Maxima and Minima of Functions of Two Variables :-

$$\nabla f = f(a+h, b+k) - f(a,b)$$

If $\nabla f < 0$, then $f(x,y)$ is said to have maximum value at $x=a, y=b$

$\nabla f > 0$, then ——— minimum - —

Conditions for $f(x,y)$ to be Maximum or Minimum

▷ A maximum or a minimum value of a function is called its extreme value.

Necessary conditions for $f(x,y)$ to have a maximum or a minimum value at (a,b) are

$$f_x(a,b) = 0 \text{ and } f_y(a,b) = 0$$

Case 1 $r = \frac{\partial^2 f}{\partial x^2} = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$

(1) $rt - s^2 > 0$, then $f(x,y)$ has a maximum or a minimum at (a,b) according $r<0$ or $r>0$.

(2) $rt - s^2 < 0$, then there is neither a maximum or a minimum value at (a,b). The pt (a,b) is a saddle pt in this case

3) If $rt - s^2 = 0$, doubtful case, hence further investigation is required

Note:- The pt (a,b) is called a stationary point if $f_x(a,b) = 0$, $f_y(a,b) = 0$. = critical pt

① Given $f(x,y,z) = \frac{5xyz}{x+2y+4z}$. Find the values of x, y, z for which $f(x,y,z)$ is ~~maximum~~ maximum, subject to the condition $xyz = 8$.

~~Examine~~

Soln:

$$u = \frac{5xyz}{x+2y+4z} \quad \text{— ①}, \quad xyz = 8 \quad \text{— ②}$$

$$\therefore u = \frac{40}{x+2y+4z} \quad \text{— ③}$$

$$\therefore du = \frac{-40}{(x+2y+4z)^2}(dx+2dy+4dz)$$

For max and min of u, $du = 0$

$$\Rightarrow dx + 2dy + 4dz = 0 \quad \text{— ④}$$

From ②; $\log x + \log y + \log z = \log 8$

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = \cancel{\log 8}\ 0 \quad \text{— ⑤}$$

④ + λ⑤; we get

$$dx\left(1+\frac{\lambda}{x}\right) + \left(2+\frac{\lambda}{y}\right)dy + \left(4+\frac{\lambda}{z}\right)dz = 0$$

$$\Rightarrow 1+\frac{\lambda}{x} = 0 \quad \Rightarrow x = -\lambda$$

$$2+\frac{\lambda}{y} = 0 \quad \Rightarrow y = -\lambda/2$$

$$4+\frac{\lambda}{z} = 0 \quad \Rightarrow z = -\lambda/4$$

$$② \Rightarrow \quad -\frac{\lambda^3}{8} = 8 \Rightarrow \lambda = -4$$

$\therefore x = 4, y = 2, z = 1$

$\therefore$ u is stationary at the pt $x = 4, y = 2, z = 1$

Now, $u = \frac{40}{\quad}$ — ⑥

$$\therefore \frac{\partial u}{\partial x} = \frac{-40}{(x+2y+4z)^2}\left(1+4\frac{\partial z}{\partial x}\right)$$

From ①, $\log x + \log y + \log z = \log 8$

$$\Rightarrow \frac{1}{x} + \frac{1}{z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{z}{x}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{-40}{(x+2y+4z)^2}\left(1-\frac{4z}{x}\right)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{80}{(x+2y+4z)^3}\left(1+4\frac{\partial z}{\partial x}\right)\left(1-\frac{4z}{x}\right) - \frac{40}{(x+2y+4z)^2}\left(\frac{4z}{x^2} - \frac{4}{x}\frac{\partial z}{\partial x}\right)$$

$$= \frac{80}{(x+2y+4z)^3}\left(1-\frac{4z}{x}\right)^2 - \frac{40}{(x+2y+4z)^2}\left(\frac{8z}{x^2}\right)$$

At the stationary pt

$$r = \frac{80}{(12)^3}(1-1)^2 - \frac{40}{144}\left(\frac{1}{4}+\frac{1}{4}\right) < 0$$

$\therefore$ u is maximum at the pt $(4,2,1)$

Qu :- Examine for extreme values :

(i) $x^2+y^2+6x+12$ (ii) $x^3+y^3-63(x+y)+12xy$