Line Integrals

Position vector of a point on the curve C can be written as $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $a \le t \le b$. —— (1)

Line integral w.r.to arc length

Let C be a simple smooth curve whose parametric eqn is as given in eqn (1). Let $f(x,y,z)$ be continuous on C. Then, we define the line integral of f over C w.r.to the arc length s by

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{x'^2+y'^2+z'^2}\,dt$$

Since $ds = \frac{ds}{dt}\cdot dt = \left|\frac{dr}{dt}\right| dt = \left(\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\right)dt$

Prob:- (1) Evaluate $\int_C x^2 y\,ds$, when C is the curve defined by $x = 3\cos t$, $y = 3\sin t$, $0 \le t \le \frac{\pi}{2}$.

Soln:- $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{(-3\sin t)^2+(3\cos t)^2} = \sqrt{9} = 3$

Therefore, $ds = \frac{ds}{dt}\cdot dt = 3dt$

$$\therefore \int_C x^2 y\,ds = \int_0^{\pi/2} (3\cos t)^2 (3\sin t)\,3dt = 81\int_0^{\pi/2} \cos^2 t\,\sin t\,dt$$

$$= -27\left[\cos^3 t\right]_0^{\pi/2} = 27.$$

~~Prob~~ ②:- Line integral of vector fields

Let C be ~~the~~ a smooth curve whose parametric ~~equation~~ representation is as given in ①. Let

$\vec{V}(x,y,z) = V_1(x,y,z)\hat{i} + V_2(x,y,z)\hat{j} + V_3(x,y,z)\hat{k}$.

be the vector field that is continuous on C.

Then, the line integral of $\vec{V}$ ~~of~~ over C is defined by

$$\int_C \vec{V}\cdot dr = \int_C V_1 dx + V_2 dy + V_3 dz.$$

Prob:- Evaluate the line integral of $\vec{V} = xy\hat{i} + y^2\hat{j} + e^z\hat{k}$ over the curve C whose parametric representation is given by $x = t^2, y = 2t, z = t, \quad 0 \le t \le 1$.

Solⁿ:- The position vector of any point on C is given by $\vec{r} = t^2\hat{i} + 2t\hat{j} + t\hat{k}$.

Then line integral is,

$$\int_C \vec{V}\cdot d\vec{r} = \int_0^1 \vec{V}\cdot\frac{d\vec{r}}{dt}\,dt$$

$$= \int_0^1 (xy\hat{i} + y^2\hat{j} + e^z\hat{k})\cdot(2t\hat{i} + 2\hat{j} + \hat{k})\,dt$$

$$= \int_0^1 (2t^3\hat{i} + 4t^2\hat{j} + e^t\hat{k})\cdot(2t\hat{i} + 2\hat{j} + \hat{k})\,dt$$

$$= \int_0^1 (4t^4 + 8t^2 + e^t)\,dt = \left[\frac{4}{5}t^5 + \frac{8}{3}t^3 + e^t\right]_0^1$$

$$= \left(\frac{4}{5} + \frac{8}{3} + e - 1\right) = \frac{37}{15} + e \quad \text{Ans}$$

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Prob:- Evaluate $\int_C (x+y)dx - x^2 dy + (y+z)dz$

where C is $x^2 = 4y$, $z = x$, $0 \le x \le 2$.

Soln:- We parametrise C as $x = t$, $y = \frac{t^2}{4}$
and $z = t$, $0 \le t \le 2$. Therefore,

$$I = \int_C (x+y)dx - x^2 dy + (y+z)dz$$

$$= \int_0^2 \left(t + \frac{t^2}{4}\right) dt - t^2 \frac{t}{2}\, dt + \left(\frac{t^2}{4} + t\right) dt$$

$$= \int_0^2 \left(2t + \frac{t^2}{2} - \frac{t^3}{2}\right) dt = \left[t^2 + \frac{t^3}{6} - \frac{t^4}{8}\right]_0^2$$

$$= \frac{10}{3}.$$

Prob:- Find the work done by the force $\vec{F} = -xy\hat{i} + y^2\hat{j} + z\hat{k}$ in moving a particle over the circular path $x^2+y^2=4$, $z=0$ from $(2,0,0)$ to $(0,2,0)$.

Soln:- The parametric representation of the given curve is
$x = 2\cos t$, $y = 2\sin t$, $z = 0$; $0 \le t \le \frac{\pi}{2}$.

Therefore, work done W is given by

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C -xy\,dx + y^2 dy + z\,dz$$

$$= \int_0^{\pi/2} (-4\sin t\cos t)(-2\sin t)\,dt + (4\sin^2 t)(2\cos t)\,dt$$

$$= 16\int_0^{\pi/2} \sin^2 t \cos t\, dt$$

Put $\sin t = z$
$\cos t\, dt = dz$
$z: 0 \to 1$

$$= 16\int_0^1 z^2 dz = \frac{16}{3}.$$ Ans

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Question Bank-5 (Solution)

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$$\int_C (x^2+yz)\,ds \quad ; \quad C: \; x = 4y,\ z=3 \text{ from } (2,\tfrac{1}{2},3) \text{ to } (4,1,3).$$

Let $x=t$. Then $y=t/4$ and $z=3$.

Therefore, the curve C is represented by

$x=t$, $y=\frac{t}{4}$, $z=3$; $2 \le t \le 4$.

Now,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2} = \sqrt{1^2+\left(\frac{1}{4}\right)^2+0}$$

$$\frac{ds}{dt} = \sqrt{\frac{17}{16}} = \frac{\sqrt{17}}{4}$$

Hence, line integral

$$\int_C (x^2+yz)\,ds = \int_2^4 \left[t^2+\left(\frac{t}{4}\right)3\right]\frac{\sqrt{17}}{4}\,dt$$

$$= \frac{\sqrt{17}}{4}\int_2^4 \left(t^2+\frac{3t}{4}\right)dt = \frac{\sqrt{17}}{4}\left[\frac{t^3}{3}+\frac{3t^2}{8}\right]_2^4$$

$$= \frac{\sqrt{17}}{4}\left[\left(\frac{64}{3}+6\right)-\left(\frac{8}{3}+\frac{3}{2}\right)\right]$$

$$= \frac{\sqrt{17}}{4}\left[\frac{128+36-16-9}{6}\right] = \frac{139\sqrt{17}}{24} \quad \text{Ans}$$

24 $\vec{V} = x^2\hat{i} - 2y\hat{j} + z^2\hat{k}$ over the straight line path from $(-1,2,3)$ to $(2,3,5)$

The parametric representation of the straight line is given by

~~$r(t) = (-\hat{i}+2\hat{j}+3\hat{k}) + t(2\hat{i}+3\hat{j}+5\hat{k})$~~

~~$= (-1+2t)\hat{i} + 2+3t)$~~

$$\vec{r}(t) = \vec{a} + t(\vec{b}-\vec{a})$$

$$= (-\hat{i}+2\hat{j}+3\hat{k}) + t(3\hat{i}+\hat{j}+2\hat{k})$$

$$\vec{r}(t) = (-1+3t)\hat{i} + (2+t)\hat{j} + (3+2t)\hat{k}; \quad 0 \le t \le 1.$$

$\therefore \frac{dr}{dt} = 3\hat{i} + \hat{j} + 2\hat{k}$

Then, line integral

$$\int_C \vec{V} \cdot d\vec{r} = \int_C \vec{V} \cdot \frac{dr}{dt}\, dt = \int_0^1 \left[(-1+3t)^{\frac{2}{3}} - 2(2+t) + 2(3+2t)^2\right] dt$$

$$= \int_0^1 (17 + 4t + 35t^2)\, dt = \frac{92}{3} \quad \text{Ans}$$

26 Evaluate $\int_C (x+y)\,dx - x^2 dy + (y+z)\,dz$;

C: $x^2 = 4y$, $z = x$, $0 \le x \le 2$.

let $x = t$, $y = t^2/4$; $z = t$; $0 \le t \le 2$

$$\therefore \int_C (x+y)\,dx - x^2 dy + (y+z)\,dz$$

$$= \int_0^2 \left(t + \frac{t^2}{4}\right) dt - t^2 \cdot \frac{2t}{4}\, dt + \left(\frac{t^2}{4} + t\right) dt$$

$$= \int_0^2 \left[t + \frac{t^2}{4} - \frac{t^3}{2} + \frac{t^2}{4} + t\right] dt$$

$$= \int_0^2 \left(2t + \frac{t^2}{2} - \frac{t^3}{2}\right) dt = \left[t^2 + \frac{t^3}{6} - \frac{t^4}{8}\right]_0^2 = \frac{10}{3}$$

Ans