§ 2·3. Trigonometric and Inverse Trigonometric Functions with their Domain and Range

S. No.	Function	Domain	Range
1.	$\sin x$	R or $x \in R$	$[-1,1]$ or $-1 \le \sin x \le 1$
2.	$\cos x$	R or $x \in R$	$[-1,1]$ or $-1 \le \cos x \le 1$
3.	$\tan x$	$x \in R - \{(2n+1)\frac{\pi}{2}; n \in I\}$	R or $-\infty < \tan x < \infty$
4.	$\cot x$	$x \in R - \{n\pi; n \in I\}$	R or $-\infty < \cot x < \infty$
5.	$\sec x$	$x \in R - \{(2n+1)\frac{\pi}{2}; n \in I\}$	$(-\infty, -1] \cup [1, \infty)$
6.	$\operatorname{cosec} x$	$x \in R - \{n\pi : n \in I\}$	$(-\infty, -1] \cup [1, \infty)$
7.	$\sin^{-1} x = \theta$	$[-1,1]$ or $-1 \le x \le 1$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ or $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$
8.	$\cos^{-1} x = \theta$	$[-1,1]$ or $-1 \le x \le 1$	$[0, \pi]$ or $0 \le \cos^{-1} x \le \pi$
9.	$\tan^{-1} x = \theta$	$(-\infty, \infty)$ or $-\infty < x < \infty$ or $x \in R$	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ or $\frac{-\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$.
10.	$\cot^{-1} x = \theta$	$(-\infty, \infty)$ or $-\infty < x < \infty$ or $x \in R$	$(0, \pi)$ or $0 < \cot^{-1} x < \pi$
11.	$\sec^{-1} x = \theta$	$(-\infty, -1] \cup [1, \infty)$ or $x \le -1, x \ge 1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ or $\theta \ne \frac{\pi}{2}$, $0 \le \theta \le \pi$
12.	$\operatorname{cosec}^{-1} x = \theta$	$(-\infty, -1] \cup [1, \infty)$ or $x \le -1, x \ge 1$	$\left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ or $\theta \ne 0, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

§ 2·4. Principal Value Branch

The graph of trigonometric function in which principal value lies is called principal value branch.

The principal value branch of the inverse trigonometric function are shown in the following table :

Function	Principal value branch
(i) $y = \sin^{-1} x$	$\frac{-\pi}{2} \le y \le \frac{\pi}{2}$, where $-1 \le x \le 1$
(ii) $y = \cos^{-1} x$	$0 \le y \le \pi$, where $-1 \le x \le 1$
(iii) $y = \tan^{-1} x$	$\frac{-\pi}{2} < y < \frac{\pi}{2}$, where $-\infty < x < \infty$
(iv) $y = \sec^{-1} x$	$\begin{cases} 0 \le y < \frac{\pi}{2}, 1 \le x < \infty \\ \frac{\pi}{2} < y \le \pi, -\infty < x \le -1 \end{cases}$
(v) $y = \operatorname{cosec}^{-1} x$	$\begin{cases} 0 < y \le \frac{\pi}{2}, 1 \le x < \infty \\ \frac{-\pi}{2} \le y < 0, -\infty < x \le -1 \end{cases}$
(vi) $y = \cot^{-1} x$	$0 < y < \pi, -\infty < x < \infty$

$$\Rightarrow \quad x = \sin y$$

$$\Rightarrow \quad \operatorname{cosec} y = \frac{1}{x}$$

$$\Rightarrow \quad y = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right) \qquad \text{...(4)}$$

From eqns. (3) and (4), we get

$$\boxed{\sin^{-1} x = \operatorname{cosec}^{-1}\frac{1}{x}}$$

Similarly, we can prove (ii) and (iii).

(C) Conversion property :

(i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$

$$= \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

(ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$

$$= \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}}$$

(iii) $\tan^{-1} x = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$

$$= \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \sec^{-1}\sqrt{1+x^2}$$

$$= \operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}.$$

Proof : (i) If $\sin\theta = x$, then $\theta = \sin^{-1} x$

$$\because \quad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2} \qquad \text{...(1)}$$

$$\therefore \quad \theta = \cos^{-1}\sqrt{1-x^2} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} \qquad \text{...(2)}$$

$$\text{and} \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1-x^2}}$$

$$\therefore \quad \theta = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x} \qquad \text{...(3)}$$

By eqns. (1), (2) and (3), we get

$$\boxed{\begin{aligned}\sin^{-1} x &= \cos^{-1}\sqrt{1-x^2} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} \\ &= \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}\end{aligned}}$$

Similarly, we can prove (ii) and (iii).

§ 2·8. Prove that :

(i) $\sin^{-1}(-x) = -\sin^{-1} x$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$

(vi) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$.

Proof : (i) Let, $\sin^{-1}(-x) = \theta$ ...(1)

$$\therefore \quad \sin\theta = -x$$

$$\Rightarrow \quad -\sin\theta = x$$

$$\Rightarrow \quad \sin(-\theta) = x, \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$\therefore \quad -\theta = \sin^{-1} x$$

$$\Rightarrow \quad \theta = -\sin^{-1} x \qquad \text{...(2)}$$

From eqns. (1) and (2), we get

$$\boxed{\sin^{-1}(-x) = -\sin^{-1} x}$$

(ii) Suppose, $\cos^{-1}(-x) = \theta$ ...(3)

$$\Rightarrow \quad -x = \cos\theta$$

$$\Rightarrow \quad x = -\cos\theta$$

$$\Rightarrow \quad x = \cos(\pi - \theta)$$

$$\Rightarrow \quad \pi - \theta = \cos^{-1} x$$

$$\therefore \quad \theta = \pi - \cos^{-1} x \qquad \text{...(4)}$$

$\therefore$ From eqns. (3) and (4), we get

$$\boxed{\cos^{-1}(-x) = \pi - \cos^{-1} x}$$

(iii) Let, $\tan^{-1}(-x) = \theta$ ...(5)

$$\Rightarrow \quad -x = \tan\theta$$

$$\Rightarrow \quad x = -\tan\theta = \tan(-\theta)$$

$$\therefore \quad -\theta = \tan^{-1} x$$

$$\Rightarrow \quad \theta = -\tan^{-1} x \qquad \text{...(6)}$$

$\therefore$ From eqns. (5) and (6),

$$\boxed{\tan^{-1}(-x) = -\tan^{-1} x}$$

Similarly, the other relations can be proved.

$$\cot(180^\circ+\theta)=\frac{OM'}{M'P'}=\frac{-OM}{-MP}=\cot\theta$$

$$\operatorname{cosec}(180^\circ+\theta)=\frac{OP'}{M'P'}=\frac{OP}{-MP}=-\operatorname{cosec}\theta$$

$$\sec(180^\circ+\theta)=\frac{OP'}{OM'}=\frac{OP}{-OM}=-\sec\theta.$$

§ 1(C)·15. Expressing the trigonometrical ratios of '$270^\circ-\theta$' in terms of trigonometrical ratios of 'θ'

Suppose that the revolving line OA starts from the initial position and assumes the final position OP, in the process makes $\angle POA=\theta$.

Again, supposing that OA moves through 270° in the positive direction and then turns through '$-\theta$' *i.e.*, turns through 'θ' in the negative direction.

Then, $\quad \angle AOP'=270^\circ-\theta$

Let $\quad OP=OP'$

Perpendiculars PM and $P'M'$ are dropped on AOA' from points P and P'.

Now, in ΔPOM and $\Delta P'OM'$, we have

$OP=OP'$

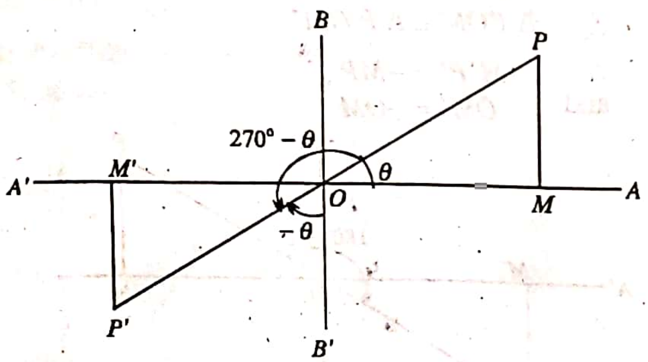
$\angle PMO=\angle P'M'O$, $\quad (\because$ each $=90^\circ)$

$\angle POM=\angle P'OM'$, $\quad (\because$ each $=\theta)$

$\therefore \quad \Delta POM\cong\Delta P'OM'$

$\therefore \quad M'P'=-OM$

and $\quad OM'=-MP$



Now, $\sin(270^\circ-\theta)=\dfrac{M'P'}{OP'}=\dfrac{-OM}{OP}=-\cos\theta$

$$\cos(270^\circ-\theta)=\frac{OM'}{OP'}=\frac{-MP}{OP}=-\sin\theta$$

$$\tan(270^\circ-\theta)=\frac{M'P'}{OM'}=\frac{-OM}{-MP}=\cot\theta$$

$$\cot(270^\circ-\theta)=\frac{OM'}{M'P'}=\frac{-MP}{-OM}=\tan\theta$$

$$\operatorname{cosec}(270^\circ-\theta)=\frac{OP'}{M'P'}=\frac{OP}{-OM}$$

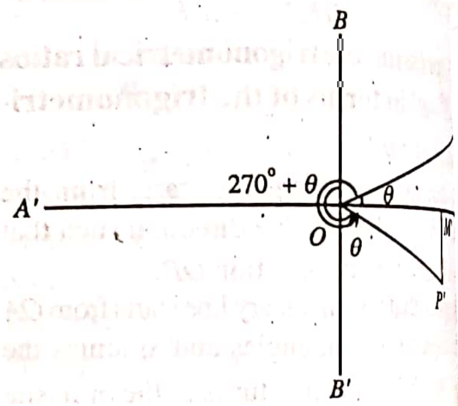
$$\sec(270^\circ-\theta)=\frac{OP'}{OM'}=\frac{OP}{-MP}$$

§ 1(C)·16. Expressing the trigonometrical ratios of '$270^\circ+\theta$' in terms of trigonometrical ratios of 'θ'

Suppose that an imaginary revolving line ... in the positive direction such that $\angle POA$ ... assumes the final position OP.

Again, suppose that the line OA turns through ... right angles and then through an angle θ ...

$\therefore \quad \angle AOP'=270^\circ+\theta$



Suppose, $\quad OP=OP'$

Perpendiculars PM and $P'M'$ are dropped ... and P' on OA.

Now, in ΔOPM and $\Delta P'OM'$,

$OP=OP'$

$\angle POM=\angle OP'M'$,

$\angle OMP=\angle OM'P'$,

$\therefore \quad \Delta POM\cong\Delta OP'M'$

$\therefore \quad OM'=MP$

and $\quad M'P'=-OM$

Now, $\sin(270^\circ+\theta)=\dfrac{M'P'}{OP'}=\dfrac{-OM}{OP}$

$$\cos(270^\circ+\theta)=\frac{OM'}{OP'}=\frac{MP}{OP}$$

$$\tan(270^\circ+\theta)=\frac{M'P'}{OM'}=\frac{-OM}{MP}$$

$$\cot(270^\circ+\theta)=\frac{OM'}{M'P'}=\frac{MP}{-OM}$$

$$\operatorname{cosec}(270^\circ+\theta)=\frac{OP'}{M'P'}=\frac{OP}{-OM}$$

$$\sec(270^\circ+\theta)=\frac{OP'}{OM'}=\frac{OP}{MP}$$

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§ 1(C)·17. Expressing the trigonometrical rations of tios of '360°$-\theta$' in terms of the trigonmetrical ratios of 'θ'

We know that an imaginary line will have the same position under the following two movements :

(i) The imaginary line is moved through 360° in +ve direction and 'θ' in the – ve direction.

(ii) The imaginary line is turned through θ (– ve direction) from its initial position. Hence, trigonometrical ratios in both the cases will be the same.

$$\sin(360^\circ - \theta) = \sin(-\theta) = -\sin\theta$$
$$\cos(360^\circ - \theta) = \cos(-\theta) = \cos\theta$$
$$\tan(360^\circ - \theta) = \tan(-\theta) = -\tan\theta$$
$$\cot(360^\circ - \theta) = \cot(-\theta) = -\cot\theta$$
$$\text{cosec}\,(360^\circ - \theta) = \text{cosec}(-\theta) = -\text{cosec}\,\theta$$
$$\sec(360^\circ - \theta) = \sec(-\theta) = \sec\theta.$$

§ 1(C)·18. Expressing the trigonometrical ratios of the angle (360° $+\theta$) in terms of the trigonometrical ratios of angle 'θ'

We know that the position in which any imaginary line will remain after making an angle of θ in the +ve direction is same as its position after turning through 360° $+\theta$ in the positive direction. Hence, the trigonometrical ratios of 360° $+\theta$ and θ will be the same.

$$\therefore \quad \sin(360^\circ + \theta) = \sin\theta$$
$$\cos(360^\circ + \theta) = \cos\theta$$
$$\tan(360^\circ + \theta) = \tan\theta$$
$$\cot(360^\circ + \theta) = \cot\theta$$
$$\text{cosec}\,(360^\circ + \theta) = \text{cosec}\,\theta$$
$$\sec(360^\circ + \theta) = \sec\theta.$$

§ 1(C)·19. Expressing the trigonometrical ratios of ($n.$360° $\pm\,\theta$) in terms of the trigonometrical ratios of 'θ'

We can say that if any imaginary line turns through θ in the positive direction or negative direction then turns through 360°, it will remain in the same position finally. It may turn in multiples of 360° (*i.e.*, 1.360°, 2.360°, 3.360° ....etc.)

Hence, the trigonometrical ratios of (360° $\pm\,\theta$) will be the same :

$$\sin(n.360^\circ + \theta) = \sin\theta$$
$$\cos(n.360^\circ + \theta) = \cos\theta$$
$$\tan(n.360^\circ + \theta) = \tan\theta$$
$$\cot(n.360^\circ + \theta) = \cot\theta$$
$$\text{cosec}(n.360^\circ + \theta) = \text{cosec}\,\theta$$
$$\sec(n.360^\circ + \theta) = \sec\theta$$

and
$$\sin(n.360^\circ - \theta) = \sin(-\theta) = -\sin\theta$$
$$\cos(n.360^\circ - \theta) = \cos(-\theta) = \cos\theta$$
$$\tan(n.360^\circ - \theta) = \tan(-\theta) = -\tan\theta$$
$$\cot(n.360^\circ - \theta) = \cot(-\theta) = -\cot\theta$$
$$\text{cosec}(n.360^\circ - \theta) = \text{cosec}(-\theta) = -\text{cosec}\,\theta$$
$$\sec(n.360^\circ - \theta) = \sec(-\theta) = \sec\theta.$$

ILLUSTRATIVE EXAMPLES

Example 1. Find the values of the following :

(i) sin (–1485°) **(ii) cos(390°)**

(iii) tan (330°).

Sol. (i) $\sin(-1485^\circ) = -\sin(1485^\circ)$, $[\because \sin(-\theta) = -\sin\theta]$

$= -\sin(4.360^\circ + 45^\circ)$, $[\because \sin(2n\pi + \theta) = \sin\theta]$

$= -\sin 45^\circ = -\dfrac{1}{\sqrt{2}}$. **Ans.**

(ii) $\cos(390^\circ) = \cos(360^\circ + 30^\circ)$

$= \cos 30^\circ = \dfrac{\sqrt{3}}{2}$, $[\because \cos(2\pi + \theta) = \cos\theta]$. **Ans.**

(iii) $\tan(330^\circ) = \tan(360^\circ - 30^\circ)$

$= \tan(-30^\circ)$, $[\because \tan(2\pi - \theta) = \tan(-\theta)]$

$= -\tan 30^\circ = -\dfrac{1}{\sqrt{3}}$, $[\because \tan(-\theta) = -\tan\theta]$. **Ans.**