ME 613: Finite Element and Boundary Element Methods
Spring 2024, Indian Institute of Technology, Bombay
Prof. R. Ganesh

Homework 1

Assigned on: 22 Jan 2024 Due date: 07 Feb 2024

Problem 1

Consider the two plane structures shown in fig. 1, where rigid blocks (shaded regions) are connected by flexible bars (red color). Assuming that only horizontal displacements are allowed, write the reduced global equilibrium equations in terms of stiffness k^e, nodal displacements u_i, and applied loads F_i. Further, assuming $F_3 = 0$ and $u_3 = u_0$, where u_0 is a specified **non-zero displacement**, write the equilibrium equations in the appropriate unknown variables u_i.

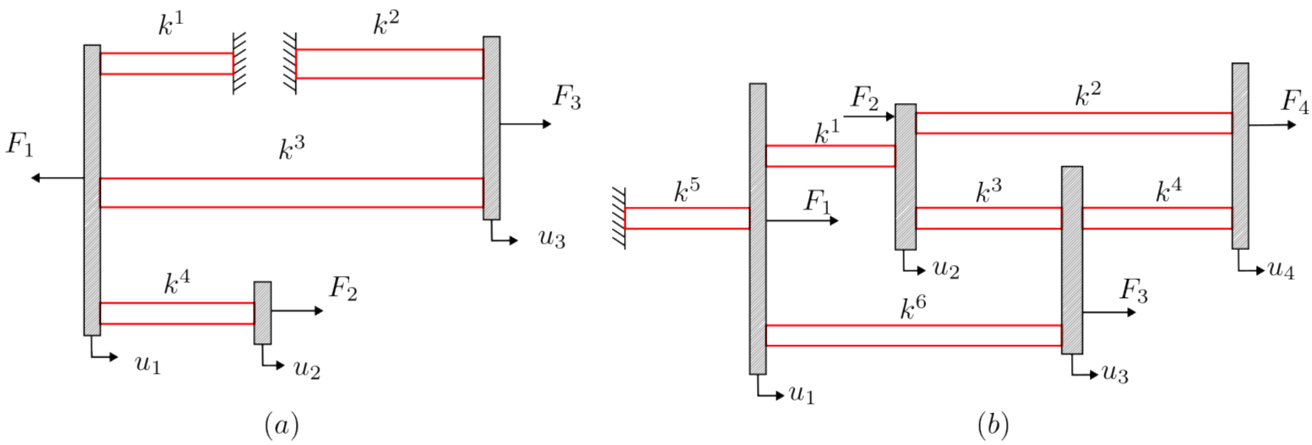
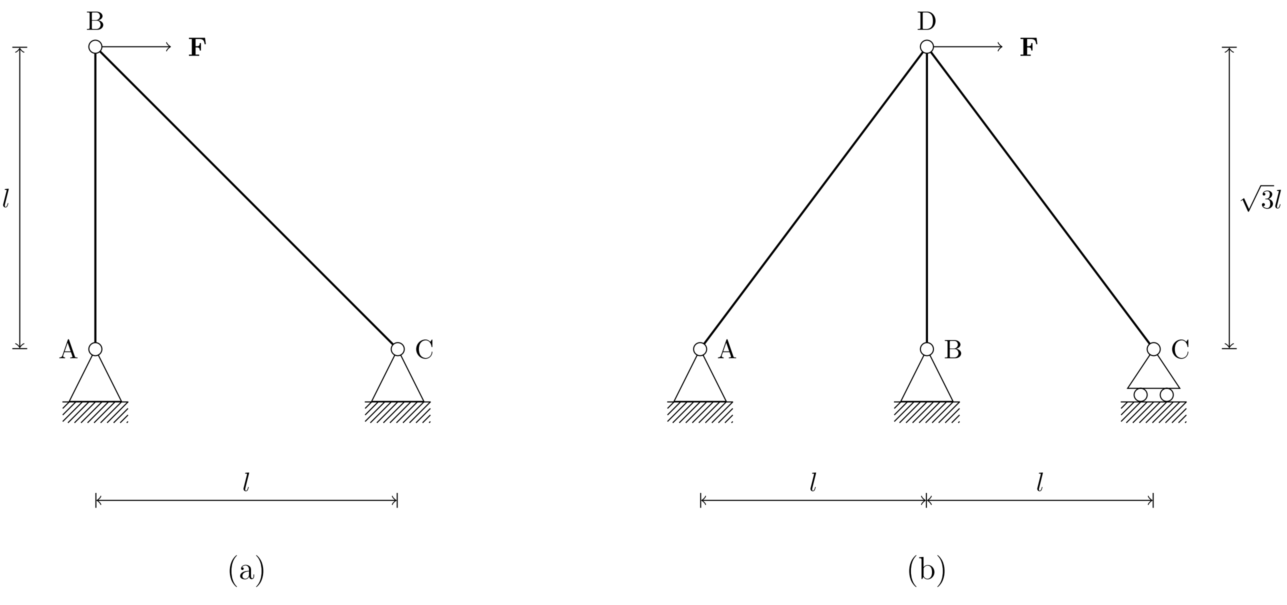


Figure 1: Two planar structures consisting of rigid blocks connected by flexible bars.

Problem 2

Consider the two truss structures shown below. Assuming that the cross-sectional area (A) and the Young's Modulus (E) of all the bars are same, determine the displacements of the free end of the truss and the reactions at the supports using the direct application of the Finite Element Method (as a function of $\mathbf{F}, l, A, E$).



Problem 3

Consider the following differential equation:

$$\frac{d^2u}{dx^2} + u = x^2 \text{ for } 0 < x < 1,$$

which is subject to the boundary conditions

$$u(0) = 0, \quad \left.\frac{du}{dx}\right|_{x=1} = 1.$$

3.a Construct an approximate two-term and three-term polynomial solution (you should have two/three constants in your approximation, respectively).
3.b Use the collocation method to determine the approximate solution for both the approximations. Assume the collocation points to be $x_i = 1/3, 2/3$ and $x_i = 1/4, 1/2, 3/4$ for the corresponding approximations.
3.c Use the method of least squares to determine the constants for the two approximations.
3.d Use the Galerkin method to determine the constants for the two approximations *(use the weighted-residual for the differential equation, not the weak form).*

Problem 4

Consider the following differential equation:

$$\frac{d^2u}{dx^2} = -\cos(\pi x) \text{ for } 0 < x < 1,$$

which is subject to the boundary conditions

$$u(0) = 0, \ \ u(1) = 0.$$

4.a Construct an approximate three-term polynomial solution, as well as a three-term trigonometric solution (you should have three constants in your approximation).
4.b Use the collocation method to determine the approximate solution for both the approximations. Assume $x_i = 1/4, 1/2, 3/4$ as the collocation points.
4.c Use the Galerkin method to determine the constants for the two approximations *(use the weighted-residual for the differential equation, not the weak form).*
4.d Would the solution change if you applied the Galerkin method to the weak form of the differential equation? Explain your answer.

Problem 5

Consider the following differential equation:

$$\frac{d}{dx}\left(u\frac{du}{dx}\right) - f(x) = 0 \text{ for } 0 < x < L,$$

which is subject to the boundary conditions

$$\left.\left(u\frac{du}{dx}\right)\right|_{x=0} = 0, \ \ u(L) = u_0.$$

Write down the weak form for the differential equation.

Problem 6

Consider the following differential equation

$$2u\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2 + 4 = 0, \quad 0 < x < 1,$$

which is subject to the boundary conditions

$$u(0) = 1, \ \ u(1) = 0.$$

6.a Write down the weak form of the weighted residual equation. What are the conditions that the weighting function must satisfy?
6.b Construct a two-constant polynomial approximation that satisfies the boundary conditions. Choose the appropriate weighting function to solve the **weak form of the weighted residual using the Galerkin approximation**, and determine the constants.
6.c Construct a two-constant trigonometric approximation for u and choose the appropriate weighting function to solve the **weak form of the weighted residual using the Galerkin approximation**.

Problem 7

Consider a hollow tube as shown in fig. 2. The tube has inner radius R_i and outer radius R_o, and carries a fluid at temperature T_i, while the outer temperature is T_o. The tube is in steady-state conditions and is governed by the following ODE:

$$r\frac{d^2T(r)}{dr^2} + \frac{dT(r)}{dr} = 0,$$

where $T(r)$ ($R_i \leq r \leq R_o$) is the temperature profile through the thickness of the tube. Write the weak form of the governing equation, and determine the expression for the stiffness matrix using a piecewise linear approximation for the variation in temperature (assume that the tube is discretized into 3 elements of equal length/thickness). How will you solve the system of equations to determine the temperature gradient along the radius (provide the appropriate equations along with your explanation)?
Hint: You can follow the procedure for the 1D bar equation from class, with the appropriate changes in the expressions/boundary conditions.

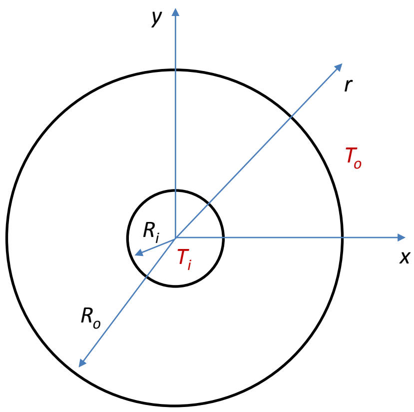


Figure 2: Cross-section of a hollow tube carrying fluid at temperature T_i. The exterior of the tube is exposed to ambient temperature (T_o).

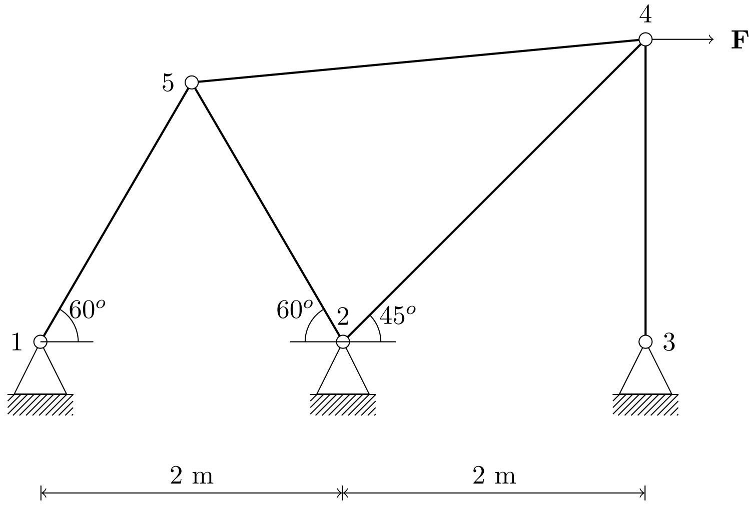
Problem 8

Write a numerical code in MATLAB/Python to solve the Truss problems given below (solve for displacements at free nodes, and support at the reactions; use $A = 0.01\,\mathrm{m}^2$, $E = 150\,\mathrm{GPa}$, $\mathbf{F} = 7\,\mathrm{kN}$). You may begin from the example(s) shared in class for the 1D problem, and modify them accordingly to solve these problems. For this case, it is sufficient to write a general function which takes the truss geometry/material data as input, and gives the global stiffness matrix as output. Then, you can apply the boundary conditions and forcing for each problem separately (you can also give this as input to the code by using binary arrays to set the fixed degrees of freedom to zero, as well as a separate force vector). The input array to the function should be of the form:

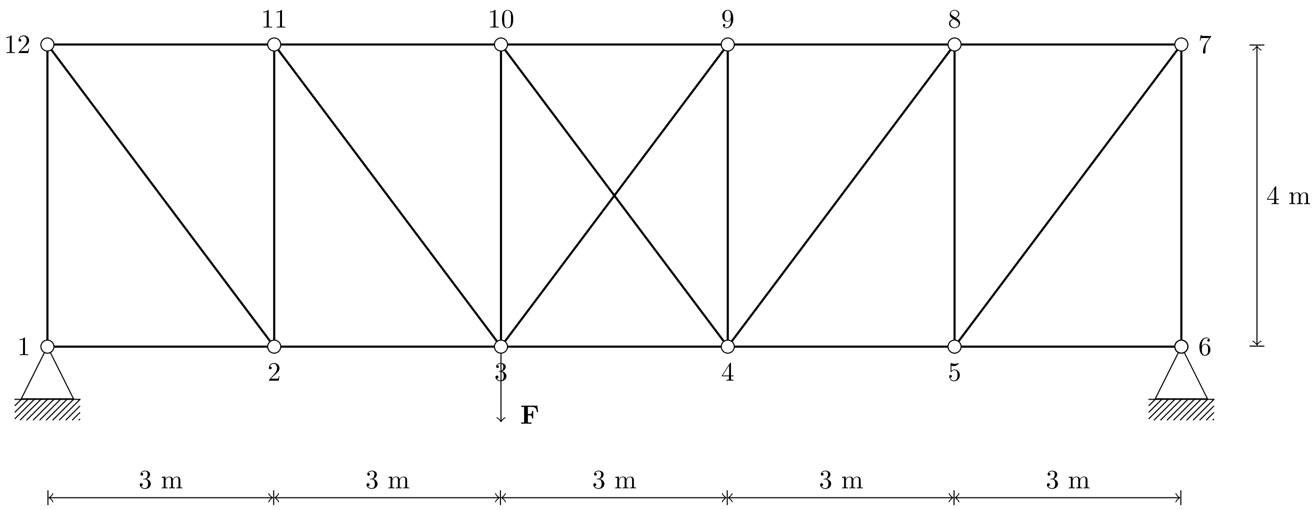
Nodal connectivity 1	Nodal connectivity 2	Angle w.r.t X axis	Element Stiffness

In other words, for each bar in the truss, specify the global node numbers that make up the bar (Nodal connectivity 1 & 2), the angle the individual bar makes w.r.t the X-axis, and the Element Stiffness value (EA/L). The table below gives the data for the first truss structure (the element number is the same as the row index in the table; fill the appropriate value for θ and k^e). Ensure that the order of your node number and the angle measurement is consistent with the coordinate system you choose. You may want to first verify your code and input for the structures you solved in Problem 2 (use appropriate numerical values).

Nodal connectivity 1	Nodal connectivity 2	Angle w.r.t X axis	Element Stiffness
1	5	60°	k^1
2	5	120°	k^2
2	4	45°	k^3
3	4	90°	k^4
5	4	θ°	k^5



(a)



(b)

Acknowledgement

The truss structures in this homework were drawn in LaTeXusing the library provided by Nicola Rainiero.