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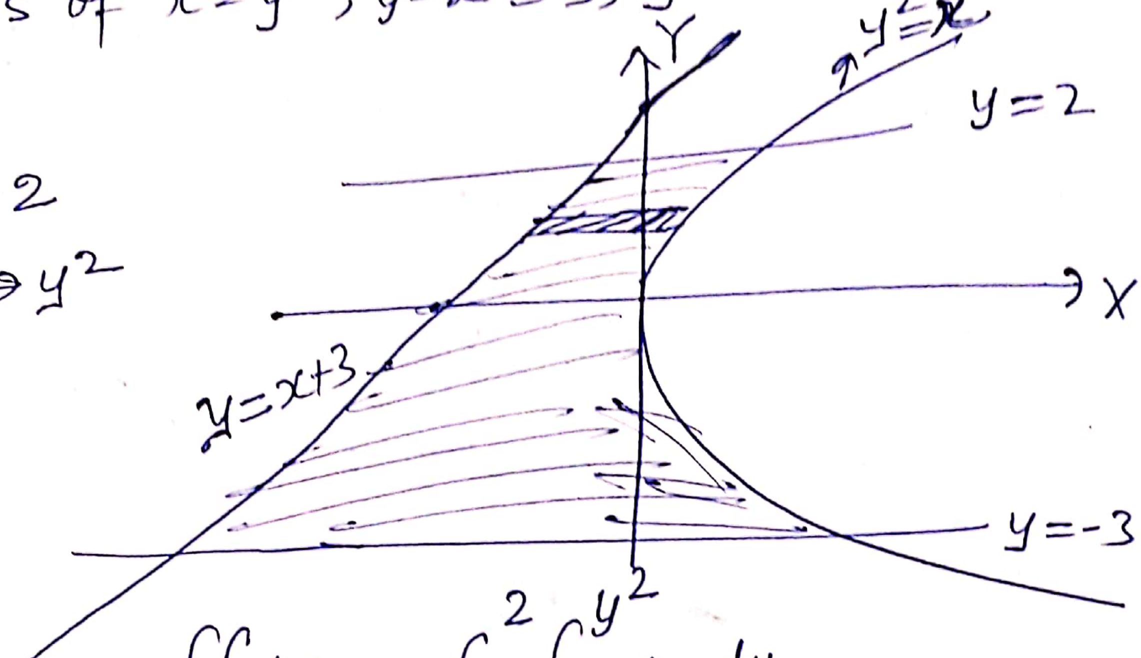
Area and Volume of a solid

Prob ① Find the area of a plane region bounded by the graphs of $x = y^2$, $y - x = 3$, $y = -3$ and $y = 2$.

Solution:-

$y : -3 \to 2$

$x : y-3 \to y^2$



$$\text{Area } A = \iint_R dA = \int_{-3}^{2}\int_{y-3}^{y^2} dx\,dy$$

$$= \int_{-3}^{2} [x]_{y-3}^{y^2}\,dy = \int_{-3}^{2} (y^2 - y + 3)\,dy$$

$$= \left(\frac{y^3}{3} - \frac{y^2}{2} + 3y\right)\Big|_{-3}^{2} = \frac{175}{6}.$$

Prob ②:- Find the volume of the tetrahedron bounded by the plane $2x + y + z = 2$ and the three coordinate planes.

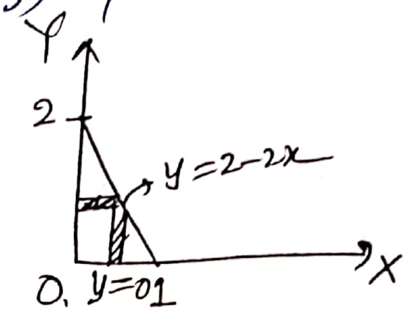
Solution:- Since the plane $2x+y+z=2$ intersects the coordinate axes at the points $(1,0,0)$, $(0,2,0)$ and $(0,0,2)$.

$Z = 2 - 2x - y.$

The trace is found by simply setting $z = 0$; $2x + y = 2$

Volume is

$$V = \int_0^1 \int_0^{2-2x} z \, dy \, dx = \int_0^1 \int_0^{2-2x} (2 - 2x - y) \, dy \, dx$$



$$= \int_0^1 \left(2y - 2xy - \frac{y^2}{2}\right) \Bigg|_{y=0}^{y=2-2x} dx$$

$$= \int_0^1 \left[2(2-2x) - 2x(2-2x) - \frac{(2-2x)^2}{2}\right] dx$$

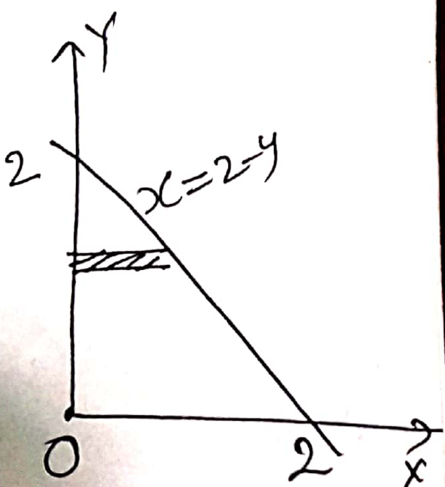
$$= 2\int_0^1 (1 + x^2 - 2x) \, dx = 2\left[x + \frac{x^3}{3} - x^2\right]_0^1$$

$$= 2\left(1 + \frac{1}{3} - 1\right) = \frac{2}{3}. \quad \text{Ans.}$$

(3) Find the volume of the solid lying in the first octant and bounded by the graphs of $z = 4 - x^2$, $x + y = 2$, $x = 0$, $y = 0$ and $z = 0$.

Solution:-

$$V = \int_0^2 \left[\int_0^{2-y} (4 - x^2) \, dx\right] dy$$



$$= \int_0^2 \left(4x - \frac{x^3}{3}\right) \Bigg|_{x=0}^{x=2-y} dy$$

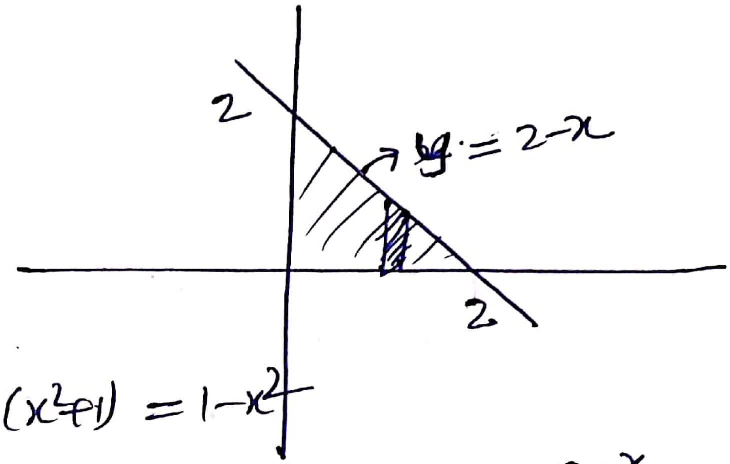
$$= \int_0^2 \left[4(2-y) - \frac{(2-y)^3}{3}\right] dy = \frac{20}{3}.$$

Prob (4) Find the volume of the solid bounded by the graphs of $z=2$, $z=x^2+1$, $y=0$ and $x+y=2$.

Solution:-

$2 = x^2+1 \Rightarrow x^2 = 1$

$\Rightarrow x = -1$ and $+1$



Limits are

$x : -1 \rightarrow 1$

$y : 0 \rightarrow 2-x$

$\cancel{z = x^2+1}$ $f(x,y) = 2-(x^2+1) = 1-x^2$

$$V = \int_{-1}^{1}\int_{0}^{2-x} (x^2+1)\,dy\,dx = \int_{-1}^{1} (x^2+1)\left[y\right]_0^{2-x} dx$$

$$= \int_{-1}^{1} (x^2+1)(2-x)\,dx = \int_{-1}^{1} (2x^2+2-x^3-x)\,dx$$

$$= \int_{-1}^{1} (-x^3+2x^2-x+2)\,dx$$

$$= -4\left[\frac{x^3}{3}\right]_0^1 + 2\left[x\right]_{-1}^{1} = -\frac{4}{3}+4 = \frac{8}{3}$$

Area and Volume in Polar Coordinate

Prob ①:- Find the area inside the curve defined by $r = 2-2\sin\theta$.

Solution:- $r: 0 \to 2-2\sin\theta$

$\theta: 0 \to 2\pi$

Area $A = \iint_R dA = \int_0^{2\pi}\int_0^{2-2\sin\theta} r\,dr\,d\theta$

$$= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{2-2\sin\theta} d\theta = \int_0^{2\pi} \frac{(2-2\sin\theta)^2}{2} d\theta$$

$$= 2\int_0^{2\pi} (1+\sin^2\theta - 2\sin\theta)\,d\theta$$

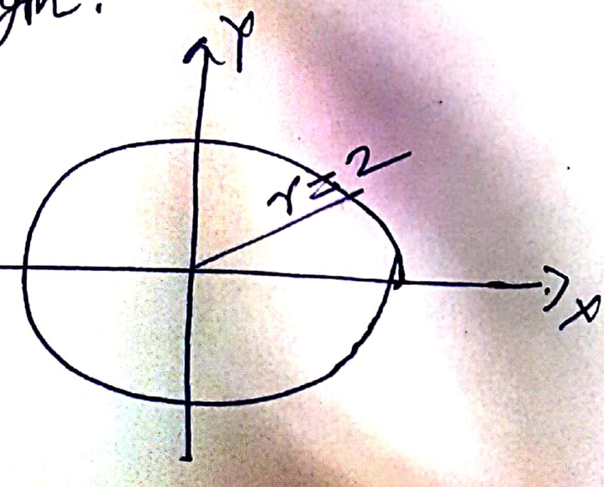
$$= 2\left[\int_0^{2\pi} 1\,d\theta + \int_0^{2\pi}\left[\frac{1-\cos 2\theta}{2}\right] d\theta - 2\int_0^{2\pi}\sin\theta\,d\theta\right]$$

$$= 2[2\pi + \pi] = 2\times 3\pi = 6\pi.$$

Prob ②:- Evaluate $\iint_R (x^2+y^2+3)\,dA$, where R is the circle of radius 2 centered at the origin.

Solution:- $r: 0 \to 2$

$\theta: 0 \to 2\pi$



$$\therefore \iint_R (x^2+y^2+3)\,dA = \int_0^{2\pi}\int_0^{2} (r^2+3)\,r\,dr\,d\theta$$

$\because r^2 = x^2+y^2$

Area by Double Integration

(a) Cartesian Co-ordinates

The area A of the region bounded by the curves $y=f_1(x), y=f_2(x)$ and the lines $x=a, x=b$ is given by $A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$

The area A of the region bounded by the curves $x=f_1(y)$, $x=f_2(y)$ and the lines $y=c, y=d$ is given by

$$A = \int_c^d \int_{f_1(y)}^{f_2(y)} dx\,dy$$

(b) Polar Coordinates:-

The area A of the region bounded by the curves $r=f_1(\theta), r=f_2(\theta)$ and the lines $\theta=\alpha, \theta=\beta$ is given by

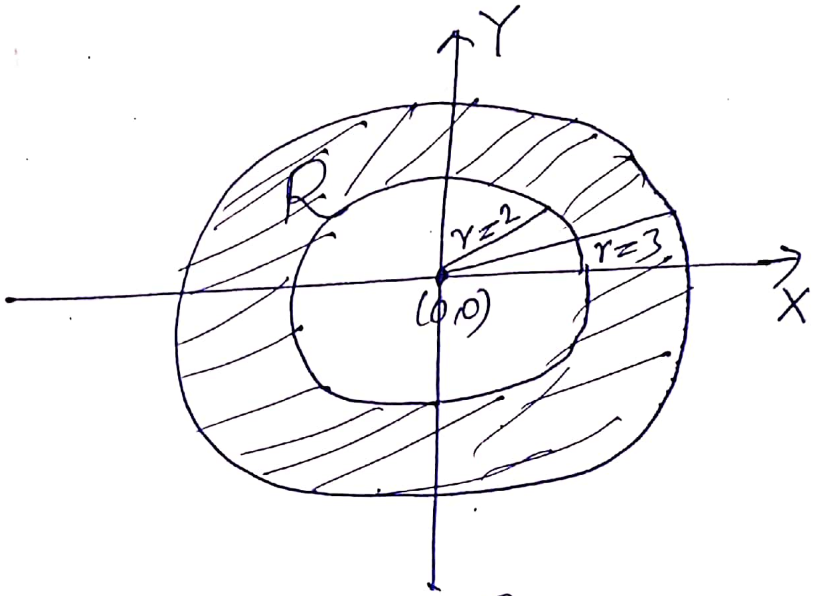
$$A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r\,dr\,d\theta .$$

In case of parametric eq^ns, ~~the~~ area is given by

$$A = \int_{t=\alpha}^{\beta} y \frac{dx}{dt} dt$$

$$= \int_{t=\alpha}^{\beta} x \frac{dy}{dt} dt$$

Prob ③:- Find the volume inside the paraboloid $z = 9-x^2-y^2$, outside the cylinder $x^2+y^2=4$ and above the xy-plane.



Solution:- $z = 9-x^2-y^2$

with $z=0$,

$9-x^2-y^2=0$

$\Rightarrow x^2+y^2=9$

$r^2=9 \Rightarrow r=3$

and $x^2+y^2=4 \Rightarrow r=2$

$r : 2 \to 3$

$\theta : 0 \to 2\pi$

$$\therefore \text{Volume} = \iint_R (9-x^2-y^2)\,dA = \int_0^{2\pi}\int_2^3 (9-r^2).r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\int_2^3 (9r-r^3)\,dr\right]d\theta = \int_0^{2\pi}\left[\frac{9r^2}{2}-\frac{r^4}{4}\right]_2^3 d\theta$$

$$= 2\pi \times \left[\left(\frac{9}{2}\times 9-\frac{81}{4}\right)-\left(\frac{9\times 4}{2}-\frac{16}{4}\right)\right]$$

$$= 2\pi \times \left[\frac{81}{2}\left(1-\frac{1}{2}\right)-(18-4)\right] = 2\pi \times \left[\frac{81}{4}-14\right]$$

$$= 2\pi \times \left(\frac{81-56}{4}\right) = \frac{2\pi\times 25}{4} = \frac{25\pi}{2}.$$

Prob ④ Evaluate the iterated integral

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} x^2(x^2+y^2)^2\,dy\,dx.$$

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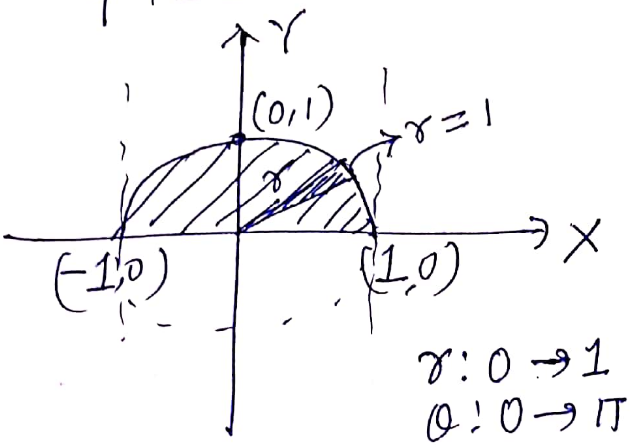
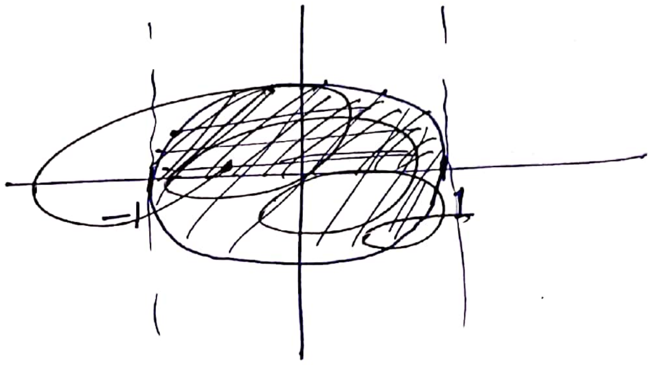
Solution:- $y: 0 \to \sqrt{1-x^2}$; $x: -1 \to 1$

$y^2 + x^2 = 1$





$r: 0 \to 1$
$\theta: 0 \to \pi$

$$\therefore I = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x^2 (x^2+y^2)^2 \, dy\, dx = \int_0^{\pi}\int_0^1 (r\cos\theta)^2 \cdot (r^2)^2 \, r \, dr\, d\theta$$

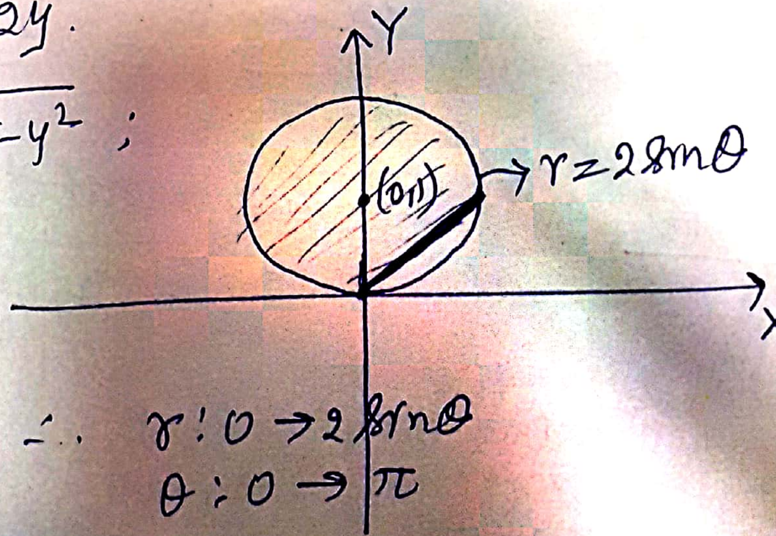
$$= \int_0^{\pi}\int_0^1 r^7 \cos^2\theta \, dr\, d\theta = \int_0^{\pi} \cos^2\theta \left[\int_0^1 r^7 dr\right] d\theta$$

$$= \int_0^{\pi} \cos^2\theta \left[\frac{r^8}{8}\right]_0^1 d\theta = \frac{1}{8}\int_0^{\pi} \cos^2\theta \, d\theta$$

$$= \frac{1}{8}\int_0^{\pi} \frac{1}{2}(1+\cos 2\theta)\, d\theta = \frac{1}{16}\int_0^{\pi} (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{16}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{1}{16}[\pi] = \frac{\pi}{16} \cdot \text{Ans}$$

5) Find the volume cut out of the sphere $x^2+y^2+z^2=4$ by the cylinder $x^2+y^2=2y$.



Solution:- $z = \sqrt{4-x^2-y^2}$;

Circle $x^2+y^2=2y$

Centre $= (0,1)$

radius $= 1$

$r^2 = 2r\sin\theta$

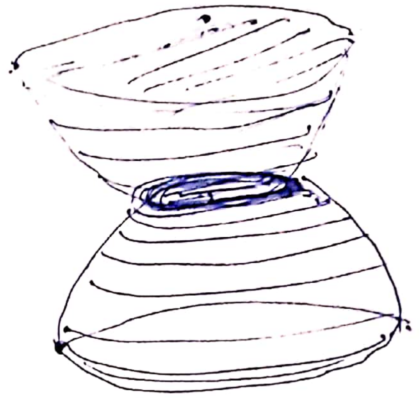
$r = 2\sin\theta$

$\therefore$ $r: 0 \to 2\sin\theta$

$\theta: 0 \to \pi$

Notice that equal partitione of the volume lie above the below the circle of radius 1 centered at (0,1)

Therefore, required volume



$$V = 2\iint_R \left(\sqrt{4-x^2-y^2}\right) dA$$

$$=2\int_0^{\pi}\left[\int_0^{2\sin\theta}\left(\sqrt{4-r^2}\right) r\,dr\right]d\theta$$

$$=2\int_0^{\pi}\left[\int_{2\cos\theta}^{2} t\cdot t\,dt\right] \cancel{\cos\theta}\, d\theta$$

put $4-r^2=t^2$

$-2r\,dr=2t\,dt$

$r\,dr=-t\,dt$

$t: 2\to 2\cos\theta$

$$=2\int_0^{\pi}\left[\int_{2\cos\theta}^{2} t^2\,dt\right]d\theta$$

$$=\frac{2}{3}\int_0^{\pi}\left[8-8\cos^3\theta\right]d\theta$$

$$=-\frac{16}{3}\int_0^{\pi}(\cos^3\theta-1)\,d\theta = -\frac{32}{3}\int_0^{\pi/2}(\cos^3\theta-1)\,d\theta$$

$$=-\frac{64}{9}+\frac{16}{3}\pi \cong 9.644 \text{ Ans}$$

Prob (6) Find the volume of the solid bounded by $z=8-x^2-y^2$ and $z=x^2+y^2$

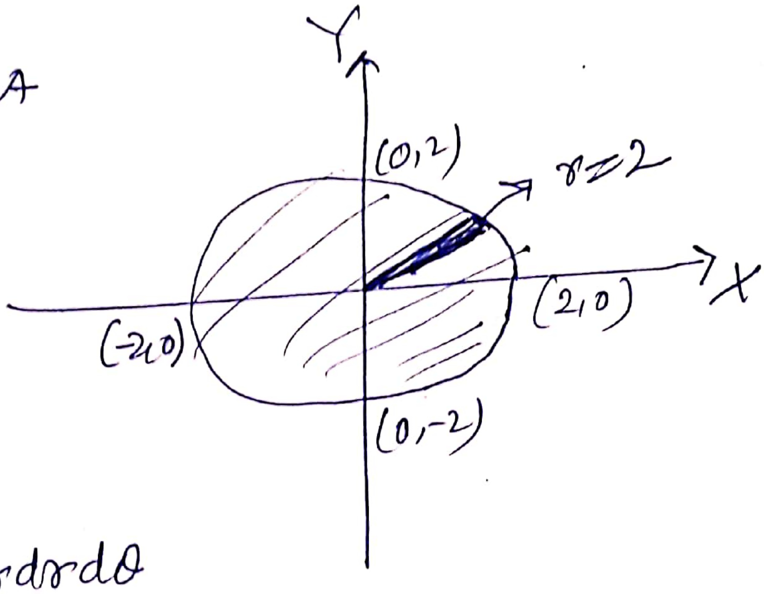
Solution: The height of solid is given by

$$(8-x^2-y^2)-(x^2+y^2)$$

$$=8-2x^2-2y^2$$

required volume

$$V = \iint_R (8-2x^2-2y^2)\,dA$$



$$8-x^2-y^2 = x^2+y^2$$
$$\Rightarrow x^2+y^2=4$$

$\theta : 0 \to 2\pi$
$r : 0 \to 2$

$$\therefore V = \int_0^{2\pi}\int_0^2 (8-2r^2)\,r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\left[\int_0^2 (4r-r^3)\,dr\right]d\theta = 2\int_0^{2\pi}\left[2r^2-\frac{r^4}{4}\right]_0^2 d\theta$$

$$= 2\int_0^{2\pi}(2\times4-4)\,d\theta = 8\int_0^{2\pi} d\theta = 8\times 2\pi = 16\pi.$$