Lecture - 3

Solution of Algebraic and Transcendental Equations

* In scientific and engineering studies, a frequently occurring problem is to find the roots of equation of the form $\boxed{f(x) = 0}$ —— (i)

* $f(x) = 0$, can be algebraic equation i.e

$3x + 5y - 21 = 0$ (linear)

$2x + 3xy - 25 = 0$

$x^3 - xy - 3y^3 = 0$

$5x^5 - 3x^3 + 2x^2 + x + 1 = 0$

$x^2 - 4x + 4 = 0$

$x^3 - 4x^2 + x = 0$

polynomial equations (simple class of algebraic eqs)

* $f(x) = 0$, can be Transcendental equations. The A non-algebraic equation is called a transcendental equation. These equations include trignometric, exponential and logarithmic functions.

Examples ⇒ $2\sin x - x = 0$, $e^x \sin x - \frac{1}{2}x = 0$, $\log x^2 - 1 = 0$.

* ~~Algebraic~~ Polynomial equations can have exactly n roots.

* Transcendental equations may have a finite or infinite number of real roots or may not have any real root.

* There are number of ways to find the roots of nonlinear equations (i) such as (i) Direct analytic methods (ii) Graphical methods (iii) Trial and error methods (iv) Iterative methods

- If $f(x)=0$ is a quadratic, cubic, or a biquadratic equation then algebraic formulae are available for expressing the roots in term of coefficients.
- On the other hand, when $f(x)$ is a polynomial of higher degree or an expression involving transcendental terms then algebraic methods are not available and a recourse must be taken to find the roots by iterative methods.

* If $f(x)$ is a polynomial of the form $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n$, then the following results from the theory of equations would be useful

(i) Every polynomial equation of the n^{th} degree has n and only n roots.

(ii) If n is odd, the polynomial equation has atleast one real root whose sign is opposite to the last term.

(iii) If n is even and the constant term is negative, then the equation has atleast one positive root and atleast one negative root.

(iv) If the polynomial equation has (a) real coefficients, then imaginary roots occur in pairs and (b) rational coefficients, then irrational roots occur in pair.

(v) <u>Des cartes Rule of sign</u> ⇒

(a) A polynomial eq^n $f(x)=0$ can not have more number of positive real roots than the number changes of sign in the coefficient of $f(x)$.

(5) ~~The~~ $f(x)=0$ can not have more number of negative real roots than the number of changes of sign in the coefficients of $f(-x)$.

* Iterative method, based on the number of guesses they use can be grouped into two catagories:-

(i) Bracketing methods
(ii) open end methods

Bracketing methods (interpolation methods) ⇒ This method start with two initial guesses that 'bracket' the root and then systematically reduce the width of the bracket until the solution is reached. Two popular method under this category are

(i) Bisection method
(ii) False position method

* These methods are based on the assumption that the function changes sign in the vicinity of a root.

open end methods (extrapolation methods) ⇒ These methods use a single starting value or two values that do not necessary bracket the root.

The three popular methods under this catagory are

(i) Newton-Raphson method
(ii) Secant Method
(iii) Fixed point method

* It may be noted that the bracketing methods require to find sign changes in the function during every iteration. Open end methods do not require this.

Bisection Method ⇒ This method is one of the simplest and most reliable method of iterative methods for the solution of non linear equation. This method is based on the idea that if a function $f(x)$ is continuous between a & b, and $f(a)$ and $f(b)$ are of opposite signs then there exists atleast one root between a & b.

* For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the root lies between a & b and let its approximate value be given by $x_0 = \frac{a+b}{2}$. If $f(x_0) = 0$, we conclude that x_0 is a root of $f(x) = 0$. Otherwise, the root lies either between x_0 & b or between a & x_0 depending on whether $f(x_0)$ is negative or positive. We design this new interval as $[a_1, b_1]$ whose length is $\frac{|b-a|}{2}$. As before, this is bisected at x_1 and the new interval will be exactly half the length of the previous one.

We repeat this process until the latest interval (which contains the root) is as small as desired, say ε. It is clear that the interval width is reduced by a factor of one-half at each step and at the end of the nth step, the new interval will be $[a_n, b_n]$ of length $\frac{|b-a|}{2^n}$. Then, we have

$$\frac{|b-a|}{2^n} \le \varepsilon$$

which gives on simplification

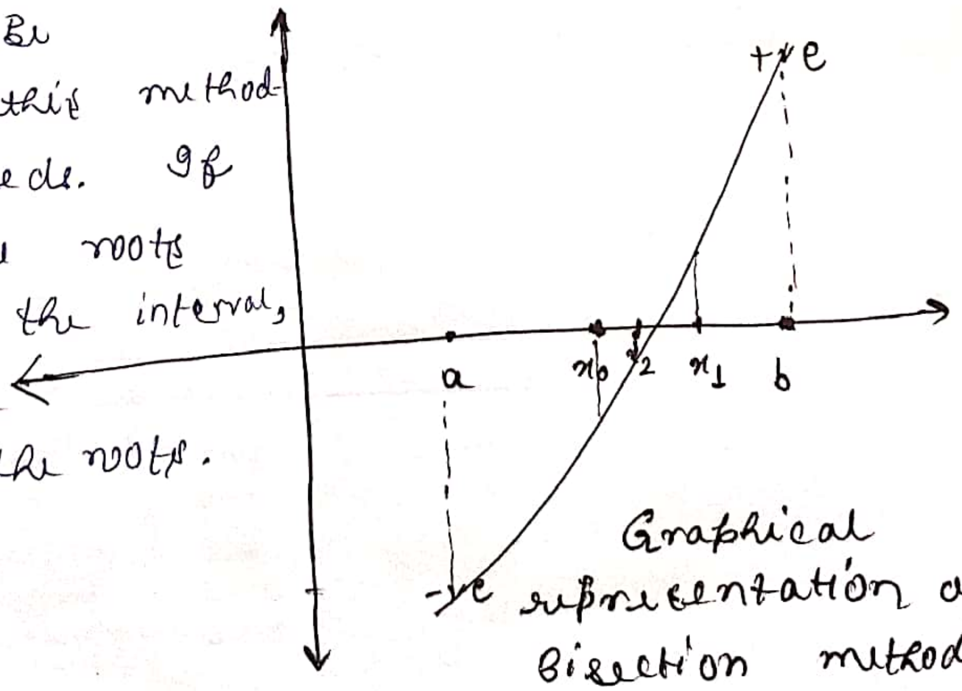
$$n \geq \frac{\log_e(|b-a|/\varepsilon)}{\log_e 2} \qquad \text{———} \; (*)$$

(*) gives the number of iterations required to achieve an accuracy ε.

For example If $|b-a| = 1$, $\varepsilon = 0.001$ then

$$n \geq 9.9657842 7 \simeq 10$$

* It should be noted that this method always succeeds. If there are more roots than one in the interval, bisection method finds one of the roots.



Graphical representation of Bisection method

Steps

(i) Choose a & b such that $f(a)$ & $f(b)$ are of opposite sign i.e $f(a)\,f(b) < 0$

(ii) Set $x_0 = \dfrac{a+b}{2}$,

(iii) If $f(x_0) = 0$, then x_0 is the required root.

If $f(x_0)\,f(a) < 0$, the root lies in the interval (a, x_0).

If $f(x_0)\,f(b) < 0$, the root lies in the interval (x_0, b).

and repeat the similar procedure.

Example ⇒ find the root of the eqn

$$x^3 - 5x + 1 = 0$$

Solution ⇒ $f(0) = 1$, $f(1) = -3$

$f(0)\,f(1) = -3 < 0$

Thus root will lies between 0 & 1

$$x_0 = \frac{0+1}{2} = 0.5$$

$$f(x_0) = .125 - 2.5 + 1 = -1.375 < 0$$

therefore root will lies in between 0 & 0.5.

For simplicity we make the table

k	a_k	b_k	x_k	$f(x_k)$
0	0	1	0.5	−1.375
1	0	0.5	0.25	−ve
2	0	0.25	0.125	+ve
3	0.125	0.25	0.1875	+ve
4	0.1875	0.25	0.21875	−ve
5	0.1875	0.21875	0.203125	−ve
6	0.1875	0.203125	0.1953125	

approx root = 0.1953125

Ans

Example ⇒ Find a real root of the equation $x^3 - 2x - 5 = 0$

Solution ⇒ Here, $f(x) = x^3 - 2x - 5$

$$f(2) = -1 \, , \quad f(3) = 16 \quad \Rightarrow \quad f(2)\, f(3) < 0$$

Thus a root lies between 2 & 3, and we take

$$x_0 = \frac{2+3}{2} = 2.5 \, ,$$

$f(x_0) = f(2.5) = 5.6250$ (+ve), Thus root lies between 2 and 2.5. Again

$$x_1 = \frac{2+2.5}{2} = \frac{4.5}{2} = 2.25$$

$$f(x_1) = 1.890625 \text{ (+ve)}$$

Thus, root lies between 2 & 2.25

$$x_2 = \frac{2+2.25}{2} = \frac{4.25}{2} = 2.125$$

$f(x_2) = 0.3457$ (+ve), Thus root lies between 2 & 2.125

$$x_3 = \frac{2+2.125}{2} = \frac{4.125}{2} = 2.0625$$

Proceeding in this way, we obtain

$$x_4 = 2.09375 \, , \quad x_5 = 2.10938 \, ,$$

$$x_6 = 2.10156 \, , \quad x_8 = 2.09766 \, ,$$

$$x_9 = 2.09570 \, , \quad x_{10} = 2.09473 \, , \quad x_{11} = 2.09424$$

Thus root correct to three decimal places is 2.094

Ans

Convergence of iterative methods

We now study the rate at which the iteration methods converge to the exact root, if the initial approximation is sufficiently close to the desired root.

* Define the error of approximation at the k^{th} iteration as $\varepsilon_K = \underset{\text{approximate root}}{x_K} - \underset{\text{exact root}}{\alpha}$, $k = 0, 1, 2, \ldots$

Definition ⇒ An iterative method is said to be of order p or has the rate of convergence p if p is largest positive real number for which there exists a finite constant $C \neq 0$ such that

$$|\varepsilon_{K+1}| \leq C|\varepsilon_K|^p.$$

The constant C, which is independent of k, is called the asymptotic error constant and it depends on the derivatives of $f(x)$ at $x = \alpha$, and ε_{K+1} & ε_K are the errors in the $(k+1)^{th}$ & k^{th} iterations.

Rate of convergence of Bisection method ⇒ In Bisection method, we bisect the interval at each iteration if we take mid point of interval as approximate root then error will not be more than half of the previous interval i.e.

$$|e_{K+1}| \leq 0.5\,|e_K|$$

comparing with $|e_{K+1}| \leq c\,|e_K|^p$, we get

$$c = 0.5, \quad p = 1$$

→ Rate of convergence of Bisection method is one (i.e. linear)