Topic: Finite Differences and Interpolation

Prerequisite

- Knowledge of basic arithmetic operations
- Knowledge of approximate value
- Knowledge of properties of Polynomials

Objectives

- To understand interpolation formulas
- To fit a polynomial by interpolation formulas
- To find the missing value(s) in the given sequence.

Introduction: Finite Differences

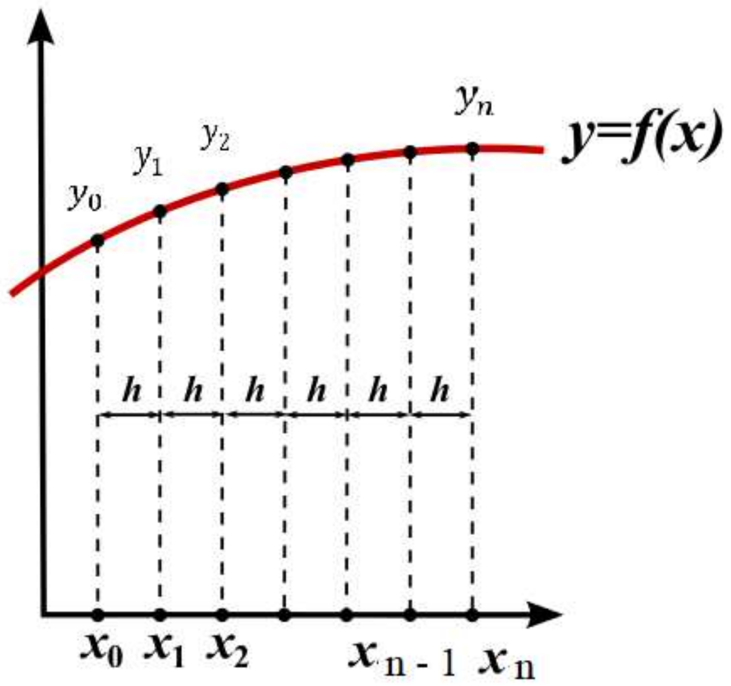
Finite differences are the back bone of Numerical Methods play a key role in the formulation of interpolating polynomials.

The interpolation is the art of calculating values between the tabular values. And interpolation formulae are used to derive formulae for numerical differentiation and integration. Thus one can solve the differential equations using finite differences.

The finite differences deals with the changes that take place in the value of the function (dependent variable), due to finite changes in the independent variable.

FINITE DIFFERENCES

Suppose that a function y = f (x) is tabulated for the equally spaced value (arguments) $x_0, x_0 + h, x_0 + 2h, \ldots\ldots, x_0 + nh$ giving the functional values(entries) $y_0, y_1, y_2, \ldots\ldots y_n$.



The difference between two consecutive values of x is called <u>differencing interval</u> and is denoted by h.

The difference between two consecutive values of y are called <u>Finite differences.</u>

Forward Differences. The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \ldots. \Delta y_{n-1}$ respectively are called the first forward differences, where Δ is the forward difference operator.

In general the **first forward differences** are $\Delta y_r = y_{r+1} - y_r$, r $= 0, 1, 2, \ldots$.

Similarly the **second forward differences** are defined by

$$\Delta^2 y_r = \Delta\, y_{r+1} - \Delta\, y_r$$

In general, $\Delta^p y_r = \Delta^{p-1}\, y_{r+1} - \Delta^{p-1}\, y_r$ defines the pth forward differences.

These differences are systematically set out as follows in

Forward Difference Table

Arguments	Entries	1st Difference	2nd Difference	3rd Diff	4th Diff	5th Diff
x_0	y_0	$y_1 - y_0 = \Delta y_0$	$\Delta\, y_1 - \Delta\, y_0 = \Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$
$x_0 + h$	y_1	$y_2 - y_1 = \Delta y_1$	$\Delta\, y_2 - \Delta\, y_1 = \Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_1$	
$x_0 + 2h$	y_2	$y_3 - y_2 = \Delta y_2$	$\Delta\, y_3 - \Delta\, y_2 = \Delta^2 y_2$	$\Delta^3 y_2$		
$x_0 + 3h$	y_3	$y_4 - y_3 = \Delta y_3$	$\Delta\, y_4 - \Delta\, y_3 = \Delta^2 y_3$			
$x_0 + 4h$	y_4	$y_5 - y_4 = \Delta y_4$				
$x_0 + 5h$	y_5					

The first entry y_0 is called the leading term and Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ etc. are called the leading differences

Observation. Any higher order forward difference can be expressed in terms of entries.

As $\Delta^2 y_0 = \Delta\, y_1 - \Delta\, y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2\, y_1 + y_0$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$

$$= (\, y_3 - 2\, y_2 + y_1) - (y_2 - 2\, y_1 + y_0)$$

$$= y_3 - 3\, y_2 + 3 y_1 - y_0$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$$

$$= (y_4 - 3\, y_3 + 3 y_2 - y_1) - (y_3 - 3\, y_2 + 3 y_1 - y_0) = y_4 - 4\, y_3 + 6 y_2 - 4 y_1 + y_0$$

The coefficients occurring on the right hand side being the binomial coefficients, we have in general

$$\Delta^n y_0 = y_n - n_{C_1} y_{n-1} + n_{C_2} y_{n-2} - \cdots + (-1)^n\, y_0$$

Example 1: Find $\Delta^2(e^x)$ if the differencing interval is 1

Solution: $\Delta^2(e^x) = \Delta(\Delta e^x)$

$$= \Delta(e^{x+1} - e^x)$$

$$= \Delta\{e^x(e - 1)\}$$

$$= (e - 1)\Delta e^x$$

$$= (e - 1)(e^{x+1} - e^x)$$

$$= (e - 1)\{e^x(e - 1)\}$$

$$= e^x (e - 1)^2$$

Example 2: Find $\Delta^3(x^4+x^3+x+4)$ if the differencing interval is 1

Solution: $\Delta^3(x^4+x^3+x+4) = \Delta^2[\{(x+1)^4+(x+1)^3+(x+1)+4\}-(x^4+x^3+x+4)]$

$= \Delta^2\ [(x^4+4\,x^3+6\,x^2+4x+1)+(x^3+3\,x^2+3x+1)+x+5-x^4-\ x^3-x-4]$

$= \Delta^2\ (4\,x^3+9\,x^2+7x+3)$

$= \Delta[\Delta(4\,x^3+9\,x^2+7x+3)]$

$= \Delta[\{4(x+1)^3+9(x+1)^2+7(x+1)+3\}-(4\,x^3+9\,x^2+7x+3)]$

$= \Delta[4(x^3+3\,x^2+3x+1)+9\,(\,x^2+2x+1)+7\mathrm{x}+10-4\,x^3-9\,x^2-7x-3]$

$= \Delta[12\,x^2+30\,x+20]$

$= [12\,(x+1)^2+30\,(x+1)+20]-[12\,x^2+30\,x+20]$

$= [12(\,x^2+2x+1)+30\,x+50]-12x^2-30\,x-20 \quad = \quad 24\,x-42$ Ans.

Observation.

The nth differences of a polynomial of the nth degree, are constant and all higher order differences are zero.

The converse is also true i.e., if the nth differences of a function tabulated at equally spaced intervals are constant, the function is enables us to approximate a function of degree n.

Example 3: Construct the forward difference table for the data below:

x:	0	1	2	3	4
f(x):	1	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 f(1)$

Solution:

Arguments	Entries	Δ	Δ^2	Δ^3
0	1			
		$\Delta f(0) = 0.5$		
1	1.5		$\Delta^2 f(0) = 0.2$	
		$\Delta f(1) = 0.7$		$\Delta^3 f(0) = 0$
2	2.2		$\Delta^2 f(1) = 0.2$	
		$\Delta f(2) = 0.9$		$\Delta^3 f(1) = 0.4$ Ans.
3	3.1		$\Delta^2 f(2) = 0.6$	
		$\Delta f(0) = 1.5$		
4	4.6			

Example 4: Find the missing values in the following table:

x: 45 50 55 60 65

y: 3 * 2 * - 2.4

Solution: Let the missing values be a & b, and construct the forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	3			
		a -3		
50	a		5 – 2 a	
		2 - a		3 a + b - 9
55	2		a + b - 4	
		b - 2		- a – 3 b + 3.6
60	b		- 0.4 – 2 b	
		- 2.4 - b		
65	- 2.4			

As only three entries are given, the function y can be represented by a second degree polynomial.

Therefore, all the 3rd order differences will be vanished, i.e., $\Delta^3 y_0 = 0 \ \ and \ \ \Delta^3 y_1 = 0$

Thus 3 a + b – 9 = 0 and - a – 3 b + 3.6 = 0

Solving these equations, we get   a = 2.925   and   b = 0.225

Shift Operator E is the operation of increasing the argument x by h so that

$$Ef(x) = f(x+h) \quad or\ Ey_x = y_{x+h}$$

$$E^2f(x) = f(x+2h) \quad or\ E^2y_x = y_{x+2h}$$

$$E^3f(x) = f(x+3h) \quad or\ E^3y_x = y_{x+3h} \quad \text{etc.}$$

And the inverse operator E^{-1} is defined by

$$E^{-1}f(x) = f(x-h) \ \ or\ \ E^{-1}y_x = y_{x-h}$$

$$E^{-2}f(x) = f(x-2h) \ \ or\ \ E^{-2}y_x = y_{x-2h}$$

$$E^{-3}f(x) = f(x-3h) \ \ or\ \ E^{-3}y_x = y_{x-3h} \quad \text{etc.}$$

Relation between Δ $\boldsymbol{and\ \ E}$

As we know,

$$\Delta \boldsymbol{y_0} = \boldsymbol{y_1} - \boldsymbol{y_0}$$

$$= \boldsymbol{E}\boldsymbol{y_0} - \boldsymbol{y_0}$$

$$= (\boldsymbol{E} - \boldsymbol{1})\boldsymbol{y_0}$$

$$\Rightarrow \Delta \equiv \boldsymbol{E} - \boldsymbol{1}$$

Meaning of Interpolation and Extrapolation

Newton's Forward Interpolation Formula

Newton's Forward Interpolation Formula, this formula is used for interpolating the values of y near the beginning of a set of tabulated values of y.

Let the function y = f (x) take the values $y_0, y_1, y_2, \ldots \ldots y_n$ corresponding to the equally spaced values $x_0, x_0 + h, x_0 + 2h, \ldots \ldots, x_0 + nh$ of x.

x	x_0	$x_0 + h$	$x_0 + 2h$	….	$x_0 + nh$
y	y_0	y_1	y_2	….	y_n

If it is required to evaluate y for $x = x_0 + p\,h$, then

$$y(x_0 + p\,h) = E^p y_0 = (1 + \Delta)^p y_0 \qquad [\because\ \Delta \equiv \boldsymbol{E} - \mathbf{1}]$$

$$= \left\{1 + p\,\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots + \frac{p(p-1)(p-2)\ldots\{p-(n-1)\}}{n!}\Delta^n\right\} y_0$$

$$= y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ \ldots . + \frac{p(p-1)(p-2)\ldots\{p-(n-1)\}}{n!}\Delta^n y_0$$

is called Newton's forward interpolation formula as it contains y_0 and the forward differences of y_0.

Example

Example 5. For the given data, evaluate y (x = 5)

x: 4 6 8 10

y: 1 3 8 16

Solution:

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1= y_0	2= Δy_0	3= $\Delta^2\, y_0$	0= $\Delta^3\, y_0$
6	3	5	3	
8	8	8		
10	16			

Here, $x_0 = 4, h = 2 \; and \; x = 5,$

Therefore, the $relation \; x = x_0 + p\, h \; gives \; p = \frac{x - x_0}{h} = \frac{5-4}{2} = 0.5$

$Hence, \; using \; Newton's \; forward \; difference \; formula, y(x = 5) = y_0 + p\, \Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \; \ldots.$

$$= 1 + 0.5(2) + \frac{0.5(-0.5)}{2!}(3) + \frac{0.5(-0.5)(-1.5)}{3!}(0) +$$

$$= 1 + 1 - \frac{3}{8} + 0 = 1.625$$

Example

Example 6. For the given data, evaluate f(3.5)

x:	3	4	5	6	7
f(x):	3	6.6	15	22	35

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	$3 = y_0$	$3.6 = \Delta y_0$	$4.8 = \Delta^2 y_0$	$-6.2 = \Delta^3 y_0$	$13.6 = \Delta^4 y_0$
4	6.6	8.4	- 1.4	7.4	
5	15	7	6		
6	22	13			
7	35				

Solution:

Here, $x_0 = 3, h = 1 \; and \; x = 3.5,$

Therefore, the $relation \; x = x_0 + p\,h \; gives \; p = \frac{x - x_0}{h} = \frac{3.5 - 3}{1} = 0.5$

$Hence, \; using \; Newton's \; forward \; difference \; formula, \; f(x) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \; \ldots.$

$$\Rightarrow \quad f(3.5) = 3 + 0.5(3.6) + \frac{0.5(-0.5)}{2!}(4.8) + \frac{0.5(-0.5)(-1.5)}{3!}(-6.2) + \frac{0.5(-0.5)(-1.5)(-2.5)}{4!}(13.6)$$

$$= 3.28125$$

Backward Differences

Backward Differences. The differences $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$

when denoted by $\nabla y_1, \nabla y_2, \dots . \nabla y_n$ respectively are called the first backward differences, where ∇ is the backward difference operator.

In general the **first backward differences** are $\nabla y_r = y_r - y_{r-1}$, r $= 1, 2, 3, \dots .$

Similarly the **second backward differences** are defined by

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$$

In general, $\nabla^p y_r = \nabla^{p-1} y_r - \nabla^{p-1} y_{r-1}$ defines the pth backward differences.

Observation: 1. $\Delta \boldsymbol{y_0} = \boldsymbol{y_1} - \boldsymbol{y_0} = \boldsymbol{\nabla y_1}$

$$2.\ \nabla^2 y_2 = \nabla y_2 - \nabla y_1 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2\,y_1 + y_0$$
$$= \Delta^2\, y_0$$

Backward Difference Table

Arguments	Entries	1st Difference	2nd Difference	3rd Diff	4th Diff	5th Diff
x_0	y_0					
		$y_1 - y_0 = \nabla y_1$				
$x_0 + h$	y_1		$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$			
		$y_2 - y_1 = \nabla y_2$		$\nabla^3 y_3$		
$x_0 + 2h$	y_2		$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$		$\nabla^4 y_4$	
		$y_3 - y_2 = \nabla y_3$		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_0 + 3h$	y_3		$\nabla y_4 - \nabla y_3 = \nabla^2 y_4$		$\nabla^4 y_5$	
		$y_4 - y_3 = \nabla y_4$		$\nabla^3 y_5$		
$x_0 + 4h$	y_4		$\nabla y_5 - \nabla y_4 = \nabla^2 y_5$			
		$y_5 - y_4 = \nabla y_5$				
$x_0 + 5h$	y_5					

Relation Between Shift Operator and Backward Difference Operator

As we know that, $\nabla y_1 = y_1 - y_0$

$$= y_1 - E^{-1} y_1$$

$$= \left(1 - E^{-1}\right) y_1$$

$$\Rightarrow \nabla \equiv 1 - E^{-1}$$

Newton's Backward Interpolation Formula

Newton's Backward Interpolation Formula, this formula is used for interpolating the values of y near the end of a set of tabulated values of y.

Let the function y = f (x) take the values $y_0, y_1, y_2, \ldots\ldots y_n$ corresponding to the equally spaced values $x_0, x_0 + h, x_0 + 2h, \ldots\ldots, x_0 + nh$ of x.

If it is required to evaluate y for $x = x_n + p\,h$, then

$$y(x_n + p\,h) = E^p y_n = (\mathbf{1} - \boldsymbol{\nabla})^{-p} y_n \qquad [\because\ \boldsymbol{\nabla} \equiv \mathbf{1} - \boldsymbol{E}^{-1} \Rightarrow \boldsymbol{E}^{-1} = \mathbf{1} - \boldsymbol{\nabla}]$$

$$\Rightarrow \boldsymbol{E} = \frac{\mathbf{1}}{\mathbf{1} - \boldsymbol{\nabla}} = (\mathbf{1} - \boldsymbol{\nabla})^{-\mathbf{1}}$$

$$= \left\{1 + p\boldsymbol{\nabla} + \frac{p(p+1)}{2!}\boldsymbol{\nabla}^2 + \frac{p(p+1)(p+2)}{3!}\boldsymbol{\nabla}^3 + \cdots + \frac{p(p+1)(p+2)\ldots\{p+(n-1)\}}{n!}\boldsymbol{\nabla}^n\right\} y_n$$

$$= y_n + p\boldsymbol{\nabla} y_n + \frac{p(p+1)}{2!}\boldsymbol{\nabla}^2 y_n + \frac{p(p+1)(p+2)}{3!}\boldsymbol{\nabla}^3 y_n + \ldots. + \frac{p(p+1)(p+2)\ldots\{p+(n-1)\}}{n!}\boldsymbol{\nabla}^n y_n$$

is called Newton's backward interpolation formula as it contains y_n and the backward differences of y_n.

Example 7. For the given data, evaluate f(7.5)

x's	f(x)
1	1
2	8
3	27
4	64
5	125
6	216
7	343
8	512

Solution: Let us construct the backward difference table with the help of given data:

x's	f(x)	∇	∇^2	∇^3	∇^4
1	1	7	12		
2	8	19		6	
3	27	37	18	6	0
4	64	61	24		0
5	125	91	30	6	0
6	216	127	36	6	0
7	343	169	42	6	
8	512				

Here, $x_n = 8, h = 1$ and $x = 7.5,$

Therefore, the $relation$ $x = x_n + p\,h$ $gives$ $p = \frac{x - x_n}{h}$

$$= \frac{7.5 - 8}{1} = -0.5$$

$Hence, using\ Newton's\ forward\ difference\ formula,$

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n +$$

$$\ldots. + \frac{p(p+1)(p+2)\ldots\{p+(n-1)\}}{n!}\nabla^n y_n$$

$$= 512 + \frac{(-0.5)(-0.5+1)}{2!}42 + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}6 + 0$$

= 512 – 5.25 – 0.375 = 506.375

Interpolation with Unequal Intervals

Lagrange's Interpolation Formula, this formula is used for interpolating the values of y when the values of x are at unequal differencing interval.

Consider a function f (x), and its corresponding values for $x_1, x_2, \ldots \ldots x_n$ are $f(x_1),\ f(x_2), \ldots \ldots, f(x_n)$, then

$$f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)\ldots(x_1-x_n)} . f(x_1) + \frac{(x-x_1)(x-x_3)(x-x_4)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)\ldots(x_2-x_n)} . f(x_2) + .$$

$$. . + \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)(x_n-x_3)\ldots(x_n-x_{n-1})} . f(x_n)$$

Example: Find f(6), for the given values,

x:	3	7	9	10
f(x):	16	10	40	25

Solution: Here, the values of x are at unequal intervals, so by Lagrange's formula

$$f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)\ldots(x_1-x_n)}.f(x_1) + \frac{(x-x_1)(x-x_3)(x-x_4)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)\ldots(x_2-x_n)}.f(x_2)+ .$$

$$..+\frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)(x_n-x_3)\ldots(x_n-x_{n-1})}.f(x_n)$$

$$\Rightarrow \quad f(6) = \frac{(6-7)(6-9)(6-10)}{(3-7)(3-9)(3-10)}.16 + \frac{(6-3)(6-9)(6-10)}{(7-3)(7-9)(7-10)}.10 + \frac{(6-3)(6-7)(6-10)}{(9-3)(9-7)(9-10)}.40 + \frac{(6-3)(6-7)(6-9)}{(10-3)(10-7)(10-9)}.25$$

$$= \frac{(-1)(-3)(-4)}{(-4)(-6)(-7)}.16 + \frac{(3)(-3)(-4)}{(4)(-2)(-3)}.10 + \frac{(3)(-1)(-4)}{(6)(2)(-1)}.40 + \frac{(3)(-1)(-3)}{(7)(3)(1)}.25$$

$$= \frac{8}{7} + 15 - 40 + \frac{75}{7} = \frac{83}{7} - 25 = -13.143$$ Ans.

Example: Use Lagrange's interpolation formula to fit a polynomial of minimum degree to the following data:

x:	-1	1	2	4
f(x):	13	15	13	33

and hence find f(3).

Solution: Here, the values of x are at unequal intervals, so by Lagrange's formula

$$f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)\dots(x-x_n)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)\dots(x_1-x_n)} . f(x_1) + \frac{(x-x_1)(x-x_3)(x-x_4)\dots(x-x_n)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)\dots(x_2-x_n)} . f(x_2) + .$$

$$\dots + \frac{(x-x_1)(x-x_2)(x-x_3)\dots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)(x_n-x_3)\dots(x_n-x_{n-1})} . f(x_n)$$

$$\Rightarrow \quad f(x) = \frac{(x-1)(x-2)(x-4)}{(-1-1)(-1-2)(-1-4)} . 13 + \frac{(x+1)(x-2)(x-4)}{(1+1)(1-2)(1-4)} . 15 + \frac{(x+1)(x-1)(x-4)}{(2+1)(2-1)(2-4)} . 13 + \frac{(x+1)(x-1)(x-2)}{(4+1)(4-1)(4-2)} . 33$$

$$= (x^3 - 7x^2 + 14\,x - 8)(-\frac{13}{30}) + (x^3 - 5x^2 + 2\,x + 8)\left(\frac{5}{2}\right) + (x^3 - 4x^2 - \,x + 4)\left(-\frac{13}{6}\right) + (x^3 - 2x^2 - \,x + 2)\left(\frac{11}{10}\right)$$

$= x^3 - 3x^2 + 17$ and $f(3) = 3^3 - 3(3)^2 + 17 = 27 - 27 + 17 = 17$ Ans.

Example: By Lagrange's formula, find f(2), for the given data:

x:	1	3	5	6
f(x):	2	10	26	37

Solution: Here, the values of x are at unequal intervals, so by Lagrange's formula

$$f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)\ldots(x_1-x_n)}.f(x_1) + \frac{(x-x_1)(x-x_3)(x-x_4)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)\ldots(x_2-x_n)}.f(x_2)+.$$

$$..+\frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)(x_n-x_3)\ldots(x_n-x_{n-1})}.f(x_n)$$

$$\Rightarrow \quad f(2) = \frac{(2-3)(2-5)(2-6)}{(1-3)(1-5)(1-6)}.(2) + \frac{(2-1)(2-5)(2-6)}{(3-1)(3-5)(3-6)}.(10) + \frac{(2-1)(2-3)(2-6)}{(5-1)(5-3)(5-6)}.(26) + \frac{(2-1)(2-3)(2-5)}{(6-1)(6-3)(6-5)}.(37)$$

$$= \frac{(-1)(-3)(-4)}{(-2)(-4)(-5)}.(2) + \frac{(1)(-3)(-4)}{(2)(-2)(-3)}.(10) + \frac{(1)(-1)(-4)}{(4)(2)(-1)}.(26) + \frac{(1)(-1)(-3)}{(5)(3)(1)}.(37)$$

$$= \frac{3}{5} + 10 - 13 + \frac{37}{5} = \frac{40}{5} - 3 = 8 - 3 = 5 \quad \text{Ans.}$$

Divided Differences

Divided Difference Table

Consider a function f (x), and its corresponding values for $x_0, x_1, x_2, \ldots\ldots x_n$ are

$f(x_0), f(x_1),\ f(x_2), \ldots\ldots, f(x_n)$, then $\frac{f(x_1)-f(x_0)}{x_1-x_0}$ is the divided difference of order one denoted

by $\underset{x_1}{\Delta}$ or by $[f(x_0), f(x_1)]$ or by $[y_0, y_1]$

The divided differences can be put in a tabular form as under:

$Values\ of\ x$	$Values\ of\ f(x) = y$	$1st\ Divided\ Difference$	$2nd\ Divided\ Difference$
x_0	$f(x_0) = y_0$		
		$\frac{y_1 - y_0}{x_1 - x_0} = [y_0, y_1]$	
x_1	$f(x_1) = y_1$		$\frac{[y_1, y_2] - [y_0, y_1]}{x_2 - x_0} = [y_0, y_1, y_2]$
		$\frac{y_2 - y_1}{x_2 - x_1} = [y_1, y_2]$	
x_2	$f(x_2) = y_2$		

Example: Construct the divided difference table for the data:

x:	0	1	2	4
y:	0	1	8	12

Divided Difference Table

$Values\ of\ x$	$Values\ of\ f(x) = y$	$1st\ Divided\ Difference$	$2nd\ Divided\ Difference$	$2nd\ Divided\ Differenc$
0	0			
		$\frac{1-0}{1-0} = 1 = [y_0, y_1]$		
1	1		$\frac{7-1}{2-0} = 3 = [y_0, y_1, y_2]$	
		$\frac{8-1}{2-1} = 7 = [y_1, y_2]$		$-\frac{7}{6} = [y_0, y_1, y_2, y_3]$
2	8		$\frac{2-7}{4-1} = -\frac{5}{3} = [y_1, y_2, y_3]$	
		$\frac{12-8}{4-2} = 2 = [y_2, y_3]$		
4	12			

Newton’s Divided Difference Formula

Newton’s Divided Difference Formula, this formula is also used for interpolating the values of y when the values of x are at unequal differencing interval.

Consider a function y = f (x), and its corresponding values for $x_0, x_1, x_2, \ldots\ldots x_n$ are $y_0, y_1, y_2, \ldots\ldots y_n$ then the value of f(x) at ant point ‘x’ is

$$f(x) = y_0 + (x - x_0)[y_0, y_1] + (x - x_0)(x - x_1)[y_0, y_1, y_2] + (x - x_0)(x - x_1)(x - x_2)[y_0, y_1, y_2, y_3] + \cdots \ldots$$

Example: Use Newton's divided difference formula to fit a polynomial of minimum degree to the following data:

x:	- 4	-1	0	2	5
y = f (x):	1245	33	5	9	1335

and hence find f (1) & f (3).

Solution: To start with, construct the divided difference table from the given data:

Values of x	*Values of* $f(x) = y$	*1st Divided Difference*	*2nd Divided Difference*	*3rd divided difference*	*4th divided difference*
-4	1245	$\frac{33-1245}{-1+4} = -404 = [y_0, y_1]$	$\frac{-28+404}{0+4} = 94 = [y_0, y_1, y_2]$ $0-(-4)$	$\frac{10-94}{2+4} = -14 = [y_0, y_1, y_2, y_3]$	$\frac{13+14}{5+4} = 3 = [y_0, y_1, y_2, y_3, y_4]$
-1 ✓	33	$\frac{5-33}{0+1} = -28 = [y_1, y_2]$	$\frac{2+28}{2+1} = 10 = [y_1, y_2, y_3]$	$\frac{88-10}{5+1} = 13 = [y_1, y_2, y_3, y_4]$	
0	5	$\frac{9-5}{2-0} = 2 = [y_2, y_3]$	$\frac{442-2}{5-0} = 88 = [y_2, y_3, y_4]$		
2 ✓	9	$\frac{1335-9}{5-2} = 442 = [y_3, y_4]$			
5	1335				

By Newton's divided difference formula,

$$f(x) = y_0 + (x - x_0)[y_0, y_1] + (x - x_0)(x - x_1)[y_0, y_1, y_2] + (x - x_0)(x - x_1)(x - x_2)[y_0, y_1, y_2, y_3]$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)[y_0, y_1, y_2, y_3, y_4] + \cdots \ldots$$

$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94) + (x + 4)(x + 1)(x - 0)(-14)$$

$$+ (x + 4)(x + 1)(x - 0)(x - 2)\ (3)$$

$$= 1245 - 404\,x - 1616 + 94\,x^2 + 470\,x + 376 - 14(x^3 + 5\,x^2 + 4\,x) + 3(x^4 + 3x^3 - 6x^2 - 8\,x)$$

$$= 3\,x^4 - 5x^3 + 6x^2 - 14\,x + 5$$

$f\,(1) = 3\,(1)^4 - 5(1)^3 + 6(1)^2 - 14(1) + 5 = 3 - 5 + 6 - 14 + 5 = -5$

$f\,(3) = 3\,(3)^4 - 5(3)^3 + 6(3)^2 - 14(3) + 5 = 3 - 5 + 6 - 14 + 5 = 12\,5$

Example: Use Newton's divided difference formula to find f (9) from the following data:

x:	5	7	11	13	17
y = f (x):	150	392	1452	2366	5202

x	y	Δy	Δ^2	Δ^3	Δ^4
5	150				
		$\frac{392-150}{7-5} = 121$			
7	392		$\frac{265-121}{11-5} = 24$		
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$	
11	1452		$\frac{457-265}{13-7} = 32$		$\frac{1-1}{17-5} = 0$
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$	
13	2366		$\frac{709-457}{17-11} = 42$		
		$\frac{5202-2366}{17-13} = 709$			
17	5202				

By Newton's divided difference formula,

$$f(9) = y_0 + (9 - x_0)[y_0, y_1] + (9 - x_0)(9 - x_1)[y_0, y_1, y_2] + (9 - x_0)(9 - x_1)(9 - x_2)[y_0, y_1, y_2, y_3]$$

$$+ (9 - x_0)(9 - x_1)(9 - x_2)(9 - x_3)[y_0, y_1, y_2, y_3, y_4] + \cdots \ldots$$

$$= 150 + (9 - 5)(121) \ + (9 - 5)(9 - 7)(24) \ + (9 - 5)(9 - 7)(9 - 11)(1)$$

$$+(9 - 5)(9 - 7)(9 - 11)(9 - 13)\ (0)$$

$$= 150 + 484 \ + 192 - 16 + 0$$

$$= 810 \qquad\qquad Ans.$$

Example: Use Newton's divided difference formula to fit a polynomial of minimum degree to the following data:

x:	0	2	3	4	7
y = f (x):	4	26	58	112	466

and hence find f (6).

Solution: To start with, construct the divided difference table from the given data:

$Values\ of\ x$	$Values\ of\ f(x) = y$	$1st\ Divided\ Difference$	$2nd\ Divided\ Difference$	$3rd\ divided\ difference$	$4th\ divided\ difference$
0	4	$\frac{26-4}{2-0} = 11 = [y_0, y_1]$	$\frac{32-11}{3-0} = 7 = [y_0, y_1, y_2]$	$\frac{11-7}{4-0} = 1 = [y_0, y_1, y_2, y_3]$	$\frac{1-1}{7-0} = 0 = [y_0, y_1, y_2, y_3, y_4]$
2	26	$\frac{58-26}{3-2} = 32 = [y_1, y_2]$	$\frac{54-32}{4-2} = 11 = [y_1, y_2, y_3]$	$\frac{16-11}{7-2} = 1 = [y_1, y_2, y_3, y_4]$	
3	58	$\frac{112-58}{4-3} = 54 = [y_2, y_3]$	$\frac{118-54}{7-3} = 16 = [y_2, y_3, y_4]$		
4	112	$\frac{466-112}{7-4} = 118 = [y_3, y_4]$			
7	466				

By Newton's divided difference formula,

$$f(x) = y_0 + (x - x_0)[y_0, y_1] + (x - x_0)(x - x_1)[y_0, y_1, y_2] + (x - x_0)(x - x_1)(x - x_2)[y_0, y_1, y_2, y_3]$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)[y_0, y_1, y_2, y_3, y_4] + \cdots \ldots$$

$$= 4 + (x - 0)(11) + (x - 0)(x - 2)(7) + (x - 0)(x - 2)(x - 3)(1)$$

$$+ (x - 0)(x - 2)(x - 3)(x - 4)\,(0)$$

$$= 4 + 11\,x + 7\,x^2 - 14\,x + (x^3 - 5\,x^2 + 6\,x)$$

$$= x^3 + 2\,x^2 + 3\,x + 4$$

$$Therefore f(6) = (6)^3 + 2\,(6)^2 + 3\,(6) + 4$$

$$= 216 + 72 + 18 + 4 = 310$$ *Ans.*

Inverse Interpolation

So far, given a set of values of x and y, we have been finding the value of y corresponding to a certain value of x. On the other hand, the process of estimating the values of x for a value of y (which is not in the table) is called inverse interpolation. When the values of x are unequally spaced Lagrange's inverse method is used and when the values of x are equally spaced, the iterative interpolation method will be used.

Lagrange's inverse method, this method is similar to Lagrange's interpolation formula, the difference being that x is assumed to be expressible as a polynomial in y.

Lagrange's inverse method is merely a relation between two variables either of which may be taken as the independent variable. Therefore, on inter-changing x and y in Lagrange's interpolation formula, we obtain Lagrange's inverse formula.

Inverse Interpolation

As we know, the Lagrange's formula is

$$y=\frac{(x-x_2)(x-x_3)(x-x_4)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)\ldots(x_1-x_n)}.y_1+\frac{(x-x_1)(x-x_3)(x-x_4)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)\ldots(x_2-x_n)}.y_2+\ ..+\frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)(x_n-x_3)\ldots(x_n-x_{n-1})}.y_n$$

On interchanging x and y, Lagrange's inverse formula is

$$x=\frac{(y-y_2)(y-y_3)(y-y_4)\ldots(y-y_n)}{(y_1-y_2)(y_1-y_3)(y_1-y_4)\ldots(y_1-y_n)}.x_1+\frac{(y-y_1)(y-y_3)(y-y_4)\ldots(y-y_n)}{(y_2-y_1)(y_2-y_3)(y_2-y_4)\ldots(y_2-y_n)}.x_2+\ ..+\frac{(y-y_1)(y-y_2)(y-y_3)\ldots(y-y_{n-1})}{(y_n-y_1)(y_n-y_2)(y_n-y_3)\ldots(y_n-y_{n-1})}.x_n$$

Question

Question: The following table gives the values of x and y:

x:	1.2	2.1	2.8	4.1	4.9
y:	4.2	6.8	9.8	13.4	15.5

Find the value of x when y = 12, using Lagrange's inverse formula.

Solution: Lagrange's inverse formula is

$$x=\frac{(y-y_2)(y-y_3)(y-y_4)\dots(y-y_n)}{(y_1-y_2)(y_1-y)(y_1-y_4)\dots(y_1-y_n)}.x_1+\frac{(y-y_1)(y-y_3)(y-y_4)\dots(y-y_n)}{(y_2-y_1)(y_2-y_3)(y_2-y_4)\dots(y_2-y_n)}.x_2+\,..+\frac{(y-y_1)(y-y_2)(y-y_3)\dots(y-y_{n-1})}{(y_n-y_1)(y_n-y_2)(y_n-y_3)\dots(y_n-y_{n-1})}.x_n$$

$$=\frac{(12-6.8)(12-9.8)(12-13.4)(12-15.5)}{(4.2-6.8)(4.2-9.8)(4.2-13.4)(4.2-15.5)}.(1.2)+\frac{(12-4.2)(12-9.8)(12-13.4)(12-15.5)}{(6.8-4.2)(6.8-9.8)(6.8-13.4)(6.8-15.5)}.(2.1)+\frac{(12-4.2)(12-6.8)(12-13.4)(12-15.5)}{(9.8-4.2)(9.8-6.8)(9.8-13.4)(9.8-15.5)}.(2.8)$$

$$+\frac{(12-4.2)(12-6.8)(12-9.8)(12-15.5)}{(13.4-4.2)(13.4-6.8)(13.4-9.8)(13.4-15.5)}.(4.1)+\frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)}{(15.5-4.2)(15.5-6.8)(15.5-9.8)(15.5-13.4)}.(4.9)$$

$$=\frac{(5.2)(2.2)(-1.4)(-3.5)}{(-2.6)(-5.6)(-9.2)(-11.3)}.(1.2)+\frac{(7.8)(2.2)(-1.4)(-3.5)}{(2.6)(-3)(-6.6)(-8.7)}.(2.1)+\frac{(7.8)(5.2)(-1.4)(-3.5)}{(5.6)(3)(-3.6)(-5.7)}.(2.8)+\frac{(7.8)(5.2)(2.2)(-3.5)}{(9.2)(6.6)(3.6)(-2.1)}.(4.1)$$

$$+\frac{(7.8)(5.2)(2.2)(-1.4)}{(11.3)(8.7)(5.7)(2.1)}.(4.9)=0.04444-0.39425+1.61423+2.78945=4.05387$$

Ans.

Question

Question: The following table gives the values of x and y:

x:	5	6	9	11
y:	12	13	14	16

Find the value of x when y = 15, using Lagrange's inverse formula.

Solution: Lagrange's inverse formula is

$$x=\frac{(y-y_2)(y-y_3)(y-y_4)\ldots(y-y_n)}{(y_1-y_2)(y_1-y)(y_1-y_4)\ldots(y_1-y_n)}.x_1+\frac{(y-y_1)(y-y_3)(y-y_4)\ldots(y-y_n)}{(y_2-y_1)(y_2-y_3)(y_2-y_4)\ldots(y_2-y_n)}.x_2+\ ..+\frac{(y-y_1)(y-y_2)(y-y_3)\ldots(y-y_{n-1})}{(y_n-y_1)(y_n-y_2)(y_n-y_3)\ldots(y_n-y_{n-1})}.x_n$$

$$=\frac{(15-13)(15-14)(15-16)}{(12-13)(12-14)(12-16)}.5+\frac{(15-12)(15-14)(15-16)}{(13-12)(13-14)(13-16)}.6+\frac{(15-12)(15-13)(15-16)}{(14-12)(14-13)(14-16)}.9+\frac{(15-12)(15-13)(15-14)}{(16-12)(16-13)(16-14)}.11$$

$$=\frac{(2)(1)(-1)}{(-1)(-2)(-4)}.5+\frac{(3)(1)(-1)}{(1)(-1)(-3)}.6+\frac{(3)(2)(-1)}{(2)(1)(-2)}.9+\frac{(3)(2)(1)}{(4)(3)(2)}.11$$

$$=\frac{5}{4}-6+\frac{27}{2}+\frac{11}{4}=\frac{5-24+54+11}{4}=\frac{23}{2}=11.5$$

Iterative Interpolation

The process of estimating the values of x for a value of y (which is not in the table) is called inverse interpolation. When the values of x are equally spaced, the Iterative Interpolation method will be used.

Newton's forward interpolation formula is

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ \ldots. + \frac{p(p-1)(p-2)\ldots\{p-(n-1)\}}{n!}\Delta^n y_0$$

From this equation, we get

$$p = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p(p-1)}{2!}\Delta^2 y_0 - \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 - \cdots\right] \ldots(1)$$

Neglecting the second and higher differences, we obtain the first approximation to p as

$$p_1 = (y_p - y_0)\frac{1}{\Delta y_0} \qquad \ldots(2)$$

To find the second approximation, retaining the term with second differences in (1) and replacing p with p_1, we get

$$p_2 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_1(p_1-1)}{2!}\Delta^2 y_0\right]$$

To find the third approximation, retaining the term with third differences in (1) and replacing p with p_2, we have

$$p_3 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_2(p_2-1)}{2!}\Delta^2 y_0 - \frac{p_2(p_2-1)(p_2-2)}{3!}\Delta^3 y_0\right] \textit{ and so on.}$$

This process is continued till two successive approximations of p agree with each other.

Example: The following values of y = f (x) are given

x:	10	15	20
f(x):	1754	2648	3564

Find the value of x for y = 3000

Solution: Taking $x_0 = 10$ *and* $h = 5$, *the difference table is*

x	y	Δy	Δ^2
10	1754		
		894	
15	2648		22
		916	
20	3564		

Here, $y_p = 3000, y_0 = 1754, \Delta y_0 = 894\ and \Delta^2 y_0 = 22$

Therefore, the successive approximations to p are

$$p_1 = (y_p - y_0)\frac{1}{\Delta y_0}$$

$$= (3000 - 1754)\frac{1}{894} = 1.39$$

$$p_2 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_1(p_1-1)}{2!}\Delta^2 y_0\right]$$

$$= \frac{1}{894}\left[3000 - 1754 - \frac{1.39(1.39-1)}{2!}22\right] = 1.387$$

$$p_3 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p_2(p_2-1)}{2!}\Delta^2 y_0 - \frac{p_2(p_2-1)(p_2-2)}{3!}\Delta^3 y_0\right]$$

$$= \frac{1}{894}\left[3000 - 1754 - \frac{1.387(1.387-1)}{2!}22 - 0\right] = 1.387$$

We, therefore, take p = 1.387 up to 3 decimal places.

Hence the value of x corresponding to y = 3000 is

$$x = x_0 + p\,h$$

$$= 10 + (1.387)\,5$$

$$= 16.935$$

Practice Questions

Q 1. For the given data, evaluate f (47) and f (59)

x:	45	50	55	60
f(x):	0.7	0.76	0.82	0.87

Q 2. Find f (1.5) and f (7.2) for the given data:

x:	1	2	3	4	5	6	7	8
f(x):	1	8	27	64	125	216	343	512

Q 3. The following values of the function f(x) for values of x are given as

$f(1) = 4$, $f(2)= 5$, $f(7) = 5$, $f(8) = 4$

Find f(6) by (i) Lagrange's method and by (ii) Newton's divided difference method

Q 4. The function $y = f(x)$ is given at the points (6, 3), (8, 1), (9, 1) and (11, 9). Find the value of y for $x = 10$ using (i) Lagrange's method (ii) Newton's divided difference method.

Q 5. For the given data, find x when $y = 5$ using iterative interpolation method

x:	1.8	2	2.2	2.4	2.6
f(x):	2.9	3.6	4.4	5.5	6.7

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Thanks