STRAIGHT LINE

Introduction:

Straight line is a locus of all the points in such way that if we join them then they define the unique direction.

- Any linear equation in two variables, which are in the form $ax+by+c=0; a\neq 0, b\neq 0$, represents the straight line in two dimensions.
- $ax+by+c=0$ is a relation between x and y, which satisfied by co-ordinate of every point lying on it.
- If $a=0$, then equation of straight line is parallel to x-axis, at a distance $-\frac{c}{b}$ from x-axis.
- If $b=0$, then equation of straight line is parallel to y-axis, at a distance $-\frac{c}{a}$ from y-axis.
- If $c=0$, then equation of straight line is passing through the origin.
- $x=0$ represents the equation of y-axis
- $y=0$ represents the equation of x-axis.

Example 1. Find the equations of the lines parallel to axes and passing through $(-2,3)$.

Example 2. Prove that every straight line has an equation of the form $ax+by+c=0$, where a, b and c are constants. [NCERT Exemplar]

Pair of Straight Line:

Consider $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are two simultaneous equations of straight line.

Pair of linear equations $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$	**Algebraic condition**	**Graphical representation**	**Algebraic interpretation**
Consistent (independent)	$\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$	Intersecting	Only one solution
Consistent (dependent)	$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$	Coincident lines	Infinite solution
Inconsistent	$\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$	Pair of parallel lines	No solution

Methods of Solving pair of Straight Lines

1. Elimination Method
2. Substitution Method
3. Cross Multiplication Method

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Example 3. Classify the following pairs of lines as coincident, parallel or intersecting:

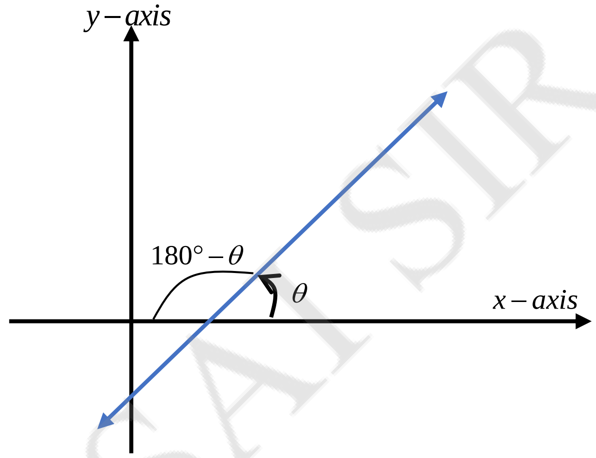
(i) $x+2y-3=0$ & $-3x-6y+9=0$

(ii) $x+2y+1=0$ & $2x+4y+3=0$

(iii) $3x-2y+5=0$ & $2x+y-5=0$

Inclination Angle

The angle θ (say) made by the line l with positive direction of x-axis, measured in anti-clockwise direction is called inclination angle.



- $0° \le \theta \le 180°$

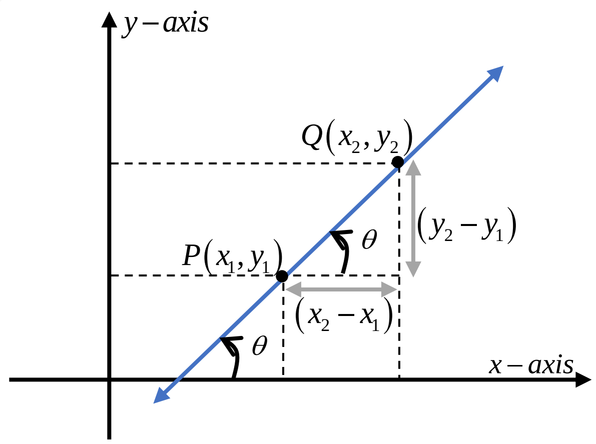
Slope of Line

If θ is the inclination angle of a line l, then $\tan\theta$ is called the slope or gradient of a line l.

- Slope of a line denoted by m.
- $\because \tan 90°$ is not defined, $\therefore$ slope of a line whose inclination angle is $90°$ is not defined.
- $\therefore m = \tan\theta$, $\theta \ne 90°$

Slope of a Line Joining two Points:

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ are given two different points in a plane. Draw a line passing through these two points. Draw perpendiculars from points P and Q on x-axis and y-axis, as shown in the figure.



$\therefore$ The slope of a line passing through P and Q is given by $\tan\theta = \dfrac{y_2-y_1}{x_2-x_1}$.

- Slope of every line is Unique.
- If $\theta = 0°$ or $\theta = 180°$, then $m = \tan 0° = \tan 180° = 0$, line is parallel to x-axis or perpendicular to y-axis.
- If $\theta = 90°$, $m = \tan 90° =$ not defined, then line is parallel to y-axis or perpendicular to x-axis.
- Slope of a line equally incline with the axes is $m = \pm 1$

Example 4. Find the slope of the lines: [NCERT]

(i) Passing through the points $(3,-2)$ and $(-1,4)$,

(ii) Passing through the points $(3,-2)$ and $(7,-2)$,

(iii) Passing through the points $(3,-2)$ and $(3,4)$,

(iv) Making inclination of $60°$ with the positive direction of x-axis

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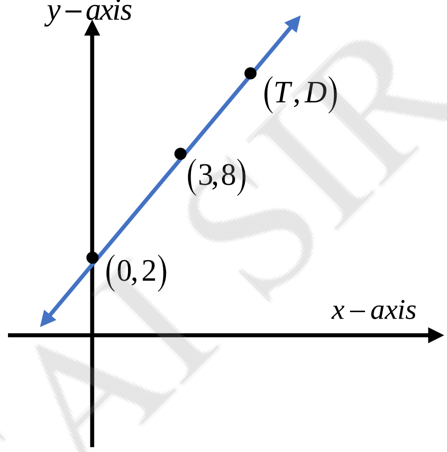
Collinearity of three Points:

If $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are three points in a plane such that A, B, C are collinear, then A, B, C lies in a line.

$\therefore$ Slope of $AB =$ Slope of $BC =$ Slope of CA.

Example 5. In fig., time and distance graph of a linear motion if given. Two positions of time and distance is recorded as, when $T = 0, D = 2$ and when $T = 3, D = 8$. Using the concept of slope, find law of motion, i.e. how distance depends upon time. [NCERT]



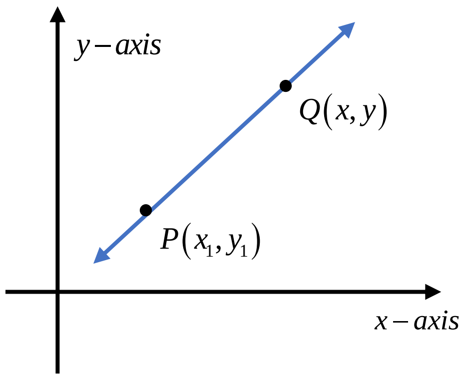
Standard Forms of the Equation of a Line

As we know that, every line in a plane contains infinitely many points on it. We are assuming a variable point P on the line, which satisfies every point on the line. So, we need to find the locus of the point using given conditions. We are going to discuss some of the forms of the lines with the help of algebraic equations using given conditions or statements.

- To find the equations of line we need at least two conditions in the following manners.

Point Slope Form:

Consider m be the slope of a line and line is passes through a point $P(x_1, y_1)$. Let $Q(x, y)$ be an any point on the line.



Using the definition of slope of the line PQ, $m = \dfrac{y - y_1}{x - x_1}$

$\therefore (y - y_1) = m(x - x_1)$

Therefore, $(y - y_1) = m(x - x_1)$ is our required equation of line having slope m and passes through a point $P(x_1, y_1)$.

Example 6. Find the equation of line through $(-2, 3)$ with slope -4. [NCERT]

Example 7. Find the equation of a line which passes through the point $(2, 3)$ and make an angle of $30°$ with the positive direction of x-axis. [NCERT Exemplar]

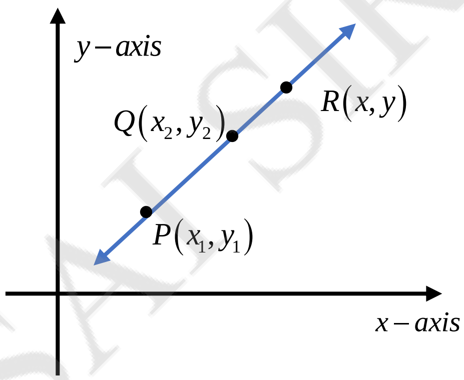
Example 8. Determine equation of line through the point $(-4,-3)$ and parallel to x-axis.

Example 9. Find the equation of line passing through the point $(0,2)$ making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also find the equation of line parallel to it and crossing y-axis at distance 2 unit below the origin.

Example 10. Mid-points of the sides of Δ are $(2,1),(-5,7)$ and $(-5,-5)$. Find the equation of the sides of the triangle.

Two Point Form:

Let the line l passing through two given points $P(x_1,y_1)$ and $Q(x_2,y_2)$, as shown in the figure. Let $R(x,y)$ be an any point on the line.



As we know that the three points on the line are collinear.

$\therefore$ Slope of $PR =$ Slope of PQ

$$\therefore \frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

$$\therefore (y-y_1)=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

Therefore, $(y-y_1)=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$ is our required equation of line passing through two given points $P(x_1,y_1)$ and $Q(x_2,y_2)$.

- Equation of line passing through two given points $P(x_1,y_1)$ and $Q(x_2,y_2)$ is expressed in the determinant form as $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$

Example 11. Find the equation of line through the points $(1,-1)$ and $(3,5)$. [NCERT]

Example 12. The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given that $K=273$ when $F=32$ and that $k=373$ when $F=212$. Express K in terms of F and find the value of F, when $K=0$. [NCERT]

Example 13. Find equation of line joining the points $(at_1^2,2at_1)$ and $(at_2^2,2at_2)$.

Example 14. If the line joining two points $A(2,0)$ and $B(3,1)$ is rotated about A in anti-clock wise direction through an angle of $15°$. Find the equation of the line in new position. [NCERT Exemplar]

Example 15. Mid-points of the sides of Δ are $(2,1),(-5,7)$ and $(-5,-5)$. Find the equation of the sides of the triangle. (repeated)

Example 16. In what ratio is line joining the points $(2,3)\&(4,1)$ divides the segment joining the points $(1,2)\&(4,3)$.

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Example 17. Prove that points $(5,1),(1,-1)$ and $(11,4)$ are collinear. Also find the equation of straight line on which these point lies.

Example 18. Find equation of internal bisector of $\angle BAC$ of ΔABC with vertices A,B,C are $(5,2),(2,3)$ and $(6,5)$ respectively.

Example 19. Rectangle has two opposite vertices at the point $(1,2)$ and $(5,5)$. If the other vertices lie on the line $x=3$, find equation of sides of rectangle.

Slope Intercept Form:

Intercept:

x-intercept is the distance of the point of intersection of line with x-axis from origin. If x-intercept is on the right side of the origin then considered as positive and on the left side of the origin is considered as negative.

y-intercept is the distance of the point of intersection of line with y-axis from origin. If y-intercept is above the origin then considered as positive and below the origin is considered as negative.

CASE I: -

Consider m be the slope of a line and c be the y-intercept of the line. Therefore, coordinates of the point where line intersects y-axis is $(0,c)$.

Now, using slope point form, $(y-y_1)=m(x-x_1)$

$\therefore (y-c)=m(x-0)$

$\therefore y=mx+c$

Therefore, $y=mx+c$ is our required equation of line having slope m and its y-intercept c.

CASE II: -

Consider m be the slope of a line and d be the x-intercept of the line. Therefore, coordinates of the point where line intersects x-axis is $(d,0)$.

Now, using slope point form, $(y-y_1)=m(x-x_1)$

$\therefore (y-0)=m(x-d)$

$\therefore y=m(x-d)$

Therefore, $y=m(x-d)$ is our required equation of line having slope m and its x-intercept d.

Example 20. Find the equation of straight line which cuts off an intercept of 5 unit on negative direction of y-axis and makes an angle of 120° degree with positive direction of x-axis.

Example 21. Write the equation of the lines for which $\theta=\tan^{-1}\dfrac{1}{2}$, where θ is the inclination of the line and x-intercept are 4.

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Example 22. Find equation of straight line cutting of an intercept -1 from y-axis and being equally inclined to the axes.

Intercept Form:

Consider a line l. If a and b are the intercepts made by a line on the x-axis and y-axis respectively.

Therefore, coordinate of a point where line intersects x-axis and y-axis are $(a,0)$ and $(0,b)$, respectively.

Now, using two-point form, $(y-y_1)=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$.

$$\therefore (y-0)=\frac{b-0}{0-a}(x-a)$$

$$\therefore y=\frac{b}{-a}(x-a)$$

$$\therefore \frac{y}{b}=\frac{x}{-a}+1$$

$$\therefore \frac{x}{a}+\frac{y}{b}=1$$

Therefore, $\frac{x}{a}+\frac{y}{b}=1$ is our required equation of line having a and b are the x-intercept and y-intercept.

Example 23. Find the equation of the line, which makes intercepts -3 and 2 on the x and y axes respectively. [NCERT]

Example 24. The equation of lines which passes through the point $P(h,k)$ and sum of its intercepts on the axes is 14 is –

(a) $4x-3y+24=0$, $x-y+7=0$

(b) $4x-3y=24$, $x-y=7$

(c) $4x+3y+24=0$, $x+y+7=0$

(d) $4x+3y=24$, $x+y=7$

Example 25. The intercept cut off by a line from y-axis is twice than that from x-axis, and the line passes through the point $(1,2)$. The equation of the line is [NCERT Exemplar]

(a) $2x+y=4$

(b) $2x+y+4=0$

(c) $2x-y=4$

(d) $2x-y+4=0$

Example 26. A line passes through $P(1,2)$ such that its intercept between the axes is bisected at P. the equation of the line is [NCERT Exemplar]

(a) $x+2y=5$

(b) $x-y+1=0$

(c) $x+y-3=0$

(d) $2x+y-4=0$

Example 27. The sum of the reciprocal of the intercepts made by a straight line is constant. Show that it always passes through a fixed point.

Example 28. Find equation of straight line which passes through origin and trisect the intercept of line $3x+4y=12$ between the axes.

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Example 29. Show that the area of the triangle formed by the lines $y=m_1x+c_1$, $y=m_2x+c_2$ and $x=0$ is $\dfrac{(c_1-c_2)^2}{2|m_1-m_2|}$. [NCERT]

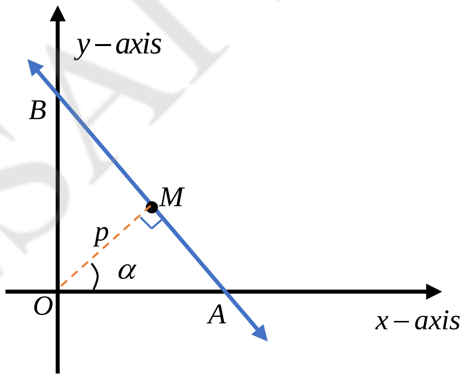
Example 30. Show that the locus of the mid-point of the distance between the axes of the variable line $x\cos\alpha+y\sin\alpha=p$ is $\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{4}{p^2}$ where p is a constant.

[NCERT Exemplar]

Example 31. A line passes through $(3,-2)$. Find the locus of the middle point of the portion of the line intercept between the axes.

Normal Form:

Let p be the perpendicular distance of line l from the origin and this perpendicular makes an angle α ($0\le\alpha<2\pi$) with positive x-axis, as shown in the figure.



In ΔOAM,

$OA=\dfrac{p}{\cos\alpha}$ & $OB=\dfrac{p}{\sin\alpha}$ (these are intercepts on the axes)

Now, using intercept form, $\dfrac{x}{a}+\dfrac{y}{b}=1$

$$\therefore \frac{x}{\left(\frac{p}{\cos\alpha}\right)}+\frac{y}{\left(\frac{p}{\sin\alpha}\right)}=1$$

$$\therefore x\cos\alpha+y\sin\alpha=p$$

Therefore, $x\cos\alpha+y\sin\alpha=p$ is our required equation line whose perpendicular distance from origin is p and perpendicular makes an angle α with positive x-axis.

Example 32. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with positive direction of x-axis is $15°$.

[NCERT]

Example 33. The length of the perpendicular from the origin to a line is 7 and makes at angle of $150°$ with the positive direction of y-axis.

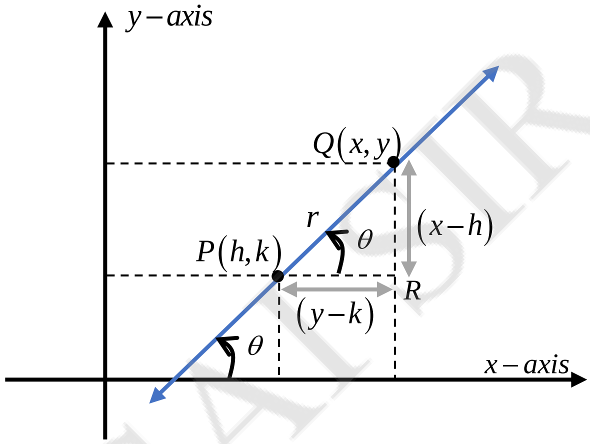
Example 34. Find equation of straight line upon which the length of the perpendicular is 5 and the slope of perpendicular is $\dfrac{3}{4}$.

Example 35. Find the equation of the straight line on which the perpendicular from origin makes an angle $30°$ with positive x-axis and which forms a triangle of area $\dfrac{50}{\sqrt{3}}$ sq. units with the coordinate axes.

Example 36. Two points A and B move on the positive direction of x-axis and y-axis respectively, such that $OA+OB=k$. Show that the locus of the foot of the perpendicular from the origin O on the line AB is $(x+y)(x^2+y^2)=kxy$.

Parametric Form:

Parametric form of a line passing through a point $P(h,k)$ and making an angle θ with the positive direction of x-axis. Now take a point $Q(x,y)$ at a distance r from the point P, as shown in the figure.



In ΔPQR

$$\cos\theta=\frac{x-h}{r} \ \ \& \ \ \sin\theta=\frac{y-k}{r}$$

On simplifying we get, $x=h+r\cos\theta$ and $y=k+r\sin\theta$

Also represent as $\dfrac{x-h}{\cos\theta}=\dfrac{y-k}{\sin\theta}=r$ … which is the parametric equation of a line.

Therefore, $x=h+r\sin\theta$ and $y=k+r\cos\theta$ are the parametric form of any point on the line whose inclination angle is θ and at a distance r from a fixed point $P(h,k)$.

- There are two cases arise. One, when r is positive (as we discuss above) and another is when r is negative, then consider Q below P.

Example 37. Find the distance of the line $4x-y=0$ from the point $P(4,1)$ measured along the line making an angle of $135°$ with positive x-axis. [NCERT]

Example 38. Equation of a line which passes through point $A(2,3)$ and makes an angle of $45°$ with x-axis. If this line meets the line $x+y+1=0$ at point P then distance AP is –

(a) $2\sqrt{3}$ (c) $5\sqrt{2}$

(b) $3\sqrt{2}$ (d) $2\sqrt{5}$

Example 39. Find the distance of a point $A(2,3)$ measured parallel to the line $x-y=5$ from the line $2x+y+6=0$.

Example 40. If the slope of a line passing through the point $A(3,2)$ is $\frac{3}{4}$, then find points on the line which are 5 units away from the point A. [NCERT Exemplar]

Example 41. The line joining two points $A(2,0)$ and $B(3,1)$ is rotated about A in anticlockwise direction through an angle of $\frac{1}{x^2}+\frac{1}{y^2}=\frac{4}{p^2}$. Find the equation of line in the new position. If B goes to C in the new position. What will be the coordinate of C?

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Example 42. Find the direction in which a straight line must be drawn through the point $(-1,2)$ so that its point of intersection with the line $x+y=4$ may be at a distance of 3 units from the point.

Example 43. Find the equation of line passing through the point $(2,3)$ and making a intercept of length 3 units between the lines $y+2x=2$ and $y+2x=5$.

Example 44. If the line $y-\sqrt{3}x+3=0$ cuts the parabola $y^2=x+2$ at A and B, then find the value of $PA.PB$. (where $P\equiv\left(\sqrt{3},0\right)$)

Example 45. A straight line through $P(-2,-3)$ cuts the pair of straight lines $x^2+3y^2+4xy-8x-6y-9=0$ in Q and R. find the equation of line if $PQ.PR=20$. (M2 – we can solve with the help of pair of straight line)

Example 46. A line through $A(-5,-4)$ meets the lines $x+3y+2=0, 2x+y+4=0$ and $x-y-5=0$ at the point B,C,D respectively if $\left(\frac{15}{AB}\right)^2+\left(\frac{10}{AC}\right)^2=\left(\frac{6}{AD}\right)^2$.

Example 47. The sides AB, AC of ΔABC are $2x+3y=29$ and $x+2y=16$ respectively. If the mid-point of BC is $(5,6)$. Then find the equation of BC.

Conversion of General Form to Other Form (& Visa-versa)

As we know, first degree equation in two variables $Ax+By+C=0, A\neq 0, B\neq 0$, represents the straight line. This equation is known as the general linear equation or general equation of a line.

Slope Intercept form:

$Ax+By+C=0$ can be written as $y=-\frac{A}{B}x+-\frac{C}{B}$, compared with $y=mx+c$.

We get, slope of the line $m=-\frac{A}{B}$ & y-intercept $c=-\frac{C}{B}$.

Or

$Ax+By+C=0$ can be written as $y=\left(-\frac{A}{B}\right)x+\left(-\frac{C}{B}\right)=-\frac{A}{B}\left(x+\frac{C}{A}\right)$, compare with $y=m(x-d)$.

We get, slope of the line $m=-\frac{A}{B}$ & x-intercept $d=-\frac{C}{A}$.

Intercept Form:

$Ax+By+C=0$ can be written as

$Ax+By=-C$

$\therefore \frac{x}{\left(\frac{-C}{A}\right)}+\frac{y}{\left(\frac{-C}{B}\right)}=1$, compare with $\frac{x}{a}+\frac{y}{b}=1$

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We get, x-intercept $a=-\frac{C}{A}$ and y-intercept $b=-\frac{C}{B}$.

Normal Form:

As we know that $Ax+By+C=0$ and $x\cos\alpha+y\sin\alpha=p$ represents the same line. So, apply the condition of two lines are collinear. (i.e. $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$)

$$\therefore \frac{A}{\cos\alpha}=\frac{B}{\sin\alpha}=\frac{C}{-p}$$

$$\therefore \cos\alpha=\frac{-pA}{C} \ \& \ \sin\alpha=\frac{-pB}{C}$$

But we know that $\sin^2\alpha+\cos^2\alpha=1$

$$\left(\frac{-pB}{C}\right)^2+\left(\frac{-pA}{C}\right)^2=1$$

On simplifying we get $p=\pm\frac{C}{\sqrt{A^2+B^2}}$ (p must be positive so, take + or – sign depends on the value of C)

Therefore, $\sin\alpha=\mp\frac{B}{\sqrt{A^2+B^2}}$ and $\cos\alpha=\mp\frac{A}{\sqrt{A^2+B^2}}$

- To convert general form to normal form, shift constant term on one of the sides of equal sign, make it positive and terms consisting x and y variable on other side.
  $$Ax+By=-C$$
- Now divide the whole equation by $\sqrt{A^2+B^2}$
  $$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y=-\frac{C}{\sqrt{A^2+B^2}}$$
- Now compare with $x\cos\alpha+y\sin\alpha=p$, we get $p=\frac{C}{\sqrt{A^2+B^2}}$ (make sure p is positive) , so accordingly $\sin\alpha=\frac{B}{\sqrt{A^2+B^2}}$ and $\cos\alpha=\frac{A}{\sqrt{A^2+B^2}}$.

Example 48. Equation of a line is $3x-4y+10=0$. Find its [NCERT]
(i) Slope,
(ii) x and y intercepts.

Example 49. The inclination of the line $x\cos\alpha+y\sin\alpha=p$ with positive direction of x-axis is [NCERT Exemplar]

(a) $45°$
(b) $135°$
(c) $-45°$
(d) $-135°$

Example 50. Reduce the equation $\sqrt{3}x+y-8=0$ into normal form. Find the values of p and α. [NCERT]

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Example 51. Reduce the line $\sqrt{3}x+y-8=0$.

(i) In slope-intercept form and hence find slope and y-intercept.

(ii) In intercept form and hence find intercept on the axes.

(iii) In normal form and hence find perpendicular distance from the origin and angle made by the perpendicular with the positive x-axis.

Example 52. Find the equation of line with slope 2 and length of perpendicular from origin is $\sqrt{5}$.

Example 53. Show that two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, where $b_1,b_2\neq 0$ are: [NCERT]

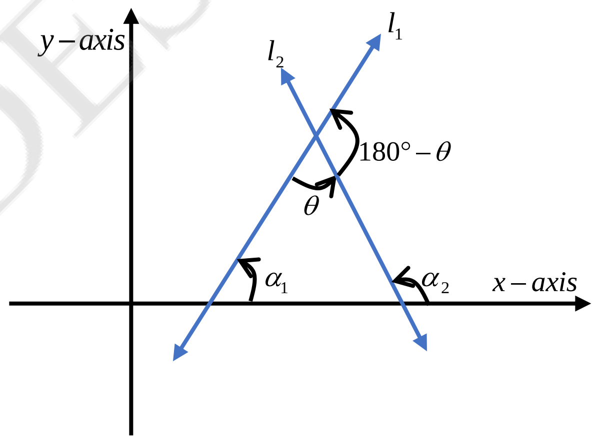
(i) Parallel if $\frac{a_1}{b_1}=\frac{a_2}{b_2}$

(ii) Perpendicular if $a_1a_2+b_1b_2=0$

Example 54. The line $2x-y=5$ turns about a point on it, whose ordinate and abscissa are equal, through an angle of $45°$ in anticlockwise direction. Find the equation of line in new position.

Angle between two Lines

Consider two non-vertical lines l_1 and l_2 with slope m_1 and m_2 respectively. Let α_1 and α_2 are the inclination angles of l_1 and l_2, respectively.



$\therefore m_1=\tan\alpha_1$ and $m_2=\tan\alpha_2$

Let θ be the acute angle between l_1 and l_2, then

$\theta=\alpha_2-\alpha_1$

$$\therefore \tan\theta=\tan(\alpha_2-\alpha_1)=\frac{\tan\alpha_2-\tan\alpha_1}{1+\tan\alpha_2\tan\alpha_1}$$

$$\therefore \tan\theta=\frac{m_2-m_1}{1+m_2m_1}$$

$\because \theta$ is acute

$$\therefore \theta=\tan^{-1}\left|\frac{m_2-m_1}{1+m_2m_1}\right|$$

- There are two angles formed between two lines but usually the acute angle is taken as the angle between the lines.
- Therefore, another angle between lines l_1 and l_2 is $180°-\theta$, which is obtuse.
- If $\theta=0°$ then lines l_1 and l_2 are **parallel** $\Rightarrow$ $m_2-m_1=0\Rightarrow m_1=m_2$
- If $\theta=90°$ then lines l_1 and l_2 are **perpendicular** $\Rightarrow$ $1+m_2m_1$
- If l_1, l_2 and l_3 are the three lines represents the sides of the triangle ΔABC, with positive slopes m_1, m_2 and m_3 respectively. Then, $\tan A=\frac{m_2-m_1}{1+m_2m_1}$, $\tan B=\frac{m_3-m_2}{1+m_3m_2}$ and $\tan C=\frac{m_1-m_3}{1+m_1m_3}$.

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Example 55. Find the angle between the lines
(i) $y-\sqrt{3}x-5=0$ and $\sqrt{3}y-x+6=0$ [NCERT]
(ii) $3x+y-7=0$ and $x+2y-9=0$

Example 56. If the angle between two lines is $\frac{\pi}{4}$ and slope of one of the lines is $\frac{1}{2}$, find the slope of the other line. [NCERT]

Example 57. Line through the points $(-2,6)$ and $(4,8)$ is perpendicular to the line through the points $(8,12)$ and $(x,24)$. Find the value of x. [NCERT]

Example 58. If $x+4y-5=0$ and $4x+ky+7=0$ are two perpendicular lines then k is –
(a) 3
(b) 4
(c) -1
(d) -4

Example 59. A line L passing through the points $(1,1)$ & $(0,2)$ and another line M which is perpendicular to L passes through the point $\left(0,-\frac{1}{2}\right)$. The area of the triangle formed by these lines with y-axis is –
(a) $\frac{25}{8}$
(b) $\frac{25}{16}$
(c) $\frac{25}{4}$
(d) $\frac{25}{32}$

Example 60. If the straight line $3x+4y+5-k(x+y+3)=0$ is parallel to y-axis, then the value of k is –
(a) 1
(b) 2
(c) 3
(d) 4

Example 61. If $A(1,2), B(-1,3)$ and $C(3,-5)$ be the vertices of Δ, then find angle B.

Example 62. Without using Pythagoras theorem, show that $A(4,4), B(3,5)$ and $C(-1,-1)$ are vertices of right angle.

Example 63. Vertices of Δ, $A(x_1, x_1\tan\theta_1), B(x_2, x_2\tan\theta_2)$ and $C(x_3, x_3\tan\theta_3)$. If circumcentre of Δ coincide with origin and $H(\bar{x},\bar{y})$ is orthocentre. Show that
$$\frac{\bar{x}}{\bar{y}}=\frac{\sin\theta_1+\sin\theta_2+\sin\theta_3}{\cos\theta_1+\cos\theta_2+\cos\theta_3}.$$

Equation of line Parallel and Perpendicular to a Given Line:

- Equation of line parallel to the line $ax+by+c=0$ is given by $ax+by+\lambda=0$, where λ is the unknown constant find using the given condition.
- Equation of line perpendicular to the line $ax+by+c=0$ is given by $bx-ay+\mu=0$, where μ is the unknown constant find using the given condition.

Example 64. Find the equation of a line perpendicular to the line $x-2y+3=0$ and passing through the point $(1,-2)$. [NCERT]

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Example 65. Find the equation of the line which has positive y-intercept 4 units and parallel to the line $3x-3y-7=0$. Also find the point where it cuts the x-axis.

Example 66. Find the value of K for which the line $(k-3)x-(4-k^2)y+k^2-7k+6=0$ is

(i) Parallel to x-axis
(ii) Parallel to y-axis
(iii)Passing through origin.

Example 67. Find equation of straight line parallel to $2x+3y+11=0$ and which is such that the sum of its intercept on the axes is 15.

Example 68. Find the equation of straight line passing through $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line $x\sec\theta+y\operatorname{cosec}\theta=a$.

Example 69. A person standing at a function of two straight path represented by equation $2x-3y-4=0$ and $3x-4y-5=0$, want to reach the path whose equation is $6x-7y+8=0$ in the least time. Find equation of path that he should follow. Also, find the distance person has to travels.

Examples based on Centroid, Incentres, Circumcentre and Orthocentres.

Example 70. Equations of two sides of a Δ are $3x-2y+6=0$ and $4x+5y-20=0$. If $(1,1)$ is orthocentre of Δ then fins the equation of third side.

Example 71. Find centroid, incentre, circumcentre and orthocentre of Δ whose sides have equation: $3x-4y=0, 12y+5x=0$ and $y-15=0$.

Example 72. Find circumcentre of Δ whose sides are $3x-y+3=0, 3x+4y+3=0$ and $x+3y+11=0$.

Straight Line Making Given Angle with a Line

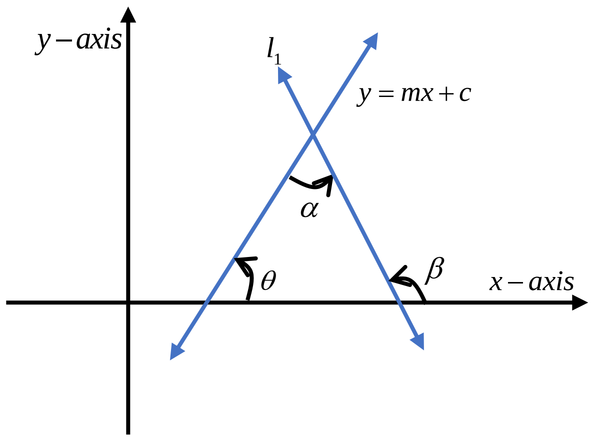
Consider an equation of line l_1, passing through a point (x_1, y_1) and making an angle α, with the line $y=mx+c$, as shown in the figure. Let $m=\tan\theta$ and Slope of the line l_1 is $\tan\beta$.

CASE I: -

$\because \beta=\theta+\alpha$

$$\therefore \tan\beta=\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1+\tan\theta\tan\alpha}$$

$$\therefore \tan\beta=\frac{m+\tan\alpha}{1+m\tan\alpha} \qquad \ldots (1)$$



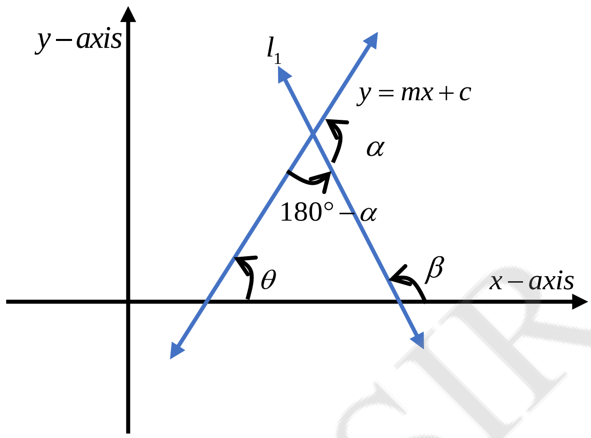
CASE II: -

$\because \beta = 180^\circ - \alpha + \theta = 180^\circ + (\theta - \alpha)$

$\therefore \tan\beta = \tan(180^\circ + (\theta - \alpha))$

$\therefore \tan\beta = \tan(\theta - \alpha) = \dfrac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$

$\therefore \tan\beta = \dfrac{m - \tan\alpha}{1 + m\tan\alpha}$ $\qquad \ldots (2)$



On combining Case I and Case II, we get, slope of the required line is $\tan\beta = \dfrac{m \pm \tan\alpha}{1 \mp m\tan\alpha}$.

Therefore the equation line passing through (x_1, y_1) and slope $\tan\beta = \dfrac{m \pm \tan\alpha}{1 \mp m\tan\alpha}$ is given by

$(y - y_1) = \dfrac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$

Example 73. Find equation of straight lines through $(7,9)$ and making an angle of 60° with the given line $x - \sqrt{2}y - 2\sqrt{3} = 0$.

Example 74. Find the equation to the sides of an isosceles right-angled triangle, the equation of whose hypotenuse is $3x + 4y = 4$ and the opposite vertex is the point $(2,2)$.

Example 75. If one diagonal of a square is along the line $8x - 15y = 0$ and one of its vertices is at $(1,2)$, then find the equation od sides of the square passing through this vertex.

[NCERT Exemplar]

Example 76. The opposite angular points of parallelogram are $(3,4)$ and $(1,-1)$. Find coordinates of other two vertices.

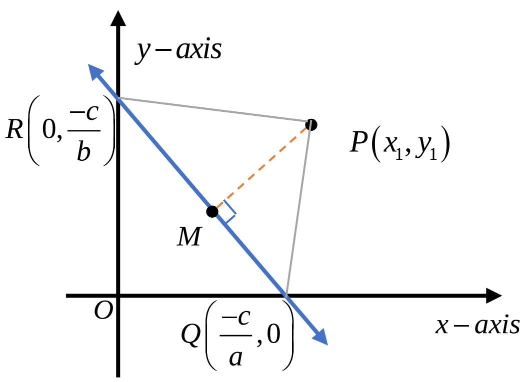
Example 77. A line $4x + y = 1$ through the point $A(2,-7)$ meets the line BC whose equation is $3x - 4y + 1 = 0$ at the point B. Find the equation to the line AC so that $AB = AC$.

Distance of a point From a Line

Consider a point $P(x_1, y_1)$ not on a line $ax + by + c = 0$. Draw a perpendicular PM from the point P to the line. Let $d = PM$ be the distance of a point $P(x_1, y_1)$ from the line $ax + by + c = 0$.

Line is intersecting x-axis at $Q\left(\dfrac{-c}{a}, 0\right)$ and y-axis at $R\left(0, \dfrac{-c}{b}\right)$.

Join PQ and PR, as shown in the figure.



In a ΔPQR,

area $[\Delta PQR] = \dfrac{1}{2} PM \times RQ$

$PM = \frac{2 \times \text{area } [\Delta PQR]}{RQ}$

& $\text{area } [\Delta PQR] = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ \frac{-c}{a} & 0 & 1 \\ 0 & \frac{-c}{b} & 1 \end{vmatrix} = \frac{1}{2}\left[x_1\left(\frac{c}{b}\right) + y_1\left(\frac{c}{a}\right) + \frac{c^2}{ab}\right] = \frac{1}{2}\frac{c}{ab}[ax_1 + by_1 + c]$

& $RQ = \sqrt{\left(0+\frac{c}{a}\right)^2 + \left(\frac{-c}{b} - 0\right)^2} = \sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2}} = \left|\frac{c}{ab}\right|\sqrt{a^2+b^2}$.

Substituting values of $\text{area } [\Delta PQR]$ and RQ in equation (1), we get

$d = PM = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$ … which is our required distance.

- Distance of a line from origin (put $(x_1, y_1) \equiv (0,0)$) is $\frac{|c|}{\sqrt{a^2+b^2}}$.

Example 78. Find the distance of the point $(3,-5)$ from the line $3x - 4y - 26 = 0$. [NCERT]

Example 79. Prove that no line can be drawn through the point $(4,-5)$ so that its distance from $(-2,3)$ will be equal to 12.

Example 80. Find point on y-axis whose perpendicular distance from $4x - 3y - 12 = 0$ is 3.

Example 81. Find all points on $x + y = 4$ that lies at a unit distance from $4x + 3y - 10 = 0$

Example 82. Equation of base of an equilateral triangle is $x + y - 2 = 0$ and the opposite vertex is $(2,-1)$. Find area of triangle.

Example 83. If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

Example 84. Find the points on the x-axis such that their perpendicular distance from the line $\frac{x}{a} + \frac{y}{b} = 1$ is a, $a, b > 0$.

Example 85. If P_1 and P_2 be the perpendicular from origin upon the straight line $x\sec\theta + y\operatorname{cosec}\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$. Prove that $4P_1^2 + P_2^2 = a^2$.

Example 86. Prove that length of perpendicular from points $P(m^2, 2m), Q(mn, m+n)$ and $R(n^2, 2n)$ to the line $x\cos^2\theta + y\sin\theta\cos\theta + \sin^2\theta = 0$ are in GP.

Distance between Two Parallel Lines:

Consider two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$. Let (x_1, y_1) lies on the line $ax + by + c_2 = 0$.

So, $ax_1 + by_1 = -c_2$ … (1)

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Now, distance between two lines is same as distance between (x_1, y_1) and $ax + by + c_1 = 0$.

So, its distance $= \frac{|ax_1 + by_1 + c_1|}{\sqrt{a^2 + b^2}}$

$\therefore d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$ … which is required formula of distance.

- To apply the above formula, make sure the coefficient of x and y in both the equations are same.
- If $y = mx + c_1$ & $y = mx + c_2$ are two parallel line, then distance between them is given by $\frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$.
- In a parallelogram, if p_1 and p_2 are the perpendicular distance between both parallel pairs and θ be the angle between two adjacent side. Then area of parallelogram $= \frac{p_1 p_2}{\sin\theta}$
- Area of parallelogram bounded by the lines $y = m_1 x + c_1$, $y = m_1 x + c_2$, $y = m_2 x + d_1$ and $y = m_2 x + d_2$ is given by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

Example 87. Find the distance between the following pair of parallel lines:
(i) $3x + 4y = 13$, $3x + 4y = 3$
(ii) $3x - 4y + 9 = 0$, $6x - 8y - 15 = 0$

Example 88. Show that the area of the parallelogram formed by the lines $2x - 3y + a = 0$, $3x - 2y - a = 0$, $2x - 3y + 3a = 0$ and $3x - 2y - 2a = 0$ is $\frac{12a^2}{5}$ sq. units.

Example 89. Find the equation of line midway between the parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

Example 90. Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form 3 sides of two squares. Find the equation of remaining sides of these squares.

Family of Lines

Consider two equation of line $L_1 : a_1 x + b_1 y + c_1 = 0$ and $L_2 : a_2 x + b_2 y + c_2 = 0$, then equation of line passing through the point of intersection of L_1 & L_2 is given by $L_1 + \lambda L_2 = 0$. The value of λ is obtained with help of the additional information given in the context.

Logic: Let (x_1, y_1) is the point of intersection of the line L_1 & L_2. Since (x_1, y_1) satisfies L_1, L_2 and the required equation.

Now, if $a_1 x_1 + b_1 y_1 + c_1 = 0$ and $a_2 x_1 + b_2 y_1 + c_2 = 0$ then $(a_1 x_1 + b_1 y_1 + c_1) + \lambda(a_2 x_1 + b_2 y_1 + c_2) = 0$. Implies (x_1, y_1) satisfies $L_1 + \lambda L_2 = 0$.

So, we can say that $L_1 + \lambda L_2 = 0$ is the family of the concurrent line with $L_1 : a_1 x + b_1 y + c_1 = 0$ and $L_2 : a_2 x + b_2 y + c_2 = 0$.

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Example 91. Find the equation of the line joining the point $(2,-9)$ and the point of intersection of lines $2x+5y-8=0$ and $3x-4y-35=0$.

Example 92. Find the equation to the straight line passing through the point of intersection of the lines $5x-6y-1=0$ & $3x+2y+5=0$ and perpendicular to the line $3x-5y+11=0$.
[NCERT Exemplar]

Example 93. Find the equation of the line which pass through the point of intersection of the lines $4x-3y=1$ & $2x-5y+3=0$ and equally inclined to the axes.

Example 94. Prove that each member of the family of straight lines $(3\sin\theta+4\cos\theta)x+(2\sin\theta-7\cos\theta)y+(\sin\theta+2\cos\theta)=0$ (θ is a parameter) passes through a fixed point.

Example 95. If t_1 and t_2 are roots of the equation $t^2+\lambda t+1=0$, where λ is the arbitrary constant then prove that the line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ always passes through a fixed point. Also find that point.

Example 96. A variable line is drawn through O, to cut two fixed straight lines L_1 and L_2 in A_1 and A_2 respectively. A point A is taken on the variable line such that $\frac{m+n}{OA}=\frac{m}{OA_1}+\frac{n}{OA_2}$.
Show that the locus of A is straight line passing through the point of intersection of L_1 and L_2, where O is being origin.

Concurrency of Lines

If two or more line intersect at a point, then they are called concurrent lines.

Method 1: -

Let $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$ are three equation with three variable. Then, to check whether the equations are concurrent or not, use following steps

STEP I: - Find the solution of any two equations.

STEP II: - If the value of x and y satisfies in the third equation, then they are concurrent or if they did not satisfies then they are not concurrent.

- On simplifying the above method, if $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$ are concurrent, if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$

Example 97. If the lines $2x+y-3=0$, $5x+ky-3=0$ and $3x-y-2=0$ are concurrent, find the value of k. [NCERT]

Example 98. Show that the lines $x-y-6=0, 4x-3y-20=0$ and $6x+5y+8=0$ are concurrent. Also find their common point of intersection.

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Example 99. If the lines $ac+by+p=0$, $x\cos\alpha+y\sin\alpha-p=0$ $(p\neq 0)$ and $x\sin\alpha-y\cos\alpha=0$ are concurrent and the first two lines include an angle $\frac{\pi}{4}$, then a^2+b^2 is equal to –

(a) 1
(c) 2
(b) p^2
(d) 4

Example 100. If the line $ax+y+1=0, x+by+1=0$ and $x+y+c=0$ are concurrent $(a\neq b\neq c\neq 1)$. Prove that $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$.

Position of Two Points with respect to a Given Line

- If $P(x_1,y_1)$ and $Q(x_2,y_2)$ are on the same side of the line $ax+by+c=0$, then $(ax_1+by_1+c)(ax_2+by_2+c)>0$,
- $P(x_1,y_1)$ and $Q(x_2,y_2)$ are on the opposite side of the line $ax+by+c=0$, then $(ax_1+by_1+c)(ax_2+by_2+c)<0$
- And, at least one of $P(x_1,y_1)$ and $Q(x_2,y_2)$ lie(s) on the line $ax+by+c=0$, then $(ax_1+by_1+c)(ax_2+by_2+c)=0$

Example 101. Examine the positions of the points $(3,4)$ and $(2,-6)$ w.r.t $3x-4y=8$.

Example 102. If $(2,9),(-2,1)$ and $(1,-3)$ are the vertices of a triangle, then prove that the origin lies inside the triangle.

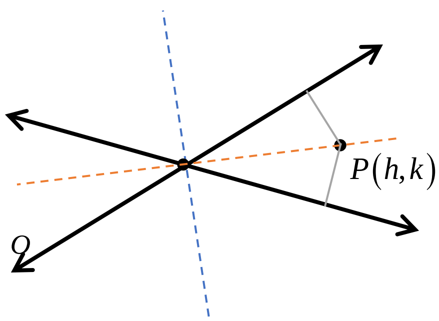
Example 103. If point $(4,7)$ and $(\cos\theta,\sin\theta)$ where $0<\theta<\pi$ lies on the same side of line $x+y-1=0$, then Prove that θ lies in the first quadrant.

Example 104. Let $P(\sin\theta,\cos\theta)$, $0\le\theta\le 2\pi$ be a point and let OAB be a triangle with vertices $(0,0),\left(\sqrt{\frac{3}{2}},0\right)$ and $\left(0,\sqrt{\frac{3}{2}}\right)$. Find θ if P lies inside ΔOAB.

Example 105. Determine all values of α for which the point (α,α^2) lies inside the Δ formed by the lines $2x+3y-1=0, 2x+2y-3=0$ and $5x-6y-1=0$.

Equation of Bisectors of Angles between Two Lines

Consider two intersecting lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$. Any point on the line of angle bisectors of these line is always equidistance from these lines.



Let $P(h,k)$ is any point equidistance from the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$. So, locus of P is our required equation of bisectors of the angle.

$$\therefore \frac{|a_1h+b_1k+c_1|}{\sqrt{a_1^2+b_1^2}}=\frac{|a_2h+b_2k+c_2|}{\sqrt{a_2^2+b_2^2}}$$

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Therefore, $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ … which is required equations of bisectors of the given lines.

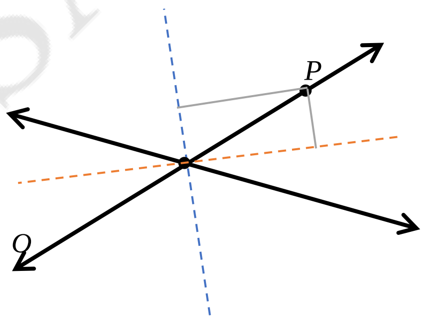
- Equation of straight lines passing through $P(x_1,y_1)$ & equally inclined with the lines $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

Some methods to find whether the angle bisector is acute angle bisector or obtuse angle bisector:

Method 1: -

- If angle between one of the angle bisectors and one of the given lines is less than 1, then chosen angle bisector is acute angle bisector, and another is obtuse angle bisector.
- If angle between one of the angle bisectors and one of the given lines is greater than 1, then chosen angle bisector is obtuse angle bisector, and another is acute angle bisector.

Method 2: -



Step I: - Chose any one of the given equations of line.

Step II: - Chose any point on it.

Step III: - Find the length of perpendicular from both the equation of bisectors.

Step IV: - Distance from the acute angle bisector is smaller than distance from the obtuse angle bisector.

Method 3: -

Step I: - Make the constant term c_1 and c_2 in the equation of lines $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$.

Step II: - Apply $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

Step III: - Equation of bisector with positive sign **contains the origin**

Step IV: - (i) If $a_1a_2+b_1b_2>0$, then equation of bisector with positive sign is obtuse angle bisector and equation of bisector with negative sign is acute angle bisector.

(ii) If $a_1a_2+b_1b_2<0$, then equation of bisector with negative sign is obtuse angle bisector and equation of bisector with positive sign is acute angle bisector.

- So, from Step III & Step IV, if $a_1a_2+b_1b_2>0$, then obtuse angle bisector is lies with the origin. And, if $a_1a_2+b_1b_2<0$, then acute angle bisector is lies with the origin.

Example 106. Show that the path of a moving point such that its distances from two lines $3x-2y=5$ and $3x+2y=5$ are equal is a straight line. [NCERT]

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Example 107. Find the equations of bisectors of the angle between the lines $4x+3y=7$ and $24x+7y-31=0$. Also find which of them is

(i) The bisector of the angle containing origin

(ii) The bisector of the acute angle.

Example 108. For the straight lines $3x+2y=5$ and $3x+2y=5$, find the equation of the

(i) Bisector of the obtuse angle between them.

(ii) Bisector of the acute angle between them.

(iii) Bisector of angle which contains origin.

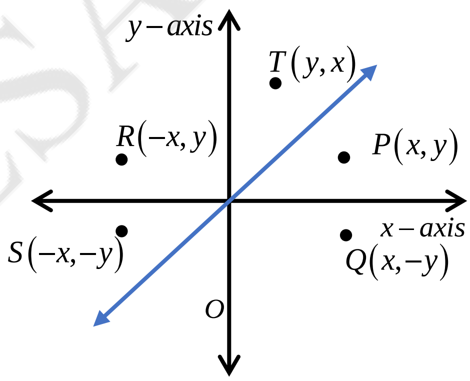
(iv) Bisector of angle which contains $(1,2)$

Example 109. A ray of light is sent along the line $x-2y-3=0$. Upon reaching the line mirror $3x-2y-5=0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

Reflection of a Point

Consider $P(x,y)$ be any point, then its image w.r.t

- X-axis is $Q(x,-y)$
- Y-axis is $R(-x,y)$
- Origin is $S(-x,-y)$
- The line $y=x$ is $T(y,x)$



Reflection of a point w.r.t a given line:

Method 1: -

Consider a line $ax+by+c=0$ and a point $P(x_1,y_1)$ not lie on the line. Let $I(h,k)$ and $F(\alpha,\beta)$ are the image and foot of point $P(x_1,y_1)$ w.r.t $ax+by+c=0$. F is the mid-point of PI. Let θ be the inclination of the line PI.

$\therefore \alpha=\dfrac{x_1+h}{2}$ & $\beta=\dfrac{y_1+k}{2}$

And α,β satisfies the equation of line $ax+by+c=0$

$$\therefore a\left(\frac{x_1+h}{2}\right)+b\left(\frac{y_1+k}{2}\right)+c=0$$

On simplifying, $ah+bk=-(ax_1+by_1+2c)$ $\quad\ldots\quad(1)$

Also, slope of PI $\times$ Slope of $(ax+by+c=0)$ $=-1$

$$\frac{y_1-k}{x_1-h}\times\frac{-a}{b}=-1$$

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$\therefore \dfrac{x_1-h}{a}=\dfrac{y_1-k}{b}$

Using proportion

$$\frac{x_1-h}{a}=\frac{y_1-k}{b}=\frac{a\left(x_1-h\right)+b\left(y_1-k\right)}{a^2+b^2}=\frac{ax_1-ah+by_1-k}{a^2+b^2}$$

$$\therefore \frac{x_1-h}{a}=\frac{y_1-k}{b}=2\left(\frac{ax_1+by_1+c}{a^2+b^2}\right)$$

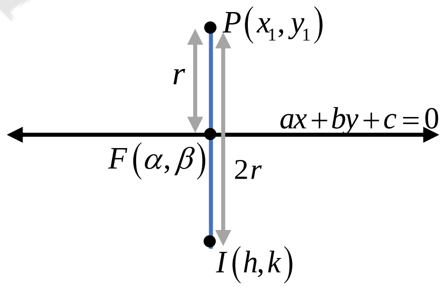
Or $\therefore \dfrac{h-x_1}{a}=\dfrac{k-y_1}{b}=-2\left(\dfrac{ax_1+by_1+c}{a^2+b^2}\right)$ … which is required equation to find the image.

As we know that $F\left(\alpha,\beta\right)$ is mid-point of PI

$\therefore \dfrac{\alpha-x_1}{a}=\dfrac{\alpha-y_1}{b}=-\left(\dfrac{ax_1+by_1+c}{a^2+b^2}\right)$ … which is required equation to find the foot of perpendicular.

Method 2: -

Consider a line $ax+by+c=0$ and a point $P\left(x_1,y_1\right)$ not lie on the line. Let $I\left(h,k\right)$ and $F\left(\alpha,\beta\right)$are the image and foot of point $P\left(x_1,y_1\right)$ w.r.t $ax+by+c=0$. Let r be the distance between P and F, therefore $PI=2r$. And, θ be the inclination of the line PI.



Here, slope of line PI $=\dfrac{-1}{\text{slope of the line } ax+by+c=0}=\dfrac{b}{a}=\tan\theta$

$\therefore \sin\theta=\dfrac{b}{\sqrt{a^2+b^2}}$ & $\cos\theta=\dfrac{a}{\sqrt{a^2+b^2}}$

Also, $r=\dfrac{\left|ax_1+by_1+c\right|}{\sqrt{a^2+b^2}}$

Also, using the parametric form of a line which is passing through $P\left(x_1,y_1\right)$ and inclination angle is θ,

$$\frac{x_1-\alpha}{\cos\theta}=\frac{y_1-\alpha}{\sin\theta}=r$$

We have $\dfrac{\alpha-x_1}{\left(\dfrac{a}{\sqrt{a^2+b^2}}\right)}=\dfrac{\beta-y_1}{\left(\dfrac{b}{\sqrt{a^2+b^2}}\right)}=-r=-\dfrac{\left|ax_1+by_1+c\right|}{\sqrt{a^2+b^2}}$

Multiply whole equation by $\sqrt{a^2+b^2}$, we get

$$\frac{\alpha-x_1}{a}=\frac{\beta-y_1}{b}=-r=-\frac{\left|ax_1+by_1+c\right|}{a^2+b^2}$$

SAURABH DESAI SIR

$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2} \quad (\because a(ax_1 + by_1 + c) \;\&\; b(ax_1 + by_1 + c) \text{ sign remain unchanged})$$

So, $\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$ is the required equation to find the foot of perpendicular.

Similarly, we have $\frac{h - x_1}{a} = \frac{k - y_1}{b} = -2\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$, to find the co-ordinate of the image.

Example 110. The point $(4,1)$ undergoes the following transformations, then the match the correct alternatives:

	Column I		Column II
(A)	Reflection about x-axis is	(p)	$(4,-1)$
(B)	Reflection about y-axis is	(q)	$(-4,-1)$
(C)	Reflection about origin is	(r)	$\left(\frac{-12}{25}, \frac{-59}{25}\right)$
(D)	Reflection about the line $y = x$ is	(s)	$(-4,1)$
(E)	Reflection about the line $4x + 3y - 5 = 0$ is	(t)	$(1,4)$

Example 111. The image of point $A(1,2)$ by the line mirror $y = x$ is the point B and the image of B by the line mirror $y = 0$ is the point (α, β). Find $\alpha + \beta$.

Example 112. The coordinates of the foot of the perpendicular from the point $(2,3)$ on the line $x + y - 11 = 0$ are [NCERT Exemplar]

(a) $(-6,5)$
(b) $(5,6)$
(c) $(-5,6)$
(d) $(6,5)$

Example 113. The reflection of the point $(4,-13)$ about the line $5x + y + 6 = 0$ is [NCERT Exemplar]

(a) $(-1,-14)$
(b) $(3,4)$
(c) $(0,0)$
(d) $(1,2)$

Example 114. Assuming that straight lines work as the plane mirror for a point, find the image of the point $(1,2)$ in the line $x - 3y + 4 = 0$. [NCERT]

Example 115. A ray of light coming from the point $(1,2)$ is reflected at a point A on the x-axis and then passes through the point $(5,3)$. Find the coordinates of the point A. [NCERT Exemplar]

Example 116. A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching $3x - 2y - 5 = 0$ the ray is reflection from it. Find the equation of line containing the reflected ray.

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Pair of Straight Lines

The general equation of second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents pair of straight lines, if $\Delta = abc+2fgh-af^2-bg^2-ch^2=0$ i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$

Proof: Let $L_1: y=m_1x+c_1$ and $L_2: y=m_2x+c_2$ represents two different straight line. Then pair of straight line is represented by $L_1.L_2=0 \Rightarrow (y-m_1x-c_1)(y-m_2x-c_2)=0$. On expanding, we get $y^2+m_1m_2x^2-(m_1+m_2)xy+(m_1c_2+c_1m_2)x-(c_1+c_2)y+c_1c_2=0$. So, in general we can write it as $ax^2+2hxy+by^2+2gx+2fy+c=0$.

On comparing the coefficient, we get $m_1m_2=\frac{a}{b}$ and $m_1+m_2=-\frac{2h}{b}$.

Homogeneous equation of second degree:

If every term of the equation has the same degree then equation is homogenous. Let $L_1: y=m_1x$ and $L_2: y=m_2x$ represents two pair of straight lines. The second-degree equation of pair of straight line is given by $ax^2+2hxy+by^2=0$.

- These straight lines are passing through the origin.

Angle Between the Pair of straight Lines

Let θ be the angle between the lines L_1 and L_2, then $\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$ and using the above relations

$$\tan\theta=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$$

- Process of calculating $\tan\theta$ is same for both heterogenous as well as homogenous.
- If the lines are real and distinct, then angle between them is real $\Rightarrow h^2>ab$.
- If the lines are real and coincide, then angle between them is zero $\Rightarrow h^2=ab$ (If the equations are homogenous).
- If the lines are imaginary, then $h^2<ab$.
- If lines are parallel, then $h^2=ab$, and
- If lines are perpendicular, then $a+b=0$, i.e. coefficient of x^2+ coefficient of $y^2=0$.
- If the lines are equally inclined to the axes, then $h=0$.

SAURABH DESAI SIR

The Combine Equation of Angle bisector of Homogeneous Equation

The combine equation of angle bisectors between the lines represented by homogenous equations of second degree is given by $\frac{x^2-y^2}{a-b}=\frac{xy}{h}, a \neq b, h \neq 0$.

- If $a=b$, then bisectors are $x^2-y^2=0 \Rightarrow x-y=0, x+y=0$.
- If $h=0$, then bisectors are $xy=0 \Rightarrow x=0, y=0$.
- These bisectors are always at the right angle.

NOTE

- To factorize $ax^2+2hxy+by^2+2gx+2fy+c=0$, consider the quadratic equation in x or y (taking other as constant), then apply Shridharacharya's formula. Discriminant is always perfect square.
- Pair of straight lines perpendicular to the lines $ax^2+2hxy+by^2=0$ and through origin are given by $bx^2-2hxy+ay^2=0$.
- The product of the perpendiculars drawn from the point (x_1, y_1) on the lines $ax^2+2hxy+by^2=0$ is $\left|\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{(a-b)^2+4h^2}}\right|$.
- The product of the perpendiculars drawn from the origin to the lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ is $\left|\frac{c}{\sqrt{(a-b)^2+4h^2}}\right|$.

Example 117. If $\lambda x^2-10xy+12y^2+5x-16y-3=0$ represents a pair of straight lines, then λ is equal to

(a) 4 (c) 2
(b) 3 (d) 1

Example 118. Prove that the equation $x^2-5xy+4y^2=0$ represents two lines passing through the origin. also find their equation.

Example 119. If the equation $3x^2-kxy-10y^2+7x+19y=6$ represents a pair of lines, find the value of k.

Example 120. If the equation $6x^2-11xy-10y^2-19y+c=0$ represents a pair of lines, find their equation. Also find the angle between the two lines.

Example 121. Find the point of intersection and the angle between the lines given by the equation: $2x^2-3xy-2y^2+10x-5y+12=0$

Example 122. Show that the two straight lines $x^2\left(\tan^2\theta+\cos^2\theta\right)-2xy\tan\theta+y^2\sin^2\theta=0$ represented by the equation are such that the difference of their slopes is 2.

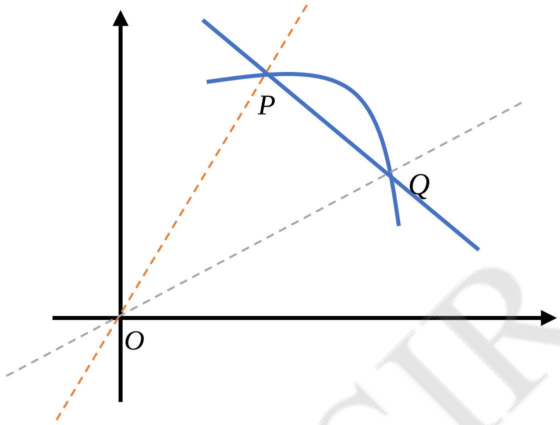
Example 123. If pairs of straight lines $x^2-2pxy-y^2=0$ and $x^2-2qxy-y^2=0$ be such that each pair bisectors the angle between the other pair, prove that $pq=-1$.

Example 124. A straight line through $P(-2,-3)$ cuts the pair of straight lines $x^2+3y^2+4xy-8x-6y-9=0$ in Q and R. find the equation of line if $PQ.PR=20$.

SAURABH DESAI SIR

Equation of Lines Joining the Points of Intersection of a Line and a Curve to the Origin

Let $ax^2+2hxy+by^2+2gx+2fy+c=0$ be the equation of curve and $lx+my+n=0$ be a straight line as shown in the figure.



Now the combine equation of lines passing through the point of intersections of line and curve and the origin is given by (i.e. combine equation of OP and OQ)

$$ax^2+2hxy+by^2+2g\left(\frac{lx+my}{-n}\right)x+2f\left(\frac{lx+my}{-n}\right)y+c\left(\frac{lx+my}{-n}\right)^2=0$$

- Any second degree curve through the four points of intersection of $f(x,y)=0$ & $xy=0$ is given by $f(x,y)+\lambda xy=0$ where $f(x,y)=0$ is also second degree curve.

Example 125. The chord $\sqrt{6}x=\sqrt{8}px+\sqrt{2}$ of the curve $py^2+1=4x$ subtends a right angle at origin then find the value of p.

Example 126. Find the angle subtended at the origin by the intercept made on the curve $x^2-xy-y^2+3x-6y+18=0$ by the line $2x-y=3$.

Example 127. Find the equation of the lines joining the origin to the points of intersection of the curve $2x^2+3xy-4x+1=0$ and the line $3x+y=1$.

Some Mixed Concept Examples

Example 128. A line is such that its segment between the lines $5x-y+4=0$ and $3x+4y-4=0$ is bisected at the point $(1,5)$. Obtain its equation. [NCERT]

Example 129. ABCD is a variable rectangle having its sides parallel to fixed directions (say m). The vertices B and D lie on $x=a$, $x=-a$ respectively and A lies on the line $y=0$. Find the locus of C.

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