LECTURE NOTES

Mathematics-II

Course Code: MA102

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Syllabus

Unit-I. Matrices: Rank of a matrix, inverse of a matrix using elementary transformations, consistency of linear system of equations, eigenvalues and eigenvectors of a matrix, Cayley Hamilton theorem, diagonalisation of matrix.

Unit-II. Ordinary differential equations: Second & higher order linear differential equations with constant coefficients, general solution of homogeneous and non-homogeneous equations, method of variation of parameters, Euler-Cauchy equation, simultaneous linear equations, applications to simple harmonic motion.

Unit-III. Special functions: Power series method, Frobenious method, Legendre polynomials, Bessel equation, Bessel functions of first kind, orthogonal property.

Unit-IV. Laplace transforms: Basic properties, Laplace transform of derivatives and integrals, inverse Laplace transform, differentiation and integration of Laplace transform, convolution theorem, unit step function, periodic function, applications of Laplace transform to initial and boundary value problems.

Unit-V. Fourier series: Fourier series, Fourier series of functions of arbitrary period, even and odd functions, half range series, complex form of Fourier series, numerical harmonic analysis.

Unit-VI. Fourier transforms: Fourier transforms, transforms of derivatives and integrals, applications to boundary value problem in ordinary differential equations (simple cases only).

Unit-I

Matrices

Consider a non-zero matrix A. A matrix obtained by leaving out some rows or columns of the matrix A is called a **sub-matrix** of the matrix A. A matrix A is a sub-matrix of itself. Every square matrix has a determinant.

If A is an $m \times n$ matrix, then the determinant of every square sub-matrix of A is called a minor of A.

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & -3 \\ 3 & 7 & -1 & 5 \\ -1 & 3 & 2 & 1 \end{pmatrix}.$$

Here, $A_1 = \begin{pmatrix} 1 & 0 \\ 3 & 7 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 7 & -1 \\ -1 & 3 & 2 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 0 & 2 & -3 \\ 3 & 7 & -1 & 5 \end{pmatrix}$ are sub-matrices of the matrix A. A_1 and A_2 are square sub-matrices. $|A_1|$ is a 2-rowed minor of A. $|A_2|$ is a 3-rowed minor of A.

Rank of a Matrix

A number p is the rank of a matrix A if

(*i*) there is at least one square sub-matrix of A of order p whose determinant is not equal to zero and

(*ii*) the determinant of every square sub-matrix of A of order $(p+1)$, if any, is equal to zero.

In short, the rank of a matrix is the order of any highest order non-vanishing minor of the matrix. We shall denote the rank of a matrix A by $\rho(A)$.

Remarks.

(*i*) If A is an $m \times n$ matrix, then $\rho(A)$ is at most equal to the smaller of the numbers m and n, but it may be less.

(*ii*) If A is a non-singular square matrix of order n, then $\rho(A) = n$.

(*iii*) The rank of a square matrix A of order n can be less than n iff A is singular i.e., $|A| = 0$.

(*iv*) The rank of every non-zero matrix is ≥ 1. If $A = 0$ (the zero matrix), then $\rho(A) = 0$.

(*v*) Consider a square matrix A. Then the rank of A, $\rho(A) \leq i$ if all $(i+1)$-rowed minors of A are equal to zero. Also, $\rho(A) \geq i$ if there is at least one i-rowed minor of A which is not equal to zero.

Examples.

1. The rank of the identity matrix of order n is n since the identity matrix is non-singular.
2. The rank of a zero matrix is zero.
3. Consider the matrix

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}.$$

Here, $|A| = 24 \neq 0$. That is, A is a non-singular matrix of order 3.
Hence, $\rho(A) = 3$.

4. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}.$$

Here, $|A| = 0$. This implies that $\rho(A) < 3$.

Now, the 2-rowed minor $A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$. Hence, $\rho(A) = 2$.

5. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Here, $|A| = 0$. This implies that $\rho(A) < 3$.

Now, the 2-rowed minor $A = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$. Hence, $\rho(A) = 2$.

6. Consider the matrix

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix}.$$

Clearly, $|A| = 0$ since the first and third rows of A are identical. This implies that $\rho(A) < 3$.
Also, every 2-rowed minor of A is equal to 0. But A is not a null matrix. Hence, $\rho(A) = 1$.

7. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & -1 & 4 \\ 3 & 5 & 4 & 1 \end{pmatrix}.$$

Since the size of the matrix is 2×4, $\rho(A) \leq 2$ (the smaller of the numbers 2 and 4).

The minor $\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 10 - 3 = 7 \neq 0$. Also, there is no minor of A of order greater than 2. Hence, $\rho(A) = 2$.

Echelon form of a matrix

A matrix A is in Echelon form if

(i) every row of A which has all its entries 0 occurs below every row which has a non-zero entry and

(ii) the number of zeros before the first non-zero entry in a row is less than the number of such zeors in the next row.

The rank of a matrix in Echelon form is equal to the number of non-zero rows of the matrix.

Using this result, one can easily determine the rank of a given matrix by first putting the matrix in Echelon form. A given matrix which is not in Echelon form may be reduced to Echelon form by applying elementary row transformations.

Examples

1. The identity matrix of any order is in Echelon form. So, the rank of an identity matrix of order n is n since it has n non-zero rows. That is, $\rho(I_n) = n$.

2. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here, A is in Echelon form. So, $\rho(A) =$ number of non-zero rows of $A = 2$.

Some Important Results

1. $\rho(A) = \rho(A^T)$, where A^T denotes the transpose of the matrix A.
2. The rank of a matrix cannot be less than the rank of every sub-matrix thereof.

3. The rank of a matrix does not alter on affixing any number of additional rows or columns of zeros.

Examples

1. Find the rank of

$$(i)\ A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \qquad (ii)\ B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix}.$$

Elementary Row Transformations

We have three elementary row transformations.

(*i*) Interchange of the ith row and the jth row. This is written as $R_i \leftrightarrow R_j$.

(*ii*) Multiplication of the ith row by a non-zero scalar k. This is written symbolically as $R_i \to kR_j$.

(*iii*) Addition of k times of the jth row to the ith row. This is symbolically denoted by $R_i \to R_i + kR_j$.

Note that elementary row transformations do not change the rank of a matrix.

Equivalence of matrices

A matrix B is said to be equivalent to a matrix A if B can be obtained from A by performing a finite number of elementary row transformations on A. If A is equivalent to B, we write $A \sim B$.

Equivalence of matrices is an equivalence relation. That is, for any three matrices A, B, C, we have (*i*) $A \sim A$ (reflexive) (*ii*) $A \sim B \Rightarrow B \sim A$ (symmetric) and (*iii*) $A \sim B$ and $B \sim C \Rightarrow A \sim C$ (transitive).
If A and B are equivalent matrices, then $\rho(A) = \rho(B)$.

Examples

1. Find the rank (by first reducing the matrix in Echelon form using elementary row transformations).

$$(i)\ A = \begin{pmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix} \qquad (ii)\ A = \begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 1 \end{pmatrix}$$

$$(iii)\ A = \begin{pmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{pmatrix} \qquad (iv)\ A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix}$$

$$(v)\ A = \begin{pmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix} \qquad (vi)\ A = \begin{pmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 5 \end{pmatrix}$$

Solution. (*i*) $\rho(A) = 3$ (*ii*) $\rho(A) = 2$ (*iii*) $\rho(A) = 3$ (*iv*) $\rho(A) = 3$ (*v*) $\rho(A) = 3$ (*vi*) $\rho(A) = 4$.

2. Are the matrices equivalent?

$$(i)\ A = \begin{pmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 5 \end{pmatrix}$$

$$(ii)\ A = \begin{pmatrix} 3 & 9 & 0 & 2 \\ 1 & -2 & 1 & 3 \\ 5 & 6 & 1 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 3 & 4 \\ 2 & 3 & 7 \end{pmatrix}$$

Solution. (i) $\rho(A) = 2$, $\rho(B) = 4$. The matrices are not equivalent.
(ii) The matrices are of different sizes. They are not equivalent.

3. Find the ranks of A, B, $A + B$, AB and BA where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{pmatrix}.$$

Solution. $\rho(A) = 2$, $\rho(B) = 1$, $\rho(A + B) = 2$, $\rho(AB) = 0$, $\rho(BA) = 1$.

Two important results

1. The rank of a product of two matrices cannot exceed the rank of either matrix. That is, for any two matrices A and B, $\rho(AB) \leq \rho(A)$ and $\rho(AB) \leq \rho(B)$.

2. Every non-singular matrix is row equivalent to a unit matrix.

Inverse of a matrix using elementary transformations

A square matrix A is said to be invertible if there exists a matrix B such that $AB = BA = I$, where I is the identity matrix.

The inverse of a square matrix exists if it is non-singular.

Examples

1. Find the inverse (using elementary row transformations).

$$(i)\ A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \quad (ii)\ A = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{pmatrix}$$

$$(iii)\ A = \begin{pmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{pmatrix}, \quad (iv)\ A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 3 & 4 \\ 2 & 3 & 7 \end{pmatrix}$$

Solution. We have, $A = IA$.
That is,

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

Applying $R_2 \longrightarrow R_2 - 3R_1$, $R_3 \longrightarrow R_3 - R_1$.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} A$$

Applying $R_2 \longrightarrow -\frac{1}{2}R_2$.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3/4 & -1/4 & 0 \\ -1 & 0 & 1 \end{pmatrix} A$$

Applying $R_1 \longrightarrow R_1 - 2R_2$, $R_3 \longrightarrow R_3 + R_2$.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 & 0 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{pmatrix} A$$

Applying $R_1 \longrightarrow R_1 - R_3$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{pmatrix} A$$

This shows that $I_3 = BA$, where $B = \begin{pmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{pmatrix}$.

Hence,

$$A^{-1} = B = \begin{pmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{pmatrix}.$$

Vectors

An ordered n-tuple $X = (x_1, x_2, \ldots, x_n)$ is called an n-vector. $x_1, x_2, \ldots, x_n$ are called the compoenents of the vector $X = (x_1, x_2, \ldots, x_n)$. A vector may be written either as a row vector or as a column vector.

If A is a matrix of the type $m \times n$, then each row of A will be an n-vector and each column of A will be an m-vector.

A vector whose components are all equal to zero is called the zero vector and is denoted by O.

If α be any number and X be any vector, then relative to the vector X, α is called a scalar.

Algebra of Vectors

Equality of vectors: Two vectors $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ are said to be equal, written $X = Y$, if the corresponding components are equal i.e., $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Addition of vectors: The sum of two vectors $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ is defined as $X + Y = (x_1 + y_1,\ x_2 + y_2,\ \ldots, x_n + y_n)$.

Multiplication of a vector by a scalar: For a given vector $X = (x_1, x_2, \ldots, x_n)$ and a scalar α, we define the scalar multiple of X by α as $\alpha X = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$.

Some Properties

Let X, Y, Z be any three vectors and α, β be two scalars. Then

(*i*) $X + Y = Y + X$ (Commutative law holds)

(*ii*) $(X + Y) + Z = X + (Y + Z)$ (Associative law holds)

(*iii*) $\alpha(X + Y) = \alpha X + \alpha Y$ (A scalar is distributive over vector addition)

(*iv*) $(\alpha + \beta)X = \alpha X + \beta X$ (Vector multiplication is ditributive over scalar addition)

(*v*) $\alpha(\beta X) = (\alpha\beta)X$.

Linear dependence and linear independence of vectors

A set of p n-vectors $X_1, X_2, \ldots, X_p$ is said to be linearly dependent if there exists p scalars $\alpha_1, \alpha_2, \ldots, \alpha_p$, not all zero such that

$$\alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_p X_p = O$$

where O denotes the n-vector whose components are all equal to zero.

A set of p n-vectors $X_1, X_2, \ldots, X_p$ is said to be linearly independent if every relation of the form

$$\alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_p X_p = O$$

implies $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_p = 0$.

Examples

1. The vectors $X_1 = (1, 1, 2)$ and $X_2 = (3, 3, 6)$ are linearly dependent vectors since there exist $\alpha_1 = 3$, $\alpha_2 = -1$ such that $\alpha_1 X_1 + \alpha_2 X_2 = O$.
2. The singleton set $\{O\}$ is linearly dependent, O being the zero vector.
3. The vectors $X_1 = (1, 2, 3)$ and $X_2 = (4, -2, 7)$ are linearly independent.
4. The vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ are linearly independent.

A vector as a linear combination of vectors

A vector X which can be expressed in the form $X = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_p X_p$ is said to be a linear combination of the vectors $X_1, X_2, \ldots, X_p$. Here, $\alpha_1, \alpha_2, \ldots, \alpha_p$ are scalars.

Two important results

(*i*) If a set of vectors is linearly dependent, then at least one member of the set can be written as a linear combination of the remaining members.

(*ii*) If a set of vectors is linearly independent, then no member of the set can be written as a linear combination of the remaining members.

Row rank and column rank of a matrix

Let $A = (a_{ij})_{m\times n}$ be an $m \times n$ matrix. Each of the m rows of A consists of n elements. Therefore, the row vectors of A are n-vectors. The maximum number of linearly independent rows of A is said to be the row rank of the matrix A.

Similarly, each of the n columns of A consists of m elements. Therefore, the column vectors of A are m-vectors. The maximum number of linearly independent columns of A is said to be the column rank of the matrix A.

The row rank, the column rank and the rank of a matrix are all equal. Thus, we can define the rank of a matrix as the maximum number of linearly independent row vectors or column vectors.

Consistency of linear system of equations

A system of linear equations may be homogeneous or non-homogeneous.

Homogeneous linear equations

Consider the following system of linear equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 +, \ldots, +a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 +, \ldots, +a_{2n}x_n &= 0 \\ \cdots \quad \cdots \quad &\cdots \\ a_{m1}x_1 + a_{m2}x_2 +, \ldots, +a_{mn}x_n &= 0 \end{aligned} \tag{1}$$

The system of linear equations (1) is a system of m homogeneous equations in n unknowns $x_1, x_2, \ldots, x_n$.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$, and $O = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{m\times 1}$.

Then the system of linear equations (1) can be written in the form

$$AX = O \tag{2}$$

The matrix A is called the coefficient matrix of the system of equations (1). Clearly, $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ i.e., $X = O$ is a solution of (1). This is a trivial solution of (1).

Remarks

1. If X_1 and X_2 are two solutions of (1.2), then their linear combination $\alpha_1 X_1 + \alpha_2 X_2$, where α_1, α_2 are arbitrary constants, is also a solution of (1.2).

2. The collection of all the solutions of the system of equations $AX = O$ forms a subspace of the n-vector space V_n.

3. The number of linearly independent solutions of m homogeneous linear equations in n variables, $AX = O$, is $(n - r)$, where r is the rank of the matrix A.

4. The set of solutions $\{X_1, X_2, \ldots, X_{n-r}\}$ forms a basis of the vector space of all the solutions of the system of equations $AX = O$.

Some important results on the nature of solutions of AX=O

Suppose we have m equations in n unknowns. Then the coefficient matrix A will be of the type $m \times n$. Let r be the rank of the matrix A. Obviously, r cannot be greater than n (the number of columns of the matrix A). Therefore, we have either $r = n$ or $r < n$.

Case I. If $r = n$, the equation $AX = O$ will have $n - n$ i.e., no linearly independent solutions.
In this case, the zero solution will be the only solution. We know that zero vector forms a linearly dependent set.

Case II. If $r < n$, we shall have $(n - r)$ linearly independent solutions. Any linear combination of these $(n - r)$ solutions will also be a solution of $AX = O$. Thus, in this case the equation $AX = O$ will have an infinite number of solutions.

Case III. Suppose $m < n$ i.e., the number of equations is less than the number of unknowns. Since $r \leq m$, therefore r is definitely less than n. Hence, in this case, the given system of equations must possess a non-zero solution. The equation $AX = O$ will have an infinite number of solutions.

Fundamental set of solutions of the equation AX=O

Suppose the rank r of the coefficient matrix A is less than the number of the unknowns n. In this case, the given equations have a set of $(n - r)$ linearly independent solutions and every possible solution is a linear combination of these $(n - r)$ solutions. This set of $(n - r)$ solutions is called a fundamental set of solutions of the equation $AX = O$.

A set of linearly independent solutions $X_1, X_2, \ldots, X_n$ of the system of homogeneous equations $AX = O$ is called the fundamental system of solutions of $AX = O$ if every solution X of $AX = O$ can be written as a linear combination of these vectors i.e., in the form $X = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_n X_n$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars.

Working rule for finding the solutions of the equation AX=O.

Reduce the coefficient matrix A to Echelon form by applying elementary row transformations only. This Echelon form will enable us to determine the rank of the matrix A. Suppose the matrix A is of the type $m \times n$ and its rank comes out to be r.

If $r < m$, then in the process of reducing the matrix A to Echelon form, $(n-r)$ equations will be eliminated. The given system of m equations will thus be replaced by an equivalent system of r equations. Solving these r equations (by Cramer's rule or otherwise) we can express the values of some r unknowns in terms of the remaining $(n-r)$ unknowns. These $(n-r)$ unknowns can be given any arbitrarily chosen values.

If $r = n$, the zero solution (trivial solution) will be the only solution.

If $r < n$, there will be an infinitely many solutions.

Examples

1. Consider the following homogeneous system of linear equations.

$$x + 2y + 3z = 0$$
$$3x + 4y + 4z = 0$$
$$7x + 10y + 12z = 0$$

The given system of equations can be written in matrix form as

$$AX = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We shall reduce A to triangular form by elementary row transformations. Applying $R_2 \longrightarrow R_2 - 3R_1$ and $R_3 \longrightarrow R_3 - 7R_1$, the given system is equivalent to

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Here, we observe that the determinant of the matrix on the left side is not equal to 0. Therefore, the rank of this matrix is 3. So, there is no need for further application of elementary transformations on A.

The rank of the coefficient matrix A is 3 and is equal to the number of unknowns.

Therefore, the given system of equations does not possess any linearly independent solution.

Thus, the zero solution: $x = 0, y = 0, z = 0$ is the only solution of the given system of equations.

2. To solve the system

$$\begin{aligned} x+2y+3z&=0\\ 3x+4y+4z&=0\\ 7x+10y+12z&=0 \end{aligned}$$

we write the equivalent matrix equation as

$$AX=\begin{pmatrix}1&3&-2\\2&-1&4\\1&-11&14\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}=O.$$

We shall reduce A to triangular form by elementary row transformations. Applying $R_2\longrightarrow R_2-2R_1$ and $R_3\longrightarrow R_3-R_1$, the given system is equivalent to

$$\begin{pmatrix}1&3&-2\\0&-7&8\\0&-14&16\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Applying $R_3\longrightarrow R_3-2R_2$

$$\begin{pmatrix}1&3&-2\\0&-7&8\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

The coefficient matrix is now triangular and it is clear that its rank is 2. Therefore, the given system possesses 3-2=1 linearly independent solution. We shall assign arbitrary values to $n-r=3-2=1$ variable and the remaining $r=2$ variables shall be found in terms of these. The given system of equations is equivalent to

$$\begin{aligned} x+3y-2z&=0\\ -7y+8z&=0 \end{aligned}$$

These give $y=\frac{8}{7}z$, $x=-\frac{10}{7}z$. Taking $z=c$, we get $y=\frac{8}{7}c$, $x=-\frac{10}{7}c$. Hence, the general solution of the given system is $x=-\frac{10}{7}c$, $y=\frac{8}{7}c$, $z=c$, where c is an arbitrary constant.

3. Solve:

$$\begin{aligned} x+y+z&=0\\ 2x-y-3z&=0\\ 3x-5y+4z&=0\\ x+17y+4z&=0 \end{aligned}$$

Solution: $x = y = z = 0$ is the only solution.

4. Solve:

$$\begin{aligned} 2x - 2y + 5z + 3w &= 0 \\ 4x - y + z + w &= 0 \\ 3x - 2y + 3z + 4w &= 0 \\ x - 3y + 7z + 6w &= 0 \end{aligned}$$

Solution: $x = \frac{5}{9}c$, $y = 4c$, $z = \frac{7}{9}c$, $w = c$ constitute the general solution of the system.

5. Solve:

$$\begin{aligned} x + y + z &= 0 \\ 2x + 5y + 7z &= 0 \\ 2x - 5y + 3z &= 0 \end{aligned}$$

Solution: $x = y = z = 0$ is the only solution.

Non-homogeneous system of linear equations

It is not always possible to solve every pair two simultaneous equations of the form

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

Consider the simultaneous equations

$$\begin{aligned} 2x + 3y &= 5 \\ 6x + 9y &= 7 \end{aligned}$$

There is no set of values of x and y which satisfies both these equations. Such equations are said to be **inconsistent**.

Consider the following simultaneous equations

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 2 \end{aligned}$$

These equations are simultaneously satisfied by $x = 1$, $y = 1$. Such equations are said to be **consistent**.

Consider the equations

$$\begin{aligned} 2x + 3y &= 5 \\ 6x + 9y &= 15 \end{aligned}$$

One may find that these equations are simultaneously satisfied by an infinite number of values of x and y.

Consider the following system of linear equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 +, \dots, + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 +, \dots, + a_{2n}x_n &= b_2 \\ \dots \quad \dots \quad \dots & \\ a_{m1}x_1 + a_{m2}x_2 +, \dots, + a_{mn}x_n &= b_m \end{aligned} \tag{3}$$

The above system of linear equations (3) is a system of m non-homogeneous equations in n unknowns $x_1, x_2, \dots, x_n$.

If we write $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m\times n}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$

and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}$,

then the system of linear equations (3) can be written in the matrix form as $AX = B$, where the matrix A is called the coefficient matrix of the system of equations (3).

Any set of values of $x_1, x_2, \dots, x_n$ which simultaneously satisfy all these equations is called a solution of the system (3).

When the system of equations has one or more solutions, the equations are said to be **consistent**, otherwise, they are said to be **inconsistent**.

The matrix

$$[A\ B] = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$

is called the **Augmented Matrix** of the given system of equations.

Condition for consistency

The system of equations $AX = B$ is consistent (i.e., possesses a solution) if and only if the coefficient matrix A and the augmented matrix $[A\ B]$ are of the same rank.

Condition for a system of n equations in n-unknowns to have a unique solution

If A be an n-rowed non-singular matrix, X be an $n \times 1$ matrix, B be an $n \times 1$ matrix, the system of equations $AX = B$ has a unique solution.

Working rule for finding the solution of the equation AX=B

Suppose the coefficient matrix A is of the type $m \times n$ i.e., we have m equations in n unknowns. Write the augmented matrix $[A\ B]$ and reduce it to an Echelon form by applying only elementary row transformations on it. This Echelon form will enable us to know the ranks of the augmented matrix $[A\ B]$ and the coefficient matrix A. Then the following different cases arise.

Case I. Rank $A <$ Rank $[A\ B]$. In this case, the equations $AX = B$ are inconsistent i.e., they have no solution.

Case II. Rank $A =$ Rank $[A\ B] = r$ (say). In this case, the equations $AX = B$ are consistent i.e., they possess a solution. If $r < m$, then in the process of reducing the matrix $[A\ B]$ to Echelon form, $(m - r)$ equations will be eliminated. The given system of m equations will then be replaced by an equivalent system of r equations. From these r equations we shall be able to express the values of some r unknowns in terms of the remaining $(n - r)$ unknowns which can be given any arbitrarily chosen values.

If $r = n$, then $n - r = 0$. So, no variable is to be assigned arbitrary values and therefore in this case there will be a unique solution.

If $r < n$, then $(n-r)$ variables can be assigned arbitrary values. So in this case, there will be an infinite number of solutions. Only $n - r + 1$ solutions will be linearly independent and the rest of the solutions will be linear combinations of them.

If $m < n$, then $r \leq m < n$. Thus, in this case, $n - r > 0$. Therefore, when the number of equations is less than the number of unknowns, the equations will always have an infinite number of solutions, provided they are consistent.

Examples

1. Apply the test of rank to examine if the system of equations is consistent. Find the solution if it is consistent.

$$\begin{aligned} x + y + z &= -3 \\ 3x + y - 2z &= -2 \\ 2x + 4y + 7z &= 7 \end{aligned}$$

Not consistent. The given system of equations is equivalent to the matrix equation $AX = B$ where $A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} -3 \\ -2 \\ 7 \end{pmatrix}$.

The augmented matrix is

$$[A\ B] = \begin{pmatrix} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{pmatrix}$$

We shall reduce the augmented matrix to Echelon form by applying elementary row transformations only.
Applying $R_2 \longrightarrow R_2 - 3R_1, R_3 \longrightarrow R_3 - 2R_1$, we get

$$[A\ B] \sim \begin{pmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{pmatrix}$$

Applying $R_3 \longrightarrow R_3 + R_2$.

$$[A\ B] \sim \begin{pmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{pmatrix}$$

This is the Echelon form of $[A\ B]$.
Therefore, rank $[A\ B] =$ the number of non-zero rows in the Echelon form $= 3$.
Also, by the same elementary row transformations, we obtain

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

Clearly, $\rho(A) = 2$. Since $\rho[A\ B] \neq \rho(A)$, the given equations are inconsistent i.e., they have no solution.

2. Apply the test of rank to examine if the system of equations is consistent. Find the solution if it is consistent.

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 7z = 30$$

Consistent. $x = a - 2, y = 8 - 2a, z = a$.
The given system of equations is equivalent to the matrix equation $AX = B$ where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 6 \\ 14 \\ 30 \end{pmatrix}$.

Augmented matrix

$$[A\ B] = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{pmatrix}$$

We shall reduce the augmented matrix to Echelon form by applying elementary row transformations only.
Applying $R_2 \longrightarrow R_2 - R_1, R_3 \longrightarrow R_3 - R_1$, we get

$$[A\ B] \sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{pmatrix}$$

Applying $R_3 \longrightarrow R_3 - 3R_2$.

$$[A\ B] \sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the Echelon form of $[A\ B]$.
Therefore, rank $[A\ B] =$ the number of non-zero rows in the Echelon form $= 2$.
Also, by the same elementary row transformations, we have

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

which is in the Echelon form.
So, $\rho(A) =$ the number of non-zero rows in the Echelon form $= 2$. Therefore, since $\rho[A\ B] = \rho(A)$, the given equations are consistent.
We now find the solution to the given system of equations.
The number of unknowns is 3 and $\rho(A) = 2$. Since $\rho(A)$ is less than the number of unknowns, the given system of equations has an infinite number of solutions.
We note that the given system of equations is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}.$$

And this matrix equation is equivalent to the system of equations

$$\begin{aligned} x + y + z &= 6 \\ y + 2z &= 8 \end{aligned}$$

These give $y = 8 - 2z$, $x = 6 - (y + z) = z - 2$. Choosing $z = a$, we have $y = 8 - 2a$, $x = a - 2$.

Hence, the general solution of the given system is $x = a - 2$, $y = 8 - 2a$, $z = a$, where a is an arbitrary constant.

3. Apply the test of rank to examine if the system of equations is consistent. Find the solution if it is consistent.

$$\begin{aligned} x + 2y - z &= 3 \\ 3x - y + 2z &= 1 \\ 2x - 2y + 3z &= 2 \\ x - y + z &= -1 \end{aligned}$$

Consistent. $x = -1, y = 4, z = 4$.
The given system of equations is equivalent to the matrix equation $AX = B$ where $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ 1 \\ 2 \\ -1 \end{pmatrix}$.
Augmented matrix

$$[A\ B] = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

We shall reduce the augmented matrix to Echelon form by applying elementary row transformations only.
Applying $R_2 \longrightarrow R_2 - 3R_1, R_3 \longrightarrow R_3 - 2R_1$, $R_4 \longrightarrow R_4 - R_1$, we get

$$[A\ B] \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{pmatrix}$$

Applying $R_2 \longrightarrow R_2 - R_3$.

$$[A\ B] \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{pmatrix}$$

Applying $R_3 \longrightarrow R_3 - 6R_2$, $R_4 \longrightarrow R_4 - 3R_2$, we get

$$[A\ B] \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 2 & 8 \end{pmatrix}$$

Applying $R_3 \longrightarrow \frac{1}{5}R_3$, $R_4 \longrightarrow \frac{1}{2}R_4$, we get

$$[A\ B] \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Applying $R_4 \longrightarrow R_4 - R_3$

$$[A\ B] \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The augmented matrix $[A\ B]$ has been reduced to Echelon form.
Therefore, rank $[A\ B]$ = the number of non-zero rows in the Echelon form $= 3$.
Also, by the same elementary row transformations, we have

$$A \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which is in the Echelon form.
So, $\rho(A)$ = the number of non-zero rows in the Echelon form $= 3$. Therefore, since $\rho[A\ B] = \rho(A)$, the given equations are consistent.
We now find the solution to the given system of equations.
The number of unknowns $= \rho(A) = 3$. Therefore, the given equations have a unique solution.
We note that the given system of equations is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 4 \\ 0 \end{pmatrix}.$$

And this matrix equation is equivalent to the system of equations

$$\begin{aligned} x + 2y - z &= 3 \\ -y &= -4 \\ z &= 4 \end{aligned}$$

Solving these, we get $x = -1$, $y = 4$, $z = 4$, which is the required solution.

4. Apply the test of rank to examine if the system of equations is consistent. Find the solution if it is consistent.

$$\begin{aligned} 2x - y + 3z &= 8 \\ -x + 2y + z &= 4 \\ 3x + y - 4z &= 0 \end{aligned}$$

Consistent. $x = 2, y = 2, z = 2$.

5. Apply the test of rank to examine if the system of equations is consistent. Find the solution if it is consistent.

$$\begin{aligned} x + y + z &= 9 \\ 2x + 5y + 7z &= 52 \\ 2x + y - z &= 0 \end{aligned}$$

Consistent. $x = 1, y = 3, z = 5$.

6. Apply the test of rank to examine if the system of equations is consistent. Find the solution if it is consistent.

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + 4z &= 1 \end{aligned}$$

Consistent. $x = -7, y = 22, z = -9$.

7. Apply the test of rank to examine if the system of equations is consistent. Find the solution if it is consistent.

$$\begin{aligned} x + y + 4z &= 6 \\ 3x + 2y - 2z &= 9 \\ 5x + y + 2z &= 13 \end{aligned}$$

Consistent. $x = 2, y = 2, z = \frac{1}{2}$.

Eigenvalues and Eigenvectors of a matrix

Matrix polynomials. An expression of the form

$$P(x) = A_0 + A_1x + A_2x^2 + \ldots + A_{m-1}x^{m-1} + A_mx^m$$

where A_0, A_1, A_2, $\ldots$, A_{m-1}, A_m are all square matrices of the same order, is called a **matrix polynomial** of degree m, provided A_m is not a zero matrix. Here, x is called the indeterminate. If A is any square matrix, we can write $A = x^0A$.

Two matrix polynomials are said to be equal iff the coefficients of like powers of x are the same.

Eigenvalues and Eigenvectors

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order n. Let

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{pmatrix}$$

be a column vector. Consider the vector equation

$$AX = \lambda X \tag{4}$$

where λ is a scalar. Clearly, $X = O$, the zero vector, is a solution of (4) for any value of λ.

We now examine whether there exist scalars λ and non-zero vectors X which satisfy (4). Equation (4) can be written as

$$\begin{aligned} AX &= \lambda IX \\ (A - \lambda I)X &= O \end{aligned} \tag{5}$$

where I is the identity matrix of order n.

The matrix equation (5) represents the following system of n homogeneous equations in n unknowns.

$$\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 +, \dots, +a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 +, \dots, +a_{2n}x_n &= 0 \\ \dots \quad \dots \quad \dots & \\ a_{n1}x_1 + a_{n2}x_2 +, \dots, +(a_{nn} - \lambda)x_n &= 0 \end{aligned} \tag{6}$$

The coefficient matrix of the equations (6) is $(A - \lambda I)$. The system (6) has a non-zero solution ($X \neq O$) if and only if the rank of the coefficient matrix $(A - \lambda I)$ is less than the number of unknowns n. But this is true if and only if $(A - \lambda I)$ is singular i.e., $|A - \lambda I| = 0$. Therefore, the scalars λ for which $|A - \lambda I| = 0$ are of special importance.

Some definitions and results

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order n and x be an indeterminate. The matrix $(A - xI)$ is called the characteristic matrix of A, where I is the identity matrix of order n. The determinant

$$|A - xI| = \begin{vmatrix} a_{11} - x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - x & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - x \end{vmatrix}$$

which is a polynomial in x of degree n, is called the **characteristic polynomial** of A.
The equation $|A - xI| = 0$ is called the **characteristic equation** of A and the roots of this characteristic equation are called the **characteristic values** or **characteristic roots** or **eigenvalues** of the matrix A.

The set of the eigenvalues of A is called the **spectrum** of A.

If λ is an eigenvalue of the matrix A, then $|A - \lambda I| = 0$ i.e., $A - \lambda I$ is singular and therefore, there exists a non-zero vector X such that $(A - \lambda I)X = O$ or, $AX = \lambda X$.

If λ is an eigenvalue of a square matrix A of order n, then a non-zero vector X such that $AX = \lambda X$ is called an eigenvector of A corresponding to the eigenvalue λ.

Let A be a square matrix of order n. A real number λ is an eigenvalue of A if and only if there is a non-zero n-vector X such that $AX = \lambda X$. Also, any non-zero vector X for which $AX = \lambda X$ is an eigenvector corresponding to the eigenvalue λ.

An eigenvalue can be zero. However, by definition, an eigenvector is never the zero vector.

Let A be a square matrix of order n and λ be an eigenvalue of A. Then the set $E_\lambda = \{X | AX = \lambda X\}$ is called the **eigenspace** of λ.

The eigenspace E_λ for a particular eigenvalue λ of A consists of the set of all eigenvectors for A associated with λ, together with the zero vector O, since $AO = O = \lambda O$, for any λ.

Certain relations between eigenvalues and eigenvectors

1. λ is an eigenvalue of a matrix A if and only if there exists a non-zero vector X such that $AX = \lambda X$.

Suppose λ is an eigenvalue of the matrix A. Then $|A - \lambda I| = 0$ i.e., the matrix $A - \lambda I$ is singular. Therefore, the equation $(A - \lambda I)X = O$ has a non-zero solution i.e., there exists a non-zero vector X such that $(A - \lambda I)X = O$ or, $AX = \lambda X$.
Conversely, assume that there exists a non-zero vector X such that $AX = \lambda X$ i.e., $(A - \lambda I)X = O$. Since the matrix equation $(A - \lambda I)X = O$ has a non-zero solution, the coefficient matrix $A - \lambda I$ is singular i.e., $|A - \lambda I| = 0$. Hence, λ is an eigenvalue of the matrix A.

2. If X is an eigenvector of a matrix A corresponding to the eigenvalue λ,then αX is also an eigenvector of A corresponding to the same eigenvalue λ, α being a non-zero scalar.,

Assume that X is an eigenvector of A corresponding to the eigenvalue λ. Then X is non-zero and $AX = \lambda X$. Also, αX is non-zero for any non-zero scalar and $A(\alpha X) = \alpha(AX) = \alpha(\lambda X) = \lambda(\alpha X)$. Hence, αX is an eigenvector of A corresponding to the eigenvalue λ.

3. For a given eigenvalue λ, there corresponds more than one eigenvector.

4. If X is an eigenvector of a matrix A, then X cannot corresponds to more than one eigenvalue of A.

Suppose, if possible, X is an eigenvector corresponding to two eigenvalues λ_1 and λ_2. Then by definition, $AX = \lambda_1 X$ and $AX = \lambda_2 X$. These give $\lambda_1 X = \lambda_2 X$ or, $(\lambda_1 - \lambda_2)X = O$. But X is non-zero. Hence, we must have $\lambda_1 = \lambda_2$.

Algebraic and geometric multiplicity of an eigenvalue

The algebraic multiplicity of an eigenvalue λ of a matrix A is the multiplicity of λ as a root of the characteristic equation of A, $|A - xI| = 0$. Thus, a number p is called the **algebraic multiplicity** of an eigenvalue λ of a matrix A if λ is of multiplicity/order p as a root of the equation $|A - xI| = 0$.

If there are q linearly independent eigenvectors corresponding to an eigenvalue λ, then q is called the **geometric multiplicity** of λ.

We may also define **algebraic multiplicity** of an eigenvalue as follows.

Let A be a square matrix of order n and let λ be an eigenvalue for A. Suppose $(x-\lambda)^k$ is the highest power of $(x-\lambda)$ that divides the characteristic polynomial of A, $|A - xI| = 0$. Then k is called the **algebraic multiplicity** of λ.

Nature of the eigenvalues of special types of matrices

1. The eigenvalues of a Hermitian matrix are real.

2. The eigenvalues of a real symmetric matrix are all real.

3.The eigenvalues of a skew-Hermitian matrix are either purely imaginary or zero.

4. The eigenvalues of a real skew-symmetric matrix are either pure imaginary or zero, for every such matrix is skew-Hermitian.

5. The eigenvalues of a unitary matrix are of unit modulus.

6. The eigenvalues of an orthogonal matrix are of unit modulus.

Process of finding the eigenvalues and eigenvectors of a matrix

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n.
Step 1. Write the characteristic equation $|A - xI| = 0$.
Step 2. Solve the characteristic equation for x to obtain n roots. These roots give the eigenvalues of A.
Step 3. If $x = \lambda$ is an eigenvalue of A, the corresponding eigenvectors of A will be given by the non-zero vectors $X = (x_1, x_2, \dots, x_n)'$ satisfying the equation

$AX = \lambda X$ or, $(A - \lambda I)X = O$.

Examples

1. To determine the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.

The characteristic equation of A is $|A - xI| = 0$.
That is,

$$\begin{vmatrix} 5-x & 4 \\ 1 & 2-x \end{vmatrix} = 0.$$

This implies $x^2 - 7x + 6 = 0$. The roots are $x = 1, 6$. Therefore, the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 6$.
We now find the eigenvectors for each of the eigenvalues of A.
The eigenvectors $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ of A corresponding to the eigenvalue $\lambda_1 = 1$ are given by the non-zero solutions of the homogeneous system $(A - 1I)X = O$.
From this, we get

$$\begin{pmatrix} 5-1 & 4 \\ 1 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or,} \quad \begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or, by applying $R_2 \longrightarrow R_2 - 4R_1$, we get

$$\begin{pmatrix} 4 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is equivalent to the system,

$$4x_1 + 4x_2 = 0 \quad \text{or,} \quad x_1 + x_2 = 0 \quad \text{or,} \quad x_1 = -x_2.$$

Taking $x_1 = 1$, we find that $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 1$. Every non-zero multiple of the vector X_1 is an eigenvector of A corresponding to the eigenvalue 1.
Similarly, the eigenvectors $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ of A corresponding to the eigenvalue $\lambda_2 = 6$ are given by the non-zero solutions of the homogeneous system $(A - 6I)X = O$.
From this, we get

$$\begin{pmatrix} 5-6 & 4 \\ 1 & 2-6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or,} \quad \begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or, by applying $R_2 \longrightarrow R_2 + R_1$, we get

$$\begin{pmatrix} -1 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The coefficient matrix of these equations is of rank 1. Therefore, these equations have 2-1=1 linearly independent solution.
The last matrix equation is equivalent to the system

$$-x_1 + 4x_2 = 0.$$

Taking $x_2 = 1$, we find that $X_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 6$. Every non-zero multiple of the vector X_2 is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 6$.

2. To determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{pmatrix}.$$

The characteristic polynomial of A is

$$\begin{aligned}
|A - xI| = |xI - A| &= \left| \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} - \begin{pmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{pmatrix} \right| \\
&= \begin{vmatrix} x-7 & -1 & 1 \\ 11 & x+3 & -2 \\ -18 & -2 & x+4 \end{vmatrix} \\
&= x^3 - 12x - 16 \\
&= (x+2)^2(x-4).
\end{aligned}$$

The eigenvalues of A are the solutions of $|xI - A| = 0$. Therefore, the eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = 4$.

To find the eigenvectors corresponding to the eigenvalue $\lambda_1 = -2$.

The eigenvectors $X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ of A corresponding to the eigenvalue $\lambda_1 = -2$ are obtained by solving the homogeneous system $(A + 2I)X = O$.
From this, we get

$$\begin{pmatrix} 7+2 & 1 & -1 \\ -11 & -3+2 & 2 \\ 18 & 2 & -4+2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or,

$$\begin{pmatrix} 9 & 1 & -1 \\ -11 & -1 & 2 \\ 18 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying the elementary row transformations, the matrix on the left reduces to

$$\begin{pmatrix}1 & 0 & -1/2\\ 0 & 1 & 7/2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

This is equivalent to the system

$$x_1-\frac{1}{2}x_3=0,\quad x_2+\frac{7}{2}x_3=0\quad \text{or,}\quad x_1=\frac{1}{2}x_3,\quad x_2=-\frac{7}{2}x_3.$$

Taking $x_3=1$, we find that $X_1=\begin{pmatrix}1/2\\ -7/2\\ 1\end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1=-2$. Every non-zero multiple of the vector X_1 is an eigenvector of A corresponding to the eigenvalue $\lambda_1=-2$.Thus, the complete solution set for this system is the eigenspace $E_{-2}=\left\{\alpha\begin{pmatrix}1/2\\ -7/2\\ 1\end{pmatrix}:\ \alpha\in\mathbb{R}\right\}$.

This is equivalent to $E_4=\left\{\alpha\begin{pmatrix}1\\ -7\\ 2\end{pmatrix}:\ \alpha\in\mathbb{R}\right\}$ obtained by multiplying by 2.

Similarly, the eigenvectors $X_2=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$ of A corresponding to the eigenvalue $\lambda_2=4$ are obtained by solving homogeneous system $(A-4I)X=O$.
From this, we get

$$\begin{pmatrix}7-4 & 1 & -1\\ -11 & -3-4 & 2\\ 18 & 2 & -4-4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

or,

$$\begin{pmatrix}3 & 1 & -1\\ -11 & -7 & 2\\ 18 & 2 & -8\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Applying the elementary transformations, the matrix on the left side row reduces to

$$\begin{pmatrix}1 & 0 & -1/2\\ 0 & 1 & 1/2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$

This is equivalent to the system,

$$x_1-\frac{1}{2}x_3=0,\quad x_2+\frac{1}{2}x_3=0.$$

Taking $x_3 = 1$, we find that $X_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 4$. Every non-zero multiple of the vector X_2 is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 4$. Thus, the complete solution set for this system is the eigenspace $E_4 = \left\{ \alpha \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$.

This is equivalent to $E_4 = \left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$ obtained by multiplying by 2.

3. To determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}.$$

The characteristic polynomial of A is

$$\begin{aligned} |A - xI| = |xI - A| &= \left| \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} - \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \right| \\ &= \begin{vmatrix} x-8 & -6 & 2 \\ -6 & x-7 & -4 \\ 2 & -4 & x-3 \end{vmatrix} \\ &= x^3 - 18x^2 + 45x \\ &= x(x-3)(x-15). \end{aligned}$$

The eigenvalues of A are the solutions of $|xI - A| = 0$. Therefore, the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = 15$.

To find the eigenvectors corresponding to the eigenvalue $\lambda_1 = 0$.

The eigenvectors $X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ of A corresponding to the eigenvalue $\lambda_1 = 0$ are given by the non-zero solutions of the homogeneous system $(A - 0I)X = O$. From this, we get

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_1 \longleftrightarrow R_3$,

$$\begin{pmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \longrightarrow R_2 + 3R_1, R_3 \longrightarrow R_3 - 4R_1$,

$$\begin{pmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_3 \longrightarrow R_3 + 2R_2$,

$$\begin{pmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The coefficient matrix of these equations is rank 2. Therefore these equations have 3-2=1 linearly independent solution. Thus there is only one linearly independent eigenvector corresponding to the eigenvalue 0.
The last matrix equation is equivalent to the system

$$2x_1 - 4x_2 + 3x_3 = 0, \quad -5x_2 + 5x_3 = 0.$$

From the last equation, we get $x_2 = x_3$.

Taking $x_3 = 1 = x_2$, the first equation gives $x_1 = 1/2$. Therefore, $X_1 = \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$. Every non-zero multiple of the vector X_1 is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$.Thus, the complete solution set for this system is the eigenspace $E_0 = \left\{ \alpha \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$. This is equivalent to $E_0 = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$ obtained by multiplying by 2 to remove fractions.

The eigenvectors $X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ of A corresponding to the eigenvalue $\lambda_2 = 3$ are given by the non-zero solutions of the homogeneous system $(A - 3I)X = O$.
From this, we get

$$\begin{pmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or,

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_1 \longrightarrow R_1 + R_2$,

$$\begin{pmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \longrightarrow R_2 - 6R_1$, $R_3 \longrightarrow R_3 + 2R_1$,

$$\begin{pmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & -8 & -4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_3 \longrightarrow R_3 + \frac{1}{2}R_2$,

$$\begin{pmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The coefficient matrix of these equations is rank 2. Therefore, these equations have 3-2=1 linearly independent solution. Thus there is only one linearly independent eigenvector corresponding to the eigenvalue $\lambda_2 = 3$.
The last matrix equation is equivalent to the system

$$-x_1 - 2x_2 - 2x_3 = 0, \quad 16x_2 + 8x_3 = 0.$$

From the second equation we get $x_2 = -\frac{1}{2}x_3$. Taking $x_3 = 4$ gives $x_2 = -2$ and from the first equation $x_1 = -4$. Therefore, $X_2 = \begin{pmatrix} -4 \\ -2 \\ 4 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 3$. Every non-zero multiple of the vector X_2 is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 3$. Thus, the complete solution set for this system is the eigenspace $E_3 = \left\{ \alpha \begin{pmatrix} -4 \\ -2 \\ 4 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$.

The eigenvectors $X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ of A corresponding to the eigenvalue $\lambda_3 = 15$ are given by the non-zero solutions of the homogeneous system $(A - 15I)X = O$.
From this, we get

$$\begin{pmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or,

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_1 \longrightarrow R_1 - R_2$,

$$\begin{pmatrix} -1 & 2 & 6 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \longrightarrow R_2 - 6R_1$, $R_3 \longrightarrow R_3 + 2R_1$,

$$\begin{pmatrix} -1 & 2 & 6 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The coefficient matrix of these equations is rank 2. Therefore, these equations have 3-2=1 linearly independent solution. Thus there is only one linearly independent eigenvector corresponding to the eigenvalue $\lambda_3 = 15$.
The last matrix equation is equivalent to the system

$$-x_1 + 2x_2 + 6x_3 = 0, \quad -20x_2 - 40x_3 = 0.$$

From the second equation we get $x_2 = -\frac{1}{2}x_3$. Taking $x_3 = 1$ gives $x_2 = -2$, and from the first equation $x_1 = 2$. Therefore, $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_3 = 15$. Every non-zero multiple of the vector X_3 is an eigenvector of A corresponding to the eigenvalue $\lambda_3 = 15$. Thus, the complete solution set for this system is the eigenspace $E_{15} = \left\{ \alpha \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$.

4. Determine the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

5. Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Remarks

1. The matrices A and A^T have the same eigenvalues.

2. The eigenvalues of A^θ are the conjugates of the eigenvalues of A.

3. 0 is an eigenvalue of a matrix if and only if the matrix is singular.

4. The eigenvalues of a triangular matrix are just the diagonal elements of the matrix.

5. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, then $\alpha\lambda_1, \alpha\lambda_2, \ldots, \alpha\lambda_n$ are the

eigenvalues of αA.

6. If A is non-singular, the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

7. If λ is an eigenvalue of the matrix A, then $\alpha+\lambda$ is an eigenvalue of the matrix $A+\alpha I$.

8. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a matrix of order n and α is a scalar, then the eigenvalues of $A-\alpha I$ are $\lambda_1-\alpha, \lambda_2-\alpha, \ldots, \lambda_n-\alpha$.

9. The two matrices A and $P^{-1}AP$ have the same eigenvalues.

10. If the eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the eigenvalues of A^2 are $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$.

Cayley-Hamilton Theorem. Every square matrix satisfies its characteristic equation. That is, if for a square matrix A of order n, $|A-xI| = (-1)^n(x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0)$, then $A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \ldots + a_0 I = O$.
Or,
Every square matrix is a root of its characteristic polynomial.

Let A be a square matrix of order n and let $p(x)$ be its characteristic polynomial, say, $p(x) = |xI - A| = (-1)^n(x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0)$.
Let $q(x)$ denote the classical adjoint of the matrix $(xI-A)$. The elements of $q(x)$ are cofactors of the matrix $(xI-A)$, and hence are polynomials in x of degree not exceeding $n-1$. Thus, $q(x) = B_{n-1}x^{n-1} + B_{n-2}x^{n-2} + \ldots + B_1 x + B_0$, where the B_i are $n \times n$ square matrices whose elements are scalars.
Using the property $A(adj\ A) = |A|\, I$, we have $(xI-A)\, q(x) = |xI-A|\, I$. Therefore,

$$(xI-A)(B_{n-1}x^{n-1} + B_{n-2}x^{n-2} + \ldots + B_0) = (-1)^n(x^n + a_{n-1}x^{n-1} + \ldots + a_0).$$

Equating coefficients of like powers of x gives
$B_{n-1} = (-1)^n I,\ \ B_{n-2} - AB_{n-1} = (-1)^n a_{n-1} I,\ \ \ldots,\ \ B_0 - AB_1 = (-1)^n a_1 I,$
$-AB_0 = (-1)^n a_0 I$.
Multiplying these equations successively by $A^n, A^{n-1}, \ldots, A, I$.
$A^n B_{n-1} = (-1)^n A^n I,\ \ A^{n-1}B_{n-2} - A^n B_{n-1} = (-1)^n a_{n-1} A^{n-1},\ \ \ldots,$
$AB_0 - A^2 B_1 = (-1)^n a_1 A,\ \ -AB_0 = (-1)^n a_0 I$.
Adding these equations, we get

$$O = (-1)^n(A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \ldots + a_0 I).$$

Hence, $A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \ldots + a_0 I = O$, showing that the square matrix A satisfies its characteristic equation.

Corollaries.

1. If A be a non-singular matrix, then $|A| \neq 0$. Also $|A| = (-1)^n a_n$ and therefore $a_n \neq 0$.
Pre-multiplying $A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \ldots + a_0 I = O$ by A^{-1}, we get

$$A^{n-1} + a_{n-1}A^{n-2} + a_{n-2}A^{n-3} + \ldots + a_1A^{-2} + a_0A^{-1} = O$$
$$\text{Or, } A^{-1} = (-1/a_0)(A^{n-1} + a_{n-1}A^{n-2} + a_{n-2}A^{n-3} + \ldots + a_1A^{-2}).$$

2. If m is a positive integer such that $m \geq n$, then multiplying $A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \ldots + a_0I = O$ by A^{m-n}, we get

$$A^m + a_{n-1}A^{m-1} + a_{n-2}A^{m-2} + \ldots + a_0A^{m-n} = O,$$

showing that any positive integral power A^m $(m \geq n)$ of A is linearly expressible in terms of those of lower order.

Examples

1. Consider the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$. We shall verify that A satisfies its characteristic equation (Cayley Hamilton Theorem) and hence find A^{-1}.

We have

$$|xI - A| = \begin{vmatrix} x-2 & -1 & 1 \\ -1 & x-2 & -1 \\ 1 & -1 & x-2 \end{vmatrix} = x^3 - 6x^2 + 9x - 4.$$

Therefore, the characteristic equation is $x^3 - 6x^2 + 9x - 4 = 0$.
Now, we have to verify that

$$A^3 - 6A^2 + 9A - 4I = O \tag{7}$$

which follows from Cayley Hamilton Theorem.
We have

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

It follows that

$$A^2 = AA = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^3 = A^2A = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}.$$

Now,

$$A^3 - 6A^2 + 9A - 4I = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$
$$+9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Multiplying (1.8) by A^{-1}, we get

$$A^2 - 6A + 9I - 4A^{-1} = O.$$

Therefore, $A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$.
Now,

$$A^2 - 6A + 9I = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

Hence, $A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$.

2. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$. Verify that A satisfies its characteristic equation (Cayley Hamilton Theorem) and hence find A^{-1}.
Proceed as in above example.

$$A^{-1} = -\frac{1}{2}(A^2 - 6A + 7I) = \begin{pmatrix} -3 & 0 & 2 \\ -1 & 1/2 & 1/2 \\ 2 & 0 & -1 \end{pmatrix}.$$

3. State Cayley-Hamilton theorem. Use it to express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A, when $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$.
Cayley-Hamilton theorem states that every square matrix satisfies its characteristic equation.
We have

$$|A - xI| = \begin{vmatrix} 3-x & 1 \\ -1 & 2-x \end{vmatrix} = x^2 - 5x + 7.$$

Therefore, the characteristic equation of A is

$$|A - xI| = x^2 - 5x + 7 = 0. \tag{8}$$

By Cayley-Hamilton theorem, the matrix A must satisfy (1.9). Therefore, we have

$$A^2 - 5A + 7I = O. \tag{9}$$

From (1.10), we get

$$A^2 = 5A - 7I. \tag{10}$$

Multiplying both sides of (1.11) by A, we get

$$A^3 = 5A^2 - 7A \tag{11}$$

$$\therefore \quad A^4 = 5A^3 - 7A^2 \tag{12}$$

$$\text{and} \quad A^5 = 5A^4 - 7A^3 \tag{13}$$

Now

$$\begin{aligned}
2A^5 - 3A^4 + A^2 - 4I &= 2(5A^4 - 7A^3) - 3A^4 + A^2 - 4I \quad \text{[by (1.14)]} \\
&= 7A^4 - 14A^3 + A^2 - 4I \\
&= 7(5A^3 - 7A^2) - 14A^3 + A^2 - 4I \quad \text{[by (1.13)]} \\
&= 21A^3 - 48A^2 - 4I \\
&= 21(5A^2 - 7A) - 48A^2 - 4I \quad \text{[by (1.12)]} \\
&= 57A^2 - 147A - 4I \\
&= 57(5A - 7I) - 147A - 4I \quad \text{[by (1.11)]} \\
&= 138A - 403I, \text{ which is a linear polynomial in A.}
\end{aligned}$$

4. Verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Find the inverse of the matrix A and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

5. Verify Cayley-Hamilton theorem for the matrices and hence find their inverses.

$$(i)\ A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix} \qquad (ii)\ A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(iii)\ A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad (iv)\ A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$$

Diagonalization of Matrices

One of the most important applications of eigenvalues and eigenvectors is in the diagonalization of matrices. Because diagonal matrices have such a simple structure, it is relatively easy to compute a matrix product when one of the matrices is diagonal. As we will see later, other important matrix computations are also easier when using diagonal matrices. Hence, if a given square matrix can be replaced by a corresponding diagonal matrix, it could greatly simplify computations involving the original matrix. Therefore, our next goal is to present a formal method for using eigenvalues and eigenvectors to find a diagonal form for a given square matrix, if possible. Before stating the method, we motivate it with an example.

Example

Consider a 3×3 matrix

$$A = \begin{pmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 0$. $X = (4,3,0)$ and $Y = (-2,0,1)$ are two linearly independent eigenvectors corresponding to the eigenvalue $\lambda_1 = 2$ and $Z = (-1,1,1)$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 0$. Using these three vectors as columns we form a 3×3 matrix

$$P = \begin{pmatrix} -4 & -2 & -1 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Here, $|P| = -1$ and so P is non-singular. The inverse of P is calculated to be

$$P^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & -4 & 7 \\ -3 & 4 & -6 \end{pmatrix}.$$

By using A, P and P^{-1}, we can now compute a diagonal matrix D as

$$\begin{aligned} D &= P^{-1}AP \\ &= \begin{pmatrix} 1 & -1 & 2 \\ 3 & -4 & 7 \\ -3 & 4 & -6 \end{pmatrix} \begin{pmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{pmatrix} \begin{pmatrix} -4 & -2 & -1 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Each main diagonal entry d_{ii} of D is an eigenvalue having an associated eigenvector in the corresponding column of P.

Similarity of matrices. A matrix B is similar to a matrix A if there exists some (non-singular) matrix P such that $P^{-1}AP = B$.

As a consequence of this definition, we see that the diagonal matrix D in the above example is similar to the original matrix A. The computation

$$(P^{-1})^{-1}D(P^{-1}) = PDP^{-1} = PP^{-1}APP^{-1} = (PP^{-1})A(PP^{-1}) = A$$

shows that A is also similar to D.

In general, for any two matrices A and B, A is similar to B if and only if B is similar to A. Thus, we will just say A and B are similar matrices without giving an 'order' to the similarity relationship.

Two properties of similarity relation between matrices.

(*i*) Similar matrices must be square, have the same size, and have equal determinants.

(*ii*) Similar matrices have identical characteristics polynomials.

The following is an important result regarding the diagonalisation process.

Let A and P be $n \times n$ matrices such that each column of P is an eigenvector for A. If P is non-singular, then $D = P^{-1}AP$ is a diagonal matrix similar to A. The ith main diagonal entry d_{ii} of D is the eigenvalue for the eigenvector forming the ith column of P.

Method for diagonalising an n × n matrix A (if possible)

Step 1. Calculate the characteristic polynomial, $p_A(x) = |xI_n - A|$.

Step 2. Find all real roots of $p_A(x)$, that is, all real solutions to $p_A(x) = 0$. These are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ for A.

Step 3. For each eigenvalue λ_m in turn:

Row reduce the augmented matrix $[\lambda_m I_n - A|O]$. Use the result to obtain a set of particular solutions to the homogeneous system $(\lambda_m I_n - A)X = O$ by setting each independent variable in turn equal to 1 and all other independent variables equal to 0. (One may eliminate fractions from these solutions by replacing them with non-zero scalar multiples). We will often refer to the particular eigenvectors that are obtained in this manner as **fundamental eigenvectors**.

Step 4. If, after repeating Step 3 for each eigenvalue, you have fewer than n fundamental eigenvectors overall for A, then A cannot be put into diagonal form. Stop.

Step 5. Otherwise, form a matrix P whose column whose column vectors are these n fundamental eigenvectors. This matrix P is non-singular.

Step 6. To check your work, verify that $D = P^{-1}AP$ is a diagonal matrix whose d_{ii} entry is the eigenvalue for the fundamental vector forming the ith column of P. Also note that $A = PDP^{-1}$.

Examples

1. Consider the 4×4 matrix

$$A = \begin{pmatrix} -4 & 7 & 1 & 4 \\ 6 & -16 & -3 & -9 \\ 12 & -27 & -4 & -15 \\ -18 & 43 & 7 & 24 \end{pmatrix}.$$

Step 1. A little lengthy calculation gives $p_A(x) = x^4 - 3x^2 - 2x = x(x-2)(x+1)^2$.

Step 2. The eigenvalues of A are the roots of $p_A(x)$, namely, $\lambda_1 = -1$, $\lambda_2 = 2$, and $\lambda_3 = 0$.

Step 3. We first compute the eigenvectors for $\lambda_1 = -1$. Row reducing $[(-1)I_4 - A|O]$ yields

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Setting the first independent variable (corresponding to column 3) equal to 1 and the second independent variable (column 4) equal to 0 gives a fundamental eigenvector $X_1 = (-2, -1, 1, 0)$. Setting the second independent variable equal to 1 and the first independent variable equal to 0 gives a fundamental eigenvector $X_2 = (-1, -1, 0, 1)$.

Similarly, we row reduce $[2I_4 - A|O]$ to obtain the eigenvector $(1/6, -1/3, -2/3, 1)$. We multiply this by 6 to avoid fractions, yielding a fundamental eigenvector $X_3 = (1, -2, -4, 6)$. Finally, from $[0I_4 - A|O]$, we obtain a fundamental eigenvector $X_4 = (1, -3, -3, 7)$.

Step 4. We have produced four fundamental eigenvectors for this 4×4 matrix, so we proceed to Step 5.

Step 5. Let

$$P = \begin{pmatrix} -2 & -1 & 1 & 1 \\ -1 & -1 & -2 & -3 \\ 1 & 0 & -4 & -3 \\ 0 & 1 & 6 & 7 \end{pmatrix},$$

the matrix whose columns are our fundamental eigenvectors X_1, X_2, X_3, X_4.

Step 6. Calculating $D = P^{-1}AP$, we verify that D is the diagonal matrix whose corresponding entries on the main diagonal are the eigenvalues -1, -1, 2 and 0, respectively.

Diagonalizable matrix

An $n \times n$ matrix A is diagonalisable if and only if there exists a non-singular $n \times n$ matrix P such that $D = P^{-1}AP$ is diagonal.

Note: Unsolved problems are left as exercises. Use this material as supplementary to class lectures. For any query, feel free to write: *jamkhongamtouthang@dtu.ac.in*