Triple Integrals

① $$\iiint_R f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

② $$\iiint_R f(x,y,z)\,dV = \iint_R \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\,dA.$$

Prob ①:- Evaluate the triple integral $\iiint_Q 2xe^y \sin z\,dV$, where Q is the rectangular box defined by

$$Q = \{(x,y,z) \mid 1 \le x \le 2,\ 0 \le y \le 1 \text{ and } 0 \le z \le \pi\}.$$

Solution:-

$$\iiint_Q 2xe^y \sin z\,dV = \int_{z=0}^{\pi} \int_{y=0}^{1} \int_{x=1}^{2} 2xe^y \sin z\,dx\,dy\,dz$$

$$= \int_0^{\pi} \int_0^1 2e^y \sin z \left[\int_1^2 x\,dx\right] dy\,dz$$

$$= \int_0^{\pi} \int_0^1 e^y \sin z \times 3\,dy\,dz$$

$$= 3\int_0^{\pi} \sin z \left[\int_0^1 e^y\,dy\right] dz = 3(e-1)\int_0^{\pi} \sin z\,dz$$

$$= -3(e-1)\left[\cos z\right]_0^{\pi} = 3(1-e)\left[\cos\pi - \cos 0^\circ\right]$$

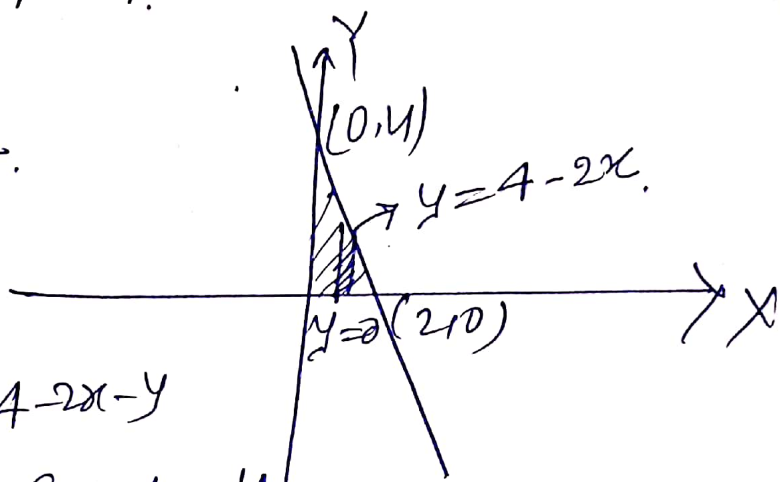
$$= 3(1-e)(-1-1) = 6(e-1).$$ Ans

Prob ②:– Evaluate $\iiint_Q 6xy\,dV$, where Q is the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $2x+y+z=4$.

Solution: $2x+y+z=4 \Rightarrow z = 4-2x-y$

$z: 0 \to 4-2x-y$

With $z=0$; $2x+y=4$.



$x: 0 \to 2$

$y: 0 \to 4-2x$

Therefore,

$$\iiint_Q 6xy\,dV = \iint_R \int_0^{4-2x-y} 6xy\,dz\,dA$$

$$= \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} 6xy\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^{4-2x} 6xy\,(4-2x-y)\,dy\,dx$$

$$= \int_0^2 6x\left[\int_0^{4-2x} (4y-2xy-y^2)\,dy\right]dx$$

$$= \int_0^2 6x\left(4\frac{y^2}{2} - 2x\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^{4-2x} dx$$

$$= \int_0^2 \left[12x(4-2x)^2 - 6x^2(4-2x)^2 - 2x(4-2x)^3\right]dx$$

$$= \frac{64}{5}.$$ (check)

No ③ :- Evaluate $\int_0^4\int_x^4\int_0^y \frac{6}{1+48z-z^3}\,dz\,dy\,dx$. (Imp)

Solution:- Given limits are

$x: 0\to 4,\ y: x\to 4;\ z: 0\to y$

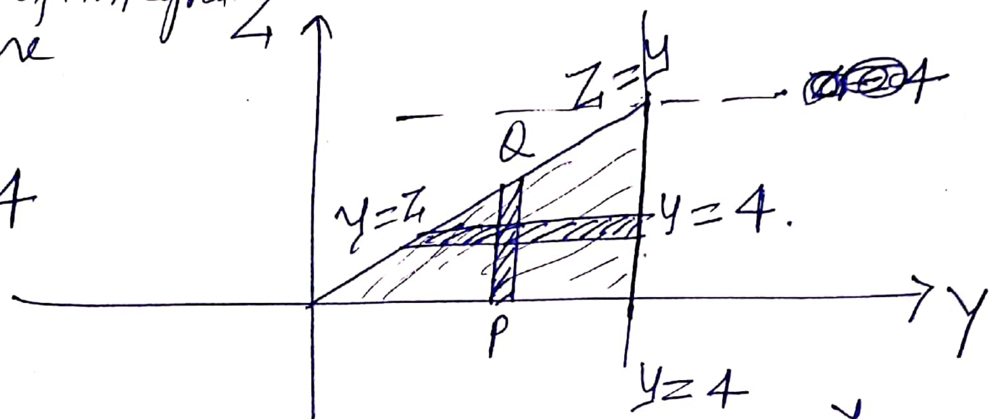
Change of order of integration

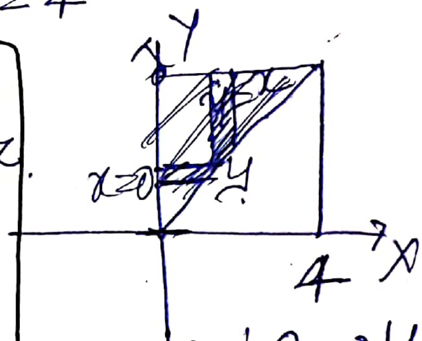
New limits are

$y: z\to 4$

$z: 0\to 4$

$x: 0\to y$





$x: 0\to y$

Therefore

$$I = \int_{x=0}^{y}\int_{z=0}^{4}\int_{y=z}^{4}\frac{6}{1+48z-z^3}\,dy\,dz\,dz$$

$$= \int_0^4\int_{z=0}^{4}\frac{6}{1+48z-z^3}\int_{y=z}^{4}dy\,dz\,dx$$

$$= \int_0^4\int_{z=0}^{4}\frac{6(4-z)}{1+48z-z^3}\,dz\,dx$$

put $1+48z-z^3$

$1+4$

Therefore,

$$I = \int_{z=0}^{4}\int_{y=z}^{4}\int_{x=0}^{y}\frac{6}{1+48z-z^3}\,dx\,dy\,dz$$

$$= \int_{z=0}^{4}\int_{y=z}^{4}\frac{6}{1+48z-z^3}\cdot y\,dy\,dz$$

$$= \int_{z=0}^{4}\frac{6}{1+48z-z^3}\left[\frac{y^2}{2}\right]_{y=z}^{4}dz = \int_{z=0}^{4}\frac{3(16-z^2)}{1+48z-z^3}\,dz$$

$$I = \int_0^1 \frac{48-3z^2}{1+48z-z^3}\,dz$$

Put $1+48z-z^3 = t$

$(48-3z^2)\,dz = dt$

$t: 1 \to 129$

$$= \int_1^{129} \frac{1}{t}\,dt = [\ln t]_1^{129}$$

$$= \ln 129 \quad \text{Ans}$$

Prob ④:- Find the volume of the solid bounded by the graphs of $z = 4-y^2$, $x+z=4$, $x=0$ and $z=0$.

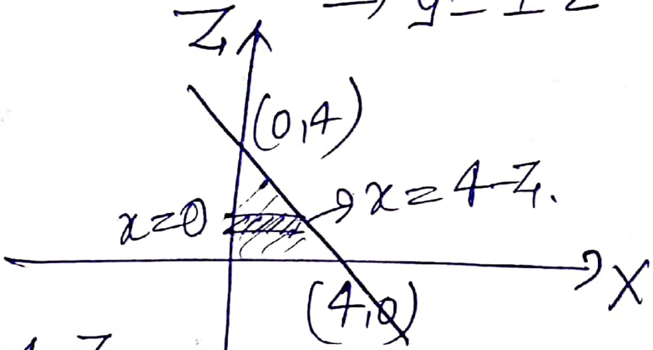
Solution:- $z = 4-y^2$ with $z=0$ gives $4-y^2=0$

$\Rightarrow y = \pm 2$

$y: -2 \to 2$

$z: 0 \to 4-y^2$

$x: 0 \to 4-z$



Volume of the solid

$$V = \iiint_Q dV = \iint_R \int_0^{4-z} dx\,dA$$

$$= \int_{-2}^{2}\int_0^{4-y^2}\int_0^{4-z} dx\,dz\,dy$$

$$= \int_{-2}^{2}\int_0^{4-y^2} (4-z)\,dz\,dy = \int_{-2}^{2}\left[4z-\frac{z^2}{2}\right]_0^{4-y^2} dy$$

$$= \int_{-2}^{2}\left[4(4-y^2)-\frac{1}{2}(4-y^2)^2\right]dy = \frac{128}{5}.$$

Alternate:- $V = 2\int_{z=0}^{4}\int_{x=0}^{4-z} y\,dx\,dz = 2\int_{z=0}^{4}\int_{x=0}^{4-z}\sqrt{4-z}\,dx\,dz$