FUNDAMENTAL OF MATHEMATICS

NUMBER SYSTEM

- Complex Number
  - Real Number
    - Rational Number
      - Fraction
      - Integers
      - Natural Number
      - Whole Number
    - Irrational Number
      - Surds
      - Transcendental number
  - Imaginary Number

1. **Natural Numbers (N)**: - Counting number $1,2,3,4...$ are known as Natural Numbers
   - $N=\{1,2,3,4\ldots\}$
2. **Whole Number (W)**: - The set of Natural Numbers including 0 is called Whole Number.
   - $W=\{1,2,3,4\ldots\}$
3. **Integers** (Z or I): - All-Natural Number and their negatives including Zero are known as Integers
   - $Z=\{...4,-3,-2,-1,0,1,2,3,4\ldots\}$

   (a) **Positive Integers (I^+)**: - $I^+=\{1,2,3,4\ldots\}$
   - Positive Integers are same as Natural Number.

   (b) **Negative Integers (I^-)**: - $I^-=\{\ldots-4,-3,-2,-1\}$

   (c) **Non-Negative Integers**: - $\{0,1,2,3,4\ldots\}$
   - Non-Negative Integers are same as Whole Number.
4. **Even Numbers**: - Integers which are divisible by 2 is called Even Number.
   - Last digits of Even Numbers are either $0,2,4,6,8$.
   - $0,\pm2,\pm4,\pm6,\ldots$
   - Even Numbers are generally denoted by $2n$
5. **Odd Numbers**: - Integers which are not divisible by 2 is called Odd Number.
   - Odd Numbers leaves remainder 1 when divided by 2.
   - $\pm1,\pm3,\pm5,\ldots$
   - Odd Numbers are generally denoted by $2n-1$ or $2n+1$.

Natural Number can be classified into 1, Prime Number, Composite Number.

6. **Prime Number**: - The set of Natural Number which has exactly two Integral factors, 1 and its self are called Prime Number.
    - $2, 3, 5, 7, 11\ldots$ etc.
    - 2 is only even Prime Number.
    - 25 Prime Numbers between 1 and 100.
7. **Composite Number**: - The set of Natural Number which has more than two Integral factors.
    - Numbers which are not Prime are Composite (except 1).
    - 1 is neither Prime nor Composite.
    - 4 is smallest Composite Number.
8. **Co-Prime Number**: - Two Natural Number Which have only 1 as the common factor are called Co-Prime or Relatively Prime.

    e.g.: - (1) 3 & 5 are Co-Prime.

    (2) 3 & 4 are Co-Prime.

    (3) 4 & 9 are Co-Prime.

9. **Rational Number** (Q): - The numbers of the form $\frac{p}{q}$, where p & q are Integers and $q \neq 0$, are known as Rational Number.

    e.g.: - $\frac{4}{7}, \frac{3}{2}, \frac{-5}{8}, \frac{0}{1}, \frac{10}{1}, \frac{125}{1}, \ldots$ etc.

    - All Integers are Rational Number.
    - Decimal expansion of Rational Number is either terminating or non-terminating repeating decimals.
    - Infinite Rational Numbers are existing between two Rational Number.

    e.g.: - (1) $\frac{5}{2} = 2.5$ $\rightarrow$ Terminating.

    (2) $\frac{10}{3} = 3.333\ldots = 3.\bar{3}$ $\rightarrow$ Non-Terminating Repeating.

10. **Irrational Numbers** (Q' or Q^c): - Irrational numbers are numbers which cannot be expressed in $\frac{p}{q}$, where p & q are Integers and $q \neq 0$, are known as Irrational Number.
    - Decimal expansion of Irrational Number is non-terminating non repeating

    (a) **Surds**: - Surds is an Irrational root of Rational Number.
    - $\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots$ etc.
    - Approx. value of $\sqrt{2} = 1.4142, \sqrt{3} = 1.7325, \sqrt{5} = 2.236, \sqrt{7} = 2.646$ (Remember)

    (b) **Transcendental Numbers**: - Constant Number such as $\pi, e\ \ldots$etc. are Irrational Number, Known as Transcendental Numbers.
    - Approx. value of $\pi = \frac{22}{7}$ or 3.14, $e = 2.71$ (Remember)

NOTE

- Sum of a rational Number and Irrational Number is always an irrational Number.
- The product of a non-zero Rational Number and an Irrational Number will always be an Irrational Number.

  i.e. If $a \in Q \,\&\, b \in Q^c$, then $ab =$ Rational Number only when $a = 0$
- Sum, difference, product and division of two Irrational Number need not be an Irrational Number. (It may be Rational Number also)
- Test to find whether a given Number is a Prime or not.

  STEP 1: - Select a least positive integer n such that $n^2 >$ given Number.
  STEP 2: - Test the divisibility of a given Number by every Prime Number less than n.

  STEP 3: - The given Number is Prime only if it is not divisible by any of these Primes.

Example 1. Check whether the given number is Prime or not
(i) 571 (ii) 923

***Explanation*:**

(i) $529 = 23^2 < 571$
$576 = 24^2 > 571$
$\therefore n = 24$
Prime Number less than 24 are $2, 3, 5, 7, 11, 13, 17, 19, 23$.
Since 571 is not divisible by any of these Number
$\therefore 571$ is Prime.

- **Distributive Law**: -
  (i) $a \times (b + c) = a \times b + a \times c$
  (ii) $(a + b) \times c = a \times c + b \times c$
- All Real Numbers follow the "law of trichotomy" i.e. if there are two real Number a and b, then either $a = b$
  $a < b$
  $a > b$

SQUARE

$a^2 = a \times a$

$1^2 = 1$	$11^2 = 121$	$21^2 = 441$
$2^2 = 4$	$12^2 = 144$	$22^2 = 484$
$3^2 = 9$	$13^2 = 169$	$23^2 = 529$
$4^2 = 16$	$14^2 = 196$	$24^2 = 576$
$5^2 = 25$	$15^2 = 225$	$25^2 = 625$
$6^2 = 36$	$16^2 = 256$	
$7^2 = 49$	$17^2 = 289$	
$8^2 = 64$	$18^2 = 324$	
$9^2 = 81$	$19^2 = 361$	
$10^2 = 100$	$20^2 = 400$	

- $1, 4, 9, 16, \ldots$ these numbers are called perfect square.
- Square of any number never ends with $2, 3, 7 \& 8$
- Square of odd number is odd.
- Square of even number is even
- Square of any number is always positive.
- Pythagorean Triplets: For every natural number $m > 1$, $2m, m^2 - 1 \& m^2 + 1$ form a Pythagorean Triplets.
- My method of Finding Pythagorean Triplets

  $3, 4, 5$

  $5, 12, 13$

  $7, 24, 25 \ldots$ and so on.
- To find the Square of Numbers from 26 to 75 (base 50).

  Let a number be 47

  STEP 1: - 47 is 3 less than 50.

  STEP 2: - $\left(\frac{50-3\times 2}{2}\right)\times 100+(3)^2 = 2209$

  i.e. Last two digits of 47^2 is 3^2, i.e. 09.
- To find the Square of Numbers from 76 to 125 (base 100).

  Let a number be 98

  STEP 1: - 98 is 2 less than 100.

  STEP 2: - $(100-2\times 2)\times 100+(2)^2 = 9604$

  i.e. Last two digits of 98^2 is 2^2, i.e. 04.
- Difference of two consecutive natural number is equal to sum of number

  $n^2-(n-1)^2 = 2n-1 = n+(n-1)$

  e.g.: - $41^2 - 40^2 = 41+40$
- Square of any number ends with 5: $A5^2 = A(A+1)/25$.

  e.g.: - (1) $25^2 = (2\times 3)5^2 = 625$

  (2) $125^2 = (12\times 13)5^2 = 13225$
- Sum of first n odd natural number: $1+3+5\ldots+(2n-1) = n^2$.
- Sum of first n natural number: $1+2+3\ldots+n = \frac{n(n+1)}{2}$.
- Sum of Square of first n natural number: $1^2+2^2+3^2\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$.
- Sum of Cube of first n natural number: $1^3+2^3+3^3\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

SQUARE ROOT

- Square root of odd number is odd.
- Square root of even number is even.

Method to find the Square root of a number

1. By Prime factorisation

Example 2. Find the square root of

(i) 2304

(ii) 4489

(iii) 207025

(iv) 5776

2. By Long division

Example 3. Find the square root of

(i) 3481 (ii) 226576 (iii) 4761

Example 4. Find the square root of

(i) 2.56 (ii) 7.2 (iii) 42.25

Example 5. Find the square root of

(i) 12.1801
(ii) 127.0129
(iii) 0.1790136
(iv) 5
(v) 7

[Ans 3.49, 11.27, 0.423, 2.236, 2,646]

CUBE AND CUBE ROOT

$a^3 = a \times a \times a$

$1^3 = 1$	$11^3 = 1331$
$2^3 = 8$	$12^3 = 1728$
$3^3 = 27$	$13^3 = 2197$
$4^3 = 64$	$14^3 = 2744$
$5^3 = 125$	$15^3 = 3375$
$6^3 = 216$	
$7^3 = 343$	
$8^3 = 512$	
$9^3 = 729$	
$10^3 = 1000$	

- $1, 8, 27, 64, \ldots$ these numbers are called perfect cubes.
- Cube Number ending with $0, 1, 4, 5, 6$ and 9 ends with digits $0, 1, 4, 5, 6$ and 9 respectively. (Same for Cube root)
- Cube of odd number is odd. (Same for Cube root)
- Cube of even number is even. (Same for Cube root)
- Cube of positive number is positive and negative number is negative. (Same for Cube root)
- Cube of a natural number never ends with two zero.
- If last digit of a number is 2 then its cube ends with digit 8. (visa-versa) (Same for Cube root)
- If last digit of a number is 3 then its cube ends with digit 7. (visa-versa) (Same for Cube root)
- If number n is greater than 1, then
  a. $n^x > n$ if $x > 1$
  b. $n^x < n$ if $x < 1$, and
- If number n is less than 1, then
  a. $n^x < n$ if $x > 1$
  b. $n^x > n$ if $x < 1$

e.g.: - Let $n = 2(>1)$

$2^2 = 4 > 2(x > 1)$

$\sqrt{2} = 2^{\frac{1}{2}} = 1.414 < 2(x < 1)$

Example 6. Find the cube root of

(i) 512 (iii) 13824 (v) 15625

(ii) 27000 (iv) 46656

EXPONENT AND POWER

a^n is exponent form of a number.

Where,

- a is called Base
- n is called Power
- $a^n = a \times a \times a \ldots n$ times.

e.g.: - 64 can be written in exponent form as 2^6.

Law of Exponents

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$
3. $\left(a^m\right)^n = a^{mn}$
4. $a^n \times b^n = (ab)^n$
5. $\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$
6. $a^0 = 1, a \neq 0$
7. $a^{-n} = \dfrac{1}{a^n}$

Example 7. Find x so that

(i) $(-5)^{x+1} \times (-5)^5 = (-5)^7$

(ii) $\left[2^{-1} + 4^{-1} + 6^{-1} + 8^{-1}\right]^x = 1$

(iii) $\left(\dfrac{5}{3}\right)^{-2} \times \left(\dfrac{5}{3}\right)^{-14} = \left(\dfrac{5}{3}\right)^{8x}$

Example 8. $\left[\left(\dfrac{2}{13}\right)^{-6} \div \left(\dfrac{2}{13}\right)^3\right]^3 \times \left(\dfrac{2}{13}\right)^{-9} =$ ________

Example 9. Standard form of

(i) $\dfrac{1}{10,00,00,000}$

(ii) 12340000

Example 10. The usual form of

(i) 3.41×10^4

(ii) 2.39461×10^6

Example 11. Simplify

(i) $\dfrac{125 \times x^{-3}}{5^{-3} \times 25 \times x^{-6}}$

(ii) $\dfrac{16 \times 10^2 \times 64}{2^4 \times 4^2}$

(iii) $\left(\dfrac{4}{13}\right)^4 \times \left(\dfrac{13}{7}\right)^2 \times \left(\dfrac{7}{4}\right)^3$

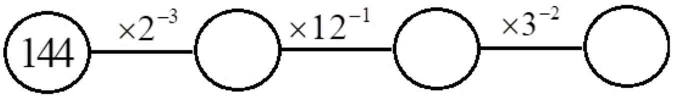
Example 12. Find the value of m

(i) $\dfrac{2^m \times 2^6}{2^{-3}} = 2^{18}$

(ii) $\dfrac{5^m \times 5^3 \times 5^{-2}}{5^{-5}} = 5^{12}$

Example 13. The cells of bacteria double itself every hour. How many cells will there be after 8 hours?

Example 14. Fill in the blanks.



Example 15. Find x.

(i) $2^{3x} = 8^{2x+1}$

(ii) $5^x + 5^{x-1} = 750$

(iii) $2^x + 2^x + 2^x = 192$

SURDS

Irrational roots of Rational Number are called Surds.

$\sqrt[n]{\ }$ is called Radical sign (n^{th} root)

- $\sqrt{\ }$ is Square root
- $\sqrt[3]{\ }$ is Cube root
- $\sqrt{49} = 7$ is not surd
- $\sqrt{48} = 4\sqrt{3}$ is surd
- Apply $\sqrt[n]{\ } = (\ )^{\frac{1}{n}}$ then solve examples
- In the expression of the form $\dfrac{a}{\sqrt{b}+\sqrt{c}}$, the denominator can be rationalised by multiplying numerator and the denominator by $\sqrt{b}-\sqrt{c}$ which is called the Conjugate of $\sqrt{b}+\sqrt{c}$.
- If $x+\sqrt{y} = a+\sqrt{b}$, where x, y, a, b are rational, then $x = a, y = b$.

Example 16. If x and y are Prime Numbers which satisfy $x^2 - 2y^2 = 1$, Solve for x and y.

[Ans $x = 3, y = 2$]

Example 17. Rationalise the denominator

(i) $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

(ii) $\dfrac{1}{\sqrt{5}+\sqrt{3}-\sqrt{8}}$

(iii) $\dfrac{12}{3+\sqrt{5}-2\sqrt{2}}$

Example 18. Find the natural number n for which the fraction $\dfrac{15n^2+8n+6}{n}$ is a natural number.

Example 19. Find the integral solutions of the equation $xy = 2x - y$.

Example 20. Find the square root of

(i) $7+2\sqrt{10}$

(ii) $7+4\sqrt{3}$

(iii) $10+\sqrt{24}+\sqrt{60}+\sqrt{40}$

(iv) $\sqrt{32}-\sqrt{24}$

(v) $12+2\sqrt{35}$

(vi) $\sqrt{18+\sqrt{120}+\sqrt{140}+\sqrt{168}}$

Example 21. $a, b \in Q$, find the value of a and b

(i) $\dfrac{3+\sqrt{5}}{3-\sqrt{5}} = a + b\sqrt{5}$

(ii) $\dfrac{3+2\sqrt{3}}{5-2\sqrt{3}} = a + b\sqrt{3}$

Example 22. Find the cube root of

(i) $72-32\sqrt{5}$

(ii) $9\sqrt{3}+11\sqrt{2}$

Some Useful results

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$
3. $\left(a^m\right)^n = \left(a^n\right)^m = a^{mn}$
4. $\left(\frac{a}{b}\right)^{-\frac{m}{n}} = \left(\frac{b}{a}\right)^{\frac{m}{n}}$
5. $a^m \div b^{-n} = a^m \times b^n$
6. $\left(\sqrt[n]{a}\right)^n = a$
7. $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$
8. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
9. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$
10. $\sqrt[n]{\sqrt[m]{\left(a^k\right)^m}} = \left(\left(\left(a^k\right)^m\right)^{\frac{1}{m}}\right)^{\frac{1}{n}} = a^{\frac{k}{n}} = \sqrt[n]{a^k}$
11. $\sqrt{a} \times \sqrt{a} = a$
12. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
13. $\left(\sqrt{a} + \sqrt{b}\right)^2 = a + b + 2\sqrt{ab}$
14. $\left(\sqrt{a} - \sqrt{b}\right)^2 = a + b - 2\sqrt{ab}$
15. If $x = n(n+1)$, then
    a. $\sqrt{x - \sqrt{x - \sqrt{x - \ldots \infty}}} = n$
    b. $\sqrt{x + \sqrt{x + \sqrt{x + \ldots \infty}}} = n+1$

Example 23. Find the value of

(i) $(243)^{0.8} \div (243)^{0.4}$

(ii) $(27)^{\frac{2}{3}} \div (64)^{-\frac{4}{3}}$

(iii) $(-3)^{(-2)^{(-2)^{(-4)}}}$

(iv) $\sqrt[3]{\sqrt[2]{729}}$

(v) $\sqrt[5]{64} \times \sqrt[5]{512}$

(vi) $\dfrac{\sqrt[7]{\sqrt[5]{\left(\left(21\right)^7\right)^5}}}{\sqrt[5]{\sqrt[3]{\left(\left(7\right)^5\right)^3}}}$

(vii) $\sqrt{5} \times \sqrt{125}$

Example 24. Find the value of $\sqrt{72 + \sqrt{72 + \sqrt{72 + \ldots \infty}}} - \sqrt{12 - \sqrt{12 - \sqrt{12 - \ldots \infty}}}$

DIVISIBILITY RULE

1. Divisible by 2:
   If a number is divisible by 2, only if its last digit is $0, 2, 4, 6, 8$ (Or we can say divisible by 2)
2. Divisible by 3:
   If a number is divisible by 3, only if sum of the digits of number should be divisible by 3.
3. Divisible by 4:
   If a number is divisible by 4, only if its last two digits is divisible by 4.
4. Divisible by 5:
   If a number is divisible by 5, only if its last digit is 0 or 5.
5. Divisible by 6:
   If a number is divisible by 6, only if it is divisible by 2 and 3.
6. Divisible by 7, 11, 13:
   A number can be divisible by 7, 11, 13, if and only if the difference of the number formed by last three digits and the number formed by the rest digits is divisible by 7, 11, 13 respectively.
   e.g.: - Check whether the following are divisible by 7 or 11 or 13.
   (i) 139125
   (ii) 12478375
7. Divisible by 8:
   If a number is divisible by 8, only if its last three digits should be divisible by 8.
8. Divisible by 9:
   If a number is divisible by 9, only if sum of the digits of number should be divisible by 9.

9. Divisible by 5:
   If a number is divisible by 10, only if its last digit is 0.
10. Divisible by 11:
    If a number is divisible by 11, only if difference between the sum of digits at odd place & even place is divisible by 11.
    e.g.: - Check 57945822 is divisible by 11 or not.

Example 25. A number of the form $10^n - 1$ is always divisible by 11 for every n is a natural number when

(a) n is odd
(b) n is even
(c) n is prime
(d) can't say

Example 26. If $653xy$ is divisible by 80, then find the value of $x+y$.

Example 27. Find the value of k, if $k35624$ is divisible by 11?

Example 28. How many numbers between 1 and 1000 are divisible by 7?

Example 29. How many numbers are there between 200 and 800 which are divisible by 5 and 7?

Example 30. How many numbers are there between 200 and 800 which are divisible by 5 or 7?

Example 31. At least what number must be subtracted from 434079, so that it becomes divisible by 137?

Example 32. Prove that

(i) The sum $ab+ba$ is multiple of 11.
(ii) $10^{25} - 7$ is divisible by 3
(iii) $n^3 + 20n$ is divisible by 48
(iv) $n^2 + 1$ is not divisible by 3
(v) $5a^3 + 13a - 30$ is divisible by 6

BODMAS RULE

BODMAS stands for

B: Brackets: $\left[\left\{\left(\overline{\ \ }\right)\right\}\right]$, they are removed strictly in the below order.

a. $\overline{\ \ }$ bar
b. ( ) curve or small bracket
c. { } curly bracket
d. [ ] square bracket

O: of

D: Division: $\div$

M: Multiplication: $\times$

A: Addition: $+$

S: Subtraction: $-$

Example 33. Simplify

(i) $8\frac{1}{2} - \left[3\frac{1}{5} \div 4\frac{1}{2} \text{ of } 5\frac{1}{3} + \left\{11 - \left(3 - \overline{1\frac{1}{4} - \frac{5}{8}}\right)\right\}\right]$ [Ans $-\frac{31}{120}$]

(ii) $5\frac{1}{3} - \left\{4\frac{1}{3} - \left(3\frac{1}{3} - \overline{2\frac{1}{3} - \frac{1}{3}}\right)\right\}$

Operation on Odd and Even Number.

1. $O \times E = E$
   $O \times O = O$
   $E \times E = E$
   $E \times O = E$
2. $O \pm E = O$
   $E \pm O = O$
   $E \pm E = E$
   $O \pm O = E$
3. $O \div E =$ not divisible
   $O \div O = O$
   $E \div E = E$ or O
   $E \div O = E$

HCF and LCM of Numbers

1. Factors: - A factor is numbers that divides a given number without remainder
    - Number of factors are finite.
    - 1 is smallest factor of every number
    - Number itself is largest factor of every number

   e.g.: - (i) Factor of $9 = 1, 3, 9$

   (ii) Factor of $18 = 1, 2, 3, 6, 9, 18$

2. Multiples: - A multiple is number that can be divided by given number without remainder.
    - Number of multiples are infinite.
    - Number itself is smallest multiple of every number

   e.g.: - (i) Multiple of $3 = 3, 6, 9, 12, 15, 18, \dots$

   (ii) Multiple of $6 = 6, 12, 18, 24, \dots$

3. Common Factor: - A Common Factor of two or more numbers is number which divides each of them exactly.
    - Number of Common factors of two or more numbers are finite.

   e.g.: - (i) 4 is a Common factor of 8 & 12.

   (ii) Common factors of 6 & 18 are $1, 2, 3, 6$

4. Common Multiple: - A Common Multiple of two or more numbers is a number which is exactly divisible by each one of them.
    - Number of Common multiples of two or more numbers are infinite.

   e.g.: - (i) 32 is a Common multiple of 8 & 16.

   (ii) Common multiples of 3 & 6 are $6, 12, 18, \dots$ etc

5. Highest Common Factor (Or GCD): - HCF of two or more number is the greatest number that divides each one of them exactly.
    - HCF of two or more given numbers is Unique.

   e.g.: - (i) 6 is HCF of 6 & 18.

   (ii) 4 is HCF 8 & 12

6. Least Common Multiple: - LCM of two or more number is the least or lowest number which is exactly divisible by each of them.
    - LCM of two or more given numbers is Unique.

   e.g.: - (i) 6 is LCM of 3 & 6. (ii) 36 is LCM of 12 & 18

Methods to find HCF and LCM are

1. Prime Factorisation
2. By method of division

Example 34. Find the HCF of

(i) $70, 90$

(ii) $3332, 3724, 4508$

(iii) $3556, 3444$

(iv) $360, 132$

[Ans 10, 196, 28,12]

Example 35. Find the greatest possible length which can be used to measure exactly the lengths 7m, 3m 85cm, 12m 95cm. [Ans 35 cm]

Example 36. Find the LCM of

(i) $32, 48, 60$ and 320 (ii) $12, 15, 20, 54$

[Ans 960, 540]

Example 37. Find LCM of $1.2, 0.24, 6$

Example 38. Find LCM & HCF of $\frac{4}{9}, \frac{10}{21}$ and $\frac{20}{63}$.

Example 39. The HCF & LCM of two number are 63 & 1260 respectively. If one number is 315, find other.

POLYNOMIALS

Polynomials is a function $P(x)$ of the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x^1 + a_0$.

Where,

- $a_n, a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are real numbers
- $a_n \neq 0$
- $a_n, a_{n-1}, a_{n-2}, \ldots a_1$ are called the coefficient of $x^n, x^{n-1}, x^{n-2}, \ldots x^1$ and a_0 is constant term
- $a_n x^n$ is called leading term
- a_n is called leading coefficient
- n is whole number

- If $a_n, a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are integers then we call it a polynomial over integers.
- If $a_n, a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are rational numbers, then it is a polynomial over rationals.

e.g.: - (1) $4x^2 + 7x - 8$ is polynomial over integers.

(2) $\frac{7}{4}x^3 + \frac{2}{3}x^2 - \frac{8}{7}x + 5$ is polynomial over rationals.

(3) $4x^2 - \sqrt{3}x + \sqrt{5}$ is polynomial over reals.

Degree of a Polynomials

'n' is a degree of polynomial. Degree is a highest power of the variable in a polynomial. Or power of a variable of leading term.

e.g.: - In the polynomial $8x^6 - 4x^5 + 7x^3 - x^2 + 3$, the term with the highest power is x^6. Hence the degree of the polynomial is 6.

Classification of polynomial

1. According to number of terms
   a. **Monomial**: - Polynomial having only one term
   b. **Binomial**: - Polynomial having exactly two term
   c. **Trinomial**: - Polynomial having exactly three term

   In general, 'Polynomial having n terms.
2. According to degree of polynomial
   a. **Zero Polynomial**: - $P(x) = 0$
   b. **Constant Polynomial**: - $P(x) = a$, where $a \neq 0$
   c. **Linear Polynomial**: - $P(x) = ax + b$, where $a \neq 0$

d. **Quadratic Polynomial**: - $P(x)=ax^2+bx+c$, where $a\neq 0$

e. **Cubic Polynomial**: - $P(x)=ax^3+bx^2+cx+d$, where $a\neq 0$

f. **Bi-Quadratic Polynomial**: - $P(x)=ax^4+bx^3+cx^2+dx+e$, where $a\neq 0$

Division of a Polynomial by a Polynomial

Let $p(x)$ and $f(x)$ be two polynomials and $f(x)\neq 0$. Then, we can find Polynomials $q(x)$ and $r(x)$ such that $p(x)=f(x).q(x)+r(x)$

Where, $q(x)$ is quotient and $r(x)$ is remainder & degree of $r(x)<$ degree of $q(x)$.

Factor Theorem

Let $p(x)$ be a polynomial of degree $n>0$. If $p(a)=0$ for a real number a, then $(x-a)$ is a factor of $p(x)$

Remainder Theorem

If $p(x)$ be a polynomial of degree $n>0$ and a is any real number. If $p(x)$ is divided by $(x-a)$, then the remainder is $p(a)$.

Some Useful results

1. $(a+b)^2=a^2+b^2+2ab$
2. $(a-b)^2=a^2+b^2-2ab$
3. $(a+b)(a-b)=a^2-b^2$
4. $(a+b)^3=a^3+b^3+3ab(a+b)$
5. $(a-b)^3=a^3-b^3-3ab(a-b)$
6. $a^3+b^3=(a+b)(a^2-ab+b^2)$
7. $a^3-b^3=(a-b)(a^2+ab+b^2)$
8. $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$
9. $a^2+b^2+c^2-ab-bc-ca=\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$
10. $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$
11. If $a+b+c=0$, then $a^3+b^3+c^3=3abc$
12. If $a^3+b^3+c^3=3abc$, then $a+b+c=0$ or $a=b=c$
13. a^n-b^n is divisible by $(a-b)$ for all value of n.
14. a^n-b^n is divisible by $(a+b)$ only when n is even.
15. a^n+b^n is never divisible by $(a-b)$.
16. a^n+b^n is divisible by $(a+b)$ only when n is odd.
17. $(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$
18. $a^4+a^2+1=(a^2+1)^2-a^2=(a^2+a+1)(a^2-a+1)$

Example 40. Determine if $(x-1)$ is a factor of $x^8-x^7+x^6-x^5+x^4-x+1$.

Example 41. Find the remainder when $(x-1)$ is divisible by $p(x)=x^5+5x^4-3x+7$.

Example 42. Determine whether $(x-1)$ is a factor of x^3-3x^2+4x+2 or not.

Example 43. Divide $f(x)=5x^3-70x^2+153x-342$ by $g(x)=x^2-10x+16$. Find the quotient and the remainder. [Ans $5x-20$, $-127x-22$]

Example 44. Find the GCD of $4+9x-9x^2$ and $9x^2-24x+16$.

Example 45. Find the remainder when $x+x^9+x^{25}+x^{49}+x^{81}$ is divide by x^3-x.

Example 46. Factorize:

(i) $(x^2+x+1)(x^2+x+2)-12$ [Ans $(x-1)(x+2)(x^2+x+5)$]

(ii) $(x+y+z)^3-x^3-y^3-z^3$

Example 47. Show that $x-2y$ is a factor of $3x^3-2x^2y-13xy^2+10y^3$.

RATIO AND PROPORTION

Ratio: - Ratio is a comparison of two quantities of same kind by division.

- Let a & b be two quantities of same kind, then $a:b$ is ratio of two quantities, denoted by $\frac{a}{b}$.
- $\frac{a}{b}$ may be an integer or fraction.
- Ratio is always is in its simplest form i.e. HCF of a and b is 1.

Properties

1. If $\frac{a}{b}$ is given ratio, then $\frac{a}{b}=\frac{ma}{mb}$.
2. If $\frac{a}{b}$ is given ratio, then $\frac{a}{b}=\frac{a\div m}{b\div m}$.
3. To compare two or more ratio, reduce them to common denominator.
4. $\frac{a}{b}:\frac{c}{d}=\frac{ad}{bc}$.

Proportion: - When two ratios are equal then the four quantities are said to be proportional.

- If $\frac{a}{b}=\frac{c}{d}$, then it is written as $a:b::c:d$ and read as "a is to be b as c is to d.
- a & d are said extremes and b & c are said to be means.
- Product of extremes is equals to product of means i.e. $ad=bc$

Properties

1. If $\frac{a}{b}=\frac{c}{d}$ is given proportion, then $\frac{a}{c}=\frac{b}{d}$. (Invertendo).
2. If $\frac{a}{b}=\frac{c}{d}$ is given proportion, then $\frac{b}{a}=\frac{d}{c}$. (Alternendo).
3. If $\frac{a}{b}=\frac{c}{d}$ is given proportion, then $\frac{a+b}{b}=\frac{c+d}{d}$. (Componendo).
4. If $\frac{a}{b}=\frac{c}{d}$ is given proportion, then $\frac{a-b}{b}=\frac{c-d}{d}$. (Dividendo).
5. If $\frac{a}{b}=\frac{c}{d}$ is given proportion, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$. (Componendo - Dividendo).
6. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$, then each ratio $=\frac{a+c+e+\ldots}{b+d+f+\ldots}$.
7. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$, then each ratio $=\frac{xa+yc+ze+\ldots}{xb+yd+zf+\ldots}$.
8. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$, then each ratio $=\left(\frac{a^n+c^n+e^n+\ldots}{b^n+d^n+f^n+\ldots}\right)^{\frac{1}{n}}$.

Example 48. If $\frac{x+y}{2}=\frac{y+z}{3}=\frac{z+x}{4}$, then find $x:y:z$.

Example 49. If $a(y+z)=b(z+x)=c(x+y)$, then show that $\frac{a-b}{x^2-y^2}=\frac{b-c}{y^2-z^2}=\frac{c-a}{z^2-x^2}$.

Example 50. If $x=\dfrac{\sqrt{2a+3b}+\sqrt{2a-3b}}{\sqrt{2a+3b}-\sqrt{2a-3b}}$, show that $3bx^2-4ax+3b=0$.

Example 51. If $\dfrac{2y+2z-x}{a}=\dfrac{2z+2x-y}{b}=\dfrac{2x+2y-z}{c}$, then show that

$$\frac{9x}{2b+2c-a}=\frac{9y}{2c+2a-b}=\frac{9z}{2a+2b-c}.$$

Example 52. Solve: $\dfrac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}-\sqrt{2-x}}=2.$

LINEAR FUNCTION

Example 53. Solve the following by substitution method, elimination method and by cross multiplication method.

(i) $5x-24y=16,$
$4x-y=31$ [Ans $x=8, y=1$]

(ii) $2x+3y-8=0,$
$3x-4y+5=0$ [Ans $x=1, y=2$]

Example 54. Solve:

$2x-3y+4z=0,$
$7x+2y-6z=0,$
$4x+3y+z=37$ [Ans $x=2, y=8, z=5$]