Lecture 8

Taylor's Theorem :- ~~If function $f(x)$ is a~~

If $f(x)$ is a function and

(i) $f^{(n-1)}$ is continuous in $[a, a+h]$

(ii) $f^{(n)}$ exists in $(a, a+h)$.

Then there exists at least one number ~~θ~~ ~~lq~~ $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + h\frac{f'(a)}{1!} + \frac{h^2}{2!}f''(a) + \cdots$$

$$\cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n \quad \text{——— (1)}$$

where, $R_n = \dfrac{h^n(1-\theta)^{n-p}}{p\,\lfloor n-1} f^{(n)}(a+\theta h)$; $0<\theta<1$.

where p is given +ve integer.

Problem (1) Verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 2 terms in $[0,1]$.

Sol^n:- $f(x) = (1-x)^{5/2}$ here $\boxed{n=2}$

$f'(x) = -\frac{5}{2}(1-x)^{3/2}$ continuous in $[0,1]$

$f''(x) = +\frac{15}{4}(1-x)^{1/2}$ differentiable in $(0,1)$.

Thus, $f(x)$ satisfies the conditions of Taylor's thm.

In Taylor's thm with Lagrange's form of remainder upto 2 terms,

n = no. of terms in the remainder $= 2 = p$

$a=0$, $x=1$ $\because$ interval is $[0,1]$.

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Thus, from Taylor thm,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x) \quad ; 0 < \theta < 1 \quad (1)$$

Now, $f(0) = 1$, $f'(0) = -\frac{5}{2}$, $f''(\theta h) = \frac{15}{4}(1-\theta)^{1/2}$

and $f(1) = 0$.

Putting $x = 1$ in eqn (1), we get

$$0 = 1 - \frac{5}{2} + \frac{15}{2!\,4}(1-\theta)^{1/2}$$

$$\Rightarrow \quad \theta = \frac{9}{25} = 0.36 \in (0,1).$$

Thus verifying the Taylor's thm.

Problem:- Expand the polynomial

$f(x) = x^5 + 2x^4 - x^2 + x + 1$ in powers of $x+1$.

OR

Obtain the Taylors series expansion of $f(x)$ about the point $x = -1$.

Solution:- $f(-1) = -1 + 2 - 1 - 1 + 1 = 0$

$f'(x) = 5x^4 + 8x^3 - 2x + 1 \quad \therefore f'(-1) = 5 - 8 + 2 + 1 = 0$

$f''(x) = 20x^3 + 24x^2 - 2 \quad \therefore f''(-1) = -20 + 24 - 2 = 2$

$f'''(x) = 60x^2 + 48x \quad \therefore f'''(-1) = 60 - 48 = 12$

$f^{IV}(x) = 120x + 48 \quad \therefore f^{IV}(-1) = -120 + 48 = -72$

$f^{V}(x) = 120$, $f^{VI}(x) = 0$

Taylors series expansion of $f(x)$ about $x = -1$ is

$$f(x) = f(-1) + (x+1) f'(-1) + \frac{(x+1)^2}{2!} f''(-1) + \frac{(x+1)^3}{3!} f'''(-1) + \frac{(x+1)^4}{4!} f^{IV}(-1) + \frac{(x+1)^5}{5!} f^{V}(-1) + 0 \cdots$$

$$= 0 + 0 + \frac{(x+1)^2}{2} \cdot 2 + \frac{(x+1)^3}{3!}(12) + \frac{(x+1)^4}{4!}(-72) + \frac{(x+1)^5}{5!}(120) + \cdots$$

$$f(x) = (x+1)^2 + 2(x+1)^3 - 3(x+1)^4 + (x+1)^5 \quad \underline{\text{Ans}}$$

Prob ②:- Write Taylor's formula for the function $y=\sqrt{x}$ with Lagrange's remainder with $a=1$, $n=3$.

Solⁿ:- $y=\sqrt{x}=f(x)$, $f(a)=f(1)=1$

$f'(x)=\frac{1}{2\sqrt{x}}$ $\therefore f'(1)=\frac{1}{2}$

$f''(x)=-\frac{1}{4}x^{-3/2}$ $\therefore f''(1)=-1/4$

$f'''(x)=\frac{3}{8}x^{-5/2}$ $\therefore f'''(1)=3/8$

$f^{IV}(x)=\frac{-15}{16}x^{-7/2}$ $\therefore f^{IV}(1)=-15/16$

Taylor's formula with Lagrange's remainder upto 4 terms (i.e. $n=3$)

$$f(x)=f(a)+(x-a)\frac{f'(a)}{1!}+(x-a)^2\frac{f''(a)}{2!}+\frac{(x-a)^3}{3!}f'''(a)$$
$$+\frac{(x-a)^4}{4!}f^{IV}(a+\theta(x-a)).$$

At $a=1$

$$f(x)=1+(x-1)\cdot\frac{1}{2}+(x-1)^2\left(-\frac{1}{4}\right)\frac{1}{2!}+(x-1)^3\frac{3}{8}\cdot\frac{1}{3!}$$
$$+(x-1)^4\left(-\frac{15}{16}\right)\frac{1}{4!}\frac{1}{[1+\theta(x-1)]^{7/2}}$$

$$f(x)=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3$$
$$-\frac{5}{128}(x-1)^4[1+\theta(x-1)]^{-7/2}$$ Ans

Exercise

~~Problem~~ ① Expand $\sin x$ in powers of $(x-\pi/2)$.

② Using Taylor's series find approx. value of $\sqrt{10}$.

③ Find Taylor series expansion of $f(x)$ about 'a' where

(i) $f(x)=\ln x$, $a=1$

(ii) $f(x)=\tan x$, $a=\pi/4$

(iii) $f(x)=\ln\cos x$, $a=\pi/3$.

Indeterminate Forms

If two functions $f(x)$ and $g(x)$ are both zero at $x = a$ the fraction $\frac{f(a)}{g(a)}$ is called indeterminate form $\frac{0}{0}$. Although the function $F(x) = \frac{f(x)}{g(x)}$ is undefined at $x = 0$.

L'Hospital's Rule (Type $\frac{0}{0}$ form)

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad ; \quad f(a) = 0 = g(a) \text{ But } g'(a) \neq 0$$

Problem:- ① Evaluate

(a) $\lim_{x \to 1} \frac{1 + \ln x - x}{1 - 2x + x^2}$ (b) $\lim_{x \to 0} \frac{e^{ax} - e^{-ax}}{\ln(1 + bx)}$

(c) $\lim_{x \to \frac{\pi}{4}} \left(\frac{\sec^2 x - 2\tan x}{1 + \cos 4x}\right)$ (d) $\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x \sin^3 x}$

(e) $\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3}$ is finite. Find 'a' and the limit.

Soln:- (a) At $x = 1$, $\frac{1 + \ln x - x}{1 - 2x + x^2} = \frac{0}{0}$ indeterminate.

Applying L'Hospital's rule

$$\lim_{x \to 1} \frac{1 + \ln x - x}{1 - 2x + x^2} = \lim_{x \to 1} \frac{0 + \frac{1}{x} - 1}{-2 + 2x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{-1/x^2}{2} \quad \text{(Applying L'Hospital rule)}$$

$$= -\frac{1}{2} \text{ Ans.}$$

(b) Ans $\frac{2a}{b}$ (c) $\frac{1}{2}$ (d) $\frac{1}{12}$

(e) $\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3} = \lim_{x \to 0} \frac{2\cos 2x + a \cos x}{3x^2} \to \infty$

But limit is finite, we choose $a = -2$, then $\frac{0}{0}$ form

$$\therefore = \lim_{x \to 0} \frac{-4\sin 2x + 2\sin x}{6x} = \lim_{x \to 0} \frac{-8\cos 2x + 2\cos x}{6}$$

$$= -\frac{6}{6} = -1$$

Type $\frac{\infty}{\infty}$ form

Problem (1) Evaluate

(a) $\lim\limits_{x\to\infty} \frac{x^n}{e^x}$ (b) $\lim\limits_{x\to\infty} \frac{x}{\sqrt{1+x^2}}$ (c) $\lim\limits_{x\to 0} \log_{\tan x} \tan 2x$

Soln:- (a) $\lim\limits_{x\to\infty} \frac{x^n}{e^x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \cancel{\lim_{x\to\infty} \frac{nx^{n-1}}{e^x}} \lim_{x\to\infty} \frac{\frac{d^n}{dx^n}x^n}{\frac{d^n}{dx^n}e^x} = \lim_{x\to\infty} \frac{n!}{e^x} = 0$$

(b) $\lim\limits_{x\to\infty} \frac{x}{\sqrt{1+x^2}}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{2}\times 2x\frac{1}{\sqrt{1+x^2}}} = \lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x} \quad : \frac{\infty}{\infty}\text{ form}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2}\times 2x\frac{1}{\sqrt{1+x^2}}}{1} = \lim_{x\to\infty} \frac{x}{\sqrt{1+x^2}} \quad : \left(\frac{\infty}{\infty}\text{ form}\right)$$

Put $z = 1/x^2$, then

$$\lim_{x\to\infty} \frac{x}{\sqrt{1+x^2}} = \lim_{z\to 0} \frac{1}{\sqrt{1+z}} = 1. \text{ Ans}$$

(c) $$\lim_{x\to 0} \log_{\tan x} \tan 2x = \lim_{x\to 0} \frac{\log \tan 2x}{\log \tan x} \quad : \frac{\infty}{\infty}$$

$$= \lim_{x\to 0} \frac{\frac{2\sec^2 2x}{\tan 2x}}{\frac{\sec^2 x}{\tan x}} \quad : \frac{0}{0}\text{ form}$$

$$= \lim_{x\to 0} \frac{2\sec^2 2x \cdot \tan x}{\sec^2 x \cdot \tan 2x} = \lim_{x\to 0} \frac{2\frac{\sin x}{\cos 2x \cos x}}{\frac{\sin 2x}{\cos 2x \cdot \cos^2 x}}$$

$$= \lim_{x\to 0} \frac{2\sin x \cdot \cos x}{\cos 2x \cdot \sin 2x}$$

$$= \lim_{x\to 0} \frac{1}{\cos 2x} = \frac{1}{1} = 1 \quad \text{Ans}$$