Green's theorem :- Let C be a piecewise smooth simple closed curve bounding a region R. If $f, g, \frac{\partial f}{\partial y}$ and $\frac{\partial g}{\partial x}$ are continuous on R, then

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy \quad \text{--- (1)}$$

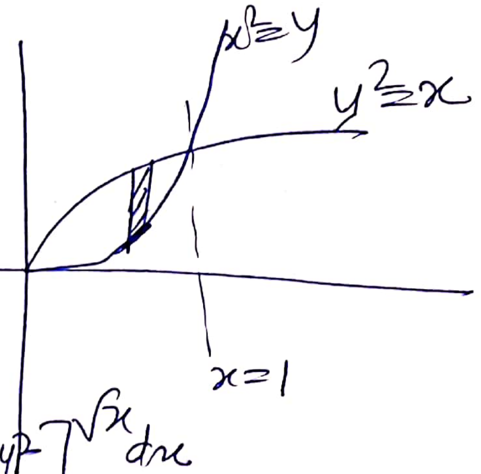
the integration being carried in the positive direction of C.

Prob :- (1) Evaluate $\oint_C (x^2+y^2)\,dx + (y+2x)\,dy$, where C is the boundary of the region in the first quadrant that is bounded by the curves $y^2=x$ and $x^2=y$.

Solⁿ :- We have $f(x,y) = x^2+y^2$ and $g(x,y) = y+2x$

$y : x^2 \to \sqrt{x}$

$x : 0 \to 1$



Using the Green's thm, we obtain

$$\oint_C (x^2+y^2)\,dx + (y+2x)\,dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dy\,dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (2-2y)\,dy\,dx = \int_0^1 \left[2y - y^2\right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 (2\sqrt{x} - x - 2x^2 + x^4)\,dx = \frac{4}{3} - \frac{1}{2} - \frac{2}{3} + \frac{1}{5} = \frac{11}{30}.$$

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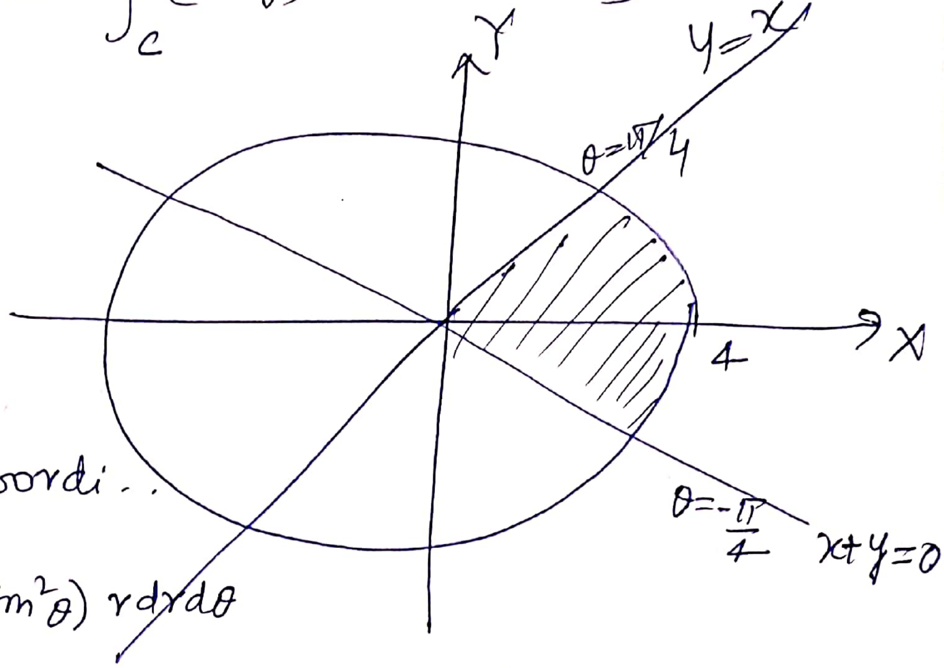
Prob:- Find the work done by the force $\vec{F} = (x^2 - y^3)\hat{i} + (x+y)\hat{j}$ in moving a particle along the closed path C containing the curve $x+y=0$, $x^2+y^2=16$ and $y=x$ in the first and fourth quadrants.

Solution:- The work done by the force is given by

$$W = \oint_C \vec{F} \cdot d\vec{r} = \oint_C (x^2 - y^3)\,dx + (x+y)\,dy$$



$r: 0 \to 4$

$\theta: -\frac{\pi}{4} \to \frac{\pi}{4}$

$$\therefore W = \iint_R (1+3y^2)\,dx\,dy$$

changing into polar coordi..

$$W = \int_{-\pi/4}^{\pi/4} \int_0^4 (1+3r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\frac{r^2}{2} + \frac{3}{4} r^4 \sin^2\theta\right]_0^4 d\theta = \int_{-\pi/4}^{\pi/4} (8 + 192\sin^2\theta)\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} [8 + 96(1-\cos 2\theta)]\,d\theta = \int_{-\pi/4}^{\pi/4} (104 - 96\cos 2\theta)\,d\theta$$

$$= 104\left(\frac{\pi}{2}\right) - 2 \times 48\left[\sin 2\theta\right]_0^{\pi/4} = 52\pi - 96.$$

Ans

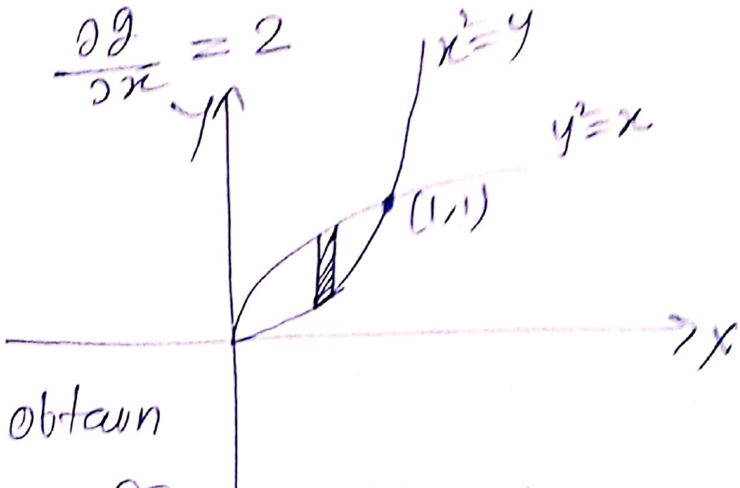
31 Based on Green's theorem

Evaluate $\oint_C (x^2+y^2)\,dx + (y+2x)\,dy$.

C: first quadrant, bdd by the curves $y^2 = x$, $x^2 = y$ by Green's thm.

We have $f(x,y) = x^2+y^2$, $g(x,y) = y+2x$

$\therefore \frac{\partial f}{\partial y} = 2y$, $\frac{\partial g}{\partial x} = 2$



Limit of bdd region

$y : x^2 \to \sqrt{x}$

$x : 0 \to 1$

Using the Green's thm, we obtain

$$\oint_C (x^2+y^2)\,dx + (y+2x)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \left[\int_{x^2}^{\sqrt{x}} (2-2y)\,dy\right] dx = \int_0^1 \left[2y - y^2\right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left\{(2\sqrt{x} - x) - (2x^2 - x^4)\right\} dx$$

$$= \int_0^1 (2\sqrt{x} - x - 2x^2 + x^4)\,dx$$

$$= \left[\frac{2x^{3/2}}{3/2} - \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{4}{3} - \frac{1}{2} - \frac{2}{3} + \frac{1}{5} = \frac{11}{30}.$$ Ans

32. Find Work done

$\vec{F} = (x^2 - y^3)\hat{i} + (x+y)\hat{j}$

$C: x^2+y^2=16,\ x+y=0,\ x-y=0$ in 1st & 4th quadrant

by Green's thm.

let $\vec{r} = x\hat{i} + y\hat{j}$. Then

Work done by the force

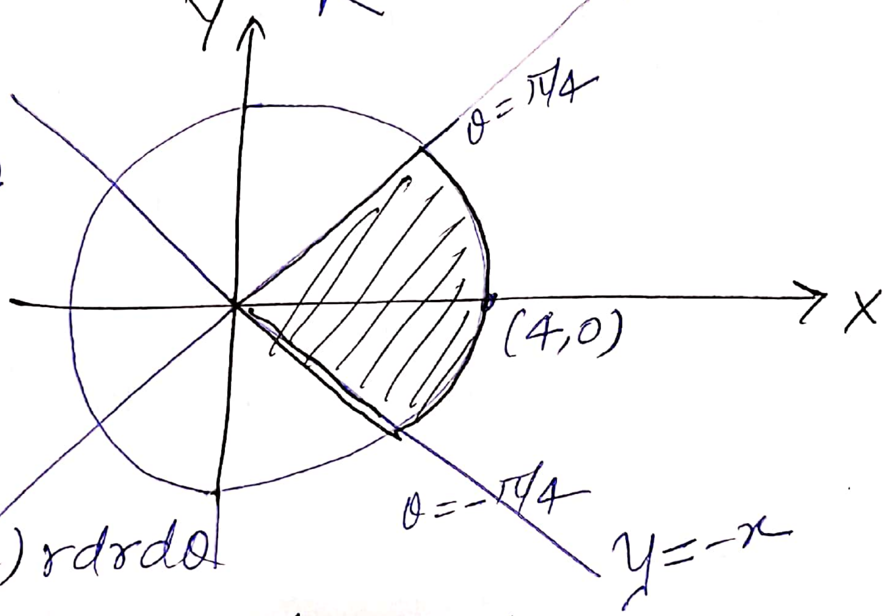
$$W = \oint_C \vec{F}\cdot d\vec{r} = \oint_C (x^2-y^3)dx + (x+y)dy \quad \text{—(1)}$$

We have $f(x,y) = x^2 - y^3$, $g(x,y) = x+y$

$$\frac{\partial f}{\partial y} = -3y^2, \quad \frac{\partial g}{\partial x} = 1$$

By using Green's theorem, we get

$$\therefore W = \oint_C (x^2-y^3)dx + (x+y)dy = \iint_R (1+3y^2)\,dx\,dy \quad \text{—(2)}$$



Changing into polar coordinate

$x = r\cos\theta,\ y = r\sin\theta$

$0 \le r \le 4,$

$-\frac{\pi}{4} \le \theta \le \pi/4$

From (2); we get

$$W = \int_{-\pi/4}^{\pi/4}\int_0^4 (1+3r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\frac{r^2}{2} + \frac{3}{4}r^4\sin^2\theta\right]_0^4 d\theta = \int_{-\pi/4}^{\pi/4} (8 + 192\sin^2\theta)\,d\theta$$

$$= 2\int_0^{\pi/4} [8 + 96(1-\cos 2\theta)]\,d\theta = 2\int_0^{\pi/4} (104 - 96\cos 2\theta)\,d\theta$$

$$= 2\left[104\theta - 48\sin 2\theta\right]_0^{\pi/4} = 52\pi - 96. \text{ Ans}$$

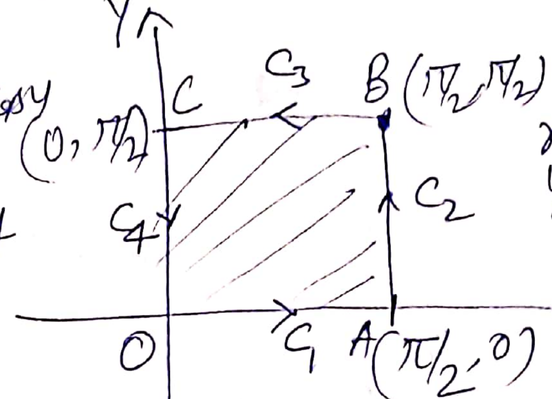
33. Evaluate $\oint_C (e^{-x}\sin y)\,dx + (e^{-x}\cos y)\,dy$,

$C: (0,\pi), (\pi/2, 0), (\pi/2, \pi/2), (0, \pi/2)$ square.

~~The line integral~~

$f(x,y) = e^{-x}\sin y,\ g(x,y) = e^{-x}\cos y$

$\frac{\partial f}{\partial y} = e^{-x}\cos y,\ \frac{\partial g}{\partial x} = -e^{-x}\cos y$



$x: 0 \to \pi/2$
$y: 0 \to \pi/2$

Using Green's theorem,

$$\oint_C (e^{-x}\sin y)\,dx + (e^{-x}\cos y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$= \iint_R (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} (-2e^{-x}\cos y)\,dx\,dy$$

$$= -2\int_0^{\pi/2} \cos y \left[\int_0^{\pi/2} e^{-x}\,dx\right] dy = 2\int_0^{\pi/2} \cos y \left[e^{-x}\right]_0^{\pi/2} dy$$

$$= 2(e^{-\pi/2} - 1)\int_0^{\pi/2} \cos y\,dy = 2(e^{-\pi/2} - 1)\left[\sin y\right]_0^{\pi/2}$$

$$= 2(e^{-\pi/2} - 1)(1 - 0) = 2(e^{-\pi/2} - 1)$$ Ans

34. $\oint_C 3x^2 y\,dx - 2xy^2\,dy$;

$C: x^2 + y^2 \le 16,\ x \geq 0, y \geq 0.$

Here, $f(x,y) = 3x^2 y$; $g(x,y) = -2xy^2$

$\frac{\partial f}{\partial y} = 3x^2$; $\frac{\partial g}{\partial x} = -2y^2$

Using Green's theorem,

$$\oint_C 3x^2 y\,dx - 2xy^2\,dy = \iint_R (-2y^2 - 3x^2)\,dx\,dy$$

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$x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$

$r: 0 \to 4,\ \theta: 0 \to \pi/2$

$$\therefore \oint_C (3x^2y\,dx - 2xy^2)\,dy = \int_0^{\pi/2}\int_0^4 (-2\,r^2\sin^2\theta - 3\,r^2\cos^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^4 r^2\sin^2\theta \cdot r\,dr\,d\theta - 3\int_0^{\pi/2}\int_0^4 r^2 \cdot r\,dr\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\sin^2\theta\,[r^4]_0^4\,d\theta - \frac{3}{4}\int_0^{\pi/4}[r^4]_0^4\,d\theta$$

$$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$$

$$= \frac{256}{8}\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta - 64\times 3\int_0^{\pi/2} d\theta$$

$$= 32(\pi/2) - 64\times 3\times \pi/2$$

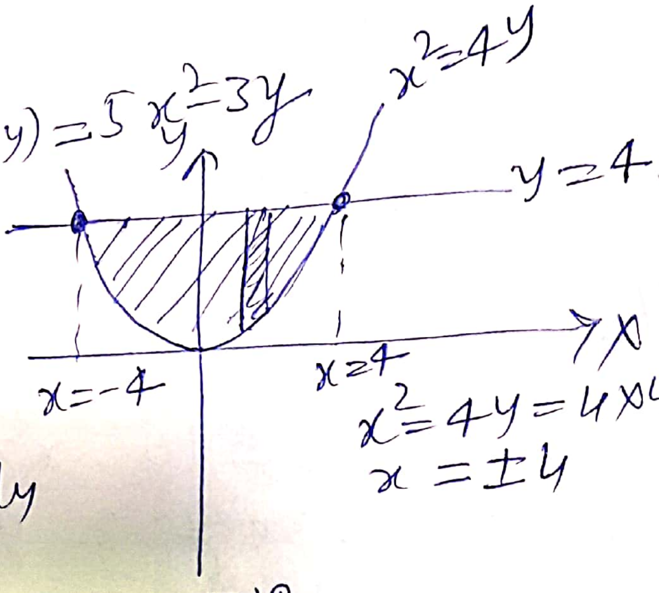
$$= \cancel{16\pi - 48\pi} = 16\pi - 96\pi = -80\pi.$$

$\boxed{35}$ $\oint_C (x^2+y^2)\,dx + (5x^2-3y)\,dy$

C: $x^2 = 4y,\ y = 4.$

Here, $f(x,y) = x^2+y^2,\ g(x,y) = 5x^2-3y$

$\frac{\partial f}{\partial y} = 2y,\ \frac{\partial g}{\partial x} = 10x$



$x^2 = 4y = 4\times 4$

$x = \pm 4$

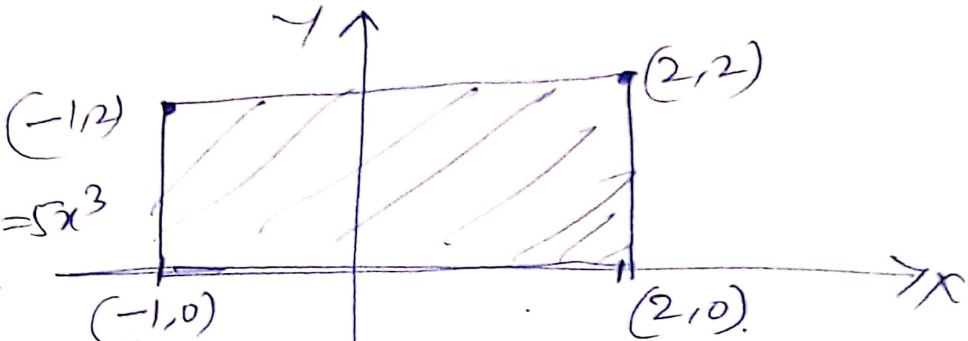
Using Green's thm

$$\oint_C (x^2+y^2)\,dx + (5x^2-3y)\,dy = \iint_R (10x - 2y)\,dx\,dy$$

$$= \int_{-4}^{4}\int_{x^2/4}^{4} (10x-2y)\,dy\,dx = \frac{-512}{5}$$

36 Evaluate $\oint_C xy^2\,dx + 5x^3\,dy$;

C : rectangle vertrices $(-1,0), (2,0), (2,2), (-1,2)$



$x : -1 \rightarrow 2$

$y : 0 \rightarrow 2$

Here, $f(x,y) = xy^2$, $g(x,y) = 5x^3$

$\frac{\partial f}{\partial y} = 2xy$, $\frac{\partial g}{\partial x} = 15x^2$

Using Green's thm, we get

$$\oint_C xy^2\,dx + 5x^3\,dy = \iint_R (15x^2 - 2xy)\,dx\,dy$$

$$= \int_{-1}^{2}\int_{0}^{2} (15x^2 - 2xy)\,dy\,dx = \underline{84}$$