ME 613: Finite Element and Boundary Element Methods

Spring 2024, Indian Institute of Technology, Bombay

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Homework 2A

Assigned on: 12 Feb 2024 Due date: 22 Feb 2024

Note: You can use symbolic manipulation to determine the local stiffness (coefficient) matrix and load (source) vector for the different problems. You have to write the local equations and assembly procedure by hand. Any matrix inversion can also be performed using the computer. For higher-order interpolation, assume the nodes to be equally spaced unless mentioned otherwise.

Problem 1

Consider the 1-D bar problem we studied in class (fig. 1):

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + f_0 = 0 \text{ for } 0 < x < L,$$

subject to the boundary conditions,

$$u(0) = 0, \quad \left(EA\frac{du}{dx}\right)\bigg|_{x=L} = P.$$

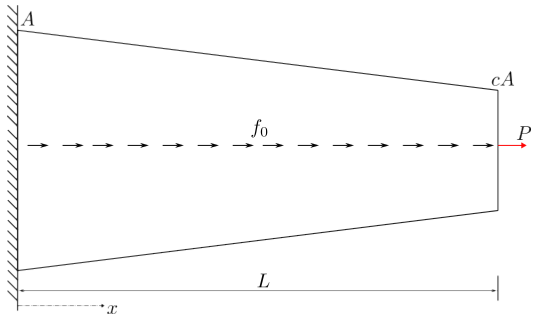

Figure 1: 1-D bar with linearly varying cross-section, subject to a distributed, as well as end load.

4.a Derive the weak form of this differential equation, and determine the stiffness and load vector for a 2-node finite element (use $f_0 = \rho g A$ and the actual variation of the cross-sectional area to obtain the expressions. Determine the tip displacement of the bar using this modified linear element (use $c = 0.5$, and discretize the bar with two elements along the length).

4.b Assuming that the Young's Modulus (E) and the cross-sectional area (A) is constant, derive the element stiffness and load vector for a 3-node finite element discretization, with the interior node places such that the finite element is divided into two halves of length $1:2$ - in other words, *use the Lagrangian polynomial framework we discussed in class during the derivation for the quadratic interpolation, with the interior node at* $\xi = l/3$.

4.c Assume that the Young's Modulus E is constant, and $c = 0.5$. Discretize this bar with two quadratic finite elements (the one you derived above) and solve for the displacement of the free end (Assume each element to have a constant cross-sectional area, which is the average of the cross-sectional area of the left and right end of the element).

4.d Derive the stiffness and load vector for a 4-node finite element, assuming that the nodes are equally spaced (assume E, A to be constant). Use a one-element discretization to solve the above problem.

4.e Tabulate (or plot) the values obtained from the above solutions and compare them with the analytical solution. Ensure that you have equal number of points in all cases (if the nodal values are different, use the appropriate element-level interpolation to find the values for comparison).

Problem 2

A steel rod of diameter $D = 2$ cm, length $L = 5$ cm, and thermal conductivity $k = 50\,\mathrm{W/m/°C}$ is exposed to ambient air at $T_\infty = 20°\mathrm{C}$, with a heat transfer coefficient $h = 100\,\mathrm{W/m^2/°C}$. If the left end of the rod is maintained at temperature $T_0 = 320°\mathrm{C}$, determine the temperatures at distances 2.5 cm and 5 cm from the left end, as well as the heat flux at the left end. Use (a) two linear elements and (b) one quadratic element to solve the problem using the finite element method (you can use the local stiffness and force vector expressions derived in class directly).

Problem 3

Consider the hollow tube shown in fig. 2, which has inner radius R_i and outer radius R_o. The tube carries a fluid at temperature T_i, while the outer temperature is T_o. The tube is in steady-state conditions and is governed by the following ODE:

$$r\frac{d^2T(r)}{dr^2} + \frac{dT(r)}{dr} = 0,$$

where $T(r)$ $(R_i \leq r \leq R_o)$ is the temperature profile through the thickness of the tube. Write the weak form of the governing equation, and determine the expression for the stiffness matrix using a piecewise **quadratic** approximation for the variation in temperature (assume that the tube is discretized into 2 elements of equal length/thickness). Write down the system of equations to determine the temperature gradient along the radius (provide the appropriate equations along with your explanation)?

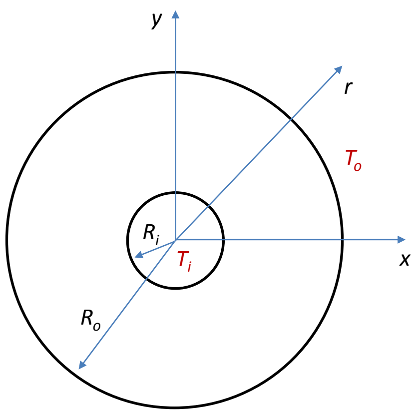

Figure 2: Cross-section of a hollow tube carrying fluid at temperature T_i. The exterior of the tube is exposed to ambient temperature (T_o).

Problem 4

Consider the steady laminar flow of a Newtonian fluid with constant density in a long annular region between two coaxial cylinders of radii R_i and R_0. The differential equation for this problem is given by

$$\frac{1}{r}\frac{d}{dr}\left(r\mu\frac{dv}{dr}\right) + f_0 = 0,$$

where v is the velocity along the axis of the cylinder (z-direction), μ is the viscosity, and f_0 is the source term due to the combined effect of pressure and gravitational loads on the cylinder. The boundary conditions for a fully developed laminar flow are

$$v = 0 \text{ at } r = R_i, R_o,$$

where R_i, R_o refer to the radius of the solid and hollow cylinders, respectively. Using the discretization developed in Problem 3 (use one/two elements), obtain the solution for the velocity field and compare the solution with the exact solution

$$v_e(r) = \frac{f_0 R_0^2}{4\mu}\left[1 - \left(\frac{r}{R_o}\right)^2 + \frac{1-k^2}{ln(1/k)}ln\left(\frac{r}{R_0}\right)\right],$$

where $k = R_i/R_o$. Determine the shear stress $\tau_{rz} = -\mu dv/dr$ at the walls using the velocity field and compare with the exact values.

Problem 5

Consider the simply supported 1-D Beam subjected to a distributed load, as shown in fig. 3. Assuming $q(x) = q_0 \sin(\pi x/L)$, determine the displacement using the finite element method, for a discretization of one/two elements along the length. Draw the shear force and bending moment diagrams for both the cases (using the determined displacement values), and compare them with the analytical solution.

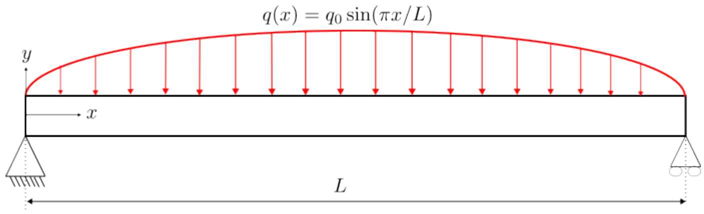

Figure 3: Euler-Bernoulli beam subjected to sinusoidally varying distributed load.

Problem 6

Consider the following differential equations governing bending of a beam using the Euler-Bernoulli Beam theory:

$$\frac{d^2w}{dx^2} - \frac{M}{EI} = 0, \quad \frac{d^2M}{dx^2} - kw + q = 0,$$

where w denotes the transverse deflection, M the bending moment, q the distributed transverse load, and k the elastic foundation modulus. Choosing M and w to be primary variables, develop the weak form of the coupled second-order differential equations, and the local matrix-vector equations over a typical element (x_k, x_{k+1}).