Lecture 4

Power Series

A power series about $x=0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

A power series about $x=a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

in which the center a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

A power series defines a function $f(x)$ on a certain interval where it converges.

Ex③ For what values of x do the following power series converge?

a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots$

b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - + \cdots$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

d) $\displaystyle\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2!\,x^2 + 3!\,x^3 + \cdots$

Sol^n:- Apply the ratio test to the series.

(a) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = \frac{n}{n+1}|x|$

$\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{n}{n+1}\,|x| = |x|$

The series converges absolutely for $|x| < 1$.

It diverges if $|x|>1$.
At $x=1$, we get series

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}---$ which converges (by ~~Leibnitz Third test~~ Alternating series)

At $x=-1$, we get $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}----$. Which diverges.

Hence series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ converges for $-1<x\leq 1$.

(b) $\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{2n+1}}{2n+1}\cdot\frac{2n-1}{x^{2n-1}}\right|=\frac{2n-1}{2n+1}\cdot x^2$

$\underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right|=1.x^2=x^2$

The series converges absolutely for $x^2<1$.
It diverges if $x^2>1$.

At $x=1$: we get

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+----$ which converges by Alternating series

At $x=-1$; we get

$-1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}--+--$, which converges by Alternating series

Hence series (b) converges for $-1\leq x\leq 1$.

(c) $u_n=\frac{x^n}{n!}$ $\therefore$ $u_{n+1}=\frac{x^{n+1}}{(n+1)!}$

$\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{n+1}\; n!}{(n+1)!\; x^n}\right|=\frac{1\;|x|}{n+1}$

$\lim_{n\to\infty}\frac{|x|}{n+1}\to 0$ for every x

The series converges absolutely for all x.

(d) $\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|=(n+1)|x|\to\infty$ unless $x=0$.

The series (d) diverges for all values of x except $x=0$

Convergence theorem for Power series:- If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely $\forall x$ with $|x| < |c|$. If the series diverges at $x = d$, then it diverges $\forall x$ with $|x| > |d|$.

The Radius of Convergence of a Power Series:-

R: called radius of convergence

It is calculated from ~~nth~~ root test or ratio test

i.e. $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

Let $\sum c_n x^n$ be a power series. Then there exists a radius R for which

a) The series converges for $|x| < R$, and

b) The series ~~con~~ diverges for $|x| > R$,

R is called the radius of convergence.

Corr:- The convergence of the series $\sum_{n=1}^{\infty} c_n (x-a)^n$ is described by one of the following three cases:

1. If R is finite, then series converges absolutely ~~for~~ ~~with $|x-a| < R$.~~ in region $|x-a| < R$.

2. If $R = \infty$. Then series converges absolutely for every x.

3. If $R = 0$. Then series converges at $x = a$, elsewhere diverges

The interval of radius R centered at $x=a$ is called the interval of convergence.

<u>Problem:-</u> Find the radius of convergence for the series

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n} x^n$ (ii) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{n^2+1} x^n$ $R=5$ (iii) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

(iv) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2^n+1)(n^2+1)}$ (v) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2^n+1)(n^2+1)}$

<u>Solⁿ:-</u> (i) Using the root test here $a_n = \dfrac{1}{n\,3^n}$

$$\lim_{n\to\infty} |a_n|^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{n3^n}\right)^{1/n} = \frac{1}{3}\underset{n\to\infty}{Lt}\left(\frac{1}{n}\right)^{1/n} = \frac{1}{3}\cdot\frac{1}{Lt\, n^{1/n}}$$

$$= \frac{1}{3}\times\frac{1}{1} = \frac{1}{3}$$

We know, $\dfrac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \dfrac{1}{3}$

$$\Rightarrow \boxed{R=3}$$

and interval of convergence is $|x|<3$.

(iii) $a_n = \dfrac{1}{n!}$ $\therefore a_{n+1} = \dfrac{1}{(n+1)!}$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \underset{n\to\infty}{Lt}\left|\frac{n!}{(n+1)!}\right| = \underset{n\to\infty}{Lt}\frac{1}{n+1} = 0$$

$\therefore \boxed{R=\infty}$ $\therefore$ series converges for $\forall x$.

(iv) $$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \underset{n\to\infty}{Lt}\ \cancel{(2^{n}+1)[(n+1)^2+1]}\ \frac{(2^n+1)(n^2+1)}{(2^{n+1}+1)[(n+1)^2+1]}$$

$$= \underset{n\to\infty}{Lt}\left(\frac{1+2^{-n}}{2+2^{-n}}\right)\left(\frac{1+n^{-2}}{(1+n^{-1})^2+n^{-2}}\right) = \frac{1}{2}$$

$\therefore \boxed{R=2}$