# Taylor's Theorem :-

Let $f(x+h)$ be a function of $h$ ($x$ being independent of $h$) which can be expanded in powers of $h$ and the expansion be differentiable any number of times, then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots$$

**Proof:-** Let $f(x+h) = B_0 + B_1 h + B_2 h^2 + \cdots \quad (1)$

where $B_0, B_1, B_2, \ldots$, are the functions of $x$ alone which are to be determined.

Now, $\frac{d}{dh}[f(x+h)] = \frac{d}{dz}f(z)\cdot\frac{dz}{dh}$ where $z = x+h$

$$= f'(z) = f'(x+h).$$

By successive differentiation of (1) w.r.to $h$, we get

$$f'(x+h) = B_1 + 2B_2 h + 3B_3 h^2 + 4B_4 h^3 + \cdots$$

$$f''(x+h) = 2B_2 + 6B_3 h + 12B_4 h^2 + \cdots$$

Put $h = 0$, we get

$f(x) = B_0$, $f'(x) = B_1$, $f''(x) = B_2 \cdot 2!$, $f'''(x) = 3!\,B_3, \ldots$

Hence,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

$$\cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots$$

## Other forms of Taylor's thm

(1) Writing $a$ for $x$ in (1), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots \quad (2)$$

(2) Putting $a+h=b$ or $h=b-a$ in (2); we get

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^n}{n!}f^n(a) + \cdots$$

(3) Putting $h=x-a$ in (2); we get

$$\boxed{f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots} \quad (3)$$

**Corr:-** Putting $a=0$ and $h=x$ in (2); we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots \quad (4)$$

which is Maclaurin's theorem.

**Prob:-** Expand $\log(1+x)$ in powers of $x$. Then find series for $\log_e\left(\frac{1+x}{1-x}\right)$ and hence determine the value of $\log_e\left(\frac{11}{9}\right)$ upto five places of decimal.

**Solⁿ:-** Let $f(x) = \log(1+x)$ $\quad \therefore f(0) = \log 1 = 0.$

$f'(x) = (1+x)^{-1}$ $\quad \therefore f'(0) = 1$

$f''(x) = -(1+x)^{-2}$ $\quad \therefore f''(0) = -1$

$f'''(x) = 2(1+x)^{-3}$ $\quad \therefore f'''(0) = 2$

$f^{iv}(x) = -6(1+x)^{-4}$ $\quad \therefore f^{iv}(0) = -6.$

and so, on.

Putting these values in

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\therefore \log(1+x) = 0 + x - \frac{x^2}{2!} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\therefore \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

Now,

$$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right]$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right]$$

Put $x = \frac{1}{10}$ in the above result, we get

$$\log\left(\frac{11}{9}\right) = 2\left[\frac{1}{10} + \frac{1}{3}\left(\frac{1}{10}\right)^3 + \frac{1}{5}\left(\frac{1}{10}\right)^5 + \cdots\right]$$

$$\log\left(\frac{11}{9}\right) = 0.20067$$

Ex:- Expand $\sin^{-1} x$ upto four terms in powers of $x$.

Ex ① Expand $\sin x$ in ascending powers of $\left(x - \frac{\pi}{2}\right)$.

② obtain the first four terms in the expansion of $\log \sin x$ in powers of $(x-3)$.

Ex!: Expand $\log x$ in powers of $(x-1)$ by Taylor's theorem and hence find the value of $\log_e(1.1)$.

Soln: Here, $f(x) = \log x = \log(1+\underbrace{x-1}_{h})$, $f(1)=0$

$a=1, h=x-1.$

$f'(x) = \frac{1}{x}$, $\quad f'(1) = 1$

$\therefore f''(1) = -1$

$f''(x) = -\frac{1}{x^2}$,

$f'''(1) = 2$

$f'''(x) = \frac{2}{x^3}$,

$\therefore f^{iv}(1) = -6$ and so on

$f^{iv}(x) = \frac{-6}{x^4}$

$\therefore f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(x) + \cdots$

$\Rightarrow \log x = 0 + (x-1)\cdot 1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}\cdot 2 + \cdots$

$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$

Putting $x = 1.1$, we get

$\log(1.1) = (1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 + \cdots$

$= 0.095305.$

Prob ① $f(x) = e^{x-1}$ about $c=1$.

② $f(x) = \frac{1}{x^2-3x+2}$ about $c=0$.

③ $f(x) = x\sin 2x$ about $c=0$.

## Taylor series expansion of $f(x,y)$ at $(a,b)$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + (y-b)^2 f_{yy}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)\right.$$
$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b)\right.$$
$$\left. + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots$$

Problem:- Find the taylor series expansion of $f(x,y) = \tan^{-1}(xy)$ about $(1,-1)$.

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