Stoke's theorem :- Let S be a piecewise smooth orientable surface bounded by a piecewise smooth simple closed curve C. Let $V(x,y,z) = V_1(x,y,z)\hat{i} + V_2(x,y,z)\hat{j} + V_3(x,y,z)\hat{k}$ be a vector function which is continuous and has continuous first order partial derivatives in a domain which contains S. If C is traversed in the positive direction, then

$$\oint_C \vec{V}\cdot d\bar{r} = \iint_S (\nabla \times \vec{V})\cdot \hat{n}\, dA$$

where $\hat{n}$ is the unit normal vector to S in the direction of orientation of C.

Prob:- Verify Stoke's thm for the vector field $\vec{V} = (3x-y)\hat{i} - 2yz^2\hat{j} - 2y^2z\hat{k}$, where S is the surface of the sphere $x^2+y^2+z^2=16$, $z>0$.

Now,

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x-y & -2yz^2 & -2y^2z \end{vmatrix}$$

$$= \hat{i}(-4yz + 4yz) + \hat{j}(0-0) + \hat{k}(0+1) = \hat{k}.$$

and $\hat{n} = \dfrac{\text{grad} f}{|\text{grad} f|} = \dfrac{2(x\hat{i}+y\hat{j}+z\hat{k})}{2\sqrt{x^2+y^2+z^2}}$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{16}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{4},$$

Now, $(\nabla \times \vec{V}) \cdot \hat{n} = \hat{k} \cdot \left(\dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{4}\right) = \dfrac{z}{4}$

Therefore, $\displaystyle\iint_S (\nabla \times \vec{V}) \cdot \hat{n}\, dA = \iint_R \frac{z}{4} \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \iint_R \frac{z}{4} \frac{dx\,dy}{(z/4)}$

$$= \iint_R dx\,dy = \pi(4)^2 = 16\pi.$$

proved

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43 Evolute $\iint_S (\text{curl } \vec{F}) . ds$, where

$\vec{F} = z^2\hat{i} - 3xy\hat{j} + x^3y^3\hat{k}$ $\quad$ $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$z : 1 \to 5 - x^2 - y^2$

By using stoke's theorem

$$\iint_S (\text{curl } \vec{F}) . ds = \oint_C \vec{F} . dr$$

$$= \oint_C (z^2\hat{i} - 3xy\hat{j} + x^3y^3) . dr$$

$$= \oint_C z^2 dx - 3xy\, dy + 0$$ {as dz becomes zero}

$f(x,y) = z^2 = 1 \therefore \frac{\partial f}{\partial y} = 0$

$g(x,y) = -3xy \therefore \frac{\partial g}{\partial x} = -3y$

$$\therefore \iint_S (\text{curl } \vec{F}) . ds = \oint_C z^2 dx - 3xy\, dy$$

$$= \iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\, dy$$ (by using Green's thm)

$$= \iint -3y\, dx\, dy$$

Changing into polar cordinate

$1 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$

$\therefore r : 0 \to 2, \quad \theta : 0 \to 2\pi$

$$\therefore \iint_S (\text{curl } \vec{F}) . ds = -3 \int_0^{2\pi} \int_0^2 r \sin\theta . r\, dr\, d\theta$$

$$= -\frac{3}{3} \int_0^{2\pi} \sin\theta \left[r^3\right]_0^2 d\theta = -8 \int_0^{2\pi} \sin\theta\, d\theta$$

$$= -16 \int_0^{\pi} \sin\theta\, d\theta = 16 \left[\cos\theta\right]_0^{\pi} = 16[-1-1]$$

$$= -32$$ Ans

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44 Evaluate $\iint_S (\text{curl } \vec{F}) \cdot ds$

Where $\vec{F} = 2yz\hat{i} + 3xz\hat{j} + xy\hat{k}$

S: paraboloid $z = x^2+y^2$ for $x^2+y^2 \le 4$

Using Stokes, theorem

$$\iint_S (\text{curl } \vec{F}) \cdot ds = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \oint_C 2yz\,dx + 3xz\,dy \quad \text{— (1)} \quad [\because dz = 0]$$

$f(x,y) = 2yz = 2y(x^2+y^2) \quad \therefore \frac{\partial f}{\partial y} = 2x^2+6y^2$

$g(x,y) = 3x(x^2+y^2) \quad \therefore \frac{\partial g}{\partial x} = 9x^2+3y^2$

Now, applying Green's theorem

$$\iint_S (\text{curl } \vec{F}) \cdot ds = \oint_C 2yz\,dx + 3xz\,dy$$

$$= \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \iint_R (7x^2 - 3y^2)\,dx\,dy$$

Changing into polar coordinate $r: 0 \to 2, \theta: 0 \to 2\pi$

$$\therefore \iint_S (\text{curl } \vec{F}) \cdot ds = \int_0^{2\pi}\int_0^2 (7r^2\cos^2\theta - 3r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} (7\cos^2\theta - 3\sin^2\theta) \frac{1}{4}[r^4]_0^2\, d\theta$$

$$= 4\int_0^{2\pi} (7\cos^2\theta - 3\sin^2\theta)\, d\theta$$

$$= 2\int_0^{2\pi} \{7(1+\cos 2\theta) + 3(\cos 2\theta - 1)\}\, d\theta$$

$$= 2\int_0^{2\pi} (4 + 10\cos 2\theta)\, d\theta = 8 \times 2\pi = 16\pi$$

Ans

45 $\vec{V} = 3y\hat{i} + 4z\hat{j} + 2x\hat{k}$

Using stoke's theorem, $\oint_C V \cdot dr = \iint_S (\nabla \times \vec{V}) \cdot \hat{n}\, dA$ —(1)

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y & 4z & 2x \end{vmatrix} = \hat{i}[-4] + \hat{j}[-2] + \hat{k}[-3]$$

$$= -4\hat{i} - 2\hat{j} - 3\hat{k}$$

$x^2 + y^2 + z^2 = 16$; $x \geq 0$. ; $y^2 + z^2 = 4$

$\therefore x^2 = 12 \Rightarrow f = x^2 - 12 = 0$ be surface

grad $f = \nabla f = 2x\hat{i}$ $\therefore |\nabla f| = \frac{\cancel{2x\hat{i}}}{\sqrt{4x^2}} = \cancel{2x\hat{i}}$ $\sqrt{(2x)^2} = 2x$

$$\therefore \hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\hat{i}}{2x} = \hat{i}$$

$$dA = \frac{dy\,dz}{\hat{n}\cdot\hat{i}} = \frac{dy\,dz}{1} = dy\,dz$$

Limits are $r : 0 \to 2$, $\theta : 0 \to 2\pi$

from (1) ; we get

$$\oint_C V \cdot dr = \int^{2\pi}\int_0^2 (-4\hat{i} - 2\hat{j} - 3\hat{k}) \cdot \hat{i}\; dy\,dz$$

$$= \int_0^{2\pi}\int_0^2 -4\, r\,dr\,d\theta = -\frac{4}{2}\int_0^{2\pi} [r^2]_0^2\, d\theta$$

$$= -\frac{16}{2}\int_0^{2\pi} d\theta = -8(2\pi) = -16\pi \text{ Ans}$$