Module No. 3: Linear Programming – Simplex Method

An optimum design problem having linear cost and constraint functions in the design variables is called a linear programming (LP) problem.

Linear programming problems arise in some fields of engineering such as water resources, systems engineering, traffic flow control, resources management, and transportation engineering. In the areas of aerospace, automotive, structural, or mechanical system design, most problems are not linear.

However, one way of solving nonlinear programming (NLP) problems is to transform them into a sequence of linear programs

In addition, some NLP methods solve an LP problem during their iterative solution processes. Thus, linear programming methods are useful in many applications and must be clearly understood.

Standard LP Definition: Linear programming problems may have equality as well as inequality constraints. Also, many problems require maximization of a function, whereas others require minimization. Although the standard LP problem can be defined and treated in several different ways, here we define it as minimization of a cost function with equality constraints and non-negativity of design variables. This definition will be used to describe the method (the Simplex method) to solve LP problems. The form is not as restrictive as it may appear since all other LP problems can be transcribed into it. We will explain the process of transcribing a given LP problem into the standard form.

LINEAR FUNCTIONS

Cost Function

Any linear function f(x) of k variables x, such as the cost function, has only first-degree terms and is written in the expanded, summation or matrix form as:

$$f(x) = c_1x_1 + c_2x_2 + \dots\dots\dots + c_kx_k = \sum_{i=1}^{k} c_ix_i = \mathbf{c}^T\mathbf{x} \qquad \text{(A)}$$

Where c_i, $i = 1$ to k are constraints

Constraints

All functions of an LP problem can be represented in the form of Eq. (A). However, when there are multiple linear functions, the constants ci must be represented by double subscripts rather than just one subscript.

We will use the symbols a_{ij} to represent constants in the constraint expressions. The i^{th} linear constraint involving k design variables, x_j, $j = 1$ to k has one of the following three possible forms, "≤," "=," or "≥" (written in expanded or summation notation):

$$a_{i1}x_1 + \dots\dots + a_{ik}\, x_k \le b_i \quad \text{or} \sum_{j=1}^{k} a_{ij}x_j \le b_i$$

$a_{i1}x_1 + \ldots\ldots + a_{ik}\ x_k = b_i$ or $\sum_{j=1}^{k} a_{ij}x_j = b_i$

$a_{i1}x_1 + \ldots\ldots + a_{ik}\ x_k \geq b_i$ or $\sum_{j=1}^{k} a_{ij}x_j \geq b_i$

where a_{ij} and b_i are known constants. The right sides b_i of the constraints are sometimes called the resource limits and assumed to be always nonnegative *i.e.* $b_i \geq 0$

b_i's can always be made nonnegative by multiplying both sides of above equations by *-1* if necessary.

Note that, multiplication by *-1* changes the sense of the origin inequality. i.e. "≤ type" becomes "≥ type" and vice-versa.

Standard LP Problem

For notational clarity, let x represent an n-vector consisting of the original design variables and any additional variables used to transcribe the problem into the standard form.

The standard LP problem is defined as finding the variables xi, I = 1 to n to –

Minimize $f(x) = c_1x_1 + c_2x_2 + \ldots\ldots\ldots.. + c_nx_n$

Subject to the m independent equality constraints

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n = b_2$

……. …… ……….. ………… ………… …….

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n = b_m$

with $b_i \geq 0$, $i = 1$ to m, and non-negativity constraints on the variables

$x_j \geq 0$; $j = 1$ to n

The quantities $b_i \geq 0$, c_j and aij ($i = 1$ to m and $j = 1$ to n) are known constants and m and n are positive integers.

Summation Form of the Standard LP Problem

The standard LP problem can also be written in the summation notation as finding the variables xi, I = 1 to n to –

Minimize

$$f = \sum_{i=1}^{n} c_i x_i$$

Subject to *m* independent equality constraints –

$$\sum_{j=1}^{n} a_{ij}x_j = b_i;\ b_i \geq 0,\ i = 1\ to\ m$$

And the non-negativity constraints –

$$x_j \geq 0; j = 1\ to\ n$$

Matrix Form of the Standard LP Problem

Matrix notation may also be used to define the standard LP problem as finding the n-vector x to –

Minimize

$$f = \mathbf{c}^T \boldsymbol{x}$$

Subject to the constraints –

$\mathbf{Ax} = \mathbf{b}$; $\mathbf{b} \geq 0$

$X \geq 0$

where $A=[a_{ij}]$ is an $m \times n$ matrix, $\boldsymbol{c}$ and $\boldsymbol{x}$ are n-vectors, and $\boldsymbol{b}$ is an m-vector

Transcription to Standard LP

The formulations given in Eqs. Of previous topic are more general than what may appear at first sight because all LP problems can be transcribed into them. "≤ type" and "≥ type" inequalities can be converted to equalities using slack and surplus variables. Unrestricted variables can be decomposed into the difference of two non-negative variables.

Maximization of functions can also be routinely treated. These transformations are explained in the following paragraphs.

Non-Negative Constraint Limits

The resource limits (right side of constraints) in the standard LP are assumed to be always non-negative (i.e., bi ≥ 0).

If any b_i is negative, it can be made non-negative by multiplying both sides of the constraint by *-1*. Note, however, that multiplication by *-1* changes the sense of the original inequality: "≤ type" becomes "≥ type" and vice versa.

For example, a constraint $x_1 + 2x_2 \leq -2$ must be transformed as $-x_1 - 2x_2 \geq 2$ to have a non-negative right side.

Treatment of Inequalities

Since only equality constraints are treated in standard linear programming, the inequalities given in following Eqs.

$a_{i1}x_1 + \ldots\ldots + a_{ik}\ x_k \leq b_i$ **and** **(A)**

$a_{i1}x_1 + \ldots\ldots + a_{ik}\ x_k \geq b_i$ (B)

must be converted to equalities. Since any inequality can be converted to equality by introducing a non-negative slack or surplus variable.

Treatment of "≤ Type" Constraints

For the i^{th} "≤ type" constraint in Eq. (A) with a non-negative right side, we introduce a non-negative slack variable $s_i \geq 0$ and convert it to an equality as –

$a_{i1}x_1 + \ldots\ldots + a_{ik}\ x_k + s_i = b_i$**;** $b_i \geq 0$; $s_i \geq 0$

Treatment of "≥ Type" Constraint

Similarly, the ith "≥ type" constraint in Eq. (B) with a non-negative right side is converted to equality by subtracting a non-negative surplus variable si ≥ 0, as –

$a_{i1}x_1 + \ldots\ldots + a_{ik}\ x_k - s_i = b_i$**;** $b_i \geq 0$; $s_i \geq 0$

The slack and surplus variables are additional unknowns that must be determined as a part of the solution for the LP problem. At the optimum point, if the slack or surplus variable si is positive, the corresponding constraint is inactive; if si is zero, it is active.

Unrestricted Variables: In addition to the equality constraints we require all design variables to be nonnegative in the standard LP problem –

i.e. $x_i \geq 0$; $i = 1$ to k If a design variable xj is unrestricted in sign, it can always be written as the difference of two non-negative variables: -

$$x_j = x_j^+ - x_j^- \ ;\ x_j^+ \geq 0,\ x_j^- \geq 0$$

This decomposition is substituted into all equations, and x_j^+ and x_j^- are treated as unknowns in the problem.

Splitting each free variable into its positive and negative parts increases the dimension of the design variable vector by one.

Maximization of a Function

Maximization of functions can be treated routinely. For example, if the objective is to maximize a function, we simply minimize its negative. That is

Maximize –

$z = (d_1x_1 + d_2x_2 + \ldots + d_nx_n) \Leftrightarrow$ minimize $f = -(d_1x_1 + d_2x_2 + \ldots + d_nx_n)$

Note that a function that is to be maximized is denoted as z.

BASIC CONCEPTS RELATED TO LP PROBLEMS

It is shown that the optimum solution for the LP problem always lies on the boundary of the feasible set. In addition, the solution is at least at one of the vertices of the convex feasible set (called the *convex polyhedral set*).

Basic Concepts:

Convexity of LP: Since all functions are linear in an LP problem, the feasible set defined by linear equalities or inequalities is *convex*. Also, the cost function is linear, so it is convex. Therefore, the LP problem is convex, and if an *optimum solution* exists, it is a *global optimum*, as stated in Theorem.

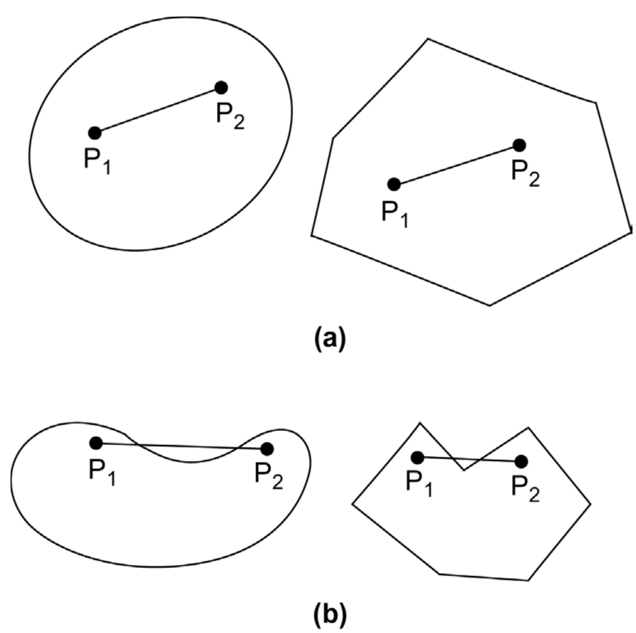
GLOBAL OPTIMALITY

In the optimum design of systems, the question of the global optimality of a solution always arises. In general, it is difficult to answer the question satisfactorily.

1. If the cost function $f(\mathbf{x})$ is continuous on a closed and bounded feasible set, then the Weierstrass Theorem guarantees the existence of a global minimum. Therefore, if we calculate all the local minimum points for the function, the point that gives the least value to the cost function can be selected as a global minimum for the function. This is called *exhaustive search* of the feasible design space.

2. If the optimization problem can be shown to be convex, then any local minimum is also a global minimum. Also the KKT necessary conditions are necessary as well as sufficient for the minimum point

Convex Sets



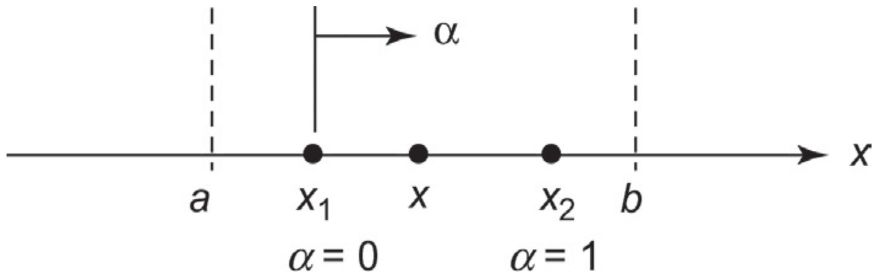
(a) Convex sets. (b) N onconvex sets.

A convex set S is a collection of points (vectors $\mathbf{x}$) having the following property: if $P1$ and

$P2$ are any points in S, then the entire line segment $P1$–$P2$ is also in S.

This is a necessary and sufficient condition for convexity of the set S. Fig. shows some examples of convex and non-convex sets.

Line Segment:



Convex interval between *a* and *b* on a line.

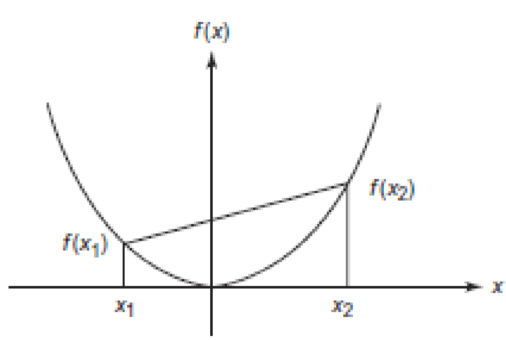
To explain convex sets further, let us consider points on a real line along the x-axis (Fig.). Points in any interval on the line represent a convex set. Consider an interval between points a and b as shown in Fig. To show that it is a convex set, let x_1 and x_2 be two points in the interval. The *line segment* between the points can be written as

$$x = \alpha x_2 + (1 - \alpha)x_1; \; 0 \leq \alpha \leq 1$$

In this equation, $a = 0$ gives $x = x_1$; and $a = 1$ gives $x = x_2$. It is clear that the line defined in above Equation is in the interval $[a, b]$. The entire line segment is on the line between a and b. Therefore, the set of points between a and b is a convex set.

Convex Functions

Consider a function of a single variable $f(x)=x^2$. A graph of the function is shown in Figure. If a straight line is constructed between any two points $(x_1,f(x_1))$ and $(x_2,f(x_2))$ on the curve, the line lies above the graph of $f(x)$ at all points between x_1 and x_2. This property characterizes convex functions.



Characterization of a convex function.

Using the geometry, the foregoing definition of a convex function can be expressed by the inequality -

$f(x) \leq \alpha f(x_2) + (1 - \alpha) f(x_1)$

Since $x = \alpha x_2 + (1 - \alpha)x_1$, the above inequality becomes –

$f(\alpha x_2 + (1 - \alpha) x_1 \leq \alpha f(x_2 + (1 - \alpha)f(x_1)$ for $0 \leq \alpha \leq 1$ (A)

The foregoing definition of a convex function of one variable can be generalized to functions of n variables. A function f(x) defined on a convex set S is convex if it satisfies the inequality –

$f(\alpha x^{(2)} + (1 - \alpha)x^{(1)} \leq \alpha f(x^{(2)} + (1 - \alpha)f(x^{(1)})$ for $0 \leq \alpha \leq 1$ (B)

for any two points $x^{(1)}$ and $x^{(2)}$ in S. Note that convex set S is a region in the n-dimensional space satisfying the convexity condition. Equations (A) and (B) give necessary and sufficient conditions for the convexity of a function.

Check for the Convexity of a Function:

A function of n variables $f(x_1, x_2, . . ., x_n)$ defined on a convex set S is convex if and only if the Hessian matrix of the function is positive semidefinite or positive definite at all points in the set *S*. If the Hessian matrix is positive definite for all points in the feasible set, then *f* is called a strictly convex function.

LP Solution on the Boundary of the Feasible Set

It is important to note that even when there are inequality constraints in an LP design problem, the optimum solution, if it exists, always lies on the boundary of the feasible set; that is, some constraints are always active at the optimum. This can be seen by writing the necessary conditions of Theorem for an unconstrained optimum. These conditions, $\partial f / \partial xi = 0$, when used for the cost function of Eq. Minimize $f = \sum_{i=1}^{n} c_i x_i$, give ci = 0 for i = 1 to n. This is not possible, as all ci are not zero. If all ci were zero, there would be no cost function.

Therefore, by contradiction, the optimum solution for any LP problem must lie on the boundary of the feasible set.

This is in contrast to general nonlinear problems, where the optimum point can be inside or on the boundary of the feasible set.

Infinite Roots of Ax = b

An optimum solution to the LP problem must also satisfy the equality constraints in

Eq. Minimize $f(x) = c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots.. + c_n x_n$

Subject to the *m* independent equality constraints

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n = b_2$

$\dots\dots \quad \dots\dots \quad \dots\dots\dots \quad \dots\dots\dots \quad \dots\dots\dots \quad \dots\dots$ *(C)*

$a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots\dots + a_{mn}x_n = b_m$

with $b_i \geq 0$, $i = 1$ to m, and non-negativity constraints on the variables

$x_j \geq 0$; $j = 1$ to n *(D)*

Only then can the solution be feasible for the problem. Therefore, to have a meaningful optimum design problem, above Eq. must have more than one solution. Only then there is a choice of feasible solutions that can have minimum cost. To have many solutions, the number of linearly independent equations in above Eq. must be less than n, the number of variables in the LP problem.

LP Terminology

- **Vertex (Extreme) Point**. This is a point of the feasible set that does not lie on a line segment joining two other points of the set. For example, every point on the circumference of a circle and each vertex of the polygon satisfy the requirements for an extreme point.
- **Feasible Solution**. Any solution of the constraint Eq. (C) satisfying the non-negativity conditions of Eq. $x_j \geq 0$; $j = 1$ *to* n is a feasible solution.
- **Basic Solution**. A basic solution is a solution of the constraint Eq. (C) obtained by setting the "redundant number" (n - m) of the variables to zero and solving the equations simultaneously for the remaining variables
- **Nonbasic Variables**. The variables set to zero in the basic solution are called nonbasic.
- **Basic Variables**. The variables that are not set to zero in the basic solution are called basic.
- **Basic Feasible Solution**. A basic solution satisfying the non-negativity conditions on the variables in Eq. (D) is called a basic feasible solution.
- **Degenerate Basic Solution**. If a basic variable has a zero value in a basic solution, the solution is a degenerate basic solution.
- **Degenerate Basic Feasible Solution**. If a basic variable has a zero value in a basic feasible solution, the solution is a degenerate basic feasible solution.
- **Optimum Solution**. A feasible solution minimizing the cost function is an optimum solution.
- **Optimum Basic Solution**. The optimum basic solution is a basic feasible solution that has an optimum cost function value.
- **Convex Polyhedron**. If the feasible set for an LP problem is bounded, it is a convex polyhedron.

Optimum Solution to LP Problems

- **Theorem 1**: Extreme Points and the Basic Feasible Solutions The collection of feasible solutions for an LP problem constitutes a convex set whose extreme points correspond to basic feasible solutions. This theorem relates extreme points of the convex polyhedron to the basic feasible solutions.
- **Theorem 2**: The Basic Theorem of Linear Programming This theorem establishes the importance of the basic feasible solutions.
- Let the m × n coefficient matrix A of the constraint equations have full row rank (i.e., rank (A) = m). Then –

  1. If there is a feasible solution, there is a basic feasible solution.

  2. If there is an optimum feasible solution, there is an optimum basic feasible solution.

Number of Basic Solutions

- As noted earlier, the LP problem has infinite feasible designs. We seek a feasible design that minimizes the cost function. Theorem 2 says that such a solution must be one of the basic feasible solutions, that is, at one of the extreme points of the convex feasible set.
- Thus, our task of solving an LP problem is reduced to the search for an optimum only among the basic feasible solutions. For a problem having *n* variables and m constraints, the maximum number of basic solutions is obtained by counting the total number of combinations where *m* variables are nonzero out of a total of *n* variables. This number is given by the formula:
- $\#\ of\ basic\ solutions = \binom{n}{m} = \frac{n!}{m!(n-m)!}$ This formula gives a finite number of basic solutions. Thus, according to Theorem 2, the optimum solution is at one of these basic solutions that is also feasible.

CALCULATION OF BASIC SOLUTIONS

In the last section, we observed the importance of basic solutions to the linear equations **Ax=b**; the optimum solution for the LP problem is at least one of the basic feasible solutions.

Therefore, it is important to generate basic solutions for the problem in a systematic way such a method, called the Gauss-Jordan elimination method.

The Simplex method, described in the next section, uses this procedure to search for the optimum solution among the basic feasible solutions to **Ax=b**.

The Tableau

It is customary to represent the linear system **Ax=b** in a tableau. A tableau is defined as the representation of a scene or a picture. It is a convenient way of representing all necessary information related to an LP problem.

In the Simplex method, the tableau consists of coefficients of the variables in the cost and constraint functions.

The Pivot Step

In the Simplex method, we want to systematically search among the basic feasible solutions for the optimum design. We must have a basic feasible solution to initiate the Simplex method. Starting from the basic feasible solution, we want to find another one that decreases the cost function. This can be done by interchanging a current basic variable with a nonbasic variable. That is, a current basic variable is made nonbasic (i.e., reduced to 0 from a positive value), and a current nonbasic variable is made basic (i.e., increased from 0 to a positive value).

The pivot step of the Gauss-Jordan elimination method accomplishes this task and results in a new canonical form (general solution).

Basic Solutions to Ax=b

Using the Gauss-Jordan elimination method, we can systematically generate all of the basic solutions for an LP problem. Then, evaluating the cost function for the basic feasible solutions, we can determine the optimum solution for the problem.

The Simplex method uses this approach with one exception: It searches through only the basic feasible solutions and stops once an optimum solution is obtained.

The Simplex

A Simplex in two-dimensional space is formed by any three points that do not lie on a straight line. In three-dimensional space, it is formed by four points that do not lie in the same plane. Three points can lie in a plane, but the fourth has to lie outside. In general, a Simplex in the n-dimensional space is a convex hull of any ($n+1$) points that do not lie on one hyperplane. A convex hull of ($n+1$) points is the smallest convex set containing all of the points. Thus, the Simplex represents a convex set.

THE SIMPLEX METHOD

Basic Steps in the Simplex Method:

The basics of the Simplex method for solving LP problems are described in this section. The ideas of canonical form, pivot row, pivot column, pivot element, and pivot step, which were introduced in the previous section, are used.

The method is described as an extension of the standard Gauss-Jordan elimination procedure for solving a system of linear equations **Ax=b**, where A is an $m \times n$ ($m < n$) matrix, $\boldsymbol{x}$ is an n-vector, and **b** is an m-vector. In this section, the Simplex method is developed and illustrated for "≤ type" constraints, since with such constraints, the method can be developed in a straightforward manner.

Theorem 2 guarantees that one of the basic feasible solutions is an optimum solution for the LP problem. The basic idea of the Simplex method is to proceed from one basic feasible solution to another in a way that continually decreases the cost function until the minimum is reached. The method never calculates basic infeasible solutions. The Gauss-Jordan elimination procedure, described in the previous section, is used to systematically find basic feasible solutions of the linear system of equations **Ax** = **b** until the optimum is reached.

The method starts with a basic feasible solution (i.e., at a vertex of the convex feasible set). A move is then made to an adjacent vertex while maintaining the feasibility of the new solution (i.e., all $x_i \geq 0$) as well as reducing the cost function. This is accomplished by replacing a basic variable with a nonbasic variable in the current basic feasible solution. Two basic questions now arise:

1. How do we choose a current nonbasic variable that should become basic?

2. Which variable from the current basic set should become nonbasic?

The Simplex method answers these questions based on theoretical considerations.

The Simplex Algorithm

The steps of the Simplex method with only "≤ type" constraints. They are summarized for the general LP problem as follows:

- **Step 1**. Problem in the standard form. Transcribe the problem into the standard LP form.
- **Step 2**. Initial basic feasible solution. This is readily available if all constraints are "≤ type" because the slack variables are basic and the real variables are nonbasic. If there are equality and/or "≥ type" constraints, then the two-phase Simplex procedure must be used. Introduction of artificial variable for each equality and "≥ type" constraint gives an initial basic feasible solution to the Phase I problem.
- **Step 3**. Optimality check: The cost function must be in terms of only the nonbasic variables. This is readily available when there are only "≤ type" constraints. For equality and/or "≥ type" constraints, the artificial cost function for the Phase I problem can also be easily transformed to be in terms of the nonbasic variables.
- If all of the reduced cost coefficients for nonbasic variables are non-negative (≥ 0), we have the optimum solution (end of Phase I). Otherwise, there is a possibility of improving the cost function (artificial cost function). We need to select a nonbasic variable that should become basic.
- **Step 4**. Selection of a nonbasic variable to become basic. We scan the cost row (the artificial cost row for the Phase I problem) and identify a column having negative reduced cost coefficient because the nonbasic variable associated with this column should become basic to reduce the cost (artificial cost) function from its current value. This is called the pivot column.
- **Step 5**. Selection of a basic variable to become nonbasic. If all elements in the pivot column are negative, then we have an unbounded problem. If there are positive elements in the pivot column, then we take ratios of the right-side parameters with the positive elements in the pivot column and identify a row with the smallest positive ratio according to Eq. –

- $\frac{b_p}{a_{pq}} = \min_i \left\{ \frac{b_i}{a_{iq}},\ a_{iq} > 0; i = 1\ to\ m \right\}$ (A)
- In the case of a tie, any row among the tying ratios can be selected. The basic variable associated with this row should become nonbasic (i.e., zero). The selected row is called the pivot row, and its intersection with the pivot column identifies the pivot element.
- **Step 6**. Pivot step. Use the Gauss-Jordan elimination procedure and the pivot row identified in Step 5. Elimination must also be performed in the cost function (artificial cost) row so that it is only in terms of nonbasic variables in the next tableau. This step eliminates the nonbasic variable identified in Step 4 from all of the rows except the pivot row; that is, it becomes a basic variable.
- **Step 7.** Optimum solution. If the optimum solution is obtained, then read the values of the basic variables and the optimum value of the cost function from the tableau. Otherwise, go to Step 3.

AN ALTERNATE OR TWO-PHASE SIMPLEX OR BIG-M METHOD

Artificial Variables: In many design problems, there are "≥ type" and "equality" constraints. For such constraints initial basic feasible solution is not readily available. To obtain such a solution, new no-negative variables are added to each "≥ type" or "equality" constraints. They are called Artificial Variables. Which are different from the Surplus variables. They have no physical meaning, however, with their addition we obtain an initial basic feasible solution by treating them as basic.

When there are "≥ type" (with positive right side) and equality constraints in the LP problem, an initial basic feasible solution is not readily available. We must use the two-phase Simplex method to solve the problem.

To define the Phase I minimization problem, we introduce an artificial variable for each "≥ type" and equality constraint.

For the sake of simplicity of discussion, let us assume that each constraint of the standard LP problem requires an artificial variable in Phase I of the Simplex method.

The constraints that do not require an artificial variable can also be treated routinely, as we saw in the example problems in the previous chapter. Recalling that the standard LP problem has n variables and m equality constraints, the constraint equations Ax5b augmented with the artificial variables are now given as –

$\sum_{j=1}^{n} a_{ij}\, x_j + x_{n+i} = b_i\,; i = 1\ to\ m$ (B)

Where x_{n+i} , I = 1 to m are the artificial variables. Thus the basic feasible solution to the Phase – I Problem is given as –

Basic variables $x_{n+i} = b_i\,; i = 1\ to\ m$

Nonbasic variables: $x_j = 0$, $j = 1\ to\ n$

Note that the artificial variables basically augment the convex polyhedron of the original problem. The initial basic feasible solution to the Phase I problem corresponds to an extreme point (vertex) located in the expanded space. The problem now is to traverse the extreme points in the expanded space until an extreme point is reached in the original space.

When the original space is reached, all artificial variables will be nonbasic (i.e., they will have zero values) and the artificial cost function will have a zero value. At this point the augmented space is literally removed so that future movements are only among the extreme points of the original space until the optimum point is reached. In short, after creating artificial variables, we eliminate them as quickly as possible.

Artificial Cost Function: To eliminate the artificial variables from the problem, we define an auxiliary function, called the artificial cost function, and minimize it subject to the constraints of Eq. (B) and the non-negativity of all of the variables. The artificial cost function is simply a sum of all of the artificial variables and will be designated as w:

$$w = x_{n+1} + x_{n+2} + \dots\dots\dots + x_{n+m} = \sum_{i=1}^{m} x_{n+i} \qquad \text{(C)}$$

The objective of the Phase I problem is to make all of the artificial variables nonbasic so that they have zero value. In that case, the artificial cost function in Eq. (C) will be zero, indicating the end of Phase I.

The Pivot Step: The pivot step, based on the Guass-Jordan elimination procedure, interchanges a basic variable with a nonbasic variable. Let a basic variable x_p ($1 \le p \le m$) be selected to replace a nonbasic variable x_q for $(n\text{-}m) \le q \le n$. The pth basic column is to be interchanged with the qth nonbasic column. That is, the q^{th} column will become a column of the identity matrix, and the p^{th} column will no longer be the identity matrix column. This is possible only when the pivot element in the p^{th} column and q^{th} row is nonzero (i.e., $a_{pq} \neq 0$).

The current nonbasic variable x_q will be basic if it is eliminated from all of the equations except the pth one. This can be accomplished by performing a Gauss-Jordan elimination step on the qth column of the tableau shown earlier in Table using the p^{th} row for elimination. This will give $a_{pq}=1$ and zeros elsewhere in the qth column.

Phase – I: Algorithm

- **Step – 1**: Introduce slack and surplus variables in the constraints and express them in the standard LP form. The RHS constants of all the constraints must be nonnegative. If j^{th} design variables y_j is unrestricted in sign, substitute $y_j = y_j^{+} - y_j^{-}$ in all th equations.
- **Step – 2**: Define artificial variables for the equality and "≥ type" constraints. This system equations then gives a canonical form of Phase –I. define an artificial cost function to express it in terms of only the nonbasic variables.
- **Step – 3**: Write simples Tableau for the problem. The artificial cost function is written in the last row and the original cost function in the second last row.

- **Step – 4**: Scan the last row of the tableau and note the most negative coefficient. Let its index be 'r'. This implies that x_r should become a basic variable.
- **Step – 5**: Calculate the ratios of the RHS parameters with the positive coefficients in the r^{th} column and select a row index 'S' according to equation –
- $\frac{b_s}{a_{s,r}} = \min_i \left\{ \frac{b_i}{a_{i,r}},\ a_{ir} > 0; i = 1\ to\ m \right\}$
- This implies that x_s should become nonbasic. If the above equation doses not yield a pivot element (i.e. all elements are negative in the rth column), the problem is Unbounded.
- **Step – 6**: Perform the pivot step on the rows of the tableau with a_{sr} as the pivot element and the S^{th} row as the pivot row.
- **Step – 7**: If there are negative entries in the last row, go to Step – 4, otherwise go to Step – 8.
- **Step – 8**: If all entries in the row are nonnegative and w is zero, Phase – I of the simplex algorithm is complete and an initial basic feasible solution is obtained for the original problem. If all entries in the last row are nonnegative and $w \neq 0$, then the problem is infeasible.

Phase – II: Algorithm

In the final tableau from Phase I, the artificial cost row is replaced by the actual cost function equation and the Simplex iterations continue based on the algorithm. The basic variables, however, should not appear in the cost function. Thus, the pivot steps need to be performed on the cost function equation to eliminate the basic variables from it.

A convenient way of accomplishing this is to treat the cost function as one of the equations in the Phase I tableau, say the second equation from the bottom. Elimination is performed on this equation along with the others. In this way, the cost function is in the correct form to continue with Phase II.

The Phase – II algorithm Steps are as follows –

Step – 1, Step – 2, and **Step – 3**, are same as **Step – 4, Step – 5,** and **Step – 6** of **Phase – I**.

Step – 4: If there are negative entries in the last row, go to Step – 1. Otherwise, the last Tableau yields optimum solution for the problem.

Features of the Simplex method

1. If there is a solution to the LP problem, the method finds it.

2. If the problem is infeasible, the method indicates it.

3. If the problem is unbounded, the method indicates it.

4. If there are multiple solutions, the method indicates it.

Exercise:

1. Convert the following problems to the standard LP form.

8.2 Minimize $f = 5x_1 + 4x_2 - x_3$
subject to $x_1 + 2x_2 - x_3 \geq 1$
$2x_1 + x_2 + x_3 \geq 4$
$x_1, x_2 \geq 0$; x_3 is unrestricted in sign

8.3 Maximize $z = x_1 + 2x_2$
subject to $-x_1 + 3x_2 \leq 10$
$x_1 + x_2 \leq 6$
$x_1 - x_2 \leq 2$
$x_1 + 3x_2 \geq 6$
$x_1, x_2 \geq 0$

8.4 Minimize $f = 2x_1 - 3x_2$
subject to $x_1 + x_2 \leq 1$
$-2x_1 + x_2 \geq 2$
$x_1, x_2 \geq 0$

8.5 Maximize $z = 4x_1 + 2x_2$
subject to $-2x_1 + x_2 \leq 4$
$x_1 + 2x_2 \geq 2$
$x_1, x_2 \geq 0$

2. Find all the basic solutions for the following LP problems using the Gauss–Jordan elimination method. Identify basic feasible solutions and show them on graph paper.

8.25 Maximize $z = 5x_1 - 2x_2$
subject to $2x_1 + x_2 \leq 9$
$x_1 - 2x_2 \leq 2$
$-3x_1 + 2x_2 \leq 3$
$x_1, x_2 \geq 0$

8.26 Maximize $z = x_1 + 4x_2$
subject to $x_1 + 2x_2 \leq 5$
$x_1 + x_2 = 4$
$x_1 - x_2 \geq 3$
$x_1, x_2 \geq 0$

8.27 Minimize $f = 5x_1 + 4x_2 - x_3$
subject to $x_1 + 2x_2 - x_3 \geq 1$
$2x_1 + x_2 + x_3 \geq 4$
$x_1, x_3 \geq 0$; x_2 is unrestricted in sign

3. Solve the following problems by the Simplex method and verify the solution graphically whenever possible.

8.32 Maximize $z = x_1 + 0.5x_2$
subject to $6x_1 + 5x_2 \leq 30$
$3x_1 + x_2 \leq 12$
$x_1 + 3x_2 \leq 12$
$x_1, x_2 \geq 0$

8.33 Maximize $z = 3x_1 + 2x_2$
subject to $3x_1 + 2x_2 \leq 6$
$-4x_1 + 9x_2 \leq 36$
$x_1, x_2 \geq 0$

4. Solve the following LP problems by the Simplex method and verify the solution graphically, whenever possible.

8.55 Maximize $z = x_1 + 2x_2$
subject to $-x_1 + 3x_2 \leq 10$
$x_1 + x_2 \leq 6$
$x_1 - x_2 \leq 2$
$x_1 + 3x_2 \geq 6$
$x_1, x_2 \geq 0$

8.56 Maximize $z = 4x_1 + 2x_2$
subject to $-2x_1 + x_2 \leq 4$
$x_1 + 2x_2 \geq 2$
$x_1, x_2 \geq 0$