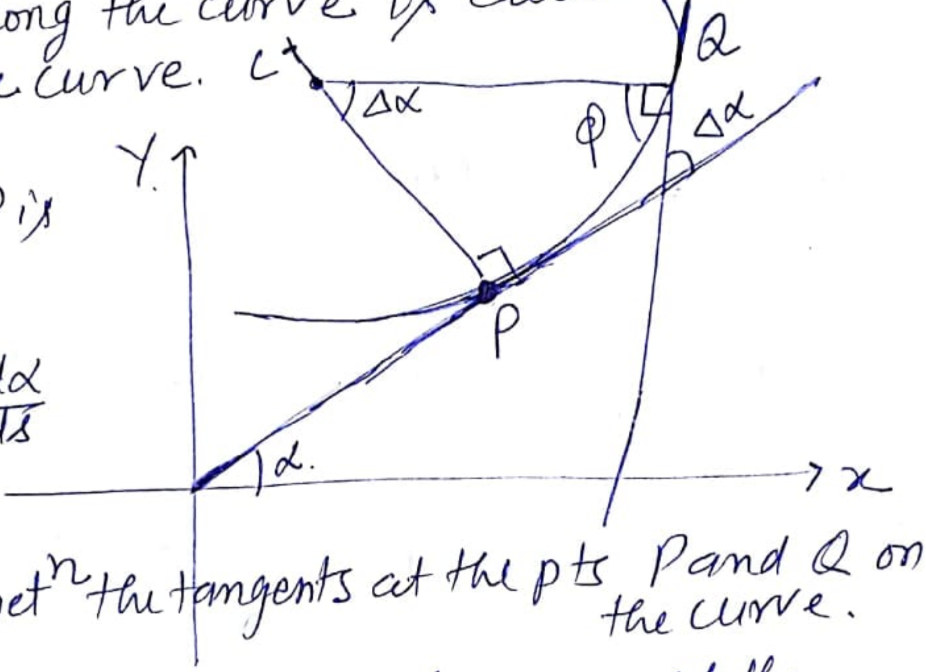
Lecture 14

Curvature (K) :- The rate of change of the direction of the tangent line, at a point on the curve, w.r.to the arc length 's' along the curve is called the curvature of the curve.



The curvature of the curve at the point P is defined as

Curvature
$$K = \lim_{\Delta s \to 0} \frac{\Delta\alpha}{\Delta s} = \frac{d\alpha}{ds}$$

Where $\Delta\alpha$: angle betⁿ the tangents at the pts P and Q on the curve.

Curvature measures the degree of sharpness of the bending of a curve at that point on the curve.

Radius of Curvature (ρ) length CP

to a curve at a point is denoted by ρ and is the reciprocal of the curvature at that point.

Thus $\boxed{\rho = \frac{1}{K}}$

Circle of curvature :- This point C is called centre of curvature,

The circle with centre at C, and radius ρ is called the circle of curvature.

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Formula:- In Cartesian form

Curvature $K = \frac{d\alpha}{ds} = \frac{d\alpha/dx}{ds/dx} = \frac{y''}{[1+(y')^2]^{3/2}}$

and $\rho = \frac{1}{K} = \frac{[1+(y')^2]^{3/2}}{y''}$ [Radius of curvature]

Centre of curvature

$$\alpha = x - \rho\sin\theta = x - \frac{y'}{y''}[1+(y')^2]$$

$$\beta = y + \rho\cos x = y + \frac{[1+(y')^2]}{y''}$$

Problem (1) Find the curvature and radius of curvature of following curves at the indicated points.

(i) $x^2+y^2=a^2$ at (x,y)

(ii) $y^2=2x(3-x^2)$ at the pts where the tangents are || to x-axis $R=(1/3)$

(iii) $a^2y^2=a^3-x^3$ at $(a,0)$ Ans $R=3/2$

(iv) $x^3+y^3=3axy$ at $\left(\frac{3a}{2},\frac{3a}{2}\right)$ Ans $\frac{3a}{8\sqrt{2}}$

Solⁿ:- (1) $x^2+y^2=a^2$ — (1)

Diff (1), we get $2x+2yy'=0 \Rightarrow x+yy'=0$ — (2)

$$\boxed{y' = -\frac{x}{y}}$$

Diffⁿ (2) again w.r.to x, we get $1+yy''+y'^2=0$

$$yy'' = -(1+y'^2) = -\left(1+\frac{x^2}{\;}\right) = -\left(\frac{x^2+y^2}{y^2}\right)$$

$$yy'' = -\frac{a^2}{y^2} \Rightarrow \boxed{y''}$$

Now, curvature $K = \frac{y''}{[1+(y' \quad x^2/y^2)^{3/2}}$

$$K = \frac{-a^2/y^3}{a^3\frac{1}{y^3}} = -\frac{1}{a}$$

ture $= |K| = \frac{1}{a}$

and ρ

Ans (iii) $(x+y)^{2/3}+(x-y)^{2/3}=2a^{2/3}$. (ii) $a^{4/3}x^{2/3}+b^{2/3}y^{2/3}=(a^2-b^2)^{2/3}$

Evolute The locus of the centre of curvature $C(\alpha,\beta)$ is called the evolute of the curve.

Involute:- The involute of a curve is a curve for which the given curve is an evolute.

Therefore, the given curve is the involute.

Problem:- ① Find the evolutes of the following curves:

(i) $y^2=4ax$, (ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (iii) $x=a\cos^3\theta, y=a\sin^3\theta$.

Sol^n (i) $2yy'=4a \Rightarrow y'=\frac{2a}{y}$

$yy''+y'^2=0 \Rightarrow y''=-\frac{y'^2}{y}=-\frac{1}{y}\left(\frac{2a}{y}\right)^2$

$y''=\frac{-4a^2}{y^3}$

Now,
$$\alpha=x-\frac{y'[1+y'^2]}{y''}=x-\frac{\left(\frac{2a}{y}\right)\left[1+\frac{4a^2}{y^2}\right]}{\frac{-4a^2}{y^3}}$$

$$=x+\frac{y^2+4a^2}{2a}=x+\left(\frac{4ax+4a^2}{2a}\right)$$

$$\alpha=x+2x+2a=\cancel{3x+2a}\ \cancel{x-2a}\ 3x+2a \quad \text{—①}$$

$$\beta=y+\frac{1+y'^2}{y''}=y+\frac{\left(1+\frac{4a^2}{y^2}\right)}{\frac{-4a^2}{y^3}}=y+\frac{y(y^2+4a^2)}{(-4a^2)}$$

$$=\frac{-4a^2y+y^3+4a^2y}{-4a^2}=\frac{y^3}{-4a^2}=\frac{(4ax)^{3/2}}{-4a^2}$$

$$\beta=-\frac{2}{\sqrt{a}}x^{3/2} \quad \text{—②} \qquad \therefore \text{centre}: \left(3x+2a, -\frac{2x^{3/2}}{\sqrt{a}}\right)$$

From ①; $x=\frac{\alpha-2a}{3}$; from ②; $\beta=-\frac{2}{\sqrt{a}}\left(\frac{\alpha-2a}{3}\right)^{3/2}$

i.e. $\beta^2=\frac{4}{a}\left(\frac{\alpha-2a}{3}\right)^3 \Rightarrow (\alpha-2a)^3=\frac{27a}{4}\beta^2$

Locus, $\boxed{(x-2a)^3=\frac{27a}{4}y^2}$: Evolute.

Problem:- Find the coordinates of the centre and radius of the circle of curvature at the indicated points. Also, obtain the equation of the circle of curvature for the following curve:

(i) $xy = 1$ at $(1,1)$ (ii) $y = e^x$ at $(0,1)$ $K = \frac{1}{2\sqrt{2}}$; Centre $= (-2,3)$

(iii) $y = \tan x$ at $\left(\frac{\pi}{4}, 1\right)$ $K = \frac{4}{5\sqrt{5}}$; Centre $= \left(\frac{\pi - 10}{4}, \frac{9}{4}\right)$

Sol^n:- (i) $xy = 1$ —— (i)

Differentiating (i) w.r.to x, we get

$$y + xy' = 0 \Rightarrow \boxed{y' = -\frac{y}{x}}$$

Diff^n again,

$$y' + y' + xy'' = 0 \Rightarrow y'' = -\frac{2y'}{x} = -\frac{2}{x}\left(-\frac{y}{x}\right) = \frac{2y}{x^2}$$

At $(1,1)$ $y' = -1$, $y'' = 2$

Hence, $$K = \frac{y''}{[1+y'^2]^{3/2}} = \frac{2}{2^{3/2}} = 2^{-1/2} = \frac{1}{\sqrt{2}}$$

Curvature $= |K| = \frac{1}{\sqrt{2}}$ Ans

Radius of curvature $\rho = \frac{1}{|K|} = \sqrt{2}$ Ans

The coordinates of the centre are

$$\alpha = x - \frac{y'[1+y'^2]}{y''} = 1 - \frac{(-1)[1+1]}{2} = 1+1 = 2$$

$$\beta = y + \frac{[1+y'^2]}{y''} = 1 + \frac{1+1}{2} = 1+1 = 2$$

∴ Centre $\equiv (2,2)$

∴ The circle of curva[illegible]

$$\boxed{(x-2)^2 + (y-2)^2 = 2}$$