Extrema of Functions of Several variables

Def ① :- $f(x,y)$ is called a local maximum at (a,b),
if $f(a,b) \geq f(x,y) \quad \forall (x,y) \in R$.

Similarly, $f(x,y)$ is called a local minimum at (a,b),
if $f(a,b) \leq f(x,y) \quad \forall (x,y) \in R$.

R : open disk.

In either case, $f(a,b)$ is called a local extremum of f.

Critical point :- The point (a,b) is a critical point of the function $f(x,y)$ if (a,b) is in the domain of f and either

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$$

or one or both of f_x and f_y do not exist at (a,b)

Theorem :- If $f(x,y)$ has a local extremum at (a,b), then (a,b) must be a critical point of f.

Saddle point:- The point $P(a, b)$ is a critical point of $z = f(x,y)$ if (a,b) is a critical point of f and if every open disk centered at (a,b) contains points (x,y) in the domain of f for which $f(x,y) < f(a,b)$ and points (x,y) in the domain of f for which $f(x,y) > f(a,b)$

Second Derivatives Test :-

Suppose that $f(x,y)$ has continuous second-order partial derivatives in some open disk containing the point (a,b) and that $f_x(a,b) = f_y(a,b) = 0$.
Define the discriminant D for the point (a,b) by

$$D(a,b) = \left[f_{xx} \cdot f_{yy} - (f_{xy})^2\right]_{(a,b)}$$
$$= rt - s^2.$$

(i) If $D(a,b) > 0$ and $r > 0$, then f has a local minimum at (a,b).

(ii) If $D(a,b) > 0$ and $r < 0$, - - - local maximum

(iii) If $D(a,b) < 0$, then f has a saddle point at (a,b).

(iv) If $D(a,b) = 0$, then no conclusion can be drawn.

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① :- Locate and classify all critical points for $f(x,y) = 3x^2 - y^3 - 6xy$.

Soln :-

$f_x = 6x - 6y$

$f_y = -3y^2 - 6x$

Now, $f_x = 0 \Rightarrow 6x - 6y = 0 \Rightarrow x = y$

$f_y = 0 \Rightarrow -3x^2 - 6x = 0$ put $x = y$

$\Rightarrow x(x+2) = 0$

$\Rightarrow x = 0, x = -2$

$\therefore y = 0, y = -2$.

The only two critical points are $(0,0)$ and $(-2,-2)$.

Classify :- $f_{xx} = 6$, $f_{yy} = -6y$ and $f_{xy} = -6$.

$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$D(x,y) = 6 \times (-6y) - (-6)^2 = -36y - 36$

$D(x,y) = -36(y+1)$

At $(0,0)$ $D(0,0) = -36(0+1) = -36 < 0$

and $D(-2,-2) = -36(-2+1) = 36 > 0$.

We conclude that there is a saddle point of f at $(0,0)$.

Since $D(-2,-2) > 0$ and $f_{xx}(-2,-2) = 6 > 0$.

Hence, there is a local minimum at $(-2,-2)$.

Prob (2) :- Locate and classify all critical points for $f(x,y) = x^3 - 2y^2 - 2y^4 + 3x^2y$.

Solution:- $f_x = 3x^2 + 6xy$

$f_y = -4y - 8y^3 + 3x^2$.

Since both f_x and f_y exist for all (x,y), the critical points are sol^{n}s of the two equations

$$f_x = 3x^2 + 6xy = 0 \quad \text{---} (1)$$

and $$f_y = -4y - 8y^3 + 3x^2 = 0 \quad \text{---} (2)$$

$(1) \Rightarrow \quad x(x+2y) = 0 \Rightarrow x = 0$ or $x = -2y$.

Putting $x=0$ in (2); we have

$$-4y - 8y^3 = 0 \Rightarrow y(1+2y^2) = 0$$

$\Rightarrow \quad y = 0$. Thus for $x = 0$.

we have only one critical point $(0,0)$.

Substituting $x = -2y$ in (2); we get

$$-4y - 8y^3 + 3(-2y)^2 = 0$$

$$\Rightarrow -4y - 8y^3 + 12y^2 = 0 \Rightarrow -4y(1+2y^2-3y) = 0$$

$$\Rightarrow \quad -4y(2y-1)(y-1) = 0$$

$$\Rightarrow \quad y = 0, y = \tfrac{1}{2}, y = 1.$$

$$\therefore x = 0, x = -1, x = -2$$

Critical points are $(0,0)$, $(-1, \frac{1}{2})$ and $(-2, 1)$

ussrfy :- $f_{xx} = 6x + 6y$

$f_{yy} = -4 - 24y^2$

$f_{xy} = 6x$

$$D(x,y) = (6x+6y)\cdot(-4-24y^2) - (6x)^2$$

At (0,0)

$D(0,0) = 0$ no local extremum at (0,0)

At $(-1, 1/2)$

$$D(-1, 1/2) = (-6+3)(-4-24\times\tfrac{1}{4}) - (-6)^2$$

$$= 30 - 36 = -6 < 0$$

and i.e. f has saddle point at $(-1, \frac{1}{2})$.

At $(-2,1)$

$$D(-2,1) = (-12+6)(-4-24) - (-12)^2$$

$$= 24 > 0$$

and $f_{xx}(-2,1) = -12+6 = -6 < 0$

$\therefore$ f has a local maximum at $(-2,1)$.

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Definition:- We call $f(a,b)$ the absolute maxim[um]
of f on the region R if $f(a,b) \geq f(x,y)$ $\forall (x,y) \in$ [R].
Similarly, $f(a,b)$ is called the absolute minimu[m]
of f on R if $f(a,b) \leq f(x,y)$ $\forall (x,y) \in R$.
In either case, $f(a,b)$ is called an absolute
~~maxi~~ extremum of f.