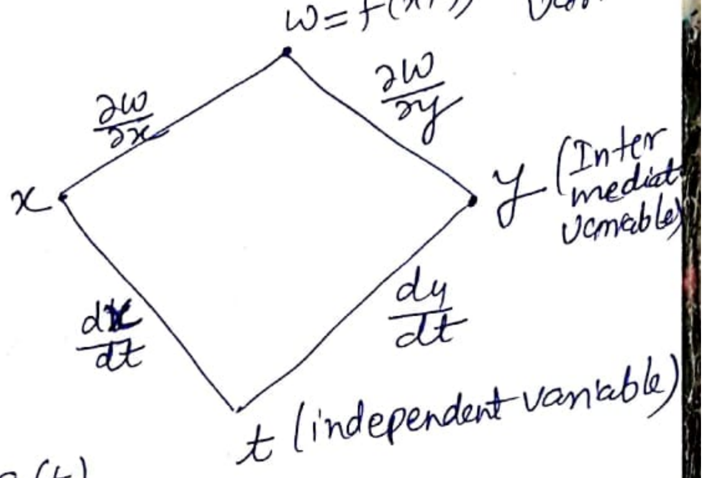
Ans $\dfrac{\partial z}{\partial x} = 0$, $\dfrac{\partial z}{\partial y} = -1$

Question Bank

(30) Draw a branch diagram and write a chain rule for derivative of a function of 1 independent variable and 2 intermediate variables.

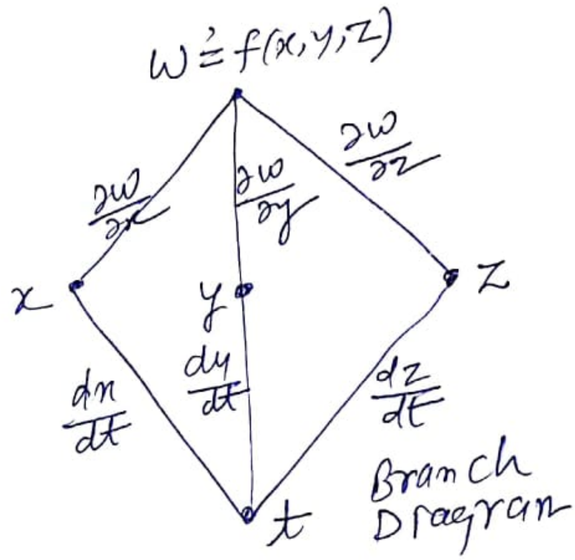
$$w = f(x,y)$$
$$x = g(t),\ y = h(t)$$
$$w = H(t)$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$



(31) $w = f(x,y,z)$; $x, y, z \equiv g(t)$
$$w = H(t)$$

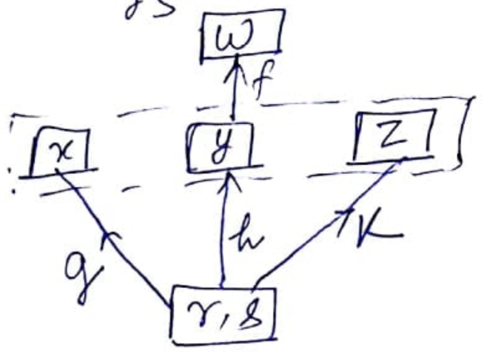
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

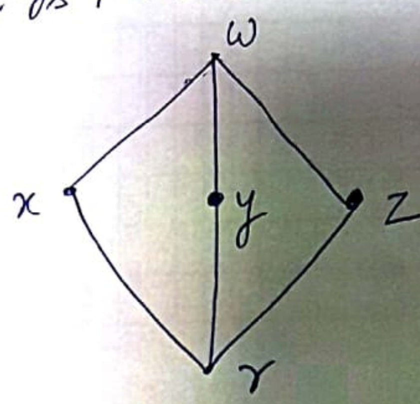


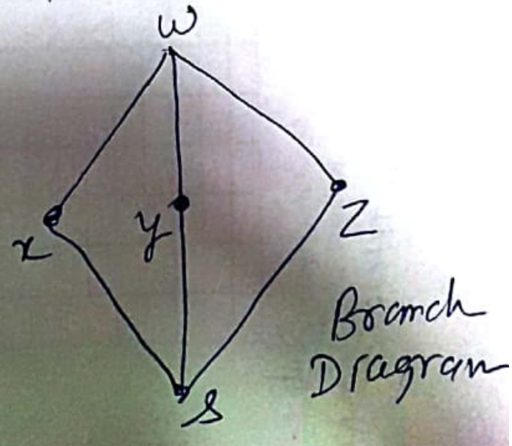
(32) $w = f(x,y,z)$;
$$x, y, z \equiv g(r,s)$$

$$\frac{\partial w}{\partial r} = w_x x_r + w_y y_r + w_z \cdot z_r$$

$$\frac{\partial w}{\partial s} = w_x \cdot x_s + w_y \cdot y_s + w_z \cdot z_s$$







(37) $f(u,v,w)$ differentiable,

$u = x-y$, $v = y-z$, $w = z-x$

To prove, $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$

Now, $u_x = 1, u_y = -1, u_z = 0$

$v_x = 0, v_y = 1, v_z = -1$

$w_x = -1, w_y = 0, w_z = 1$

Now, By Chain Rule

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$\frac{\partial f}{\partial x} = 1\cdot f_u + 0 - 1\cdot f_w = f_u - f_w \quad \text{— (i)}$$

$$\frac{\partial f}{\partial y} = f_u\cdot u_y + f_v\cdot v_y + f_w\cdot w_y$$
$$= -f_u + f_v + 0 = -f_u + f_v \quad \text{— (ii)}$$

Next, $\frac{\partial f}{\partial z} = f_u\cdot u_z + f_v\cdot v_z + f_w\cdot w_z$

$$= 0 - f_v + f_w = -f_v + f_w \quad \text{— (iii)}$$

Adding (i), (ii) & (iii) ; we obtain

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = f_u - f_w - f_u + f_v - f_v + f_w = 0$$

Proved

(38) $w = f(u,v)$ satisfies $f_{uu} + f_{vv} = 0$ — (i)

$u = \frac{x^2 - y^2}{2}$; $v = xy$

$u_x = x,\ u_y = -y$ | $v_x = y.\ v_y = x$; $v_{xx} = 0 = v_{yy}$

$u_{xx} = 1,\ u_{yy} = -1$ | $v_{xy} = 1,\ v_{yx} = 1$

$u_{xy} = 0 = u_{yx}$

$$w_x = f_u \cdot u_x + f_v \cdot v_x = x f_u + y f_v$$

Then $w_{xx} = 1 \cdot f_u + x\left(\frac{\partial}{\partial x} f_u\right) + y \cdot \frac{\partial}{\partial x}(f_v)$

$$= f_u + x\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right) \cdot \frac{\partial v}{\partial x}\right]$$

$$+ y\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right) \cdot \frac{\partial v}{\partial x}\right]$$

$$= f_u + x\left[x f_{uu} + y f_{uv}\right]$$

$$+ y\left[x f_{uv} + y f_{vv}\right]$$

$$w_{xx} = f_u + x^2 f_{uu} + y^2 f_{vv} + 2xy f_{uv} \text{ — (ii)}$$

Now, $w_y = f_u \cdot u_y + f_v \cdot v_y = -y f_u + x f_v$

Then, $w_{yy} = -f_u - y\left[\frac{\partial}{\partial y} f_u\right] + x\left[\frac{\partial}{\partial y} f_v\right]$

$$= -f_u - y\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) \cdot \frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right) \cdot \frac{\partial v}{\partial y}\right]$$

$$+ x\left[\frac{\partial}{\partial u}(f_v) \cdot \frac{\partial u}{\partial y} + \frac{\partial}{\partial v}(f_v) \cdot \frac{\partial v}{\partial y}\right]$$

$$= -f_u - y\left[-y f_{uu} + x f_{uv}\right]$$

$$+ x\left[-y f_{uv} + x f_{vv}\right]$$

$$= -f_u + y^2 f_{uu} + x^2 f_{vv} - 2xy f_{uv} \text{ — (iii)}$$

Adding (ii) and (iii); we get

$w_{xx} + w_{yy} = f_u - f_u + x^2(f_{uu} + f_{vv}) + y^2(f_{uu} + f_{vv}) + 0$ (Using (i))

$= 0$ proved