Lecture 5

Taylor Series:- Taylor series of function $f(x)$ about $x=a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n f^{(n)}(a)}{n!} + \cdots$$

where f be a function with derivatives of all order.

Maclaurin Series:- Maclaurin series of f is the Taylor series of f at $x=0$ or

$$f(0) + f'(0)x + \frac{x^2 f''(0)}{2!} + \cdots + \frac{x^n f^{(n)}(0)}{n!} + \cdots$$

Ex ① Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a=2$. Where, if anywhere, does the series converge to $\frac{1}{x}$?

Solⁿ:- We need to find $f(2), f'(2), f''(2) \cdots$

~~Taking~~ $f(x) = \frac{1}{x}$ $\Rightarrow$ $f(2) = \frac{1}{2}$

$f'(x) = -\frac{1}{x^2}$ $\Rightarrow$ $f'(2) = -\frac{1}{2^2}$,

$f''(x) = \frac{2}{x^3}$ $\Rightarrow$ $f''(2) = \frac{2}{2^3}$

$f'''(x) = -\frac{6}{x^4}$ $\Rightarrow$ $f'''(2) = -\frac{6}{2^4}$

The Taylor series at $a=2$ is

$$f(2) + f'(2)(x-2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots$$

$$= \frac{1}{2} - \frac{1}{2^2}(x-2) + \frac{(x-2)^2}{2!}\left(\frac{2}{2^3}\right) + \frac{(x-2)^3}{3!}\left(-\frac{6}{2^4}\right) + \cdots$$

$$= \frac{1}{2} - \frac{x-2}{2^2} + \frac{(x-2)^2}{2^3} - \frac{(x-2)^3}{2^4} + \cdots \quad \text{Ans}$$

... is a geometric series with

$a = \frac{1}{2}$ and $r = -\frac{(x-2)}{2}$.

It converges absolutely $|r| < 1$

i.e. $|x-2| < 2$

and its sum is $= \frac{a}{1-r} = \frac{1/2}{1+\left(\frac{x-2}{2}\right)} = \frac{1/2}{\frac{x}{2}} = \frac{1}{x}$

i.e. Taylor series of $f(x) = 1/x$ at $a = 2$ converges to $\frac{1}{x}$ for $|x-2| < 2$ or $0 < x < 4$.

<u>Taylor Polynomial</u>:-

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a).$$

<u>Ex ②</u> Find the Taylor series and Taylor polynomial generated by $f(x) = e^x$ at $x = 0$.

<u>Sol^n</u>:- $f(x) = e^x$ $\therefore f(0) = e^0 = 1$

$f'(x) = e^x$ $\therefore f'(0) = 1$ and so on

The Taylor series of e^x at $x = 0$ is

$$f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$: Maclaurin series

Now, Taylor Polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

<u>Ex ③</u> Find the Taylor series and Taylor polynomial generated by $f(x) = \cos x$ at $x = 0$.

Taylor's formula:-

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n(x)$$

where $R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c) \qquad c \in (a, x)$

$\downarrow$

Remainder of order n.

Problem ① The Maclaurin series for e^x.

Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

Sol^n:- Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

where $R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n)}(c) = \frac{e^c x^{n+1}}{(n+1)!} \qquad c \in [0, x]$

$\lim_{n \to \infty} |R_n(x)| = 0$ for every x,

$\therefore$ Series converges to e^x for every x.

Prob ②:- The Maclaurin series for $\sin x$

Show that the Maclaurin series for $\sin x$ converges to $\sin x$ $\forall x$.

Sol^n:-

$f(x) = \sin x$ | $f(0) = \sin 0 = 0$

$f'(x) = \cos x$ | $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ | $f''(0) = 0$

$f'''(x) = -\cos x$ | $f'''(0) = -1$

$f^{IV}(x) = \sin x$ | $f^{IV}(0) = 0$

$f^{(V)}(x) = \cos x$ | $f^{V}(0) = 1$

Hence, Maclaurin series for $\sin x$ is

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots$$

Hence, we write as

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + R_{2n+1}(x)$$

and $R_{2n+1}(x) = (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!} \to 0$ as $n \to \infty$ $\forall x$.

Hence, Maclaurin series for $\sin x$ converges to $\sin x$ for every x.

Ex ③ The Maclaurin series for $\cos x$

Ans $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots = \sum \frac{(-1)^n x^{2n}}{(2n)!}$

Prob ④ Find the Maclaurin series for $\cos 2x$

Soln:-

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + - \cdots$$

$$= 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$$ Ans

Prob ⑤ Find the Maclaurin series for $x \sin x$.

Soln:-

$$x \sin x = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots\right)$$

$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + - \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{(2n-1)!}$$ Ans

Example ④ Using known series, find the first few terms of the Taylor series for the given function using power series operations.

(a) $\frac{1}{3}(2x + x\cos x)$ (b) $e^x \cos x$

Sol^n:- (a) $\frac{1}{3}(2x + x\cos x) = \frac{2}{3}x + \frac{x}{3}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$

$$= \frac{2x}{3} + \frac{x}{3} - \frac{x^3}{3!} + \frac{x^5}{3 \cdot 4!} - + \cdots$$

$$= x - \frac{x^3}{6} + \frac{x^5}{72} - + \cdots \quad \text{Ans.}$$

(b) $e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$

$$\times \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$

$$- \left(\frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!\,2!} + \frac{x^5}{2!\,3!}\right)$$

$$+ \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!\,4!} + \cdots\right) + \cdots$$

$$= 1 + x + x^3\left(\frac{1}{6} - \frac{1}{2}\right) + x^4\left(\frac{1}{24} + \frac{1}{24} - \frac{1}{4}\right) + \cdots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

Ans