Partial differential equations

Basic of PDE :-

Partial differential equations :- A differential eqn which contain two or more independent variables and partial derivatives with respect to them is called a partial differential equation (PDE).

Order of a PDE :- The order of a PDE is the order of highest order derivative appearing in equation.

Degree of PDE :- The degree of a PDE is the power of highest order derivative accurring in the equation when differential coefficient are made free from radicals and fractions.

Example :- The PDE $\frac{\partial^3 z}{\partial x^3} - 6\left(\frac{\partial z}{\partial y}\right)^2 - 4z = 0$ is of order-3 and degree-1.

Solution of a PDE :- A solution of PDE is an explicit or implicit relation between the variables involved that does not contain derivatives and satisfies the PDE.

Natuarally (2), the Solution of a PDE in one dependent variable z and two independent variables x and y are of the form $z = \phi(x,y)$ (explicit form) or of the form $f(x,y,z)=0$ (implicit form). Geometrically speaking both forms represents the surface in xyz-space. hence forth, the Solution of such a PDE are also called "integral surface of the Solution of equation.

Notation :- In the entire course, for dependent variable z and independent variables x and y, we shall adopt the following notations

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t$$

formulation of PDE :- There are two way to form a PDE

1. By eliminating the arbitary Constants
2. By eliminating the arbitrary functions

formulation of PDE by eliminating the arbitrary Constant

Example (1) By eliminating the Constants, obtain the PDE of the relation $2z = \frac{x^2}{a^2} + y^2/b^2$ — (1)

(1)

(3)

Solution:- Differentiating above partially w.r.to x, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} + 0$$

or $p = \frac{x}{a^2}$

or $\frac{1}{a^2} = p/x.$

Similarly, differentiating partially w.r.to y, we get

$$2\frac{\partial z}{\partial y} = 0 + \frac{2y}{b^2}.$$

or $q = y/b^2$

or $\frac{1}{b^2} = \frac{q}{y}.$

putting these value of $1/a^2$ and $1/b^2$ in given relation (1), we obtain

$$2z = x^2 \times \frac{p}{x} + y^2 \times \frac{q}{y}$$

or $2z = px + qy$

which is the required PDE of order one.

Example (2) form a PDE by eliminating the constants h and K from the relation.

$$(x-h)^2 + (y-K)^2 + z^2 = c^2$$

Solution. Differentiating above partially w.r.to x and y, we get

(4) $2(x-h)+0+2z\frac{\partial z}{\partial x}=0 \Rightarrow (x-h)=-zp.$

and

$$0+2(y-k)+2z\frac{\partial z}{\partial y}=0 \Rightarrow (y-k)=-zq.$$

putting these values in given relation, we obtain

$$(-zp)^2+(-zq)^2+z^2=c^2$$

$$z^2(p^2+q^2+1)=c^2$$

which is the required PDE.

Formulation of PDEs by eliminating the arbitrary functions:—

Example (1) By eliminating the arbitrary function, obtain the PDE from $z=f(x^2+y^2)$.

Solution:— The given equation is

$$z=f(x^2+y^2) \quad \text{— (2)}$$

Differentiating above partially w.r.to x, we get

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}f(x^2+y^2)$$

$$=\partial/_{\partial x}f(u) \quad (\text{where } u=x^2+y^2)$$

$$=\frac{d}{du}f(u)\frac{\partial u}{\partial x} \quad (\text{using chain rule})$$

$$=f'(x^2+y^2)\,2x.$$

$\Rightarrow p = 2xf'(x^2+y^2)$. —— (3) (5)

Similarly, differentiating partially w.r.to y, we get

$$q = 2yf'(x^2+y^2) \text{ —— (4)}$$

On dividing eqⁿ (3) and eqⁿ (4), we obtain

$$\frac{p}{q} = x/y \quad \text{or} \quad py = xq$$

which is the required PDE.

Example (2) By eliminating the arbitrary functions, obtain the PDE from $z = f(x+ct) + g(x-ct)$.

Solution:— Differentiating twice above partially w.r.to x and t, we get

$$\frac{\partial z}{\partial x} = f'(x+ct) + g'(x-ct)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct) \text{ —— (5)}$$

and $$\frac{\partial z}{\partial t} = cf'(x+ct) - cg'(x-ct) = c[f'(x+ct) - g'(x-ct)]$$

$$\frac{\partial^2 z}{\partial t^2} = c^2[f''(x+ct) + g''(x-ct)]. \text{ —— (6)}$$

from (5) and (6), we get the required PDE

$$\frac{\partial^2 z}{\partial \cancel{x^2}} = c^2 \frac{\partial^2 z}{\partial x^2}.$$

Solution of PDE for four Standard form:—

Form I: if $f(p,q)=0$, i.e equations containing only p and q.

A Complete Solution of such an equation is

$$z = ax + by + c \quad —① $$

Where a and b are such that

$$f(a,b) = 0 \quad —②$$

Write ② as $b=\phi(a)$ and use it in ① to obtain the required Solution, i.e $z = ax+\phi(a)y+c$

Where a and c are arbitrary Constants.

Example ① Solve $\sqrt{p}+\sqrt{q}=1$

Solution. The Complete Solution is

$$z = ax+by+c$$

such that $\quad f(a,b)=0$

i.e $\sqrt{a}+\sqrt{b}-1=0$ as $f(p,q)=\sqrt{p}+\sqrt{q}-1=0$

$$\Rightarrow \sqrt{b} = 1-\sqrt{a}$$

$$\Rightarrow b = (1-\sqrt{a})^2$$

Therefore, from ③, the required Solution is

$$z = ax+(1-\sqrt{a})^2y+c$$

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Example 2 Solve $x^2p^2+y^2q^2=z^2$

Solution. Rewriting the given equation as

$$x^2\left(\frac{\partial z}{\partial x}\right)^2+y^2\left(\frac{\partial z}{\partial y}\right)^2=z^2$$

$$\left(\frac{x\,\partial z}{z\,\partial x}\right)^2+\left(\frac{y\,\partial z}{z\,\partial y}\right)^2=1 \quad \text{—— (4)}$$

Now, set

$$\left.\begin{cases}\dfrac{\partial x}{x}=\partial X \Rightarrow \log x = X\\[2mm] \dfrac{\partial y}{y}=\partial Y \Rightarrow \log y = Y\\[2mm] \dfrac{\partial z}{z}=\partial Z \Rightarrow \log z = Z\end{cases}\right\} \text{—— (5)}$$

So that (4) reduces to Standard form I, i.e

$$\left(\frac{\partial Z}{\partial X}\right)^2+\left(\frac{\partial Z}{\partial Y}\right)^2=1$$

i.e., $\quad P^2+Q^2=1$ —— (6)

Thus the Complete Solution (6) is

$$Z = aX+bY+c \text{ — (7)}$$

such that

$$f(a,b)=0$$

i.e $\quad a^2+b^2-1=0,$

$$\Rightarrow b^2=1-a^2$$

$$\Rightarrow b=\sqrt{1-a^2}$$

Therefore, from (7), the Solution becomes

$$Z = aX+\sqrt{1-a^2}\,Y+c$$

Now, using the Setting ⑤, the required Solution is

$$\log z = a\log x + \sqrt{1-a^2}\log y + c$$

Example ③ Solve the partial differential equation:

$$(y-x)(qy-px) = (p-q)^2$$

Solution:- Let $(x+y) = X$ and $xy = Y$ — ⑧

then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{\partial z}{\partial X}\times 1 + \frac{\partial z}{\partial Y}\times y$$

$$= P + yQ \quad \text{— ⑨}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial X}\times 1 + \frac{\partial z}{\partial Y}\times x$$

Where $P = \frac{\partial z}{\partial X}$ and $Q = \frac{\partial z}{\partial Y}$, from ⑨ and ⑩, we have

$$= P + xQ \quad \text{— ⑩}$$

$$p-q = (y-x)Q \quad \text{and} \quad qy - px = (y-x)P \quad \text{— ⑪}$$

using ⑪, the given equation takes the form

$$(y-x)(qy-px) = (p-q)^2$$

$$(y-x)(y-x)P = (y-x)^2Q^2$$

$$\Rightarrow P = Q^2 \quad \text{— ⑫}$$

Which is of the form $f(P,Q)=0$, Thus the solution is

$$z = ax + by + c \quad \text{--- (13)}$$

Such that $f(a,b)=0$, i.e $a-b^2=0 \Rightarrow a=b^2$

Therefore, from (13), the Solution becomes

$$z = b^2x + by + c$$

using eqn (8), the required Solution is

$$z = b^2(x+y) + bxy + c$$

Example (4) solve the partial differential equation

$$(x^2+y^2)(p^2+q^2) = 1$$

Solution. Consider the transformation

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$ and $\theta = \tan^{-1} y/x$ --- (14)

then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} \quad \text{--- (15)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y} \quad \text{--- (16)}$$

using above, the given equation takes the form

$$(x^2+y^2)(p^2+q^2) = 1$$

$$\left(r\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = 1 \quad \text{--- (17)}$$

put $\partial R = \frac{\partial r}{r} \Rightarrow R = \log r$, then eqn (17) can be expressed a

$$\left(\frac{\partial z}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = 1$$

i.e

$$P^2 + Q^2 = 1 \quad \text{--- (18)}$$

(10) which is of the form $f(P, Q) = 0$, where $P = \frac{\partial z}{\partial R}$ and $Q = \frac{\partial z}{\partial \Theta}$. Thus, the the solution is

$$z = aR + b\Theta + c \quad \text{--- (19)}$$

such that $f(a, b) = 0$ i.e, $a^2 + b^2 = 1 \Rightarrow b = \sqrt{1-a^2}$

Therefore, from (19), the solution becomes

$$z = aR + \sqrt{1-a^2}\,\Theta + c$$

using $R = \log r$ and eqn (14), the required solⁿ is

$$z = \frac{1}{2} a \log(x^2 + y^2) + \sqrt{1-a^2} \tan^{-1} y/x + c$$

Standard form II :—

Let $f(z, p, q) = 0$, i.e equations containing z, p and q.

To solve such equation we use the following steps:—

Steps ①:- Assume $z = \phi(x + ay) = \phi(u)$, where $u = x + ay$

then $p = \frac{\partial z}{\partial x} = \phi'(x+ay) = \phi'(u) = \frac{dz}{du}$

and $q = \frac{\partial z}{\partial y} = a\phi'(x+ay) = a\phi'(u) = a\frac{dz}{du}$

Step ② Substitute these value of p and q in the given equation, we get

$$f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0,$$ Integrating it, we

get the Complete Solution.

Example (1) Solve $z^2(p^2+q^2+1)=c^2$

Let $u=x+ay$, so that $p=\frac{dz}{du}$, $q=a\frac{dz}{du}$, substitute these value of p and q in the given equation, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2+a^2\left[\frac{dz}{du}\right]^2+1\right]=c^2$$

$$z^2(1+a^2)\left(\frac{dz}{du}\right)^2=c^2$$
$$+z^2$$

$$\Rightarrow \quad \left(z\frac{dz}{du}\right)^2=\frac{c^2-z^2}{1+a^2}$$

Now, seperate the variables, we get

$$\pm\frac{z\,dz}{\sqrt{c^2-z^2}}=\frac{du}{\sqrt{1+a^2}}$$

$$\pm\sqrt{1+a^2}\,\frac{z\,dz}{\sqrt{c^2-z^2}}=du$$

$$\Rightarrow \quad \pm\sqrt{1+a^2}\sqrt{c^2-z^2}=u+c$$

Now, replace $u=x+ay$, we get

$$\sqrt{1+a^2}\sqrt{c^2-z^2}=x+ay+c.$$

Example (2) Solve $z^2(p^2x^2+q^2)=1$

Solution:- The given eq^n can be written as

(12)

$$z^2\left\{\left(\frac{\partial z}{\partial x}\right)^2 x^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\} = 1$$

putting $\frac{\partial x}{x} = \partial X \Rightarrow \log x = X$, thus

$$z^2\left\{\left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\} = 1 \quad \text{— ①}$$

Which is of the form $f(z, p, q) = 0$

Now, let $u = X + ay$ so that $\frac{\partial z}{\partial X} = \frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y} = a\frac{\partial z}{\partial u}$

then eqn ① becomes

$$z^2\left\{\left(\frac{\partial z}{\partial u}\right)^2 + a^2\left(\frac{\partial z}{\partial u}\right)^2\right\} = 1$$

$$z^2\left\{(1+a^2)\left(\frac{\partial z}{\partial u}\right)^2\right\} = 1$$

Seperate the variables, we get

$$\left(\frac{dz}{du}\right)^2 = \frac{1}{z^2(1+a^2)}$$

$$\frac{\partial z}{\partial u} = \pm\frac{1}{z\sqrt{1-a^2}}$$

$$\sqrt{1+a^2}\int z\,dz = \pm\int \partial u$$

$$\sqrt{1+a^2} \times \frac{z^2}{2} = \pm u + b$$

$$(\sqrt{1+a^2})z^2 = \pm 2(X + ay) + b$$

since $X = \log x$ i.e

$$(\sqrt{1+a^2})z^2 = \pm 2(\log x + ay) + b.$$

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Standard form-III :-

given $f_1(x,p) = f_2(y,q)$, i.e the equation in which the variable z does not appear and the terms containing x and p can be separated from those containing y and q.

To obtain a Solution of such an equation, we proceed as follows:

1. Put $f_1(x,p) = f_2(y,q) = a$, where a is an arbitrary constant.
2. Solving these equation for p and q. Let $p = f_1(x,a)$ and $q = f_2(y,a)$.
3. Since $dz = p\,dx + q\,dy$

$$\Rightarrow dz = f_1(x,a)dx + f_2(y,a)dy$$

Integrating above equation, we get

$$z = \int f_1(x,a)dx + \int f_2(y,a)dy + b$$

which is the required Complete Solution.

Example① Solve $q - p + x - y = 0$

Solution① The given eqn can be written as

$$q - y = p - x$$

Let $q - y = p - x = a$, so that $p = x + a$ and $q = y + a$

Now, we have

$$dz = (x+a)dx + (y+a)dy$$

Integrating the above eq^n, we have

$$\int dz = \int (x+a)dx + \int (y+a)dy$$

$$z = \frac{(x+a)^2}{2} + \frac{(y+a)^2}{2} + b_1$$

$$2z = (x+a)^2 + (y+b)^2 + 2b_1$$

Thus, the required sol^n is

$$2z = (x+a)^2 + (y+a)^2 + b, \text{ where } b = 2b_1$$

Example ② Solve $z^2(p^2+q^2) = x^2+y^2$

Solution ② The given equation can be written as

$$z^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = x^2+y^2$$

$$\left(\frac{z\partial z}{\partial x}\right)^2 + \left(\frac{z\partial z}{\partial y}\right)^2 = x^2+y^2 \quad \text{— ①}$$

Put $z\partial z = \partial Z$ So that $\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} = \frac{z\partial z}{\partial x} = P$

$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial y} = \frac{z\partial z}{\partial y} = Q$$

Now, eq^n ① becomes

$$P^2 + Q^2 = x^2+y^2$$

i.e $P^2 - x^2 = -Q^2 + y^2$

Put $P^2 - x^2 = y^2 - Q^2 = a$, Then $P = \sqrt{a+x^2}$, $Q = \sqrt{y^2-a}$

therefore, $dz = p\,dx + q\,dy$

$$dz = \sqrt{a+x^2}\,dx + \sqrt{y^2-a}\,dy$$

integrating, we get

$$z = \int \sqrt{a+x^2}\,dx + \int \sqrt{y^2-a}\,dy$$

$$\frac{2\times z^2}{2} = x\sqrt{a+x^2} + y\sqrt{y^2-a} + a\log\left\{\frac{x+\sqrt{a+x^2}}{y+\sqrt{y^2-a}}\right\} + b$$

The Complete Solution is

$$z^2 = x\sqrt{x^2+a} + y\sqrt{y^2+a} + a\log\frac{x+\sqrt{x^2+a}}{y+\sqrt{y^2-a}} + b.$$

Standard form IV :-

Let $z = px + qy + f(p,q)$, i.e equation analogous to Clairaut's equation.

The Complete Solution for such an eqn is

$$z = ax + by + f(a,b).$$

Which obtained by writting a for p and b for q in the given eqn.

Example ① solve $z = px + qy + pq$

The given eqn is Clairaut's equation, so the Complete Solution for such an eqn as

$$z = ax + by + ab. \quad \text{Where } f(p,q) = pq$$

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Example (2) find the singular Solution of the partial differential eq^n.

$$z = px + qy + \sqrt{1+p^2+q^2}$$

Solution (2) The given eq^n is Clairaut's equation, so the Complete Solution for such as equation as

$$z = ax + by + \sqrt{1+a^2+b^2} \quad \text{as } f(p,q) = \sqrt{1+p^2+q^2} \quad (1)$$

To find the singular Solution, differentiate (1) with respect to a and b, we get

$$x + \frac{a}{\sqrt{1+a^2+b^2}} = 0, \quad y + \frac{b}{\sqrt{1+a^2+b^2}} = 0 \quad (2)$$

from (2), we have

$$x^2 = \frac{a^2}{1+a^2+b^2}, \quad y^2 = \frac{b^2}{1+a^2+b^2}$$

$$\Rightarrow \quad x^2+y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$x^2+y^2 = \frac{(a^2+b^2+1)-1}{(1+a^2+b^2)}$$

$$x^2+y^2 = 1 - \frac{1}{(1+a^2+b^2)}$$

$$\Rightarrow \quad \frac{1}{1+a^2+b^2} = 1-x^2-y^2$$

$$1+a^2+b^2 = \frac{1}{1-x^2-y^2}$$

$$\Rightarrow \quad \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

put this value in (2), So that after Simplification, we get

$$a = -\frac{x}{\sqrt{1-x^2-y^2}}, \quad b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

use this value in eqⁿ (1), we get

$$z = ax + by + \sqrt{1+a^2+b^2}$$

$$z = -\frac{x}{\sqrt{1-x^2-y^2}} \times x - \frac{y}{\sqrt{1-x^2-y^2}} \times y + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2 \Rightarrow z^2+x^2+y^2 = 1$$

Example (3) using Some transformation, reduce the PDE $2y + 2zq = q(xp+yq)$ to clairaut's form and obtain the Complete Solution.

Solⁿ (3) The given partial differential equation can be written as

$$2y + 2zq = q(xp+yq)$$

$$2zq = xpq + yq^2 - 2y$$

$$z = \frac{1}{2}xp + \frac{1}{2}yq - \frac{y}{q}$$

$$z = \frac{1}{2}x\frac{\partial z}{\partial x} + \frac{1}{2}y\frac{\partial z}{\partial y} - \frac{1}{2}\frac{2y}{\frac{\partial z}{\partial y}}$$

$$z = x^2\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) - \frac{1}{2}\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right)^{-1} \quad (3)$$

Now, use the transformation $u = x^2$ and $v = y^2$

So that
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$
$$= \frac{\partial z}{\partial u} \times 2x + \frac{\partial z}{\partial v} \times 0$$
$$\frac{\partial z}{\partial x} = 2x\frac{\partial z}{\partial u}$$
$$\Rightarrow \frac{1}{2x}\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}$$

and
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$
$$= \frac{\partial z}{\partial u} \times 0 + \frac{\partial z}{\partial v} \times 2y$$
$$\Rightarrow \frac{1}{2y}\frac{\partial z}{\partial y} = \frac{\partial z}{\partial v}$$

using these transformation, (3) reduce to
$$z = Pu + Qv - \frac{1}{2Q} \quad \text{--- (4)}$$

where $P = \frac{\partial z}{\partial u}$, $Q = \frac{\partial z}{\partial v}$, eqn (4) is of Clairaut form.

Thus the Solution is $z = au + bv - \frac{1}{2b}$

Thus the required Soln is (as $u = x^2$, $v = y^2$)
$$z = ax^2 + by^2 - \frac{1}{2b}$$

Charpit Method:-

for a non-linear PDE of order one there is general Method for finding the Complete Solution. This method is known as Charpit's Method and is described as follows write the given equation of the form

$$f(x,y,z,p,q)=0 \quad \text{--- (1)}$$

Consider the simultaneous total differential equations

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} \quad \text{--- (2)}$$

eqn (2) are called Charpit's subsidary equation.

taking the last two fractions of eqn (2) and find the either the value of p or q. if we have obtained the value of p, then the value of q is obtained from (1). put these value of p and q in the following equation

$$dz = p\,dx + q\,dy$$

and integrate the above equation to get the required soln.

Example (1) solve $(p^2+q^2)y = qz$

Soln:- Let $f(x,y,p,q,z) = (p^2+q^2)y - qz$ --- (1)

from (1) $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y} = p^2+q^2$, $\frac{\partial f}{\partial z} = -q$, $\frac{\partial f}{\partial p} = 2py$, $\frac{\partial f}{\partial q} = 2qy-z$ --- (2)

Consider the subsidiary equations

$$\frac{dx}{-2py} = \frac{dy}{-z+2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2} \quad \text{--- (3)}$$

taking last two fraction, we have

$$dp/-pq = dq/p^2$$

(20) $\Rightarrow p\,dp + q\,dq = 0$ —— (4)

On integration, we get

$$p^2 + q^2 = c^2 \quad \text{—— (5)}$$

Now, using (5) in eqn (1), we get

$$q = \frac{c^2 y}{z}$$

Now, substitute this value of q in (5), we get

$$p = \frac{c}{z}\sqrt{z^2 - c^2 y^2}$$

hence,

$$dz = p\,dx + q\,dy$$

$$dz = \frac{c}{z}\sqrt{z^2 - c^2y^2}\,dx + \frac{c^2 y}{z}\,dy$$

$$z\,dz = c\sqrt{z^2 - c^2y^2}\,dx + c^2 y\,dy$$

$$z\,dz - c^2 y\,dy = c\sqrt{z^2 - c^2y^2}\,dx$$

OR $$\frac{\frac{1}{2}\,d(z^2 - c^2y^2)}{\sqrt{z^2 - c^2y^2}} = c\,dx$$

Integrating, we have

$$\sqrt{z^2 - c^2y^2} = cx + a$$

$$z^2 = (cx+a)^2 + c^2y^2$$

which is the required soln.

Example (2) Solve $p(q^2+1) + (b-z)q = 0$

Soln:- Let $f(x,y,z,p,q) = 0 \Rightarrow p(q^2+1) + (b-z)q = 0$ —— (1)

from eqn (1), we have

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = -q, \quad \frac{\partial f}{\partial p} = q^2+1, \quad \frac{\partial f}{\partial q} = 2pq + b - z \quad \text{—— (2)}$$

consider the subsidiary equations

$$\frac{dx}{\frac{\partial f}{\partial p}} = \frac{dy}{\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} \quad \text{--- (3)}$$

using (2) in (3), we have

$$\frac{dx}{-(q^2+1)} = \frac{dy}{-(2pq+b-z)} = \frac{dz}{-3pq^2-p-(b-z)q} = \frac{dp}{pq} = \frac{dq}{q^2} \quad (4)$$

taking the last two fraction, we have

$$\frac{dp}{pq} = \frac{dq}{q^2} \Rightarrow \log p = \log q + \log a$$

$$\Rightarrow \quad p = aq \quad \text{--- (5)}$$

putting this value in eq^n (1), we get

$$p(q^2+1) + (b-z)q = 0$$

$$p = \frac{(z-b)q}{(q^2+1)}$$

$$\frac{\cancel{q}}{a} = \frac{(z-b)\cancel{q}}{(q^2+1)}$$

$$\Rightarrow \quad q^2 = a(z-b)-1$$

$$q = \sqrt{a(z-b)-1}$$

Thus, from (5), $p = q/a = \dfrac{\sqrt{a(z-b)-1}}{a}$

substituting the values of p and q in $dz = pdx + qdy$, we obtain

$$dz = \left(\frac{\sqrt{a(z-b)-1}}{a}\right)dx + \left(\sqrt{a(z-b)-1}\right)dy$$

$$\frac{dz}{\sqrt{a(z-b)-1}} = \frac{dx}{a} + dy \Rightarrow 2\sqrt{a(z-b)-1} = x + ay + b$$

which is the required sol^n.

Example (3) (22) $p^2x + pqy + pqx + q^2y = 1$

Solⁿ :- Let $f(x,y,z,p,q) = p^2x + pqy + pqx + q^2y - 1 = 0$ —— (1)

from eqⁿ (1), we have

$\frac{\partial f}{\partial x} = p^2 + pq$, $\frac{\partial f}{\partial y} = q^2 + pq$, $\frac{\partial f}{\partial z} = 0$, $\frac{\partial f}{\partial p} = 2px + qy + qx$

$\frac{\partial f}{\partial q} = 2qy + py + px = 2qy + p(x+y)$ —— (2)

Consider the subsidiary equations

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} \quad \text{—— (3)}$$

using (2), we get, the subsidiary equation as

$$\frac{dx}{-(2px + q(x+y))} = \frac{dy}{-(2qy + p(x+y))} = \frac{dz}{-(2p^2x + 2pq(x+y) + 2q^2y)} = \frac{dp}{p} = \frac{dq}{q}$$

taking last to fraction, we get

$$\frac{dp}{p} = \frac{dq}{q}$$ and integrate, we get

$$p = qa$$

using $p = aq$ in eqⁿ (1), we get after Simlification

$q = \frac{1}{\sqrt{(a+1)(ax+y)}}$ So, that $p = \frac{a}{\sqrt{(a+1)(ax+y)}}$. Now, substituting the values of p and q in equation $dz = pdx + qdy$

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we get $$dz = \frac{a\,dx + dy}{\sqrt{(a+1)(ax+y)}}$$

OR $$dz = \frac{1}{\sqrt{a+1}} \frac{d(ax+y)}{\sqrt{(ax+y)}}$$

integrating the above equation, we get

$$z = \frac{1}{\sqrt{1+a}} (ax+y)^{1/2} + b$$

Which is the required Solution.

Lagrange's Method :—

Quasi linear PDE of first order and Method for Solving such equations :— A partial differential equation of the first order is of the form

$$Pp + Qq = R \quad \text{—} \ (1)$$

Where P, Q and R are the functions of x, y and z. This eqn is known as Quasi-linear partial differential eqn or Lagrange's equation and the Solution of the above equation can be obtained by Lagrange's Method as $\phi(u,v) = 0$ or $u = f(v)$.

Working Rule :— To obtain the solution (1), we apply the following rules

(1) Write eqn (1) in the forms $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(2) Solve these equations by Method of grouping or

Method (24) of Multiplier, given $u=a$ and $v=b$ as its Solutions.

(3) Write the Solution as $\phi(u,v)=0$ or $u=f(v)$.

In the same way, we can obtain the Solution of the linear Partial differential equation involving more than two variables.

Example ① Solve $(y+z)p+(x+z)q=x+y$

Soln:- Compare given eqn with $Pp+Qq=R$

here, $P=y+z$, $Q=x+z$, $R=x+y$

The subsidiary eqn is given by

$$\frac{dx}{y+z}=\frac{dy}{x+z}=\frac{dz}{x+y}$$

Which can be written as

$$\frac{dx+dy+dz}{2x+2y+2z}=\frac{dx-dy}{-(x-y)}=\frac{dx-dz}{-(x-z)}$$

then

$$\frac{dx+dy+dz}{2x+2y+2z}=\frac{dx-dy}{-(x-y)}$$

$$\frac{dx+dy+dz}{x+y+z}=\frac{2(dx-dy)}{(x-y)}$$

$$\log(x+y+z)+2\log(x-y)=\log a$$

$$\log(x+y+z)(x-y)^2=\log a$$

$$(x+y+z)(x-y)^2=a$$

$$\frac{dx-dy}{+(x-y)} = \frac{dx-dz}{+(x-z)}$$

$$\log(x-y) - \log(x-z) = \log b$$

$$\log\left(\frac{(x-y)}{(x-z)}\right) = \log b$$

$$\frac{(x-y)}{(x-z)} = b$$

hence, the required Solution is

$$\phi\left[(x+y+z)(x-y)^2, \frac{x-y}{x-z}\right] = 0$$

or

$$(x-y) = (x-z) f\left[(x+y+z)(x-y)^2\right]$$

Example (2) Solve $(mz-ny)p + (nx-lz)q = ly-mx$

Solution. The subsidiary equation is

$$\frac{dx}{(mz-ny)} = \frac{dy}{(nx-lz)} = \frac{dz}{ly-mx}$$

using multiplier x, y, z, we have

$$\text{each } f^n = \frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow \quad xdx+ydy+zdz = 0$$

which gives $\quad x^2+y^2+z^2 = a$

and again, using multipliers l, m, n, we get

$$\text{each} f^n = \frac{ldx+mdy+ndz}{0}$$

$$ldx + mdy + ndz = 0$$

on integration, which gives

$$lx + my + nz = b$$

hence, the required Solution is

$$x^2+y^2+z^2 = f(lx+my+nz).$$

Example 3 solve $(x^2-y^2-z^2)p + 2xyq = 2xz$

Solution:- The subsidiary equation is

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

from last two fraction, we have

$$\frac{dy}{\cancel{2x}y} = \frac{dz}{\cancel{2x}z} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

on integration, we get $y/z = a$

Now, using Multipliers x, y, z, we have

$$\frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2} = \frac{dz}{z}$$

on integration, we get $\frac{x^2+y^2+z^2}{z} = b$

hence, the required Sol^n is

$$x^2+y^2+z^2 = z f(y/z).$$

Example (4) Solve $(w+y+z)\frac{\partial w}{\partial x}+(w+x+z)\frac{\partial w}{\partial y}+(w+x+y)\frac{\partial w}{\partial z}=x+y+z$.

Solution (4) The subsidiary eq^n is

$$\frac{dw}{x+y+z}=\frac{dx}{y+z+w}=\frac{dy}{z+w+x}=\frac{dz}{w+x+y}$$

Which gives

$$\frac{dw+dx+dy+dz}{3(w+x+y+z)}=\frac{dw-dx}{(x-w)}=\frac{dw-dy}{(y-w)}=\frac{dw-dz}{(z-w)}$$

taking first two fraction, we get

$$\frac{dw+dx+dy+dz}{3(w+x+y+z)}+\frac{dw-dx}{(w-x)}=0 \text{ by integrating}$$

$$\frac{1}{3}\log(w+x+y+z)+\log(w-x)=\log a$$

$$\Rightarrow (w+x+y+z)^{1/3}(w-x)=a$$

Similarly, the other two solution are

$$(w-y)(w+x+y+z)^{1/3}=b$$

$$(w-z)(w+x+y+z)^{1/3}=c$$

Therefore, the required Solution is

$$\phi\left[(w-x)u,(w-y)u,(w-z)u\right]=0$$

Where $u=(x+y+z+w)^{1/3}$.

(28)

Homogeneous linear partial differential equation with Constant Coefficients :- An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + k_2 \frac{\partial^n z}{\partial x^n \partial y^2} + \cdots + k_n \frac{\partial^n z}{\partial y^n} = f(x,y) \quad \text{--- (1)}$$

Where $k_1, k_2, \ldots, k_n$ are Constant, is called linear homogeneous PDE of order n with Constant Coefficient.

putting $D = \partial/\partial x$, $D' = \partial/\partial y$ in eqn (1), then eqn (1) reduced to

$$(D^n + k_1 D^{n-1} D' + k_2 D^{n-2} D'^2 + \cdots + k_n D'^n) z = f(x,y).$$

this can be written as

$$f(D, D') z = f(x,y)$$

As in the case of ODE with Constant Coeffients, here also the Complete solution of (1) Consists of the Complementary function (C.f) and Particular integral (P.I).

Method for finding Complementary functions :- To find Complementary function, Consider a second order PDE

$$\frac{\partial^2 z}{\partial x^2} + k_1 \frac{\partial^2 z}{\partial x \partial y} + k_2 \frac{\partial^2 z}{\partial y^2} = 0$$

i.e $$(D^2 + k_1 DD' + k_2 D'^2) z = 0 \quad \text{--- (1)}$$

The auxiliary equation (by subtituting $D = m$ and $D' = 1$) in above eqn (1), we have

$$m^2 + k_1 m + k_2 = 0$$

After Simplification, we get two roots m_1 and m_2

There will be two cases

1. (Case-I):- if the roots are real and distinct. i.e, $m_1 \neq m_2$, then the Complementary function is

$$C.f = f(y+m_1x) + \phi(y+m_2x)$$

2. (Case-II):- if the roots are equal. i.e $m_1 = m_2$, then the Complementary function is

$$C.f = f(y+m_1x) + x\phi(y+m_1x)$$

Remark. for higher order PDE, we have the auxilary equations

$$m^n + k_1 m^{n-1} + k_2 m^{n-2} + \cdots + k_n = 0 \quad \text{--- (2)}$$

After Simplification we get n roots as $m_1, m_2, --, m_n$.

Now,

1. (Case-I):- if all the roots of eqn (2) are real and distinct then

$$C.f = f_1(y+m_1x) + f_2(y+m_2x) + \cdots + f_n(y+m_nx)$$

2. (Case-II):- if two roots of eqn (2) are equal, then

$$C.f = f_1(y+m_1x) + xf_2(y+m_1x) + f_3(y+m_3x) + \cdots + f_n(y+m_nx)$$

3. (Case-III):- if three roots of eqn (2) are equal, then

$$C.f = f_1(y+m_1x) + xf_2(y+m_1x) + x^2f_3(y+m_1x) + f_4(y+m_4x) + \cdots + f_n(y+m_nx)$$

Example (1) $2\frac{\partial^2 z}{\partial x^2}+5\frac{\partial^2 z}{\partial x\partial y}+2\frac{\partial^2 z}{\partial y^2}=0$ — (3)

Solution. eqn (3) can be written as

$$(2D^2+5DD'+2D'^2)z=0$$

Then, the auxiliary eqn is

$$2m^2+5m+2=0$$
$$\Rightarrow (m+2)(2m+1)=0$$
$$\Rightarrow m=-2,-1/2$$

Since the roots are real and distinct, then the Complementary function is

$$C.f = f(y-2x)+\phi(y-\tfrac{1}{2}x).$$

Example (2) solve $\frac{\partial^2 z}{\partial x^2}+6\frac{\partial^2 z}{\partial x\partial y}+9\frac{\partial^2 z}{\partial y^2}=0$ — (4)

Solution. eqn (4) can be written as

$$(D^2+6DD'+9D'^2)z=0$$

Then, the auxiliary eqn is

$$m^2+6m+9=0$$
$$(m+3)^2=0 \Rightarrow m=-3,-3$$

here two roots are equal, So, the Complementary fn is $C.f = f(y-3x)+x\,\phi(y-3x)$

Example (3) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0$

Solution :- The auxiliary eqⁿ is $(D^2 + DD' - 2D'^2)z = 0$

$$m^2 + m - 2 = 0$$
$$m^2 + 2m - m - 2 = 0$$
$$m(m+2) - 1(m+2) = 0$$
$$(m+2)(m-1) = 0$$
$$\Rightarrow m = -2, 1$$

then $C.f = f(y-2x) + \phi(y+x)$

Solution of Non-homogeneous linear PDE with Constant Coefficients :- The Complete Solution of second order Non-homogeneous linear PDE with Constant Coefficient is $z = C.f + P.I.$

Method for finding Particular integral :- To find the Particular integral, Consider a second order linear PDE

$$\frac{\partial^2 z}{\partial x^2} + k_1 \frac{\partial^2 z}{\partial x \partial y} + k_2 \frac{\partial^2 z}{\partial y^2} = f(x,y)$$

i.e $(D^2 + k_1 DD' + k_2 D'^2)z = f(x,y)$

Then $P.I = \frac{1}{f(D,D')} f(x,y)$

(32) The Concept is Same as in ODE, i.e $P.I = \frac{1}{f(D)} Q(x)$.

There are Some Cases to find the Particular integral.

1. Case-I :- When $f(x,y) = e^{ax+by}$, then

$$P.I = \frac{e^{ax+by}}{f(a,b)}, \text{ provided } f(a,b) \neq 0$$

if $f(a,b) = 0$, then

$$P.I = \frac{1}{f(D,D')} f(x,y) = \frac{1}{f(D+a)\,D'+b)} e^{ax+by}$$

2. Case-II, When $f(x,y) = \sin(mx+ny)$ or $\cos(mx+ny)$ then

$$P.I = \frac{1}{f(D,D')} f(x,y) = \frac{1}{f(D^2, DD', D'^2)} f(x,y) = \frac{1}{f(-m^2, -mn, -n^2)} \sin(mx+ny) \text{ or } \cos(mx+ny)$$

3. Case-III :- When $f(x,y) = x^m y^n$, where m, n are Constants then

$$P.I = \frac{1}{f(D,D')} f(x,y) = \frac{1}{f(D,D')} x^m y^n = \left[f(D,D')\right]^{-1} (x^m y^n)$$

Now, evaluate, $\left[f(D,D')\right]^{-1}$ by expanding in ~~ascending~~ powers of D or D' using the binomial theorem and then operate on $x^m y^n$ term by term.

4 Case-IV :— When $f(x,y)$ is any function, then

$$P.I = \frac{1}{f(D,D')} f(x,y)$$

Now, resolve, $\frac{1}{f(D,D')}$ into partial fraction considering $f(D,D')$ as a function of D only and operate each partial fractions on $f(x,y)$ as

$$P.I = \frac{1}{f(D,D')} f(x,y) = \frac{1}{(D-mD')} f(x,y) = \int f(x,c-mx)\,dx$$

Where c is replaced by $y+mx$ after integration.

5. Case-V :— When $f(x,y) = e^{ax+by} V(x,y)$, where $V(x,y)$ is a f^n of x and y. then

$$P.I = \frac{1}{f(D,D')} e^{ax+by} V(x,y) = e^{ax+by} \frac{1}{f(D+a, D'+b)} V(x,y)$$

6. Case-VI :— When $f(x,y) = \phi(ax+by)$.

here, we have two cases:

(i) Replace D by a and D' by b in $f(D,D')$, we get $f(a,b) \neq 0$

Moreover, if $f(D,D')$ is homogeneous function of degree n, then

$$P.I = \frac{1}{f(D,D')} f(x,y) = \frac{1}{f(D,D')} \phi(ax+by) = \frac{1}{f(a,b)} \int\int \cdots \int \phi(u)\,du^n$$

Where u is replaced by $ax+by$ after integrating the right hand side n times with respect to u.

(ii) Replace D by a and D' by b in $f(D,D')$, we get $f(a,b)=0$. In such cases, $f(D,D')$ must be factorized and in general, there are two types of factors i.e

$$f(D,D') = (bD-aD')^m g(D,D'), \text{ where } g(a,b) \neq 0$$

Then the Particular integral can be obtained by as in (i) as well as by making use of the formula

$$\frac{1}{(bD-aD')^m}\phi(ax+by) = \frac{x^m}{b^m . m!}\phi(ax+by).$$

Example ① find the Complete Solution of the following Non-homogeneous PDE $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

Solution ① The auxiliary eqn of the given eqn $D^3-3D^2D'+4D'^3=0$ replace $D=m$, $D'=1$ then we have

$$m^3-3m^2+4=0$$

$$m^2(m+1)-4m(m+1)+4(m+1)=0$$

$$\Rightarrow (m+1)(m^2-4m+4)=0$$

$$(m+1)(m-2)^2=0$$

$$\Rightarrow m=-1,2,2$$

hence, the Complementary f^n $C.f = f_1(y-x)+f_2(y+2x)+x f_3(y+2x)$

To find Particular integral

$$P.I = \frac{1}{f(D,D')} f(x,y) \quad \text{as} \quad \frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}$$

$$P.I = \frac{1}{(D^3 - 3D^2D' + 4D'^3)} e^{x+2y} = \frac{1}{[(1)^3 - 3(1)^2 \times 2 + 4(2)^3} e^{x+2y}$$ here $a=1, b=2$

$$P.I = \frac{1}{27} e^{x+2y}$$

hence, the complete sol^n is

$$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}.$$

Example ② solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$

Solution:- The given PDE can be written as

$$(D^2 - DD')z = \cos x \cos 2y \quad \because 2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

i.e, $(D^2 - DD')z = \frac{1}{2}[\cos(x+2y) + \cos(x-2y)]$

The auxilary eq^n of given PDE is

$$m^2 - m = 0$$

$$m(m-1) = 0$$

$$\Rightarrow m = 0, 1$$

So, the complementary function is

$$C.f = f_1(y) + f_2(y+x)$$

To find Particular integral,

$$P.I = \frac{1}{f(D,D')} f(x,y) \text{ as } \frac{1}{f(D^2, DD', D'^2)} \cos(mx+ny) = \frac{1}{f(-m^2, -mn, -n^2)} \cos(mx+ny)$$

$$P.I = \frac{1}{(D^2-DD')} \frac{1}{2}\left[\cos(x+2y) + \cos(x-2y)\right]$$

$$P.I = \frac{1}{2}\left[\frac{1}{(D^2-DD')}\cos(x+2y) + \frac{1}{(D^2-DD')}\cos(x-2y)\right]$$

$$P.I = \frac{1}{2}\left[\frac{1}{[-(1)^2-\{-(1)(2)\}]}\cos(x+2y) + \frac{1}{[-(1)^2-\{-(1)(-2)\}]}\cos(x-2y)\right]$$

$$P.I = \frac{1}{2}\left[\frac{1}{-1+2}\cos(x+2y) + \frac{1}{-1-2}\cos(x-2y)\right]$$

$$P.I = \frac{1}{2}\left[\cos(x+2y) - \frac{1}{3}\cos(x-2y)\right]$$

hence, the Complete Solution is

$$z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - 1/6\cos(x-2y)$$

Example (3) solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$

Solution:- The given PDE can be written as

$$(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$$

The auxilary eq^n of given PDE is

$$m^3 - 2m^2 = 0$$

$$m^2(m-2) = 0$$

$$\Rightarrow m = 0, 0, 2$$

Therefore, the Complementary f^n is $C.f = f_1(y) + xf_2(y) + f_3(y+2x)$.

$$P.I = \frac{1}{f(D,D')} f(x,y)$$

$$P.I = \frac{1}{(D^3-2D^2D')} 2e^{2x} + 3x^2y$$

$$P.I = \frac{1}{(D^3-2D^2D')} 2e^{2x} + \frac{1}{(D^3-2D^2D')} 3x^2y$$

$$P.I = \frac{1}{[(2)^3-2(2)^2\times 0]} 2e^{2x} + \frac{1}{D^3\left(1-\frac{2D'}{D}\right)} 3x^2y$$

$$P.I = \frac{1}{8} 2e^{2x} + \frac{3}{D^3}\left(1-\frac{2D'}{D}\right)^{-1} x^2y$$

$$P.I = \frac{e^{2x}}{4} + \frac{3}{D^3}\left[1+\frac{2D'}{D}+\frac{4D'^2}{D^2}+\cdots\right] x^2y$$

$$P.I = \frac{e^{2x}}{4} + \frac{3}{D^3}\left[x^2y+\frac{2}{D}D'(x^2y)+\frac{4}{D^2}D'^2(x^2y)+\cdots\right]$$

$$P.I = \frac{e^{2x}}{4} + \frac{3}{D^3}\left[x^2y+\frac{2}{D}x^2+\frac{4}{D^2}\times 0\right], \quad D = d/dx, \ D' = \frac{d}{dy}, \ \frac{1}{D}f(x) = \int f(x)\,dx$$

$$P.I = \frac{e^{2x}}{4} + \frac{3}{D^3}\left[x^2y+\frac{2x^3}{3}\right]$$

$$P.I = \frac{e^{2x}}{4} + 3\frac{1}{D^3}(x^2y) + 2\cdot\frac{1}{D^3}x^3$$

$$P.I = \frac{e^{2x}}{4} + 3\times\frac{1}{D^2}\frac{x^3}{3}y + 2\frac{1}{D^2}\frac{x^4}{4}$$

$$P.I = \frac{e^{2x}}{4} + \frac{x^5y}{20} + \frac{x^6}{60}$$, hence, the Complete Solⁿ is

$$z = f_1(y) + xf_2(y) + f_3(y+2x) + \frac{e^{2x}}{4} + \frac{x^5y}{20} + \frac{x^6}{60}$$

Example (4) (39) $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

Sol^n:- The given PDE can be written as

$$(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$$

The auxilary eq^n of given PDE is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$\Rightarrow m = 2, 2$$

hence, the Complementary f^n is

$$C.F = f_1(y+2x) + x f_2(y+2x)$$

To find particular integral,

$$P.I = \frac{1}{f(D,D')} f(x,y) = \frac{1}{(D-2D')^2} e^{2x+y}$$

here $f(a,b) = 0$, so, use $\frac{1}{(bD-aD')^m} \phi(ax+by) = \frac{x^m}{b^m m!} \phi(ax+by)$

then $P.I = \frac{1}{(D-2D')^2} e^{2x+y}$

$$P.I = \frac{x^2}{(1)^2 2!} e^{2x+y}$$

$$P.I = \frac{x^2}{2} e^{2x+y}$$

Hence, the Complete Solution is

$$z = f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}$$

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Example 5 Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$

Solution. The given PDE can be written as

$$(D^2 + DD' - 6D'^2)z = y\cos x$$

The auxiliary equation of given PDE is

$$m^2 + m - 6 = 0$$
$$\Rightarrow m = 2, -3$$

Hence, the Complementary f^n is

$$C.f = f_1(y+2x) + f_2(y-3x)$$

To find the particular integral

$$P.I = \frac{1}{f(D,D')} f(x,y) = \frac{1}{(D^2+DD'-6D'^2)}(y\cos x)$$

$$P.I = \frac{1}{(D-2D')(D+3D')}(y\cos x)$$

$$P.I = \frac{1}{(D-2D')}\left[\frac{1}{(D+3D')}(y\cos x)\right], \quad \text{using } \frac{1}{(D-mD')}f(x,y) = \int f(x, c-mx)\,dx$$

$$P.I = \frac{1}{(D-2D')}\int (c+3x)\cos x\,dx$$

$$P.I = \frac{1}{(D-2D')}\left[(c+3x)\int \cos x\,dx - \int\left[\frac{d}{dx}(c+3x)\cdot\int \cos x\,dx\right]dx\right]$$

$$P.I = \frac{1}{(D-2D')}\left[(c+3x)\sin x + 3\cos x\right]$$

$$P.I = \frac{1}{(D-2D')} y\sin x + 3\cos x \qquad \because y = c - mx \Rightarrow y = c+3x$$

$P.I = \int \left[(c-2x)\sin x + 3\cos x \right] dx$

$P.I = \int (c-2x)\sin x\, dx + 3\int \cos x\, dx$

$P.I = \left[(c-2x)(-\cos x) - \int -2(-\cos x)\, dx \right] + 3\sin x$

$P.I = \left[-y\cos x - 2\sin x + 3\sin x \right] \quad \because y = c - mx \Rightarrow y = c - 2x$

$P.I = -y\cos x + \sin x$

Hence, the Complete Sol^n is

$$z = f_1(y+2x) + f_2(y-3x) + \sin x - y\cos x$$

Example 6 solve $(D^2 + 3DD' + 2D'^2)z = x + y$

Sol^n:- The auxiliary eq^n of given PDE is

$$m^2 + 3m + 2 = 0$$

$$\Rightarrow m = -1, -2$$

hence, the Complementary f^n is

$$C.f = f_1(y-x) + f_2(y-2x).$$

To find the particular integral,

$$P.I = \frac{1}{f(D,D')} f(x,y) = \frac{1}{(D^2 + 3DD' + 2D'^2)}(x+y)$$

here, $f(D,D')$ is a homogeneous f^n of degree 2 and $f(a,b) \neq 0$

Thus, using Case VI

$$\frac{1}{f(D,D')}\phi(ax+by) = \frac{1}{f(a,b)}\iint\cdots\int \phi(u)\,du^n$$

Where n is the degree of homogeneous f^n.

$$P.I = \frac{1}{(D^2+3DD'+2D'^2)}(x+y)$$

$$P.I = \frac{1}{[(1)^2+3\times1\times1+2\times(1)^2]}\iint u\,du^2$$

$$P.I = \frac{1}{6}\int\frac{u^2}{2}du = \frac{1}{12}\cdot\frac{u^3}{3} = \frac{1}{36}u^3$$

$$P.I = \frac{1}{36}(x+y)^3$$

hence, the Complete Sol^n is $z = C.f + P.I$

$$z = f_1(y-x)+f_2(y-2x)+\frac{1}{36}(x+y)^3$$

Example 7 Solve $(4D^2-4DD'+D'^2)z = \log(x+2y)$

Solution:- The auxiliary eq^n of given PDE is

$$4m^2-4m+1=0$$

$$\Rightarrow m = \frac{1}{2}, \frac{1}{2}$$

Hence, the Complementary f^n is

$$C.f = f_1\left(y+\frac{1}{2}x\right)+x f_2\left(y+\frac{1}{2}x\right)$$

To find (42) particular integral,

$$P.I = \frac{1}{f(D,D')} f(x,y)$$

$$P.I = \frac{1}{(4D^2 - 4DD' + D'^2)} \log(x+2y)$$

$$P.I = \frac{1}{(2D-D')^2} \log(x+2y)$$

here, $f(a,b) = 0$ and $f(D,D')$ can be factorize and thus using Case VI,

$$\frac{1}{(bD-aD')^m} \phi(ax+by) = \frac{x^m}{b^m \, m!} \phi(ax+by)$$

we have

$$P.I = \frac{1}{(2D-D')^2} \log(x+2y)$$

$$P.I = \frac{x^2}{(2)^2 \cdot 2!} \log(x+2y)$$

$$P.I = \frac{x^2}{8} \log(x+2y)$$

Therefore, the Complete Sol^n is

$$z = f_1\left(y+\frac{1}{2}x\right) + x f_2\left(y+\frac{x}{2}\right) + \frac{x^2}{8}\log(x+2y)$$

Which is the required Solution.

Home Assignment

Question ① solve $(D^2+DD'-2D'^2)z = e^x(y-1)$

Answer $z = f_1(y+x) + f_2(y-2x) + e^x(y-2)$

Question ② solve $(D^2-2DD'+D'^2)z = \sin x$

Answer $z = f_1(y+x) + x f_2(y+x) - \sin x$

Question ③ solve $(D^2+4DD'-D'^2)z = y^2$

Answer $z = f_1(y+x) + f_2(y-5x) + \frac{x^2y^2}{2} - \frac{4x^2y}{3} + \frac{7x^4}{4}$

Question ④ solve $(D^2-DD'-6D'^2)z = xy$

Answer $z = f_1(y-2x) + f_2(y+3x) + \frac{x^3y}{6} + \frac{x^4}{24}$

Question ⑤ solve $(2D^2-5DD'+2D'^2)z = 5\sin(2x+y)$

Answer $z = f_1(y+2x) + f_2(2y+x) - \frac{5x}{3} x\cos(2x+y)$

Question ⑥ solve $(D^2+2DD'+D'^2)z = 2(y-x)+\sin(x-y)$

Answer $z = f_1(y-x) + x f_2(y-x) + x^2y - x^3 + \frac{x^2}{2}\sin(x-y)$

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Non-homogeneous linear PDE :-

Consider $f(D,D')z = f(x,y)$ — ①

if the polynomial $f(D,D')$ is not homogeneous, then eqⁿ ① is called non-homogeneous linear PDE.

To obtain Complementary fⁿ, we factorize $f(D,D')$ into factors of the form $(D-mD'-c)$, where c is any Constant.

Then different cases arises

Case-I, When $f(D,D')$ is resolved into non-repeated factors of the form $(lD-mD'-c)$ then

$$C.f = e^{cx/l}\phi(ly+mx)$$

When $f(D,D')$ is resolved into repeated factor of the form $(lD-mD'-c)^n$, then C.f is

$$C.f = e^{cx/l}.\{\phi(ly+mx)+x\phi_2(ly+mx)+\dots+x^{n-1}\phi_n(ly+mx)\}$$

Case-II Corresponding to a factor $(mD'+c)$, then C.f is

$$C.f = e^{-cy/m}\phi(mx)$$

if the factor $(mD'+c)$ is repeated n times, then

$$C.f = e^{-cy/m}.\{\phi_1(mx)+x\phi_2(mx)+\dots+x^{n-1}\phi_n(mx)\}$$

Case-III for a non-repeated factor of the form $(lD - mD')$, then C.f is

$$C.f = \phi(ly + mx)$$

When $(lD - mD')$ is repeated ~~in~~ n times then C.f is

$$C.f = \phi_1(ly+mx) + x\phi_2(ly+mx) + x^2\phi_3(ly+mx) + \cdots + x^{n-1}\phi_n(ly+mx)$$

Case-IV Corresponding to a factor D,

$$C.f = \phi(y)$$

While for a factor of the form D^n, then

$$C.f = \phi_1(y) + x\phi_2(y) + \cdots + x^{n-1}\phi_n(y)$$

Case-V for a factor of the form D', then C.f is

$$z = \phi(x)$$

and Corresponding to a factor D'^n, then

$$C.f = \phi_1(x) + y\phi_2(x) + \cdots + y^{n-1}\phi_n(x).$$

Case-VI Moreover, if $f(D, D')$ can not be factorized into linear factors, then the Method of finding C.f is as follows:-

Step① Let $z = ce^{hx+ky}$ be a Solution of the equation $f(D,D')z = 0$

Step② substituting this value of z in the given PDE, we get

$$C.f(h,k)\, e^{hx+ky} = 0$$

Which (46) is zero only when $f(h,k)=0$

Step ③ Solve $f(h,k)=0$ for h and k, then $z = ce^{hx+ky}$ will be a part of the C.F.

Step ④ if D' in $f(D,D')z=0$ is of degree n, then the solution of $f(h,k)=0$, will gives $f_1(h), f_2(h), \dots, f_n(h)$.

Step ⑤ Corresponding to $k = f_1(h)$, the part of the solution of $f(D,D')z=0$ is $\sum c_1 e^{hx+f_1(h)y}$, where $\sum$ denotes the infinite series obtained by assigning c and h all possible values.

$$\therefore z = \sum c_1 e^{hx+f_1(h)y} + \sum c_2 e^{hx+f_2(h)y} + \dots + \sum c_n e^{hx+f_n(h)y}$$

if $k = ah+b$ i.e, $f(h)$ is linear in h, then the solution can take a simple form.

equation $f(h,k)=0$ can also be solved for h in terms of k, and in such case, we get another form of solution.

Remark. Methods for finding P.I are the same as in homogeneous case.

Example ① Solve $(D^2+2DD'+D'^2-2D-2D')z = \sin(x+2y)$

Solution here $f(D,D') = D^2+2DD'+D'^2-2D-2D'$

$$= \{(D+D')^2 - 2(D+D')\}$$

$f(D,D') = (D+D')(D+D'-2)$

As, we know, Solution corresponding to $(lD-mD')$ is $\phi(ly+mx)$ and for $(lD-mD'-c)$ is $e^{cx/l}\phi(ly+mx)$, then

$$(D+D') \rightarrow \phi_1(y-x)$$

$$(D+D'-2) \rightarrow e^{2x/1}\phi_2(y-x)$$

Therefore $C.f = \phi_1(y-x) + e^{2x}\phi_2(y-x)$

and the particular integral is

$$P.I = \frac{1}{f(D,D')} f(x,y) = \frac{1}{(D^2+2DD'+D'^2-2D-2D')} \sin(x+2y)$$

As, we know, if $\frac{1}{f(D,D')}\sin(mx+ny)$ then $D^2 \rightarrow -m^2$

$DD' \rightarrow -mn$

$D'^2 \rightarrow -n^2$

Therefore,

$$P.I = \frac{1}{(-(1)^2+2(-2)-(2)^2-2D-2D')} \sin(x+2y)$$

$$P.I = \frac{1}{(-9-2D-2D')} \sin(x+2y)$$

$$P.I = -\frac{1}{2(D+D')+9} \sin(x+2y)$$

$$P.I = -\frac{1}{2(D+D')+9} \times \frac{2(D+D')-9}{2(D+D')-9} \sin(x+2y)$$

$$P.I = -\frac{2(D+D')-9}{4(D+D')^2-81} \sin(x+2y)$$

$P.I = \frac{2(D+D')-9}{4(D^2+2DD'+D'^2)-81} \sin(x+2y)$ 36-81

$P.I = \frac{2(D+D')-9}{4[-(1)^2+2x-2-(2)^2]-81} \sin(x+2y)$

$P.I = \frac{2(D+D')-9}{-117} \sin(x+2y)$

$P.I = +\frac{1}{117}\, 2D(\sin(x+2y)) + 2D'\sin(x+2y) - 9\sin(x+2y))$

$P.I = \frac{1}{117}\, 2\cos(x+2y) + 4\cos(x+2y) - 9\sin(x+2y))$

$P.I = \frac{3}{117}\left[2\cos(x+2y) - 3\sin(x+2y))\right]$

$P.I = \frac{1}{39}\left[2\cos(x+2y) - 3\sin(x+2y))\right]$

Then, the complete solⁿ is

$z = \phi_1(y-x) + e^{2x}\phi_2(y-x) + \frac{1}{39}\left[2\cos(x+2y) - 3\sin(x+2y))\right]$

Example ② solve $(DD' + D - D' - 1)z = xy$

Solution ② The given PDE can be expressed as

$$(D-1)(D'+1)z = xy$$

As we know, solution corresponding to $(lD - mD' - c)$ is $e^{cx/l}\phi(ly+mx)$ and for $(mD'+c)$ is $e^{-cy/m}\phi(mx)$, then

$(D-1) \to e^{1\cdot x/1}\phi_1(y) = e^{x}\phi_1(y)$

$(D'+1) \to e^{-1\cdot y/1}\phi_2(x) = e^{-y}\phi_2(x)$

Therefore, $C.f = e^{x}\phi_1(y) + e^{-y}\phi_2(x)$

and the Particular Integral is

$$P.I = \frac{1}{f(D,D')} f(x,y)$$

$$P.I = \frac{1}{(D-1)(D'+1)} xy = -(1-D)^{-1}(1+D')^{-1} xy$$

$$P.I = -\left[(1+D+D^2+\cdots)(1+D'+D'^2+\cdots)\right](xy)$$

$$P.I = -\left[(1+D+D^2+\cdots)(xy+D'(xy)+D'^2(xy))\right]$$

$$P.I = -\left[(1+D+D^2+\cdots)(xy-x)\right]$$

$$P.I = -\left[xy-x+y-1+0\right]$$

$$P.I = -xy+x-y+1$$

$$P.I = -xy+x-y+1$$

Thus, the Complete Soln is

$$z = e^{x}\phi_1(y) + e^{-x}\phi_2(y) - xy + x - y + 1$$

Example (3) $(D^2-6DD'+9D'^2-4D+12D'+4)z = 2e^{2x}\tan(y+3x)$

Soln:- The given equation can be written as

$$(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$$

As, we know, Solution Corresponding to $(lD-mD'-c)^n$ is $e^{cx/l}\{\phi_1(ly+mx) + x\phi_2(ly+mx) + \cdots + x^{n-1}\phi_n(ly+mx)\}$

Then $(D-3D'-2)^2 \longrightarrow e^{2x/1}\{\phi_1(y+3x) + x\phi_2(y+3x)$

and (b) the Particular integral is

$$P.I = \frac{1}{f(D,D')} f(x,y)$$

$$P.I = \frac{1}{(D-3D'-2)^2} 2e^{2x} \tan(y+3x) \text{ using } \frac{1}{f(D,D')} e^{ax+by} V(x,y) = e^{ax+by} \frac{1}{f(D+a,D'+b)} V(x,y)$$

$$P.I = 2e^{2x} \frac{1}{[(D+2)-3(D'+0)-2]^2} \tan(y+3x)$$

$$P.I = 2e^{2x} \frac{1}{(D-3D')^2} \tan(y+3x), \text{ using } \frac{1}{(bD-aD')^m} \phi(ax+by) = \frac{x^m}{b^m m!} \phi(ax+by)$$

$$P.I = \cancel{2} e^{2x} \times \frac{x^2 \tan(y+3x)}{\cancel{1^2 \times 2!}}$$

$$P.I = x^2 e^{2x} \tan(y+3x)$$

Thus, the Complete Soln is

$$z = e^{2x} \phi_1(y+3x) + x e^{2x} \phi_2(y+3x) + x^2 e^{3x} \tan(y+3x)$$

Example 4 Solve $(D^2 - D')z = 0$

Solution. Assume that a Solution of the given PDE be of the form $z = c e^{hx+ky}$ ——— (1)

So that $\quad D'z = ck e^{hx+ky}$

$$Dz = ch e^{hx+ky}, \quad D^2 z = ch^2 e^{hx+ky}$$

Then the given equation reduces to

$$(D^2 - D')z = 0$$

$$ch^2 e^{hx+ky} - ck e^{hx+ky} = 0 \qquad c(h^2-k) e^{hx+ky} = 0$$

which lead to $h^2-k=0$ i.e, $h^2=k$. with this value of k, the eqn ① becomes

$$z = ce^{hx+h^2y}$$

, therefore, the general soln is

$$z = \sum ce^{hx+h^2y}$$

, where c and h are arbitrary constants.

Example ⑤ solve $(D^2-D'^2-1)z = 0$

Solution. Let the solution of the given PDE be of the form

$$z = ce^{hx+ky} \quad \text{--- ①}$$

So that

$$Dz = ch\,e^{hx+ky}, \quad D^2z = ch^2e^{hx+ky}$$

$$D'z = ck\,e^{hx+ky}, \quad D'^2z = ck^2e^{hx+ky}$$

Thus, the given eqn reduces to

$$(D^2-D'^2-1)z=0$$

$m=\sqrt{2}$ $m=\pm\sqrt{2}$

$$ch^2e^{hx+ky} - ck^2e^{hx+ky} - 1 = 0$$

$$c(h^2-k^2-1)\,e^{hx+ky} = 0$$

Which gives

$$h^2-k^2-1=0 \quad \text{--- ②}$$

if we choose $h = \sec\alpha$ and $k=\tan\alpha$, then eqn ② is satisfied.

Therefore, the general solution is

$$z = \sum ce^{x\sec\alpha + y\tan\alpha}$$

, where, c and α are arbitrary constants.

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Example (6) Solve $(D-2D')(D^2-D')z = e^{2x+y}+xy$

Solution. here, Corresponding to the linear factor $(D-2D')$, the Complementary f^n is $f(y+2x)$. While (D^2-2D') can not be resolved into linear factor in D and D' and for this factor (D^2-D'), the Complementary function is $\sum ce^{hx+ky}$, where h and K are related through the equation $h^2-K=0$, so that $h^2=K$. Then, for the given eqn

$$C.F = f(y+2x) + \sum ce^{hx+h^2y}$$

Also,

$$P.I = \frac{1}{(D-2D')(D^2-D')} e^{2x+y} + \frac{1}{(D-2D')(D^2-D')} xy = P_1 + P_2$$

Now, $P_1 = \frac{1}{(D-2D')(D^2-D')} e^{2x+y}$

$$P_1 = \frac{1}{(D-2D')(4-1)} e^{2x+y}$$

$$= \frac{1}{3} \frac{e^{2x+y}}{(D-2D')}, \text{ using } \frac{1}{(bD-aD')^m}\phi(ax+by) = \frac{x^m}{b^m m!}\phi(ax+by)$$

$$= \frac{1}{3} \frac{x e^{2x+y}}{1^{1} 1!} = \frac{xe^{2x+y}}{3}$$

Now, $P_2 = \frac{1}{(D-2D')(D^2-D')} (xy)$

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$$P_2 = \frac{1}{(-2D')\left(1-\frac{D}{2D'}\right)(-D')\left(1-\frac{D^2}{D'}\right)} xy$$

$$P_2 = \frac{1}{2D'^2}\left[\left(1-\frac{D}{2D'}\right)^{-1}\left(1-\frac{D^2}{D'}\right)^{-1}\right] xy$$

$$P_2 = \frac{1}{2D'^2}\left[\left(1+\frac{D}{2D'}+\cdots\right)\left(1+\frac{D^2}{D'}+\cdots\right)\right] xy$$

$$P_2 = \frac{1}{2D'^2}\left[\left(1+\frac{D}{2D'}+\cdots\right)(xy+0)\right]$$

$$P_2 = \frac{1}{2D'^2}\left[\left(xy+\frac{y^2}{4}\right)\right]$$

$$P_2 = \frac{1}{2D'}\left[\frac{xy^2}{2}+\frac{y^3}{12}\right]$$ here $D' = d/dy$ $\frac{1}{D'} = \int dy$

$$P_2 = \frac{1}{2}\left[\frac{xy^3}{6}+\frac{y^4}{48}\right]$$

Thus, the Particular integral is

$$P.I = \frac{1}{3} x e^{2x+y} + \frac{1}{2}\left(\frac{1}{6}xy^3 + \frac{1}{48}y^4\right)$$

Therefore, the Complete soln is

$$z = f(y+2x) + \sum c e^{hx+ky} + \frac{1}{3} x e^{2x+y} + \frac{1}{2}\left[\frac{1}{6}xy^3 + \frac{1}{48}y^4\right]$$

Home Assignment

Question ① Solve $(D-1)(D-D'+1)z = e^{y}$

Answer $z = e^{x}\phi_1(y) + e^{-x}\phi_2(y-x) - xe^{y}$

Question ② Solve $(D^3 - 3DD' + D' + 4)z = e^{2x+y}$

Answer $z = \sum c e^{hx + f(h)y} + \frac{1}{7}e^{2x+y}$

Question ③ Solve $(2DD' + D'^2 - 3D')z = 3\cos(3x-2y)$

Answer $z = f_1(x) + e^{3y} f_2(2y-x) + \frac{3}{50}\left[4\cos(3x-2y) + 3\sin(3x-2y)\right]$

Question ④ Solve $(D+D'-1)(D+2D'-3)z = 4+3x+6y$

Answer $z = e^{x} f_1(y-x) + e^{3x} f_2(y-2x) + x + 2y + 6$

Question ⑤ Solve $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^{y}$

Answer $z = e^{x} f_1(y) + e^{-x} f_2(x+y) + \frac{1}{2}\sin(x+2y) - xe^{y}$

Question ⑥ Solve $(2D^2 - D')(D^2 - D')z = 0$

Answer $z = \sum c_1 e^{h_1 x + h_1^2 y} + \sum c_2 e^{h_2 x + 2h_2^2 y}$

Question ⑦ Solve $(D - D'^2)z = 0$

Answer $z = \sum c e^{np}$, where $p = x\cos\alpha + y\sin\alpha$

here c and α are Constant.

Method of Separation of Variables :-

Consider a PDE of the form

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

Where A, B, C are Continuous function of x and y, the derivatives are also Continuous and f denotes a polynomial function of $x, y, z, \frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Remark (1) The above equation is hyperbolic, parabolic and elliptic according as B^2-4AC is positive, zero and negative, respectively.

(2) In physical problems, we usually look for a Solution of a PDE which Satisfies Certain specified Condition, known to be the boundary Conditions.

(3) The differential equation, together with these boundary Conditions, Constitutes a Boundary Value problem.

(4) A process used in finding the Solution of boundary Value problem involving partial differential equations is known as Separation of Variables.

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Example ① Solve $\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial y^2} = 0$ — ①

Solution. Let the solution of the form

$$z = X(x)\,Y(y) \quad \text{— ②}$$

Where X is a f^n of x only and Y is a function of y only.

Since, from ②, we have $\frac{\partial z}{\partial x} = X'Y$, $\frac{\partial^2 z}{\partial x^2} = X''Y$,

Similarly, $\frac{\partial^2 z}{\partial y^2} = XY''$, So eq^n ① becomes

$$X''Y + 4XY'' = 0$$

$$\Rightarrow \frac{X''}{X} = -4\frac{Y''}{Y} = k \text{(say)}$$

Therefore, $\frac{X''}{X} = k$ and $-4\frac{Y''}{Y} = k$

$$\Rightarrow X'' = kX \quad \text{and} \quad -4Y'' = kY \quad \text{— ③}$$

There will be three posibilities k is negative, positive and zero.

Case–I When $k>0$, then A.E is

$$m^2 = k \qquad\qquad -4m^2 = k$$

$$m = \pm\sqrt{k} \qquad\qquad m^2 = \frac{ki^2}{4} = \pm\frac{\sqrt{k}\,i}{2}$$

then $X = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$ and $Y = C_3 \sin\frac{\sqrt{k}y}{2} + C_4 \cos\frac{\sqrt{k}y}{2}$

Case-II When $K<0$ suppose $k=-n^2, n>0$ (57)

then eqn (3) becomes

$$X'' + n^2 X = 0 \quad \text{and} \quad Y'' - \frac{n^2}{4} Y = 0$$

therefore A.E is $m^2 + n^2 = 0$ and $m^2 - \frac{n^2}{4} = 0$

$\Rightarrow m = \pm ni$ and $m = \pm \frac{n}{2}$

then Soln is

$$X = C_1 \cos nx + C_2 \sin nx \quad \text{and} \quad Y = C_3 e^{n/2 y} + C_4 e^{-n/2 y}$$

Case-III when $K=0$ then eqn (3) becomes

$$X'' = 0 \quad \text{and} \quad Y'' = 0$$

therefore, A.E is $m^2 = 0$ and $m^2 = 0$

$m = 0, 0$ and $m = 0, 0$

then soln are

$$X = (C_1 + x C_2) \quad \text{and} \quad Y = (C_3 + y C_4)$$

Example (2) solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ — (1)

Soln:- we assume that the solution is of the form

$$z = X(x) Y(y) \quad \text{— (2)}$$

where X is a function of x only and Y is a function of y only, therefore, equation (1) reduce to

$$X''Y - 2X'Y + XY' = 0$$

$$(X'' - 2X')Y + XY' = 0$$

$$\Rightarrow \frac{X'' - 2X'}{X} = \frac{-Y'}{Y} = k \text{ (say)}$$

$$\Rightarrow X'' - 2X' - kX = 0 \quad \text{and} \quad Y' + kY = 0$$

$\therefore$ the Solution of these ODE is

$$X = c_1 e^{(2(1+\sqrt{1+k})x} + c_2 e^{2(1-\sqrt{1+k})x}$$

$$Y = c_3 e^{-ky}$$

Therefore the required solⁿ is

$$z = X(x)\,Y(y)$$

$$z = \left[c_1 e^{(2(1+\sqrt{1+k})x} + c_2 e^{2(1-\sqrt{1+k})x}\right] c_3 e^{-ky}$$

$$z = \left[c_1 c_3 e^{2(1+\sqrt{1+k})x} + c_3 c_2 e^{2(1-\sqrt{1+k})x}\right] e^{-ky}$$

Home Assignment

Question (1) $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, when $u(x,0) = 6e^{-3x}$

Answer $u = 6e^{-3x-2t}$

Question (2) $4\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 3z$, when $x = 0$, $z = 3e^{-y} - e^{-5y}$

Answer $= 4e^{x-y} - e^{2x-5y}$