Partial Derivative

Definition:- The partial derivative of $f(x,y)$ with respect to x, written $\frac{\partial f}{\partial x}$, is defined by

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

for any values of x and y for which the limit exists.

The partial derivative of $f(x,y)$ w.r. to y, is defined by

$$\frac{\partial f}{\partial y}(x,y) = \lim_{K \to 0} \frac{f(x,y+K) - f(x,y)}{K},$$

for any values of x and y for which the limit exists.

Geometrical Meaning:-

$\frac{\partial f}{\partial x}(a,b)$ gives the slope of the tangent line to this curve at $x=a$.

$\frac{\partial f}{\partial y}(a,b)$ gives the slope of the tangent line to the curve at $y=b$.

Examples:- (1) For $f(x,y) = 3x^2 + x^3y + 4y^2$, Compute $\frac{\partial f}{\partial x}(x,y)$, $\frac{\partial f}{\partial y}(x,y)$, $f_x(1,0)$ and $f_y(2,-1)$

Solution:- $\frac{\partial f}{\partial x} = 6x + 3x^2y + 0 = 6x + 3x^2y = f_x(x,y)$

$f_y = \frac{\partial f}{\partial y} = 0 + x^3 + 4 \times 2y = x^3 + 8y$

Now, $f_x(1,0) = 6 \times 1 + 3 \times (1)^2 \times 0 = 6 + 0 = 6$

and $f_y(2,-1) = (2)^3 + 8(-1) = 8 - 8 = 0$.

(2) For $f(x,y) = e^{xy} + \frac{x}{y}$, Compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Soln:- $\frac{\partial f}{\partial x} = e^{xy} \times y + \frac{1}{y} = ye^{xy} + \frac{1}{y}$

and $\frac{\partial f}{\partial y} = e^{xy} \times x - \frac{x}{y^2} = xe^{xy} - \frac{x}{y^2}$.

Que :- For a real gas, vander Waal's equation states that $\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$, — (*)

where P is the pressure of the gas, V is the volume of the gas, T is the temperature (in degrees Kelvin), n is the number of moles of gas, R is the universal gas constant and a and b are constants. Compute and interpret $\frac{\partial P(V,T)}{\partial V}$ and $\frac{\partial T(P,V)}{\partial P}$.

Solⁿ:- from (*);

$$P = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2} \quad \text{— (1)}$$

$$\frac{\partial P}{\partial V} = \frac{-nRT}{(V-nb)^2} + \frac{2an^2}{V^3} \quad \underline{\text{Ans}}$$

Now, from (*), $T = \frac{1}{nR}\left(P + \frac{n^2 a}{V^2}\right)(V - nb)$

$$\therefore \frac{\partial T}{\partial P} = \frac{V - nb}{nR} \quad \underline{\text{Ans}}.$$

Note:-

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x \partial y}.$$

Prob ① Find all second-order partial derivati[ves] of $f(x,y) = x^2y - y^3 + \ln x$.

Solution:- $\frac{\partial f}{\partial x} = 2xy + \frac{1}{x}$, $\frac{\partial f}{\partial y} = x^2 - 3y^2$.

We then have.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xy + \frac{1}{x}\right) = 2y - \frac{1}{x^2}.$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy + \frac{1}{x}\right) = 2x,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2 - 3y^2\right) = 2x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^2 - 3y^2\right) = -6y.$$

Prob ②:- For $f(x,y) = \cos(xy) - x^3 + y^4$, compute f_{xyy}.

Solⁿ:- $f_{xyy} = (f_{xy})_y$

Now, $\frac{\partial f}{\partial x} = f_x = -(\sin xy) \times y - 3x^2 = -y\sin xy - 3x^2$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(-y\sin xy - 3x^2\right)$$

$$= -\sin xy - y(\cos xy) \times x = -\sin xy - xy\cos xy$$

Then, $f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}(f_{xy}) = \frac{\partial}{\partial y}\left(-\sin xy - xy\cos xy\right)$

$$= -x\cos xy - x\left[\cos xy - xy\sin xy\right]$$

$$= -x\cos xy - x\cos xy + x^2y\sin xy$$

$$\therefore f_{xyy} = -2x\cos xy + x^2y\sin xy.$$

rob ③:- For $f(x,y,z) = \sqrt{xy^3z} + 4x^2y$, defined for $x,y,z \geq 0$. Compute f_x, f_{xy} and f_{xyz}.

Solution:- $f(x,y,z) = x^{1/2}y^{3/2}z^{1/2} + 4x^2y$ —— ①

$$f_x = \frac{1}{2}x^{-1/2}y^{3/2}z^{1/2} + 8xy$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{1}{2}x^{-1/2}y^{3/2}z^{1/2} + 8xy\right) = \frac{1}{2}x^{-1/2}\left(\frac{3}{2}y^{1/2}\right)z^{1/2} + 8x$$

$$f_{xy} = \frac{3}{4}x^{-1/2}y^{1/2}z^{1/2} + 8x.$$

Finally, treating x and y as constants, we get

$$f_{xyz} = \frac{\partial}{\partial z}\left(\frac{3}{4}x^{-1/2}y^{1/2}z^{1/2} + 8x\right) = \frac{3}{8}x^{-1/2}y^{1/2}z^{-1/2}$$

Prob ④:- The sag in a beam of length L, width w and height h is given by $S(L,w,h) = c\frac{L^4}{wh^3}$ —— (*) for some constant c. Show that $\frac{\partial S}{\partial L} = \frac{4}{L}S$, $\frac{\partial S}{\partial w} = -\frac{1}{w}S$ and $\frac{\partial S}{\partial h} = -\frac{3}{h}S$. Use this result to determine which variable has the greatest proportional effect on the sag.

Solution:- $\frac{\partial S}{\partial L} = \frac{4cL^3}{wh^3} = \frac{4}{L}\left(\frac{cL^4}{wh^3}\right) = \frac{4}{L}S.$

Now, $\frac{\partial S}{\partial w} = \frac{-cL^4}{h^3w^2} = -\frac{1}{w}\left(\frac{cL^4}{wh^3}\right) = -\frac{S}{w}.$

Next, $\frac{\partial S}{\partial h} = \frac{-3cL^4}{wh^4} = -\frac{3}{h}\left(\frac{cL^4}{wh^3}\right) = -\frac{3S}{h}.$

Change in the length has the greatest effect on the amount of sag.

6① If $x^x y^y z^z = c$, show that at $x=y=z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}!$$

Solution:- Given $x^x y^y z^z = c$, Assume $z = f(x,y)$.

Taking log on both sides, we get

$$x \log x + y \log y + z \log z = \log c \quad \text{---} ①$$

Diff. ~~w.r.t~~ (1) partially w.r.t. x, we get

$$\left(\frac{x}{x} + \log x\right) + 0 + \left(\frac{z}{z}\cdot\frac{\partial z}{\partial x} + \log z \cdot \frac{\partial z}{\partial x}\right) = 0.$$

$$\Rightarrow (1+\log x) + (1+\log z)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z} \quad \text{---} (2)$$

Sly, $$\frac{\partial z}{\partial y} = -\frac{1+\log y}{1+\log z} \quad \text{---} (3)$$

Now, Differentiating (3) partially w.r.t to x, we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\frac{1+\log y}{1+\log z}\right]$$

$$= -(1+\log y)\frac{(-1)}{(1+\log z)^2}\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}$$

$$= \frac{1+\log y}{z(1+\log z)^2}\cdot\left(-\frac{1+\log x}{1+\log z}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-(1+\log x)(1+\log y)}{z(1+\log z)^3} \quad \text{---} (4)$$

$\therefore$ at $x = y = z$, we have

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)(1+\log x)}{x(1+\log x)^3} = \frac{-1}{x(1+\log x)}$$

$$= -(x \log ex)^{-1}!$$

Prob(2):- If $u = \log(x^3+y^3+z^3-3xyz)$, show that (*)

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Sol^n:- Diff (1) partially w.r.to 'x', we get

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} \quad \text{——— (1)}$$

Sly,
$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} \quad \text{——— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xy} \quad \text{——— (3)}$$

On adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-zx-\cancel{z}y)}{x^3+y^3+z^3-3xyz} = \frac{3}{x+y+z}$$

$$\Rightarrow \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z}$$

Now,
$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{3}{x+y+z}\right]$$

$$= -3(x+y+z)^{-2} - 3(x+y+z)^{-2} - 3(x+y+z)^{-2}$$

$$= \frac{-9}{(x+y+z)^2} \quad \text{Ans.}$$

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Problem on Partial Derivatives

(1) Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4,-5)$ if $f(x,y) = x^2 + 3xy + y - 1$

(2) Find $\frac{\partial f}{\partial y}$ as a function if $f(x,y) = y \sin xy$

(3) Find f_x and f_y as function if $f(x,y) = \frac{2y}{y + \cos x}$.

(4) Find $\frac{\partial z}{\partial x}$ if the equation $yz - \ln z = x + y$

Ans $z_x = \frac{z}{yz-1}$

(6) If x, y, and z are independent variables and $f(x,y,z) = x\sin(y+3z)$. Find $f_z = ?$

Ans $3x\cos(y+3z)$

(9) If $f(x,y) = x\cos y + ye^x$, find the second-order derivatives f_{xx}, f_{xy}, f_{yy}, f_{yx}

(10) Find $\frac{\partial^2 w}{\partial x \partial y}$ if $w = xy + \frac{e^y}{y^2+1}$

(11) Find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2z + x^2y$

Ans -4