Lecture 10

Improper Integral of Type I

Definition:- Integrals with infinite limits of integration are improper integrals of type I.

1. If $f(x)$ is continuous on $[a,\infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

2. If $f(x)$ is continuous on $(-\infty,b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

3. If $f(x)$ is continuous on $(-\infty,\infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$, where c is any real number

In each case, if the limit is finite we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, then improper integral diverges.

Problem ① Is the area under the curve $y = \frac{\ln x}{x^2}$ from $x = 1$ to $x = \infty$ finite? If so, what is its value.

Sol:-
$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\frac{1}{x}\,dx$$
$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1$$

$$\text{Area} = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right)$$
$$= -\lim_{b\to\infty}\left(\frac{1}{b}\right) - 0 + 1 = 0 + 1 = 1. \quad \text{Ans}$$

The improper integral converges and the area has finite value

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Ex ② Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Solⁿ:- $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} = I_1 + I_2$

Now, $I_1 = \int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1+x^2} = \lim_{a \to -\infty} \left[\tan^{-1}x\right]_a^0$

$= \lim_{a \to -\infty} (-\tan^{-1}a) = -\tan^{-1}(-\infty) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$

and $I_2 = \int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b \to \infty} \tan^{-1}(b) = \frac{\pi}{2}$

Thus, $I = I_1 + I_2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ Ans

The integral $\int_{1}^{\infty} \frac{dx}{x^p}$

Problem:-③ For what values of p does the integral $\int_{1}^{\infty} \frac{dx}{x^p}$ converges? When the integral does converge, what is its value?

Solⁿ:- $\int_{1}^{\infty} \frac{dx}{x^p} = \lim_{b \to \infty} \int_{1}^{b} \frac{dx}{x^p} = \lim_{b \to \infty} \frac{1}{1-p}\left[x^{1-p}\right]_1^b$

$= \lim_{b \to \infty} \frac{1}{1-p}\left[b^{1-p} - 1\right] = \lim_{b \to \infty} \frac{1}{(1-p)}\left[\frac{1}{b^{p-1}} - 1\right]$

$$\int_{1}^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & ; \ p > 1 \\ \infty & ; \ p < 1 \end{cases}$$

If $p = 1$ $\int_{1}^{\infty} \frac{dx}{x^p} = \int_{1}^{\infty} \frac{dx}{x} = \lim_{b \to \infty} \int_{1}^{b} \frac{dx}{x}$

$= \lim_{b \to \infty} (\ln b - \ln 1) = \infty$

Hence, integral converges to the value $\frac{1}{p-1}$ if $p > 1$ and it diverges if $p \le 1$.

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