useful rules for estimating errors :-

If the approximate value of a number, X having n decimal digits in X', then

(1) Absolute error due to truncation to k digits $= |X - X'| < 10^{n-k}$

(2) Absolute error due to rounding off to k digits $= |X - X'| < \frac{1}{2} 10^{n-k}$

(3) Relative error due to truncation to k digits $= \left|\frac{X - X'}{X}\right| < 10^{1-k}$

(4) Relative error due to rounding off to k digits $= \left|\frac{X - X'}{X}\right| < \frac{1}{2} 10^{1-k}$

observation (1) If number is correct to n significant digits, then the maximum relative error $\leq \frac{1}{2} 10^{-n}$. If a number is correct to d decimal places, then the absolute error $\leq \frac{1}{2} 10^{-d}$

observation (2) ⇒ If the first significant figure of a number is k and number is correct to n significant figures, then the relative error

$$= \frac{1}{k \times 10^{n-1}}$$

ex ⇒ Let us varify the above result by finding the relative error in the number 864.32, correct to five significant figures.

Example ⇒ Round off the number 865250 and 37.46235 to four significant figures, compute E_a, E_r, E_p in each case.

Ans 865250 number rounded off to four significant figures

$= 865200$

$E_a = |x - x'| = |865250 - 865200| = 50$

$E_r = \left|\frac{x - x'}{x}\right| = \left|\frac{50}{865250}\right| = 5.78 \times 10^{-5}$

$E_p = 100 \times E_r = 5.78 \times 10^{-3}$

37.46235 rounded off to four significant figures

$= 37.46$

$E_a = |x - x'| = |37.46235 - 37.46|$

$= |00.00235|$

$= 0.00235$

$E_r = \left|\frac{E_a}{x}\right| = \left|\frac{0.01235}{37.46235}\right| = 6.27 \times 10^{-5}$

$E_p = 100 \times E_r = 6.27 \times 10^{-3}$

Ans

Find the absolute error if the number (8)

$x = 0.00545828$ is

(i) truncated to three decimal places (digits).

(ii) rounded off to three decimal places (digits).

Solution ⇒ $x = 0.00545828$

$= 0.545828 \times 10^{-2}$

(i) After truncating to three decimal places, its approximate value $x' = 0.545 \times 10^{-2}$

Therefore absolute error $= |x - x'|$

$= 0.000828 \times 10^{-2}$

$= 0.828 \times 10^{-5}$

(ii) After rounding off to three decimal places

$x' = 0.546 \times 10^{-2}$

Absolute error $= |x - x'|$

$= |0.545828 \times 10^{-2} - 0.546 \times 10^{-2}|$

$= 0.000172 \times 10^{-2}$

$= 0.172 \times 10^{-5}$ Ans

.546000
.545828
000172

Find the relative error if the number $x = 0.004997$ is

(i) truncated to three decimal places

(ii) rounded off to three decimal places.

$x = 0.4997 \times 10^{-2}$

Ans ⇒ $x' = 0.499 \times 10^{-2}$

Relative error $= \left|\frac{x-x'}{x}\right| = \left|\frac{0.0007 \times 10^{-2}}{0.499 \times 10^{-2}}\right|$

$= 0.140 \times 10^{-2}$

After rounding off to three decimal places, the approximate value of the given number

$x' = 0.5000 \times 10^{-2}$

Relative error $= \cancel{\frac{0.360 \times 10^{-2}}{0.1111111}} = \frac{.0003 \times 10^{-2}}{0.4997 \times 10^{-2}}$

$= 0.600 \times 10^{-3}$

Ans

Cropping ⇒ In cropping, the extra digits are dropped. This is called truncating the number. Suppose, we are using a computer with a fixed word length of four digits. Then a number like 42.7893 will be stored as 42.78 and the digits 93 will be dropped.

* We can express the number 42.7893 in floating point form as

$x = 0.427893 \times 10^2$

$= (0.4278 + 0.000093) \times 10^2$

$= (0.4278 + (0.93 \times 10^{-4})) \times 10^2$

This can be expressed in general form as

True $x = (f_x + g_x \times 10^{-d})\, 10^E$

$= f_x \times 10^E + g_x \times 10^{E-d}$ = Approximate x + Error

Error Propagation ⇒ Number of computational steps are carried out for the solution of a problem. It is necessary to understand the way the error propagates with progressive computation.

* If the approximate values of x & y are x' & y' respectively then the ~~absolute~~ error

$$E_{ax} = x - x' \quad \& \quad E_{ay} = y - y'$$

1. ~~Absolute~~ Error in addition operation ⇒

$$x + y = x' + E_{ax} + y' + E_{ay}$$

$$= (x' + y') + (E_{ax} + E_{ay})$$

Thus Total error $= E_{x+y} = E_{ax} + E_{ay}$

2. Error in subtraction ⇒

$$x - y = x' + E_{ax} - y' - E_{ay} = (x' - y') + (E_{ax} - E_{ay})$$

Therefore, total error in subtraction is

$$\text{Total Error} = E_{x-y} = E_{ax} - E_{ay}$$

* $E_{ax} + E_{ay}$ (i.e. error in addition) does not mean error will increase in all cases of addition. It depends on the sign of individual errors. Similarly in the case with subtractions.

* Since we do not normally know the sign of errors, we can only estimate error bounds.

* Thus we can say that

$$\boxed{|E_{x\pm y}| \le |E_{ax}| + |E_{ay}|}$$

* Thus, magnitude of the absolute error of a sum (or difference) is equal to or less than the sum of the magnitudes of the absolute errors of the operands.

* The equality applies when operands have same signs.

* Inequality applies if the signs are different.

Multiplication ⇒ $XY = (X' + E_{ax})(Y' + E_{ay})$

$$= X'Y' + X'E_{ay} + Y'E_{ax} + E_{ax}E_{ay}$$

Errors are normally small and therefore their product will be much smaller, so we can ignore $E_{ax}E_{ay}$ term

$$\Rightarrow \quad XY = X'Y' + \underbrace{X'E_{ay} + Y'E_{ax}}_{\text{error term}}$$

Thus error in multiplication

$$= X'E_{ay} + Y'E_{ax}$$

$$= X'Y'\left(\frac{E_{ay}}{Y'} + \frac{E_{ax}}{X'}\right)$$

approximately

Division $\Rightarrow$ $$\frac{X}{Y} = \frac{X' + E_{ax}}{Y' + E_{ay}}$$

multiplying both numerator & denominator by $Y' - E_{ay}$, we get

$$\frac{X}{Y} = \frac{(X' + E_{ax})(Y' - E_{ay})}{(Y' + E_{ay})(Y' - E_{ay})} = \frac{X'Y' + Y'E_{ax} - X'E_{ay} - E_{ax}E_{ay}}{Y'^2 - E_{ay}^2}$$

$$= \frac{X'Y' + Y'E_{ax} - X'E_{ay}}{Y'^2}$$ (dropping terms involving product of error)

$$= \frac{X'}{Y'} + \frac{E_{ax}}{Y'} - \frac{X'E_{ay}}{Y'^2}$$

$$= \frac{X'}{Y'} + \underbrace{\frac{X'}{Y'}\left[\frac{E_{ax}}{X'} - \frac{E_{ay}}{Y'}\right]}_{\text{error}}$$

Thus total error $= \frac{X'}{Y'}\left[\frac{E_{ax}}{X'} - \frac{E_{ay}}{Y'}\right]$

* Applying the triangle inequality theorem,

$$E_{x/y} \le \left|\frac{X'}{Y'}\right| \left\{ \left|\frac{E_{ax}}{X'}\right| + \left|\frac{E_{ay}}{Y'}\right| \right\}$$

$$E_{xy} \le |X'Y'| \left\{ \left|\frac{E_{ax}}{X'}\right| + \left|\frac{E_{ay}}{Y'}\right| \right\}$$

Question $\Rightarrow$ Find absolute & relative error in $\sqrt{6} + \sqrt{7} + \sqrt{8}$ correct to four ~~decimal places~~ significant digits.

Solution $\Rightarrow$ $X = 2.4495$, $Y = 2.6458$, $Z = 2.8284$

$X' = 2.449$, $Y' = 2.645$, $Z' = 2.828$

$E_{absolute} = 0.0005 + 0.0008 + 0.0004 = 0.0017$