TRIGONOMETRIC RATIO AND IDENTITIES

The word "Trigonometry" is derived from the Greek words 'trigon' and metron' and it means 'Measuring the sides of a triangle'.

Measurement of angles

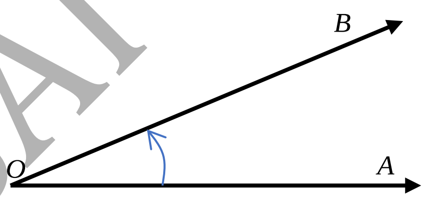
Angle: Angle is a measure of rotation of a given ray about a fixed point.

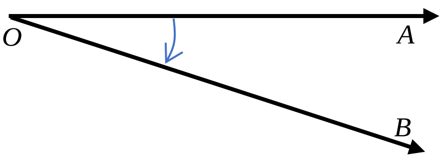
If a ray $\overrightarrow{OA}$ is rotated from $\overrightarrow{OA}$ to $\overrightarrow{OB}$ through a fixed point O.

Then, $\overrightarrow{OA}$ is called initial side.

$\overrightarrow{OB}$ is called terminal side.

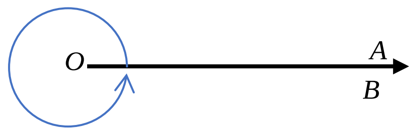
O is called vertex.





If angle is measured anti-clock wise direction, then it is considered positive. And if angle is measured clock wise direction, then it is considered negative.

If initial side is rotated and coincide with terminal side is called **one complete revolution.**



There are several units to measuring angles.

1. **Degree measurement** (or sexagesimal system or British system)

- $\left(\frac{1}{360}\right)^{th}$ of a complete revolution is called one degree.
- It is denoted by $1°$
- $1° = \left(\frac{1}{360}\right)^{th}$ of complete revolution.
- $360° =$ complete revolution
- 1 right angle $= \left(\frac{1}{4}\right)^{th}$ of complete revolution $= 90°$
- $\left(\frac{1}{60}\right)^{th}$ of a degree is called minute

  $\therefore 1° = 60'$
- $\left(\frac{1}{60}\right)^{th}$ of a minute is called second

  $\therefore 1' = 60''$

2. **Grade measurement** (or Centesimal system or French system)

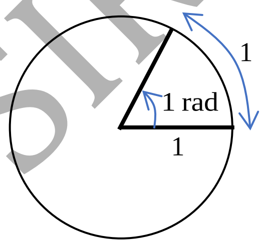
- $\left(\dfrac{1}{400}\right)^{th}$ of a complete revolution is called one grade.
- It is denoted by 1^g
- $1^g = \left(\dfrac{1}{400}\right)^{th}$ of complete revolution.
- $400^g =$ complete revolution
- 1 right angle $= \left(\dfrac{1}{4}\right)^{th}$ of complete revolution $= 100^g$
- $\left(\dfrac{1}{100}\right)^{th}$ of a degree is called minute

  $\therefore 1^g = 100'$
- $\left(\dfrac{1}{100}\right)^{th}$ of a minute is called second

  $\therefore 1' = 100''$

3. **Radian measurement** (or Circular system)

Angle subtended by an arc of length 1 unit in a circle of radius 1 unit (unit circle) at the centre is called 1 radian.



- One complete revolution subtends an angle 2π.
- If angle subtended at centre is θ, radius of circle is r and length of arc is l, then $\theta = \dfrac{l}{r}$.

Relation between degree, grade & radian

If angle θ is written as D if measured in degree, G if measured in grade & R if measured in radian., then

$$\frac{D}{360} = \frac{G}{400} = \frac{R}{2\pi}$$

- $D = \dfrac{180}{\pi} \times R$ & $R = \dfrac{\pi}{180} \times D$

Example 1. Convert the following degree measure into radian measure.

(a) $\left(\dfrac{2\pi}{15}\right)^c$ (b) $(6)^c$ (c) $\left(\dfrac{1}{4}\right)^c$ (d) $(-2)^c$

Example 2. Convert the following radian measure into degree measure.

(i) $340°$ (ii) $40°20'$ (iii) $-37°30'$ (iv) $5°37'30''$

Example 3. Find the length of an arc of a circle of radius $5cm$ subtending a central angle measuring $15°$

Example 4. If the arcs of same length in two circles subtend angles of $60°$ and $75°$ at them centres. Find ratio of their radii.

Example 5. Express in three system of angular measurement, the magnitude of the angle of a regular decagon.

Example 6. The radius of a circle is $30cm$. Find the length of an arc of this circle if the length of the chord of the arc is $30cm$.

Example 7. The minute hand of a watch is $1.5cm$ long. How far does its tip move in 40 minute? (Use $\pi = 3.14$)

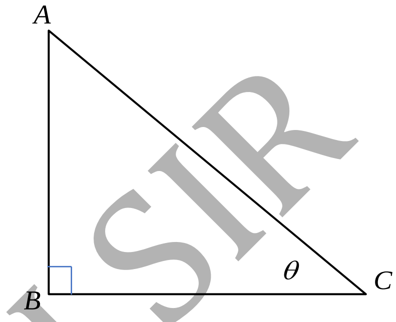
Example 8. A circular wire of radius $3cm$ is cut and bent so as to lie along the circumference of a hoop whose radius is $48cm$. Find the angle in degree which is subtended at the centre of hoop.

Example 9. Find the angle between the minute hand of a clock and the hour hand when the time is $7:20$ AM.

[If time is H:M, then angle between hour and minute hand is $\theta = 30\text{H} - \left(\frac{11}{2}\right)\text{M}$]

Basic Trigonometric Ratio (or Function)

In ΔABC, $\angle B = 90^\circ$ and $\angle C = \theta$, then



Sine of angle $\theta = \sin\theta = \dfrac{\text{length of leg } \textbf{opposite} \text{ to angle } \theta}{\textbf{hypogenous}} = \dfrac{AB}{AC}$

Cosine of angle $\theta = \cos\theta = \dfrac{\text{length of leg } \textbf{adjacent} \text{ to angle } \theta}{\textbf{hypogenous}} = \dfrac{BC}{AC}$

Tangent of angle $\theta = \tan\theta = \dfrac{\text{length of leg } \textbf{opposite} \text{ to angle } \theta}{\text{length of leg } \textbf{adjacent} \text{ to angle } \theta} = \dfrac{AB}{BC}$

Cosecant of angle $\theta = \operatorname{cosec}\theta = \dfrac{\textbf{hypogenous}}{\text{length of leg } \textbf{opposite} \text{ to angle } \theta} = \dfrac{AC}{AB}$

Secant of angle $\theta = \sec\theta = \dfrac{\textbf{hypogenous}}{\text{length of leg } \textbf{adjacent} \text{ to angle } \theta} = \dfrac{AC}{BC}$

Cotangent of angle $\theta = \cot\theta = \dfrac{\text{length of leg } \textbf{adjacent} \text{ to angle } \theta}{\text{length of leg } \textbf{opposite} \text{ to angle } \theta} = \dfrac{BC}{AB}$

- $1 - \cos\theta =$ versed $\sin\theta$ (cosine falls short of unity)
- $1 - \sin\theta =$ coversed $\sin\theta$ (sine falls short of unity)
- All T-Ratios are real numbers.

Basic Trigonometric Identities

(a) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} \Rightarrow \operatorname{cosec}\theta.\sin\theta = 1$

(b) $\sec\theta = \dfrac{1}{\cos\theta} \Rightarrow \sec\theta.\cos\theta = 1$

(c) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(d) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta} \Rightarrow \cot\theta.\tan\theta = 1$

(e) $\sin^2\theta + \cos^2\theta = 1$

(f) $1 + \tan^2\theta = \sec^2\theta \;\Rightarrow\; 1 = \sec^2\theta - \tan^2\theta = (\sec\theta - \tan\theta)(\sec\theta + \tan\theta)$

(if the product of two quantities is 1, then they are reciprocal of each other)

i.e. $\sec\theta - \tan\theta = \dfrac{1}{\sec\theta + \tan\theta}$ & $\sec\theta + \tan\theta = \dfrac{1}{\sec\theta - \tan\theta}$

(g) $1 + \cot^2\theta = \operatorname{cosec}^2\theta \;\Rightarrow\; 1 = \operatorname{cosec}^2\theta - \cot^2\theta = (\operatorname{cosec}\theta - \cot\theta)(\operatorname{cosec}\theta + \cot\theta)$

$\therefore \operatorname{cosec}\theta - \cot\theta = \dfrac{1}{\operatorname{cosec}\theta + \cot\theta}$ & $\operatorname{cosec}\theta + \cot\theta = \dfrac{1}{\operatorname{cosec}\theta - \cot\theta}$

- If angle $\in$ I Quadrant i.e. $\left[0, \dfrac{\pi}{2}\right]$, then use a right-angled triangle to convert one T-Ratio from another T-Ratio. (learnt some Pythagorean triplets).

Example 10. If $\cot\theta = \frac{4}{3}$, then find the value of $\sin\theta$, $\cos\theta$ and $\operatorname{cosec}\theta$.

Example 11. If $\sin\theta + \operatorname{cosec}\theta = 2$, then find the value of $\sin^8\theta + \operatorname{cosec}^8\theta$.

Example 12. If $\sin\theta$ and $\cos\theta$ are the roots of the equation $ax^2 - bx + c = 0$, then a, b and c satisfy the relation

(a) $a^2 + b^2 + 2ac = 0$
(b) $a^2 - b^2 + 2ac = 0$
(c) $a^2 + c^2 + 2ab = 0$
(d) $a^2 - b^2 - 2ac = 0$

Example 13. If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, then $m^2 - n^2 = 4\sqrt{mn}$.

Example 14. If $\sin\theta + \sin^2\theta = 1$, then prove that $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1 = 0$.

Example 15. $2\left(\sin^6\theta + \cos^6\theta\right) - 3\left(\sin^4\theta + \cos^4\theta\right) + 1$ is equal to

(a) 0
(b) 1
(c) -2
(d) None of these

Example 16. Prove that: $2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta = \frac{1-\tan^8\theta}{\tan^4\theta}$

Example 17. If $\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1$, prove that

(i) $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$

(ii) $\frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} = 1$

Example 18. If $\frac{ax}{\cos\theta} + \frac{by}{\sin\theta} = a^2 - b^2$ and $\frac{ax\sin\theta}{\cos^2\theta} - \frac{by\cos\theta}{\sin^2\theta} = 0$, prove that $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = \left(a^2 - b^2\right)^{\frac{2}{3}}$

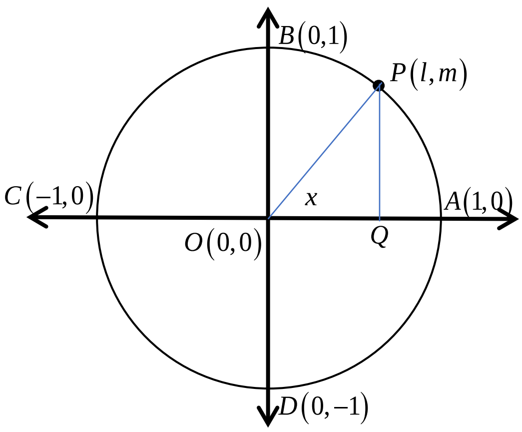
Example 19. If $\frac{\sin A}{\sin B} = p$ and $\frac{\cos A}{\cos B} = q$, find $\tan A$ and $\tan B$.

Example 20. If $\frac{\sin^4\theta}{a} + \frac{\cos^4\theta}{b} = \frac{1}{a+b}$, prove that

(i) $\frac{\sin^8\theta}{a^3} + \frac{\cos^8\theta}{b^3} = \frac{1}{(a+b)^3}$

(ii) $\frac{\sin^{4n}\theta}{a^{2n-1}} + \frac{\cos^{4n}\theta}{b^{2n-1}} = \frac{1}{(a+b)^{2n-1}}, n \in N$

Trigonometric Functions Using Unit circles



Consider a unit circle with centre at origin. Circle intersect x-axis at $A(1,0)$ & $C(-1,0)$ and y-axis at $B(0,1)$ & $D(0,-1)$.

Let a point $P(l,m)$ on the circle as shown in figure. Let $\angle AOP = x$ (it may be radian or degree).

Now, In ΔOPQ

$\cos x = l$ & $\sin x = m$

So, we write co-ordinate of $P(l,m)$ as $P(\cos x, \sin x)$.

Here, $OP = 1$, use distance formula.

$\therefore l^2 + m^2 = 1$

All angles which are integral multiples of $\frac{\pi}{2}$ are called Quadrant angles.

So, for Quadrant angles,

$$\sin 0^\circ = 0 \qquad \cos 0 = 1$$
$$\sin\frac{\pi}{2} = 1 \qquad \cos\frac{\pi}{2} = 0$$
$$\sin \pi = 0 \qquad \cos \pi = -1$$
$$\sin\frac{3\pi}{2} = -1 \qquad \cos\frac{3\pi}{2} = 0$$
$$\sin 2\pi = 0 \qquad \cos 2\pi = 1$$

Now, we can observe that values repeated after one complete revolution and one complete revolution is 2π .

$$\therefore \sin(2n\pi + x) = \sin x \ \& \ \cos(2n\pi + x) = \cos x$$

We can define other T-ratios using properties (a), (b), (c) & (d) from basic trigonometric identities.

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, x \neq n\pi, n \in I \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}, x \neq (2n+1)\frac{\pi}{2}, n \in I$$

$$\sec\theta = \frac{1}{\cos\theta}, x \neq (2n+1)\frac{\pi}{2}, n \in I \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}, x \neq n\pi, n \in I$$

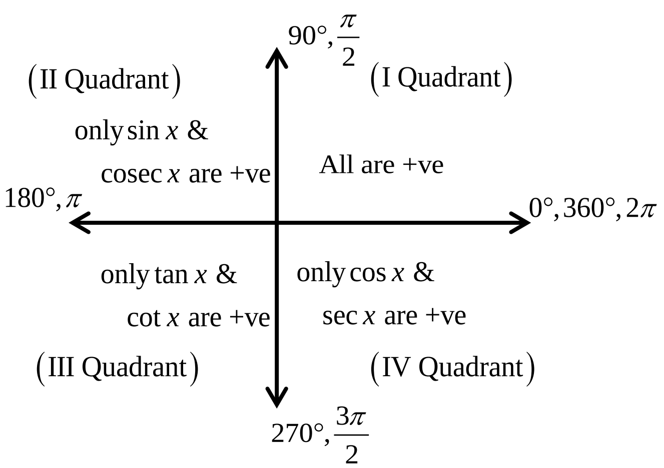
As we know that $\cos x = l$ & $\sin x = m$, then

(i) $\sin x$ & $\cos x$ are positive in first Quadrant. So, all other T-Ratios are positive.

(ii) $\sin x$ is positive & $\cos x$ is negative in second Quadrant. So, $\operatorname{cosec}\theta$ is positive and $\sec\theta$, $\tan\theta$, $\cot\theta$ are negative.

(iii) $\sin x$ & $\cos x$ are negative in third Quadrant. So, $\tan\theta$ & $\cot\theta$ are positive and $\operatorname{cosec}\theta$ & $\sec\theta$ are negative.

(iv) $\sin x$ is negative & $\cos x$ is positive in fourth Quadrant. So, $\sec\theta$ is positive and $\operatorname{cosec}\theta$, $\tan\theta$ & $\cot\theta$ are negative.



We can combine the above result, to memorise, as ASTC.

A all trigonometric function (I quad.)

S $\sin\theta$ & its reciprocal (II quad.)

T $\tan\theta$ & its reciprocal (III quad.)

C $\cos\theta$ & its reciprocal (IV quad.)

Domain & Range of T-Ratio

From the concept of unit circle, we observed that $\sin x$ & $\cos x$ is defined for all real number x and its value vary from $[-1,1]$, using domain and range of $\sin x$ & $\cos x$ we can derived the domain and range of other T-Ratio.

	Domain of T-Ratio	Range of T-Ratio	Period
$\sin x$	R	$[-1,1]$	2π
$\cos x$	R	$[-1,1]$	2π
$\tan\theta$	$R-\left\{x:(2n+1)\frac{\pi}{2}, n\in I\right\}$	R	π
$\cot\theta$	$R-\{x:\mathrm{n}\pi, n\in I\}$	R	π
$\operatorname{cosec}\theta$	$R-\{x:\mathrm{n}\pi, n\in I\}$	$R-(-1,1)$	2π
$\sec\theta$	$R-\left\{x:(2n+1)\frac{\pi}{2}, n\in I\right\}$	$R-(-1,1)$	2π

Trigonometric Functions of allied angles

If θ is any angle, then $-\theta$, $\frac{\pi}{2}\pm\theta$, $\pi\pm\theta$, $\frac{3\pi}{2}\pm\theta$, $2\pi\pm\theta$ are called allied angles.

	$-\theta$	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$ $(\text{or } -\theta)$
$\sin x$	$-\sin x$	$\cos x$	$\cos x$	$\sin x$	$-\sin x$	$-\cos x$	$-\cos x$	$-\sin x$
$\cos x$	$\cos x$	$\sin x$	$-\sin x$	$-\cos x$	$-\cos x$	$-\sin x$	$\sin x$	$\cos x$
$\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$
$\cot\theta$	$-\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$
$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$\sec\theta$	$\sec\theta$	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$
$\sec\theta$	$\sec\theta$	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$	$\operatorname{cosec}\theta$	$\sec\theta$

We can **prove all allied angles** using Trigonometric Function of Sum and Difference of Two Angles.

To convert allied angles, remember these steps

Step I: - we first know that if allied angled is of the form

(i) $\frac{\pi}{2}\pm\theta$ and $\frac{3\pi}{2}\pm\theta$, then T-Ratio convert into its complementary T-ratio. And,

(ii) $\pi\pm\theta$ and $2\pi\pm\theta$, then T-Ratio does not change.

Step II: - Use sign ($+$ or $-$), by using ASTC.

Values of some Standard Angles

	0	$30°$ or $\frac{\pi}{6}$	$45°$ or $\frac{\pi}{4}$	$60°$ or $\frac{\pi}{3}$	$90°$ or $\frac{\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ND

$\cot\theta$	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\operatorname{cosec}\theta$	ND	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	ND

ND means not defined

Example 21. If $\cos x = -\frac{3}{5}$, $x \in$ III quadrant, find the values of other five trigonometric functions.

Example 22. Find the value of

(i) $\cos\frac{31\pi}{3}$ (ii) $\sin(-1710°)$

Example 23. If $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$, then θ is equal to

(a) 30°
(b) 150°
(c) 210°
(d) None of these

Example 24. If $\cos\theta = -\frac{1}{2}$ and $\pi < \theta < \frac{3\pi}{2}$, then find the value of $4\tan^2\theta - 3\operatorname{cosec}^2\theta$.

Example 25. Prove that:

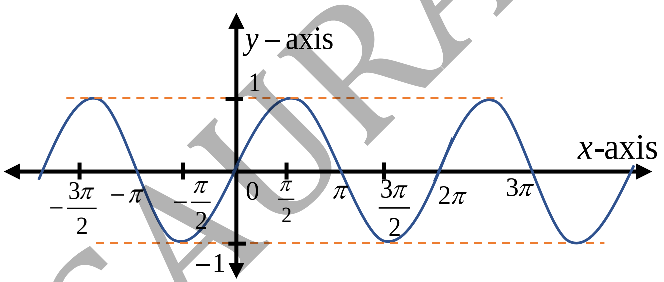
(i) $\cos 570° \sin 510° + \sin(-330°)\cos(-390°) = 0$

(ii) $\cot^2\frac{\pi}{6} + \operatorname{cosec}\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6} = 6$

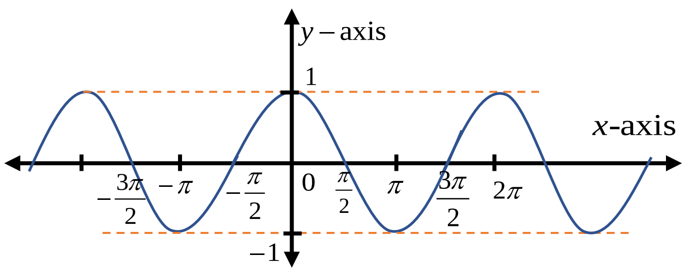
(iii) $\tan\frac{11\pi}{3} - 2\sin\frac{9\pi}{3} - \frac{3}{4}\operatorname{cosec}^2\frac{\pi}{4} + 4\cos^2\frac{17\pi}{6} = \frac{3-2\sqrt{3}}{2}$

Graph of Trigonometric Function

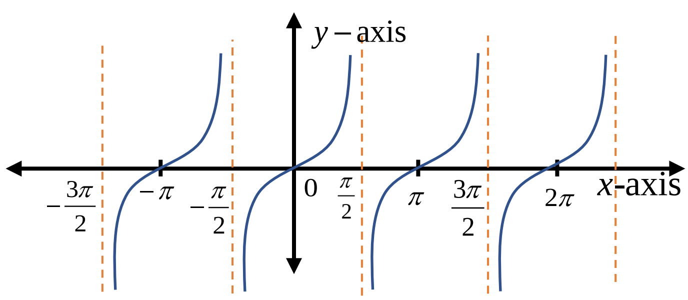
(1) $y = \sin x$



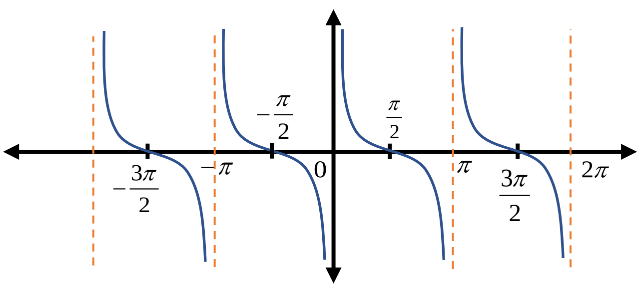
(2) $y = \cos x$



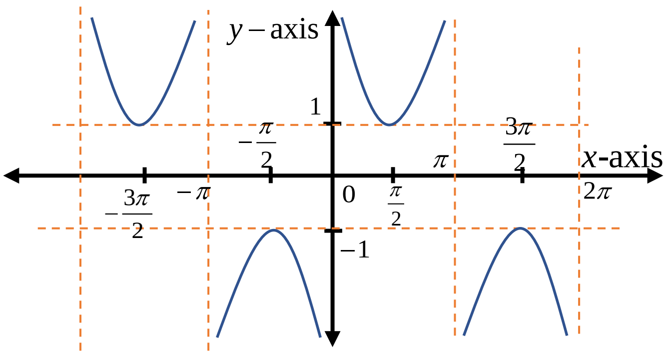
(3) $y = \tan x$



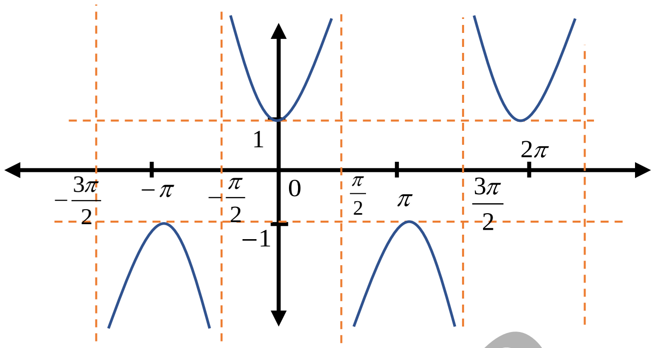
(4) $y = \cot x$



(5) $y = \operatorname{cosec} x$



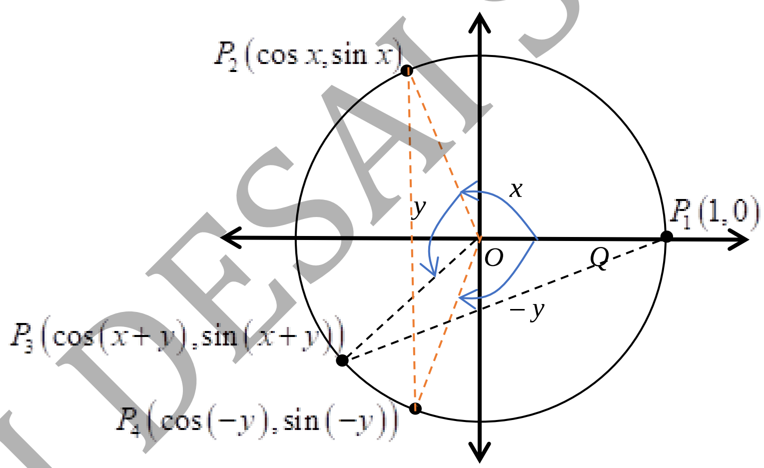
(6) $y = \sec x$



Trigonometric Function of Sum and Difference of Two Angles

Consider a unit circle with centre at origin. Let $\angle P_1OP_2 = x$, $\angle P_2OP_3 = y$ and $\angle P_1OP_4 = (-y)$.

Then, $P_1(1,0)$, $P_2(\cos x, \sin x)$, $P_3(\cos(x+y), \sin(x+y))$ and $P_4(\cos(-y), \sin(-y))$ as shown in the figure.



Consider the ΔP_1OP_3 and ΔP_2OP_4

$\Delta P_1OP_3 \cong \Delta P_2OP_4$

$\therefore P_1P_3 = P_2P_4$

$\therefore (P_1P_3)^2 = (P_2P_4)^2$

$(\cos(x+y)-1)^2 + (\sin(x+y)-0)^2 = (\cos x - \cos(-y))^2 + (\sin x - \sin(-x))^2$

$(\cos(x+y)-1)^2 + (\sin(x+y)-0)^2 = (\cos x - \cos y)^2 + (\sin x + \sin x)^2$

$\cos^2(x+y) - 2\cos(x+y) + 1 + \sin^2(x+y) = \cos^2 x + \cos^2 y - 2\cos x\cos y + \sin^2 x + \sin^2 y + 2\sin x \sin y$

$(\cos^2(x+y) + \sin^2(x+y)) - 2\cos(x+y) + 1 = (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) - 2\cos x\cos y + 2\sin x\sin y$

$1 - 2\cos(x+y) + 1 = 1 + 1 - 2\cos x\cos y + 2\sin x \sin y$

$\therefore \cos(x+y) = \cos x\cos y - \sin x \sin y$

We can derive following properties from this property only.

(1) $\cos(x+y) = \cos x\cos y - \sin x\sin y$

(2) $\cos(x-y) = \cos x\cos y + \sin x\sin y$

(3) $\sin(x+y) = \sin x\cos y + \cos x\sin y$

(4) $\sin(x-y) = \sin x\cos y - \cos x\sin y$

(5) $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

(6) $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x\tan y}$

(7) $\cot(x+y) = \dfrac{\cot x\cot y - 1}{\cot y + \cot x}$

(8) $\cot(x-y) = \dfrac{\cot x\cot y + 1}{\cot y - \cot x}$

Example 26. Prove that: $\sqrt{3}\operatorname{cosec} 20° - \sec 20° = 4$

Example 27. Prove that $\tan 70° = \cot 70° + 2\cot 40°$

Example 28. If $\sin A=\frac{3}{5}$ and $\cos B=\frac{9}{41}$, $0<A\,\&\,B<\frac{\pi}{2}$, then find the value of the following

(i) $\sin(A+B)$

(ii) $\sin(A-B)$

(iii) $\cos(A+B)$

(iv) $\cos(A-B)$

Example 29. Prove that:

(i) $\sin(45°+A)\cos(45°-B)+\cos(45°+A)\sin(45°-B)=\cos(A-B)$

(ii) $\tan\left(\frac{\pi}{4}+\theta\right)\tan\left(\frac{3\pi}{4}+\theta\right)=-1$

Example 30. If $x+y=45°$, then prove that:

(i) $(1+\tan x)(1+\tan y)=2$

(ii) $(\cot x-1)(\cot y-1)=2$

Example 31. If $\sin B=3\sin(2A+B)$, prove that $2\tan A+\tan(A+B)=0$

Example 32. Prove: $1+\tan A\tan\frac{A}{2}=\tan A\cot\frac{A}{2}-1=\sec A$

Example 33. Show that $2\sin^2\beta+4\cos(\alpha+\beta)\sin\alpha\sin\beta+\cos 2(\alpha+\beta)=\cos 2\alpha$.

Example 34. If angle θ is divided into two parts such that the tangent of one part is k times the tangents of the other, and ϕ is their difference, then show that $\sin\theta=\frac{k+1}{k-1}\sin\phi$

Example 35. If $x\cos\theta=y\cos\left(\theta+\frac{2\pi}{3}\right)=z\cos\left(\theta+\frac{4\pi}{3}\right)$, then find the value of $xy+yz+zx$.

Example 36. Prove the following **identities**:

(i) $\sin(A+B+C)$

$=\sin A\cos B\cos C+\sin B\cos A\cos C+\sin C\cos A\cos B-\sin A\sin B\sin C$

$=\cos A\cos B\cos C[\tan A+\tan B+\tan C-\tan A\tan B\tan C]$

(ii) $\cos(A+B+C)$

$=\cos A\cos B\cos C-\sin A\sin B\cos C-\sin A\cos B\sin C-\cos A\sin B\sin C$

$=\cos A\cos B\cos C[1-\tan A\tan B-\tan B\tan C-\tan C\tan A]$

(iii) $\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}=\frac{S_1-S_3}{1-S_2}$

Sum or Difference into Product

Let $x+y=A$ and $x-y=B \Rightarrow x=\frac{A+B}{2}$ & $y=\frac{A-B}{2}$,

Now, $\cos(x+y)+\cos(x-y)=\cos x\cos y-\sin x\sin y+\cos x\cos y+\sin x\sin y=2\cos x\cos y$

Replace above results, we get

(1) $\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

Similarly, we get

(2) $\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

(3) $\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

(4) $\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

Example 37. $\cos 7A+\cos 8A=2\cos\left(\frac{15A}{2}\right)\cos\left(\frac{A}{2}\right)$

Example 38. Find the value of $2\sin 3\theta\sin\theta-\cos 2\theta+\cos 4\theta$

Example 39. $\dfrac{\sin 5\theta+\sin 2\theta-\sin\theta}{\cos 5\theta+2\cos 3\theta+2\cos^2\theta+\cos\theta}$ is equal to

(a) $\tan\theta$ (c) $\cot\theta$

(b) $\cos\theta$ (d) None of these

Example 40. Simplify: $\dfrac{\sin 75°-\sin 15°}{\cos 75°+\cos 15°}$

Example 41. Prove that:

(i) $\sin 12°\sin 48°\sin 54°=\frac{1}{8}$

(ii) $\dfrac{\cos A-\cos 3A}{\sin A-\sin 3A}=-\tan 2A$

(iii) $\dfrac{\sin 2A+\sin 4A+\sin 6A+\sin 8A}{\cos 2A+\cos 4A+\cos 6A+\cos 8A}=\tan 5A$

(iv) $(\sin 3A+\sin A)\sin A+(\cos 3A-\cos A)\cos A=0$

Product into Sum or Difference

$$\cos(x+y)+\cos(x-y)=\cos x\cos y-\sin x\sin y+\cos x\cos y+\sin x\sin y=2\cos x\cos y$$

$$\therefore \cos(x+y)+\cos(x-y)=2\cos x\cos y$$

(1) $2\cos x\cos y=\cos(x+y)+\cos(x-y)$

Similarly, we get

(2) $-2\sin x\sin y=\cos(x+y)-\cos(x-y)$

(3) $2\sin x\cos y=\sin(x+y)+\sin(x-y)$

(4) $2\cos x\sin y=\sin(x+y)-\sin(x-y)$

Some more results

(5) $\sin(A+B).\sin(A-B)=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$

(6) $\cos(A+B).\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$

Example 42. If $\sin 2A=\lambda\sin 2B$, then prove that $\dfrac{\tan(A+B)}{\tan(A-B)}=\dfrac{\lambda+1}{\lambda-1}$.

Example 43. Prove that:

(i) $\cos A\sin(B-C)+\cos B\sin(C-A)+\cos C\sin(A-B)=0$

(ii) $\cos 20°\cos 40°\cos 60°\cos 80°=\frac{1}{16}$

(iii) $\dfrac{\sin 8\theta\cos\theta-\sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta-\sin 3\theta\sin 4\theta}=\tan 2\theta$

Example 44. Prove that:

(i) $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)=\frac{1}{\sqrt{2}}\sin A$

(ii) $\cos^2(A-B)+\cos^2 B-2\cos(A-B)\cos A\cos B=\sin^2 A$

Trigonometric Ratios of Multiple angles

(1) $\sin 2x = 2\sin x\cos x = \dfrac{2\tan x}{1+\tan^2 x}$

[$\sin\left(x+y\right) = \sin x\cos y + \cos x\sin y$, put $x = y$]

(2) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1-\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

[$\cos\left(x+y\right) = \cos x\cos y - \sin x\sin y$, put $x = y$]

(3) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(4) $\sin 3x = 3\sin x - 4\sin^3 x$

(5) $\cos 3x = 4\cos^3 x - 3\cos x$

(6) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$

Example 45. Prove that

(i) $\tan A + \cot A = 2\operatorname{cosec} 2A$

(ii) $\dfrac{2\cos 2A+1}{2\cos 2A-1} = \tan\left(60^\circ + A\right)\tan\left(60^\circ - A\right)$

(iii) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

(iv) $\dfrac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta} = \cot\theta$

(v) $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$

(vi) $\tan A + \tan\left(60^\circ + A\right) + \tan\left(120^\circ + A\right) = 3\tan 3A$

(vii) $\cot\theta\cot\left(60^\circ - A\right)\cot\left(60^\circ + A\right) = \cot 3\theta$

(viii) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(ix) $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

Example 46. Find the value of $\left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\dfrac{5\pi}{8}\right)\left(1+\cos\dfrac{7\pi}{8}\right)$

Trigonometric Ratios of Sub-multiple angles

(1) $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

(2) $\cos\theta = \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2} = 2\cos^2\dfrac{\theta}{2} - 1 = 1 - 2\sin^2\dfrac{\theta}{2}$

(3) $1+\cos\theta = 2\cos^2\dfrac{\theta}{2}$

(4) $1-\cos\theta = 2\sin^2\dfrac{\theta}{2} \Rightarrow \sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}} = \begin{cases} + & \text{if } \dfrac{\theta}{2} \text{ lies in I or II quadrants} \\ - & \text{if } \dfrac{\theta}{2} \text{ lies in III or IV quadrants} \end{cases}$

(5) $1+\cos\theta = 2\cos^2\dfrac{\theta}{2} \Rightarrow \cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}} = \begin{cases} + & \text{if } \dfrac{\theta}{2} \text{ lies in I or IV quadrants} \\ - & \text{if } \dfrac{\theta}{2} \text{ lies in II or III quadrants} \end{cases}$

(6) $\tan\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \begin{cases} + & \text{if } \dfrac{\theta}{2} \text{ lies in I or III quadrants} \\ - & \text{if } \dfrac{\theta}{2} \text{ lies in II or IV quadrants} \end{cases}$

Trigonometric Functions of an angle 18°

Let $\theta = 18^\circ$, then $5\theta = 90^\circ$ and $2\theta = 90^\circ - 3\theta$

$\therefore \sin(2\theta) = \sin(90^\circ - 3\theta) = \cos(3\theta)$

$\therefore 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

$\cos\theta \neq 0 \;\Rightarrow\; 2\sin\theta = 4\cos^2\theta - 3$

$\therefore 2\sin\theta = 4\left(1-\sin^2\theta\right) - 3 \quad \Rightarrow$

$4\sin^2\theta + 2\sin\theta - 1 = 0$

On solving, we get $\sin\theta = \dfrac{-1\pm\sqrt{5}}{4}$,

$\because$ $\theta \in$ I Quadrant $\Rightarrow$ $\sin\theta = \dfrac{\sqrt{5}-1}{4}$

Example 47. Match the column

Column I		Column II	
(1)	$\dfrac{1-\cos\theta}{\sin\theta}$	(a)	$\cot^2\dfrac{\theta}{2}$
(2)	$\dfrac{1+\cos\theta}{1-\cos\theta}$	(b)	$\cot\dfrac{\theta}{2}$
(3)	$\dfrac{1+\cos\theta}{\sin\theta}$	(c)	$\lvert\cos\theta+\sin\theta\rvert$
(4)	$\sqrt{1+\sin 2\theta}$	(d)	$\tan\dfrac{\theta}{2}$

Example 48. Find the value of:

(i) $\sin 15^\circ$

(ii) $\tan\dfrac{13\pi}{12}$

(iii) $\sin\dfrac{\pi}{8}$

(iv) $\cos\dfrac{\pi}{8}$

Example 49. Find the value of

(i) $\sin 67\dfrac{1}{2}^\circ + \cos 67\dfrac{1}{2}^\circ$

(ii) $\sin 78^\circ - \sin 66^\circ - \sin 42^\circ + \sin 6^\circ$

(iii) $\sin\dfrac{\pi}{10} + \sin\dfrac{13\pi}{10}$

(iv) $\cos^2 48^\circ - \sin^2 12^\circ$

Example 50. If θ lies in the second Quadrant, then find the value of $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} + \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}$

Example 51. Prove that

(i) $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$

(ii) $\dfrac{\sin 4\theta + \sin 2\theta}{1 + \cos 4\theta + \cos 2\theta} = \tan 2\theta$

(iii) $\dfrac{1+\sin A - \cos A}{1+\sin A + \cos A} = \tan\dfrac{A}{2}$

(iv) $\dfrac{1-\cos A + \cos B - \cos(A+B)}{1+\cos A - \cos B - \cos(A+B)} = \tan\dfrac{A}{2}\cot\dfrac{B}{2}$

Example 52. Show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$

Example 53. If α and β are the solutions of the equation $a\tan\theta + b\sec\theta = c$, then show that $\tan(\alpha+\beta) = \dfrac{2ac}{a^2-b^2}$

Example 54. If $\tan\dfrac{\theta}{2} = \sqrt{\dfrac{1-e}{1+e}}\tan\dfrac{\phi}{2}$, then prove that $\cos\phi = \dfrac{\cos\theta - e}{1 - e\cos\theta}$.

Trigonometric Ratios of Some more Standard Angles

	$\frac{\pi}{12}$	$\frac{\pi}{10}$	$\frac{\pi}{8}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{3\pi}{8}$	$\frac{2\pi}{5}$	$\frac{5\pi}{12}$
	15°	18°	22.5°	36°	54°	67.5°	72°	75°
$\sin\theta$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$		$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
$\cos\theta$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$		$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
$\tan\theta$	$\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$		$\sqrt{2}+1$		$\frac{\sqrt{3}+1}{\sqrt{3}-1}=2+\sqrt{3}$

Example 55. Find the value of $\tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ$.

Conditional Trigonometric Identities

If $A+B+C=180^\circ$, then

(1) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(2) $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$

(3) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$

(4) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(5) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

(6) $\cos 2A+\cos 2B+\cos 2C=1-4\cos A\cos B\cos C$

(7) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(8) $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Example 56. In any ΔABC, $\sin A-\cos B=\cos C$, then angle B is

(a) $\frac{\pi}{2}$

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$

(d) $\frac{\pi}{6}$

Example 57. If $A+B+C=\frac{3\pi}{2}$, then $\cos 2A+\cos 2B+\cos 2C$ is equal to

(a) $1-4\cos A\cos B\cos C$

(b) $4\sin A\sin B\sin C$

(c) $1+2\cos A\cos B\cos C$

(d) $1-4\sin A\sin B\sin C$

Example 58. If $ABCD$ is a cycle quadrilateral, then find the value of $\sin A+\sin B-\sin C-\sin D$.

Example 59. If $A+B+C=\frac{\pi}{2}$, then find the value of $\tan A\tan B+\tan B\tan C+\tan C\tan A$.

Example 60. If $A+B+C=180^\circ$, prove that

$$\sin\left(B+2C\right)+\sin\left(C+2A\right)+\sin\left(A+2B\right)=4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$$

Example 61. If $x+y+z=xyz$, then prove that $\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}.\frac{2y}{1-y^2}.\frac{2z}{1-z^2}$.

Maximum & Minimum values of Trigonometric Expression

- $a\cos\theta+b\sin\theta$ will always lie in the interval $\left[-\sqrt{a^2+b^2},\sqrt{a^2+b^2}\right]$
- Minimum value of $a^2\tan^2\theta+b^2\cot^2\theta=2ab$, where $a,b>0$ {$\because$ AM $\geq$ GM }
- If $A+B+C=\pi$, then $\sin A+\sin B+\sin C$ and $\sin A\sin B\sin C$ maximum only when $A=B=C=60°$
- In case a quadratic in $\sin\theta$ and $\cos\theta$ is given then the maximum or minimum values can be obtained by making perfect square.
- $\alpha,\beta\in\left(0,\frac{\pi}{2}\right)$ and $\alpha+\beta=\lambda$, then

  (i) Maximum value of $\sin\alpha\sin\beta$, $\cos\alpha\cos\beta$, $\sin\alpha+\sin\beta$ and $\cos\alpha+\cos\beta$, when $\alpha=\beta=\frac{\lambda}{2}$

  (ii) Minimum value of $\sec\alpha+\sec\beta$, $\operatorname{cosec}\alpha+\operatorname{cosec}\beta$ and $\tan\alpha+\tan\beta$, when $\alpha=\beta=\frac{\lambda}{2}$

Example 62. Find the minimum values of $\cos\theta+\cos2\theta$ for all real values of θ.

Example 63. Find the maximum and minimum values of following for all real values of θ.

(i) $1+2\sin\theta+3\cos^2\theta$

(ii) $5\cos\theta+3\sin\left(\theta+\frac{\pi}{6}\right)$

(iii) $\cos^2\theta-6\sin\theta\cos\theta+3\sin^2\theta+2$

Example 64. If $\alpha+\beta=90°$, then find the maximum value of $\sin\alpha\sin\beta$.

Example 65. If $A=\cos^2\theta+\sin^4\theta$ for all values of θ, then prove that $\frac{3}{4}\leq A\leq1$

Example 66. Prove that: $-4\leq5\cos\theta+3\cos\left(\theta+\frac{\pi}{3}\right)\leq10$, for all values of θ.

Example 67. Find the maximum value of $1+\sin\left(\frac{\pi}{4}+\theta\right)+2\cos\left(\frac{\pi}{4}-\theta\right)$.

Important Results

(1) $\sin\theta\sin\left(60°-\theta\right)\sin\left(60°+\theta\right)=\frac{1}{4}\sin3\theta$

(2) $\cos\theta\cos\left(60°-\theta\right)\cos\left(60°+\theta\right)=\frac{1}{4}\cos3\theta$

(3) $\tan\theta\tan\left(60°-\theta\right)\tan\left(60°+\theta\right)=\tan3\theta$

(4) $\cot\theta\cot\left(60°-\theta\right)\cot\left(60°+\theta\right)=\cot3\theta$

(5) $\sin^2\theta+\sin^2\left(60°-\theta\right)+\sin^2\left(60°+\theta\right)=\frac{3}{2}$

(6) $\cos^2\theta+\cos^2\left(60°-\theta\right)+\cos^2\left(60°+\theta\right)=\frac{3}{2}$

(7) $\cos\theta\cos2\theta\cos4\theta\ldots\cos\left(2^{n-1}\theta\right)=\frac{\sin\left(2^n\theta\right)}{2^n\sin\theta}$

(8) $\sin\alpha+\sin\left(\alpha+\beta\right)+\sin\left(\alpha+2\beta\right)+\ldots+\sin\left(\alpha+\left(n-1\right)\beta\right)=\frac{\sin\left(\alpha+\left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n}{2}\beta\right)}{\sin\left(\frac{\beta}{2}\right)}$

(9) $\cos\alpha+\cos\left(\alpha+\beta\right)+\cos\left(\alpha+2\beta\right)+\ldots+\cos\left(\alpha+\left(n-1\right)\beta\right)=\frac{\cos\left(\alpha+\left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n}{2}\beta\right)}{\sin\left(\frac{\beta}{2}\right)}$

Example 68. Find the value of $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5}$.

Example 69. Prove that:

(i) $\sin\alpha+\sin 3\alpha+\sin 5\alpha+\ldots+\sin(2n-1)\alpha=\frac{\sin^2 n\theta}{\sin\theta}$

(ii) $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}=\frac{1}{2}$

Example 70. Prove that: $\tan A+2\tan 2A+4\tan 4A+8\tan 8A=\cot A$

Example 71. Evaluate: $\sum_{r=1}^{n-1}\cos^2\left(\frac{r\pi}{n}\right); n\geq 2$

Example 72. Prove that: $(1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)=\tan 2^n\theta.\cot\theta$.

Example 73. Evaluate: $\sin\frac{\pi}{n}+\sin\frac{3\pi}{n}+\sin\frac{5\pi}{n}+\ldots$ to n terms.

Example 74. If $(2^n+1)\theta=\pi$, then find the value of: $2^n\cos\theta.\cos 2\theta.\cos 2^2\theta\ldots\cos 2^{n-1}\theta$.

Mixed Concept Examples

Example 75. Prove that:

(i) $\tan\alpha+2\tan 2\alpha+2^2\tan 2^2\alpha+\ldots+2^n\tan 2^n\alpha=\cot\alpha$

(ii) $\operatorname{cosec} x+\operatorname{cosec} 2x+\operatorname{cosec} 4x+\ldots+\operatorname{cosec} 2^n x=\cot\frac{x}{2}-\cot 2^n x$

Example 76. Show that $2^{\sin\theta}+2^{\cos\theta}\geq 2^{\frac{\sqrt{2}-1}{\sqrt{2}}}$ for all real values of θ