Module – 4: Linear Programming – Revised Simplex Method

DUALITY IN LINEAR PROGRAMMING:

Associated with every LP problem is another problem called the ***dual problem***.The original LP is called the *primal problem*. Dual variables are related to the Lagrange multipliers of the primal constraints. The solution to the dual problem can be recovered from the final primal solution, and vice versa

Standard Primal LP Problem: There are several ways of defining the primal and the corresponding dual problems. We will define a standard primal problem as finding x1, x2, . . .,xnto maximize a primal objectivefunction:

$$z_p = d_1x_1 + \cdots + d_nx_n = \sum_{i=1}^{n} d_ix_i = \mathbf{d}^{\mathbf{T}}\,\mathbf{x} \qquad 4.1$$

Subject to the constraints –

$$a_{11}x_1 + \cdots + a_{1n}x_n \le e_1$$

$$\ldots\ldots \qquad 4.2$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n \le e_m$$

$$x_j \ge 0;\ \ j = 1\ to\ n$$

Ax ≤ e

We will use a subscript p on z to indicate the primal objective function. Also, z is used as the maximization function. It must be understood that in the standard LP problem defined in

Expanded Form of the Standard LP Problem

Minimize $f(x) = c_1x_1 + c_2x_2 + \ldots\ldots\ldots.. + c_nx_n$

Subject to the m independent equality constraints

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots. + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots. + a_{2n}x_n = b_2$

……. …… ……….. ………… ………… …….

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots. + a_{mn}x_n = b_m$

with$b_i \ge 0$, $i = 1$ to m, and non-negativity constraints on the variables

$x_j \ge 0$; $j = 1$ to n

all constraints were equalities and right-side parameters bi were non-negative. However, in the definition of the standard primal problem, all constraints must be "≤type" and there is no restriction on the sign of the right-side parameters ei. Therefore, "≥type" constraints must be multiplied by -1 to convert them to "≤type." Equalities should be also converted to "≤type" constraints.

Dual LP Problem:

The dual for the standard primal is defined as follows: Find the dual variables y1, y2, ...,ymto minimize a dual objective function:

$$f_d = e_1y_1 + \cdots + ey_m = \sum_{i=1}^{m} e_iy_i = \mathbf{e}^{\mathbf{T}}\,\mathbf{y} \qquad 4.3$$

Subject to the constraints –

$$a_{11}y_1 + \cdots + a_{m1}y_m \ge d_1$$

$$\ldots\ldots \qquad 4.4$$

$$a_{1n}y_1 + \cdots + a_{mn}y_m \ge d_n$$

$y_j \geq 0; \; i = 1 \; to \; m$

$\mathbf{A^T y \leq d}$

We use a subscript d on f to indicate that it is the cost function for the dual problem. Note the following relations between the primal and dual problems:

1. The number of dual variables is equal to the number of primal constraints. Each dual variable is associated with a primal constraint. For example, $y_i$is associated with the i^{th} primal constraint.
2. The number of dual constraints is equal to the number of primal variables. Each primal variable is associated with a dual constraint. For example, xi is associated with the i^{th} dual constraint.
3. Primal constraints are "≤ type" inequalities, whereas dual constraints are "≥ type."
4. The maximization of the primal objective function is replaced by the minimization of the dual cost function.
5. The coefficients di of the primal objective function become the right side of the dual constraints. The right-side parameters $e_i$of the primal constraints become coefficients for the dual cost function.
6. The coefficient matrix $[a_{ij}]$ of the primal constraints is transposed to $[a_{ji}]$ for the dual constraints.
7. The non-negativity condition applies to both primal and dual variables.

Treatment of Equality Constraints:

Many design problems have equality constraints. Each equality constraint can be replaced by a pair of inequalities. For example, $2x_1 + 3x_2 = 5$ can be replaced by the pair $2x_1 + 3x_2 \geq 5$ and $2x_1 + 3x_2 \leq 5$. We can multiply the "≥ type" inequality by -1 to convert it into the standard primal form.

EXAMPLE 9.3 THE DUAL OF AN LP PROBLEM WITH EQUALITY AND "≥ TYPE" CONSTRAINTS

Write the dual for the problem

Maximize

$$z_p = x_1 + 4x_2 \tag{a}$$

subject to

$$x_1 + 2x_2 \leq 5 \tag{b}$$

$$2x_1 + x_2 = 4 \tag{c}$$

$$x_1 - x_2 \geq 1 \tag{d}$$

$$x_1, x_2 \geq 0 \tag{e}$$

Solution

The equality constraint $2x_1 + x_2 = 4$ is equivalent to the two inequalities $2x_1 + x_2 \geq 4$ and $2x_1 + x_2 \leq 4$. The "≥ type" constraints are multiplied by −1 to convert them into the "≤" form. Thus, the standard primal for the given problem is

Maximize

$$z_p = x_1 + 4x_2 \tag{f}$$

subject to

$$x_1 + 2x_2 \leq 5 \tag{g}$$

$$2x_1 + x_2 \leq 4 \tag{h}$$

$$-2x_1 - x_2 \leq -4 \tag{i}$$

$$-x_1 + x_2 \leq -1 \tag{j}$$

$$x_1, \; x_2 \geq 0 \tag{k}$$

Dual of the given primal is –

Minimize

$$f_d = 5y_1 + 4(y_2 - y_3) - y_4 \qquad \text{(l)}$$

subject to

$$y_1 + 2(y_2 - y_3) - y_4 \geq 1 \qquad \text{(m)}$$

$$2y_1 + (y_2 - y_3) + y_4 \geq 4 \qquad \text{(n)}$$

$$y_1,\ y_2,\ y_3,\ y_4 \geq 0 \qquad \text{(o)}$$

Alternate Treatment of Equality Constraints: We will show that it is not necessary to replace an equality constraint by a pair of inequalities to write the dual problem. Note that there are four dual variables for above Example. The variables y_2 and y_3 correspond to the second and third primal constraints written in the standard form. The second and third constraints are actually equivalent to the original equality constraint. Note also that the term (y_2-y_3) appears in all of the expressions of the dual problem.
We define –

$$y_5 = y_2 - y_3$$

Which can be positive, negative, or zero, since it is the difference of two non-negative variables ($y_2 \geq 0$, $y_3 \geq 0$). Substituting for y_5, the dual problem in above Example is rewritten as –

Minimize

$f_d = 5y_1 + 4y_5 - y_4$

subject to

$y_1 + 2y_5 - y_4 \geq 1$

$2y_1 + y_5 - y_4 \geq 4$

$y_1;\ y_4 \geq 0;\ y_5 = y_2 - y_3$ is unrestricted in sign

The number of dual variables is now only three. Since the number of dual variables is equal to the number of the original primal constraints, the dual variable y_5 must be associated with the equality constraint $2x_1 + x_2 = 4$. Thus, we can draw the following conclusion: If the i^{th} primal constraint is left as an equality, the i^{th} dual variable is unrestricted in sign. In a similar manner, we can show that if a primal variable is unrestricted in sign, then the i^{th} dual constraint is an equality.

Determination of the Primal Solution from the Dual Solution

It remains to be determined how the optimum solution to the primal is obtained from the optimum solution to the dual, or vice versa. First, let us multiply each dual inequality in Eq. (4.4) by x1, x2, . . .,xnand add them. Since x_j'sare restricted to be non-negative, we get the inequality

$$x_1(a_{11}y_1 + \ldots + a_{m1}y_m) + x_2(a_{12}y_1 + \ldots + a_{m2}y_m) + \ldots + x_n(a_{1n}y_1 + \ldots + a_{mn}y_m) \geq d_1x_1 + d_2x_2 + \ldots + d_nx_n \qquad 4.5$$

Or in matrix form –

$$x^TA^Ty \geq x^Td$$

Rearranging the equation by collecting terms with $y_1, y_2, \ldots y_m$(or taking the transpose of the left side as y^TAx), we obtain –

$$y_1(a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n) + y_2(a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n) + \ldots + y_m(a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n) \geq d_1x_1 + d_2x_2 + \ldots + d_nx_n \qquad 4.6$$

In the matrix form, the preceding inequality can be written as $\mathbf{y}^T\mathbf{A}\mathbf{x} \geq \mathbf{x}^T\mathbf{d}$. Each term inparentheses in Eq. (4.6) is less than the corresponding value of e on the right side ofInequalities (4.2). Therefore, replacing these terms with the corresponding e fromInequalities (4.2) preserves the inequality in Eq. (4.6):

$$y_1e_1 + y_2e_2 + \ldots + y_me_m \geq d_1x_1 + d_2x_2 + \ldots + d_nx_n; \text{ or } \mathbf{y}^T\mathbf{e} \geq \mathbf{x}^T\mathbf{d} \qquad 4.7$$

Note that in Inequality (4.7) the left side is the dual cost function and the right side is the primal objective function.

Theorem 4.1

The Relationship between Primal and Dual Problems Let x and y be in the feasible sets of the primal and dual problems, respectively (as defined in Eqs. (4.1) through (4.4)). Then the following conditions hold:

1. $f_d(y) \geq z_p(x)$.
2. If $f_d = z_p$, then x and y are the solutions to the primal and the dual problems, respectively.
3. If the primal is unbounded, the corresponding dual is infeasible, and vice versa.
4. If the primal is feasible and the dual is infeasible, then the primal is unbounded, and vice versa.

Theorem 4.2

Primal and Dual Solutions Let both the primal and the dual have feasible points. Then both have optimum solutions in x and y, respectively, and $f_d(y) = z_p(x)$.

Theorem 4.3

Solution to Primal From Dual: If the i^{th} dual constraint is a strict inequality at optimum, then the corresponding i^{th} primal variable is nonbasic (i.e., it vanishes). Also, if the i^{th} dual variable is basic, then the i^{th} primal constraint is satisfied at equality.