Unit-III : Vector Calculus

Scalar and Vector fields :-

A scalar function $f(x,y,z)$ is a function defined at each point in a certain domain D in space. Its values is real and depends only on the point $P(x,y,z)$ in space, but not on any particular coordinate system being used. For every point $(x,y,z) \in D$, f has a real value. We say that a scalar field f is defined in D.

Ex:- The distance b/w the points $P(x,y,z)$ and $P_0(x_0,y_0,z_0)$

$$d = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$$

defines a scalar field.

Vector field:- A function $V = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ defined at each point $P \in D$ is called a vector function. We say that a vector field is defined in D.

In cartesian system, we can write

$$V = V_1(x,y,z)\hat{i} + V_2(x,y,z)\hat{j} + V_3(x,y,z)\hat{k}.$$

For ex:- the velocity field $V(P)$ defined at any point P on a rotating body defines a vector field.

Parametric representation of curves :-

$$r(t) = x\hat{i} + y\hat{j} \quad ; \quad x = x(t),\ y = y(t)$$

$$r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

Tangent vector :- Let $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$.

Then $\frac{dr}{dt} = r' = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$ is called the tangent vector.

Unit vector :- The unit vector in the direction of the tangent is given by

$$\hat{r} = \frac{r'(t)}{|r'(t)|}$$

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$

Prob ② :- Represent the parabola $y = 1 - 2x^2$, $-1 \le x \le 1$ in parametric form. Hence, find $r'(0)$ and $r'(\frac{\pi}{4})$.

Solution :- Let $x = \sin t$. Then $y = 1 - 2\sin^2 t = \cos 2t$.

$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

Hence, $r(t) = (\sin t)\hat{i} + (\cos 2t)\hat{j}$; $-\frac{\pi}{2} \le t \le \pi/2$.

$r'(t) = (\cos t)\hat{i} - 2\sin 2t\,\hat{j}$

$r'(0) = (\cos 0^\circ)\hat{i} - 2\sin 0^\circ\hat{j} = \hat{i}$

$r'(\frac{\pi}{4}) = (\cos\frac{\pi}{4})\hat{i} - (2\sin\pi/2)\hat{j} = \frac{1}{\sqrt{2}}\hat{i} - 2\hat{j}$.

Prob ③ :- Find the tangent vector to the curve whose parametric representation is $x = \cos t$, $y = \sin t$, $z = t$, $-\pi \le t \le \pi$. Hence, find the unit tangent vector.

Solution :- $r(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$

Tangent vector $r'(t) = (-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}$

The unit tangent vector

$$\hat{r}'(t) = \frac{r'(t)}{|r'(t)|} = \frac{(-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}}{\sqrt{2}}$$

Gradient :- The vector operator del denoted by $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$ and $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

The gradient of a scalar field $f(x,y,z)$ denoted by ∇f or grad (f) is defined as

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}.$$

Prob(1):- Find the gradient of the following scalar fields

(i) $f(x,y) = y^2 - 4xy$ at $(1,2)$

(ii) $x^2y^2 + xy^2 - z^2$ at $(3,1,1)$.

Solution:- (i) $\nabla f(x,y) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)(y^2 - 4xy)$

$$= \hat{i}\frac{\partial}{\partial x}(y^2 - 4xy) + \hat{j}\frac{\partial}{\partial y}(y^2 - 4xy)$$

$$\nabla f(x,y) = -4y\hat{i} + (2y - 4x)\hat{j}$$

At $(1,2)$, we obtain

$$\nabla f(1,2) = -4\times 2\,\hat{i} + (2\times 2 - 4\times 1)\hat{j} = -8\hat{i}.$$

(ii) $\nabla f(x,y,z) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^2 + xy^2 - z^2)$

$$= \hat{i}\frac{\partial}{\partial x}(x^2y^2 + xy^2 - z^2) + \hat{j}\frac{\partial}{\partial y}(x^2y^2 + xy^2 - z^2) + \hat{k}\frac{\partial}{\partial z}(x^2y^2 + xy^2 - z^2)$$

$$\nabla f(x,y,z) = \hat{i}(2xy^2 + y^2) + \hat{j}(2x^2y + 2xy) + \hat{k}(-2z)$$

At $(3,1,1)$

$$\nabla f(3,1,1) = (6+1)\hat{i} + (18+6)\hat{j} - 2\hat{k}$$

$$\nabla f(3,1,1) = 7\hat{i} + 24\hat{j} - 2\hat{k}$$ Ans

Properties of Gradient:-

$$\nabla(f+g) = \nabla f + \nabla g$$

$$\nabla(c_1 f + c_2 g) = c_1 \nabla f + c_2 \nabla g \; ; \; c_1 \; \& \; c_2 \text{ are arbitrary constants.}$$

$$\nabla(fg) = f\nabla g + g\nabla f.$$

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \; ; \; g \neq 0.$$

Directional Derivative:-

The direction derivative of f in the direction $\hat{b}$ is given by

$$D_b(f) = \nabla f \cdot \hat{b} \quad \text{where } \hat{b} = \frac{\vec{u}}{|\vec{u}|}$$

Prob ①:- Find the directional derivative of $f(x,y,z) = xy^2 + 4xyz + z^2$ at the point $(1,2,3)$ in the direction $3\hat{i} + 4\hat{j} - 5\hat{k}$.

Solution:- We have

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2 + 4xyz + z^2)$$

$$= (y^2 + 4yz)\hat{i} + (2xy + 4xz)\hat{j} + (4xy + 2z)\hat{k}$$

At the point $(1,2,3)$, we have

$$\nabla f = 28\hat{i} + 16\hat{j} + 14\hat{k}.$$

The unit vector in the given direction is

$$\hat{b} = \frac{3\hat{i} + 4\hat{j} - 5\hat{k}}{\sqrt{9+16+25}} = \frac{3\hat{i} + 4\hat{j} - 5\hat{k}}{5\sqrt{2}}$$

Hence, directional derivative of $f(x,y,z)$ in direction $3\hat{i}+4\hat{j}-5\hat{k}$

$$D_b(1,2,3) = \frac{1}{5\sqrt{2}}(28\hat{i} + 16\hat{j} + 14\hat{k}) \cdot (3\hat{i} + 4\hat{j} - 5\hat{k}) = \frac{78}{5\sqrt{2}}.$$

Q(2):- If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $|\vec{r}| = r$ and $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$.
then show that $grad\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$.

<u>Proof:-</u> We have $r^2 = x^2 + y^2 + z^2$

Therefore
$$grad\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$
$$= \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$
$$= -\frac{1}{r^2}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right) = -\frac{1}{r^2}\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right)$$
$$= -\frac{1}{r^2}\left(\frac{\vec{r}}{r}\right) = -\frac{1}{r^2}(\hat{r}) = \frac{-\hat{r}}{r^2}.$$ <u>proved</u>

<u>Geometrical ~~Dalex~~ representation of the gradient</u>

Consider now a smooth curve C on the surface passing through a point P on the surface. Let $x = x(t)$, $y = y(t)$, $z = z(t)$ be the parametric representation of the curve C. Any point P on C has the position vector $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$. Since the curve lies on the surface, we have

$$f(x(t), y(t), z(t)) = K.$$

Then $\frac{d}{dt} f(x, y, z) = 0$

By chain rule $\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$

or $\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)\cdot\left(\hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}\right) = 0$

or $\nabla f \cdot r'(t) = 0.$

Remark:- The unit normal vector is

$$\hat{n} = \frac{\text{grad } f}{|\text{grad } f|} = \frac{\nabla f}{|\nabla f|}$$

Prob ① Find a unit normal vector to the surface $xy^2+2yz=8$ at the point $(3,-2,1)$.

Sol^n:- Let $f(x,y,z) = xy^2+2yz-8=0$.

Then $\frac{\partial f}{\partial x} = y^2$, $\frac{\partial f}{\partial y} = 2xy+2z$, $\frac{\partial f}{\partial z} = 2y$

Therefore,

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = y^2\hat{i} + (2xy+2z)\hat{j} + 2y\hat{k}$$

At $(3,-2,1)$

$$\nabla f = 4\hat{i} - 10\hat{j} - 4\hat{k}.$$

$$|\nabla f| = \sqrt{4^2+(-10)^2+(-4)^2} = \sqrt{16+100+16} = \sqrt{132}$$

$$= 2\sqrt{33}$$

The unit ~~vector~~ normal vector at $(3,-2,1)$ is given by

$$\hat{n} = \frac{4\hat{i}-10\hat{j}-4\hat{k}}{2\sqrt{33}} = \frac{2\hat{i}-5\hat{j}-2\hat{k}}{\sqrt{33}}$$

Prob ②:- Find the normal vector and the equation of the tangent plane to the surface $z=\sqrt{x^2+y^2}$ at the point $(3,4,5)$.

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