Divergence :-

let $\vec{V} = V_1(x,y,z)\hat{i} + V_2(x,y,z)\hat{j} + V_3(x,y,z)\hat{k}$ define a vector field.

$$\text{div}\,\vec{V} = \nabla\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Prob ① :- Find the divergence of the vector field

$$\vec{V} = (x^2y^2 - z^3)\hat{i} + 2xyz\hat{j} + e^{xyz}\hat{k}.$$

Solⁿ :- ~~Curr~~ divergence of $\vec{V}$ is

$$\nabla\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left((x^2y^2 - z^3)\hat{i} + 2xyz\hat{j} + e^{xyz}\hat{k}\right)$$

$$\nabla\cdot\vec{V} = \frac{\partial}{\partial x}(x^2y^2 - z^3) + \frac{\partial}{\partial y}(2xyz) + \frac{\partial}{\partial z}(e^{xyz})$$

$$\nabla\cdot\vec{V} = 2xy^2 + 2xz + xye^{xyz}$$ Ans

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Curl of a vector field $\vec{V}$

~~Eve~~ Curl of vector field $\vec{V}$, denoted by curl $\vec{V}$, is defined as

$$\operatorname{curl}\vec{V} = \nabla\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix}$$

if $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$.

Prob ①:- Find the curl of the vector field

$$\vec{V} = (x^2y^2 - z^3)\hat{i} + 2xyz\hat{j} + e^{xyz}\hat{k}.$$

Solution:- curl $\vec{V}$

$$= \nabla\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2y^2-z^3) & 2xyz & e^{xyz} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(e^{xyz}) - \frac{\partial}{\partial z}(2xyz)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x^2y^2-z^3) - \frac{\partial}{\partial x}(e^{xyz})\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(2xyz) - \frac{\partial}{\partial y}(x^2y^2-z^3)\right]$$

$$= (xze^{xyz} - 2xy)\hat{i} - (3z^2 + zye^{xyz})\hat{j}$$

$$+ (2yz - 2x^2y)\hat{k}.$$

Properties ① Curl of gradient

Let f be a differentiable scalar field. Then

$$\operatorname{curl}(\operatorname{grad} f) = 0 \text{ or } \nabla\times(\nabla f) = 0.$$

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(2) Divergence of curl:- Let $\vec{V}$ be a differentiable vector field. Then

$$\text{div}(\text{curl}\,\vec{V})=0 \quad \text{or} \quad \nabla\cdot(\nabla\times\vec{V})=0.$$

(3) If $\text{div}(\vec{V})=0$ ie $\nabla\cdot\vec{V}=0$, then the fluid is saido to be incompressible.

In electromagnetic theory, if $\text{div}(\vec{V})=0$, then the vector field $\vec{V}$ is said to be solenoidal.

(4) If $\text{curl}(\vec{V})=0$ ie. $\nabla\times\vec{V}=0$, then $\vec{V}$ is said to be an irrotational field.

(5) A force field F is said to be conservative if $\text{curl}(F) = \text{curl}(\text{grad}\,f)=0$

ie $\nabla\times(\nabla f)=0$,

and there exists a scalar potential function f such that $F = \text{grad}\,f$.

Prob:- Prove that

(1) $\text{div}(f\vec{V}) = f(\text{div}\,\vec{V}) + (\text{grad}\,f)\cdot\vec{V}$, where f is scalar function

(2) $\text{curl}(f\vec{V}) = (\text{grad}\,f)\times\vec{V} + f(\text{curl}\,\vec{V})$

(3) $\text{div}(\text{grad}\,f) = \nabla^2 f$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator.

(4) $\text{curl}(\text{curl}\,\vec{V}) = \nabla(\nabla\cdot\vec{V}) - \nabla^2\vec{V}$

or $\text{grad}(\text{div}\,\vec{V}) = \nabla\times(\nabla\times\vec{V}) + \nabla^2\vec{V}$.

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Proof (1):- LHS $\nabla\cdot(f\vec{V}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(fV_1\hat{i}+fV_2\hat{j}+fV_3\hat{k})$

$$= \frac{\partial}{\partial x}(fV_1)+\frac{\partial}{\partial y}(fV_2)+\frac{\partial}{\partial z}(fV_3)$$

$$= \left(V_1\frac{\partial f}{\partial x}+f\frac{\partial V_1}{\partial x}\right)+\left(V_2\frac{\partial f}{\partial y}+f\frac{\partial V_2}{\partial y}\right)+\left(V_3\frac{\partial f}{\partial z}+f\frac{\partial V_3}{\partial z}\right)$$

$$= f\left(\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}\right)+\left(V_1\frac{\partial f}{\partial x}+V_2\frac{\partial f}{\partial y}+V_3\frac{\partial f}{\partial z}\right)$$

$$= f(\nabla\cdot\vec{V})+\vec{V}\cdot\nabla f$$ proved

(2) $\text{Curl}(f\vec{V}) = \nabla\times f\vec{V} = \nabla\times(fV_1\hat{i}+fV_2\hat{j}+fV_3\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fV_1 & fV_2 & fV_3 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(fV_3)-\frac{\partial}{\partial z}(fV_2)\right]+\hat{j}\left[\frac{\partial}{\partial z}(fV_1)-\frac{\partial}{\partial x}(fV_3)\right] + \hat{k}\left[\frac{\partial}{\partial x}(fV_2)-\frac{\partial}{\partial y}(fV_1)\right]$$

$$= f(\nabla\times\vec{V}) + (\nabla f)\times\vec{V}$$

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Prob :- Find the curl and divergence of the functions:

(i) $f(x,y,z) = x^2y\,\hat{i} + (3x - yz)\,\hat{j} + z^3\,\hat{k}$

(ii) $f(x,y,z) = (x^3 - y)\,\hat{i} + y^5\,\hat{j} + e^z\,\hat{k}$.

Prob :- Determine whether the following vector fields are conservative:

(i) $F(x,y,z) = (\cos x - z)\,\hat{i} + y^2\,\hat{j} + xz\,\hat{k}$

(ii) $F(x,y,z) = 2xz\,\hat{i} + 3z^2\,\hat{j} + (x^2 + 6yz)\,\hat{k}$.

Then evaluate potential functions.

Solution :- For conservative, we need to show that

$$\nabla \times F = 0.$$

(i) $$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 3x - yz & z^3 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial z^3}{\partial y} - \frac{\partial}{\partial z}(3x - yz)\right] + \hat{j}\left[\frac{\partial (x^2y)}{\partial z} - \frac{\partial (z^3)}{\partial x}\right] + \hat{k}\left[\frac{\partial (3x - yz)}{\partial x} - \frac{\partial (x^2y)}{\partial y}\right]$$

$$= y\hat{i} + (3 - x^2)\hat{k}$$

(i) $$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x - z & y^2 & xz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial (xz)}{\partial y} - \frac{\partial y^2}{\partial z}\right] + \hat{j}\left[\frac{\partial}{\partial z}(\cos x - z) - \frac{\partial}{\partial x}(xz)\right] + \hat{k}\left[\frac{\partial y^2}{\partial x} - \frac{\partial}{\partial y}(\cos x - z)\right]$$

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$\therefore F$ is not conservative.

(ii) $$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & 3z^2 & x^2+6yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial(x^2+6yz)}{\partial y} - \frac{\partial(3z^2)}{\partial z}\right] + \hat{j}\left[\frac{\partial(2xz)}{\partial z} - \frac{\partial(x^2+6yz)}{\partial x}\right] + \hat{k}\left[\frac{\partial(3z^2)}{\partial x} - \frac{\partial(2xz)}{\partial y}\right]$$

$$= (6z-6z)\hat{i} + (2x-2x)\hat{j} + 0$$

$$= 0$$

$\Rightarrow F$ is conservative

then, there exists a potential function

$$\text{grad} f = F$$

$$\text{grad} f = 2xz\hat{i} + 3z^2\hat{j} + (x^2+6yz)\hat{k}$$

$$\Rightarrow \nabla f = \nabla(x^2z + 3yz^2)$$

$\Rightarrow f = x^2z + 3yz^2$ is the potential function.

Prob ①:- Find the constants a, b and c such that $\vec{V} = (3x+ay+z)\hat{i} + (2x-y+bz)\hat{j} + (x+cy+z)\hat{k}$ is irrotational.

Soln:- The vector field $\vec{V}$ is irrotational i.e.

$$\nabla \times \vec{V} = 0$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x+ay+z & 2x-y+bz & x+cy+z \end{vmatrix} = 0$$

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$$\Rightarrow \hat{i}\left[\frac{\partial}{\partial y}(x+cy+z)-\frac{\partial}{\partial z}(2x-y+bz)\right]$$

$$+\hat{j}\left[\frac{\partial}{\partial z}(3x+ay+z)-\frac{\partial}{\partial x}(x+cy+z)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(2x-y+bz)-\frac{\partial}{\partial y}(3x+ay+z)\right]=0$$

$$\Rightarrow (c-b)\hat{i}+(1-1)\hat{j}+(2-a)\hat{k}=0$$

$$\Rightarrow c-b=0 \Rightarrow b=c$$

$$2-a=0 \Rightarrow a=2$$

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