Lagrange's Method of Undetermined Multipliers

Let $f(x,y,z)$ be a function of x,y,z which is to be examined for maximum or minimum value. Let the variables x,y,z be connected by the relation $\phi(x,y,z)=0$.

We consider auxiliary function of the form

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z) \quad \text{---(1)}$$

where λ are said to be Lagrange multipliers.

For stationary value of $F(x,y,z)$,

$$dF = 0$$

$$\Rightarrow \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$

Ex 1: Find the minimum value of $x^2+y^2+z^2$, given that $ax+by+cz=p$.

Soln: Let $u = x^2+y^2+z^2$ ——— (1)

where $\phi(x,y,z) \equiv ax+by+cz-p=0$ ——— (2)

Consider Lagrange's Function

$$F(x,y,z) = (x^2+y^2+z^2) + \lambda(ax+by+cz-p) \quad \text{——— (3)}$$

For stationary values, $dF=0$

$\Rightarrow (2x+\lambda a)dx+(2y+\lambda b)dy+(2z+\lambda c)dz=0$

$\Rightarrow \quad 2x+\lambda a=0$ ——— (4) $\Rightarrow x=-\frac{\lambda a}{2}$

$2y+\lambda b=0$ ——— (5) $\Rightarrow y=-\frac{\lambda b}{2}$

$2z+\lambda c=0$ ——— (6) $\Rightarrow z=-\frac{\lambda c}{2}$

From (2), $a\left(-\frac{\lambda a}{2}\right)+b\left(-\frac{\lambda b}{2}\right)+c\left(-\frac{\lambda}{2}c\right)=p$

$\Rightarrow \quad \lambda(a^2+b^2+c^2)=-2p \Rightarrow \lambda=\frac{-2p}{a^2+b^2+c^2}$

$\therefore x=\frac{ap}{a^2+b^2+c^2}, y=\frac{bp}{a^2+b^2+c^2}, z=\frac{cp}{a^2+b^2+c^2}$

$\therefore u=x^2+y^2+z^2=\frac{(a^2+b^2+c^2)p^2}{(a^2+b^2+c^2)^2}=\frac{p^2}{a^2+b^2+c^2}$

$\therefore$ Minimum value of u is $\frac{p^2}{a^2+b^2+c^2}$.

Problems on Lagrange's method of undetermined multipliers :

① Find the closest point on the curve $y = 3x - 4$ to the origin.

Solution:- The distance of any point (x, y) to the origin $(0,0)$ is

$$d(x,y) = \sqrt{x^2+y^2}.$$

Hence, we take function is

$$d^2 = f(x,y) = x^2+y^2 \qquad \text{——①}$$

given, $\phi(x,y) = 3x - y - 4 = 0$ ——②

We consider, the auxiliary function as

$$F(x,y) = f(x,y) + \lambda\, \phi(x,y)$$

$$F(x,y) = x^2+y^2 + \lambda(3x-y-4) \qquad \text{——③}$$

For stationary points $dF = 0$

i.e. $\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + 3\lambda = 0 \Rightarrow x = \frac{-3\lambda}{2}$

$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y - \lambda = 0 \Rightarrow y = \frac{\lambda}{2}$

Substituting values of x and y in eqⁿ ②; we have

$$3\left(-\frac{3\lambda}{2}\right) - \frac{\lambda}{2} = 4 \Rightarrow \frac{-9\lambda-\lambda}{2} = 4$$

$$\Rightarrow -10\lambda = 8 \Rightarrow \lambda = \frac{-4}{5}$$

$\therefore x = \frac{-3}{2}\left(\frac{-4}{5}\right) = \frac{6}{5}$ and $y = \frac{-2}{5}$ $\therefore$ closest point is $\left(\frac{6}{5}, \frac{-2}{5}\right)$ to the origin.

(2) Find the maximum and minimum of the funct. $f(x,y) = 4xy$ subject to the $x^2+y^2=8$.

Solution:- Given $f(x,y) = 4xy$ —(1)

and $\phi(x,y) = x^2+y^2-8 = 0$ —(2)

Consider the auxiliary function is

$$F(x,y) = f(x,y) + \lambda\,\phi(x,y)$$

$$F(x,y) = 4xy + \lambda(x^2+y^2-8) \quad \text{—(3)}$$

For maxima or minima $\quad df = 0$

$$\Rightarrow \quad \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$

$$\Rightarrow \quad 4y + 2\lambda x = 0 \quad \text{and} \quad 4x + 2\lambda y = 0$$

$$\Rightarrow \quad x = -\frac{2y}{\lambda} \quad \text{and} \quad y = -\frac{2x}{\lambda}$$

Substituting these values in eq^n (2); we have

$$\left(-\frac{2y}{\lambda}\right)^2 + \left(-\frac{2x}{\lambda}\right)^2 = 8 \Rightarrow \frac{4}{\lambda^2}(x^2+y^2) = 8$$

$$\Rightarrow \quad \lambda^2 = 4 \Rightarrow \lambda = \pm 2$$

For $\lambda = 2$

$$\left.\begin{array}{l} x = -\frac{2y}{2} \Rightarrow x = -y \\ \text{and } y = -\frac{2x}{2} = -x \end{array}\right\} \Rightarrow \boxed{x=-y}$$

From (2); $\quad (-y)^2 + y^2 = 8 \Rightarrow 2y^2 = 8 \Rightarrow y^2 = 4$

$$\Rightarrow \quad y = \pm 2$$

If $y = 2$, then $x = -2$ $\qquad (-2,2)$

If $y = -2$, then $x = 2$ $\qquad (2,-2)$

Critical points are $(-2,2)$, $(2,-2)$.

$\lambda = -2$: $\left.\begin{array}{l} x = y \\ \text{and } y = x \end{array}\right\} \Rightarrow x = y$

from (2):

$$y^2 + y^2 = 8 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

If $y = 2$, then $x = 2$ $\quad (2,2)$

If $y = -2$, then $x = -2$ $\quad (-2,-2)$

Hence critical points are $(2,2)$, $(-2,-2)$, $(2,-2)$ and $(-2,2)$.

~~Now,~~ $\cancel{f_{xx} = 2, \; f_{yy} = 2, \; f_{xy} = 0}$

$\cancel{rt - s^2 = 2 \times 2 - 0 = 4 > 0}$

$\cancel{f_{xx} = 0, \; f_{yy} = 0, \; f_{xy} = 4}$

$\cancel{\therefore D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 0 - 16}$

~~for $\lambda = 2$,~~ $\cancel{f(x,y) = 4xy + 2(x^2 + y^2 - 8)}$

$\cancel{F_x = 4y + 4x, \; F_y = 4x + 4y}$

$\cancel{f_{xx} = 4, \; f_{yy} = 4}$

$\cancel{F_{xy} = 4}$

$\cancel{\therefore D(x,y) = F_{xx} \cdot F_{yy} - (F_{xy})^2}$

$\cancel{= 4 \times 4 - (4)^2 =}$

$f(2,2) = 4(2)(2) = 16 \qquad f_{max} = 16$

$f(-2,-2) = 4(-2)(-2) = 16$

$f(2,-2) = 4(2)(-2) = -16 \qquad f_{min} = -16$

$f(-2,2) = 4(-2)(2) = -16$

Hence $f(x,y)$ is maximum at $(2,2)$ and $(-2,2)$ and $f(x,y)$ is minimum at $(2,-2)$ and $(-2,2)$.

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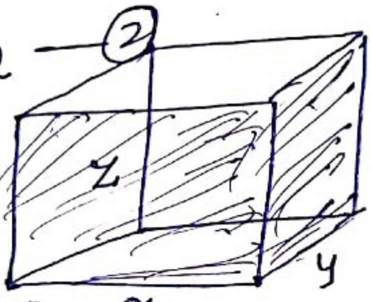
③ A rectangular box with no top is to be constructed from 432 ft^2 of material. What should be the dimensions of the box if it is to enclose maximum volume?

<u>Solution:-</u> Let x, y, z be the length, breadth and height of rectangular box respectively. Then

volume, $V = xyz$ ——①

and surface area of open rectangular box is

$$S = xy + 2(yz + zx) = 432 \quad —— ②$$



$\Rightarrow \cancel{\frac{\partial S}{\partial x} = y + 2z, \frac{\partial S}{\partial y} =}$

$\Rightarrow \phi(x,y,z) = xy + 2(yz + zx) - 432 = 0$

By Lagrange's method,

$$\frac{\partial V}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow yz + (y + 2z)\lambda = 0 \quad —— ③$$

$$\frac{\partial V}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow \lambda(x + 2z) + xz = 0 \quad —— ④$$

$$\frac{\partial V}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow xy + \lambda(2y + 2x) = 0 \quad —— ⑤$$

$x \times ③ - y \times ④$; we get

$$2z(x - y)\lambda = 0 \Rightarrow x = y \quad —— ⑥$$

$y \times ④ - z \times ⑤$; we get

$$x(y - 2z)\lambda = 0 \Rightarrow y = 2z \quad —— ⑦$$

from ⑥, ⑦; we get $\boxed{x = y = 2z}$ —— ⑧

...stituting these values in ②; we get

$$2z \times 2z + 2(2z \cdot z + z \cdot 2z) = 432$$

$$\Rightarrow 4z^2 + 8z^2 = 432 \Rightarrow 12z^2 = 432$$

$$\Rightarrow z^2 = 36 \Rightarrow z = 6$$

Hence, $x = 2 \times 6 = 12 = y$

Hence dimension of the box is $x = 12, y = 12, z = 6$.

i.e. 12, 12, 6. Ans

Prob ④:- Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum.

Solution:- ~~let $f(x,y,z) =$~~

let 24 be divided into x, y, z. Then

$$x + y + z = 24 \quad \text{—— (1)}$$

Consider the function $f(x,y,z) = xy^2z^3$ —— (2)

from (1); $\phi(x,y,z) = x + y + z - 24$ —— (3)

Consider the auxiliary function is

$$F(x,y,z) = f(x,y,z) + \lambda(x + y + z - 24) \quad \text{—— (4)}$$

For maximum $dF = 0$

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow y^2z^3 + \lambda = 0 \quad \text{—— (5)}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 2xyz^3 + \lambda = 0 \quad \text{—— (6)}$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow 3xy^2z^2 + \lambda = 0 \quad \text{—— (7)}$$

$2x \times (5) - y \times (6)$; we get

$(2x-y)\lambda = 0 \Rightarrow \; 2x = y \Rightarrow x = \frac{y}{2}$ — (8)

$3y \times (6) - 2z \times (7)$;

$(3y-2z)\lambda = 0 \Rightarrow \; 3y = 2z$ — (9)

Hence, $x = \frac{y}{2} = \frac{1}{2}\left(\frac{2}{3}z\right) = \frac{z}{3}$

i.e. $x = \frac{y}{2} = \frac{z}{3}$

Substituting these values

$\frac{z}{3} + \frac{2z}{3} + z = 24$

$\Rightarrow \quad \frac{6z}{3} = 24 \Rightarrow 2z = 24 \Rightarrow z = 12$

$\therefore \; y = \frac{2}{3} \times 12 = 8$

$x = \frac{12}{3} = 4$

Hence, divison is 4, 8, 12.