Unit 1	**Non-Linear Equations and system of linear equation:** Introduction, error and error propagation. Bisection method, False position Method, Method of Iteration, Newton-Raphson Method, Secant Method, Gauss Elimination method, Gauss –Seidel method

Introduction to Error

Exact Numbers

Numbers with which no uncertainty is associated or no approximation is taken, are known as **exact numbers** e.g., 3, 8, 11 , … are exact numbers.

Approximate Numbers

There are numbers which are not exact e.g., $\sqrt{2} = 1.41421356\ldots, e = 2.71828182\ldots.\ etc.$ as they contain infinitely many digits. The numbers obtained by retaining a few digits, are called **approximate numbers.** Consider the numbers $0.33,\ 3.14,\ 2.718$ are the approximate values of $\frac{1}{3}, \pi,$ $e\ respectively, so\ are\ the\ approximate\ numbers$.

Rounding off Numbers

If we divide 2 by 7, we get $\frac{2}{7} = 0.285714\ldots.$, a quotient which is a non-terminating decimal fraction. For using such a number in practice, it is to be cut-off to a manageable size such as 0.29, 0.286, 0.2857, …

The process of cutting-off superfluous digits and retaining as many digits as desired is known as **rounding-off a number.**

Significant Figures

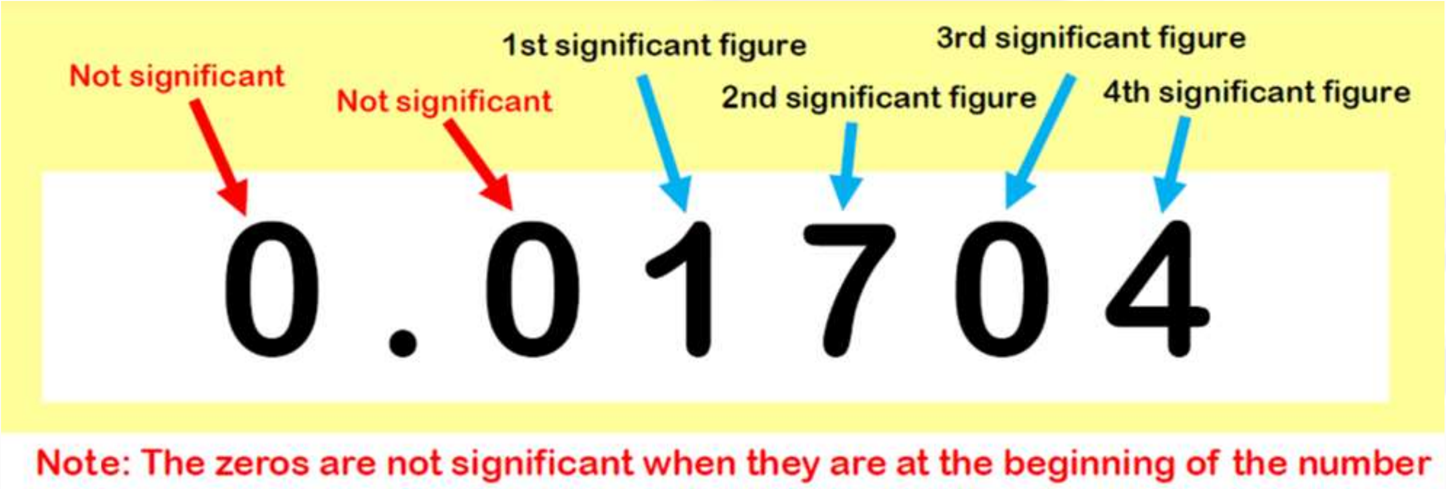
The digits which are used to express or represent a number are called significant figures. All the digits 1, 2, 3,….9 are significant figures. Zero may or may not be a significant figure. It depends on the context in which zero has been used.

When zero is used to fix up the decimal point or fill up the places of discarded digits, it is not a significant figure.

e.g., the numbers 0.00056 and 3200 (correct to two significant figure) zeros are insignificant

But in the number 320 (correct to three significant figures), zero is significant figure.

And, zero used between two non-zero digits is a significant figure.



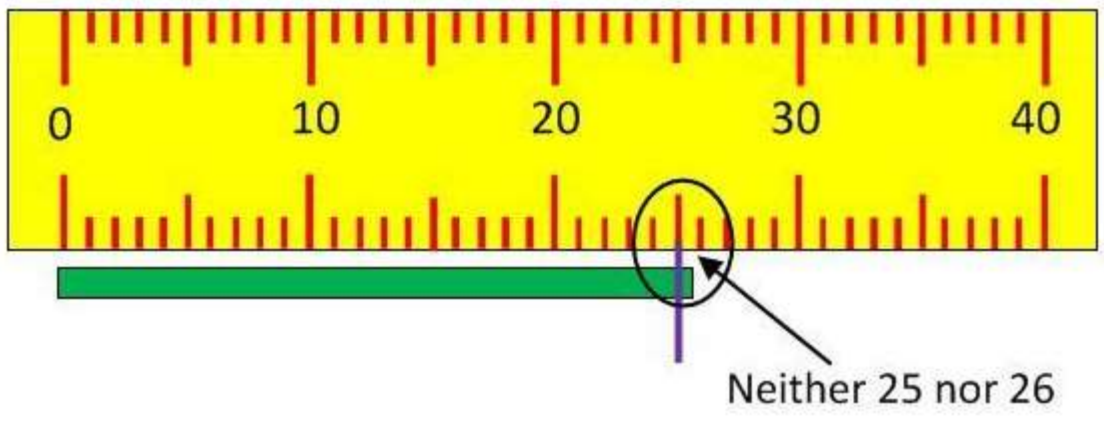
Rules for Rounding off Numbers

If the first digit to be dropped is:	Then the last significant digit should:	Examples (Rounded off to 3 sig. figs.)
Greater than 5	Be increased by 1	42.68 → 42.7 5.3999 → 5.40
Less than 5	Stay the same	17.326 → 17.3 0.9994 → 0.999
5, followed by nonzero digit(s)	Be increased by 1	2.7852 → 2.79 15.555 → 15.6

Error

The solutions of mathematical problems are of two types: analytical and numerical. The analytical solutions can be expressed in closed form and these solutions are error free. On the other hand, numerical method which solves problems using computational machines and provides approximate values.

Thus, Errors may occur at any stage of the process of solving a problem.



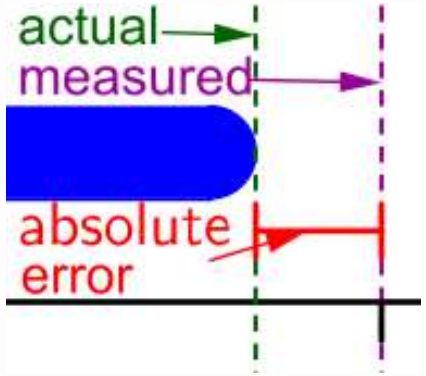
Error - Definition

The **error** is defined as the difference between the exact value and the approximate value.

i.e., Error = Exact Value – Approximate Value

e.g., approximate value of 5/3 = 1.667 up to 3 decimal places.

Therefore, Error, $\mathrm{E} = \frac{5}{3} - 1.667 = \frac{5-5.001}{3} = -\frac{0.001}{3}$



Absolute Error

$If\ x'\ is\ the\ approximate\ value\ of\ exact\ number\ x, then\ the\ absolute\ error, is$

$$E_a = \Delta_x = |x - x'|$$

Relative Error

$If\ x'\ is\ the\ approximate\ value\ of\ exact\ number\ x, then\ the\ relative\ error, is$

$$E_r = \left|\frac{x - x'}{x}\right|\ i.e. E_r = \left|\frac{Error}{exact\ value}\right|$$

Percentage Error, $E_p = E_r \times 100$

Example

Example: If 0.333 is the approximate value of 1/3, find absolute, relative and percentage errors.

Solution: (i) $Here, actual\ value, x = \frac{1}{3}, approximate\ value, x' = 0.333$

$$\therefore\ Absolute\ error, E_a = |x - x'| = \left|\frac{1}{3} - 0.333\right|$$

$$= \left|\frac{1}{3} - \frac{333}{1000}\right| = \frac{1}{3000} = 0.00033$$

(ii) $Relative\ error, E_r = \left|\frac{x - x\prime}{x}\right| = \left|\frac{0.00033}{\frac{1}{3}}\right| = 0.00099$

(iii) Percentage Error, $E_p = E_r \times 100 = 0.00099 \times 100 = 0.099$

Types of Errors

Inherent Errors: Errors which exist in the problem itself, before determining its solution. Inherent errors occur due to approximate given data or limitations of computing aids.
e.g., the area of a circle is calculated by taking the value of π (as 3.14 which is a approximate value of π.

Round-off Errors: These errors are machine dependent, arise due to rounding off the numbers during the process of computations. The round off errors can be reduced by doing the computations to more significant digits at each stage of computation.
e.g., if a number with large number of digits such as 8.26758467351, is rounded off to a significant figure 8.2676, so in this process, the round-off error is
8.26758467351 - 8.2676 = -0.00001532649
Some numbers viz. 1/3, 2/3, 1/7 etc. have infinite no. of decimal digits. Thus, to get the result, the numbers are to be rounded-off into some finite number of digits.
e.g., $\frac{1}{3} \approx .333333\ldots$ Round-off error can be reduced by using higher precision (64 bit rather than 32)

Truncation Errors: These errors are method dependent, arise by truncating the infinite series to some approximate terms. e.g., the use of an infinite series to compute the value of $\cos x, \sin x, e^x$, etc.

The Maclaurin's series of the value of $cos\, x$ is $cos\, x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

This is an infinite series expansion but if we use only first three terms in the expansion then this approximate value will lead to truncation error.

Truncation error can be reduced by using smaller steps $\Delta t,$ higher order algorithm

Propagation of Errors

By propagation of errors, we mean an error occurring in the succeeding steps of a process due to the occurrence of an earlier error. Suppose we are given two numbers and these numbers are rounded off to some significant digits then these numbers will contain some roundoff error and if we perform any arithmetic operation like addition, subtraction, multiplication or division on the numbers then the resultant will also be erroneous. Similarly, if we have a function of one or more independent variables, and if there exist any error in independent variables, then dependent variable will also be affected, due to accumulation of errors. This is called as propagation of errors.

Error Accumulation

Consider the exact numbers $X_1, X_2, \dots, X_n$ and their approximate values respectively $x_1, x_2, \dots, x_n$.

Let $\Delta x_1, \Delta x_2, \dots, \Delta x_n$ be the errors in $x_1, x_2, \dots, x_n$

i.e. $X_i = x_i \pm \Delta x_i$, $\text{i} = 1,2, \dots \text{n}$. And let, $X = X_1 + X_2 \dots + X_n$ and $x = x_1 + x_2 + \cdots + x_n$.

Therefore, the total absolute error is

$|X - x| = |(X_1 - x_1) + (X_2 - x_2) + \cdots + (X_n - x_n)|.$

$|\Delta x| \leq |\Delta x_1| + |\Delta x_2| + \cdots |\Delta x_n|.$

Similarly in case of subtraction, multiplication and division of numbers containing errors, the total absolute error can be calculated.

Example: Estimate the absolute error in $\sqrt{6}+\sqrt{7}+\sqrt{8}$ correct to 4 significant digits.

Solution: True Values are $\sqrt{6} = 2.44948 \ldots$,$\sqrt{7} = 2.64575 \ldots$, $\sqrt{8} = 2.82842 \ldots$

Approximate values correct to 4 significant digits are $\sqrt{6} = 2.449$, $\sqrt{7} = 2.646$, $\sqrt{8} = 2.828$

Let $X = \sqrt{6}+\sqrt{7}+\sqrt{8}$ and $x = 2.449 + 2.646 + 2.828$

Absolute error $|X - x| \leq |2.44948 - 2.449| + |2.64575 - 2.646| + |2.82842 - 2.828|$

$$\Rightarrow \quad |X - x| \leq 0.00115$$

Error in the approximation of function

$Let\ y = f(x_1, x_2)\ \ldots(1)\ \ be\ a\ function\ of\ of\ two\ variables\ \ x_1\ \ \&\ \ x_2.$

$If\ \delta x_1, \delta x_2\ be\ the\ errors\ in\ x_1\ \ \&\ \ x_2, then\ the\ error\ in\ y\ is\ given\ by$

$$y + \ \delta y = f(x_1 + \delta x_1, x_2 + \delta x_2)$$

$$= f(x_1, x_2) + \left(\frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2\right)\ \ \ldots.(2) \qquad [By\ applying\ Yaylor's\ series]$$

$terms\ involving\ higher\ powers\ of\ \delta x_1\ and\ \ \delta x_2\ (may\ be\ neglected)$

$$(2) - (1)\ gives\ \ \delta y = \frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2$$

$$And\ the\ relative\ error\ in\ y\ is\ \frac{\delta y}{y} = \frac{\partial f}{\partial x_1}.\frac{\delta x_1}{y} + \frac{\partial f}{\partial x_2}.\frac{\delta x_2}{y}$$

$$In\ general, the\ error\ in\ \delta y\ in\ the\ functiony = f(x_1, x_2, x_3, ..x_n)\ is\ \delta y = \frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2 + \frac{\partial f}{\partial x_3}\delta x_3 + \cdots.\frac{\partial f}{\partial x_n}\delta x_n$$

$Example: Find\ the\ percentage\ error\ in\ the\ area\ of\ a\ rectangle\ when\ an\ error\ of$
$+1\ \%$ in $measuring\ its\ length\ and\ breadth.$

$Solution: Let\ the\ area\ of\ a\ rectangle, A = x.\ y,$ $where\ x\ \&\ y\ are\ length\ and\ breadth\ respectively.$

It is given % age errors in x & y as $\frac{\delta x}{x} \times 100 = 1\ and\ \frac{\delta y}{y} \times 100 = 1$

$By\ using\ the\ formula, error\ in\ A, dA = \frac{\partial A}{\partial x}\delta x + \frac{\partial A}{\partial y}\delta y$

$$= y.\delta x + x.\delta y$$ [since $\frac{\partial A}{\partial x} = y$ & $\frac{\partial A}{\partial y} = x$]

$Dividing\ by\ A = xy\ on\ both\ side,$ $\frac{dA}{A} = \frac{y.\delta x + x.\delta y}{xy} = \frac{\delta x}{x} + \frac{\delta y}{y}$

$$\Rightarrow \frac{dA}{A} \times 100 = \frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100 = 1 + 1 = 2$$

$Hence, \%\ age\ error\ in\ area\ of\ rectangle = 2\ \%$

Solution of Non-Linear Equations

(Algebraic & Transcendental Equations)

In many engineering problems, it is required to find the solution of the equation of the form f(x) = 0, where f(x) = 0 may be algebraic or transcendental equation.

Polynomial: An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$is called a polynomial in x of degree n, provided $a_0 \neq 0, where\ a_0, a_1, a_2,\ ..a_n$ are constants(real or complex)

Algebraic Equation: A polynomial in x of degree n, when equated to zero i.e., f(x) = 0 is called an algebraic equation of degree n.

Transcendental Equation: If the polynomial f(x) involves the functions of the form such as trigonometric, logarithmic, exponential etc., then f(x) = 0 is called a transcendental equation.

e.g., $x^3 + 3x^2 + 7 = 0\ is\ an\ algebraic\ equation,\ \ x^3 + logx = 3\ is\ a\ transcedental\ equation$

& $x + cosx + logx = 2\ is\ a\ transcedental\ equation$

Location of Root

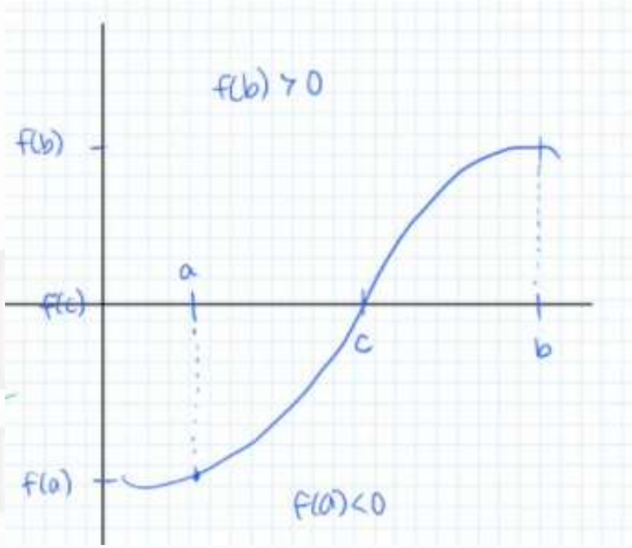
Numerical methods often a repetitive nature. These consist in repeated execution of the same process where at each step the result of the proceeding step is used. This is known as “Iteration Process” and is repeated till the result obtained to desired degree of accuracy.

Before to apply numerical methods to solve an equation, let us discuss a basic rule:

Location of Root (Intermediate Value Theorem)

If f(x) is continuous in the closed interval [a, b]

And f(a), f(b) are of opposite signs, then the equation f(x) = 0 has at least one root within [a, b]



Example: Find the interval in which the any one root of the equation, $x^3 + 3x^2 + 7 = 0$ lies.

Solution: $Let\ f(x) = x^3 + 3x^2 + 7$

$By\ trial, f(-4) = -64 + 48 + 7 = -9\ (-ve)$

$and\quad f(-3) = -27 + 27 + 7 = 7\ (+ve)$

$So, by\ intermediate\ value\ thm,\ one\ root\ of\ eqn\ x^3 + 3x^2 + 7 = 0\ lies\ in\ between\ [-4, -3]$

Example: Find the root lying intervals for the equation, $x^6 - 8\,x^4 + 10\,x^2 - 1 = 0.$

Solution: $Let\ f(x) = x^6 - 8\,x^4 + 10\,x^2 - 1$

$By\ trial, f(0) = -1\ (-ve)$

& $f(1) = 1 - 8 + 10 - 1 = 2\ (+ve)$

$So, one\ of\ the\ roots\ lies\ in\ (0, 1)$

Similarly, $f(2) = 64 - 128 + 40 - 1 = -25(-ve)$

$So, the\ another\ root\ of\ given\ eqn\ lies\ in\ (1, 2)$

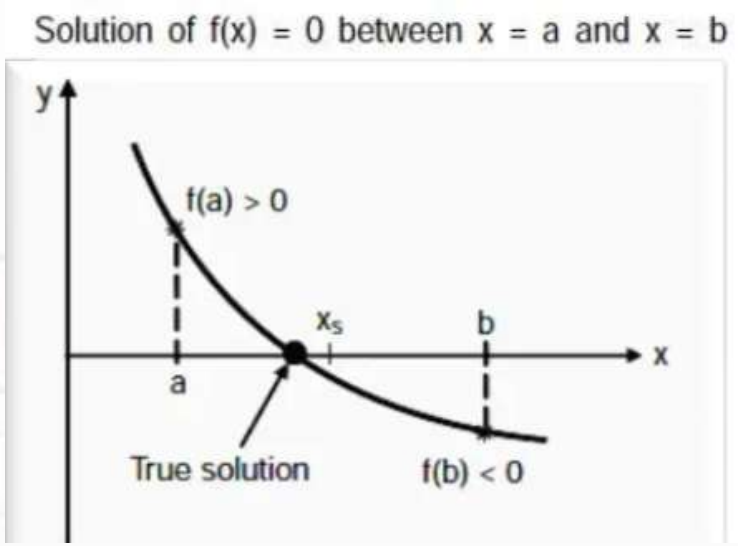
Bisection Method (Bolzano Method)

This method is to solve an equation f(x) = 0 is based on the repeated application of the intermediate value theorem. In this method consider two points ‘a’ and ‘b’ such that f(a) and f(b) are of opposite signs, so at least one real root of f(x) = 0 lies between a and b.

The approximation of the required root is given by $x_1, as\ x_1 = \frac{a+b}{2}$ $(bisection\ of\ the\ interval)$

If $f(x_1) = 0, then\ x_1 is\ a\ root\ of$ f(x)=0 otherwise, the root lies between $a\ \&\ x_1\ or\ \ x_1\ \&\ b$ according as $f(x_1)$ is $+ve\ \ or\ \ -ve$.

Then bisect the root lying interval as before and continue the process until the root is found to desired accuracy.



Example: Find the real root of the equation $x \log_{10} x = 1.2$ by bisection method up to four iterations.

Solution: Let $f(x) = x \log_{10} x - 1.2$

Here, f(2) = - ve

f(3) = + ve , so root lies between 2 & 3

By bisection method, first approximation of the root is, $x_1 = \frac{2+3}{2} = 2.5$

Now, f(2.5) = - ve, i.e., root lies in (2.5, 3), therefore 2nd approximation of the root is, $x_2 = \frac{2.5+3}{2} = 2.75$

And now, f(2.75) = + ve so root lies in (2.5, 2.75)

3rd approximation of the root is, $x_3 = \frac{2.5+2.75}{2} = 2.625$

And f(2.625) = - ve, so root lies in (2.625, 2.75)

4th approximation of the root is, $x_4 = \frac{2.625+2.75}{2} = 2.6875$ is the required root up to 4th iteration.

Example: Find a real root of the equation $x^3 - 4x - 9 = 0$ by bisection method up to three iterations.

Solution: Let f$(x) = x^3 - 4x + 9$

Here, f(-1) = -1+ 4 + 9 = + ve

f(-2) = - 8 + 8 + 9 = + ve,

f(-3) = - 27 + 12 + 9 = - ve so root lies between - 3 & - 2

By bisection method, first approximation of the root is, $x_1 = \frac{-3\ -2}{2} = -2.5$

Now, f(- 2.5) = + ve, i.e., root lies in (- 3, - 2.5), so 2nd approximation of the root is, $x_2 = \frac{-3\ -\ 2.5}{2} = -2.75$

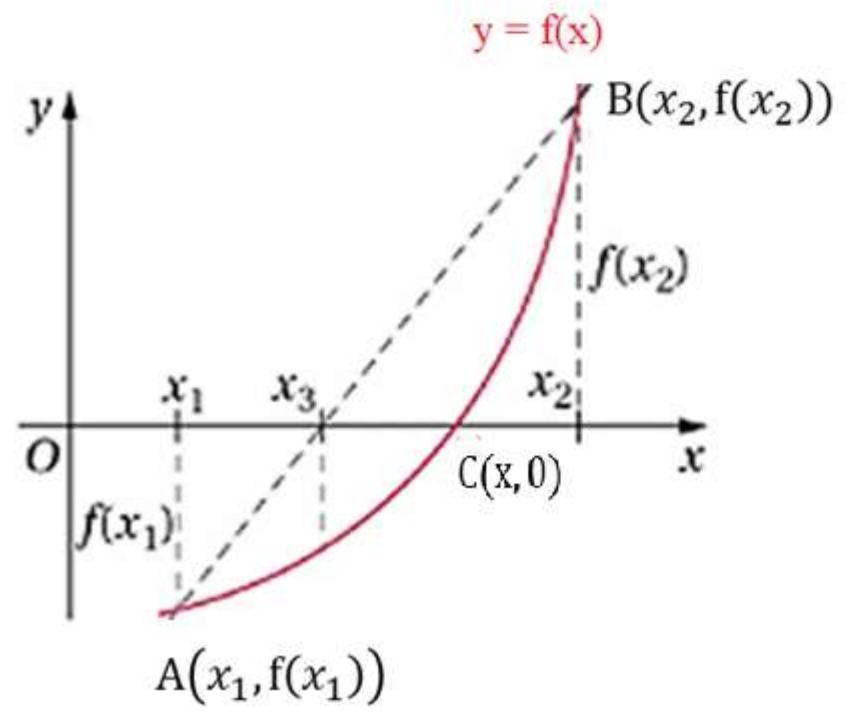
And now, f(- 2.75) = - 20.796875 + 11 + 9 = - ve so root lies in (- 2.75, - 2.5)

3rd approximation of the root is, $x_3 = \frac{-2.75-2.5}{2} = -2.625$

is the required root up to 3rd iteration.

Method of Regula Falsi or False Position

Let the equation y = f(x) = 0 .... (1)
passes through the points $A(x_1, f(x_1))$ & $B(x_2, f(x_2))$
such that $f(x_1).f(x_2) < 0$
If the secant(chord) AB meets x-axis at C, then abscissa of C is an approximation of root of f(x) = 0



Equation of straight line AB is

$$y - f(x_1) = \frac{f(x_2)-f(x_1)}{x_2-x_1}(x - x_1) \quad \ldots (2)$$

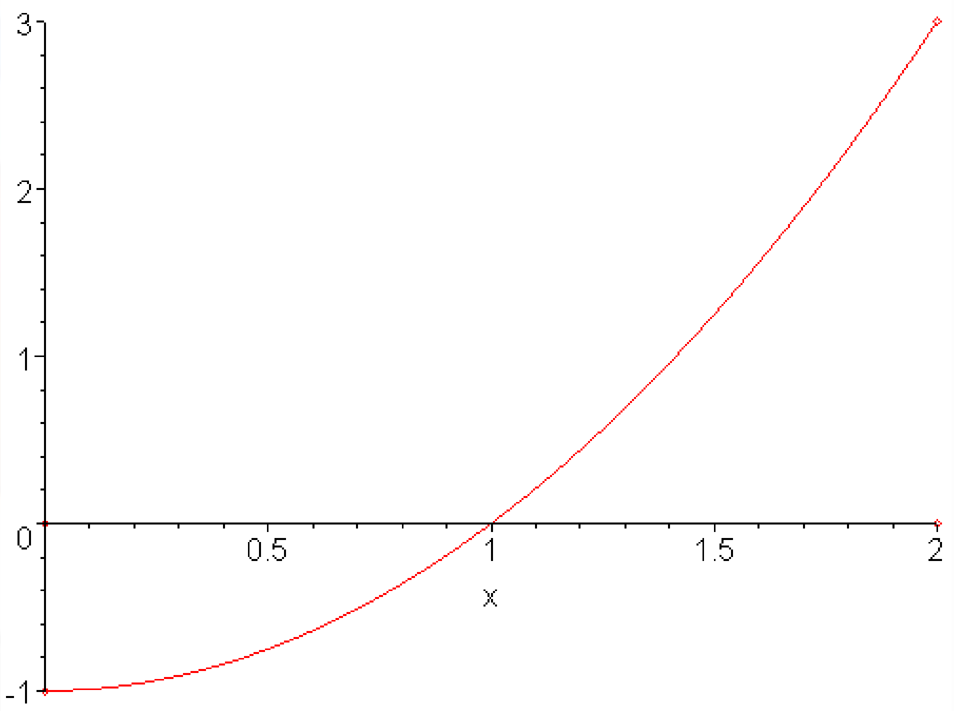
At C, $y = 0$, so eqn. (2) becomes

$$-f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} . f(x_1) \qquad = \frac{x_1 f(x_2) - x_1 f(x_1) - x_2 f(x_1) + x_1 f(x_1)}{f(x_2) - f(x_1)} = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$The\ Regula\ Falsi\ Itreration\ Algoritm\ is$$

$$x_{i+1} = \frac{x_{i-1}f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$



Remember the numerator by the following operations:

$$\begin{vmatrix} x_{i-1} & x_i \\ f(x_{i-1}) & f(x_i) \end{vmatrix}$$

Example: Find a real root of the equation $x\, e^{x} - 3 = 0$ correct to three decimal places by false position method.

Solution: Let $f(x) = x\, e^{x} - 3$

Here, f(1) = - 0.2817 = - ve

f(1.1) = 0.3046 = + ve

so root lying interval is (1, 1.1) [$x_1 = 1$ & $x_2 = 1.1$]

First approximation of the root, $x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

$= \frac{1(0.3046) - 1.1(-0.2817)}{0.3046 + 0.2817} = \frac{0.61447}{0.5863} = 1.048$ and f(1.048) = - 0.0112 (-ve)

Thus the root lying interval shrunk to (1.048, 1.1)

So, next approximation of the root, $x_4 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

$= \frac{1.048(0.3046) - 1.1(-0.0112)}{0.3046 + 0.0112} = \frac{0.33154}{0.3158} = 1.0498$ Ans.

Newton –Raphson Method

Solution of equation f(x) = 0 ….(1) by Newton-Raphson Method

$Let\ x_0\ be\ an\ approximate\ root\ of\ f(x) = 0\ and\ let\ x_1 = x_0 + h \quad ....(2)\ be\ the\ correct\ root$

$so\ that\ f(x_1) = 0$

$i.e.,\ \ f(x_0 + h) = 0$

$\Rightarrow f(x_0) + h\, f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ .... = 0$ [By Taylor's Expansion]

$\Rightarrow f(x_0) + h\, f'(x_0) = 0 \quad [h\ is\ small\ so\ neglect\ higher\ powers\ h^2,\ \ h^3\ \ and\ so]$

$\Rightarrow h = -\frac{f(x_0)}{f\prime(x_0)}, \quad f'(x_0) \neq 0$

$\therefore\ \ x_1 = x_0 - \frac{f(x_0)}{f\prime(x_0)}$

Newton –Raphson Method

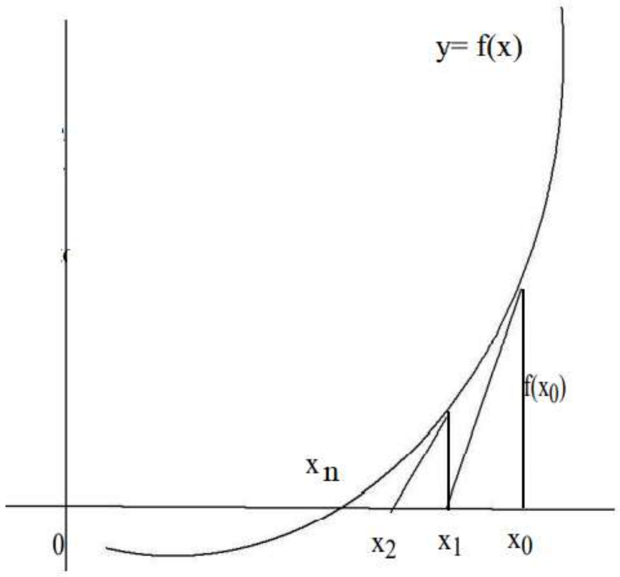
$\sim ly, 2nd\ approximation\ of\ the\ root\ is$

$$x_2 = x_1 - \frac{f(x_1)}{f\prime(x_1)}\ ,\ f'(x_1) \neq 0$$

By successive approximations, nth approximation of the root is

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f\prime(x_{n-1})},\quad f'(x_{n-1}) \neq 0$$

is the Newton-Raphson formula



NEWTON-RAPHSON FIGURE

Note 1: This method fails if f'(x) = 0

Note 2: The initial approximation should be taken very close to the root, otherwise the method may diverges.

Example: Find the real root of the equation $x^3 - x - 2 = 0$ by Newton-Raphson Method.

Solution: $Let\ f(x) = \ x^3 - x - 2$

Here, f(1.5) = 3.375 – 1.5 – 2 = - 0.125
and f(1.6) = 4.096 – 1.6 – 2 = + 0.496 therefore root of given equation lies between (1.5, 1.6)

Let the initial value of the root be $x_0 = 1.5$

$Also\ f'(x) = 3x^2 - 1\ and\ f'(1.5) = 5.75 \neq 0$

$Thus\ by\ Newton\ Raphson\ method, First\ approximate\ root\ is$

$$x_1 = x_0 - \frac{f(x_0)}{f\prime(x_0)} = 1.5 - \frac{-0.125}{5.75} = 1.5 + 0.0217 = 1.5217$$

$Second\ approximate\ root, x_2 = x_1 - \frac{f(x_1)}{f\prime(x_1)} = 1.5217 - \frac{0.0019}{5.9467} = 1.5214$

Thus the required root of given equation upto three decimal places is 1.521

Example: Find a real root of equation $2x - \log_{10} x - 7 = 0$ by Newton-Raphson Method.

Solution: Let $f(x) = 2x - \log_{10} x - 7 = 0$

Here, f(3.7) = 7.4 – 0.5682 – 7 = - 0.1682 (- ve)
and f(3.8) =7.6 – 0.5798 – 7 = 0.0202 (+ ve)   therefore root of given equation lies between (3.7, 3.8)

Let the initial value of the root be $x_0 = 3.7$

$$Also\ f'(x) = 2 - \frac{1}{x}\log_{10} e \ \ and\ f'(3.7) = 2 - \frac{0.4343}{3.7} = 1.8828 \neq 0$$

Thus by Newton Raphson method, First approximate root is

$$x_1 = x_0 - \frac{f(x_0)}{f\prime(x_0)} = 3.7 - \frac{-0.1682}{1.8828} = 3.7893$$

$$Second\ approximate\ root, x_2 = x_1 - \frac{f(x_1)}{f\prime(x_1)} = 3.7893 - \frac{0.00004}{1.88539} = 3.7893$$

Thus the required root of given equation up to four decimal places is 3.7893

Example: Find a real root of the equation $x\ cosx - 1.3 = 0$ by Newton-Raphson Method.

Solution: Let f(x) $= x\ cosx - 1.3$
And f(1) = 1(0.5403) – 1.3 = - 0.7597 (-ve)
f(2) = 2(- 0.41615) – 1.3 = - 2.1323 (-ve)
f(4.9) = 4.9(0.1865) – 1.3 = - 0.38615
f(5) = 5(0.28366) – 1.3 = 0.1183

Therefore root of the given equation lies between (4.9, 5)

Let us start with initial approximation of the root, $x_0 = 4.9$

Derivative of f(x), is f $'$(x) $= x(-\sin x) + \ cosx$
$f'(4.9) = 4.9(0.9824) + 0.1865 = 5.00026$

$f(4.9772) = 4.9772(0.2617) - 1.3 = 0.0025$
$f'(4.9772) = 4.9772(0.9651) + 0.2617 = 5.0652$

Thus by Newton Raphson method,
First approximation of root is

$$x_1 = x_0 - \frac{f(x_0)}{f\prime(x_0)}$$

$$= 4.9 - \frac{-0.38615}{5.00026} = 4.9 + 0.0772 = 4.9772$$

Second approximation of root is

$$x_2 = x_1 - \frac{f(x_1)}{f\prime(x_1)}$$

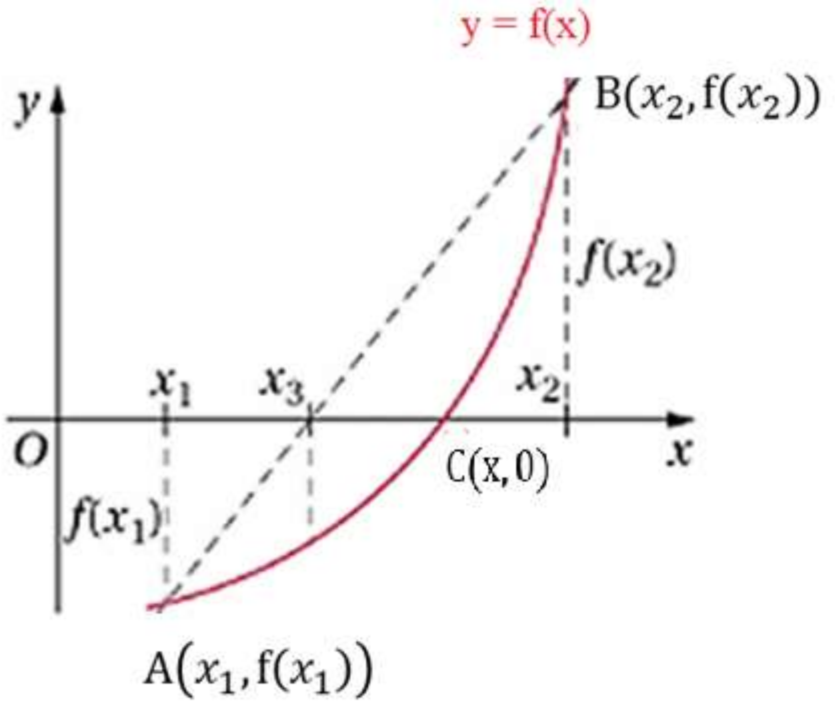
$$= 4.9772 - \frac{f(4.9772)}{f'(4.9772)}$$

$$= 4.9772 - \frac{0.0025}{5.0652} = 4.977$$

Ans.

Secant Method

Let the equation y = f(x) = 0 .... (1)
passes through the points $A(x_1, f(x_1))$ & $B(x_2, f(x_2))$
such that $f(x_1).f(x_2) < 0$
If the secant(chord) AB meets x-axis at C, then abscissa of C is an approximation of root of f(x) = 0



Equation of straight line AB is

$$y - f(x_2) = \frac{f(x_2)-f(x_1)}{x_2-x_1}(x - x_2) \quad \dots (2)$$

At C, $y = 0$, so eqn. (2) becomes

$$-f(x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_2)$$

$$\Rightarrow x = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} . f(x_2)$$

Let x_3 *be the approximated root of* (1),
Consider $x = x_3$ $So, x_3 = x_2 - \frac{x_2-x_1}{f(x_2)-f(x_1)} . f(x_2)$

The general formula for successive approximations is,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} . f(x_n) \quad , \qquad n \geq 1$$

Example: Find the root of the equation cos x $= x\, e^x$ using the secant method.

Solution: Let $f(x) =$ cos x $- x\, e^x$

Here, f(0.5) = 0.8776 – 0.5 $e^{0.5}$ = + 0.0532 (+ ve)
and f(0.6) = 0.8253 – 0.6 $e^{0.6}$ = - 0.2683 (- ve)

Taking the initial approximations $x_1 = 0.5, x_2 = 0.6, f(x_1) = 0.0532, f(x_2) = -0.2683$

First approximation of the root is $x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} . f(x_2)$

$$= 0.6 - \frac{0.6 - 0.5}{-0.2683 - 0.0532} . (-0.2683)$$

= 0.6 – 0.0834 = 0.5166

Second approximation of the root, $x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} . f(x_3)$

$x_2 = 0.6, f(x_2) = -0.2683$

$x_3 = 0.5166, f(x_3) = 0.0035$

$$= 0.5166 - \frac{0.5166 - 0.6}{0.0035 + 0.2683} . (0.0035)$$

$$= 0.5166 + \frac{0.0834}{0.2718} . (0.0035)$$

= 0.5177 Ans.

Iteration Method

To find the roots of the equation $f(x) = 0$ by Iteration Method, write the given equation as

$x = \phi(x), such\ that\ |\phi'(x)| < 1\ \forall\ x \epsilon I, where\ I\ be\ the\ root\ containing\ interval.$

$Let\ x = x_0\ be\ an\ initial\ approximation\ of\ the\ desired\ root\ \alpha\ then\ the\ first\ approximation\ x_1\ is$

$$x_1 = \phi(x_0)$$

$2nd\ approximation,\ x_2 = \phi(x_1)$

$3rd\ approximation,\ x_3 = \phi(x_2)$

$Proceeding\ in\ this\ way, the\ nth\ approximation\ is\ given\ by$

$$x_n = \phi(x_{n-1})$$

Example (Iteration Method)

Example: Find a real root of equation $\cos x = 3x - 1$ using iteration method.

Solution: Let $f(x) = \cos x - 3x + 1$

Here, f(0) = 2 (+ ve)

and $f\left(\frac{\pi}{2}\right) = -3\frac{\pi}{2} + 1 = -4.7124 + 1 = -3.7124$ (- ve)

Therefore, root of the given equation lies between $0\ and\ \frac{\pi}{2}$

Rewriting the given equation as $x = \frac{1}{3}(\cos x + 1) = \phi(x)$ ...(1)

To check whether $|\phi'(x)| < 1\ or\ \ not.$

$Here, \phi'(x) = \frac{1}{3}(-\sin x) and\ |\phi'(x)| = \frac{1}{3}|\sin x| < 1\ in\ \left(0, \frac{\pi}{2}\right)$

Hence iteration method can be applied and we start with $x_0 = 0$

$The\ successive\ approximations\ are\ x_1 = \phi(x_0) = \phi(0) = 0.6667$

$[\ \phi(x) = \frac{1}{3}(\cos x + 1)\ ]$

$$x_2 = \phi(x_1) = \phi(0.6667) = \frac{1}{3}\{\cos(0.6667) + 1\} = 0.5953$$

$$x_3 = \phi(x_2) = \phi(0.5953) = \frac{1}{3}\{\cos(0.5953) + 1\} = 0.6093$$

$$x_4 = \phi(x_3) = \phi(0.6093) = \frac{1}{3}\{\cos(0.6093) + 1\} = 0.6067$$

$$x_5 = \phi(x_4) = \phi(0.6067) = \frac{1}{3}\{\cos(0.6067) + 1\} = 0.6072$$

$$x_6 = \phi(x_5) = \phi(0.6072) = \frac{1}{3}\{\cos(0.6072) + 1\} = 0.6071$$

$Thus\ the\ root\ of\ given\ equation\ is\ 0.607\ correct\ to\ 3\ decimal\ places$

Example: Find a real root of equation $2x - \log_{10} x = 7$ using iteration method.

Solution: Let $f(x) = 2x - \log_{10} x - 7$

Here, f(3) = - 1.4471 (- ve)

and $f(4) = 0.398$ (+ ve)

Therefore, root of the given equation lies between $3\ and\ 4$

Rewriting the given equation as $x = \frac{1}{2}(\log_{10} x + 7) = \phi(x)$

To check whether $|\phi'(x)| < 1\ or\ not$.

$Here, \phi'(x) = \frac{1}{2}\left(\frac{1}{x}\log_{10} e\right)\ and\ |\phi'(x)| = \frac{1}{2}\left|\frac{1}{x}\log_{10} e\right| < 1 \forall\ x\ \epsilon(3, 4)$

Hence iteration method can be applied and we start with $x_0 = 3.6$

$$The\ successive\ approximations\ are\ x_1 = \phi(x_0) = \phi(3.6)$$

$$[\ \phi(x) = \frac{1}{2}(\log_{10} x + 7)]$$

$$= \frac{1}{2}(\log_{10} 3.6 + 7) = 3.77815$$

$$x_2 = \frac{1}{2}(\log_{10} 3.77815 + 7) = 3.7886$$

$$x_3 = \frac{1}{2}(\log_{10} 3.7886 + 7) = 3.7892$$

$$x_4 = \frac{1}{2}(\log_{10} 3.7892 + 7) = 3.7893$$

$$x_5 = \frac{1}{2}(\log_{10} 3.7893 + 7) = 3.7893$$

Hence the approximated root of the given equation is 3.7893

Practice Question

Question: Find the cube root of 15 using iteration method.

Solution: $Let\ \sqrt[3]{15} = x$

$\Rightarrow 15 = x^3 \quad \dots (1)$

$Consider, f(x) = 15 - x^3$

$Here, f(2.4) = 15 - (2.4)^3 = 15 - 13.824 = 1.176\ (+ve)$

$and\ f(2.5) = 15 - (2.5)^3 = 15 - 15.625 = -0.625(-ve)$

Hence root of the $given\ equation\ (1)\ lies\ between\ (2.4, 2.5)$

$For\ applying\ iteration\ method, rewrite\ eqn\ (1)\ as$

$$x = \frac{15 - x^3 + 20\,x}{20} = \phi(x)$$

$$Here, \phi'(x) = \frac{1}{20}(-3\,x^2 + 20) = 1 - \frac{3}{20}x^2$$

$$and\ |\phi'(x)| < 1\ for\ all\ x\ \epsilon\ (2.4, 2.5)$$

Hence iteration method can be applied and we start with $x_0 = 2.4$

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. (Complete yourself)

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Solution of System of Linear Equations by Gauss Elimination Method

A system of linear equations is a group of linear equations with various unknown factors. Solving a system involves finding the value for the unknowns to verify all the equations that make up the system. Here is an example of linear equations in 3 unknowns x, y & z

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Gauss Elimination method can be adopted to find the solution of linear simultaneous equations. In the method, equations are solved by elimination procedure of the unknowns successively. The method overall reduces the system of linear simultaneous equations to an upper triangular matrix. Then backward substitution is used to derive the unknowns.

Gauss Elimination Method

Gauss elimination method is known as the row reduction algorithm for solving linear system of equations. It consists of a sequence of elementary row operations performed on the corresponding matrix of coefficients to reduce it to upper triangular matrix.

Example: Solve the following system of equations using Gauss elimination method.

$$x + y + z = 9$$
$$2x + 5y + 8z = 51$$
$$2x + y - z = 0$$

Solution: Let us write these equations in matrix form as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 5 & 8 & 51 \\ 2 & 1 & -1 & 0 \end{array}\right]$$

Gauss Elimination Method

As system of equations is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 5 & 8 & 51 \\ 2 & 1 & -1 & 0 \end{array}\right]$$

Subtracting $twice\ of\ R_1\ from\ R_2\ to\ get\ new\ elements\ of\ R_2\ \ i.e., R_2 \to R_2 - 2\,R_1$
Subtracting $twice\ of\ R_1\ from\ R_3\ to\ get\ new\ elements\ of\ R_3\ \ i.e., R_3 \to R_3 - 2\,R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 6 & 33 \\ 0 & -1 & -3 & -18 \end{bmatrix}$$

$$Apply, R_2 \to \frac{1}{3} R_2 \quad \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 11 \\ 0 & -1 & -3 & -18 \end{bmatrix}$$

Gauss Elimination Method

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 11 \\ 0 & -1 & -3 & -18 \end{bmatrix}$$

$Operate, R_3 \rightarrow R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & -1 & -7 \end{array}\right]$$

Here, the coefficient matrix is transformed into upper triangular form, we can write linear equations as under:

$$x + y + z = 9 \quad \ldots(1)$$

$$y + 2z = 11 \quad \ldots(2)$$

$$-z = -7 \;\Rightarrow z = 7, by\ back\ substitution,\ we\ get\ y = -3\ from\ (2)$$

$$and\ (1)\ gives\ x - 3 + 7 = 9 \quad i.e, x = 5$$

Thus the required Answer is $x = 5, y = -3, z = 7$

Gauss-Seidal Method

Steps to solve a system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$ by Gauss- Seidal method.

1. Rewrite the system of given equations as

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$

$$y = \frac{1}{b_2}(d_2 - a_2x - c_2\,z)$$

$$z = \frac{1}{c_3}(d_3 - a_3\,x - b_3\,y)$$

-------- (*)

2. Start with the initial approximations x_0, y_0, z_0 for x, y, z respectively. Substituting $y = y_0, z = z_0$ in first of the equations of (*), to get

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

3. Start with the initial approximations x_0, y_0, z_0 for x, y, z respectively. Substituting $y = y_0, z = z_0$ in first of the equations of (*), to get

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

4. Then put $x = x_1, z = z_0$ in the second of the equations of (*) to get

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_0)$$

5. Next substitute $x = x_1, y = y_1$ in the third of the equations of (*) to get

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$ and so on..

i.e., as soon as a new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is repeated till the values of x, y, z are obtained to desired degree of accuracy.

Example

Example: Solve the system of following linear equations by Gauss- Seidal method:

$$5\,x + 2\,y + \ z = 12 \qquad \text{-----} \qquad (1)$$
$$x + 4\,y + 2\,z = 15 \qquad \text{-----} \qquad (2)$$
$$x + 2\,y + 5\,z = 20 \qquad \text{------} \qquad (3)$$

Solution: Rewrite the given equations as

$$(1) \Rightarrow x = \frac{1}{5}(12 - 2y - z)$$
$$(2) \Rightarrow y = \frac{1}{4}(15 - x - 2z)$$
$$(1) \Rightarrow z = \frac{1}{5}(20 - x - 2y)$$

Let the initial approximations are $x_0 = 0\,, y_0 = 0, z_0 = 0$

First approximation, $x_1 = \frac{1}{5}(12 - 2(0) - (0)) = \frac{12}{5} = 2.4$

$$y_1 = \frac{1}{4}(15 - (2.4) - (0)) = \frac{12.6}{4} = 3.15$$

and $z_1 = \frac{1}{5}(20 - (2.4) - 2(3.15)) = \frac{11.3}{5} = 2.26$

Second approximation, $x_2 = \frac{1}{5}(12 - 2(3.15) - (2.26)) = \frac{1}{5}(12 - 6.3 - 2.26) = \frac{3.44}{5} = 0.688$

$$y_2 = \frac{1}{4}(15 - (0.688) - 2(2.26)) = \frac{9.792}{4} = 2.448$$

and $z_2 = \frac{1}{5}(20 - 0.688 - 2(2.448)) = \frac{14.416}{5} = 2.8832$

Third approximation, $x_3 = \frac{1}{5}(12 - 2(2.448) - 2.8832)) = \frac{4.2208}{5} = 0.84416$

$$y_3 = \frac{1}{4}(15 - 0.84416) - 2(2.8832)) = \frac{8.3894}{4} = 2.0974$$

and $z_3 = \frac{1}{5}(20 - 0.84416 - 2(2.0974)) = \frac{14.961}{5} = 2.9922$

Fourth approximation, $x_4 = \frac{1}{5}(12 - 2(2.0974) - 2.9922)) = \frac{4.813}{5} = 0.9626$

$$y_4 = \frac{1}{4}(15 - 0.9626) - 2(2.9922)) = \frac{8.053}{4} = 2.013$$

and $z_4 = \frac{1}{5}(20 - 0.9626 - 2(2.013)) = \frac{15.011}{5} = 3.0023$

The approximated value of x , y & z are 1, 2 & 3 respectively

Example: Solve the system of following linear equations by Gauss- Seidal method:

$$3\,x + y + \ z = 4 \qquad \text{-----} \qquad (1)$$

$$x + 2\,y + 2\,z = 3 \qquad \text{-----} \qquad (2)$$

$$2\,x + y + 3\,z = 4 \qquad \text{------} \qquad (3)$$

Solution: Rewrite the given equations as

$$(1) \Rightarrow x = \frac{1}{3}(4 - y - z)$$

$$(2) \Rightarrow y = \frac{1}{2}(3 - x - 2z)$$

$$(1) \Rightarrow z = \frac{1}{3}(4 - 2\,x - y)$$

Let the initial approximations are $x_0 = 0\,, y_0 = 0, z_0 = 0$

First approximation, $x_1 = \frac{1}{3}(4 - (0) - (0)) = \frac{4}{3} = 1.3333$

$$y_1 = \frac{1}{2}(3 - (1.3333) - 2(0)) = \frac{1.6667}{2} = 0.8333$$

and $z_1 = \frac{1}{3}(4 - 2(1.3333) - (0.8333)) = \frac{0.5001}{3} = 0.1667$

Second approximation, $x_2 = \frac{1}{3}(4 - 0.8333 - 0.1667) = \frac{3}{3} = 1$

$$x = \frac{1}{3}(4 - y - z)$$

$$y_2 = \frac{1}{2}(3 - 1 - 2(0.1667)) = \frac{1.6666}{2} = 0.8333$$

$$y = \frac{1}{2}(3 - x - 2z)$$

and $z_2 = \frac{1}{3}(4 - 2(1) - (0.8333)) = \frac{1.1667}{3} = 0.3889$

$$z = \frac{1}{3}(4 - 2\,x - y)$$

Third approximation, $x_3 = \frac{1}{3}(4 - 0.8333 - 0.3889) = \frac{2.7778}{3} = 0.9259$

$$y_3 = \frac{1}{2}(3 - 0.9259 - 2(0.3889)) = \frac{1.2963}{2} = 0.6482$$

and $z_3 = \frac{1}{3}(4 - 2(0.9259) - (0.6482)) = \frac{1.5}{3} = 0.5$

Fourth approximation, $x_4 = \frac{1}{3}(4 - 0.6482 - 0.5) = \frac{2.8518}{3} = 0.9506$

$$y_4 = \frac{1}{2}(3 - 0.9506 - 2(0.5)) = \frac{1.0494}{2} = 0.5247$$

and $z_4 = \frac{1}{3}(4 - 2(0.9506) - (0.5247)) = \frac{1.5741}{3} = 0.5247$

The approximated values of x , y & z are approaching 1, 0.5 & 0.5 respectively   Ans.

Practice Questions

Q 1. The radius of a circle is found to be 100 cm. Find the relative error in the area of the circle due to an error of 1 mm.

Q 2. The radius of a sphere is found to be 10 cm with a possible error of 0.02 cm. What is the relative error in computing the volume?

Q 3. Find a real root of the equation $3x - 1 - \sin x = 0$ by bisection method up to three iterations.

Q 4. Find square root of 10 by false position method up to three decimal points.

Q 5. Find cube root of 28 by Newton-Raphson Method.

Q 6. Find a real root of equation $x = 0.5 + \sin \mathrm{x}$ by secant Method.

Q 7. Using Gauss elimination method, solve for x, y & z:

$$2x - y + 3z = 9, \quad x + y + z = 6, \quad x - y + z = 2$$

Q 8. Solve the system of following linear equations by Gauss- Seidal method:

$$10\,x + 2\,y + z = 9$$
$$2\,x + 20\,y - 2\,z = -44$$
$$-2\,x + 3\,y + 10\,z = 22$$

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Thanks