Lecture 11

Improper Integrals of Type II

Definition:- Integrals of functions that become infinite at a point within the interval of integration are improper integrals of Type II.

1. If $f(x)$ is continuous on $(a,b]$ and discontinuous at a, then $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$

2. If $f(x)$ is continuous on $[a,b)$ and discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

3. If $f(x)$ is discontinuous at c, where $a<c<b$, and continuous on $[a,c) \cup (c,b]$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example ④ Investigate the convergence of $I = \int_0^1 \frac{dx}{1-x}$.

Sol:- The integrand $f(x) = \frac{1}{1-x}$ is continuous on $[0,1)$ but is discontinuous at $x=1$ and become infinite as $x \to 1^-$. We evaluate the integral as

$$\lim_{b \to 1^-} \int_0^b \frac{dx}{1-x} = \lim_{b \to 1^-} \left[-\ln(1-x)\right]_0^b = \lim_{b \to 1^-} -\ln(1-b)$$
$$= -(-\infty) = \infty$$

The limit is infinite, so the $\int_0^1 \frac{dx}{1-x}$ diverges.

Example (5) Evaluate $\int_0^3 \frac{dx}{(x-1)^{2/3}} = I.$

Sol^n:- The integrand has a vertical asymptote at $x=1$ and is continuous on $[0,1)$ and $(1,3]$. Thus

$$I = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} = I_1 + I_2$$

Now, $I_1 = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \left[3(x-1)^{1/3}\right]_0^b$

$$= \lim_{x\to 1^-} \left[3(b-1)^{1/3}+3\right] = 3.$$

$$I_2 = \lim_{a\to 1^+} \int_a^3 \frac{dx}{(x-1)^{2/3}} = \lim_{a\to 1^+} \left[3(x-1)^{1/3}\right]_a^3$$

$$= \lim_{a\to 1^+} \left[3.2^{1/3} - 3(a-1)^{1/3}\right] = 3(2)^{1/3}$$

Hence, $I = 3 + 3\cdot 2^{1/3} = 3(1+2^{1/3})$ Ans