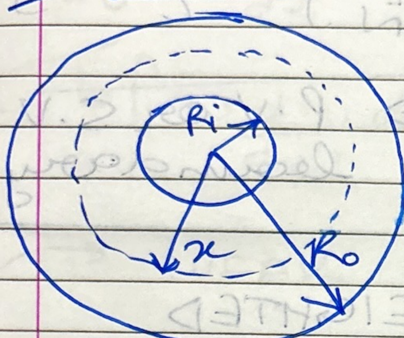
SOC 2024

Important Notes

Heat Conduction (Basic Example)

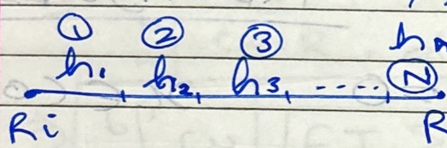


BCs: ① $Kr\frac{dT}{dr} = 0$ @ $r = R_i$

② $T(R_o) = T_o$

G.E: $-\frac{1}{r}\left[\frac{d}{dr}\left(Kr\frac{dT}{dr}\right)\right] = q(r)$

thermal load / heat generation.



$\boxed{T_e(r) = \sum_{j=1}^{n} T_j^e \psi_j^e(r)}$

→ for e^{th} element

$\boxed{T_j^e : T @ j^{th} \text{ pt in } e^{th} \text{ element}}$

n unknowns/element

⇓

put in G.E → GET R(T)

$\boxed{R(T) \Downarrow \text{Residue}}$

⇓

$\int_{he=1}^{he} R(T)\cdot w_1 \, d\Omega = 0$ → ①

" " $\cdot w_2$ " " → ②

⋮

w_n " " n eqns using n w(s)

$\underbrace{\hspace{3cm}}_{\text{single element}}$

ASSEMBLE

⇑

$$\begin{bmatrix} n \times n \end{bmatrix} \underbrace{\begin{Bmatrix} T_1^e \\ T_2^e \\ \vdots \\ T_n^e \end{Bmatrix}}_{n \times 1} = \underbrace{\begin{Bmatrix} \\ \\ \\ \end{Bmatrix}}_{n \times 1}$$

N times for all elements

For all elements ⇒

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{Bmatrix} = \begin{Bmatrix} \\ g \\ \\ \end{Bmatrix}$$

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* **Homogenous B.C** ⇒ P.V or S.V set to 0 at a boundary

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* NEED FOR WEIGHTED INTEGRAL RESIDUES

→ $-\frac{d}{dx}\left[a\frac{du}{dx}\right] + u = 0 \qquad x \in (0,1)$

B.C: $u = 1$ @ $x = 0$ & $\left[a\frac{du}{dx}\right]_{x=1} = 0$

→ $u \sim u_N = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x)$

c_j (unknown) ϕ_0 satisfy BCs (known)

VARIATIONAL METHOD

PUT INTO THE G.E

get $R(x)$

$\int_0^1 \omega \cdot R(x) \cdot dx = 0$

→ c_1 & c_2 found (if $N=2$)

e.g. ⇒ $\omega_1(x) = 1 \rightarrow \int_0^1 1 \cdot R(x)\,dx = 0$

$\omega_2(x) = x \rightarrow \int_0^1 x \cdot R(x)\,dx = 0$

→ solve for c_1 & c_2

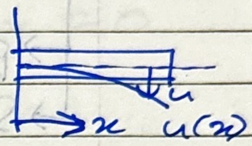
Integration By Parts

→ $\int_a^b w \frac{dv}{dx}\, dx = -\int_a^b v\, dw + [w.v]_a^b$

→ $\int_a^b w \frac{d^2 v}{dx^2}\, dx = -\int_a^b \frac{dw}{dx}\frac{du}{dx}\cdot dx + \left[w\frac{du}{dx}\right]_a^b$

→ WEAK FORMULATION.

e.g. $(EIu'')'' = q(x)$



$\int_a^b w\cdot R\cdot dx = 0$

⇒ $\boxed{\int_a^b w\left[\left[EI\sum_{j=1}^{n} c_j\phi_j''\right]'' - q\right]dx = 0}$

↳ weighted integral of residue.

→ $\int_a^b w v''''\, dx = \int_a^b v'' w''\, dx - \left[w'\frac{d^2 v}{dx^2}\right]_a^b + \left[w\frac{d^3 v}{dx^3}\right]_a^b$

Gradient Theorem

$$\int_\Omega \bar{\nabla} F \cdot \underbrace{dx\,dy}_{d\Omega} = \oint_\gamma \hat{n} F\, d\gamma$$

Divergence Theorem

$$\int_\Omega \bar{\nabla}\cdot \bar{G}\, d\Omega = \oint_\gamma \hat{n}\cdot \bar{G}\, d\gamma$$

2)

$$\int_\Omega w\bar{\nabla} G\, dx\,dy = -\int_\Omega G\bar{\nabla} w\, dx\,dy + \oint_\gamma \hat{n}(Gw)\, ds$$

$$\int_\Omega w\nabla^2 G\, dx\,dy = \oint_\gamma \hat{n}\cdot(w\bar{\nabla} G)\, ds - \int \bar{\nabla} w\cdot \bar{\nabla} G\, dx\,dy$$

Functionals

$$I(u) = \int_{x_A}^{x_B} F(x, u, u', u'', \ldots)\, dx$$

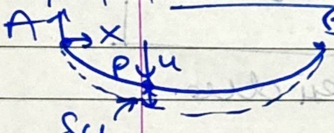
→ functional → operator mapping u to a scalar $I(u)$

→ linear functional: $l(\alpha u + \beta v) = \alpha\, l(u) + \beta\, l(v)$

→ Bilinear functional → linear in u & v both

→ Symmetric Bilinear ⇒ $B(u, v) = B(v, u)$

Variations

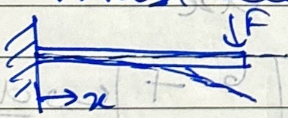


At P ⇒ $u + \delta u$

$$\boxed{\delta u = \alpha \cdot v(x)}$$

$v(x)$ ⇒ BCs related to u must be respected!!

e.g.



At $x = 0$, $u = u' = 0$ fixed

$\Rightarrow \delta u = \delta(u')\big|_{x=0} = 0$

$\Rightarrow v(x) = v'(x)\big|_{x=0} = 0.$

Props:

1) $\delta F = \cancelto{0}{\frac{\partial f}{\partial x}\delta x} + \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial u'}\delta u'$

2) $\delta(F_1 \cdot F_2) = F_2\,\delta F_1 + F_1\,\delta F_2$

3) $\delta(F_1/F_2) = \frac{\delta F_1}{F_2} - \frac{\delta F_2\, F_1}{F_2^2}$

4) $\delta(F_1)^n = n F_1^{n-1} \cdot \delta F_1$

5) $\frac{d}{dx}(\delta u) = \delta\left(\frac{du}{dx}\right)$

Weighted Integrals & Weak Form

$$-\frac{d}{dx}\left[a\frac{du}{dx}\right] = q(x) \qquad x \in (0, L)$$

$u(0) = u_0$ at $x = 0$ → Essential BC

& $a\frac{du}{dx}\Big|_{x=L} = Q_0$ → Natural BC.

Weak Formulation

Step ① ⇒ $\int_0^L w \cdot \left[-\frac{d}{dx}[a(x)u'] - q\right] dx = 0$

→ STRONG FORM

Step ② : By parts ⇒

$$\int_0^L \left[(dw/dx)\, a(x) u' - wq\right] dx - \left[w\, a(x)\, u'\right]_0^L = 0$$

→ WEAK FORM

Step ③ : Choose : $u = \sum_{j=1}^{N} c_j \phi_j(x)$

$$\Rightarrow \int w'\left[a(x) \sum_{j=1} c_j \phi_j'(x)\right] dx - \int wq\, dx - \left[w\, a^*(x)\, u'\right]_0^L = 0.$$

Coeff of w & derivatives ⇒ Secondary variables (S.V)

$a(x)\frac{du}{dx}$ → (NBC)

Dependent variable in same form as w & derivatives ⇒ Primary var. (PV)

→ EBCs.

Choice of wt. functions

↳ s.t $\boxed{w = \delta u}$

$$\Rightarrow \int_0^L (w' a(x) u' - wq)\,dx - [w a u']_0^L = 0$$

$$\hookrightarrow \boxed{\int_0^L (\delta u' a(x) u' - \delta u q)\,dx - [\delta u\, a u']_0^L = 0}$$

↳ VARIATIONAL FORM.

Note : If diff. eqn is linear & of even order, WEAK FORM is symmetric & Bilinear in u, w.

Linear, Bilinear & Quadratic Functionals

Ex: $-\dfrac{d}{dx}\left[a\dfrac{du}{dx}\right] = q$ $\quad u|_{x=0} = u_0 \quad (0, L)$

$\quad a u'|_{x=L} = Q_0$.

$\Rightarrow$ Weak Form $\Rightarrow \underbrace{\int_0^L a w' u'\,dx}_{B(w,u)} - \underbrace{\int_0^L q w\,dx}_{l_1(w)} - \underbrace{[w a u']_0^L}_{l_2(w)} = 0$

$$l(w) = l_1(w) + l_2(w)$$

$$\Rightarrow \boxed{B(w, u) = l(w)}$$

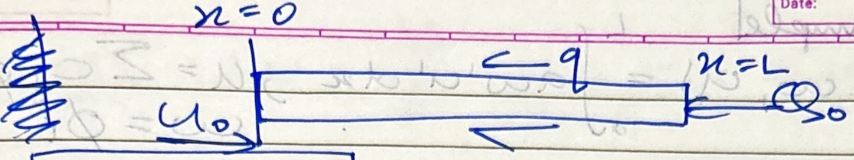
$\Rightarrow$

$$B(\delta u, u) - l(\delta u) = 0$$

$$\Rightarrow \int_0^L a\,\delta u' u'\,dx - \int_0^L q\,\delta u\,dx - [\delta u\, Q_0]_{x=L}$$

$$= \delta \underbrace{\int_0^L \frac{1}{2} a (u')^2\,dx}_{T_1} - \delta \underbrace{\int_0^L q u\,dx}_{T_2} - \delta \underbrace{[u Q_0]_{x=L}}_{T_3} = 0$$

T_1: strain energy $\quad T_2$: work done by traction $\quad T_3$: work by Ext. force.



$\boxed{a = EA}$

$\Rightarrow \; au' = \sigma$ (stress) & $u' = \epsilon$ (strain)

$\Rightarrow \; \boxed{\delta \Pi = 0} \rightarrow$ Eqbm, Π: PE of system

* VARIATIONAL METHODS

① Rayleigh Ritz Method

- → weak form
- → $u = \sum c_j \phi_j$ → $\boxed{\phi_j \text{ linearly independent}}$
  - c_j → Ritz coeffs
- → $\boxed{w = \phi_i}$
- → ϕ_j only need to satisfy EBCs
- → works well for Bilinear & symmetric

Example: $B(w,u) = l(w)$ → Bilinear & symmetric

$I = \frac{1}{2} B(u,u) - l(u) \;\Rightarrow\; \delta I = 0$

$u_N = \sum c_j \phi_j + \phi_0$

$\Rightarrow B[\phi_i, \sum c_j \phi_j + \phi_0] = l(\phi_i)$

$\Rightarrow \sum c_j B[\phi_i, \phi_j] = l(\phi_i) - B(\phi_i, \phi_0)$

$\Rightarrow \; \boxed{\underbrace{[B]}_{\text{known}} \underbrace{\{c\}}_{\text{unknown}} = \{F\}} \rightarrow$ known.

P.T.O

N = 2

Example

$$B(\omega, u) = \int_0^L a\omega' u' \, dx \quad , u = \sum c_j \phi_j$$
$$\omega = \phi_i$$

$$\Rightarrow B(\phi_i, u) = \int_0^L \sum a \phi_i' \phi_j' c_j \, dx$$

$$= \int_0^L a \phi_i' [\phi_1' c_1 + \phi_2' c_2] \, dx$$

when $i = 1 \Rightarrow$

$$B(\phi_1, u) = c_1 \underbrace{\int_0^L a\phi_1' \cdot \phi_1' \, dx}_{B_{11}} + c_2 \underbrace{\int_0^L a\phi_1' \phi_2' \, dx}_{B_{12}} \hookrightarrow EQ①$$

when $i = 2 \Rightarrow$

$$B(\phi_2, u) = c_1 \underbrace{\int_0^L a\phi_2' \phi_1' \, dx}_{B_{21}} + c_2 \underbrace{\int_0^L a\phi_2' \phi_2' \, dx}_{B_{22}} \hookrightarrow EQ②$$

$$\Rightarrow \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\& \; l(\omega) = \int q \omega \, dx \longrightarrow \{F\}$$

$$\Rightarrow F_1 = \int_0^L q \cdot \phi_1 \, dx \;\&\; F_2 = \int_0^L q \cdot \phi_2 \, dx.$$

One more example whole lecture (18)

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* METHOD OF ~~OF~~ WEIGHTED RESIDUALS

- → No weakening of differentiability
- → STRONG FORM
- → Approx. functions should satisfy EBCs & NBCs.
- → $\omega = \phi$ or $\omega \neq \phi$ both possible

Step ①: $u_N = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0$

Step ②: $R = A(u_N) = A[\sum c_j \phi_j(x) + \phi_0]$

↳ A is an operator

like $A(u) = -\frac{d}{dx}\left[a \frac{du}{dx}\right] + cu$

Step ③: $\int_{\Omega} \sum_{j=1}^{N} \psi_i R(x, y, c_j)\, dx\, dy = 0$

↳ wt. function.

↳ $N\ \psi_i \Rightarrow N$ eqns $\Rightarrow N\ c_j$(s) solved.

\# $\phi_j \rightarrow$ satisfies homogeneous form of all NBCs & EBCs.

$\phi_0 \rightarrow$ satisfy all BCs

↳ linearly independent.

I $\boxed{\text{PETROV + GALERKIN METHOD}}$

$\psi_i \Rightarrow$ wt. function.

$\sum c_j \phi_j$ — Primary variable.

$\boxed{\psi_i \neq \phi_i} \Rightarrow$ independent set of fn(s)

$\Rightarrow \int_{\Omega} \psi_i c_j A(\phi_j)\, dx\, dy = \int_{\Omega} \psi_i [f - A(\phi_0)]\, dx\, dy$

$\Rightarrow [A]\{c\} = \{f\}$

↳ not symmetric.

II $\boxed{\text{GALERKIN METHOD}}$

↳ $\psi_i = \phi_i$, $[A] \Rightarrow$ not symmetric

For linear systems → Rayleigh Ritz & Galerkin ⇒ Same answer

III Method of Least Squares

Ensure $\int_\Omega R^2 \, d\Omega$ is minimum.

$\Rightarrow \frac{\partial}{\partial c_i} \int_\Omega R^2(x, y, c_j) \, d\Omega = 0$

$\Rightarrow \int_\Omega 2R(x, y, c_j) \underbrace{\left(\frac{\partial R}{\partial c_i}\right)}_{\psi_i} d\Omega = 0$ $\quad i, j \to 1$ to N

$\Rightarrow \psi_i = \frac{\partial R}{\partial c_i} = \frac{\partial}{\partial c_i} \underbrace{\left[A(\sum_{i,N} c_j \phi_j + \phi_0) - f \right]}_{R} d\Omega$

→ If $A \Rightarrow$ linear $\Rightarrow$

$\psi_i = \frac{\partial [A \sum c_j \phi_j]}{\partial c_i} + \cancelto{0}{\frac{\partial A(\phi_0)}{\partial c_i}} - \cancelto{0}{\frac{\partial f}{\partial c_i}}$

$\Rightarrow \psi_i = \frac{\partial}{\partial c_i} A [c_1 \phi_1 + c_2 \phi_2 + \dots c_N \phi_N]$

& $\frac{\partial c_j}{\partial c_i} = \delta_{ij} \to \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$\Rightarrow \boxed{\psi_i = A(\phi_i)} \Rightarrow [A] \Rightarrow$ Symmetric

IV Collocation Method