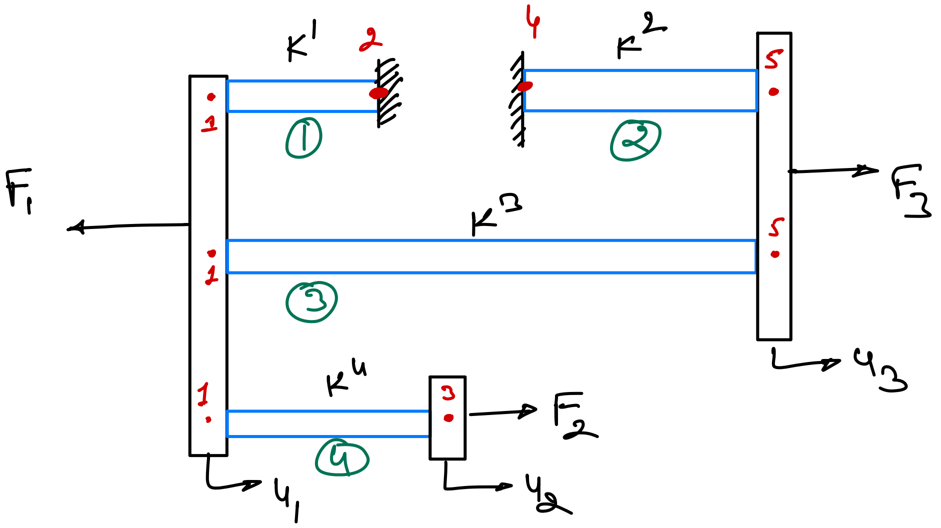
*. (ME613 - FEM and BEM : HW-01 Solⁿ) *

* Problem 1 a Solⁿ



where

• → Node Number

① → element Number

ⓘ Direct Assembly approach:

$\underline{K}_{(i)}$ = Element stiffness matrix for element i

• for element ①

$$\underline{K}_{(1)} = K^1 \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} U_1 & U_2 \end{array}$$

$$\underline{K}_{(1)} = K^1 \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] \begin{array}{c} U_1 \\ U_2 \end{array}$$

(columns: U_1, U_2)

• for element ②

$$\underline{K}_{(2)} = K^2 \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] \begin{array}{c} U_4 \\ U_5 \end{array}$$

(columns: U_4, U_5)

• for element ③

$$\underline{K}_{(3)} = K^4 \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] \begin{array}{c} U_1 \\ U_5 \end{array}$$

(columns: U_1, U_5)

• for element ④

$$\underline{K}_{(4)} = K^4 \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] \begin{array}{c} U_1 \\ U_3 \end{array}$$

(columns: U_1, U_3)

$\underline{K}^g$ = Global stiffness matrix for Assembly

$$
\begin{array}{c}
\\
\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{Bmatrix}
\end{array}
=
\begin{array}{c}
\begin{array}{ccccc} U_1 & U_2 & U_3 & U_4 & U_5 \end{array} \\
\begin{bmatrix}
K^1+K^3+K^4 & -K^1 & -K^4 & & -K^3 \\
-K^1 & K^1 & 0 & 0 & 0 \\
-K^4 & 0 & K^4 & 0 & 0 \\
0 & 0 & 0 & K^2 & -K^2 \\
-K^3 & 0 & 0 & -K^2 & K^2+K^3
\end{bmatrix}
\end{array}
\begin{array}{c}
\\
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}
\end{array}
$$

• Applying BCs i.e. $f_1 = -F_1$, $f_5 = F_3 = 0$, $f_3 = F_2$, $U_2 = U_4 = 0$ i.e. fixed

$U_5 = u_3 = u_0$, $U_1 = u_1$, $U_3 = u_2$

$$
\begin{array}{c}
\\
\begin{Bmatrix} -F_1 \\ f_2 \\ F_2 \\ f_4 \\ 0 \end{Bmatrix}
\end{array}
=
\begin{array}{c}
\begin{array}{ccccc} U_1 & U_2 & U_3 & U_4 & U_5 \end{array} \\
\begin{bmatrix}
K^1+K^3+K^4 & -K^1 & -K^4 & & -K^3 \\
-K^1 & K^1 & 0 & 0 & 0 \\
-K^4 & 0 & K^4 & 0 & 0 \\
0 & 0 & 0 & K^2 & -K^2 \\
-K^3 & 0 & 0 & -K^2 & K^2+K^3
\end{bmatrix}
\end{array}
\begin{array}{c}
\\
\begin{Bmatrix} u_1 \\ 0 \\ u_2 \\ 0 \\ u_0 \end{Bmatrix}
\end{array}
\quad \text{— eq}^n\,(1)
$$

$$
\underset{3\times 1}{\begin{Bmatrix} -F_1 \\ F_2 \\ 0 \end{Bmatrix}}
=
\underset{3\times 3}{\begin{bmatrix}
K^1+K^3+K^4 & -K^4 & -K^3 \\
-K^4 & K^4 & 0 \\
-K^3 & 0 & K^2+K^3
\end{bmatrix}}
\underset{3\times 1}{\begin{Bmatrix} u_1 \\ u_2 \\ u_0 \end{Bmatrix}}
\quad \longrightarrow \text{eq}^n\,(2)
$$

$$\begin{Bmatrix} f_2 \\ f_4 \end{Bmatrix} = \begin{bmatrix} -K^1 & 0 & 0 \\ 0 & 0 & -K^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_0 \end{Bmatrix} \quad \text{— } Eq^n(3)$$

first calculate u_1, u_2 & u_0 for given F_1, F_2 from $Eq^n(2)$

then find f_2, f_4 from $Eq^n(3)$

Ans

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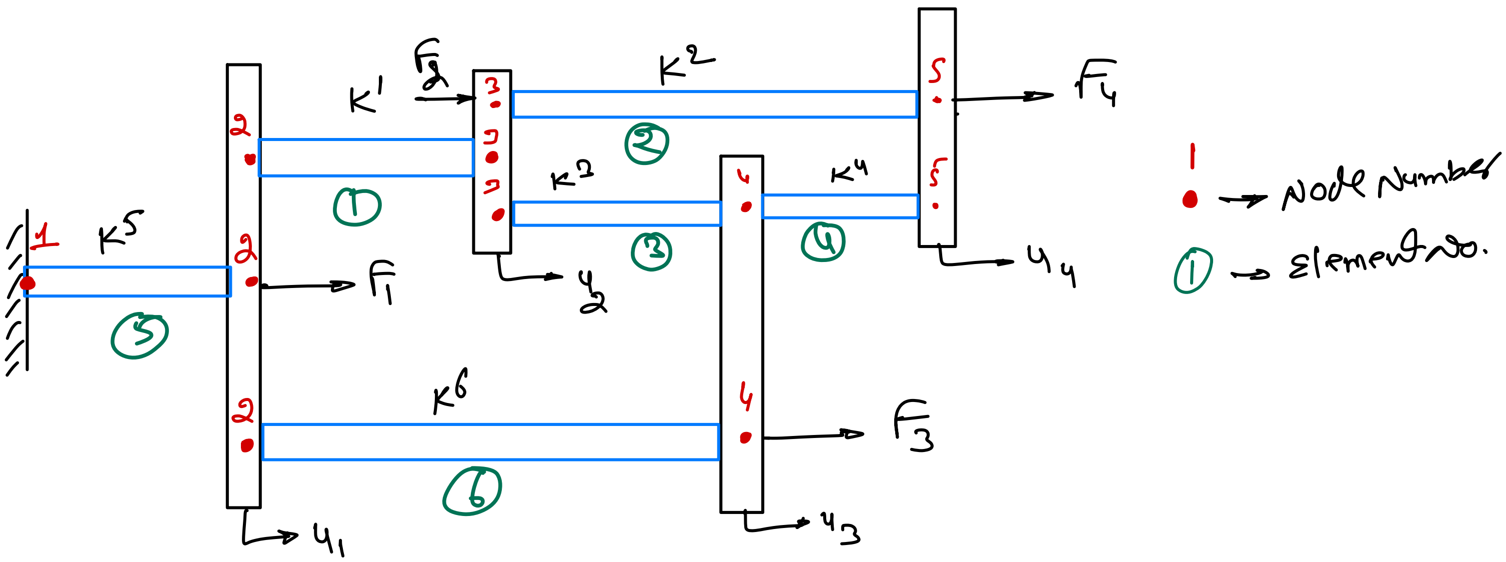
* Problem 1b Sol^n



⊙ Direct Assembly Approach

$\underline{K}_{(i)}$ = Element stiffness matrix

$\underline{K}^g$ = Global stiffness matrix

- for Element 1

$$\underline{K}_{(1)} = K^1 \begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array}$$

- for Element 2

$$\underline{K}_{(2)} = K^2 \begin{array}{c} \begin{array}{cc} U_3 & U_5 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_3 \\ U_5 \end{array}$$

- for Element 3

$$\underline{K}_{(3)} = K^3 \begin{array}{c} \begin{array}{cc} U_3 & U_4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_3 \\ U_4 \end{array}$$

- for Element ④

$$\underline{K}_{(4)} = K^4 \begin{array}{c} \begin{array}{cc} u_4 & u_5 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_4 \\ u_5 \end{array}$$

- for Element ⑤

$$\underline{K}_{(5)} = K^5 \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \end{array}$$

- for Element ⑤

$$\underline{K}_{(6)} = K^6 \begin{array}{c} \begin{array}{cc} u_2 & u_4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ u_4 \end{array}$$

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{Bmatrix} = \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} K^5 & -K^5 & 0 & 0 & 0 \\ -K^5 & K^1+K^5+K^6 & -K^1 & -K^6 & 0 \\ 0 & -K^1 & K^1+K^2+K^3 & -K^3 & -K^2 \\ 0 & -K^6 & -K^3 & K^3+K^4+K^6 & -K^4 \\ 0 & 0 & -K^2 & -K^4 & K^2+K^4 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

- Applying $\underline{\text{BCs}}$: $u_1 = 0$, $u_2 = u_1$, $u_3 = u_2$, $u_4 = u_3 = u_6$, $u_5 = u_4$

$f_2 = F_1$, $f_3 = F_2$, $f_4 = F_3 = 0$, $f_5 = F_4$

$$\begin{Bmatrix} f_1 \\ F_1 \\ F_2 \\ 0 \\ F_4 \end{Bmatrix} = \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} K^5 & -K^5 & 0 & 0 & 0 \\ -K^5 & K^1+K^5+K^6 & -K^1 & -K^6 & 0 \\ 0 & -K^1 & K^1+K^2+K^3 & -K^3 & -K^2 \\ 0 & -K^6 & -K^3 & K^3+K^4+K^6 & -K^4 \\ 0 & 0 & -K^2 & -K^4 & K^2+K^4 \end{bmatrix} \end{array} \begin{Bmatrix} 0 \\ u_1 \\ u_2 \\ u_0 \\ u_4 \end{Bmatrix} \quad \text{— eq}^n \text{ ①}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ 0 \\ F_4 \end{Bmatrix}_{4\times 1} = \begin{bmatrix} K^1+K^5+K^6 & -K^1 & -K^6 & 0 \\ -K^1 & K^1+K^2+K^3 & -K^3 & -K^2 \\ -K^6 & -K^3 & K^3+K^4+K^6 & -K^4 \\ 0 & -K^2 & -K^4 & K^2+K^4 \end{bmatrix}_{4\times 4} \begin{Bmatrix} u_1 \\ u_2 \\ u_0 \\ u_4 \end{Bmatrix}_{4\times 1} \quad \text{— eq}^n \text{ (2)}$$

from eq^n (2) find $u_1, u_2, u_0, \& u_4$ for given F_1, F_2, F_4

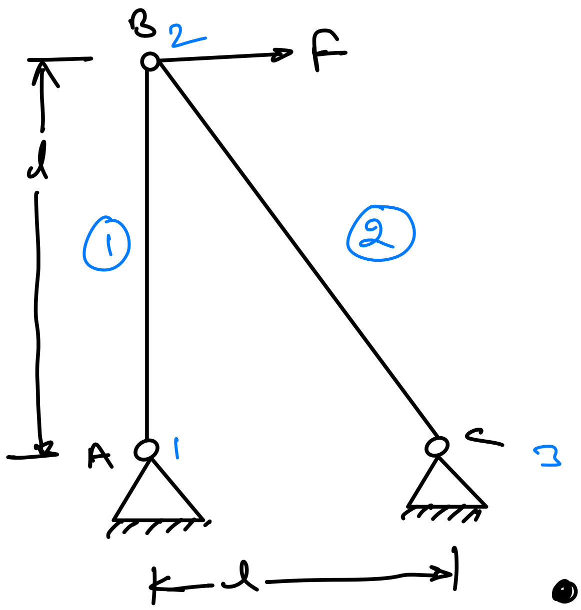
$$\{f_1\}_{1\times 1} = \begin{bmatrix} -K^5 & 0 & 0 & 0 \end{bmatrix}_{1\times 4} \begin{Bmatrix} u_1 \\ u_2 \\ u_0 \\ u_4 \end{Bmatrix}_{4\times 1} \quad \text{— eq}^n \text{ (2)}$$

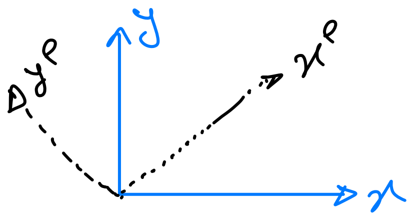
or

$$f_1 = -K^5 u_1$$

from eq^n two find unknown f_1 Ans.

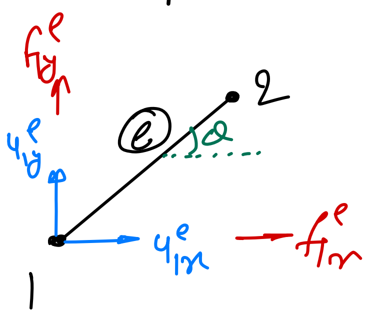
* Problem 2a Sol^n



- 

x-y : Global coordinate system

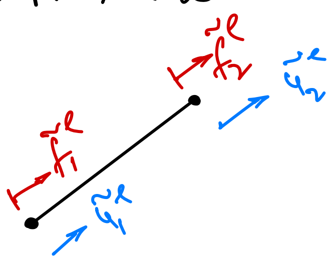
x^e-y^e: coordinate system oriented along the bar/member



- In the global system, the bar has

$$\underline{u}^e = \begin{Bmatrix} u^e_{1x} \\ u^e_{1y} \\ u^e_{2x} \\ u^e_{2y} \end{Bmatrix} ; \quad \underline{f}^e = \begin{Bmatrix} f^e_{1x} \\ f^e_{1y} \\ f^e_{2x} \\ f^e_{2y} \end{Bmatrix}$$

- in the local system



$$\underline{\tilde{u}}^e = \begin{Bmatrix} \tilde{u}^e_1 \\ \tilde{u}^e_2 \end{Bmatrix} \qquad \underline{\tilde{f}}^e = \begin{Bmatrix} \tilde{f}^e_1 \\ \tilde{f}^e_2 \end{Bmatrix}$$

- matrix eq^n in local coordinate system

$$\underbrace{K^e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\underline{\underline{K}}^e} \underbrace{\begin{Bmatrix} \tilde{u}^e_1 \\ \tilde{u}^e_2 \end{Bmatrix}}_{\underline{\tilde{u}}^e} = \underbrace{\begin{Bmatrix} \tilde{f}^e_1 \\ \tilde{f}^e_2 \end{Bmatrix}}_{\underline{\tilde{f}}^e} \quad \ldots \text{Eq}^n (1)$$

- Let θ be the orientation of the bar wrt x-y

$$\underbrace{\begin{Bmatrix} \tilde{u}^e_1 \\ \tilde{u}^e_2 \end{Bmatrix}}_{\underline{\tilde{u}}^e} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}}_{\underline{\underline{T}}^e} \underbrace{\begin{Bmatrix} u^e_{1x} \\ u^e_{1y} \\ u^e_{2x} \\ u^e_{2y} \end{Bmatrix}}_{\underline{u}^e} \quad \ldots \text{Eq}^n (2)$$

from eqⁿ(1) $\quad K^e \underline{\underline{K}}^e \underset{\sim}{\tilde{u}}^e = \tilde{f}^e$

Substituting $\underline{\tilde{u}}^e$ from eqⁿ(2), we get

$$K^e \underline{\underline{K}}^e \underline{\underline{T}}^e \underset{\sim}{u}^e = \tilde{f}^e$$

Pre-multiply it by $(\underline{\underline{T}}^e)^t$ $\quad$ i.e transpose of $\underline{\underline{T}}^e$

$$K^e (\underline{\underline{T}}^e)^t \underline{\underline{K}}^e \underline{\underline{T}}^e \underline{u}^e = (\underline{\underline{T}}^e)^t \underline{\tilde{f}}^e$$

$$(\underline{\underline{T}}^e)^t \tilde{f}^e = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{Bmatrix} \tilde{f}_1^e \\ \tilde{f}_2^e \end{Bmatrix} = \begin{Bmatrix} f_{1x}^e \\ f_{1y}^e \\ f_{2x}^e \\ f_{2y}^e \end{Bmatrix}$$

(Hint: $f_{1x}^e = \tilde{f}_1^e \cos\theta$; $f_{1y}^e = \tilde{f}_1^e \sin\theta$)

$$\boxed{K^e (\underline{\underline{T}}^e)^t \underline{\underline{K}}^e \underline{\underline{T}}^e \underline{u}^e = \underline{f}^e} \qquad \text{——— eq}^n(3)$$

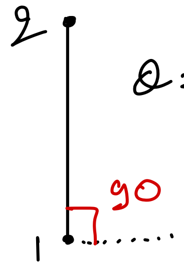
* local element eqⁿ used to form the global stiffness matrix

$$K^e (\underline{\underline{T}}^e)^t \underline{\underline{K}}^e \underline{\underline{T}}^e = \frac{A_e E_e}{l_e} \underbrace{\begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}}_{4\times 2} \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}}_{2\times 4}$$

$$\boxed{K^e (\underline{\underline{T}}^e)^t \underline{\underline{K}}^e \underline{\underline{T}}^e = \frac{A_e E_e}{l_e} \underbrace{\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}}_{4\times 4} \quad \ldots \text{eq}^n(4)}$$

- Local matrix eqⁿ for every element

→ Element (1)

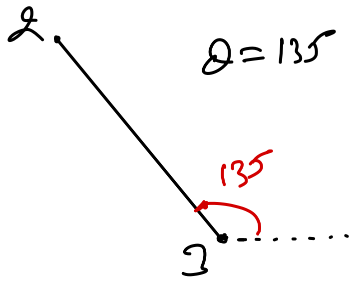


$\theta = 90$

$$\boxed{K^1 = \frac{AE}{l}}$$

$$\begin{array}{c c} & \begin{array}{cccc} 1x & 1y & 2x & 2y \end{array} \\ \begin{array}{c} 1x \\ 1y \\ 2x \\ 2y \end{array} & K^1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u^1_{1x} \\ u^1_{1y} \\ u^1_{2x} \\ u^1_{2y} \end{Bmatrix} = \begin{Bmatrix} f^1_{1x} \\ f^1_{2y} \\ f^1_{2x} \\ f^1_{2y} \end{Bmatrix} \end{array}$$

→ Element 2



$\theta = 135$

$$\boxed{K^2 = \frac{AE}{\sqrt{2}\, l}}$$

$$\begin{array}{c c} & \begin{array}{cccc} 2x & 2y & 3x & 3y \end{array} \\ \begin{array}{c} 2x \\ 2y \\ 3x \\ 3y \end{array} & K^2 \begin{bmatrix} .5 & -.5 & -.5 & .5 \\ -.5 & .5 & .5 & -.5 \\ -.5 & .5 & .5 & -.5 \\ .5 & -.5 & -.5 & .5 \end{bmatrix} \begin{Bmatrix} u^2_{2x} \\ u^2_{2y} \\ u^2_{3x} \\ u^2_{3y} \end{Bmatrix} = \begin{Bmatrix} f^2_{2x} \\ f^2_{2y} \\ f^2_{3x} \\ f^2_{3y} \end{Bmatrix} \end{array}$$

- Global matrix By Direct Assembly approach

$$\begin{array}{c c} \begin{array}{cccccc} 1x & 1y & 2x & 2y & 3x & 3y \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K^1 & 0 & -K^1 & 0 & 0 \\ 0 & 0 & 0+.5K^2 & 0-.5K^2 & -.5K^2 & .5K^2 \\ 0 & -K^1 & 0-.5K^2 & K^1+.5K^2 & .5K^2 & -.5K^2 \\ 0 & 0 & -.5K^2 & .5K^2 & .5K^2 & -.5K^2 \\ 0 & 0 & .5K^2 & -.5K^2 & -.5K^2 & .5K^2 \end{bmatrix} & \begin{array}{c} 1x \\ 1y \\ 2x \\ 2y \\ 3x \\ 3y \end{array} \end{array} \quad \ldots \text{eq}^n(5)$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K' & 0 & -K' & 0 & 0 \\ 0 & 0 & .5K^2 & -.5K^2 & -.5K^2 & .5K^2 \\ 0 & -K' & -.5K^2 & K'+.5K^2 & .5K^2 & -.5K^2 \\ 0 & 0 & -.5K^2 & .5K^2 & .5K^2 & -.5K^2 \\ 0 & 0 & .5K^2 & -.5K^2 & -.5K^2 & .5K^2 \end{bmatrix} \begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} \quad \ldots \text{eq}^n(6)$$

Applying $\underline{\underline{BC}}$. i.e. $u_{1x} = u_{1y} = u_{3x} = u_{3y} = 0 \leftarrow \underline{UBCs}$

$f_{2x} = F \quad , f_{2y} = 0 \leftarrow \underline{FBCs}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K' & 0 & -K' & 0 & 0 \\ 0 & 0 & .5K^2 & -.5K^2 & -.5K^2 & .5K^2 \\ 0 & -K' & -.5K^2 & K'+.5K^2 & .5K^2 & -.5K^2 \\ 0 & 0 & -.5K^2 & .5K^2 & .5K^2 & -.5K^2 \\ 0 & 0 & .5K^2 & -.5K^2 & -.5K^2 & .5K^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{2x} \\ u_{2y} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ F \\ 0 \\ f_{3x} \\ f_{3y} \end{Bmatrix} \quad \ldots \text{eq}^n(7)$$

$$\begin{bmatrix} .5K^2 & -.5K^2 \\ -.5K^2 & K'+.5K^2 \end{bmatrix} \begin{Bmatrix} u_{2x} \\ u_{2y} \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \quad \ldots \text{eq}^n(8)$$

$$\begin{Bmatrix} u_{2x} \\ u_{2y} \end{Bmatrix} = \begin{bmatrix} .5K^2 & -.5K^2 \\ -.5K^2 & K'+.5K^2 \end{bmatrix}^{-1} \begin{Bmatrix} F \\ 0 \end{Bmatrix}$$

$$= \frac{1}{.5K'K^2} \begin{bmatrix} K'+.5K^2 & .5K^2 \\ .5K^2 & .5K^2 \end{bmatrix} \begin{Bmatrix} F \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{K'+.5K^2}{.5K'K^2} F \\ \frac{1}{K'} F \end{Bmatrix}$$

$\underline{\underline{Ans}}$.

<u>**Note**</u>

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

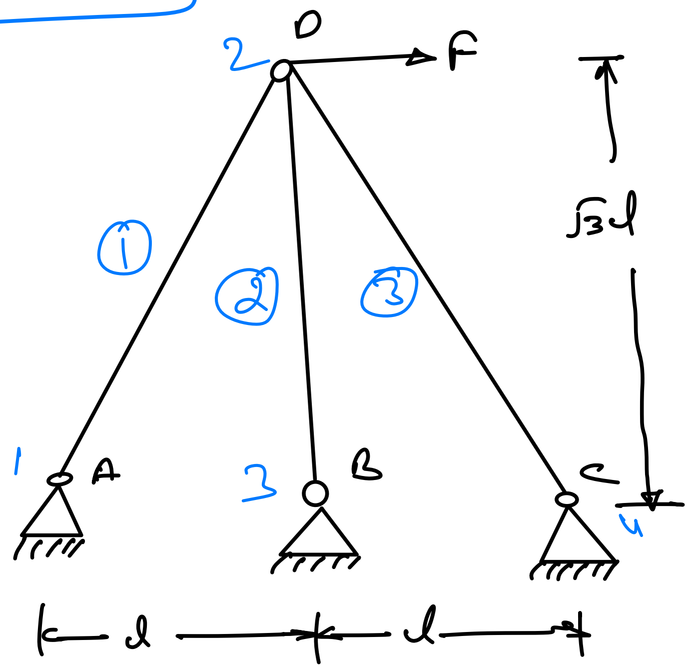
$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

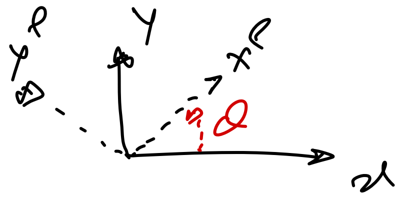
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K' & 0 & -K' & 0 & 0 \\ 0 & 0 & -.5K^2 & .5K^2 & .5K^2 & -.5K^2 \\ 0 & 0 & .5K^2 & -.5K^2 & -.5K^2 & .5K^2 \end{bmatrix}_{4\times 6} \begin{Bmatrix} 0 \\ 0 \\ \left(\frac{K'+.5K^2}{.5K'K^2}\right)F \\ F/K' \\ 0 \\ 0 \end{Bmatrix}_{6\times 1} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix}_{4\times 1} \quad \ldots \text{eq}^n (9)$$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -F \\ -F \\ F \end{Bmatrix}$$

Ans.

* Problem 2b Sol^n

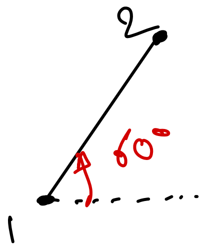




$$K^e (\underline{\underline{T}}^e)^t \underline{\underline{K}}^e \underline{\underline{T}}^e = \frac{A_e E_e}{l_e}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

i.e. Element stiffness matrix.

- Element 1

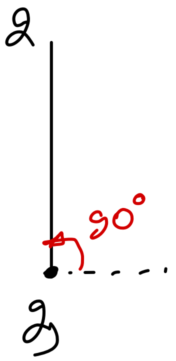


$$\underline{\underline{K}} = K^1 \begin{array}{c} \begin{array}{cccc} 1x & 1y & 2x & 2y \end{array} \\ \begin{bmatrix} .2500 & .4330 & -.2500 & -.4330 \\ .4330 & .7500 & -.4330 & -.7500 \\ -.2500 & -.4330 & .2500 & .4300 \\ -.4330 & -.7500 & .4300 & .7500 \end{bmatrix} \end{array} \begin{array}{c} 1x \\ 1y \\ 2x \\ 2y \end{array}$$

where

$$\boxed{K^1 = \frac{A_1 E_1}{l_1} \qquad K^1 = \frac{AE}{2l}}$$

- Element 2

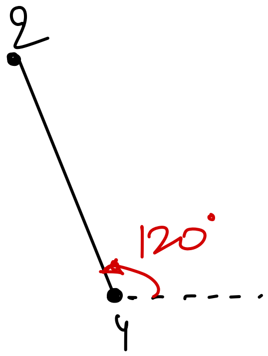


$$\underline{\underline{K}} = K^2 \begin{array}{c} \begin{array}{cccc} 3x & 3y & 2x & 2y \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} 3x \\ 3y \\ 2x \\ 2y \end{array}$$

where

$$\boxed{K^2 = \frac{A_2 E_2}{l_2} \qquad K^2 = \frac{AE}{\sqrt{3}l}}$$

- Element (3)



$$\underline{\underline{K}} = k^3 \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccc} 4x & 4y & 2x & 2y \\ \end{array}$$

$$\underline{\underline{K}} = k^3 \left[\begin{array}{cccc} .2500 & -.4330 & -.2500 & .4330 \\ -.4330 & .7500 & .4300 & -.7500 \\ -.2500 & .4330 & .2500 & -.4330 \\ .4330 & -.7500 & -.4330 & .7500 \end{array}\right]\begin{array}{c} 4x \\ 4y \\ 2x \\ 2y \end{array}$$

where

$$\boxed{\begin{array}{l} k^3 = \dfrac{A_3 E_3}{l_3} \\ k^3 = \dfrac{AE}{2l} \end{array}}$$

- Global stiffness matrix by direct Assembly approach

$$\begin{array}{cccccccc} 1x & 1y & 2x & 2y & 3x & 3y & 4x & 4y \end{array}$$

$$\left[\begin{array}{cccccccc} .2500k' & .4330k' & -.2500k' & -.4300k' & 0 & 0 & 0 & 0 \\ .4330k' & .7500k' & -.4330k' & -.7500k' & 0 & 0 & 0 & 0 \\ -.2500k' & -.4330k' & .2500k' +0 +.2500k^3 & .4300k' +0 -.4330k^3 +0 & 0 & 0 +0 & 0 -.2500k^3 & 0 +.4330k^3 \\ -.4330k' & -.7500k' & .4330k' +0 -.4330k^3 & .7500k' +k^2 +.7500k^3 +0 & 0 & 0 -k^2 & 0 +.4330k^3 & 0 -.7500k^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k^2 & 0 & k^2 & 0 & 0 \\ 0 & 0 & -.2500k^3 & .4330k^3 & 0 & 0 & .2500k^3 & -.4330k^3 \\ 0 & 0 & .4330k^3 & -.7500k^3 & 0 & 0 & -.4330k^3 & .7500k^3 \end{array}\right]\begin{array}{c} 1x \\ 1y \\ 2x \\ 2y \\ 3x \\ 3y \\ 4x \\ 4y \end{array}$$

$$\begin{array}{cccccccc} 1x & 1y & 2x & 2y & 3x & 3y & 4x & 4y \end{array}$$

$$\left[\begin{array}{cccccccc} .2500k' & .4330k' & -.2500k' & -.4300k' & 0 & 0 & 0 & 0 \\ .4330k' & .7500k' & -.4330k' & -.7500k' & 0 & 0 & 0 & 0 \\ -.2500k' & -.4330k' & .25k' + .25k^3 & .433k' - .433k^3 & 0 & 0 & -.2500k^3 & +.4330k^3 \\ -.4330k' & -.7500k' & .433k' - .433k^3 & .75k' + k^2 + .75k^3 & 0 & -k^2 & +.4330k^3 & -.7500k^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k^2 & 0 & k^2 & 0 & 0 \\ 0 & 0 & -.2500k^3 & .4330k^3 & 0 & 0 & .2500k^3 & -.4330k^3 \\ 0 & 0 & .4330k^3 & -.7500k^3 & 0 & 0 & -.4330k^3 & .7500k^3 \end{array}\right]\begin{array}{c} 1x \\ 1y \\ 2x \\ 2y \\ 3x \\ 3y \\ 4x \\ 4y \end{array}$$

$$
\begin{bmatrix}
.2500k' & .4330k' & -.2500k' & -.4300k' & 0 & 0 & 0 & 0 \\
.4330k' & .7500k' & -.4330k' & -.7500k' & 0 & 0 & 0 & 0 \\
-.2500k' & -.4330k' & .25k'+.25k^3 & .433k'-.433k^3 & 0 & 0 & -.2500k^3 & +.4330k^3 \\
-.4330k' & -.7500k' & .433k'-.433k^3 & .75k'+k^2+.75k^3 & 0 & -k^2 & +.4330k^3 & -.7500k^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -k^2 & 0 & k^2 & 0 & 0 \\
0 & 0 & -.2500k^3 & .4330k^3 & 0 & 0 & .2500k^3 & -.4330k^3 \\
0 & 0 & .4330k^3 & -.7500k^3 & 0 & 0 & -.4330k^3 & .7500k^3
\end{bmatrix}
\begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \end{Bmatrix}
=
\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{Bmatrix}
\quad \ldots \text{Eq}^n (1)
$$

Apply BCs i.e. $u_{1x} = u_{1y} = u_{3x} = u_{3y} = u_{4x} = u_{4y} = 0 \leftarrow$ UBCs

$f_{2x} = F, \quad f_{2y} = 0 \leftarrow$ FBCs

$$
\begin{bmatrix}
.2500k' & .4330k' & -.2500k' & -.4300k' & 0 & 0 & 0 & 0 \\
.4330k' & .7500k' & -.4330k' & -.7500k' & 0 & 0 & 0 & 0 \\
-.2500k' & -.4330k' & .25k'+.25k^3 & .433k'-.433k^3 & 0 & 0 & -.2500k^3 & +.4330k^3 \\
-.4330k' & -.7500k' & .433k'-.433k^3 & .75k'+k^2+.75k^3 & 0 & -k^2 & +.4330k^3 & -.7500k^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -k^2 & 0 & k^2 & 0 & 0 \\
0 & 0 & -.2500k^3 & .4330k^3 & 0 & 0 & .2500k^3 & -.4330k^3 \\
0 & 0 & .4330k^3 & -.7500k^3 & 0 & 0 & -.4330k^3 & .7500k^3
\end{bmatrix}
\begin{Bmatrix} 0 \\ 0 \\ u_{2x} \\ u_{2y} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
=
\begin{Bmatrix} f_{1x} \\ f_{1y} \\ F \\ 0 \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{Bmatrix}
\quad \ldots \text{Eq}^n (2)
$$

$$
\begin{bmatrix}
.25k'+.25k^3 & .433k'-.433k^3 \\
.433k'-.433k^3 & .75k'+k^2+.75k^3
\end{bmatrix}
\begin{Bmatrix} u_{2x} \\ u_{2y} \end{Bmatrix}
=
\begin{Bmatrix} F \\ 0 \end{Bmatrix}
\quad \ldots \text{Eq}^n (3)
$$

$$\begin{Bmatrix} u_{2x} \\ u_{2y} \end{Bmatrix} = \begin{bmatrix} .25K' + .25K^3 & .433K' - .433K^3 \\ .433K' - .433K^3 & .75K' + K^2 + .75K^3 \end{bmatrix}^{-1} \begin{Bmatrix} F \\ 0 \end{Bmatrix} \quad \ldots \text{eq}^n (4)$$

Ans

from eq^n (4) we get u_{2x}, u_{2y}

$$\underbrace{\begin{bmatrix} .2500K' & .4330K' & -.2500K' & -.4300K' & 0 & 0 & 0 & 0 \\ .4330K' & .7500K' & -.4330K' & -.7500K' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -K^2 & 0 & K^2 & 0 & 0 \\ 0 & 0 & -.2500K^3 & .4330K^3 & 0 & 0 & .2500K^3 & -.4330K^3 \\ 0 & 0 & .4330K^3 & -.7500K^3 & 0 & 0 & -.4330K^3 & .7500K^3 \end{bmatrix}}_{6\times 8} \underbrace{\begin{Bmatrix} 0 \\ 0 \\ u_{2x} \\ u_{2y} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}}_{8\times 1} = \underbrace{\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{Bmatrix}}_{6\times 1}$$

eq^n (5)

from eq^n (5) we find $f_{1x}, f_{1y}, f_{3x}, f_{3y}, f_{4x}, f_{4y}$

$$\left.\begin{aligned} f_{1x} &= -.25K' u_{2x} - .433K' u_{2y} = -K'(.25u_{2x} + .433u_{2y}) \\ f_{1y} &= -.433K' u_{2x} - .75K' u_{2y} = -K'(.433u_{2x} + .75u_{2y}) \\ f_{3x} &= 0 \text{ and } f_{3y} = -K^2 u_{2y} \\ f_{4x} &= -.25K^3 u_{2x} + .433K^3 u_{2y} = K^3(.433u_{2y} - .25u_{2x}) \\ f_{4y} &= +0.433K^3 u_{2x} - .75K^3 u_{2y} = K^3(.433u_{2x} - .75u_{2y}) \end{aligned}\right\}$$

Ans.

* problem 3 sol^n

given differential equation (D.E^n):

$$\frac{d^2u}{dx^2} + u = x^2 \quad \text{for } 0<x<1 \quad \longrightarrow Eq^n(1)$$

subjected to Boundary Conditions (BCs)

$$u(0)=0, \quad \left.\frac{du}{dx}\right|_{x=1} = 1$$

3.a Let consider approximate sol^n such that that satisfies all specified BCs.

- Assume three term trial function:

$$\tilde{u}(x) = ax + bx^2 + cx^3$$

$$\tilde{u}(0)=0 \quad ; \quad \tilde{u}'(1)=1 \quad \text{for} \quad a+2b+3c=1$$

$$\therefore \quad \tilde{u}(x) = (1-2b-3c)x + bx^2 + cx^3$$

$$\boxed{\tilde{u}(x) = x + b(x^2-2x) + c(x^3-3x)} \quad \cdots Eq^n(3a.1)$$

- Assume 4-term trial function

$$\tilde{u}(x) = ax + bx^2 + cx^3 + dx^4$$

$$\tilde{u}(0)=0 \quad ; \quad \tilde{u}'(1)=1 \quad \text{for} \quad a+2b+3c+4d=1$$

$$\therefore \quad \tilde{u}(x) = (1-2b-3c-4d)x + bx^2 + cx^3 + dx^4$$

$$\boxed{\tilde{u}(x) = x + b(x^2-2x) + c(x^3-3x) + d(x^4-4x)} \quad \cdots Eq^n(3a.2)$$

* Case 2: Three-term polynomial (or two constant) soln.

$$\boxed{\tilde{u}(x) = x + b(x^2 - 2x) + c(x^3 - 3x)}$$

(3b) Point Collocation method

$$R(x) = \tilde{u}'' + \tilde{u} - x^2 \neq 0$$

$$\tilde{u}''(x) = 2b + c(6x)$$

Substituting $\tilde{u}''$, $\tilde{u}$ into $R(x)$, we get

$$R(x) = 2b + 6cx + x + b(x^2 - 2x) + c(x^3 - 3x) - x^2$$

$$R(x) = b(2 - 2x + x^2) + c(3x + x^3) + (x - x^2) \quad \text{... eq}^n\text{(3b.1)}$$

* Point collocation method force $R(x)$ to zero at collocation points

from eq^n (3b.1)

$$R(1/3) = \frac{13b}{9} + \frac{28c}{27} + 2/9 \quad \text{.... eq}^n\text{(3b.3a)}$$

$$R(2/3) = \frac{10b}{9} + \frac{62c}{27} + \frac{2}{9} \quad \text{.... eq}^n\text{(3b.3b)}$$

Eq^n (3b.3a) & (3b.3b) can be written in matrix form as

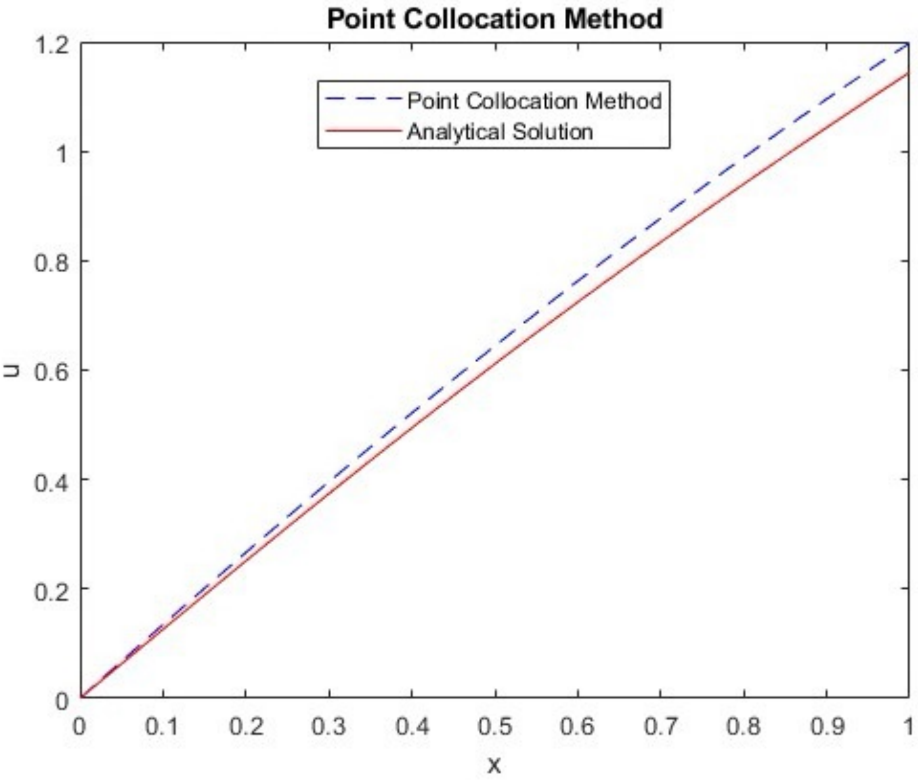
$$\begin{bmatrix} 13/9 & 28/27 \\ 10/9 & 62/27 \end{bmatrix} \begin{Bmatrix} b \\ c \end{Bmatrix} = \begin{Bmatrix} -2/9 \\ -2/9 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} b \\ c \end{Bmatrix} = \begin{bmatrix} 13/9 & 28/27 \\ 10/9 & 62/27 \end{bmatrix}^{-1} \begin{Bmatrix} -2/9 \\ -2/9 \end{Bmatrix} = \begin{Bmatrix} -0.1291 \\ -0.0342 \end{Bmatrix} \quad \text{.... eq}^n\text{(3b.3c)}$$

$$\boxed{\tilde{u}(x) = x - 0.1293\,(x^2-2x) - 0.0342\,(x^3-3x)}$$ Ans.

* $\boxed{\text{NOTE}}$ Analytical Sol^n of given ODE with given BCs.

$$\boxed{u^e = 2\cos x + \left(\frac{2\sin 1 - 1}{\cos 1}\right)\sin x + x^2 - 2}$$ Ans



3c Least square method

Consider three-term trial function as in 3a

$$\tilde{u} = x + b\,(x^2-2x) + c\,(x^3-3x) \quad \text{..... eq}^n (3c.1)$$

$$R(x) = \tilde{u}'' + \tilde{u} - x^2 \neq 0 \quad \text{.... eq}^n (3c.2)$$

substituting $\tilde{u}''$, $\tilde{u}$ into eq^n(3c.2), we get

$$R(x) = b\,(2-2x+x^2) + c\,(3x+x^3) + (x-x^2) \quad \text{... eq}^n (3c.3)$$

- method of least square

$$\boxed{\int_0^1 R \frac{\partial R}{\partial a_i} dx = 0 \qquad i = 1, 2 \ldots} \quad \text{--- } eq^n (3C.4)$$

where $R \longrightarrow$ Residual Error

$a_i \longrightarrow$ unknown constants

from eq^n (3C.4)

$$\int_0^1 R \frac{\partial R}{\partial b} dx = 0 \quad \ldots \; eq^n (3C.5)$$

$$\int_0^1 R \frac{\partial R}{\partial c} dx = 0 \quad \ldots \; eq^n (3C.6)$$

from eq^n (3C.5)

$$\int_0^1 \left[\left\{ b(2-2x+x^2) + c(3x+x^3) + (x-x^2) \right\} (2-2x+x^2) \right] dx = 0$$

$$\frac{28}{15} b + \frac{121}{60} c + \frac{13}{60} = 0 \quad \ldots \; eq^n (3C.7)$$

from eq^n (3C.6)

$$\int_0^1 \left[\left\{ b(2-2x+x^2) + c(3x+x^3) + (x-x^2) \right\} (3x+x^3) \right] dx = 0$$

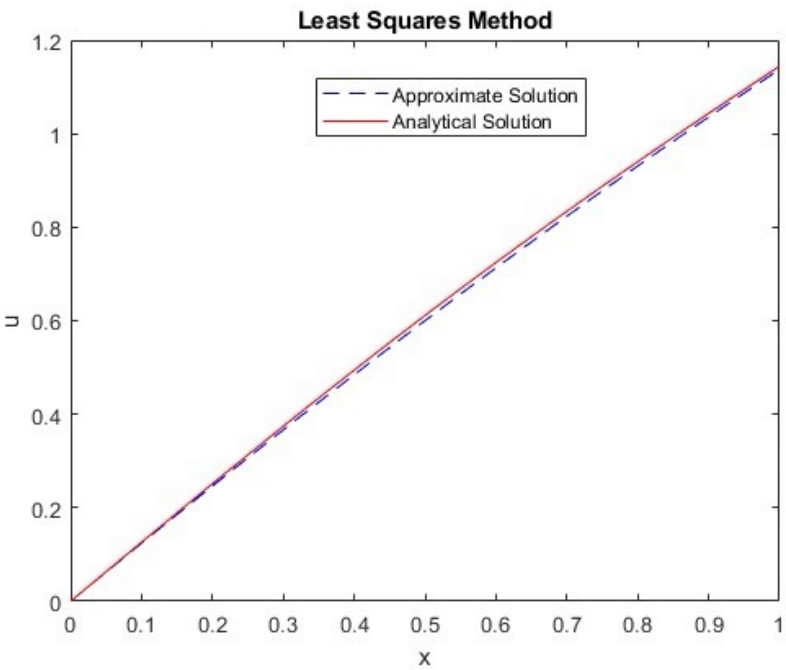
$$\frac{121}{60} b + \frac{152}{35} c + \frac{17}{60} = 0 \quad \ldots \; eq^n (3C.8)$$

writing eq^n (3C.7) & eq^n (3C.8) into matrix form as

$$\begin{bmatrix} 28/15 & \frac{121}{60} \\ \frac{121}{60} & 152/35 \end{bmatrix} \begin{Bmatrix} b \\ c \end{Bmatrix} = \begin{Bmatrix} -13/60 \\ -17/60 \end{Bmatrix}$$

$$\begin{Bmatrix} b \\ c \end{Bmatrix} = \begin{bmatrix} 28/15 & \frac{121}{60} \\ \frac{121}{60} & \frac{152}{35} \end{bmatrix}^{-1} \begin{Bmatrix} -13/60 \\ -17/60 \end{Bmatrix} = \begin{Bmatrix} -0.0915 \\ -0.0228 \end{Bmatrix} \quad \ldots \text{eq}^n (3c.5)$$

$$\therefore \boxed{\tilde{u}(x) = x - 0.0195(x^2 - 2x) - 0.0228(x^3 - 3x)} \quad \underline{\underline{\text{Ans}}}$$



3d Galerkin method

- Three-term trial function:

$$\tilde{u} = x + b(x^2 - 2x) + c(x^3 - 3x) \quad \ldots \text{eq}^n (3d.1)$$

- Residual error $R(x) = \tilde{u}'' + \tilde{u} - x^2 \neq 0 \quad \ldots \text{eq}^n (3d.2)$

Substituting $\tilde{u}''$, $\tilde{u}$ into eq^n (3d.2), we get

$$R(x) = b(2 - 2x + x^2) + c(3x + x^3) + (x - x^2) \quad \ldots \text{eq}^n (3d.3)$$

- Galerkin method

$$\boxed{\int_0^1 R \frac{\partial \tilde{u}}{\partial a_i} \, dx = 0 \qquad i = 1, 2 \ldots} \quad \ldots \text{eq}^n (3d.4)$$

from eq^n (3d.4)

$$\boxed{\begin{aligned}&\int_0^1 R\frac{\partial\tilde{u}}{\partial b}\,dx=0 \\ &\int_0^1 R\frac{\partial\tilde{u}}{\partial c}\,dx=0\end{aligned}}$$

.... eq^n (3d.5)

... eq^n (3d.6)

from eq^n (3d.5)

$$\int_0^1 \left\{\left[b(2-2x+x^2)+c(3x+x^3)+(x-x^2)\right](x^2-2x)\right\}dx=0$$

$$\frac{4}{5}b+\frac{89}{60}c=-7/60 \quad \text{... eq}^n\text{(3d.7)}$$

from eq^n (3d.6)

$$\int_0^1 \left\{\left[b(2-2x+x^2)+c(3x+x^3)+(x-x^2)\right](x^3-3x)\right\}dx=0$$

$$\frac{89}{60}b+\frac{20}{7}c=-13/60 \quad \text{... eq}^n\text{(3d.8)}$$

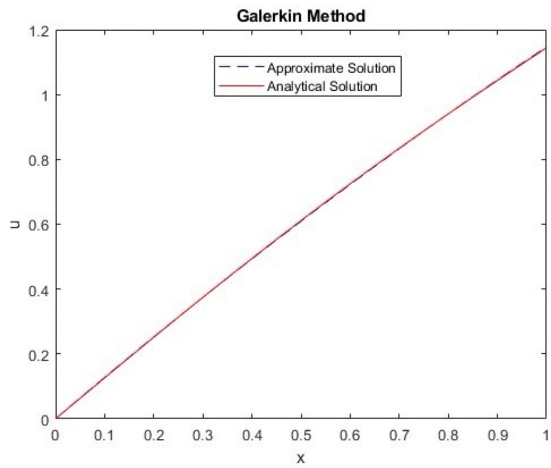
eq^n (3d.7) & eq^n (3d.8) can be written as in matrix form

$$\begin{bmatrix}4/5 & 89/60\\ 89/60 & 20/7\end{bmatrix}\begin{Bmatrix}b\\c\end{Bmatrix}=\begin{Bmatrix}-7/60\\-13/60\end{Bmatrix} \Rightarrow \begin{Bmatrix}b\\c\end{Bmatrix}=\begin{bmatrix}4/5 & 89/60\\ 89/60 & 20/7\end{bmatrix}^{-1}\begin{Bmatrix}-7/60\\-13/60\end{Bmatrix}$$

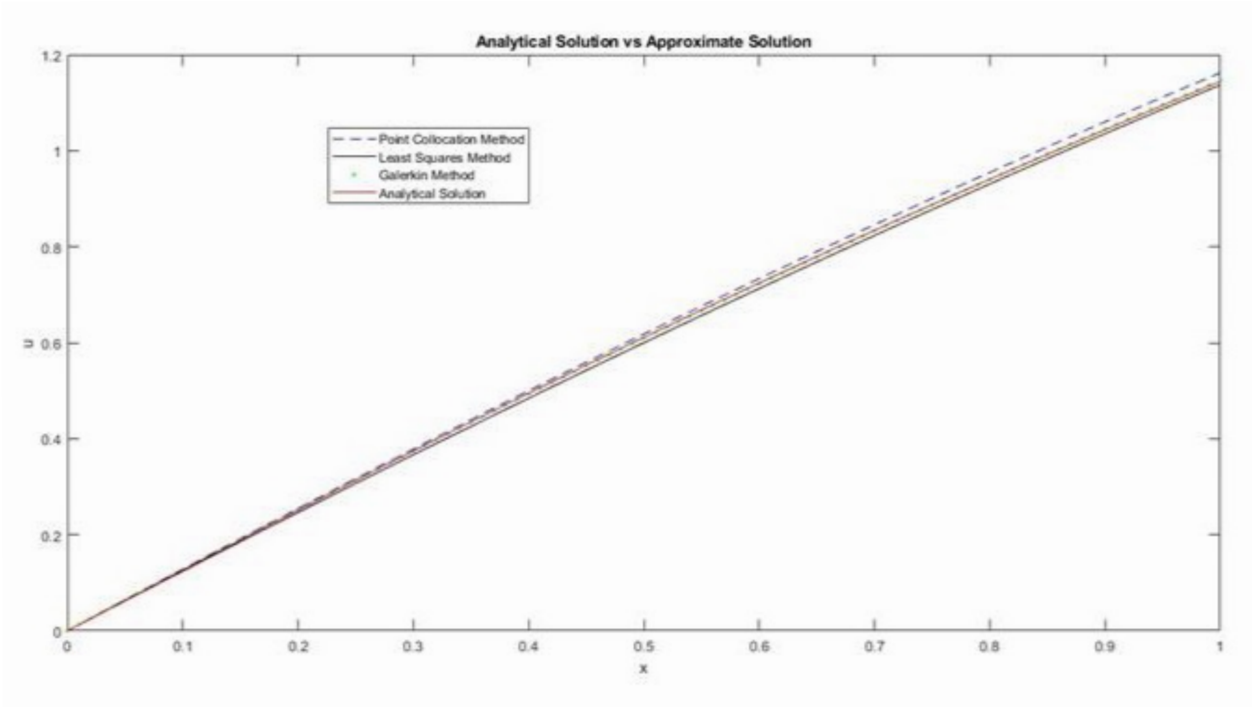
$$\begin{Bmatrix}b\\c\end{Bmatrix}=\begin{Bmatrix}-0.1398\\-0.0023\end{Bmatrix}$$

Thus $\boxed{\hat{u}(x)=x-0.1398(x^2-2x)-0.0023(x^3-3x)}$

Ans.



- Comparision of point collocation method, least square, Galerkin method and Analytical method for problem 3



* **Case- II** : 4-term polynomial (3 constant) trial sol^n

$$\boxed{\hat{u}(x) = x + b(x^2-2x) + c(x^3-3x) + d(x^4-4x)}$$

Note: we can solve this similarly as in case-I. All steps are same.

(3b) Point Collocation method:

$$\hat{u}(x) = x + \frac{279}{80765}(x^2-2x) - \frac{17808}{80765}(x^3-3x) + \frac{7504}{80765}(x^4-4x)$$

or

$$= \frac{7504x^4}{80765} - \frac{17808x^3}{80765} + \frac{279x^2}{80765} + \frac{20403x}{16153} \quad \underline{\underline{\text{Ans}}}$$

(3c) method of least squares:

$$\hat{u}(x) = x + \left(\frac{185023}{1429583280}\right)(x^2-2x) - \left(\frac{3850971}{17872910}\right)(x^3-3x) + \frac{6508307}{71491640}(x^4-4x)$$

$$= \frac{6908307x^4}{71491640} - \frac{3890971x^3}{17872910} + \frac{185023x^2}{142983280} + \frac{90365041x}{71491640}$$ Ans

(3d) Galerkin method

$$\hat{u}(x) = x + \left(\frac{1559}{758672}\right)(x^2-2x) - \left(\frac{20741}{94834}\right)(x^3-3x) + \left(\frac{36813}{379336}\right)(x^4-4x)$$

$$= \frac{36813x^4}{379336} - \frac{20741x^3}{94834} + \frac{1559x^2}{758672} + \frac{475417x}{379336}$$ Ans

* Problem 4 Solⁿ

given DE : $\frac{d^2u}{dx^2} = -\cos(\pi x)$ for $0<x<1$ — Eqⁿ(1)

BCs : $u(0) = 0$, $u(1) = 0$

(4a) Approximate (trial) solⁿ:

- Three-term trigonometric approximate solⁿ

$$\boxed{\tilde{u}(x) = \sum_{i=1}^{3} a_i \sin(i\pi x)}$$ — Eqⁿ(3a)

$\tilde{u}(0) = 0$; $\tilde{u}(1) = 0$ i.e. trial function satisfies BCs.

- Three constant polynomial solⁿ that satisfies BCs.

$\hat{u}(x) = ax + bx^2 + cx^3 + dx^4$

$\hat{u}(0) = 0$; $\hat{u}(1) = 0$ for $a+b+c+d = 0$

$$\therefore \boxed{\hat{u}(x) = a(x-x^4) + b(x^2-x^4) + c(x^3-x^4)}$$ ... Eqⁿ(3b)

* Case-I: Consider Approximate 3-term trigonometric solⁿ

(4b) Point collocation method:

- Assume three-term trial solⁿ as

$$\tilde{u}(x) = \sum_{i=1}^{3} a_i \sin(i\pi x)$$ — Eqⁿ(4b.1)

where $\sin(i\pi x) \longrightarrow$ Assumed (trial function)

$a_i \longrightarrow$ unknowns to be determined.

Let $R(x) \longrightarrow$ Residual Error

$$R(x) = \tilde{u}'' + \cos(\pi x) \neq 0 \quad \text{— Eq}^n\ (4b.2)$$

$$\tilde{u}(x) = a_1 \sin \pi x + a_2 \sin 2\pi x + a_3 \sin 3\pi x$$

$$\tilde{u}'(x) = \pi \left(a_1 \cos \pi x + 2a_2 \cos 2\pi x + 3a_3 \cos 3\pi x \right)$$

$$\tilde{u}''(x) = -\pi^2 \left(a_1 \sin \pi x + 4a_2 \sin 2\pi x + 9a_3 \sin 3\pi x \right)$$

Substituting into Eq^n (4b.2)

$$R(x) = -\pi^2 \left(a_1 \sin \pi x + 4a_2 \sin 2\pi x + 9a_3 \sin 3\pi x \right) + \cos \pi x \quad \text{... Eq}^n\ (4b.3)$$

* force Residual Error to zero at collocation points

i.e. $R(1/4) = R(1/2) = R(3/4) = 0$

$$\left.\begin{aligned}
\pi^2 \left(a_1 \sin \tfrac{\pi}{4} + 4a_2 \sin \tfrac{2\pi}{4} + 9a_3 \sin \tfrac{3\pi}{4} \right) &= \cos \tfrac{\pi}{4} \\
\pi^2 \left(a_1 \sin \tfrac{\pi}{2} + 4a_2 \sin \tfrac{2\pi}{2} + 9a_3 \sin \tfrac{3\pi}{2} \right) &= \cos \tfrac{\pi}{2} \\
\pi^2 \left(a_1 \sin \tfrac{3\pi}{4} + 4a_2 \sin \tfrac{6\pi}{4} + 9a_3 \sin \tfrac{9\pi}{4} \right) &= \cos \tfrac{3\pi}{4}
\end{aligned}\right\}$$

In matrix form

$$\pi^2 \begin{bmatrix} \sin \frac{\pi}{4} & 4 \sin \frac{\pi}{2} & 9 \sin \frac{3\pi}{4} \\ \sin \frac{\pi}{2} & 4 \sin \pi & 9 \sin \frac{3\pi}{2} \\ \sin \frac{3\pi}{4} & 4 \sin \frac{3\pi}{2} & 9 \sin \frac{9\pi}{4} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} \cos \frac{\pi}{4} \\ \cos \frac{\pi}{2} \\ \cos \frac{3\pi}{4} \end{Bmatrix} \quad \text{— Eq}^n\ (4b.4)$$

$$\pi^2 \begin{bmatrix} \frac{1}{\sqrt{2}} & 4 & 9/\sqrt{2} \\ 1 & 0 & -9 \\ \frac{1}{\sqrt{2}} & -1 & 9/\sqrt{2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{Bmatrix} \quad \text{— Eq}^n\ (4b.5)$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{\pi^2} \begin{bmatrix} 1/\sqrt{2} & 4 & 9/\sqrt{2} \\ 1 & 0 & -9 \\ 1/\sqrt{2} & -1 & 9/\sqrt{2} \end{bmatrix}^{-1} \begin{Bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{Bmatrix} \quad \text{— eq}^n \text{ (46.6)}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} -0.0304 \\ 0.0287 \\ -0.0034 \end{Bmatrix}$$

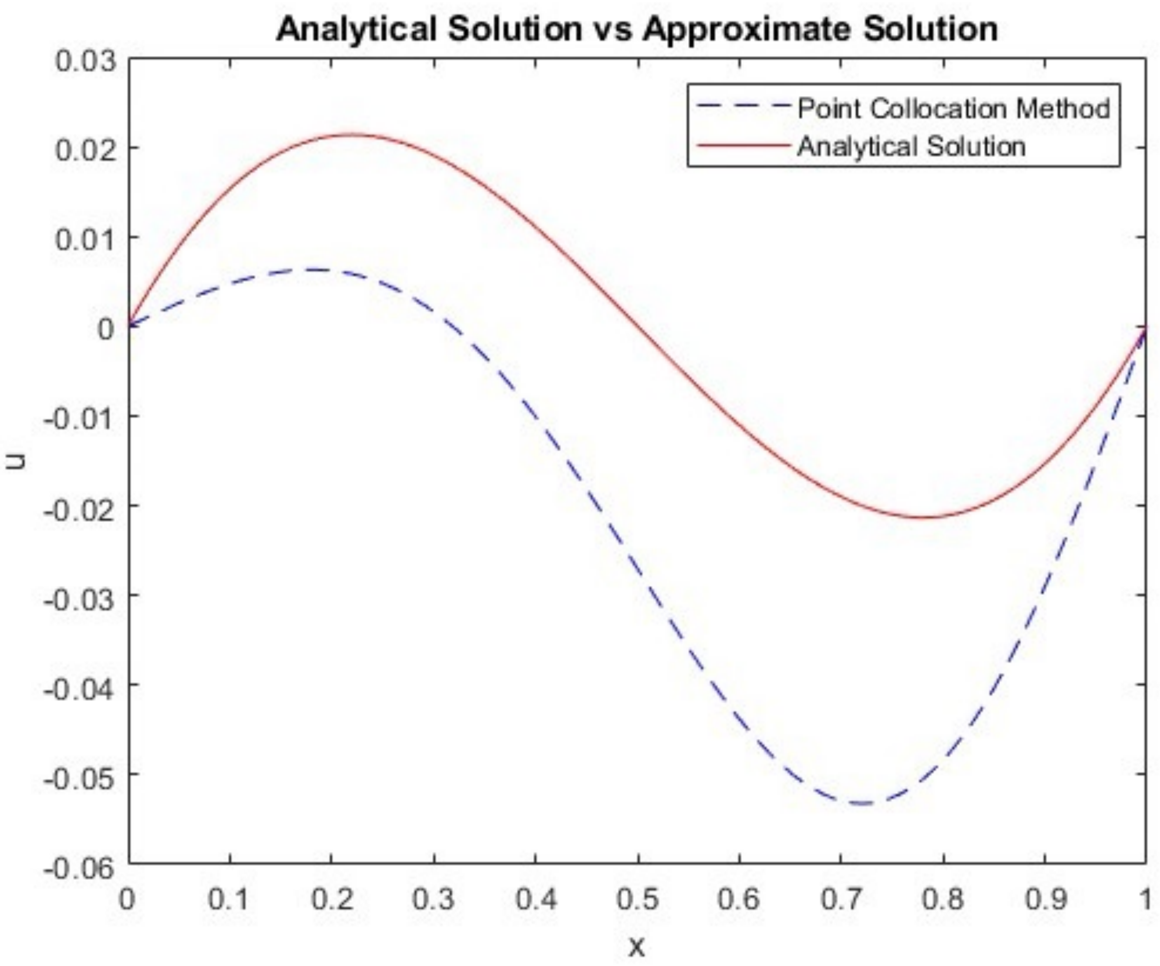
i.e. $a_1 = -0.0304$
$a_2 = 0.0287$
$a_3 = -0.0034$

$$\tilde{u}(x) = -0.0304 \sin \pi x + 0.0287 \sin 2\pi x - 0.0034 \sin 3\pi x \quad \text{... eq}^n \text{ (46.7)}$$

Ans

$\boxed{\text{NOTE}^n}$ Analytical sol^n of given ODE^n with given BCs.

$$u^e = \frac{1}{\pi^2}\left(\cos \pi x + 2x - 1\right)$$



4C

Galerkin's method of weighted residual

- Assume 3 term trial function $\tilde{u} = \sum_{i=1}^{3} a_i \sin(i\pi x)$ .... eqn (4C.1)
- Residual error $R(x) = \tilde{u}'' + \cos(\pi x) \neq 0$ --- eqn (4C.2)

$$R(x) = -\pi^2\left(a_1 \sin \pi x + 4a_2 \sin 2\pi x + 9a_3 \sin 3\pi x\right) + \cos \pi x \quad \text{... eqn (4C.3)}$$

- $\boxed{\int_0^1 R(x) \frac{\partial \tilde{u}}{\partial a_i} dx = 0 \qquad i = 1, 2, 3}$ .... eqn (4C.4)

from eqn (4C.4)

$$\boxed{\begin{aligned} &\int_0^1 R(x) \sin \pi x \, dx = 0 \quad &\text{.... eqn (4C.5a)} \\ &\int_0^1 R(x) \sin 2\pi x \, dx = 0 \quad &\text{.... eqn (4C.5b)} \\ &\int_0^1 R(x) \sin 3\pi x \, dx = 0 \quad &\text{.... eqn (4C.5c)} \end{aligned}}$$

from eqn (4C.5a)

$$\int_0^1 \left[\cos \pi x - \pi^2\left(a_1 \sin \pi x + 4a_2 \sin 2\pi x + 9a_3 \sin 3\pi x\right)\right] \sin \pi x \, dx = 0$$

$$0 - \frac{\pi^2}{2} a_1 - 0 - 0 = 0 \Rightarrow a_1 = 0$$

from eqn (4C.5b)

$$\int_0^1 \left[\cos \pi x - \pi^2\left(a_1 \sin \pi x + 4a_2 \sin 2\pi x + 9a_3 \sin 3\pi x\right)\right] \sin 2\pi x \, dx = 0$$

$$\frac{4}{3\pi} - \frac{4\pi^2 a_2}{2} = 0 \Rightarrow a_2 = \frac{4}{3\pi} \times \frac{2}{4\pi^2} = \frac{2}{3\pi^3}$$

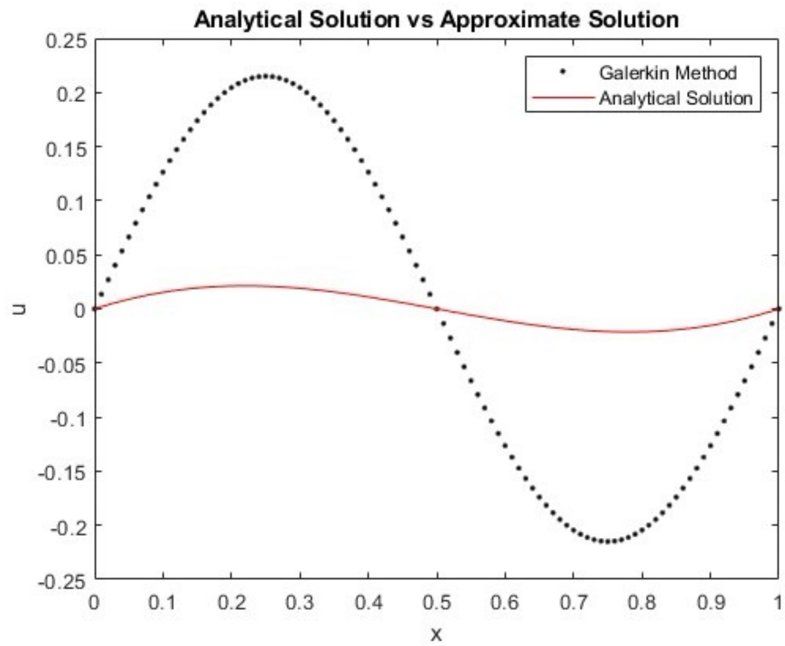
$a_2 = 0.215$

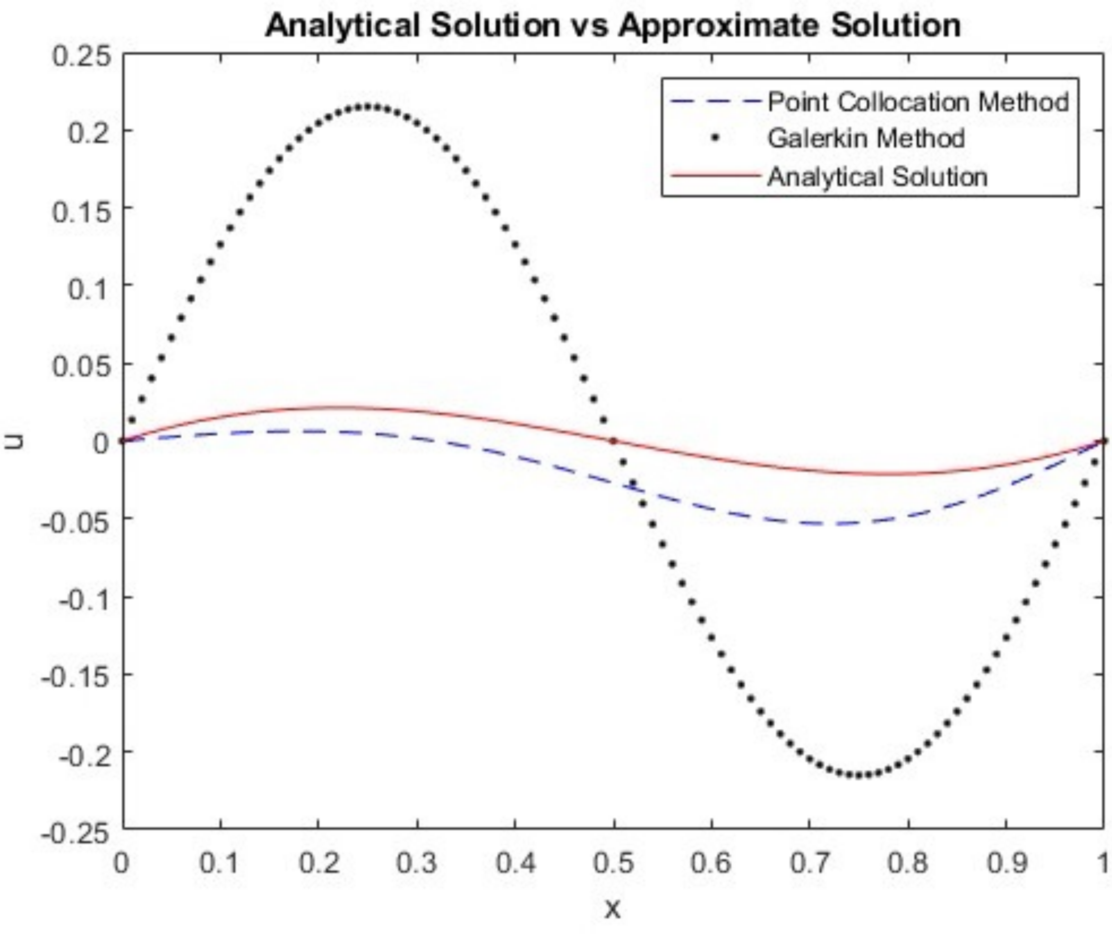
from Eqn (4c.5c)

$$\int_0^1 \left[\cos\pi x - \pi^2\left(q_1 \sin 1\pi x + 4q_2 \sin 2\pi x + 9q_3 \sin 3\pi x\right)\right] \sin 3\pi x \, dx = 0$$

$$0 - \frac{9\pi^2}{2} q_3 = 0 \Rightarrow q_3 = 0$$

Thus, $\tilde{u}(x) = \sum_{i=1}^{3} q_i \sin(i\pi x) = 0.215 \sin 2\pi x$ Ans





* Case-II: Consider Approximate 3 constant polynomial solⁿ.

i.e. $\tilde{u}(x) = a(x-x^4) + b(x^2-x^4) + c(x^3-x^4)$

(4a) Point collocation method:

$$\tilde{u}(x) = \frac{\sqrt{2}}{6}(x-x^4) - \frac{\sqrt{2}}{2}(x^2-x^4) + \frac{\sqrt{2}}{3}(x^3-x^4)$$

Ans

(4b) Galerkin method.

$$\tilde{u}(x) = a(x-x^4) + b(x^2-x^4) + c(x^3-x^4)$$

Ans

where $a = \frac{80}{\pi^4}(\sqrt{\pi}-12) - \frac{45}{\pi^4}(2\sqrt{\pi}-24)$

$b = \frac{240}{\pi^4}(2\sqrt{\pi}-24) - \frac{450}{\pi^4}(\sqrt{\pi}-12)$

$c = \frac{720}{\pi^4}(\sqrt{\pi}-12) - \frac{370}{\pi^4}(2\sqrt{\pi}-24)$

Ans (4d) No, since the specified BCs of the problem are all of the essential type, and therefore the requirement on ϕ_i in two method becomes same and weighted-integral form reduces to the weak form.

* Problem 5 Sol^n

given ODE: $\frac{d}{dx}\left(u\frac{du}{dx}\right) - f(x) = 0$ for $0 < x < L$

or $u\frac{d^2u}{dx^2} + \left(\frac{du}{dx}\right)^2 - f(x) = 0$

Subject to BCs: $\left(u\frac{du}{dx}\right)\Big|_{x=0} = 0$, $u(L) = u_0$

$$\int_0^L w(x)\left\{\frac{d}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right) - f(x)\right\} = 0 \quad \ldots \text{Eq}^n(1)$$

where $\hat{u}$ = trial function soln (Assumed), $w(x)$ = weighting function

$$\int_0^L w(x)\left\{\frac{d}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)\right\}dx = \left[w(x)\left(\hat{u}\frac{d\hat{u}}{dx}\right)\right]_0^L - \int_0^L \frac{dw}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx$$

$$= w(L)\left(\hat{u}\frac{d\hat{u}}{dx}\right)\Big|_{x=L} - w(0)\left(\hat{u}\frac{d\hat{u}}{dx}\right)\Big|_{x=0} - \int_0^L \frac{dw}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx \quad \ldots \text{Eq}^n(2)$$

Substituting Eq^n(2) into Eq^n(1), we get

$$\boxed{w(L)\left(\hat{u}\frac{d\hat{u}}{dx}\right)\Big|_{x=L} - w(0)\left(\hat{u}\frac{d\hat{u}}{dx}\right)\Big|_{x=0} - \int_0^L \frac{dw}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx = \int_0^L w(x)f(x)\,dx}$$

$$\text{— Eq}^n(3)$$

Applying BC. Neuman (natural) & Drichlet (essential)

$\left(\hat{u}\frac{d\hat{u}}{dx}\right)_{x=0} = 0$ i.e. natural BC & $\hat{u}(L) = u_0$ i.e. essential BC

$$w(L)\,u_0\left(\frac{d\hat{u}}{dx}\right)\Big|_{x=L} - \int_0^L \frac{dw}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx = \int_0^L w(x)f(x)\,dx \quad \ldots \text{Eq}^n(4)$$

Ans.

* Problem 6 Sol^n

Given DE^n: $2u\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2 + 4 = 0, \quad 0 < x < 1$ — Eq^n(1)

BCs: $u(0) = 1, \quad u(1) = 0$

(6a)

$$\int_0^1 w(x)\left\{2\hat{u}\frac{d^2\hat{u}}{dx^2} - \left(\frac{d\hat{u}}{dx}\right)^2 + 4\right\}dx = 0 \quad \ldots \text{Eq}^n(2)$$

$$\int_0^1 w(x)\left[2\left\{\frac{d}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right) - \left(\frac{d\hat{u}}{dx}\right)^2\right\} - \left(\frac{d\hat{u}}{dx}\right)^2 + 4\right]dx = 0$$

$$\int_0^1 2w(x)\frac{d}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx - \int_0^1 3w(x)\left(\frac{d\hat{u}}{dx}\right)^2dx + \int_0^1 4w(x)dx = 0 \quad \ldots \text{Eq}^n(3)$$

$$\int_0^1 w(x)\frac{d}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx = \left[w(x)\left(\hat{u}\frac{d\hat{u}}{dx}\right)\right]_0^1 - \int_0^1 \frac{dw}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx \quad \text{— Eq}^n(4)$$

Substituting Eq^n (4) into Eq^n(3), we get

$$2\left[w(x)\left(\hat{u}\frac{d\hat{u}}{dx}\right)\right]_0^1 - 2\int_0^1 \frac{dw}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx - \int_0^1 3w(x)\left(\frac{d\hat{u}}{dx}\right)^2dx + \int_0^1 4w(x)dx = 0$$

$$\boxed{2\left[w(1)\left(\hat{u}\frac{d\hat{u}}{dx}\right)\bigg|_{x=1} - w(0)\left(\hat{u}\frac{d\hat{u}}{dx}\right)\bigg|_{x=0}\right] - 2\int_0^1 \frac{dw}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx - 3\int_0^1 w(x)\left(\frac{d\hat{u}}{dx}\right)^2dx = -4\int_0^1 w(x)dx}$$

— Eq^n(5)

substituting Dirichlet (essential) BC at $x=0$, $x=1$ i.e.

$\hat{u}(0)=1$, $\hat{u}(1)=0$ and requiring $w(0)=0$ in view of prescribed essential BCs. $\hat{u}(0)=1$. Thus eqn (5) converted into

$$\boxed{\begin{gathered} 2\int_0^1 \frac{dw}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx + 3\int_0^1 w(x)\left(\frac{d\hat{u}}{dx}\right)^2 dx = 4\int_0^1 w(x)\,dx \\ \text{subjected } \hat{u}(0)=1,\ w(0)=0 \end{gathered}} \quad \text{— eq}^n (6)$$

Ans.

$w(0)=0$ i.e. condition that weighting function must satisfies.

(6b) Let $\hat{u}(x) = 1 + c_1x + c_2x^2 + c_3x^3$

$\hat{u}(0) = 1$ satisfied

$\hat{u}(1)=0 \Rightarrow 1+c_1+c_2+c_3=0 \Rightarrow c_3=(1-c_1-c_2)$

$$\hat{u}(x) = 1 + c_1x + c_2x^2 + (1-c_1-c_2)x^3$$

$$\hat{u}(x) = (1-x^3) + c_1(x-x^3) + c_2(x^2-x^3)$$

$$w_1 = (x-x^3) \quad \& \quad w_2=(x^2-x^3)$$

$$w_1(0)=0, \quad w_2(0)=0$$

$$\frac{d\hat{u}}{dx} = -3x^2 + c_1(1-3x^2) + c_2(2x-3x^2)$$

$$\frac{dw_1}{dx} = 1-3x^2, \quad \frac{dw_2}{dx} = 2x-3x^2$$

- weak form w.r.t w_1 from eq^n (6)

$$2\int_0^1 \frac{dw_1}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx + 3\int_0^1 w_1(x)\left(\frac{d\hat{u}}{dx}\right)^2 dx = 4\int_0^1 w_1(x)\,dx \quad \text{— eq}^n (7)$$

$$2\int_0^1 \frac{dw_1}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx = 2\int_0^1 (1-3x^2)\{(1-x^3)+c_1(x-x^3)+c_2(x^2-x^3)\} *$$
$$\{-3x^2+c_1(1-3x^2)+c_2(2x-3x^2)\}dx$$

$$= \frac{c_1^2}{4} + \frac{23c_1*c_2}{140} + \frac{11c_1}{10} + \frac{3c_2^2}{140} + \frac{11c_2}{28} + \frac{7}{20} \quad \ldots \text{eqn (7a)}$$

$$3\int_0^1 w_1(x)\left(\frac{d\hat{u}}{dx}\right)^2 dx = 3\int_0^1 (x-x^3)\{-3x^2+c_1(1-3x^2)+c_2(2x-3x^2)\}^2 dx$$

$$= \frac{3c_1^2}{8} + \frac{8*c_1*c_2}{35} + \frac{3c_1}{4} + \frac{7c_2^2}{40} - \frac{12c_2}{35} + \frac{9}{8} \quad \ldots \text{eqn (7b)}$$

$$4\int_0^1 w_1(x)dx = 4\int_0^1 (x-x^3)dx = 4\left(\frac{x^2}{2}-\frac{x^4}{4}\right)_0^1 = 4\left(\frac{1}{2}-\frac{1}{4}\right)$$
$$= 1 \quad \ldots \text{eqn (7c)}$$

substituting eqn 7a, 7b & 7c into eqn 7, we get

$$\frac{5c_1^2}{8} + \frac{11c_1c_2}{28} + \frac{37c_1}{20} + \frac{11c_2^2}{56} + \frac{c_2}{20} + \frac{59}{40} = 1 \quad \ldots \text{eqn (8)}$$

- Similarly, weak form wrt w_2 from eqn (6)

$$2\int_0^1 \frac{dw_2}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx + 3\int_0^1 w_2(x)\left(\frac{d\hat{u}}{dx}\right)^2 dx = 4\int_0^1 w_2(x)dx \quad \ldots \text{eqn (9)}$$

$$2\int_0^1 \frac{dw_2}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx = 2\int_0^1 (2x-3x^2)\{1-x^3+c_1(x-x^3)+c_2(x^2-x^3)\} *$$
$$\{-3x^2+c_1(1-3x^2)+c_2(2x-3x^2)\}dx$$

$$= \frac{41c_1^2}{420} + \frac{37c_1c_2}{420} + \frac{23c_1}{70} + \frac{c_2^2}{60} + \frac{11c_2}{60} + \frac{9}{140} \quad \ldots \text{eqn (9a)}$$

$$3\int_0^1 w_2(x)\left(\frac{du}{dx}\right)^2 dx = 3\int_0^1 (x^2-x^3)\left(-3x^2+c_1(1-3x^2)+c_2(2x-3x^2)\right)dx$$

$$=\frac{37c_1^2}{280}+\frac{9c_1c_2}{140}+\frac{51c_1}{140}+\frac{2c_2^2}{35}-\frac{3c_2}{28}+\frac{27}{56} \quad \ldots \text{eqn (9b)}$$

$$4\int_0^1 w_2(x)dx = 4\int_0^1 (x^2-x^3)\,dx = 4\left(\frac{x^3}{3}-\frac{x^4}{4}\right) = 4\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$=\frac{1}{3} \quad \ldots \text{eqn (9c)}$$

Substituting eqn 9a, 9b & 9c into eqn (1), we get

$$\frac{193c_1^2}{840}+\frac{16c_1c_2}{105}+\frac{97c_1}{140}+\frac{31c_2^2}{420}+\frac{8c_2}{105}+\frac{153}{280}=\frac{1}{3}$$

$$\ldots \text{eqn (10)}$$

eqn (9) & (10) rewritten as

$$\frac{5c_1^2}{8}+\frac{11c_1c_2}{28}+\frac{37c_1}{20}+\frac{11c_2^2}{56}+\frac{c_2}{20}+\frac{19}{40}=0 \quad \ldots \text{eqn (11)}$$

$$\frac{193c_1^2}{840}+\frac{16c_1c_2}{105}+\frac{97c_1}{140}+\frac{31c_2^2}{420}+\frac{8c_2}{105}+\frac{179}{840}=0 \quad \ldots \text{eqn (12)}$$

Solving eqn (11) & (12) for c_1 & c_2, we get

$c_1 = -0.3809$ & $c_2 = -0.6251$ Ans

Thus $\hat{u}(x) = (1-x^3) - 0.3809(x-x^3) - 0.6251(x^2-x^3)$

Ans

6c) Let Assumed trignometric two constant trial function that satisfies given BC

$$\hat{u}(x) = 1 + \sum_{i=1}^{3} c_i \sin\left(\frac{i\pi x}{2}\right)$$

$$= 1 + c_1 \sin\frac{\pi x}{2} + c_2 \sin \pi x + c_3 \sin\frac{3\pi x}{2} \quad \ldots \text{Eq}^n (13)$$

$\hat{u}(0) = 1$

$\hat{u}(1) = 1 + c_1 - c_3 = 0 \Rightarrow c_3 = 1 + c_1$

$$\therefore \hat{u}(x) = 1 + c_1 \sin\frac{\pi x}{2} + c_2 \sin \pi x + (1 + c_1)\sin\frac{3\pi x}{2}$$

$$\hat{u}(x) = \left(1 + \sin\frac{3\pi x}{2}\right) + c_1\left(\sin\frac{\pi x}{2} + \sin\frac{3\pi x}{2}\right) + c_2 \sin \pi x \quad \ldots \text{Eq}^n (14)$$

or

$$\hat{u}(x) = \left(1 - \sin\frac{\pi}{2}x\right) + \sum_{i=1}^{2} c_i \sin(i\pi x) \quad \ldots \text{Eq}^n (15)$$

$\hat{u}(0) = 1$ ✓

$\hat{u}(1) = 1 - 1 + 0 = 0$ ✓

or

$$\hat{u}(x) = \cos\frac{\pi}{2}x + \sum_{i=1}^{2} c_i \sin(i\pi x) \quad \ldots \text{Eq}^n (16)$$

$\hat{u}(0) = 1$ ✓

$\hat{u}(1) = 0$ ✓

Note: Eq^n (14), (15), (16) are trignometric trial function that satisfies given BCs so we can consider anyone. Let consider Eq^n (16) as for easy calculation

$$\hat{u}(x) = \cos\frac{\pi}{2}x + \sum_{i=1}^{2} c_i \sin(i\pi x) \quad \ldots \text{Eq}^n (17)$$

$$\frac{d\hat{u}}{dx} = -\frac{\pi}{2}\sin\frac{\pi}{2}x + c_1 \pi \cos \pi x + c_2 2\pi \cos 2\pi x$$

$$w_1 = \sin \pi x \quad ; \quad w_2 = \sin 2\pi x$$

$$\frac{dw_1}{dx} = \pi \cos \pi x \quad ; \quad \frac{dw_2}{dx} = 2\pi \cos 2\pi x$$

Thus, weak form wrt w_1 from eq^n (6)

$$2\int_0^1 \frac{dw_1}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx + 3\int_0^1 w_1(x)\left(\frac{d\hat{u}}{dx}\right)^2 dx = 4\int_0^1 w_1(x)\,dx \quad \ldots \text{Eq}^n (18)$$

$$2\int_0^1 \frac{dw_1}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx = 2\int_0^1 \pi\cos\pi x\left\{\cos\frac{\pi}{2}x + c_1\sin\pi x + c_2\sin 2\pi x\right\}\left\{-\frac{\pi}{2}\sin\frac{\pi}{2}x + c_1\pi\cos\pi x + 2c_2\pi\cos 2\pi x\right\}dx$$

$$= \frac{4\pi}{105}\left(35c_1^2 + 56c_1 + 28c_2^2 + 16c_2\right) \quad \ldots \text{Eq}^n (19a)$$

$$3\int_0^1 w_1(x)\left(\frac{d\hat{u}}{dx}\right)^2 dx = 3\int_0^1 \sin\pi x\left\{-\frac{\pi}{2}\sin\frac{\pi}{2}x + c_1\pi\cos\pi x + 2c_2\pi\cos 2\pi x\right\}dx \quad \ldots \text{Eq}^n (19b)$$

$$= \frac{\pi}{140}\left(280c_1^2 + 112c_1 + 1568c_2^2 + 416c_2 + 105\right)$$

$$4\int_0^1 w_1(x)\,dx = 4\int_0^1 \sin\pi x\,dx = 4\left(-\frac{1}{\pi}\cos\pi x\right)_0^1 = 4\left(-\frac{\cos\pi}{\pi} + \frac{\cos 0}{\pi}\right)$$

$$= 4\left(\frac{1}{\pi} + \frac{1}{\pi}\right) = \frac{8}{\pi} \quad \ldots \text{Eq}^n (19c)$$

Substituting Eq^n 19a, 19b, 19c into Eq^n (18)

$$\frac{4\pi}{105}\left(35c_1^2 + 56c_1 + 28c_2^2 + 16c_2\right) - \frac{8}{\pi} + \frac{\pi}{140}\left(280c_1^2 + 112c_1 + 1568c_2^2 + 416c_2 + 105\right) \quad \ldots \text{Eq}^n (20)$$

Similarly, weak form wrt w_2 from Eq^n (6)

$$2\int_0^1 \frac{dw_2}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx + 3\int_0^1 w_2\left(\frac{d\hat{u}}{dx}\right)^2 dx = 4\int_0^1 w_2\,dx \quad \ldots \text{Eq}^n (21)$$

$$2\int_0^1 \frac{dw_2}{dx}\left(\hat{u}\frac{d\hat{u}}{dx}\right)dx = 2\int_0^1 (2\pi\cos 2\pi x)\left\{\cos\frac{\pi}{2}x + c_1\sin\pi x + c_2\sin 2\pi x\right\}\left\{-\frac{\pi}{2}\sin\frac{\pi}{2}x + \pi c_1\cos\pi x + 2\pi c_2\cos 2\pi x\right\}dx$$

$$= \frac{2\pi}{315}\left(384c_1 + 1280c_2 + 1344c_1c_2 + 105\right) \quad \ldots \text{eq}^n(22a)$$

$$3\int_0^1 w_2\left(\frac{d\hat{u}}{dx}\right)^2 dx = 3\int_0^1 \sin 2\pi x\left\{-\frac{\pi}{2}\sin\frac{\pi}{2}x + \pi c_1\cos\pi x + 2\pi c_2\cos 2\pi x\right\}^2 dx$$

$$= \frac{-\pi}{210}\left(528c_1 - 160c_2 - 672c_1c_2 + 105\right) \quad \ldots \text{eq}^n(22b)$$

$$4\int_0^1 w_2\,dx = 4\int_0^1 \sin 2\pi x\,dx = 4\left(-\frac{\cos 2\pi x}{2\pi}\right)_0^1 = 4\left(\frac{-1}{2\pi} + \frac{1}{2\pi}\right) = 0 \quad \ldots \text{eq}^n(22c)$$

substituting eq^n (22a, 22b, 22c) into eq^n 22, we get

$$\frac{2\pi}{315}\left(384c_1 + 1280c_2 + 1344c_1c_2 + 105\right) - \frac{\pi}{210}\left(528c_1 - 160c_2 - 672c_1c_2 + 105\right) \quad \ldots \text{eq}^n(23)$$

Thus solving eq^n(20) & eq^n (23) for c_1 & c_2

$c_1 = 0.0390$ & $c_2 = -0.0175$

Hence,

$$\hat{u}(x) = \cos\frac{\pi}{2}x + \sum_{i=1}^{2} c_i \sin(i\pi x)$$

$$\hat{u}(x) = \cos\frac{\pi}{2}x + .0390\sin\pi x - 0.0175\sin 2\pi x \quad \ldots \text{eq}^n(24)$$

Ans.