SOC-2024: Finite Element Analysis

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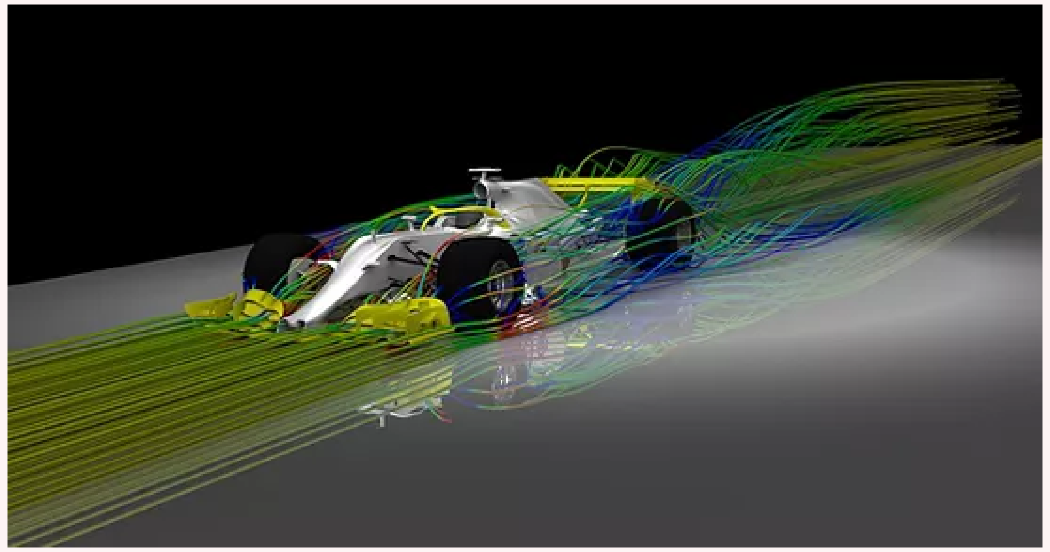
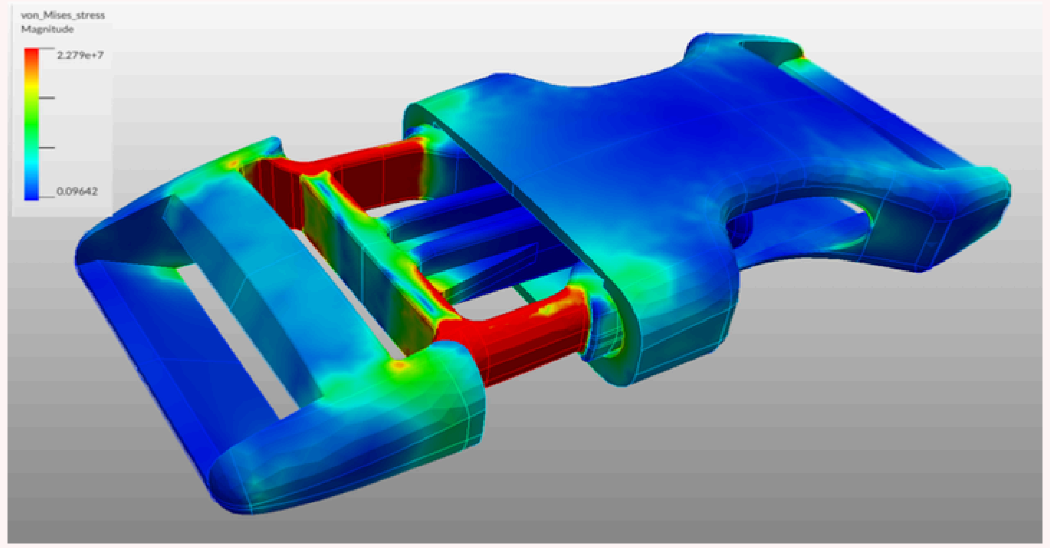
About FEM

- Numerical integration of differential equations in the form of governing equations
- Applications: Fluid mechanics, Solid mechanics, Thermal analysis, Electro-magnetodynamics
- Highly systematic and strustured process - can be automated

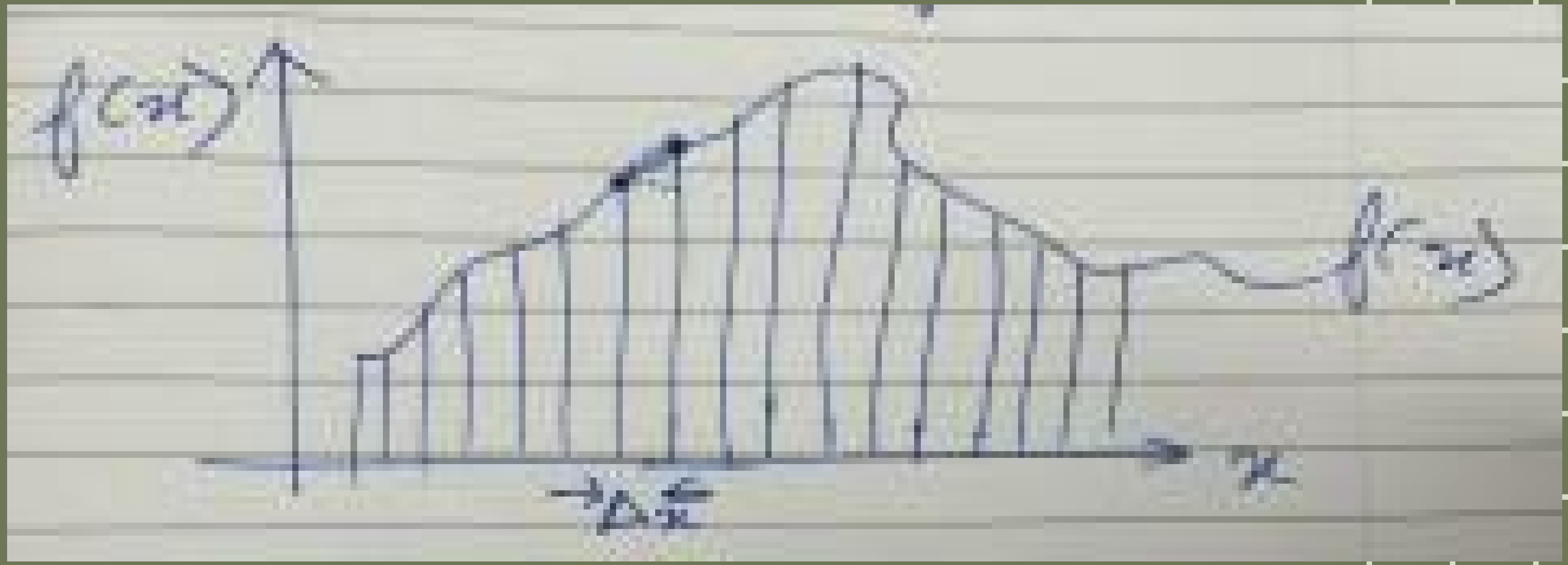




Ideology Involved

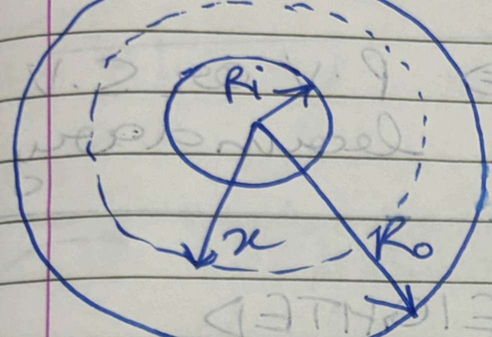
- Exact Solutions: error = 0
- Approximate solutions: involve non zero error
  - Analytic solutions: we get formulae as solutions
  - Numerical Solutions: we get numerical values in the domain
- Involves discretization of the domain

$$\int_{\Omega} (\text{Error} \cdot \text{Wt. function})\, d\Omega = 0$$



Heat conduction example

*Heat conduction (Basic Example)



BCs: ① $K r \frac{dT}{dr} = 0$ @ $r = R_i$

② $T(R_o) = T_o$

G.E: $-\frac{1}{r}\left[\frac{d}{dr}\left(K r \frac{dT}{dr}\right)\right] = q(r)$

thermal load / heat generation

① ② ③ h_N

$h_1, h_2, h_3, \ldots$ (N)

R_i R_o

$\boxed{T_j^e : T \text{ @ } j^{th} \text{ pt in } e^{th} \text{ element}}$

$\boxed{T_e(r) = \sum_{j=1}^{n} T_j^e \psi_j^e(r)}$

→ for e^{th} element

n unknowns/element

⇓

put in G.E → GET R(T)

$\boxed{R(T) \Downarrow \text{Residue}}$

⇓

$\int_{h_e=1}^{h_e} R(T) \cdot w_1 \, d\Omega = 0$ → ①

" w_2 " → ②

⋮

w_n " " n eqns using n w(s)

single element

ASSEMBLE

$$\begin{bmatrix} n \times n \end{bmatrix} \begin{Bmatrix} T_1^e \\ T_2^e \\ \vdots \\ T_n^e \end{Bmatrix}_{n \times 1} = \begin{Bmatrix} w \\ \vdots \end{Bmatrix}_{n \times 1}$$

N times for all elements

Methods of FEM

- Variational Methods: Rayleigh Ritz Method
- Method of weighted Residuals:
  - Petrov Galerkin Method
  - Galerkin
  - Method of Least Squares
  - Collocation Method
- Different in the form of integral form used and the type of shape functions used

Weighted Integrals

* Weighted Integrals & Weak form

↳ $-\frac{d}{dx}\left[a\frac{du}{dx}\right] = q(x)$ $\quad x \in (0, L)$

$u(0) = u_0$ at $x=0$
↳ Essential BC

& $a\frac{du}{dx}\Big|_{x=L} = Q_0$ → Natural BC.

Weak formulation

↳ Step ① ⇒ $\int_0^L w\cdot\left[-\frac{d}{dx}[a(x)u'] - q\right]dx = 0$

↳ STRONG FORM

Step ② : By parts ⇒

$$\int_0^L \left[\left(\frac{dw}{dx}\right)a(x)u' - wq\right]dx - \left[w\,a(x)\,u'\right]_0^L = 0$$

↳ WEAK FORM

Step ③ : Choose : $u = \sum_{j=1}^{N} u_j \phi_j(x)$

$$\Rightarrow \int w'\left[a(x)\sum_{j=1} c_j\phi_j'(x)\right]dx - \int wq\,dx - \left[w\,a^*(x)\,u'\right]_0^L = 0.$$

\# Coeff of w & derivatives ⇒ Secondary variables (S.V)

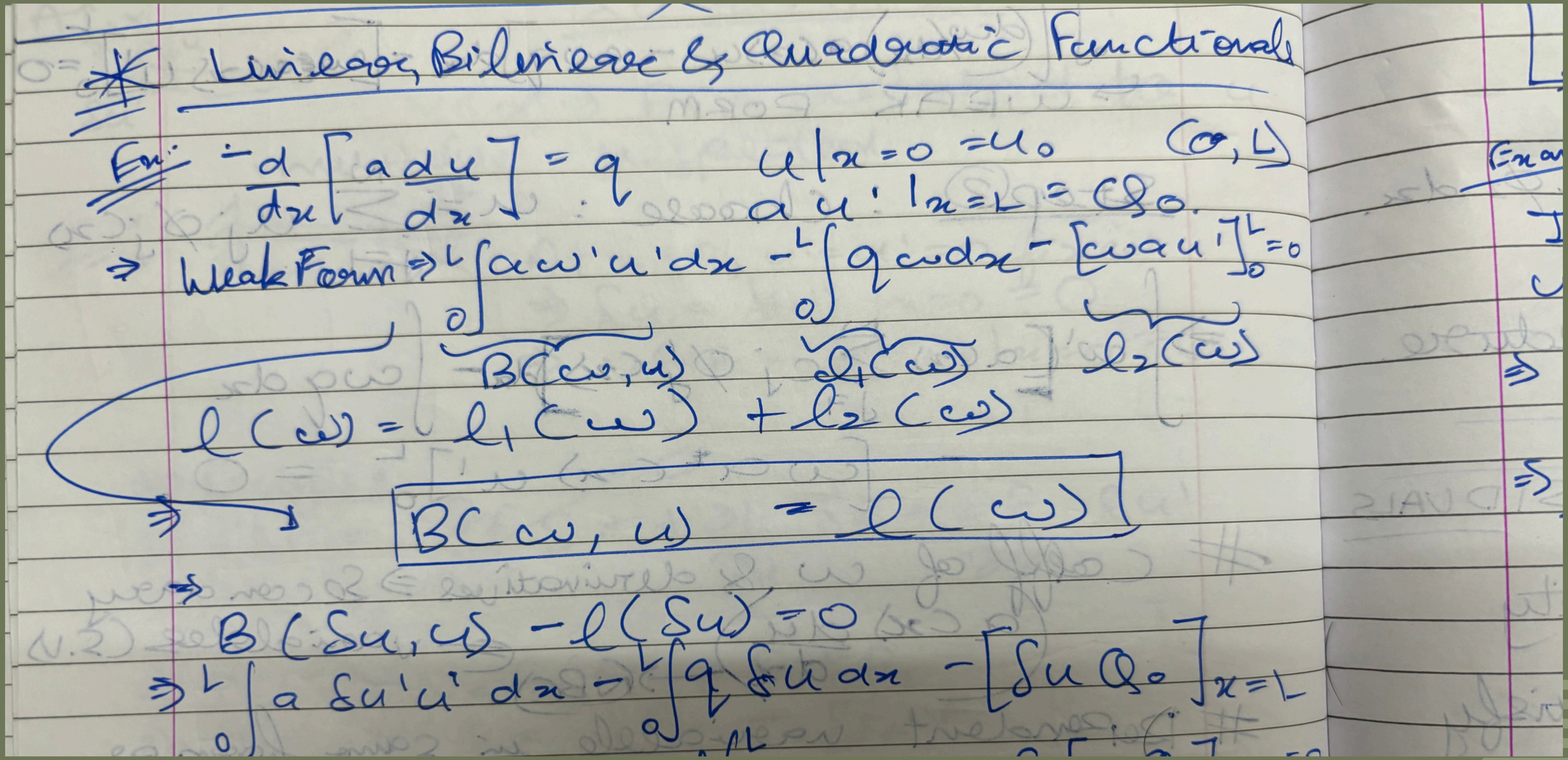
$\left(a(x)\frac{du}{dx}\right)$ → (NBC)

\# Dependent variable in same form as w & derivatives ⇒ Primary var. (PV)

↳ EBCs.

Functionals

Map dependent variables to a scalar



Linear, Bilinear & Quadratic Functionals

Ex: $-\frac{d}{dx}\left[a\frac{du}{dx}\right] = q$ $\quad u|_{x=0} = u_0 \quad (0, L)$

$a u'|_{x=L} = Q_0$

⇒ Weak Form ⇒ $\underbrace{\int_0^L a w' u' dx}_{B(w,u)} - \underbrace{\int_0^L q w dx}_{l_1(w)} - \underbrace{[w a u']_0^L}_{l_2(w)} = 0$

$l(w) = l_1(w) + l_2(w)$

⇒ $\boxed{B(w, u) = l(w)}$

⇒ $B(\delta u, u) - l(\delta u) = 0$

⇒ $\int_0^L a\, \delta u' u'\, dx - \int_0^L q\, \delta u\, dx - [\delta u\, Q_0]_{x=L}$

Matrix Form

$N=2$

$\boxed{\text{Example}}$

$$B(w,u) = \int_0^L a w' u' \, dx \,, \quad u = \sum c_j \phi_j \,, \quad w = \phi_i$$

$$\Rightarrow B(\phi_i, u) = \int_0^L \sum a \phi_i' \phi_j' c_j \, dx$$

$$= \int_0^L a \phi_i' [\phi_1' c_1 + \phi_2' c_2] \, dx$$

when $i = 1 \Rightarrow$

$$B(\phi_1, u) = c_1 \underbrace{\int_0^L a \phi_1' \cdot \phi_1' \, dx}_{B_{11}} + c_2 \int_0^L \underbrace{a \phi_1' \phi_2' \, dx}_{B_{12}} \hookrightarrow EQ①$$

when $i = 2 \Rightarrow$

$$B(\phi_2, u) = c_1 \int_0^L \underbrace{a \phi_2' \phi_1'}_{B_{21}} \, dx + c_2 \int_0^L \underbrace{a \phi_2' \phi_2' \, dx}_{B_{22}} \hookrightarrow EQ②$$

$$\Rightarrow \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\& \; l(w) = \int q w \, dx \longrightarrow \{F\}$$

$$\Rightarrow F_1 = \int_0^L q \cdot \phi_1 \, dx \;\&\; F_2 = \int_0^L q \cdot \phi_2 \, dx$$

Element Analysis

* Element level equations

$$\int_{x_A}^{x_B} [a w' u' + c w u]\, dx = \int_{x_A}^{x_B} w q \, dx + \sum_{j=1}^{n} w(x_j^e) Q_j^e$$

n = no. of nodes in element

$w = \psi_i$

& $u = \sum u_j^e \psi_j^e$

$$\Rightarrow \int_{x_A}^{x_B} \sum \left[a \psi_i'^e \psi_j'^e u_j^e + c \psi_i^e \psi_j^e u_j^e \right] = \int_{x_A}^{x_B} \psi_i q \, dx + \sum w(x_j^e) Q_j^e$$

For $i=1 \Rightarrow$ a linear element

$$\int_{x_A}^{x_B} \sum_{j=1}^{2} \left[a \psi_1'^e \psi_j'^e u_j^e + c \psi_1^e \psi_j^e u_j^e \right] dx = \int_{x_A}^{x_B} \psi_1 q \, dx + \sum_{j=1}^{2} w(x_j^e) Q_j^e$$

$L.H.S \equiv K_{11}^e u_1^e + K_{12}^e u_2^e$

$$K_{11}^e = \int_{x_A}^{x_B} \left[a \frac{d\psi_1^e}{dx} \cdot \frac{d\psi_{j=1}^e}{dx} + c \psi_1^e \psi_{j=1}^e \right] dx$$

$$K_{12}^e = \int_{x_A}^{x_B} \left[a \frac{d\psi_1^e}{dx} \frac{d\psi_{j=2}^e}{dx} + c \psi_1^e \psi_{j=2}^e \right] dx$$

$$R.H.S = \int_{x_A}^{x_B} \psi_1 q \, dx = f_1^e \ \& \ Q_1^e$$

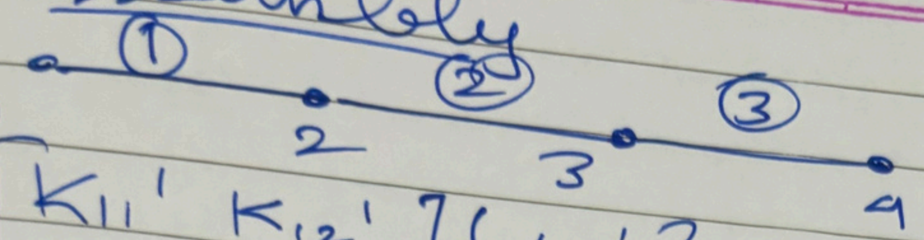
$$\Rightarrow K_{11}^e u_1^e + K_{12}^e u_2^e = f_1^e + Q_1^e$$

$$K_{21}^e u_1^e + K_{22}^e u_2^2 = f_2^e + Q_2^e$$

$$\equiv [K]\{u^e\} = \{f^e\} + \{Q^e\}$$

Assembly

Assembly



$$\begin{bmatrix} K_{11}^1 & K_{12}^1 \\ K_{21}^1 & K_{22}^1 \end{bmatrix}\begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = \begin{Bmatrix} f_1^1 + Q_1^1 \\ f_2^1 + Q_2^1 \end{Bmatrix}$$

$$\begin{bmatrix} K_{11}^2 & K_{12}^2 \\ K_{21}^2 & K_{22}^2 \end{bmatrix}\begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix} = \begin{Bmatrix} f_1^2 + Q_1^2 \\ f_2^2 + Q_2^2 \end{Bmatrix}$$

$$\begin{bmatrix} K_{11}^3 & K_{12}^3 \\ K_{21}^3 & K_{22}^3 \end{bmatrix}\begin{Bmatrix} u_1^3 \\ u_2^3 \end{Bmatrix} = \begin{Bmatrix} f_1^3 + Q_1^3 \\ f_2^3 + Q_2^3 \end{Bmatrix}$$

→ Continuity: $u_2^1 = u_1^2 = U_2$ & $u_2^2 = u_1^3 = U_3$

6 EQNs. & 3 elements (linear)

→ Force balance ⇒ $F_2 = f_2^1 + f_1^2$ & $F_3 = f_2^2 + f_1^3$

& common nodes ⇒ $\Sigma Q = 0$ ⇒ $Q_B^1 = Q_A^2$ & $Q_B^2 = Q_A^3$

$\Downarrow$

$$\underbrace{\begin{bmatrix} K_{11} & K_{12} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & 0 & K_{43} & K_{44} \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} f_1 + Q_1 \\ f_2 \\ f_3 \\ f_4 + Q_4 \end{Bmatrix}}_{\text{Assembly level.}}$$

Applications

- Heat Transfer problems
- Euler - Bernoulli Beam
- Shear Deformable Beams
- EigenValue Problems
- Time Dependent Problems

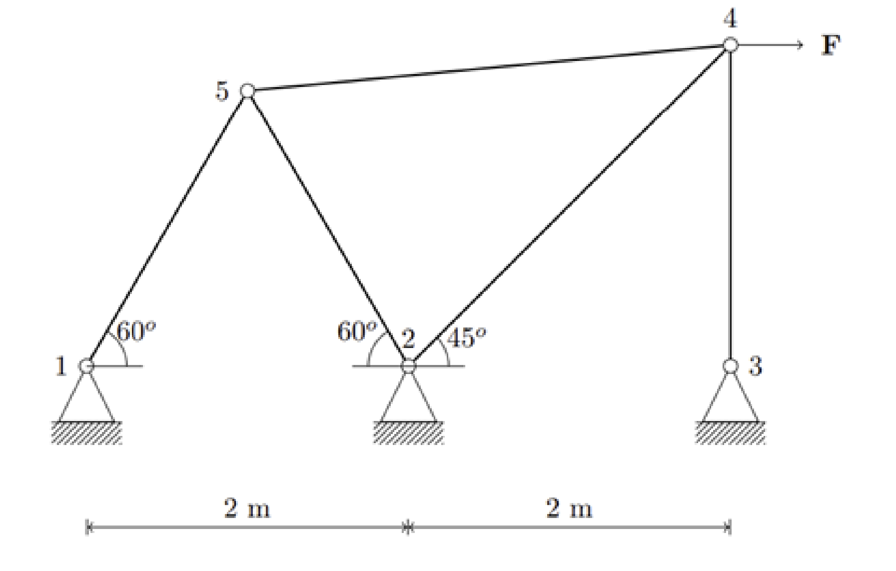
Problem Statement

Write a numerical code in MATLAB/Python to solve the Truss problems given below (solve for displacements at free nodes, and support at the reactions; use $A = 0.01\,\mathrm{m}^2$, $E = 150\,\mathrm{GPa}$, $\mathbf{F} = 7\,\mathrm{kN}$). You may begin from the example(s) shared in class for the 1D problem, and modify them accordingly to solve these problems. For this case, it is sufficient to write a general function which takes the truss geometry/material data as input, and gives the global stiffness matrix as output. Then, you can apply the boundary conditions and forcing for each problem separately (you can also give this as input to the code by using binary arrays to set the fixed degrees of freedom to zero, as well as a separate force vector). The input array to the function should be of the form:

Nodal connectivity 1	Nodal connectivity 2	Angle w.r.t X axis	Element Stiffness

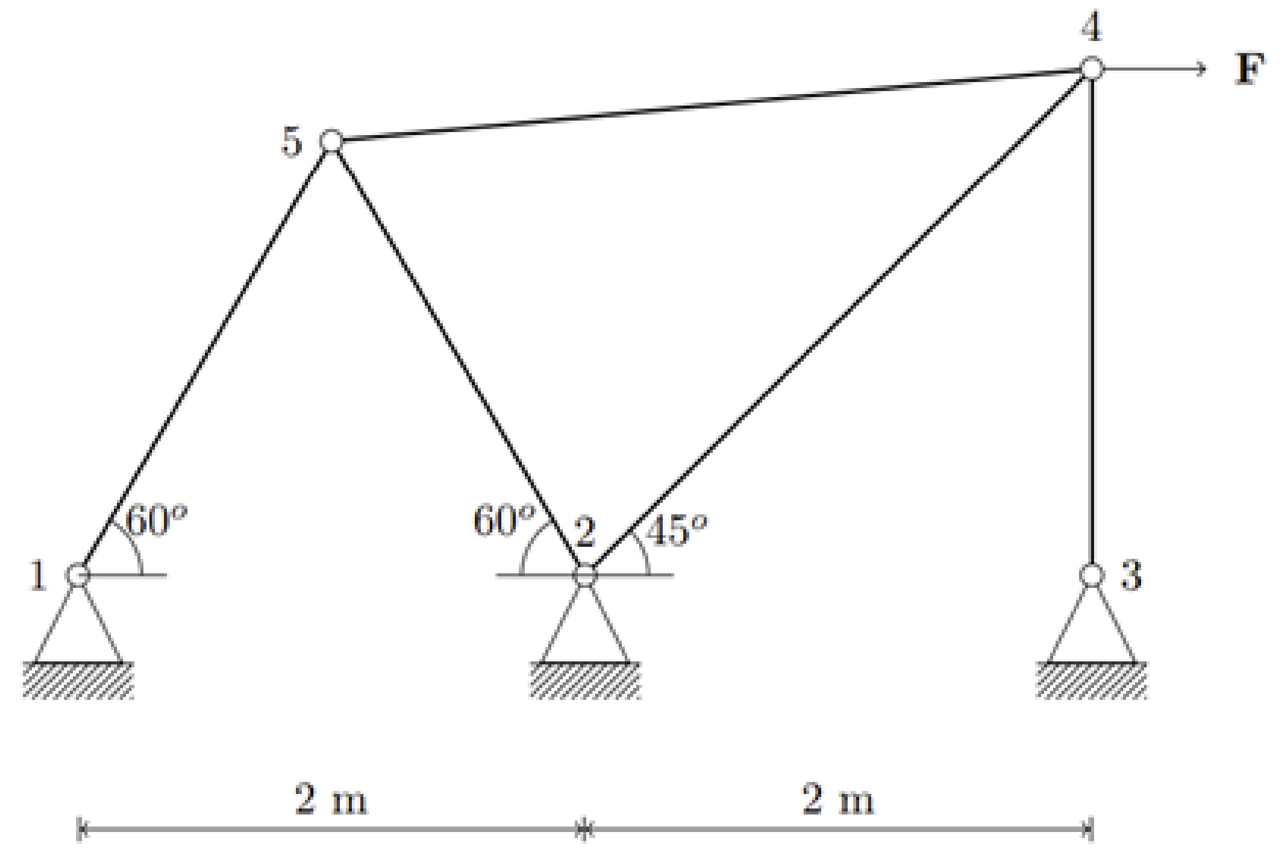
In other words, for each bar in the truss, specify the global node numbers that make up the bar (Nodal connectivity 1 & 2), the angle the individual bar makes w.r.t the X-axis, and the Element Stiffness value (EA/L). The table below gives the data for the first truss structure (the element number is the same as the row index in the table; fill the appropriate value for θ and k^e). Ensure that the order of your node number and the angle measurement is consistent with the coordinate system you choose. You may want to first verify your code and input for the structures you solved in Problem 2 (use appropriate numerical values).

Nodal connectivity 1	Nodal connectivity 2	Angle w.r.t X axis	Element Stiffness
1	5	$60°$	k^1
2	5	$120°$	k^2
2	4	$45°$	k^3
3	4	$90°$	k^4
5	4	$\theta°$	k^5

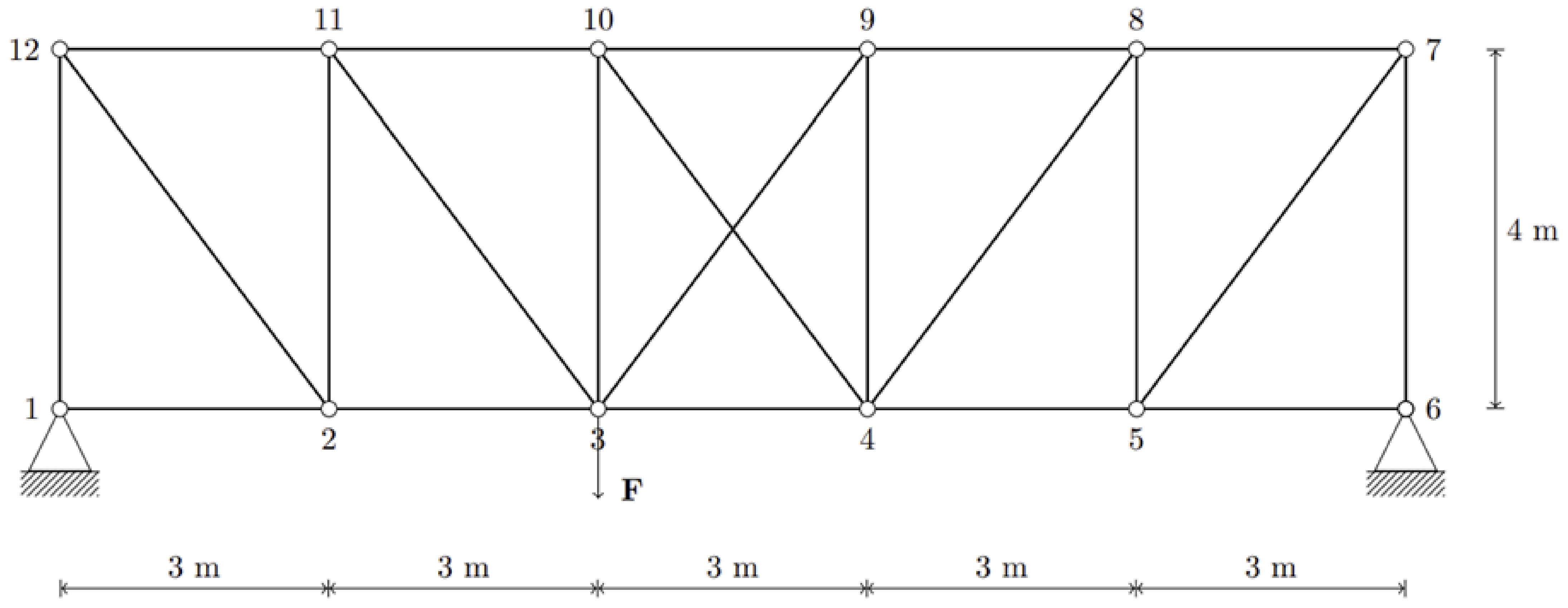


(a)

Problem Statement



(a)



(b)

Thank You

Resource Page

- https://www.youtube.com/playlist?list=PLSGws_74K018SmggufD-pbzG3thPIpF94
- Introduction to Finite Element Method by J.N Reddy