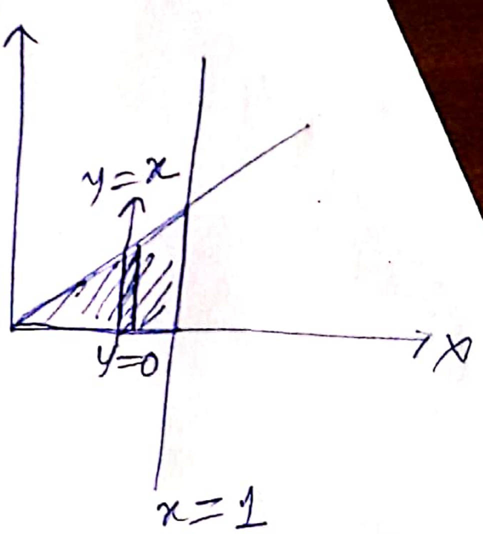
33. $y=x,\ x=1,\ x\text{-axis},\ z=3-x-y$

Limits are $x: 0 \to 1$
$y: 0 \to x$



$$\text{Volume} = \int_0^1 \int_0^x z\, dy\, dx$$

$$= \int_0^1 \left[\int_0^x (3-x-y)\, dy\right] dx$$

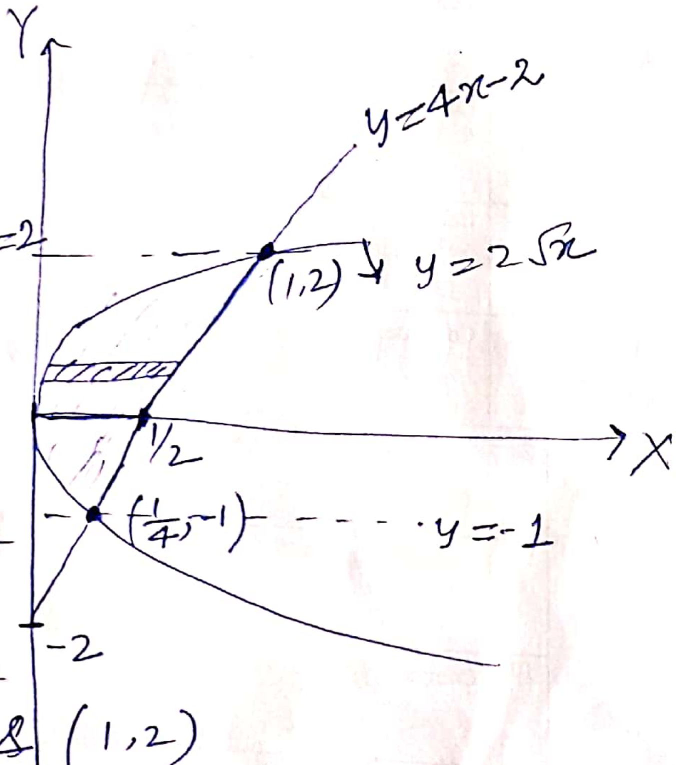
$$= \int_0^1 \left[(3-x)y - \frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left[(3-x)x - \frac{x^2}{2}\right] dx = \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = 3\int_0^1 \left(x - \frac{x^2}{2}\right) dx$$

$$= 3\left[\frac{x^2}{2} - \frac{x^3}{6}\right]_0^1 = 3\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{3(3-1)}{6} = \frac{6}{6} = 1$$

34. $z = 16 - x^2 - y^2,\ y = 2\sqrt{x},\ y = 4x-2,\ x\text{-axis}$

$y^2 = 4x,\quad \frac{x}{1/2} + \frac{y}{-2} = 1$



$(4x-2)^2 = 4x$

$\Rightarrow 16x^2 + 4 - 16x = 4x$

$\Rightarrow 16x^2 - 20x + 4 = 0$

$\Rightarrow 4x^2 - 5x + 1 = 0$

$\Rightarrow (4x^2 - 4x) - x + 1 = 0$

$\Rightarrow 4x(x-1) - 1(x-1) = 0$

$\Rightarrow (4x-1)(x-1) = 0 \Rightarrow x = 1, 1/4$

$\therefore y = 4(1) - 2 = 2$ for $x = 1$

$y = 4\left(\frac{1}{4}\right) - 2 = -1$ for $x = 1/4$

Two intersection points $\left(\frac{1}{4}, -1\right)$ & $(1, 2)$

Limits are:- $x: \frac{y^2}{4} \to \frac{1}{4}(y+2)$
$y: -1 \to 2$

Volume of bounded region

$$V = \int_{-1}^{2} \int_{y^2/4}^{\frac{1}{4}(y+2)} z\, dx\, dy = \int_{-1}^{2} \int_{y^2/4}^{\frac{1}{4}(y+2)} (16 - x^2 - y^2)\, dx\, dy$$

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$$V = \int_{-1}^{2} \left[(16-y^2)x - \frac{x^3}{3} \right]_{y^2/4}^{\frac{1}{4}(y+2)} dy$$

$$= \int_{-1}^{2} \left\{ (16-y^2)\left[\frac{y+2}{4} - \frac{y^2}{4} \right] - \frac{1}{3}\left[\left(\frac{1}{4}(y+2)\right)^3 - \left(\frac{y^2}{4}\right)^3 \right] \right\} dy$$

$$= \int_{-1}^{2} \left\{ \frac{(16-y^2)(y+2-y^2)}{4} - \frac{1}{3}\left[\frac{y^3+8+3y^2+6y}{64} - \frac{y^6}{64} \right] \right\} dy$$

$$= \int_{-1}^{2} \left\{ \frac{[16y+32-16y^2-y^3-2y^2+y^4]}{4} - \frac{1}{3\times 64}(-y^6+y^3+3y^2+6y+8) \right\} dy$$

$$= \int_{-1}^{2} \left\{ \frac{(y^4-y^3-18y^2+16y+32)}{4} - \frac{1}{3\times 64}(-y^6+y^3+3y^2+6y+8) \right\} dy$$

$$= \frac{1}{3\times 64} \int_{-1}^{2} (48y^4 - 48y^3 - 48\times 18y^2 + 48\times 16y + 48\times 32 + y^6 - y^3 - 3y^2 - 6y - 8)\, dy$$

$$= \frac{1}{192} \int_{-1}^{2} (y^6 + 48y^4 - 52y^3 - 867y^2 + 762y + 1528)\, dy$$

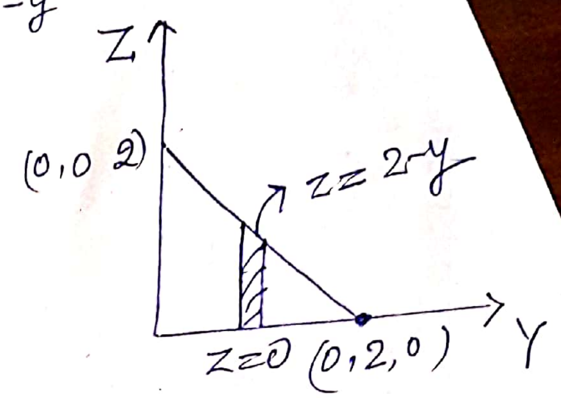
$$= \frac{1}{192} \left[\frac{y^7}{7} + \frac{48}{5}y^5 - \frac{52}{4}y^4 - \frac{867}{3}y^3 + \frac{762}{2}y^2 + 1528y \right]_{-1}^{2}$$

$$= \frac{1}{192} \left[\frac{129}{7} + \frac{48}{5}(33) - 13(15) - 289(9) + 381(3) + 1528(3) \right]_{-1}$$

50 $y+z=2$ and cylinder $x=4-y^2$

For $x=0$: $4-y^2=0 \Rightarrow y=2$

Limits for in first octant are



$x: 0 \to 4-y^2$

$y: 0 \to 2$

$z: 0 \to 2-y$

Volume of bounded region is

$$V = \int_0^2 \int_0^{4-y^2} \int_0^{2-y} dz\,dx\,dy$$

$$= \int_0^2 \int_0^{4-y^2} (2-y)\,dx\,dy$$

$$= \int_0^2 (2-y)(4-y^2)\,dy$$

$$= \cancel{\int_0^2 8dy} \int_0^2 (8-4y-2y^2+y^3)\,dy$$

$$= \left[8y-\frac{4}{2}y^2-\frac{2}{3}y^3+\frac{y^4}{4}\right]_0^2$$

$$= \left[8y-2y^2-\frac{2}{3}y^3+\frac{y^4}{4}\right]_0^2$$

$$= 8(2)-2(2^2)-\frac{2}{3}(2^3)+\frac{1}{4}(2^4)$$

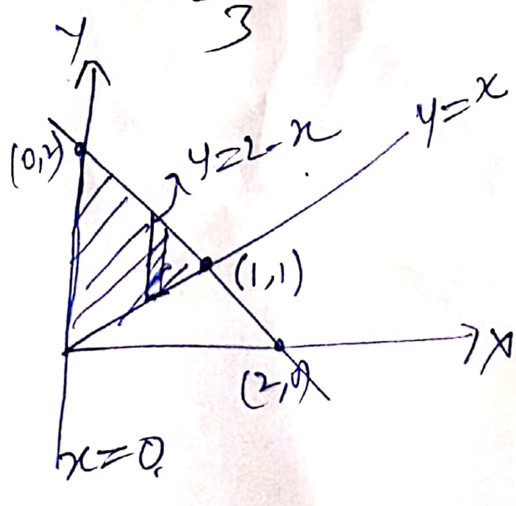
$$= 16-8-\frac{16}{3}+4 \quad = 12-\frac{16}{3} = \frac{36-16}{3}$$

$$\cancel{= \frac{24-16}{3} = \frac{72-16}{3}} \qquad = \frac{20}{3}$$

35 $y=x, x=0, x+y=2, z=x^2+y^2$

Limits for $x: 0 \to 1$

$y: x \to 2-x$



Volume $V = \int_0^1 \int_x^{2-x} (x^2+y^2)\,dy\,dx$

$$= \int_0^1 \left[x^2y+\frac{y^3}{3}\right]_x^{2-x} dx$$

$$V=\int_0^1 \left\{x^2[(2-x)-x]+\frac{1}{3}\left[(2-x)^3-x^3\right]\right\}dx$$

$$=\int_0^1\left\{2x^2-2x^3+\frac{1}{3}\left[8-x^3-12x+6x^2-x^3\right]\right\}dx$$

$$=\int_0^1\left\{2x^2-2x^3-\frac{2x^3}{3}+2x^2-4x+\frac{8}{3}\right\}dx$$

$$=\int_0^1\left(-\frac{8}{3}x^3+4x^2-4x+\frac{8}{3}\right)dx=\left[-\frac{8}{3\times4}x^4+\frac{4}{3}x^3-\frac{4}{2}x^2+\frac{8}{3}x\right]_0^1$$

$$=-\frac{2}{3}+\frac{4}{3}-2+\frac{8}{3}=\frac{-2+4-6+8}{3}=\frac{4}{3}$$

---

49. $$V=\int_0^1\int_{-1}^1\int_0^{y^2}dz\,dy\,dx$$

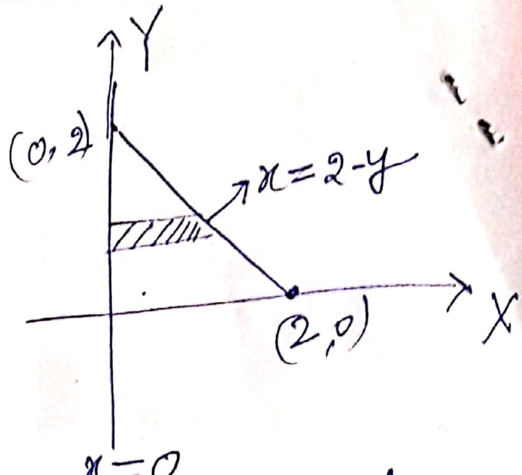
$$=\int_0^1\left[\int_{-1}^1 y^2dy\right]dx$$

$$=\int_0^1\frac{1}{3}\left[y^3\right]_{-1}^1dx=\frac{2}{3}\int_0^1dx=2/3$$

52. $x+y=2$, cylinder $y^2+4z^2=16$

$\Rightarrow \frac{x}{2}+\frac{y}{2}=1,\quad z=\frac{1}{2}\sqrt{16-y^2}$

$x=2-y$



For $z=0$, $16-y^2=0 \Rightarrow y=4$

Limits are :-

$x: 0 \to 2-y$

$y: 0 \to 4$

$z: 0 \to \frac{1}{2}\sqrt{16-y^2}$

in first octant.

Volume of bounded surface is

$$V=\int_0^4\int_0^{2-y}\int_0^{\frac{1}{2}\sqrt{16-y^2}} dz\,dx\,dy$$

$$=\int_0^4\int_0^{2-y}\frac{1}{2}\sqrt{16-y^2}\,dx\,dy = \int_0^4 \frac{(2-y)}{2}\sqrt{16-y^2}\,dy$$

Put $16-y^2=t^2$

$-2y\,dy=2t\,dt$

$-y\,dy=t\,dt$

$t: 4\to 0$

$$=\int_0^4\sqrt{16-y^2}\,dy-\frac{1}{2}\int_0^4 y\sqrt{16-y^2}\,dy$$

$$=\left[\frac{y}{2}\sqrt{16-y^2}+\frac{16}{2}\sin^{-1}\left(\frac{y}{4}\right)\right]_0^4+\frac{1}{2}\int_4^0 t.t\,dt$$

$$=0+8\sin^{-1}(1)-(0)-\frac{1}{2}\int_0^4 t^2\,dt$$

$$=8\left(\frac{\pi}{2}\right)-\frac{1}{2\times 3}\left[t^3\right]_0^4 = 4\pi-\frac{4\times4\times4}{2\times3}$$

$$\boxed{V=4\pi-\frac{32}{3}}$$ Ans

Question Bank 4

48 Find the volume of the region enclosed by the surfaces $z = x^2+3y^2$ and $z = 8-x^2-y^2$.

Solⁿ:- $x^2+3y^2 = 8-x^2-y^2$

$\Rightarrow 2x^2+4y^2 = 8 \Rightarrow x^2+2y^2 = 4$ ellipse

$y = \pm\frac{1}{2}\sqrt{4-x^2}$

$\therefore \quad y : -\frac{1}{2}\sqrt{4-x^2} \to \frac{1}{2}\sqrt{4-x^2}$

$x : -2 \to 2$

$z : x^2+3y^2 \to 8-x^2-y^2$

The volume of D is

$$V = \iiint_D dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} (8-2x^2-4y^2)\,dy\,dx$$

$$= \int_{-2}^{2} \left[(8-2x^2)y - \frac{4}{3}y^3\right]_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \frac{4\sqrt{2}}{3}(4-x^2)^{3/2}\,dx = 8\pi\sqrt{2}$$