Physical interpretation of gradient

grad U has the property that the rate of change of U w.r.t. distance in a particular direction (d) is the projection of grad U onto that direction.

The quantity $\frac{dU}{ds}$ is called ~~to~~ a directional derivative.

OR,

At any point P, grad U points in the direction of greatest ~~co~~ change of U at P, and has magnitude equal to the rate of change of U w.r.t. distance in that direction.

Physical interpretation of divergence

The divergence of a vector field represents the flux generation per unit volume at each point of the field.

Physical interpretation of Curl:-

The curl of the vector field a represents the vorticity (or circulation per unit area) of the field.

Properties:-

(1) A vector field with zero divergence is said to be "solenoidal".

(2) A vector field with zero curl is said to be "irrotational".

(3) A scalar field with zero gradient is said to be constant.

Identity 1: curl(grad f) $= 0 = \nabla \times \nabla f$

Where $f(x,y,z)$ is differentiable scalar field.

Proof:-

curl(grad f) $= \nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$

$= \hat{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) + \hat{j}(\quad) + \hat{k}(\quad)$

$= 0$

Identity 2: div curl $\vec{V} = 0 = \nabla \cdot (\nabla \times \vec{V})$

Where $\vec{V} = a(x,y,z)\hat{i} + b(x,y,z)\hat{j} + c(x,y,z)\hat{k}$ is a differentiable vector field.

Proof:-

div curl $\vec{V} = \nabla \cdot (\nabla \times \vec{V}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$

$= 0$ proved

Prob:- If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that $\text{div}\left(\frac{\vec{r}}{r^3}\right) = 0$.

Proof:- $\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{r^3}\hat{i} + \frac{y}{r^3}\hat{j} + \frac{z}{r^3}\hat{k}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)$$

$$= \frac{r^3 \cdot 1 - x\frac{\partial}{\partial x}(r^3)}{r^6} + \frac{r^3 - y\frac{\partial}{\partial y}(r^3)}{r^6} + \frac{r^3 - z\frac{\partial}{\partial z}(r^3)}{r^6}$$

$$= \frac{3}{r^3} - \frac{1}{r^6}\left[x\frac{\partial}{\partial r}(r^3) \cdot \frac{\partial r}{\partial x} + y\frac{\partial}{\partial r}(r^3)\frac{\partial r}{\partial y} + z\frac{\partial}{\partial r}(r^3)\frac{\partial r}{\partial z}\right]$$

$$= \frac{3}{r^3} - \frac{3r^2}{r^6}\left[(x)\frac{x}{r} + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)\right]$$

$$= \frac{3}{r^3} - \frac{3}{r^4}\left[\frac{x^2 + y^2 + z^2}{r}\right] = \frac{3}{r^3} - \frac{3}{r^4}\left(\frac{r^2}{r}\right)$$

$$= \frac{3}{r^3} - \frac{3}{r^3} = 0$$

proved