Volume as a Triple Integral

In cartesian co-ordinates,

$$V = \iiint_V dx\,dy\,dz$$

In cylindrical co-ordinates,

$$V = \iiint_V r\,dr\,d\phi\,dz$$

In spherical polar co-ordinates

$$V = \iiint_V r^2 \sin\theta\,dr\,d\theta\,d\phi$$

(1) Calculate the volume of the solid bounded by the surface $x=0, y=0, x+y+z=1$ and $z=0$,

Soln.:

$x : 0 \to 1$

$y : 0 \to 1-x$

$z : 0 \longrightarrow 1-x-y$

Required volume

$$V = \int_0^1 \int_0^{1-x} \int^{1-x-y} dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

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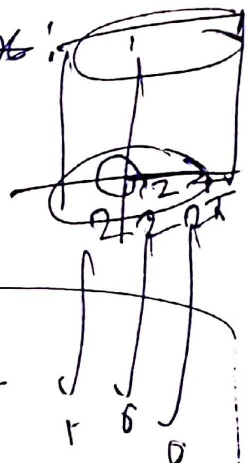
$$V = \int_0^1 \left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[(1-x)^2 - \frac{(1-x)^2}{2}\right] dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx$$

$$= -\frac{1}{6}\left[(1-x)^3\right]_0^1 = \frac{1}{6}.$$

(2) Evaluate $\iiint_D e^{x^2+y^2}\,dv$, where D is the solid ~~sphere~~ inside $x^2+y^2=4$ and between $z=1$ and $z=2$.

Sol^n: $z: 1 \to 2$, ~~$y: 0 \to \sqrt{4-x^2}$, x:~~



$y: -\sqrt{4-x^2} \to \sqrt{4-x^2}$

$x: -2 \to 2$

∴ ~~Volume,~~

$$V = \int_{z=1}^{2}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx\,dz$$

$$= 2\int_1^2\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx\,dz$$

$$= 2\int_1^2 e^{y^2}\int_{-2}^{2}\sqrt{4-x^2}\,dx = 4e^{4}\int_1^2\int_0^2\sqrt{4-x^2}\,dx$$

$$= 2\int_1^2\int_{-2}^{2} e^{x^2}\int_0^{\sqrt{4-x^2}} e^{y^2}\,dy\,dx\,dz$$

Put

$$V = \int_{z=1}^{2}\int_0^2\int_{\theta=0}^{2\pi} e^{r^2}\,r\,dr\,d\theta\,dz = 2\pi\int_1^2\left[\int_0^2 re^{r^2}\,dr\right]dz$$

$$= \pi\int_1^2 (e^4-1)\,dz = \pi(e^4-1)$$

Spherical Coordinates ~~$(\rho, \phi$~~, (r, θ, ϕ)

$x = r\sin\theta\cos\phi$, $y = r\sin\theta \cdot \sin\phi$, ~~$z = r\cos\phi$~~ $z = r\cos\theta$.

~~$y = r\sin\theta \sin$~~

$$V = \iiint_V r^2 \sin\theta \, dr\, d\theta\, d\phi.$$

Prob ① Find rectangular coordinates for the point described by $(8, \frac{\pi}{4}, \frac{\pi}{3})$ in spherical coordinates.

Solution:- $r = 8, \theta = \frac{\pi}{4}, \phi = \frac{\pi}{3}$

$x = r\sin\theta\cos\phi = 8\sin\frac{\pi}{4}\cos\frac{\pi}{3} = 8\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{4}{\sqrt{2}}$

$x = 2\sqrt{2}$

$y = r\sin\theta \cdot \sin\phi = 8\sin\frac{\pi}{4}\sin\frac{\pi}{3} = 8\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{6}$

$z = 8\cos\frac{\pi}{4} = 8\left(\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}$

$\therefore$ spherical coordinates $(2\sqrt{2}, 2\sqrt{6}, 4\sqrt{2})$.

Prob ②:- Rewrite the eqn of the cone $z^2 = x^2 + y^2$ in spherical coordinates.

Solution:- $(r\cos\theta)^2 = (r\sin\theta\cos\phi)^2 + (r\sin\theta \cdot \sin\phi)^2$

$\Rightarrow \boxed{r^2\cos^2\theta = r^2\sin^2\theta}$

Q1 Evaluate the triple integral $\iiint_Q \cos(x^2+y^2+z^2)^{3/2}\, dV$, where Q is the unit ball:

$$x^2+y^2+z^2 \le 1.$$

Solution:- $r^2 \le 1 \Rightarrow r: 0 \to 1;\ \theta: 0 \to \pi$

$\phi: 0 \to 2\pi$

$$I = \iiint_Q \cos(r^2)^{3/2}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} (\cos r^3) \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta \left[\int_{r=0}^{1} (\cos r^3) \cdot r^2 dr\right] d\theta\, d\phi$$

Put $r^3 = t$

$r^2 dr = \frac{dt}{3}$

$t: 0 \to 1$

$$= \frac{1}{3}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta \left[\int_0^1 \cos t\, dt\right] d\theta\, d\phi$$

$$= \frac{\sin 1}{3}\int_{\phi=0}^{2\pi}\left[\int_{\theta=0}^{\pi} \sin\theta\, d\theta\right] d\phi = -\frac{\sin 1}{3}\int_0^{2\pi}\left[\cos\theta\right]_0^{\pi} d\phi$$

$$= \frac{2}{3}\sin 1 \int_0^{2\pi} d\phi = 2\pi \times \frac{2}{3}\sin 1 = \frac{4\pi}{3}\sin 1$$

$\simeq \underline{3.525}$ Ans

Prob ②:- Find the volume lying inside the sphere $x^2+y^2+z^2=2z$ and inside the cone $z^2=x^2+y^2$.

Solution:- $\cancel{z^2=x^2+y^2=2z}$

$x^2+y^2+z^2=2z \Rightarrow x^2+y^2+(z^2-2z+1)=1$

$\Rightarrow x^2+y^2+(z-1)^2=1$

Sphere centered at $(0,0,1)$ radius $=1$.

Now, $x^2+y^2+z^2=2z \Rightarrow r^2=2r\cos\theta$

i.e. $r=2\cos\theta$.

$z^2=x^2+y^2 \Rightarrow r^2\cos^2\theta=r^2\sin^2\theta \Rightarrow \tan\theta=\pm 1$

$\Rightarrow \theta=\pm\frac{\pi}{4} \Rightarrow \theta=\frac{\pi}{4}$

Limits in spherical coordinates.

$r: 0 \to 2\cos\theta$

$\theta: 0 \to \pi/4$

$\phi: 0 \to 2\pi$

Required volume $V=\iiint dv$

$$V=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\theta} r^2\sin\theta\, dr\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi/4} \frac{r^3}{3}\Big|_0^{2\cos\theta} \sin\theta\, d\theta\, d\phi$$

$$=\frac{8}{3}\int_0^{2\pi}\left[\int_0^{\pi/4}\cos^3\theta\sin\theta\, d\theta\right]d\phi$$

Put $\cos\theta=t$
$\sin\theta\, d\theta=-dt$
$t: 1 \to \frac{1}{\sqrt{2}}$

$$=\frac{8}{3}\int_0^{2\pi}\left[\int_{1/\sqrt{2}}^{1} t^3\, dt\right]d\phi = \frac{2}{3}\int_0^{2\pi}\left(1-\left(\frac{1}{\sqrt{2}}\right)^4\right)d\phi$$

$=\frac{1}{2}\times 2\pi = \underline{\pi}$ Ans.

Q. (3) Evaluate the triple integral

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dy\, dx$$ by using changing into spherical coordinates.

Soln :-

$z: 0 \to \sqrt{4-x^2-y^2}$ i.e. $z^2 = 4-x^2-y^2$
$\Rightarrow r^2 = 4 \Rightarrow r = 2$

$y: 0 \to \sqrt{4-x^2}$
$\Rightarrow x^2+y^2 = 4 \Rightarrow r^2\sin^2\theta = 4 \Rightarrow r\sin\theta = 2$
$\Rightarrow \sin\theta = 1 \Rightarrow \theta = \pi/2$

$x: -2 \to 2$ i.e. $r\sin\theta\cos\phi = 2$
$\Rightarrow \cos\phi = 1 \Rightarrow \phi = 0$

$r\sin\theta\cos\phi = -2 \Rightarrow \cos\phi = -1 \Rightarrow \phi = \pi$

Limits in spherical coordinates

$r: 0 \to 2$; $\theta: 0 \to \pi/2$; $\phi: 0 \to \pi$.

$$\therefore I = \int_0^{\pi}\int_0^{\pi/2}\int_0^{2} r^2 \cdot (r^2\sin\theta)\, dr\, d\theta\, d\phi$$

$$= \frac{1}{5}\int_0^{\pi}\int_0^{\pi/2} \sin\theta \times 32\, d\theta\, d\phi = \frac{-32}{5}\int_0^{\pi} \left[\cos\theta\right]_0^{\pi/2} d\phi$$

$$= \frac{32}{5}\int_0^{\pi} d\phi = \frac{32\pi}{5}.$$ Ans