Unit IV

Numerical Solution of Ordinary Differential Equations

Prerequisite

- Finite Differences and Interpolation
- Numerical Differentiation.
- Numerical Integration.

Objective

To solve ordinary differential equations by

- Taylor’s series method
- Euler’s method, Modified Euler’ Method
- Runge-Kutta’s methods
- Milne’ Predictor-Corrector Method

Introduction: Ordinary Differential Equations

Differential equations are among the most important mathematical tools used in producing models in science and engineering. The differential equations that have only one independent variable are called ordinary differential equations.

For example: $y' = -2y$, $y(0) = 1$ has an analytic solution $y = e^{-2x}$.

The solution of an ordinary differential equation means finding an explicit expression for y in terms of x. Such a solution of a differential equation is known as the closed or finite form of solution. In the absence of such a solution, we have numerical methods to calculate approximate solution.

Numerical Solution of Ordinary Differential Equations

We will discuss, here some important methods of solving ODE of first / second order.

i) Taylor's series method,

ii) Euler's method,

iii) Modified Euler's method

iv) Runge-Kutta's Methods

Let us consider the first order differential equation $\frac{dy}{dx} = f(x, y)$ given y($x = x_0) = y_0$ …(1)

In Taylor's series method, y in (1) is approximated by a truncated series, each term of which is a function of x.

The information about the curve at one point is utilized and the solution is not iterated.

An ordinary differential equation of the nth order is of the form $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots \frac{d^ny}{dx^n}\right) = 0$

Its general solution contains n arbitrary constants and is of the form $\phi(x, y, c_1, c_2 \ldots\ldots c_n) = 0$

To obtain its particular solution, n conditions must be given so that the constants $c_1, c_2 \ldots\ldots c_n$ can be determined.

If these conditions are prescribed at one point only (say, x_0), then the differential equation together with the conditions constitute an **initial value problem** of the nth order.

If the conditions are prescribed at two or more points, then the problem is termed as **boundary value problem**.

Taylor’s Series Method

Consider the first order differential equation $\frac{dy}{dx} = f(x, y)$ $given\ that\ y(x_0) = y_0$ ... (1)

Differentiating (1), we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$\Rightarrow y'' = f_x + f_y . f$$

Differentiating this successively, we get, $y''', y^{iv}\ etc.$

Putting $x = x_0\ and\ y = y_0, the\ values\ of\ y', y'', y''', \dots.\ can\ be\ obtained\ at\ x = x_0$

Hence the Taylor's series

$y = y_0 + (x - x_0)\left(y'_{x=x_0}\right) + \frac{(x-x_0)^2}{2\,!}\left(y''_{x=x_0}\right) +$ ... gives the value of y for every value of x.

Example: Find by Taylor's series method, the values of y at x = 0.1 and x = 0.2 from

$$\frac{dy}{dx} = x + y,\ \ y(0) = 1.$$

Solution: Here, $x_0 = 0\ and\ y_0 = 1\ and\ y' = x + y$

$$\Rightarrow y'' = 1 + y',$$
$$y''' = y'',$$
$$y^{iv} = y''',\ldots.$$

$$\therefore\ \ y'\,(0) = 0 + y(0) = 1$$

$$y''(0) = 1 + y'(0) = 1 + 1 = 2$$

$$y'''(0) = y''(0) = 2$$

Putting all the values in Taylor's series

$$y(x=0.1) = y_0 + (x-x_0)\left(y'_{x=x_0}\right) + \frac{(x-x_0)^2}{2\,!}\left(y''_{x=x_0}\right) + \frac{(x-x_0)^3}{3\,!}\left(y'''_{x=x_0}\right) + \cdots$$

$$= 1 + (0.1-0)(1) + \frac{(0.1)^2}{2\,!}(2) + \frac{(0.1)^3}{3\,!}(2) + \frac{(0.1)^4}{4\,!}(2) + \cdots$$

$$= 1 + 0.1 + 0.01 + 0.00033 + 0.000008 = 1.110338$$ Ans.

$$And, y(x=0.2) = 1 + (0.2)(1) + \frac{(0.2)^2}{2\,!}(2) + \frac{(0.2)^3}{3\,!}(2) + \frac{(0.2)^4}{4\,!}(2) + \cdots$$

$$= 1 + 0.2 + 0.04 + 0.00267 + 0.000133 + .. = 1.242803$$ Ans.

$$\therefore\ y'(0) = 0 + y(0) = 1$$

$$y''(0) = 1 + y'(0) = 1 + 1 = 2$$

$$y'''(0) = y''(0) = 2$$

Example: Find by Taylor's series method, the value of y at x = 0.5 from

$$\frac{dy}{dx} = x^2y - 1,\ y(0) = 1.$$

Solution: Here, $x_0 = 0\ and\ y_0 = 1\ and\ y' = x^2y - 1$

$$\Rightarrow y'' = x^2y' + 2xy,$$

$$y''' = x^2y'' + 2xy' + 2xy' + 2y$$

$$= x^2y'' + 4xy' + 2y$$

$$y^{iv} = x^2y''' + 2xy'' + 4xy'' + 4y' + 2y'$$

$$= x^2\ y''' + 6xy'' + 6y'$$

$$\therefore\ \ y'\ (0) = 0 - 1 = -1$$

$$y''(0) = 0$$

$$y'''(0) = 2$$

$$y'^{v}(0) = -6$$

Putting all the values in Taylor's series

$$y(x=0)=1$$
$$y'(0)=0-1=-1$$
$$y''(0)=0$$
$$y'''(0)=2$$
$$y'^{v}(0)=-6$$

$$y(x=0.5)=y_0+(x-x_0)\left(y'_{x=x_0}\right)+\frac{(x-x_0)^2}{2\,!}\left(y''_{x=x_0}\right)+\frac{(x-x_0)^3}{3\,!}\left(y'''_{x=x_0}\right)+\cdots$$

$$=1+(0.5-0)(-1)+\frac{(0.5)^2}{2\,!}(0)+\frac{(0.5)^3}{3\,!}(2)+\frac{(0.5)^4}{4\,!}(-6)+\cdots$$

$$=1-0.5+0+0.04167-0.015625$$

$$=0.526045$$ Ans.

Practice Question

Question: Using Taylor's series method, find y at x = 1.1 by solving $\frac{dy}{dx} = x^2 + y^2,\ y(x = 1) = 2.3$

Solution: Here, $x_0 = 1 \; and \; y_0 = 2.3 \; and \; y' = x^2 + y^2$

$$\Rightarrow y'' = 2\,x + 2y.y',$$

$$y''' = 2 + 2\,[yy'' + y'.y']$$

$$= 2 + 2\,y.y'' + 2(y')^2$$

$$y^{iv} = 0 + 2[yy''' + y'y''] + 4y'y''$$

$$= 2\,y\,y''' + 6y'\,y''$$

$$\therefore \;\; y'\,(1) = 1 + (2.3)^2 = 6.29$$

$$y''(1) = 2 + 2(2.3)(6.29) = 30.934$$

$$y'''(1) = 2 + 2(2.3)(30.934) + 2(6.29)^2$$

$$= 2 + 142.2964 + 79.1282 = 223.4246$$

$$y'^{v}(1) = 2(2.3)(223.4246) + 6(6.29)(30.934)$$

$$= 2195.2024$$

Now putting $x_0 = 1, y_0 = 2.3, x = 1.1,$

$y'(1) = 6.29$

$y''(1) = 30.934$

$y'''(1) = 223.4246$

$y'^{v}(1) = 2195.2024$

in the Taylor's series, we get

$$y(x) = y_0 + (x - x_0)\left(y'_{x=x_0}\right) + \frac{(x-x_0)^2}{2\,!}\left(y''_{x=x_0}\right) + \frac{(x-x_0)^3}{3\,!}\left(y'''_{x=x_0}\right) + \cdots$$

$$y(1.1) = 2.3 + (0.1)(6.29) + \frac{(0.1)^2}{2\,!}(30.934) + \frac{(0.1)^3}{3\,!}(223.4246) + \frac{(0.1)^4}{4\,!}(2195.2024) + \cdots$$

$$= 2.3 + 0.629 + 0.15467 + 0.03724 + 0.00915 = 3.13006$$ Ans.

Question: Using Taylor's series method, find y at x = 0.2 by solving $\frac{dy}{dx} = 2y + 3\,e^x,\ y(x=0) = 0$

Solution: Here, $x_0 = 0\ and\ y_0 = 0\ and\ y' = 2y + 3\,e^x$

$$\Rightarrow y'' = 2\,y' + 3\,e^x,$$

$$y''' = 2\,y'' + 3\,e^x,$$

$$y^{iv} = 2\,y''' + 3\,e^x$$

$$\therefore\ \ y'\,(0) = 0 + 3\,e^0 = 3$$

$$y''(0) = 2(3) + 3\,e^0 = 9$$

$$y'''(0) = 2(9) + 3\,e^0 = 21$$

$$y'^{v}(0) = 2(21) + 3\,e^0 = 45$$

Now using all the above calculated values in the Taylor's series,

$y(x) = y_0 + (x - x_0)\left(y'_{x=x_0}\right) + \frac{(x-x_0)^2}{2\,!}\left(y''_{x=x_0}\right) + \frac{(x-x_0)^3}{3\,!}\left(y'''_{x=x_0}\right) + \cdots$, we have

$$y(0.2) = 0 + (0.2)(3) + \frac{(0.2)^2}{2\,!}(9) + \frac{(0.2)^3}{3\,!}(21) + \frac{(0.2)^4}{4\,!}(45)$$

$$= 0 + 0.6 + 0.18 + 0.028 + 0.003 = 0.811$$ Ans.

Euler's Method

As the Taylor's series method to solve given differential equation is

$$y = y_0 + h\,y' + \frac{h^2}{2\,!}y'' + \frac{h^3}{3\,!}y''' + \cdots$$

Euler's method assumes the solution by truncating the above series up to 2nd term

i.e., $y = y_0 + h\,y'$

i.e., $y_1 = y_0 + h\,f(x_0, y_0)$

Where, y_1 is the next estimated solution value;

y_0 is the given value at $x = x_0$, h is the interval between steps and

$f(x_0, y_0)\ is\ y'\ at\ starting\ point$

Similarly, to get the next value y_2 we would use the value, just found for y_1 as follows:

$$y_2 = y_1 + h\,f(x_1, y_1)$$

Repeating, this process n times, we get the value of y at nth step,

$y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})$ which is the Euler's method

Example: Find by Euler's method, the value of y in 5 steps at x = 0.5 from

$$\frac{dy}{dx} = x + y,\ \ y(0) = 1.$$

Solution: Here, $x_0 = 0\ and\ y_0 = 1\ and\ y' = x + y = f(x, y)$

We take n = 5 and h = 0.1, i.e., we calculate required value of y in 5 steps as under:

$$y_1 = y(x = 0.1) = y_0 + h\, f(x_0, y_0)$$

$$= 1 + 0.1\, f(0,1)$$

$$= 1 + 0.1\,(0 + 1) = 1.1$$

$$y_2 = y(x = 0.2) = \mathrm{y}_1 + h\,f(x_1, y_1)$$

$$= 1.1 + 0.1(0.1 + 1.1) = 1.1 + 0.12 = 1.22$$

$$y_3 = y(x = 0.3) = \mathrm{y}_2 + h\,f(x_2, y_2)$$

$$= 1.22 + 0.1(0.2 + 1.22) = 1.22 + 0.142 = 1.362$$

$$y_4 = y(x = 0.4) = \mathrm{y}_3 + h\,f(x_3, y_3)$$

$$= 1.362 + 0.1(0.3 + 1.362) = 1.362 + 0.166 = 1.528$$

$$y_5 = y(x = 0.5) = \mathrm{y}_4 + h\,f(x_4, y_4)$$

$$= 1.528 + 0.1(0.4 + 1.528) = 1.528 + 0.193 = 1.721$$ Ans.

Example: Find by Euler's method, the value of y at x = 0.1 from

$$\frac{dy}{dx} = \frac{y-x}{y+x},\ y(0) = 1.$$

Solution: Here, $x_0 = 0\ and\ y_0 = 1\ and\ y' = \frac{y-x}{y+x} = f(x,y)$

We divide the interval (0, 0.1) into 5 steps i.e., n = 5 and h = 0.02, The various values of y in 5 steps are as follows:

$$y_1 = y(x = 0.02) = \mathrm{y}_0 + h\, f(x_0, y_0)$$
$$= 1 + 0.02\, f(0,1)$$
$$= 1 + 0.02\frac{1-0}{1+0} = 1.02$$

$$y_2 = y(x = 0.04) = \mathrm{y}_1 + h\, f(x_1, y_1)$$
$$= 1.02 + 0.02\, f(0.02,1.02)$$
$$= 1.02 + 0.02\frac{1.02-0.02}{1.02+0.02} = 1.0392$$

$$y_3 = y(x = 0.06) = y_2 + h\,f(x_2, y_2)$$
$$= 1.0392 + 0.02\,f(0.04, 1.0392)$$
$$= 1.0392 + 0.02\,\frac{1.0392-0.04}{1.0392+0.04} = 1.0392 + 0.02(0.9259) = 1.0577$$

$$y_4 = y(x = 0.08) = y_3 + h\,f(x_3, y_3)$$
$$= 1.0577 + 0.02\,f(0.06, 1.0577)$$
$$= 1.0577 + 0.02\,\frac{1.0577-0.06}{1.0577+0.06} = 1.0577 + 0.02(0.8926) = 1.0755$$

$$y_5 = y(x = 0.1) = y_4 + h\,f(x_4, y_4)$$
$$= 1.0755 + 0.02\,f(0.08, 1.0755)$$
$$= 1.0755 + 0.02\,\frac{1.0755-0.08}{1.0755+0.08} = 1.0755 + 0.02(0.8615) = 1.0923 \qquad \text{Ans.}$$

Question: Find by Euler's method, the value of y at x = 0.6 from

$$\frac{dy}{dx} = 1 - 2xy,\ \ y(0) = 0 \text{ (Take h = 0.2)}$$

Solution: Here, $x_0 = 0$ *and* $y_0 = 0$ *and* $y' = 1 - 2xy = f(x, y)$

As h = 0.2, so we will find the required solution in 3 steps are as follows:

$$y_1 = y(x = 0.2) = \mathrm{y}_0 + h\, f(x_0, y_0)$$
$$= 0 + 0.2\, f(0{,}0)$$
$$= 0 + 0.2\{1 - 2(0)(0)\} = 0.2$$

$$y_2 = y(x = 0.4) = \mathrm{y}_1 + h\, f(x_1, y_1)$$
$$= 0.2 + 0.2\, f(0.2{,}0.2)$$
$$= 0.2 + 0.2\{1 - 2(0.2)(0.2)\} = 0.384$$

$$y_3 = y(x = 0.6) = \mathrm{y}_2 + h\, f(x_2, y_2)$$
$$= 0.384 + 0.2\, f(0.4{,}0.384)$$
$$= 0.384 + 0.2\{1 - 2(0.4)(0.384)\}$$
$$= 0.52256$$ Ans.

Euler's Modified Method

For better approximation, the solution obtained by Euler's method, will be modified at each step and then apply this modified value to find y for the next step, and again will be modified at each step. This scheme is called Modified Euler's method.

i. e., The improved Euler's method , starts from initial value (x_0, y_0),

and find y at $x = x_1$ *by Euler's method*

i. e., $y_1 = y_0 + h\, f(x_0, y_0)$

Now correct this value, by modified formula as under

$y_1^{(1)} = y_0 + \frac{h}{2}\{ f(x_0, y_0) + f(x_1, y_1)\}$ is the first modified value of y_1

Similarly, $y_1^{(2)} = y_0 + \frac{h}{2}\left\{f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right\}$ is the 2^{nd} modified value of y_1

We repeat this step, till two consecutive values of y agree. This is then taken as the starting point for the next step to find y_2

at $x = x_2$ *by Euler's method*

i.e., $y_2 = y_1 + h\,f(x_1, y_1)$, where y_1 is latest corrected value of y_1

Now correct this value, by modified formula as under

$y_2^{(1)} = y_1 + \frac{h}{2}\{f(x_1, y_1) + f(x_1, y_2)\}$ is the first modified value of y_2

Similarly, $y_2^{(2)} = y_1 + \frac{h}{2}\left\{f(x_1, y_1) + f\left(x_1, y_2^{(1)}\right)\right\}$ is the 2nd modified value of y_2

Repeat this step until y_2 becomes stationary. Then we proceed to calculate y_3 as above and so on.

Example: Using modified Euler's method, find an approximate value of y when x = 0.3, given that $\frac{dy}{dx} = x + y \text{ and } y = 1$ when x = 0

Solution: Here, $x_0 = 0 \text{ and } y_0 = 1 \text{ and } y' = x + y = f(x, y)$

Taking h = 0.1, so we will find the required solution in 3 steps are as follows:

By Euler's method, $y_1 = y(x = 0.1) = y_0 + h\, f(x_0, y_0)$ (Step 1)

$$= 1 + 0.1\, f(0,1)$$

$$= 1 + 0.1\{0 + 1\} = 1.1$$

Now modify this value, by modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\{\, f(x_0, y_0) + f(x_1, y_1)\} = 1 + \frac{0.1}{2}\{(0 + 1) + (0.1 + 1.1)\} = 1.11$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left\{\, f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right\} = 1 + \frac{0.1}{2}\{(0 + 1) + (0.1 + 1.11)\} = 1.1105$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left\{\, f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right\} = 1 + \frac{0.1}{2}\{(0 + 1) + (0.1 + 1.1105)\} = 1.1105$$

Step 2

By Euler's method, $y_2 = y(x = 0.2) = \mathrm{y}_1 + h\, f(x_1, y_1)$ $[y_1 = 1.1105]$

$= 1.1105 + 0.1\, f(0.1, 1.1105)$

$= 1.1105 + 0.1\{0.1 + 1.1105\} = 1.2316$

Now modify this value, by modified Euler's formula

$$y_2^{(1)} = \mathrm{y}_1 + \frac{h}{2}\{ f(x_1, y_1) + f(x_2, y_2)\} = 1.1105 + \frac{0.1}{2}\{(0.1 + 1.1105) + (0.2 + 1.2316)\} = 1.2426$$

$$y_2^{(2)} = \mathrm{y}_1 + \frac{h}{2}\left\{ f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right\} = 1.1105 + \frac{0.1}{2}\{(0.1 + 1.1105) + (0.2 + 1.2426)\} = 1.2432$$

$$y_2^{(3)} = \mathrm{y}_1 + \frac{h}{2}\left\{ f(x_1, y_1) + f\left(x_2, y_2^{(2)}\right)\right\} = 1.1105 + \frac{0.1}{2}\{(0.1 + 1.1105) + (0.2 + 1.2432)\} = 1.2432$$

Step 3

By Euler's method, $y_3 = y(x = 0.3) = \mathrm{y}_2 + h\, f(x_2, y_2)$ $[y_2 = 1.2432]$

$$= 1.2432 + 0.1\, f(0.2, 1.2432)$$

$$= 1.2432 + 0.1\{0.2 + 1.2432\} = 1.3875$$

Now modify this value, by modified Euler's formula

$$y_3^{(1)} = \mathrm{y}_2 + \frac{h}{2}\{\, f(x_2, y_2) + f(x_3, y_3)\} = 1.2432 + \frac{0.1}{2}\{(0.2 + 1.2432) + (0.3 + 1.3875)\} = 1.3997$$

$$y_3^{(2)} = \mathrm{y}_2 + \frac{h}{2}\left\{\, f(x_2, y_2) + f\left(x_3, y_3^{(1)}\right)\right\} = 1.2432 + \frac{0.1}{2}\{(0.2 + 1.2432) + (0.3 + 1.3997)\} = 1.4003$$

$$y_3^{(3)} = \mathrm{y}_2 + \frac{h}{2}\left\{\, f(x_2, y_2) + f\left(x_3, y_3^{(2)}\right)\right\} = 1.2432 + \frac{0.1}{2}\{(0.2 + 1.2432) + (0.3 + 1.4003)\} = 1.4004$$

$$y_3^{(4)} = \mathrm{y}_2 + \frac{h}{2}\left\{\, f(x_2, y_2) + f\left(x_3, y_3^{(3)}\right)\right\} = 1.2432 + \frac{0.1}{2}\{(0.2 + 1.2432) + (0.3 + 1.4004)\} = 1.4004$$

Ans.

Question: Obtain y at x = 0.2 in two steps of 0.1 each, from the differential equation

$$\frac{dy}{dx} = -x\,y^2, y = 2 \text{ at } x = 0 \text{ by modified Euler's method.}$$

Solution: Here, $x_0 = 0$ *and* $y_0 = 2$ *and* $y' = -x\,y^2 = f(x,y)$

also h = 0.1, so the required solution in 2 steps is as follows:

By Euler's method, $y_1 = y(x = 0.1) = \mathrm{y}_0 + h\,f(x_0, y_0)$ (Step 1)

$$= 2 + 0.1\,f(0,2)$$

$$= 2 + 0.1\{0\} = 2$$

Now modify this value, by modified Euler's formula

$$y_1^{(1)} = \mathrm{y}_0 + \frac{h}{2}\{\,f(x_0, y_0) + f(x_1, y_1)\} = 2 + \frac{0.1}{2}\{(0) - 0.1(4)\} = 1.98$$

$$y_1^{(2)} = \mathrm{y}_0 + \frac{h}{2}\left\{\,f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right\} = 2 + \frac{0.1}{2}\{(0 - 0.1(3.9204))\} = 1.9804$$

$$y_1^{(3)} = \mathrm{y}_0 + \frac{h}{2}\left\{\,f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right\} = 2 + \frac{0.1}{2}\{(0 - 0.1(3.922)\} = 1.9804$$

Step 2

By Euler's method, $y_2 = y(x = 0.2) = \mathrm{y}_1 + h\, f(x_1, y_1)$ $[y_1 = 1.9804]$

$$= 1.9804 + 0.1\, f(0.1, 1.9804)$$

$$= 1.9804 + 0.1\{-0.1(3.922)\} = 1.9412$$

Now modify this value, by modified Euler's formula

$$y_2^{(1)} = \mathrm{y}_1 + \frac{h}{2}\{\, f(x_1, y_1) + f(x_2, y_2)\} =$$

$$y_2^{(2)} = \mathrm{y}_1 + \frac{h}{2}\left\{\, f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right\} =$$

$$y_2^{(3)} = \mathrm{y}_1 + \frac{h}{2}\left\{\, f(x_1, y_1) + f\left(x_2, y_2^{(2)}\right)\right\} =$$

Runge-Kutta 4th Order Method

Runge-Kutta 4th order method gives more accurate solution of differential equations.

Working rule for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ is as follows:

Calculate successively $k_1 = h\, f(x_0, y_0)$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

and $k_4 = h\, f(x_0 + h, y_0 + k_3)$

Finally compute $k = \frac{1}{6}\,(k_1 + 2\,k_2 + 2\,k_3 + k_4)$

which gives the required approximate value as $y_1 = y_0 + k$

Example: Using Runge-Kutta fourth order method to find an approximate value of y when x = 0.2, given that $\frac{dy}{dx} = x + y\ and\ y = 1$ when $x = 0$

Solution: Here, $x_0 = 0, y_0 = 1, h = 0.2,\ f(x_0, y_0) = 1$

$$\therefore\ k_1 = h\,f(x_0, y_0) = 0.2(1) = 0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0.1, 1.1) = 0.24$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f(0.1,1.12) = 0.244$$

and $$k_4 = h\,f(x_0 + h, y_0 + k_3) = 0.2\,f(0.2,1.244) = 0.2888$$

$$\therefore\ k = \frac{1}{6}\,(k_1 + 2\,k_2 + 2\,k_3 + k_4) = \frac{1}{6}\,(0.2 + 2\,(0.24) + 2(0.244) + 0.2888) = 0.2428$$

Thus the required approximate value of y = y$(x = 0.2) = y_1 = y_0 + k$

$$= 1 + 0.2428 = 1.2428$$

Example: Using Runge-Kutta fourth order method, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ $with\ y(0) = 1$ at $x = 0.2, 0.4$

Solution: We have $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$

$To\ find\ y(0.2)$:

$$Here,\ x_0 = 0, y_0 = 1, h = 0.2$$

$$\therefore\ k_1 = h\,f(x_0, y_0) = 0.2\left(\frac{1-0}{1+0}\right) = 0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0.1, 1.1) = 0.2\,\frac{1.21 - 0.01}{1.21 + 0.01} = 0.19672$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f(0.1, 1.09836) = 0.1967$$

and $$k_4 = h\,f(x_0 + h, y_0 + k_3) = 0.2\,f(0.2, 1.1967) = 0.1891$$

$$\therefore\ k = \frac{1}{6}\,(k_1 + 2\,k_2 + 2\,k_3 + k_4) = \frac{1}{6}\,(0.2 + 2\,(0.19672) + 2(0.1967) + 0.1891) = 0.19599 = 0.196$$

Hence $y(x = 0.2) = y_1 = y_0 + k = 1.196$ Ans.

$Now, to\ find\ y(0.4)$:

$$Here,\ x_1 = 0.2, y_1 = 1.196, h = 0.2$$

$$\therefore\ \ k_1 = h\,f(x_1, y_1) = 0.2\left(\frac{1.43042-0.04}{1.43042+0.04}\right) = 0.1891$$

$$k_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2f(0.3, 1.2906) = 0.1795$$

$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2f(0.3, 1.2858) = 0.1793$$

and $k_4 = h\,f(x_1 + h, y_1 + k_3) = 0.2\,f(0.4, 1.3753) = 0.1688$

$$\therefore\ k = \frac{1}{6}(k_1 + 2\,k_2 + 2\,k_3 + k_4) = \frac{1}{6}(0.1891 + 2\,(0.1795) + 2(0.1793) + 0.1688) = 0.1792$$

Hence $y(x = 0.4) = y_2 = y_1 + k = 1.196 + 0.1792 = 1.3752$ Ans

Example: Apply Runge-Kutta fourth order method to find approximate value of y for x = 0.2, in the steps of 0.1, if $\frac{dy}{dx} = x + y^2 \; given \; that \; y(0) = 1$

Solution: We have $\mathrm{f(x, y)} = x + y^2$

$Here \; we \; take \; h = 0.1 \; and \; carry \; out \; the \; calculations \; in \; 2 \; steps$

Step I $\quad x_0 = 0, y_0 = 1, h = 0.1$

$$\therefore \; k_1 = h\,f(x_0, y_0) = 0.1f(0, 1) = 0.1(0 + 1) = 0.1$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(0.05, 1.05) = 0.1152$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(0.05, 1.0576) = 0.1168$$

and $\quad k_4 = h\,f(x_0 + h, y_0 + k_3) = 0.1\,f(0.1, 1.1168) = 0.1347$

$$\therefore \; k = \frac{1}{6}\,(k_1 + 2\,k_2 + 2\,k_3 + k_4) = \frac{1}{6}\,(0.1 + 2\,(0.1152) + 2(0.1168) + 0.1347) = 0.1165$$

Hence $\mathrm{y(x} = 0.1) = y_0 + k = 1.1165$

Step II $x_1 = x_0 + h = 0.1,\ y_1 = 1.1165, h = 0.1$

$$\therefore\ k_1 = h\, f(x_1, y_1) = 0.1 f(0.1, 1.1165) = 0.1(0.1 + 1.2466) = 0.1347$$

$$k_2 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1 f(0.15, 1.1138) = 0.1551$$

$$k_3 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1 f(0.15, 1.194) = 0.1576$$

and $k_4 = h\, f(x_1 + h, y_1 + k_3) = 0.1\, f(0.2, 1.2741) = 0.1823$

$$\therefore\ k = \frac{1}{6}\,(k_1 + 2\,k_2 + 2\,k_3 + k_4) = \frac{1}{6}\,(0.1347 + 2\,(0.1551) + 2(0.1576) + 0.1823) = 0.1571$$

Hence $\mathrm{y}(\mathrm{x} = 0.2) = y_1 + k = 1.1165 + 0.1571 = 1.2736$ Ans.

Milne's Predictor and Corrector Method

Milne's Predictor Formula is applicable if four consecutive values of y at 4 equally spaced points of x, as shown in the adjoining table:

x	x_0	$x_1 = x_0 + h$	$x_2 = x_1 + h$	$x_3 = x_2 + h$	$x_4 = x_3 + h$
y	y_0	y_1	y_2	y_3	?

and the general predictor formula is

$$y_{n+1\,(p)} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_{n}\right)$$

Putting n = 3, we get

$$y_{4\,(p)} = y_0 + \frac{4h}{3}\left(2y'_{1} - y'_{2} + 2y'_{3}\right) \text{ .....(1)}$$

The general corrector formula is

$$y_{n+1\,(c)} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_{n} + y'_{n+1}\right)$$

Putting n = 3, we get

$$y_{4\,(c)} = y_2 + \frac{h}{3}\left(y'_{2} + 4y'_{3} + y'_{4}\right) \text{ ......(2)}$$

Note:
First we predict the value y_4 by Milne's predictor formula(1) and then correct it by Milne's predictor formula (2).

Example: Find y(2) by Milne's Predictor corrector method from the following data:

$\boldsymbol{y' = \frac{x+y}{2}}$,

x	$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$	$x_3 = 1.5$
y	$y_0 = 2$	$y_1 = 2.636$	$y_2 = 3.595$	$y_3 = 4.968$

Solution: Milne's Predictor formula is

$$\boldsymbol{y_{4\,(p)} = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right)} \quad ....(1)$$

As the given diff. equation is $\boldsymbol{y' = \frac{x+y}{2}}$,

Therefore, $y'_1 = \frac{\boldsymbol{x_1+y_1}}{\boldsymbol{2}} = \frac{\boldsymbol{0.5+2.636}}{\boldsymbol{2}} = 1.568$

$y'_2 = \frac{x_2+y_2}{2} = \frac{1+3.595}{2} = 2.2975$

$y'_3 = \frac{x_3+y_3}{2} = \frac{1.5+4.968}{2} = 3.234$

Putting the values in (1), we get

$$\boldsymbol{y_{4\,(p)} = 2 + \frac{4(0.5)}{3}\{2(1.568) - 2.2975 + 2(3.234)\}}$$

$$\boldsymbol{= 2 + \frac{2}{3}\{3.136 - 2.2975 + 6.468\}}$$

$\boldsymbol{= 6.871}$ is the predicted value

Now, by using $x_4 = 2$ *and* $y_4 = 6.871$, calculate $y'_4 = \frac{x_4+y_4}{2} = \frac{2 + 6.871}{2} = 4.4355$

we will correct it by Milne's corrector formula to get the final value:

$$\boldsymbol{y_{4\,(c)} = y_2 + \frac{h}{3}\left(y'_2 + 4y'_3 + y'_4\right)}$$

$$= 3.595 + \frac{0.5}{3}\{2.2975 + 4\,(3.234)\ + 4.4355\}$$

$$= 3.595 + \frac{0.5}{3}\{2.2975 + 12.936\ + 4.4355\}$$

$$= 3.595 + \frac{1}{6}\{19.669\} = 3.595 + 3.2782 = 6.8732$$

Therefore $y(2) = 6.8732$ Ans.

Example: Find y(0.4) by Milne's Predictor corrector method, given that

$\frac{dy}{dx} = \frac{(1+x^2)\,y^2}{2}$,

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 1.06$	$y_2 = 1.12$	$y_3 = 1.21$

Solution: Milne's Predictor formula is

$$y_{4\,(p)} = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right) \quad ....(1)$$

As the given diff. equation is $y' = \frac{(1+x^2)\,y^2}{2}$,

Therefore, $y'_1 = \frac{(1+x^2)\,y^2}{2} = \frac{(1.01)1.1236}{2} = 0.5674$

$$y'_2 = \frac{(1+x^2)\,y^2}{2} = \frac{(1.04)1.2544}{2} = 0.6523$$

$$y'_3 = \frac{(1+x^2)\,y^2}{2} = \frac{(1.09)1.4641}{2} = 0.7979$$

Putting the values in (1), we get

$$y_{4\,(p)} = 1 + \frac{4(0.1)}{3}\{2(0.5674) - 0.6523 + 2(0.7979)\}$$

$$= 2 + \frac{0.4}{3}\{\,1.1348 - 0.6523 + 1.5958\}$$

$= 2.2771$ is the predicted value

Now, by using $x_4 = 0.4 \ and \ y_4 = 1.2771$, calculate $y'_4 = \frac{(1+x^2)\,y^2}{2} = \frac{(1.16)1.63098}{2} = 0.94597$

we will correct it by Milne's corrector formula to get the final value:

$$\boldsymbol{y_{4\,(c)} = y_2 + \frac{h}{3}\left(y'_2 + 4y'_3 + y'_4\right)}$$

$$= 1.12 + \frac{0.1}{3}\{0.6523 + 4\,(0.7979)\ + 0.94597\}$$

$$= 1.12 + \frac{0.1}{3}\{0.6523 + 3.1916\ + 0.94597\}$$

$$= 1.12 + \frac{0.1}{3}\{4.78987\} = 1.12 + 0.15966 = 1.27966$$

Therefore y$(0.4) = 1.27966$ Ans.

Practice Questions

1. Apply Runge-Kutta fourth order method to find approximate value of y for x = 0.8,

   if $\frac{dy}{dx} = \sqrt{(x+y)}$ $given\ that\ y(0.4) = 0.41$

2. Apply Runge-Kutta fourth order method to find approximate value of y for x = 0.2,

   if $\frac{dy}{dx} = 3x + \frac{1}{2}y$ $given\ that\ y(0) = 1$ taking $h = 0.1$

3. Using Runge-Kutta's method of order 4, find y for x = 0.1, 0.2, 0.3 given that

   $\frac{dy}{dx} = xy + y^2, y(0) = 1.$

4. Find y(1.0) accurate up to four decimal places using Modified Euler's method by solving the IVP $y' = -2xy^2$, $y(0) = 1$ with step length 0.2.

5. If $\frac{dy}{dx} = 2e^x - y, y(0) = 2, y(0.1) = 2.010, y(0.2) = 2.040\ and\ y(0.3) = 2.090.$

   Find y(0.4) by employing the Milne's predictor-corrector method

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Thanks