Unit-1

evaluation of double integral :-

The double integral can be expressed in the form

$$\iint_R f(x,y)\,dx\,dy = \int_{x_1}^{x_2}\int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x,y)\,dx\,dy$$

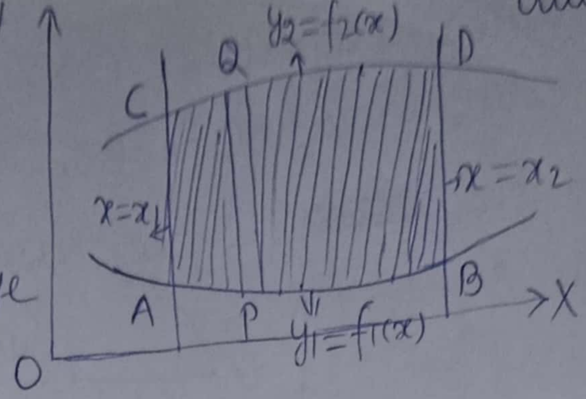
The above double integral can be evaluate in three way and its value is found as follows.

(i) When $y_1=f_1(x)$ and $y_2=f_2(x)$ are functions of x and x_1, x_2 are constants, then $f(x,y)$ is first integrated w.r.t. y keeping x fixed or constant b/w limits y_1, y_2 and then resulting expression is integrated w.r.t x within the limits x_1, x_2 i.e

$$\boxed{I_1 = \int_{x_1}^{x_2}\boxed{\int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x,y)\,dy}\,dx}$$

Where integration is carried from the inner to the outer rectangle. Fig1.1 illustrates this process. Here AB and CD are two curves whose equations are $y_1=f_1(x)$ and $y_2=f_2(x)$. PQ is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.



Thus the whole region of integration is the area ABDC.

(ii) When $x_1 = f_1(y)$, $x_2 = f_2(y)$ are functions of y and y_1, y_2 are constants, then $f(x,y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y b/w the limits y_1, y_2, i.e

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1=f_1(y)}^{x_2=f_2(y)} f(x,y)\,dx}\; dy}$$

The above integral represented geometrically as show in fig 1.2. Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy.

Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while

The outer rectangle corresponds to the sliding of this edge from AC to BD.
Thus the whole region of integration is the area ABDC.

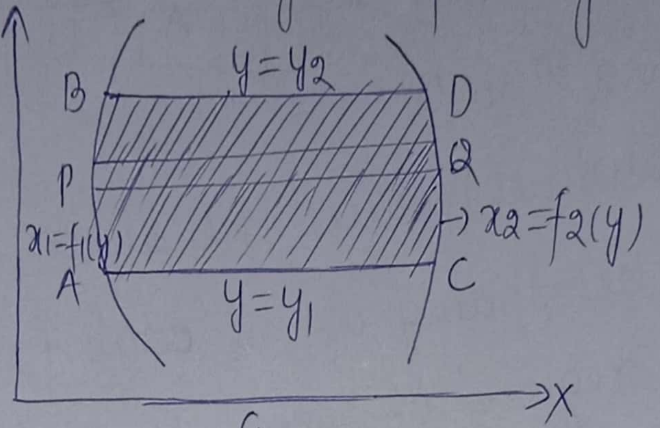


fig 1.2

(iii) When both pairs of limits are constants, the region of integration is the rectangle ABDC (fig 1.3)

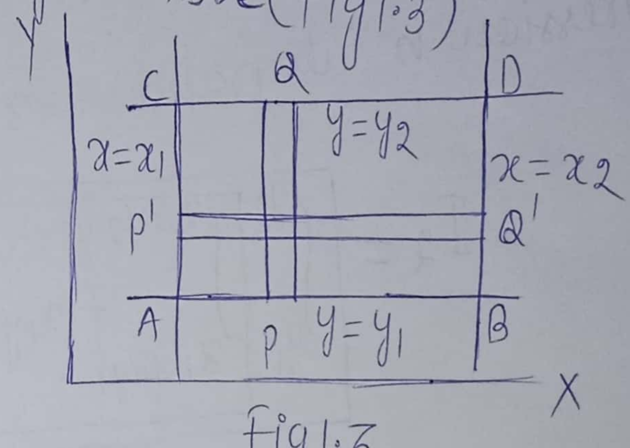


Fig 1.3

Example ① Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$

Solution:- $I = \int_0^5 dx \int_{y=0}^{y=x^2} (x^3+xy^2)\,dy$

$$\Rightarrow I = \int_0^5 \left[x^3 y + x.\frac{y^3}{3}\right]_0^{x^2} dx = \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx$$

$$= \left[\frac{x^6}{6} + \frac{x^8}{24}\right]_0^5 = 5^6\left[\frac{1}{6} + \frac{5^2}{24}\right] = 18880.2$$

Example ② Evaluate $\iint_A xy\,dx\,dy$, where A is the domain bd by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$!

Solution. The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. Fig 1.4 show that the domain A which is the area OML.

Integrating first over a vertical strip PQ, i.e, w.r.t. y from $P(y=0)$ to $Q(y = x^2/4a)$ on the parabola and then w.r.t. x from $x = 0$ to $x = 2a$, we have

$$\iint_A xy\,dx\,dy = \int_0^{2a} dx \int_0^{x^2/4a} xy\,dy = \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{a^4}{3}$$

$$\iint_A xy\,dx\,dy = \frac{a^4}{3}$$

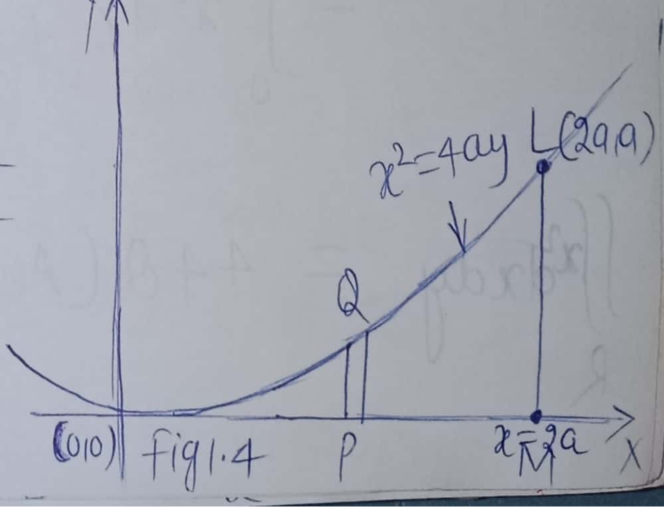


Fig 1.4

Example (3) Evaluate $\iint_R x^2\,dx\,dy$, where R is the region in the first quadrant bd by lines $x=y$, $y=0$, $x=8$ and the curve $xy=16$.

Solution. The line AL ($x=8$) intersects the hyperbola $xy=16$ at A(8,2) while the line $y=x$ intersects this hyperbola at B(4,4). Fig 1.5 shows the region R of integration which is the area OLAB. To evaluate the given integral, we divide this area into two parts OMB and MLAB.

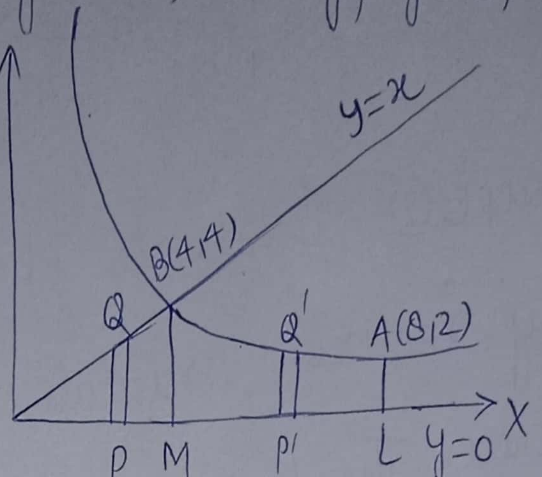


Fig 1.5

$$\Rightarrow \iint_R x^2\,dx\,dy = \int_0^4 \int_0^x x^2\,dx\,dy + \int_4^8 \int_0^{16/x} x^2\,dx\,dy$$

$$= \int_0^4 x^2\,dx \Big[y\Big]_0^x + \int_4^8 x^2\,dx \Big[y\Big]_0^{16/x}$$

$$= \int_0^4 x^3\,dx + \int_4^8 16x\,dx = \left[\frac{x^4}{4}\right]_0^4 + 16\left[\frac{x^2}{2}\right]_4^8$$

$$\iint_R x^2\,dx\,dy = 448 \text{ (Answer)}$$

Evaluation of double integrals in Polar Coordinate

The double integral in Polar Coordinate system is of the form $\int_{\theta_1}^{\theta_2}\int_{r_1=f_1(\theta)}^{r_2=f_2(\theta)} f(r,\theta)\,dr\,d\theta$, To evaluate this integral, we first integrate w.r.t. r b/w limits $r=r_1$ and $r=r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are Constants.

geometrically the double integral ~~of~~ represented in polar Coordinate system as Fig 1.6.

Here AB and CD are the curves $r_1=f_1(\theta)$ and $r_2=f_2(\theta)$ bd by lines $\theta=\theta_1$ and $\theta=\theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

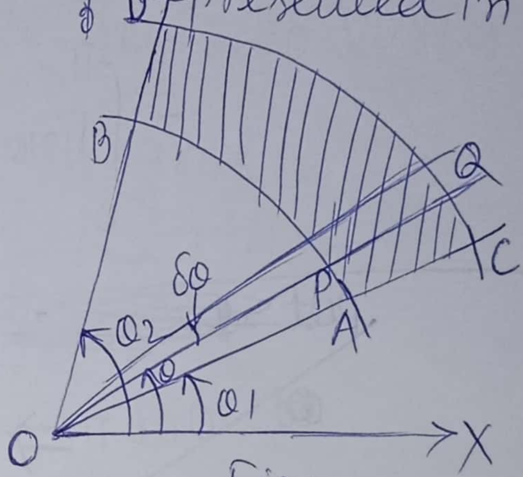


Fig 1.6

Then $\int_{r_1}^{r_2} f(r,\theta)\,dr$ indicates that the integration is along PQ from P to Q, while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD. Thus the whole region of integration is the area ACDB. The order of integration may be Changed with appropriate Changes in the limits.

Example ① Evaluate $\iint r\sin\theta \, dr\, d\theta$ over the cardioid $r = a(1-\cos\theta)$ above the initial line.

Solution:- To integrate first w.r.t. r, the limits are from $O\,(r=0)$ to $P\,(r = a(1-\cos\theta))$ and to cover the region of integration R. θ varies from 0 to π (as show fig 1.7).

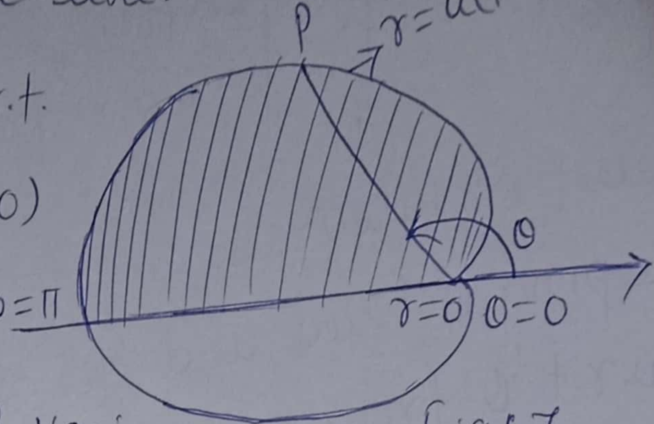


fig 1.7

$$\Rightarrow \iint_R r\sin\theta \, dr\, d\theta = \int_0^{\pi} \sin\theta \left[\int_0^{r=a(1-\cos\theta)} r\, dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \, d\theta \left[\frac{r^2}{2} \right]_0^{a(1-\cos\theta)} = \frac{a^2}{2} \int_0^{\pi} (1-\cos\theta)^2 \sin\theta \, d\theta$$

put ~~$1-\cos\theta = t$~~ $\Rightarrow$ ~~$\sin\theta\, d\theta = dt$~~

θ at $\pi \Rightarrow t = 0$, θ at 0 then $t = -1$

$$\iint_R r\sin\theta\, dr\, d\theta = \frac{a^2}{2}\int_{-1}^{0} (1+t)^2 dt = \frac{a^2}{2}\int_{-1}^{0} (1+t^2+2t)\, dt$$

$$= \frac{a^2}{2}\left[t + \frac{t^3}{3} + \frac{2t^2}{2} \right]_{-1}^{0} = \frac{a^2}{2}\left[+1 + \frac{1}{3} + 1 \right]$$

$$\iint_R r\sin\theta\, dr\, d\theta = \frac{a^2}{2}\left[\frac{(1-\cos\theta)^3}{3} \right]_0^{\pi} = \frac{a^2}{2} \cdot 8/3 = \frac{4a^2}{3}$$

Example② Evaluate $\iint r^3 dr d\theta$ over the area included b/w the circles $r = 2\sin\theta$ and $r = 4\sin\theta$ — unit-I

Solution:- Given circles $r = 2\sin\theta$

$r = 4\sin\theta$

are shown in fig 1.8. The shaded area b/w these circles is the region of integration.

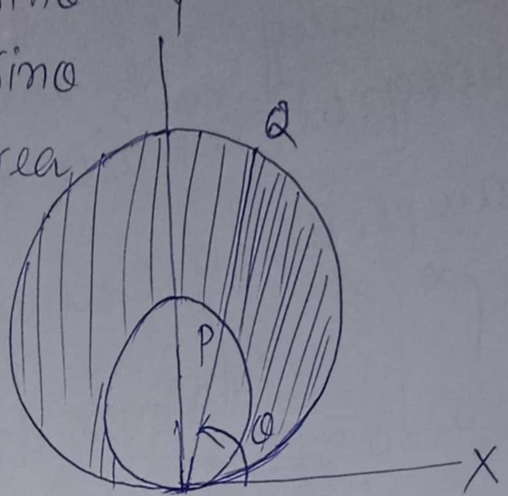


fig 1.8

if we integrate first w.r.t. r, then its limits are from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is

$$\iint_R r^3 dr d\theta = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 dr = \int_0^{\pi} d\theta \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta}$$

$$= 60\int_0^{\pi} \sin^4\theta d\theta = 60 \times 2\int_0^{\pi/2} \sin^4\theta d\theta$$

$$= 120 \times \frac{3 \cdot \frac{1}{2} \cdot \frac{1}{2}\pi}{2 \cdot 2} = \frac{360\pi}{16} = 22.5\pi$$

by using Gamma fn formula $\int_0^{\pi/2} \sin^n\theta \cos^m\theta d\theta$

$$= \frac{\Gamma\frac{n+1}{2}\,\Gamma\frac{m+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

Change of order of integration:- In a double integral with variable limits, the change of order of integration changes the limits of integration. If we change the order of integration, the value of integral remain same.

Example ① By changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx\, dy$, Show that $\int_0^\infty \frac{\sin px}{x} dx = \pi/2$

Solution:- $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx\, dy = \int_0^\infty \left(\int_0^\infty e^{-xy} \sin px \, dx \right) dy$

$$= \int_0^\infty \left[-\frac{e^{-xy}}{p^2+y^2} (p\cos px + y \sin px) \right]_0^\infty dy$$

$$= \int_0^\infty \frac{p}{p^2+y^2} dy = \left[\tan^{-1}\left(\frac{y}{p}\right) \right]_0^\infty = \pi/2 \quad — ①$$

On changing order of integration, we have

$$\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx\, dy = \int_0^\infty \sin px \left\{ \int_0^\infty e^{-xy} dy \right\} dx$$

$$= \int_0^\infty \sin px \left[\frac{e^{-xy}}{-x} \right]_0^\infty dx = \int_0^\infty \frac{\sin px}{x} dx \quad — ②$$

Thus from ① and ②, we hence

$$\int_0^\infty \frac{\sin px}{x}\,dx = \pi/2.$$

Example ② Change the order of integration in the integral $I = \int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

Solution:- Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x=0$ to $x=\sqrt{a^2-y^2}$ (i.e, to the circle $x^2+y^2=a^2$ and this strip slides from $y=-a$ to $y=-a$. This shaded semi-circular area is, therefore the region of integration of given integral as shown fig 1.9.

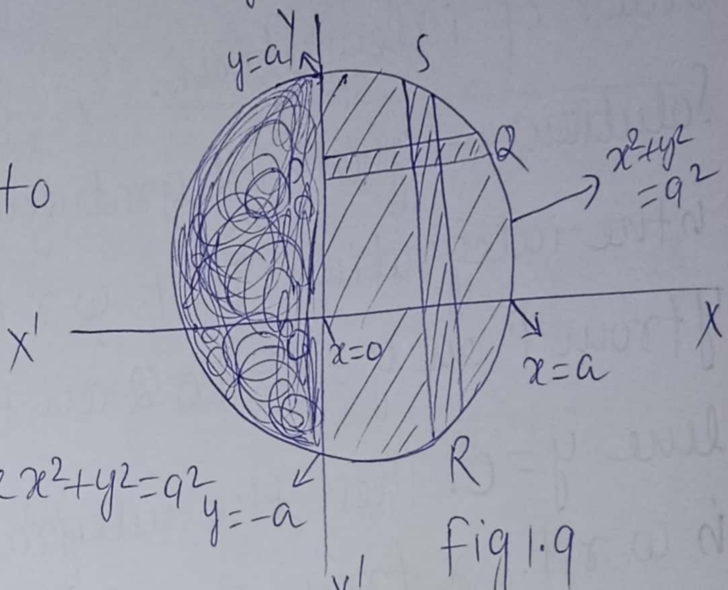


fig 1.9

On changing the order of integration, we first integrate w.r.t. y along a vertical strip RS which extends from R($y=-\sqrt{a^2-x^2}$ to S($y=\sqrt{a^2-x^2}$) To cover the given region, we then integrate w.r.t.

x from $x=0$ to $x=a$.

$$I=\int_0^a dx\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy=\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx.$$

Example ③ Evaluate $\int_0^1\int_{e^x}^{e}\frac{dy\,dx}{\log y}$ by changing the order of integration.

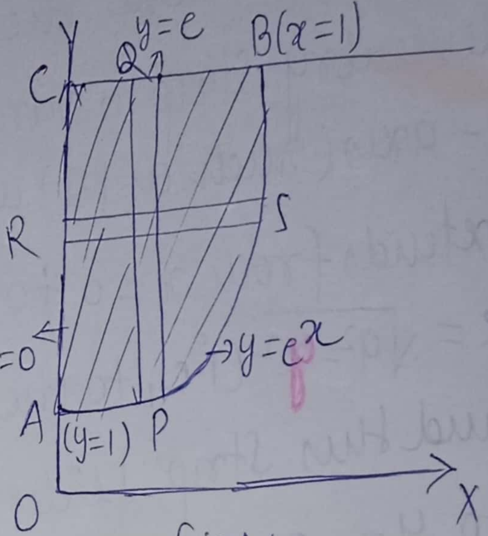


fig 1.10

Solution:— given integration is the integration first w.r.t y from P on $y=e^x$ to Q on the line $y=e$. Then the integration is w.r.t. x from $x=0$ to $x=1$, shows the shaded region ABC (fig 1.10).

On changing the order of integration, we first integrate w.r.t. x from R on $x=0$ to S on $x=\log y$ and then integrate w.r.t. y from $y=1$ to $y=e$.

Thus $\int_0^1\int_{e^x}^{e}\frac{dy\,dx}{\log y}=\int_1^e\int_0^{\log y}\frac{dx\,dy}{\log y}$

$$=\int_1^e\frac{1}{\log y}\Big[x\Big]_0^{\log y}dy=\int_1^e dy=e-1$$

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Example (4) Change the order of integration in $I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$ and hence evaluate

Solution:- given integration is first w.r.t. y and P on the Parabola $x^2 = 4ay$ to Q on the parabola $y^2 = 4ax$ and then integrate w.r.t. x from $x=0$ to $x=4a$

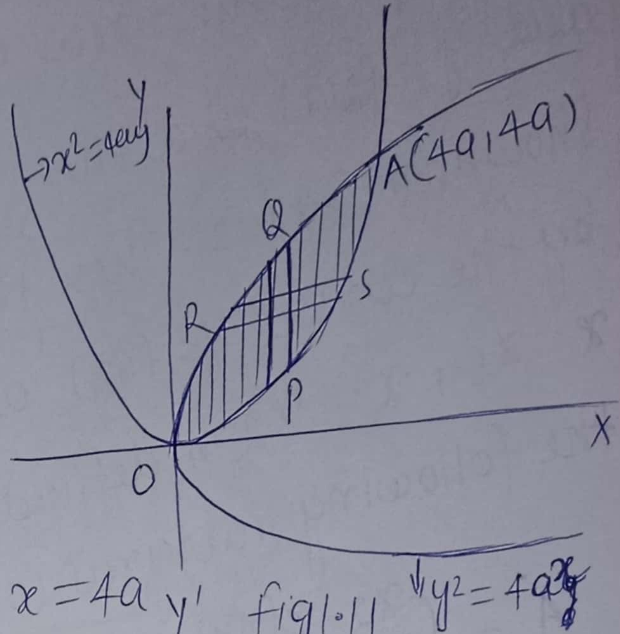

fig 1.11

Shows the shaded region of integration as fig 1.11.

On changing the order of integration, we first integrate w.r.t. x from R $(x = y^2/4a)$ to S $(x = 2\sqrt{ay})$, then w.r.t. y from $y=0$ to $y=4a$, therefore

$$I = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_0^{4a} dy\,\Big[x\Big]_{y^2/4a}^{2\sqrt{ay}} = \int_0^{4a}\left(2\sqrt{ay} - y^2/4a\right)dy$$

$$I = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy = \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

Area Enclosed by Planes curves:—

(1) Cartesian Coordinates:—

We defined the area enclosed by the curve $y=f_1(x)$ and $y=f_2(x)$ and the ordinates $x=x_1$, $x=x_2$ as shown in fig 1.12. Then the required area enclosed by the curve $y=f_1(x)$ and $y=f_2(x)$ and the ordinates $x=x_1$, $x=x_2$ is defined Mathematically by evaluate the following integral (or Area ABCD)

$$A = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(y)} dy = \int_{x_1}^{x_2}\int_{f_1(x)}^{f_2(x)} dy\,dx$$

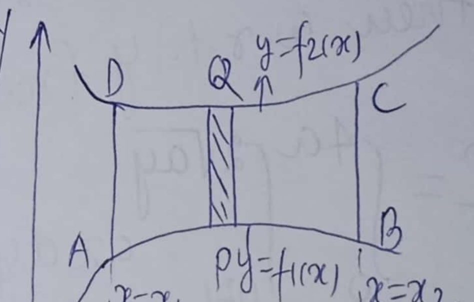

fig 1.12

Similarly, dividing the area A'B'C'D' as shown fig 1.13 into horizontal strip of width dy, we get the area A'B'C'D' enclosed by the curve $x=f_1(y)$ and $x=f_2(y)$ and the ordinates $y=y_1$, $y=y_2$ is defined as

$$\text{Area} = \int_{y_1}^{y_2}\int_{f_1(y)}^{f_2(y)} dx\,dy.$$

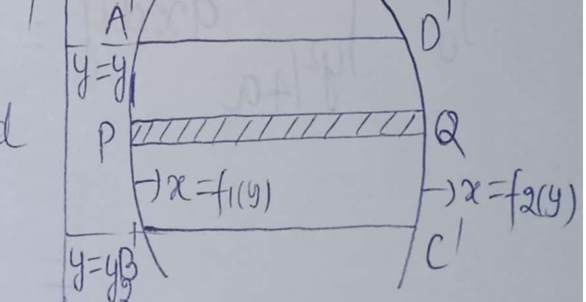

fig 1.13

unit-II

Polar Coordinates:- In Polar Coordinate system Area $A = \int_{\theta=\theta_1}^{\theta=\theta_2}\int_{r=r_1}^{r=r_2} r\,dr\,d\theta$

Example ① find the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution:- Dividing the area into vertical strips of width dx, y varies from $K(y=0)$ to L $\left(y=b\sqrt{1-x^2/a^2}\right)$ and then x from 0 to a (Shown as fig 1.4)

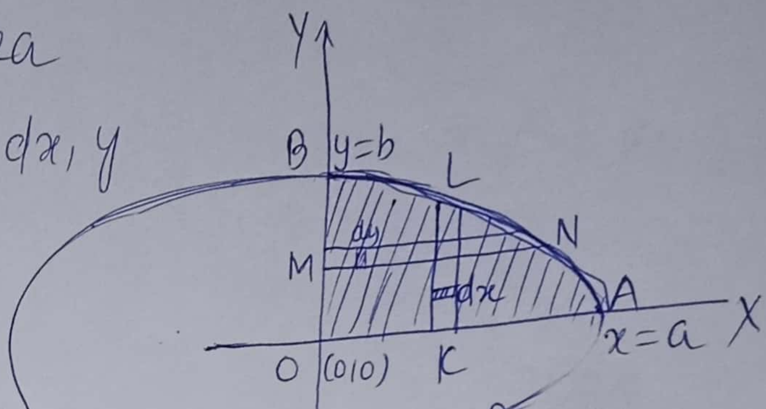

fig 1.4

therefore required Area

$$A=\int_0^a dx\int_0^{b\sqrt{1-x^2/a^2}} dy = \int_0^a dx\,\left[y\right]_0^{b\sqrt{1-x^2/a^2}}$$

$$= b/a\int_0^a \sqrt{a^2-x^2}\,dx$$

put $x=a\sin\theta$

$\Rightarrow dx = a\cos\theta\,d\theta$

limit are change when $x=0$ then $\theta=0$

when $x=a$ then $\theta=\pi/2$

$$A = b/a\int_0^{\pi/2} a^2\cos^2\theta\,d\theta = b/a\int_0^{\pi/2} a^2\frac{(1+\cos 2\theta)}{2}\,d\theta$$

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$$A = ab\int_0^{\pi/2} \frac{(1+\cos 2\theta)}{2}\, d\theta$$

$$A = \frac{ab}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{\pi ab}{4}$$

if we change the order of integration, dividing this area into horizontal strips of width dy, x varies from $M(x=0)$ to $N(x = a\sqrt{1-y^2/b^2}$ and then y varies from 0 to b.

therefore, the required area is

$$A = \int_0^b dy \int_0^{a\sqrt{1-y^2/b^2}} dx = \int_0^b dy\,[x]_0^{a\sqrt{1-y^2/b^2}}$$

$$A = a/b\int_0^b \sqrt{(b^2-y^2)}\,dy = \frac{\pi ab}{4}.$$

Note:- The change of the order of integration in any way affect the value of Area. does not

problem related to the Area enclosed by the plane curve in polar coordinate system.

Example ① Calculate the area included b/w the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.

Solution:- The curve is symmetrical about the initial line and has an asymptote $r = a\sec\theta$ (fig 1.15).

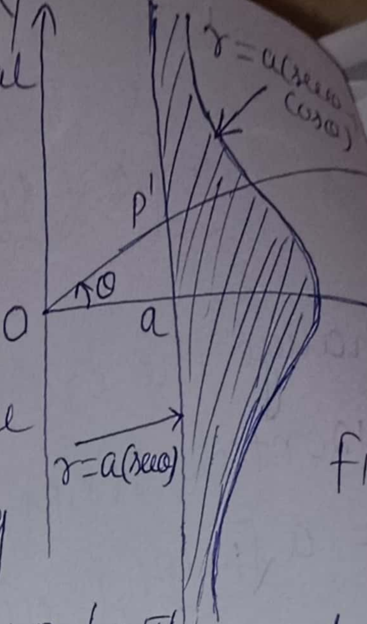


fig 1.15

Draw any line OP cutting the curve at P and its asymptote at P'. Along this line, θ is constant varies from 0 to π/2 and r varies from $a\sec\theta$ at P' to $a(\sec\theta + \cos\theta)$ at P. Then to get upper half of the area $A = \int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$, and the required Area

$$A = 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2}(2+\cos^2\theta)\,d\theta = \frac{5\pi a^2}{4}.$$

Example ② find the area lying inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$

Solution:- In fig 1.16, ABODA represented the cardioid $r = a(1+\cos\theta)$ and CBA'DC is the circle $r = a$.

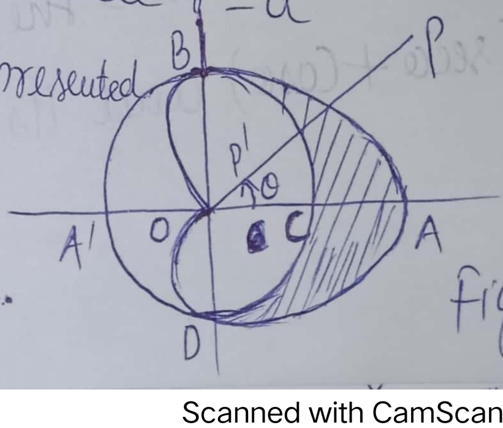


fig 1.16

Required Area = 2(Area ABCA)

$$= 2\int_0^{\pi/2}\int_{r=op'}^{r=op} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_{r=a}^{r=a(1+\cos\theta)} (r\,dr)\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$= \frac{2}{2}\int_0^{\pi/2} a^2\left[(1+\cos\theta)^2-1\right]d\theta$$

$$= a^2\int_0^{\pi/2}(\cos^2\theta+2\cos\theta)\,d\theta = \frac{a^2}{4}(\pi+8)$$

Triple integrals:-

evaluation of Triple integrals:- The triple integral of $f(x,y,z)$ over the region V and is denoted by

$$\iiint f(x,y,z)\,dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz$$

our purposes is, How to evaluate the above triple integral. The above triple integral evaluate in the following way. if x_1, x_2 are Constants; y_1, y_2 are either Constants or functions of x and z_1, z_2 are either Constants or function of x and y, then this integral is evaluated as follows:

first $f(x,y,z)$ is integrated w.r.t. z b/w the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.to y b/w the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.to x from x_1 to x_2. Thus

$$I = \int_{x_1}^{x_2}\left[\int_{y_1(x)}^{y_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\right]dy\right]dx$$

The order of integration may be different for different types of limits.

Example Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

Solution:- Integrating first w.r.t. y keeping x and z constant, we have

$$I = \int_{-1}^{1}\int_{0}^{z}\left[\int_{x-z}^{x+z}(x+y+z)\,dy\right]dx\,dz = \int_{-1}^{1}\int_{0}^{z}\left[xy+\frac{y^2}{2}+yz\right]_{x-z}^{x+z}dz\,dx$$

unit-II

$$I = \int_{-1}^{1}\int_{0}^{z}\left((x+z)(2z)+\frac{1}{2}4xz\right)dx\,dz$$

$$I = 2\int_{-1}^{1}\left[\frac{x^2 z}{2}+z^2x+\frac{x^2}{2}z\right]_0^z dz$$

$$I = 2\int_{-1}^{1}\left(\frac{z^3}{2}+z^3+\frac{z^3}{2}\right)dz = 4\left|\frac{z^4}{4}\right|_{-1}^{1} = 0$$

Example ② Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

Solution:-

$$I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz = \int_0^1 x\left[\int_0^{\sqrt{1-x^2}} y\left\{\int_0^{\sqrt{1-x^2-y^2}} z\,dz\right\}dy\right]dx$$

$$I = \int_0^1 x\left\{\int_0^{\sqrt{1-x^2}} y\cdot\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\right\}dx$$

$$I = \int_0^1 x\left\{\int_0^{\sqrt{1-x^2}} y\cdot\frac{1}{2}(1-x^2-y^2)\,dy\right\}dx$$

$$I = \frac{1}{2}\int_0^1 x\left[\frac{(1-x^2)y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$I = \frac{1}{8}\int_0^1 \left[(1-x^2)^2 \cdot 2x - (1-x^2)^4 \cdot x\right] dx$$

$$I = \frac{1}{8}\int_0^1 (x - 2x^3 + x^5)\, dx = \frac{1}{8}\left[\frac{x^2}{2} - \frac{2x^4}{4} +\right.$$

$$I = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{48}$$

Application of triple integral :—

Divide the given solid by planes parallel to the Coordinate planes into rectangular parallelopipeds of Volume $\delta x\,\delta y\,\delta z$ (fig 7.23)

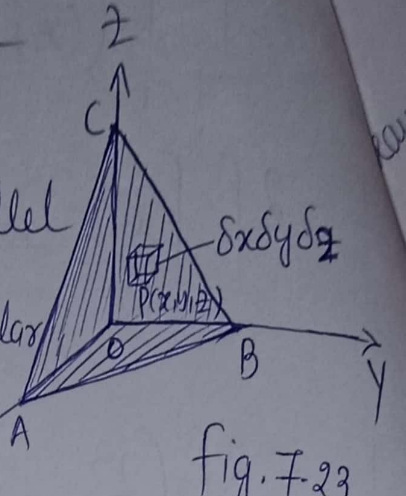


fig. 7.23

∴ the total volume $= \lim_{\substack{\delta x\to 0\\ \delta y\to 0\\ \delta z\to 0}} \sum\sum\sum \delta x\,\delta y\,\delta z$

$$= \iiint dx\,dy\,dz$$

with appropriate limits of integration.

Example ① Calculate the volume of the solid bd by the planes $x=0,\ y=0,\ x+y+z=a$ and $z=0$.

Solution:- Volume required $= \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\,dy\,dx$

$$= \int_0^a \int_0^{a-x} (a-x-y)\,dy\,dx = \int_0^a \left|(a-x)y - \frac{y^2}{2}\right|_0^{a-x} dx$$

$$= \int_0^a \left\{(a-x)^2 - \frac{(a-x)^2}{2}\right\} dx = \frac{1}{2}\int_0^a (a-x^2)^2\,dx = \frac{1}{2}\left|-\frac{(a-x)^3}{3}\right|_0^a$$

$= a^3/6$

Example ② find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solution:- Let OABC be the positive octant of the given ellipsoid, which is bd by the planes OAB$(z=0)$, OBC$(x=0)$, OCA$(y=0)$ and the surface ABC i.e $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

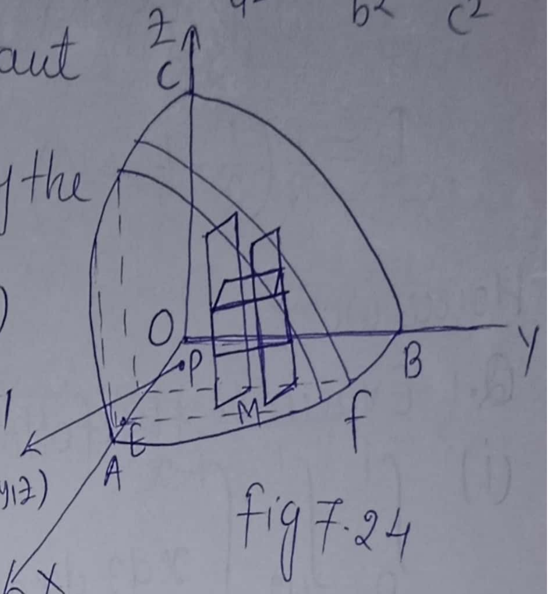


fig 7.24

divide this region R into rectangular parallelopipeds of volume $\delta x\delta y\delta z$. Consider such an element at $P(x,y,z)$. (fig 7.24).

In this region R,

(i) z varies from 0 to MN where

$$MN = c\sqrt{1-x^2/a^2-y^2/b^2}.$$

(ii) y varies from 0 to Ef, where $Ef = b\sqrt{1-\frac{x^2}{a^2}}$ from the eqn of the ellipse OAB, i.e, $x^2/a^2+y^2/b^2=1$

(iii) x varies from 0 to $OA=a$, hence the volume of the whole ellipsoid

$$= 8\int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dx\,dy\,dz = 8\int_0^a dx\int_0^{b\sqrt{1-x^2/a^2}} dy\,\Big|z\Big|_0^{c\sqrt{1-x^2/a^2-y^2/b^2}}$$

$$= 8c\int_0^a dx\int_0^{b\sqrt{1-x^2/a^2}}\sqrt{1-x^2/a^2-y^2/b^2}\,dy = \frac{8c}{b}\int_0^a dx\int_0^{\rho}\sqrt{\rho^2-y^2}\,dy \qquad \rho=b\sqrt{1-x^2/a^2}$$

$$= \frac{8c}{b}\int_0^a dx\left[\frac{y\sqrt{\rho^2-y^2}}{2}+\frac{\rho^2}{2}\sin^{-1}\frac{y}{\rho}\right]_0^{\rho} = \frac{8c}{b}\int_0^a \frac{b^2}{2}\left(1-\frac{x^2}{a^2}\right)\frac{\pi}{2}\,dx$$

$$= 2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx = 2\pi bc\left|x-\frac{x^3}{3a^2}\right|_0^a = 4\pi abc/3.$$

Home work

Q.1 Evaluate the following triple integral.

(i) $$\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\,dz\,dx\,dy$$

(ii) $$\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$$

(iii) $$\int_1^{e}\int_1^{\log y}\int_1^{e^x} \log z\,dz\,dx\,dy$$

Change of Variables

In this topic, we study the evaluation of a double or a triple integral, by changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

(i) Change of variable in a double integral :-

Let the variable x, y be changed to the new variables u, v by the transformation, $x = \phi(u,v)$, $y = \psi(u,v)$ where $\phi(u,v)$ and $\psi(u,v)$ are continuous and have continuous first order derivatives in some region R'_{uv} in the uv-plane which corresponds to the region R_{xy} in the xy-plane. Then

$$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R'_{uv}} f(\phi(u,v), \psi(u,v))\,|J|\,du\,dv \quad \text{—} \; (A)$$

Where $J = \dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$ is the Jacobian of transformation from (x,y) to (u,v) coordinates.

(ii) Change of variable for triple integrals :- the formula corresponding to (A) is

$$\iiint_{R_{xyz}} f(x,y,z)\,dx\,dy\,dz = \iiint_{R'_{uvw}} f[x(u,v,w), y(u,v,w), z(u,v,w)]\,|J|\,du\,dv\,dw \quad \text{—} \; (B)$$

where $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$ is the Jacobian of transformation from (x,y,z) to

(u, v, w) Coordinates.

Particular Cases:—

(i) To change Cartesian Coodinates (x, y) to polar Coordinates (r, θ), we have $x = r\cos\theta$, $y = r\sin\theta$ and Calculate

Jacobian $J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$

Therefore, $$\iint_{R_{xy}} f(x, y)\, dx\, dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta) \cdot r\, dr\, d\theta \quad \text{—— (C)}$$

(ii) To change rectangular Coordinates (x, y, z) to Cylindrical Coordinates (ρ, ϕ, z), put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and the value of Jacobian $J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$

Then $$\iiint_{R_{xyz}} f(x, y, z)\, dx\, dy\, dz = \iiint_{R'_{\rho\phi z}} f(\rho\cos\phi, \rho\sin\phi, z)\, \rho\, d\rho\, d\phi\, dz. \quad \text{—— (D)}$$

(iii) To change rectangular Coordinates (x, y, z) to Spherical polar Coordinates (r, θ, ϕ), put

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

and $J = \dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

Then $\displaystyle\iiint_{R_{xyz}} f(x,y,z)\,dx\,dy\,dz = \iiint_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\ r^2\sin\theta\,dr\,d\theta\,d\phi.$

(E)

Example ① Evaluate $\iint_R (x+y)^2\,dx\,dy$, where R is the parallelogram in the xy-plane with vertices $(1,0), (3,1), (2,2), (0,1)$ using the transformation $u = x+y$ and $v = x-2y$

Solution:- The region R, i.e., Parallelogram $ABCD$ in the xy-plane becomes the region R', i.e., rectangle $A'B'C'D'$ in the uv-plane as shown in fig 1.19, by taking

$$u = x+y \quad \text{and} \quad v = x-2y \qquad ①$$

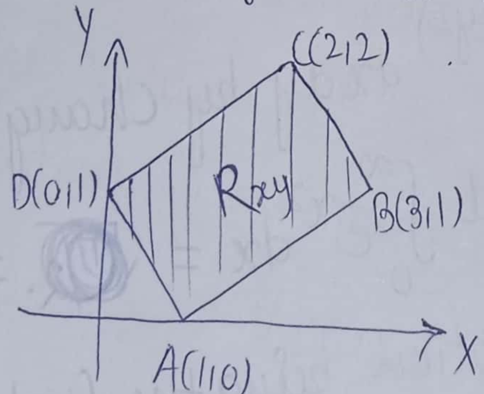
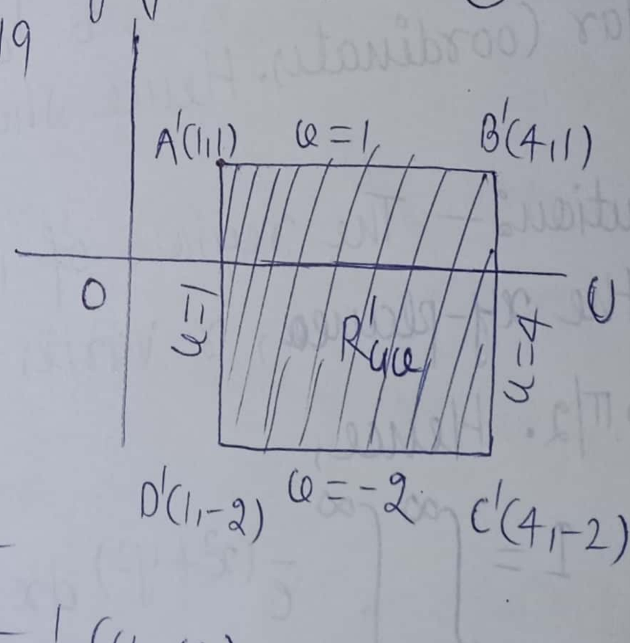


fig 1.19



Solve eqn ① for x, y, we have

$$x = \frac{1}{3}(2u+v), \quad y = \frac{1}{3}(u-v)$$

therefore, $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = -1/3$

using defn Ⓐ, Hence, the given integral

$$\iint_{R_{xy}} (x+y)^2\,dx\,dy = \iint_{R'_{uv}} u^2 |J|\,du\,dv = \int_1^4 \int_{-2}^{1} \frac{u^2}{3}\,du\,dv$$

$$= \frac{1}{3}\left[\frac{u^3}{3}\right]_1^4 \left[v\right]_{-2}^{1} = 21.$$

H.W

Example ② Evaluate $\iint_D xy\sqrt{1-x-y}\,dx\,dy$ where D is the region bd by $x=0$, $y=0$ and $x+y=1$ using the transformation $x+y=u$, $y=uv$.

Example ③ Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar Coordinates. Hence show that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

Solution:- The region of integration being the first quadrant of the xy-plane, r varies from 0 to ∞ and θ varies from 0 to $\pi/2$. Hence,

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2} \int_{r=0}^{r=\infty} e^{-r^2}\,r\,dr\,d\theta$$

$\theta=0_2, r=r_2$

$$I=-\frac{1}{2}\int_0^{\pi/2}\left\{\int_0^{\infty} e^{-r^2}(-2r)\,dr\right\}d\theta=-\frac{1}{2}\int_0^{\pi/2}\left|e^{-r^2}\right|_0^{\infty}d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}d\theta=\pi/4. \quad \text{——} (i)$$

Also, $$I=\int_0^{\infty}e^{-x^2}dx\times\int_0^{\infty}e^{-y^2}dy=\left\{\int_0^{\infty}e^{-x^2}dx\right\}^2 \quad \text{——} (ii)$$

Thus, from (i) and (ii), we have $\int_0^{\infty}e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

Example④ find, by triple integration, the volume of the sphere $x^2+y^2+z^2=a^2$.

Solution:- Changing Certisian to polar spherical coordinates by putting $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$ and we have $dx\,dy\,dz=r^2\sin\theta\,dr\,d\theta\,d\phi$

Also the volume of the sphere is 8 times the volume of its portion in the positive octant for which r varies from 0 to a, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$. therefore

Volume of the sphere $=8\int_0^{a}\int_0^{\pi/2}\int_0^{\pi/2}r^2\sin\theta\,dr\,d\theta\,d\phi$

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$$= 8\left[\frac{r^3}{3}\right]_0^a \left[-\cos\theta\right]_0^{\pi/2} \cdot \pi/2 = \frac{4\pi a^3}{3}$$

Example ⑤ find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ay$.

Solution:— The required volume is easily found by changing to cylindrical coordinates (ρ, ϕ, z). put $x=\rho\cos\phi$, $y=\rho\sin\phi$, $z=z$, and the value of Jacobian

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho.$$

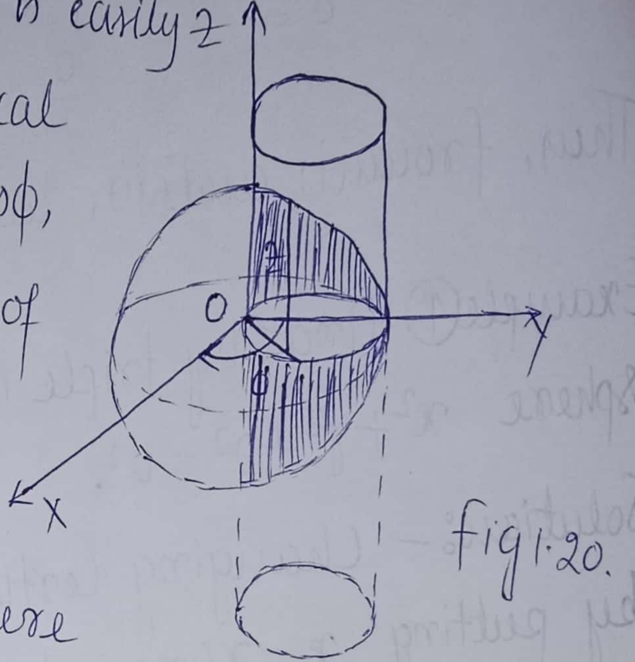

fig 1.20.

Then the equation of the sphere becomes $\rho^2+z^2=a^2$ and that of cylinder becomes $\rho = a\sin\phi$.

The volume inside the cylinder bd by the sphere is twice the volume shown shaded in the fig 1.20. for which z varies from 0 to $\sqrt{a^2-\rho^2}$, ρ varies from 0 to $a\sin\phi$ and ϕ varies from 0 to π. Hence the required volume

$$= 2\int_0^{\pi}\int_0^{a\sin\phi}\int_0^{\sqrt{a^2-\rho^2}} \rho\, dz\, d\rho\, d\phi$$

unit-II

$$= 2\int_0^{\pi}\int_0^{a\sin\phi} \rho\sqrt{a^2-\rho^2}\,d\rho\,d\phi = 2\int_0^{\pi}\left[-\frac{1}{3}(a^2-\rho^2)^{3/2}\right]_0^{a\sin\phi} d\phi$$

$$= \frac{2a^3}{3}\int_0^{\pi}(1-\cos^3\phi)\,d\phi = \frac{2a^3}{9}(3\pi-4).$$

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unit-1 last topic

Volume of Solids of revolution:—

Consider an elementary area $\delta x \delta y$ at the pt $P(x, y)$ of a plane area A as fig 1.25.

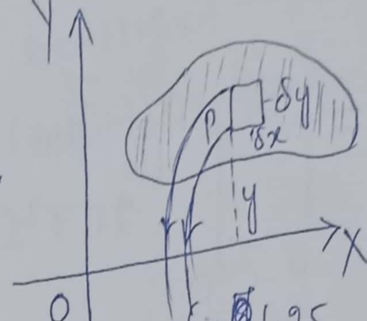

fig 1.25

As this elementary area revolves about x-axis, we get a ring of volume

$$= \pi\left((y+\delta y)^2 - y^2\right)\delta x = 2\pi y \delta x \delta y,$$ nearly to the first power of δy.

Hence the total volume of the solid formed by the revolution of the area A about x-axis

$$= \iint_A 2\pi y \, dx dy$$

In Polar Coordinates, the above formula for the volume becomes

$$\iint_A 2\pi r \sin\theta . r d\theta dr \text{ i.e } \iint_A 2\pi r^2 \sin\theta \, d\theta dr$$

Similarly, the volume of the solid formed by the revolution of the area A about y-axis $= \iint_A 2\pi x \, dx dy$.

Example (1) find the volume generated by the revolution of the cardioid $r = a(1-\cos\theta)$ about its axis.

Solution:- Required volume

$$= \int_0^{\pi}\int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

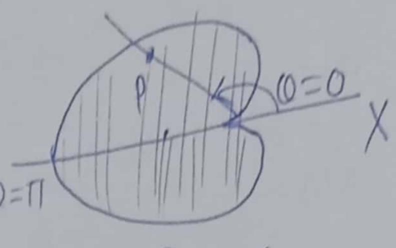

Fig 1.26

$$= 2\pi\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{a(1-\cos\theta)} \sin\theta \, d\theta = \frac{2\pi a^3}{3}\int_0^{\pi}(1-\cos\theta)^3 . \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3}\left|\frac{(1-\cos\theta)^4}{4}\right|_0^{\pi} = \frac{8\pi a^3}{3}.$$

Calculation of Mass :-

(a) for a plane lamina :- if the surface density at the pt $P(x,y)$ be $\rho = f(x,y)$ then the elementary mass at $P = \rho \delta x \delta y$

$\therefore$ total mass of the lamina $= \iint \rho \, dx dy$.

In Polar Coordinates, taking $\rho = \phi(r,\theta)$ at the pt $P(r,\theta)$,

$\therefore$ total mass of the lamina $= \iint \rho r \, d\theta \, dr$

(b) For a Solid :- If the density at the pt $P(x,y,z)$ be $\rho = f(x,y,z)$, then total mass of the solid

$$= \iiint \rho \, dx\, dy\, dz$$ with appropriate limits of integration.

Example ① Find the mass of the tetrahedron bd by the coordinate planes and the plane $\frac{x}{a} + y/b + z/c = 1$, the variable density $\rho = \mu xyz$

Soln :- Elementary mass at $P = \mu xyz\, \delta x\, \delta y\, \delta z$

$\therefore$ the whole mass $= \iiint \mu xyz\, dx\, dy\, dz$

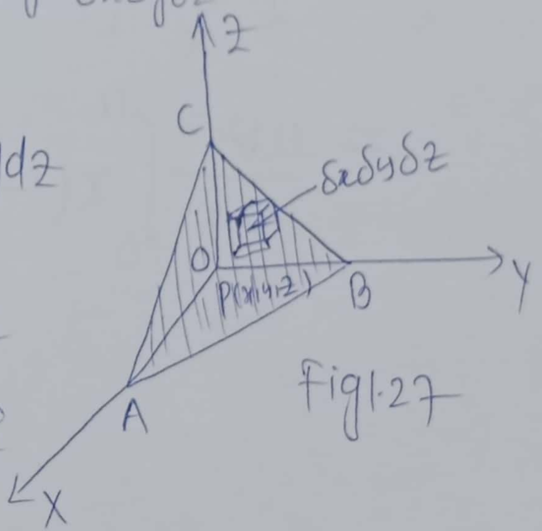

Fig 1.27

The integrals embracing the whole Volume OABC (fig 1.27). The limit of z are from 0 to $z = c(1 - x/a - y/b)$.

The limits for y are from 0 to $y = b(1 - x/a)$ and the limits for x are from 0 to a.

Hence the required Mass $= \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} \mu xyz \, dz\, dy\, dx$

$$= \mu \int_0^a \int_0^{b(1-x/a)} xy \left|\frac{z^2}{2}\right|_0^{c(1-x/a-y/b)} dy\,dz$$

$$= \mu \int_0^a \int_0^{b(1-x/a)} xy \cdot \frac{c^2}{2}(1-x/a-y/b)^2 dy\,dx$$

$$= \frac{\mu c^2}{2} \int_0^a \int_0^{b(1-x/a)} x \cdot \left[(1-x/a)^2 y - 2(1-x/a)\frac{y^2}{b} + \frac{y^3}{b^2}\right] dy\,dx$$

$$= \frac{\mu c^2}{2} \int_0^a x\left[(1-x/a)^2 \frac{y^2}{2} - 2(1-x/a)\frac{y^3}{3b} + \frac{y^4}{4b^2}\right]_0^{b(1-x/a)} dx$$

$$= \frac{\mu c^2}{2} \int_0^a x\left[\frac{b^2}{2}(1-x/a)^4 - \frac{2b^2}{3}\left(1-\frac{x}{a}\right)^4 + \frac{b^2}{4}(1-x/a)^4\right] dx$$

$$= \frac{\mu b^2 c^2}{24} \int_0^a x(1-x/a)^4 dx = \frac{\mu a^2 b^2 c^2}{720}.$$