TRIGONOMETRIC EQUATION

Definition: An equation involving trigonometric functions as a variable is called Trigonometric Equations.

Solution of Trigonometric Equation

1. Principal Solution: The solution of the trigonometric equation lying in the interval $[0, 2\pi)$.
2. General Solution: All the trigonometric functions are Periodic and Many-one, hence there are infinite values of θ for which trigonometric functions have same value. The expression involving integer n which gives all solutions of a trigonometric equation is called general solution.
3. Particular Solution: The solution of the trigonometric equation lying in the given interval.

- For all trigonometric function, we two principal values in $[0, 2\pi)$.
- General solution for both principal values will be same although these may apparently look different.

General Solutions of Some Standard Trigonometric Equation

(1) If $\sin\theta = 0 \Rightarrow \theta = n\pi$, $n \in I$.

(2) If $\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}$, $n \in I$.

(3) If $\tan\theta = 0 \Rightarrow \theta = n\pi$, $n \in I$.

(4) **Theorem 1**: - For $x, y \in R$, $\sin x = \sin y \Rightarrow x = n\pi + (-1)^n y$, $n \in I$.

(5) **Theorem 2**: - For $x, y \in R$, $\cos x = \cos y \Rightarrow x = 2n\pi \pm y$, $n \in I$.

(6) **Theorem 3**: - For $x, y \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$, $\tan x = \tan y \Rightarrow x = n\pi + y$, $n \in I$.

(7) If $\sin\theta = 1 \Rightarrow \theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$, $n \in I$.

(8) If $\cos\theta = 1 \Rightarrow \theta = 2n\pi$, $n \in I$.

(9) If $\sin^2\theta = \sin^2\alpha$, $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$, $n \in I$.

Example 1. Find the Principal solutions of the equation

(i) $\sin\theta = \frac{\sqrt{3}}{2}$

(ii) $\cot\theta = -1$

(iii) $\tan\theta = -\frac{1}{\sqrt{3}}$

Example 2. Find the solution of

(i) $\sin\theta = -\frac{\sqrt{3}}{2}$

(ii) $\cos\left(\frac{3\theta}{2}\right) = 0$

(iii) $\tan\left(\frac{3\theta}{4}\right) = 0$

(iv) $\frac{1}{3}\sin^3(-\theta) = 0$

(v) $\cos\theta = \frac{1}{2}$

(vi) $\text{cosec}\left(\frac{\theta}{2}\right) = -1$

(vii) $\sec 2\theta = -\frac{2}{\sqrt{3}}$

(viii) $\cos 4\theta = -\frac{\sqrt{3}}{2}$

(ix) $\tan^2 \theta = 1$

(x) $\cos^2 2\theta = 1$

Example 3. Solve:

(i) $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$

(ii) $\tan\theta = \frac{3}{4}$

(iii) $\tan 2x + \tan x + \tan 2x \tan x = 1$

(iv) $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$

Example 4. Solve the following simultaneous trigonometric equations

(i) $\sin x = -\frac{1}{2}, \cos x = -\frac{\sqrt{3}}{2}$

(ii) $\tan\theta = -\sqrt{3}, \sec\theta = -2$

(iii) $\tan x = -1, \cos = \frac{1}{\sqrt{2}}$

(iv) $7\cos^2\theta + 3\sin^2\theta = 4$

Important Points to be Considered

(a) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \le 1$.

(b) Avoid squaring the equations, if possible, because it may lead to extraneous solution. Reject extra solutions if they do not satisfy the given equation.

(c) Do not cancel the common variable factor from the two sides of the equations which are in the product because it may lose some solutions.

(d) The answer should not contain such values of θ, which make any of the terms undefined or infinite.

(i) Check that denominator is not zero at any stage while solving equations.

(ii) If $\tan\theta$ or $\sec\theta$ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$

(iii)If $\cot\theta$ or $\operatorname{cosec}\theta$ is involved in the equations, θ should not be odd multiple of π or 0

Different Types of Solving Trigonometric Equations

Type I: - By Factorisation

Example 5. Solve:

(i) $2\sin^2\theta - 3\sin\theta - 2 = 0$

(ii) $3\sin^2 x - 7\sin x + 2 = 0$, $x \in [0, 5\pi]$

(iii) $\sin 2x = \cos x$

(iv) $\sin 3x = \sin x$

(v) $\frac{\sin^3\frac{x}{2} - \cos^3\frac{x}{2}}{2 + \sin x} = \frac{\cos x}{3}$

(vi) $\cos^3 x + \cos^2 x - 4\cos^2\frac{x}{2} = 0$

(vii) $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

Example 6. If $\sin\alpha, 1$ & $\cos 2\alpha$ are in GP then find general solution for α.

Example 7. If $\frac{1}{6}\sin\theta, \cos\theta$ and $\tan\theta$ are in GP then general solution for θ is –

(a) $2n\pi \pm \frac{\pi}{3}$

(b) $2n\pi \pm \frac{\pi}{6}$

(c) $n\pi \pm \frac{\pi}{3}$

(d) None of these

Type II: - By Reducing it to a Quadratic Equation

Example 8. Solve:

(i) $\sin^2\theta - \cos\theta = \frac{1}{4}$, $0 \le \theta \le 2\pi$

(ii) $\cos 2x + 3\sin x = 2$, $x \in [0, \pi]$

(iii) $\tan x + \sec x = 2\cos x$, $0 \le x \le 2\pi$

(iv) $2\sin^2\theta + \sin^2 2\theta = 2$, $\theta \in (-\pi, \pi)$

Example 9. Solve:

(i) $6 - 10\cos x = 3\sin^2 x$

(ii) $2\cos^2 x + 3\sin x = 0$

(iii) $4\sec^2\theta = 5 + \tan^2\theta$

(iv) $\sec^2 2\theta = 1 - \tan 2\theta$

(v) $5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4$

(vi) $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x\cos x$

Type III: - By Introducing an Auxiliary Argument

Trigonometric of the form $a\sin\theta + b\cos\theta = c$, where $a, b, c \in R$ can be solved by dividing both sides of the equation by $\sqrt{a^2 + b^2}$

Consider $a\sin\theta + b\cos\theta = c$ …. (1)

$$\therefore \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

Equation (1) has a solution only when if $|c| \le \sqrt{a^2 + b^2}$

Let $\frac{a}{\sqrt{a^2+b^2}} = \cos\alpha$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\alpha$ & $\alpha = \tan^{-1}\frac{b}{a}$ by introducing this auxiliary argument α, equation (1) reduces to

$\sin(\theta + \alpha) = \frac{c}{\sqrt{a^2+b^2}}$, this equation can solve easily.

Example 10. Find the number of distinct solutions of $\sec x + \tan x = \sqrt{3}$, where $0 \le x \le 3\pi$

Example 11. Solve:

(i) $\sin x + \sqrt{2} = \cos x$

(ii) $\sqrt{3}\sin x + \cos x = \sqrt{3}$

(iii) $\operatorname{cosec} x = 1 + \cot x$

(iv) $\sin\theta + \cos\theta = 1$

(v) $\left(\sqrt{3} - 1\right)\sin x + \left(\sqrt{3} + 1\right)\cos x = 2$

Example 12. Prove that the equation $k\cos x - 3\sin x = k + 1$ posses' aa solution iff $k \in (-\infty, 4]$

Example 13. If the equation $7\cos x + 5\sin x = 2k + 1$ has a solution, then find the number of integral values of k for which.

Type IV: - By Transforming Sum of Trigonometric Functions into Product

Example 14. Solve:

(i) $\cos 2\theta + \cos 4\theta = 0$

(ii) $\cos 3x + \sin 2x - \sin 4x = 0$

(iii) $\sin x + \sin 3x + \sin 5x = 0$

(iv) $\sin 2x - \sin 4x + \sin 6x = 0$

(v) $\sqrt{3}\sin 2x + \cos 5x - \cos 9x = 0$

(vi) $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$

Type V: - By Transforming Product into Sum

Example 15. Solve:

(i) $\sin 5x \cos 3x = \sin 6x \cos 2x$

(ii) $4\sin\theta \sin 2\theta \sin 4\theta = \sin 3\theta$

(iii) $\cos\theta \cos 2\theta \cos 3\theta = \dfrac{1}{4}$, where $0 \le x \le \pi$

Type VI: - By Change of Variable

- Trigonometric equations of the form $P(\sin x \pm \cos x, \sin x \cos x) = 0$, where $P(x, y)$ is a polynomial, can be solved by using the substitution $\sin x \pm \cos x = t$
- Trigonometric of the form $a\sin\theta + b\cos\theta = c$ can also solved by changing $\sin\theta$ and $\cos\theta$ into their corresponding tangent of half the angle.

Example 16. Solve:

(i) $\sin x + \cos x - 2\sqrt{2}\sin x \cos x = 0$

(ii) $\sin 2x + 3\sin x = 1 + 3\cos x$

(iii) $\sin^4 x + \cos^4 x = \sin x \cos x$

Example 17. Solve:

(i) $4\sin\theta + 3\cos\theta = 5$

(ii) $\sin x + \tan\dfrac{x}{2} = 0$

Type VII: - Use of the Boundness of the Function Involved

Example 18. Solve: $\sin x\left(\cos\dfrac{x}{4} - 2\sin x\right) + \left(1 + \sin\dfrac{x}{4} - 2\cos x\right)\cos x = 0$

Example 19. Solve: $(\sin x + \cos x)^{1+\sin 2x} = 2$, where $0 \le x \le \pi$.

Example 20. Solve for x and y: $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + \dfrac{1}{2}} \le 1$

Example 21. The number of solution(s) of $2\cos^2\left(\dfrac{x}{2}\right)\sin^2 x = x^2 + \dfrac{1}{x^2}, 0 \le x \le \dfrac{\pi}{2}$, is/are –

(a) 0

(b) 1

(c) Infinite

(d) None of these

Example 22. If $x^2 - 4x + 5 - \sin y = 0$, $y \in [0, 2\pi)$, then –

(a) $y \in [0, 2\pi)$

(b) $y \in [0, 2\pi)$

(c) $y \in [0, 2\pi)$

(d) $y \in [0, 2\pi)$

Example 23. If $\sin x+\cos x=\sqrt{y+\frac{1}{y}}$, $y>0$, $x\in[0,\pi]$, then find the least positive value of x satisfying the giving condition.

Trigonometric Inequalities

Example 24. Find the solution set of inequality

(i) $\sin\theta>\frac{1}{2}$ (ii) $\cos x\geq-\frac{1}{2}$

Example 25. Find the value of x in the interval $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$ for which

$\sqrt{2}\sin 2x+1\leq 2\sin x+\sqrt{2}\cos x$.

Example 26. Find values of α lying between 0 and π for which the inequality: $\tan\alpha>\tan^3\alpha$ is valid.

Example 27. Find the value of x in the interval $[0,2\pi]$ for which $4\sin^2 x-8\sin x+3\leq 0$.

Mixed Concept Examples

Example 28. Solve:

(i) $\tan^2\theta+\sec^2\theta+3=2\left(\sqrt{2}\sec\theta+\tan\theta\right)$

(ii) $5^{\left(1+\log_5\cos x\right)}=\frac{5}{2}$

(iii) $(\cos x)^{\sin^2 x-3\sin x+2}=1, x\neq\frac{n\pi}{2}, n\in I$

Example 29. If the set of all values of x in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ satisfying $\left|4\sin x+\sqrt{2}\right|<\sqrt{6}$ is $\left(\frac{a\pi}{24},\frac{b\pi}{24}\right)$ then find the value of $\left|\frac{a-b}{3}\right|$

Example 30. Find the values of x in the interval $[0,2\pi]$ which satisfy the inequality:

$3\left|2\sin x-1\right|\geq 3+4\cos^2 x$

Example 31. Find the value of θ, for which $\cos 3\theta+\sin 3\theta+(2\sin 2\theta-3)(\sin\theta-\cos\theta)$ is always positive.

Example 32. Find general solution of $1+|\cos x|+|\cos x|^2+|\cos x|^3+\ldots\infty=4$.

Example 33. The number of values of x in the interval $[0,5\pi]$ satisfying the equation

$3\sin^2 x-7\sin x+2=0$ is –

(a) 0
(c) 6
(b) 5
(d) 10