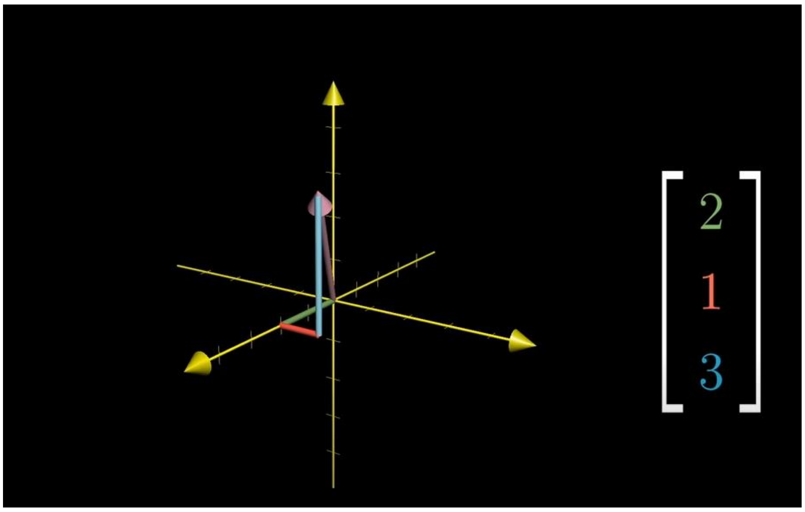
Linear Algebra

Vectors refers to an ordered list in CS

It can also refer to how to move so as to reach head from tail in a coordinate system.

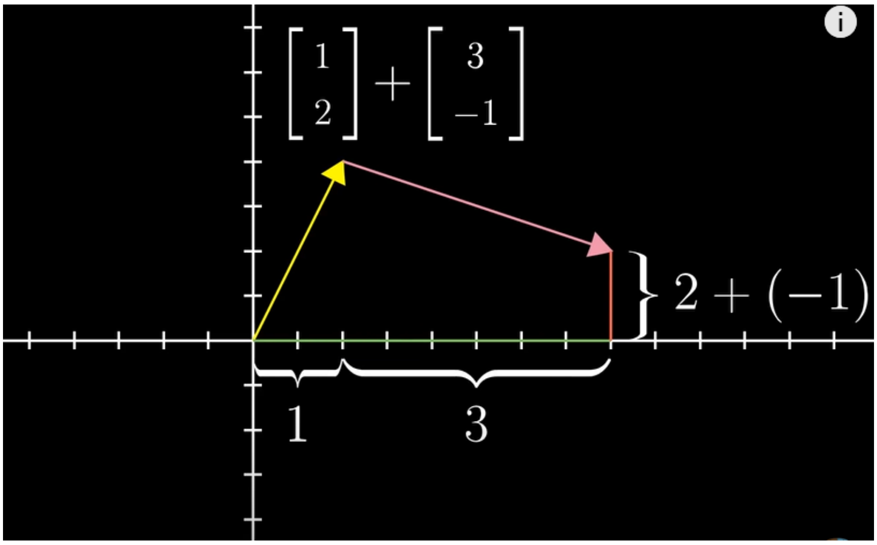
Vector is defined as follows : first the distance moved along X axis, then along y axis and then along z axis.



We can add vectors by adding their individual ordered elements.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

It can be visualised as follows :

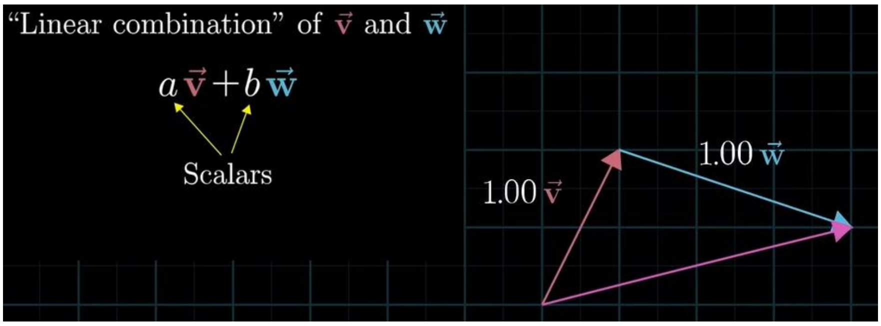


Icap and jcap are called basis vectors

Scalar multiplication of vectors means multiplying the all elements of vectors by that number. Increasing or decreasing a vector by a value is also called as scaling

$$2 \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Combining two vectors we can achieve many more vectors.



Span of a vector can be defined as set of all linear combinations of the given vectors

If two vectors in 2D are in same line then their span is limited to only their line. Similarly if three vectors in 3D are in same plane their span is only limited to their plane

Basis of vector space is a set of linearly independent vectors that span the full space

Transformations involve shifting of the coordiantes of a poin to a newer coordinate system

$$\hat{\imath} \rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \hat{\jmath} \rightarrow \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow x \begin{bmatrix} 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$$

First line shows how i and j have changed after transformation. We multiply them with x and y coordinates of original system to get a newer coordinate in vector form.

Can be also expressed as :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Where a c and b d are transformed i and j respectively .

If we are also shearing the new coordinate system after rotation, we multiply the original coordinates by what is called as a composition matrix, obtained by multiplication of shear and rotation matrices

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{Shear}} \left(\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation}} \begin{bmatrix} x \\ y \end{bmatrix} \right) = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Composition}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Transformation in a 3D matrix can be shown as follows :

$$\underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}}_{\text{Transformation}} \overbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}^{\text{Input vector}} = \underbrace{x \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + y \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + z \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}}_{\text{Output vector}}$$

Determinant of a transformation can be described as the factor by which area of a grid square increases or decreases after a transformation.

A negative determinant signals the inversion of orientation of space after transformation.

In 2D, det = 0 implies squishing the plane into a line.

In 3D, det = 0 implies squishing the volume into a plane.

Determinant can be computed as :

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc$$

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

Inverse of a matrix is that particular matrix when multiplied with the matrix gives an identity matrix as the product. If a matrix is said to squish the coordinate system in a way, inverse matrix unsquishes the coordinate system.

A matrix with determinant zero has no inverse matrix because a line can be unsquished in infinite ways.

Rank of matrix is defined as number of dimensions in the column space.

Column space is defined as span of columns of output matrix.

A matrix is full rank if its rank is the highest possible for a matrix of the same size

If a particular transformation squishes the coordinate system into a line, then there is a set of points that are squished into the origin. These set of points are called null space or kernel.

If there are nonsquare matrices, the number of columns indicate the number of base vectors that are being transformed, and number of rows tell their landing spots. Thus a 3*2 matrix implies that a 3D space is squished into a 2D space.

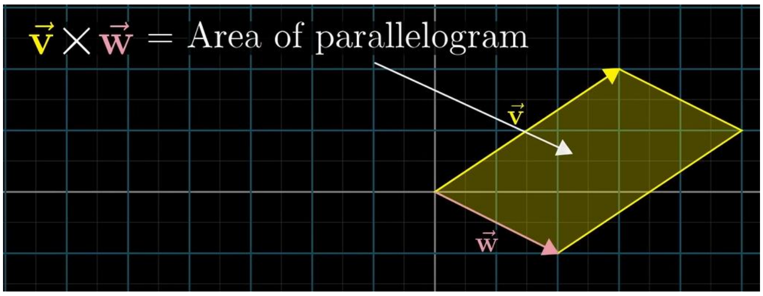
Dot products can be mathematically be expressed as follows :

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \underset{\substack{\uparrow \\ \text{Dot product}}}{\cdot} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 1\cdot 3 + 2\cdot 4$$

Their geometrical representation can give us the product of one vector and the projection of other vector on the first vector.

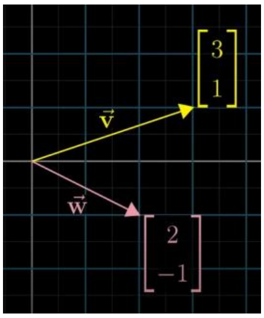
Dot product also guides us to understand whether the two vectors are perpendicular or in same or opposing directions

A cross product can be described as the area enclosed between two vectors.



Area is positive if the vector anticlockwise of the other, is placed before in the multiplication .

If we have two vectors as :



Their cross product is :

$$\vec{v}\times\vec{w}=\det\left(\begin{bmatrix}3 & 2\\ 1 & -1\end{bmatrix}\right)$$

If two vectors are given we can calculate their cross product as :

$$\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}\times\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix}=\det\left(\begin{bmatrix}\hat{\imath} & v_1 & w_1\\ \hat{\jmath} & v_2 & w_2\\ \hat{k} & v_3 & w_3\end{bmatrix}\right)$$

$$\hat{\imath}(v_2w_3-v_3w_2)+\hat{\jmath}(v_3w_1-v_1w_3)+\hat{k}(v_1w_2-v_2w_1)$$

When there is a linear transformation to the number line we can match it to a vector called dual vector , so that doing the linear transformation is same as taking dot product with the vector

$$\overbrace{\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}}^{\vec{\mathbf{p}}} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = \det\left(\begin{array}{c} \\ \left[\begin{array}{ccc} x & \overbrace{v_1}^{\vec{\mathbf{v}}} & \overbrace{w_1}^{\vec{\mathbf{w}}} \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{array}\right]\end{array}\right)$$

This trick helps us to find the area of a parallelopiped

Eigenvector is that vector which doesn't change orientation after transformation and remains along the same line s before

$$\overbrace{A}^{\text{Transformation matrix}}\vec{\mathbf{v}} = \overbrace{\lambda}^{\text{Eigenvalue}}\vec{\mathbf{v}}$$

Eigenvector

This can be simplified as :

$$\underbrace{(A-\lambda I)}\vec{\mathbf{v}}=\vec{0}$$

Where the first matrix's determinant needs to be zero.