Gauss Divergence Theorem — Let D be a closed and bounded region in the three dim. space whose boundary is a piecewise smooth surface S that is oriented outward. Let $\vec{V}(x,y,z) = V_1(x,y,z)\hat{i} + V_2(x,y,z)\hat{j} + V_3(x,y,z)\hat{k}$ be a vector field for which V_1, V_2 and V_3 are continuous and have continuous first order partial derivatives in some domain containing D, Then.

$$\iint_S (\vec{V}\cdot\hat{n})\,dA = \iiint_D \nabla\cdot\vec{V}\,dV$$

where $\hat{n}$ is the outer unit normal vector to S.

4o Prob:- Let D be the region bounded by the closed cylinder $x^2+y^2=16$, $z=0$ and $z=4$. Verify the divergence theorem if $\vec{V} = 3x^2\hat{i} + 6y^2\hat{j} + z\hat{k}$.

Solution:- $\nabla\cdot\vec{V} = 6x+12y+1$ Therefore

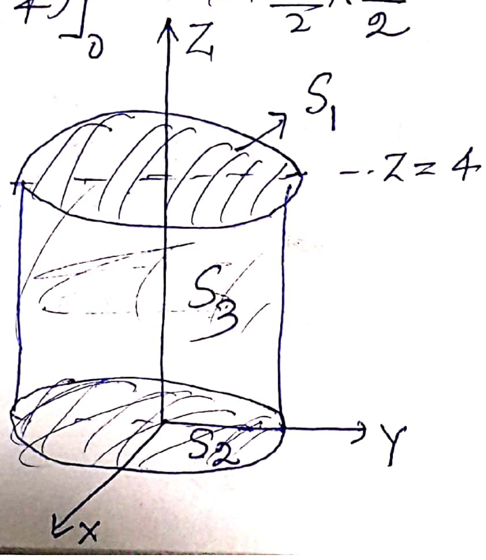
$$\iiint_D \nabla\cdot\vec{V}\,dV = \int_{z=0}^{4}\int_{x=-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (6x+12y+1)\,dy\,dx\,dz$$

$$= 4(2\times 2)\int_{x=0}^{4}\int_{0}^{\sqrt{16-x^2}} dy\,dx = 16\int_0^4 \sqrt{16-x^2}\,dx$$

$$= 16\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4 = 16\times\frac{16}{2}\times\frac{\pi}{2}$$

$$= 64\pi. \quad \text{——} \; (1)$$

The surface consists of three parts, S_1 (top), S_2 (bottom) and S_3 (vertical)



Based on Gauss Divergence thm

$$\iint_S (\vec{V}\cdot\hat{n})\,dA = \iiint_D (\nabla\cdot\vec{V})\,dV$$

39 Evaluate $\iint_S \vec{V}\cdot\hat{n}\,dA$; where

$$\vec{V} = x^2 z\hat{i} + y\hat{j} - xz^2\hat{k}$$

and S: paraboloid $z = x^2+y^2$, plane $z = 4y$

$$\nabla\cdot\vec{V} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(-xz^2)$$

$$= 2xz + 1 - 2xz$$

Hence, by Gauss divergence theorem

$$\iint_S (\vec{V}\cdot\hat{n})\,dA = \iiint_D (\nabla\cdot\vec{V})\,dV$$

$$= \iiint_D (2xz+1-2xz)\,dx\,dy\,dz \quad \text{——— (i)}$$

$$= \iiint_D dx\,dy\,dz$$

D: $z = x^2+y^2$; $z = 4y$

$4y = x^2+y^2 \Rightarrow x^2 = 4y-y^2 \Rightarrow x = \pm\sqrt{4y-y^2}$

For $x=0$ $\quad 4y-y^2 = 0 \Rightarrow y(4-y) = 0 \Rightarrow y = 4, 0$

Limit for Region D are

x : $-\sqrt{4y-y^2} \longrightarrow \sqrt{4y-y^2}$

y : $0 \longrightarrow 4$

z : $x^2+y^2 \longrightarrow 4y$

From (1); we get

$$\iint_S (\vec{V}\cdot\hat{n})\,dA = \int_0^4 \int_{-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} \int_{x^2+y^2}^{4y} 1.(\cancel{2xz+1-2xz})\,dz\,dx\,dy$$

$\cancel{= 8\pi}$

$$= \int_0^4 \int_{-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} (4y - x^2 - y^2)\, dx\, dy$$

$$= \int_0^4 \left[(4y-y^2)\, 2\sqrt{4y-y^2} - \frac{2}{3}(4y-y^2)^{3/2} \right] dy$$

$$= \frac{4}{3} \int_0^4 (4y-y^2)^{3/2} dy = \frac{4}{3} \int_0^4 \left[4-(y-2)^2\right]^{3/2} dy$$

Put $y-2 = 2\sin t$

$dy = 2\cos t . dt$

$t : -\pi/2 \to \pi/2$

$$= \frac{4}{3} \int_{-\pi/2}^{\pi/2} 16 \cos^4 t\, dt$$

$$= \frac{4}{3}(32)\left(\frac{3.1.}{4.2}\right)\frac{\pi}{.2} = 8\pi \quad \text{Ans}$$

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41 Given

$V = xy\hat{i} + yz\hat{j} + xz\hat{k}$

S: boundary of cube cut from the first octant by the planes $x=1, y=1, z=1$

Using Gauss Divergence theorem

$$\iint_S (\vec{V}\cdot\hat{n})\, dA = \iiint_D (\nabla\cdot\vec{V})\, dv \quad \text{--- (i)}$$

Now, $\nabla\cdot\vec{V} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz)$

$\nabla\cdot\vec{V} = y+z+x$

Limits are $\quad x: 0\to 1,\ y: 0\to 1,\ z: 0\to 1$

Then from (i);

$$\iint (\vec{V}\cdot\hat{n})\, dA = \int_0^1\int_0^1\int_0^1 (x+y+z)\, dz\, dy\, dx$$

$$= \int_0^1\int_0^1 \left. (x+y)z + \frac{z^2}{2} \right|_0^1 dy\, dx$$

$$= \int_0^1\int_0^1 \left[(x+y) + \frac{1}{2}\right] dy\, dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} + \frac{1}{2}y\right]_0^1 dx$$

$$= \int_0^1 \left(x + \frac{1}{2} + \frac{1}{2}\right) dx = \int_0^1 (x+1)\, dx$$

$$= \left[\frac{x^2}{2} + x\right]_0^1 = \frac{1}{2} + 1 = \frac{3}{2} \text{ Ans}$$

42 Given, $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$

S: boundary of sphere $x^2+y^2+z^2 = 4$

changing into spherical coordinate as

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\cos\phi,\ z = r\cos\theta$

$x^2+y^2+z^2 = r^2;\quad r: 0\to 2,\ \theta: 0\to\pi$

$\phi: 0\to 2\pi$

Now, $\nabla \cdot \vec{V} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$

By using Gauss divergence theorem,

$$\iint_S (\vec{V} \cdot \hat{n})\, dA = \iiint (\nabla \cdot \vec{V})\, dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{2} 3 \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 3 \int_0^{2\pi} \int_0^{\pi} \left[\frac{r^3}{3}\right]_0^2 \sin\theta\, d\theta\, d\phi$$

$$= 8 \int_0^{2\pi} \left[-\cos\theta\right]_0^{\pi} d\phi = 8 \times 2 \int_0^{2\pi} d\phi$$

$$= 16(2\pi) = 32\pi$$ Ans.

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