Double Integrals in Polar Coordinates

$$I = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, dr\, d\theta.$$

(1) Evaluate $\displaystyle\int_0^{\pi/2} \left[\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\, dr\right] d\theta$

Put $a^2-r^2=t^2$

$-2r\,dr = 2t\,dt$

$r\,dr = -t\,dt$

$t=a$

$t=a\sin\theta$

Sol^n:
$$I = -\int_0^{\pi/2}\left[\int_a^{a\sin\theta} t^2\,dt\right] d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2} \left[t^3\right]_a^{a\sin\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2} \left(a^3\sin^3\theta - a^3\right) d\theta.$$

$$= -\frac{a^3}{3}\left[\frac{2}{3}-\frac{\pi}{2}\right] = \frac{a^3}{18}(3\pi-4)$$

(2) $$I = \int_0^{\pi}\int_0^{a(1-\cos\theta)} r^2\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left[\int_0^{a(1-\cos\theta)} r^2\,dr\right] d\theta = \frac{4}{3}a^2$$

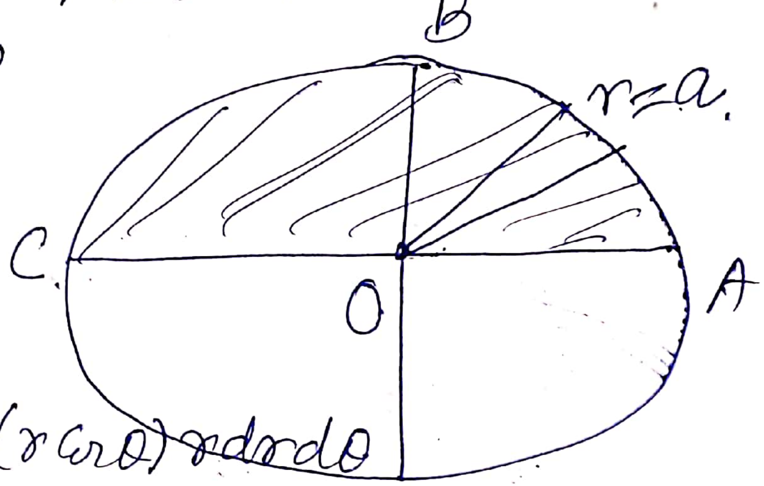
(3) $$\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\, d\theta = \frac{1}{8}(\pi-2)$$

(4)

(4) Transform the integral to cartesian form and hence evaluate $\int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta \, dr \, d\theta$

Sol^n:

$$I = \int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta \, dr \, d\theta$$



$r = 0, a$

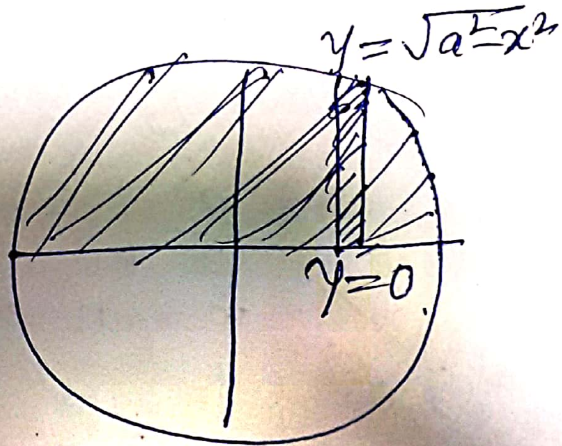
$\theta = 0, \pi$

Hence the region i.e. semicircle ABC is bdd by r=a.

$$I = \iint (r\sin\theta)(r\cos\theta)\, r\, dr\, d\theta$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,d\theta\,dr$

$$I = \int_{-a}^{a} x\,dx \int_0^{\sqrt{a^2-x^2}} y\,dy$$



$= 0$.