Change of Order of Integration

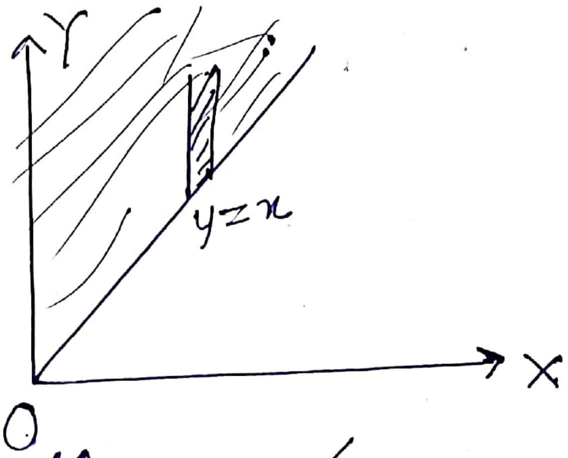
In changing the order of integration, the limits of integration change. To find the new limits, we draw the rough sketch of the region of integration.

Que (1) Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$

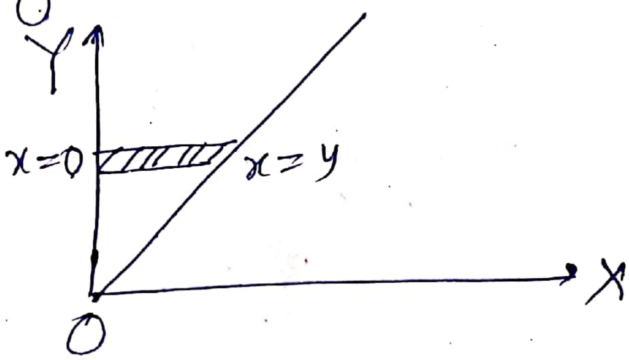
Solⁿ:-

$y : x \to \infty$

$x : 0 \to \infty$



On changing the order of integration, we first integrate w.r.t. x from $x=0$ to y, then integrate w.r.t. y from $y=0$ to $y=\infty$.



Thus

$$I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\, dy \int_0^y dx = \int_0^\infty \frac{e^{-y}}{y} [x]_0^y\, dy$$

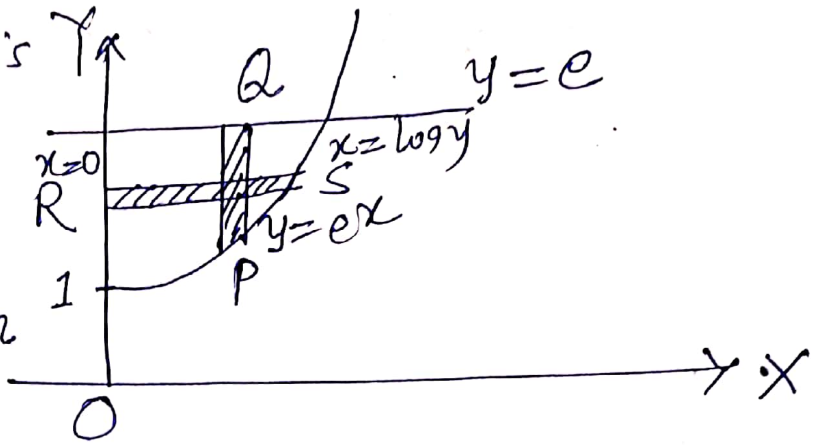
$$= \int_0^\infty \frac{e^{-y}}{y} \times y\, dy = \int_0^\infty e^{-y}\, dy$$

$$= -[e^{-y}]_0^\infty = 1.$$

(2) Evaluate $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}$ by changing the order of integration.

Solⁿ:- $y : e^x \to e$ then $x : 0 \to 1$

Here the integration is first w.r.t. y from P on $y=e^x$ to Q on the line $y=e$.



Then the integration is w.r.t. x from $x=0$ to $x=1$.

On changing the order of integration, we first integrate w.r.t. x from R on $x=0$ to S on $x=\log y$ and then w.r.t. y from $y=1$ to $y=e$

Thus $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y} = \int^{e} \int_0^{\log y} \frac{dx\,dy}{\log y}$

$$= \int_1^e \frac{1}{\log y} dy \int_0^{\log y} dx = \int_1^e dy = e-1.$$

) Change of the order of integration in

$$I = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx \text{ and hence evaluate.}$$

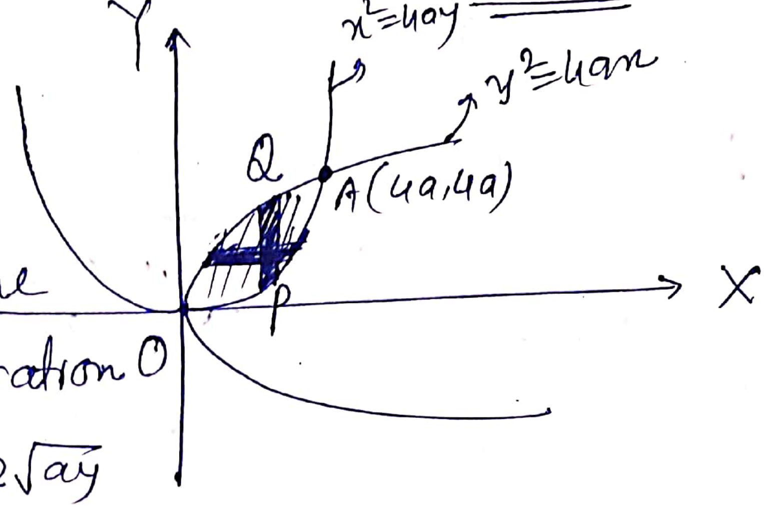
Soln:- $x: 0 \to 4a, \quad y: \frac{x^2}{4a} \to 2\sqrt{ax}$

$y = \frac{x^2}{4a} \Rightarrow \underline{\underline{x^2 = 4ay}}, \quad y = 2\sqrt{ax}$

$\underline{\underline{y^2 = 4ax}}$

$x = \frac{y^2}{4a}$

$x = \sqrt{4ay}$



On changing the order of integration

$x: \frac{y^2}{4a} \to 2\sqrt{ay}$

and $y: 0 \to 4a$

$$\therefore I = \int_0^{4a} \left[\int_{y^2/4a}^{2\sqrt{ay}} dx\right] dy = \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

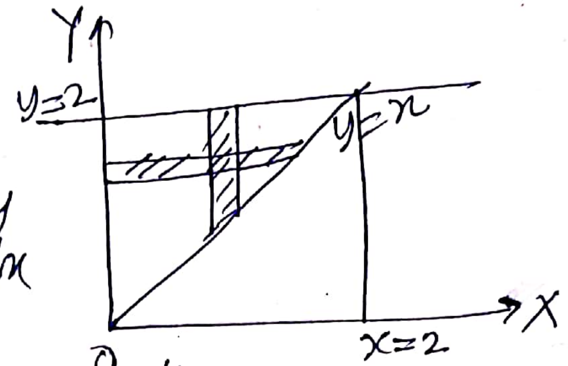
$$= \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a}$$

$$= \frac{32}{3}a^2 - \frac{16a^2}{3} = \frac{16a^2}{3}$$

(4) Evaluate by changing the order of integration

(i) $\int_0^2 \int_x^2 2e^{y^2} dy\, dx$ (ii) $\int_0^1 \int_{\sqrt{y}}^1 \cos x^3\, dx\, dy.$

Sol^n (i) $y: x \to 2,\ x: 0 \to 2$

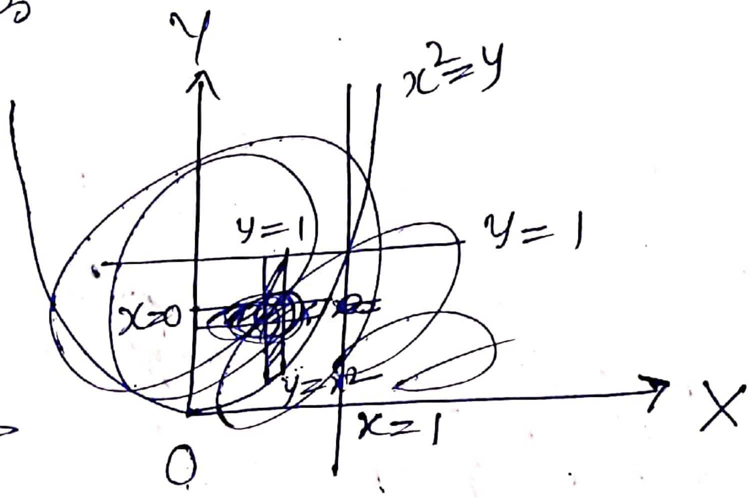


$x: 0 \to y$
$y: 0 \to 2$

$$I = \int_0^2 \int_0^y 2e^{y^2}\,dy\,dx = \int_0^2 2e^{y^2}dy \int_0^y dx$$

$$= 2\int_0^2 y\,e^{y^2}dy = \int_0^4 e^t dt = e^4 - 1.$$

(ii) $$I = \int_0^1 \int_{\sqrt{y}}^1 \cos x^3\,dx\,dy$$

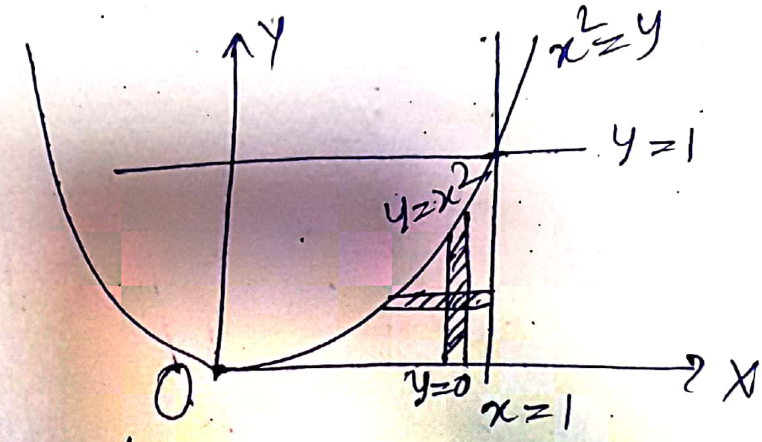


$x: \sqrt{y} \to 1$ $\quad x^2 = y$
$y: 0 \to 1.$

~~On changing order of integration~~
~~$y: x^2 \to 1$ then $x: 0 \to 1$~~

$$\cancel{I = \int_0^1 \int_{x^2}^1 \cos x^3\,dx\,dy = \int_0^1 \cos x^3\,dx \int_{x^2}^1 dy}$$

On changing order of integration
$y: 0 \to x^2$
$x: 0 \to 1$



$$I = \int_0^1 \int_0^{x^2} \cos x^3\,dx\,dy$$

$$= \int_0^1 \cos x^3\,dx \int_0^{x^2} dy = \int_0^1 x^2 \cos x^3\,dx \qquad \text{Put } x^3 = t,\ x^2 dx = \tfrac{1}{3}dt$$

$$= \frac{1}{3}\int_0^1 \cos t\,dt = +\frac{1}{3}[\sin t]_0^1$$

$$= \frac{1}{3}\sin 1.$$