Lecture 12

Gamma Function

Gamma function is an improper integral and is denoted and defined by

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx \; ; \quad n > 0.$$

Properties:- (i) $\Gamma 1 = \int_0^\infty e^{-x} dx = 1$

(ii) $\Gamma n+1 = \begin{cases} n\Gamma n & ; \text{ otherwise} \\ n! & ; \text{ if } n \text{ is any positive integer} \end{cases}$

(iii) $\Gamma \frac{1}{2} = \sqrt{\pi}$

(iv) $\Gamma -\frac{1}{2} = -2\sqrt{\pi}$

Proof (iii) We know, $\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$

Put $x = u^2$
$dx = 2u\,du$
$u : 0 \to \infty$

$$\therefore \Gamma \tfrac{1}{2} = \int_0^\infty x^{-1/2} e^{-x} dx$$

$$= \int_0^\infty u^{-1} e^{-u^2} \times 2u\,du$$

$$\Gamma \tfrac{1}{2} = 2\int_0^\infty e^{-u^2} du \quad \text{——— (1)}$$

We write $\left(\Gamma \tfrac{1}{2}\right)^2 = 2\int_0^\infty e^{-u^2} du \times 2\int_0^\infty e^{-v^2} dv$

$$= 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} du\,dv \quad \text{——— (2)}$$

Changing into polar coordinates $u = r\cos\theta$, $v = r\sin\theta$

$\Rightarrow \quad du\,dv = r\,dr\,d\theta$.

$\theta : 0 \to \pi/2$
$r : 0 \to \infty$

from (2); we obtain

$$\left[\Gamma\tfrac{1}{2}\right]^2 = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} r\,e^{-r^2}\,dr\,d\theta = 2\pi\int_0^{\infty} r\,e^{-r^2}\,dr$$

$$= -\pi\left[e^{-r^2}\right]_0^{\infty} = \pi$$

Hence, $\boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$ proved

Proof (iv) we know, $\Gamma(n+1) = n\,\Gamma n$

Put $n = -\tfrac{1}{2}$, then

$$\Gamma\tfrac{1}{2} = -\tfrac{1}{2}\,\Gamma\left(-\tfrac{1}{2}\right) \Rightarrow \sqrt{\pi} = -\tfrac{1}{2}\,\Gamma\left(-\tfrac{1}{2}\right)$$

$$\Rightarrow \boxed{\Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}}\ \text{Ans}$$

Problem (1) Evaluate the following improper integrals

(i) $\int_0^{\infty}\sqrt{x}\,e^{-x^2}\,dx$ (ii) $\int_0^{\infty} e^{-x^3}\,dx$ in terms of Gamma function.

Sol^n (i)

$$I = \int_0^{\infty}\sqrt{x}\,e^{-x^2}\,dx$$

Put $x^2 = t$

$x = \sqrt{t}$

$dx = \frac{1}{2\sqrt{t}}\,dt$

$t : 0 \to \infty$

$$= \int_0^{\infty} t^{1/4}\,e^{-t}\,\frac{1}{2t^{1/2}}\,dt = \frac{1}{2}\int_0^{\infty} t^{-1/4}\,e^{-t}\,dt$$

$$= \frac{1}{2}\int_0^{\infty} t^{(3/4-1)}\,e^{-t}\,dt = \frac{1}{2}\,\Gamma\tfrac{3}{4}\ \text{Ans}$$

(ii)

$$I = \int_0^{\infty} e^{-x^3}\,dx$$

Put $x^3 = t$

$x = t^{1/3}$

$dx = \frac{1}{3}\,t^{-2/3}\,dt$

$t : 0 \to \infty$

$$= \int_0^{\infty} e^{-t}\cdot\frac{1}{3}\,t^{-2/3}\,dt$$

$$= \frac{1}{3}\int_0^{\infty} t^{(1/3-1)}\,e^{-t}\,dt = \frac{1}{3}\,\Gamma\tfrac{1}{3}\ \text{Ans}$$

$-\frac{2}{3} = n-1$

$n = 1/3$

Problem:- Evaluate: By using Gamma function

(i) $\int_0^\infty 2^{-9x^2} dx$ (ii) $\int_0^\infty t^4 e^{-2t^2} dt$ (iii) $\int_0^\infty x^{1/3} e^{-x^2} dx$.

Solⁿ (i) $I = \int_0^\infty 2^{-9x^2} dx = \int_0^\infty e^{-9x^2 \ln 2} dx$

Put $9x^2 \ln 2 = y$

$x = \frac{1}{3}\sqrt{\frac{y}{\ln 2}}$

$dx = \frac{1}{3\sqrt{\ln 2}} \cdot \frac{1}{2\sqrt{y}} dy$

$y : 0 \to \infty$

$$I = \int_0^\infty e^{-y} \frac{1}{6\sqrt{\ln 2}\cdot y^{1/2}} dy$$

$$= \frac{1}{6\sqrt{\ln 2}} \int_0^\infty y^{-1/2} e^{-y} dy$$

$$= \frac{1}{6\sqrt{\ln 2}} \int_0^\infty y^{(1/2 - 1)} e^{-y} dy$$

$-\frac{1}{2} = n-1$

$n = \frac{1}{2}$

$$= \frac{1}{6\sqrt{\ln 2}} \Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{6\sqrt{\ln 2}} \text{ Ans.}$$

Beta Function:- If $m>0, n>0$, Beta function is an improper integral and it is denoted and defined by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad ; \quad m>0, n>0 \qquad (1)$$

Problem:- Using Beta & Gamma function. Evaluate:

(i) $\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}}$ (ii) $\int_0^a x\sqrt{a^3 - x^3}\, dx$ (iii) $\int_0^1 \frac{dx}{\sqrt[3]{1-x^3}}$

(iv) $\int_0^1 x^n (\ln x)^m dx$ (v) $\int_0^a \frac{x^{3/2}}{\sqrt{a^2-x^2}} dx$

(vi) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$ (vii) $\int_0^\infty \frac{dx}{1+x^4}$ (viii) $\int_0^\infty \frac{x^a}{a^x} dx,\ a>1$.