Q1) Gauss–Legendre

① 2 point Technique $\qquad P_2(x) = \frac{3x^2-1}{2}$

Nodal points: $x_1 = -1/\sqrt{3}$, $x_2 = 1/\sqrt{3}$

Weight,

$$\lambda_1 = \int_0^1 \left(\frac{x - x_1}{x_2 - x_1}\right) dx = \int_{-1}^{1} \left(\frac{x - 1/\sqrt{3}}{2/\sqrt{3}}\right) dx = \frac{1}{2}\int_{-1}^{1} \left(\sqrt{3}x - 1\right) dx$$

$$= 1$$

$$\lambda_2 = 1$$

DATE:

(2) 3 point Technique $\quad P_2(x) = \dfrac{5x^3-3x}{2}$

Nodal points : $x_1 = 0, \quad x_2 = -\sqrt{\frac{3}{5}}, \quad x_3 = \sqrt{\frac{3}{5}}$

Weight

$$\lambda_1 = \int_{-1}^{1} \left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right) dx$$

$$= \frac{-5}{3}\int_{-1}^{1} \underline{\left(x+\sqrt{3/5}\right)\left(x-\sqrt{3/5}\right)}\, dx = \frac{-5}{3}\int_{-1}^{1} \left(x^2-3/5\right) dx$$

$$= -\frac{5}{3}\left(\frac{2}{3}\right) + 2 = \frac{8}{9}$$

$$\lambda_2 = \int_{-1}^{1} \left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right) = \int_{-1}^{1} \frac{x\left(x-\sqrt{3/5}\right)}{6/5}\, dx$$

$$= \frac{5}{6}\int_{-1}^{1} \left(x^2 - x\sqrt{\frac{3}{5}}\right) dx = \frac{5}{9}$$

$$\lambda_3 = 5/9$$

Q3) Weight and Nodal points of Gauss - Laguerre 2 and 3 point technique.

→ $L_2(x) = x^2 - 4x + 2$ • for two point technique.

We know,

For Gauss - Laguerre,

$$\int_0^\infty e^{-x} f(x)\,dx = \sum_{K=0}^{n-1} \lambda_K f(x_K)$$

① Nodal Points,

$$x_{1/2} = \frac{4 \pm \sqrt{16 - 4(2)(1)}}{2}$$

$$x_1 = 2 - \sqrt{2}, \quad x_2 = 2 + \sqrt{2}$$

② Weights,

$$\lambda_1 = \int_0^{+\infty} e^{-x}\left(\frac{x - x_2}{x_1 - x_2}\right) = \int_0^\infty e^{-x}\left(\frac{x - (2+\sqrt{2})}{-2\sqrt{2}}\right)$$

$$\left\{ \therefore \Gamma(n) = \int_0^\infty e^{-x} t^{n-1}\,dx \quad \& \quad \Gamma(n+1) = n\Gamma(n) \right\}$$

$$\frac{-1}{2\sqrt{2}}\left[\int_0^\infty e^{-x} x - \int_0^\infty e^{-x}(2+\sqrt{2})\right] = \frac{-1}{2\sqrt{2}}\left(\Gamma(2) - \Gamma(1)(2+\sqrt{2})\right)$$

$$\{\Gamma(2) = 1! \;\&\; \Gamma(1) = 0!\}$$

$$\lambda_1 = \frac{-1}{2\sqrt{2}}\left(1 - (2+\sqrt{2})\right) = \frac{\sqrt{2}+1}{2\sqrt{2}}$$

$$\lambda_2 = \int_0^\infty e^{-x}\left(\frac{x - x_1}{x_2 - x_1}\right)$$

$$= \int_0^\infty e^{-x}\left(\frac{x-(2-\sqrt{2})}{2\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}\left[\Gamma(2) - (2-\sqrt{2})\Gamma(1)\right]$$

$$= \frac{2-\sqrt{2}}{4}$$

$L_3(x) = x^3 - 9x^2 + 18x - 6$ • For three point technique.

- Nodal Points:- $x_1 = 0.416$, $x_2 = 2.294$, $x_3 = 6.29$
- Weights,

$$\lambda_1 = \int_0^\infty e^{-x}\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)$$

$$= \int_0^\infty e^{-x}\left(\frac{x-2.294}{-1.878}\right)\left(\frac{x-6.29}{-5.874}\right) = \frac{1}{11.03}\int_0^\infty \left(x^2-8.584x+14.429\right)e^{-x}$$

$$= 0.711$$

$$\lambda_2 = \int_0^\infty e^{-x}\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_3}{x_2-x_3}\right) = \int_0^\infty e^{-x}\left(\frac{x-0.416}{1.878}\right)\left(\frac{x-6.29}{-3.996}\right)$$

$$= \frac{-1}{7.50}\int_0^\infty \left(x^2-6.706x+2.62\right)e^{-x} = \frac{-1}{7.5}\left[\Gamma(3)-6.706\,\Gamma(2)+2.62\,\Gamma(1)\right]$$

$\lambda_2 = 0.278$

$$\lambda_3 = \int_0^\infty e^{-x}\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right) = \int_0^\infty e^{-x}\left(\frac{x-0.416}{5.874}\right)\left(\frac{x-2.294}{3.996}\right)$$

$$= \frac{1}{23.47}\int_0^\infty (x^2-2.71x+0.954)e^{-x}$$

$$= \frac{1}{23.47}\left[\Gamma(3) - 2.71\,\Gamma(2) + 0.954\,\Gamma(1)\right] = \frac{0.244}{23.47}$$

$\lambda_3 = 0.0103$

Q6) Gauss - Hermite $\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx$

- 2 point technique $H_2(x) = 2(2x^2-1)$

Nodal points = $x_1 = -1/\sqrt{2}$, $x_2 = 1/\sqrt{2}$

Weight :

$$\cancel{\lambda_1 = \int_{-\infty}^{\infty} e^{-x^2}(2(2x_1^2-1)) = \int_{-\infty}^{\infty} e^{-x^2}(2(}$$

$$\lambda_1 = \int_{-\infty}^{\infty} e^{-x^2}\left(\frac{x-x_2}{x_1-x_2}\right)dx = \int_{-\infty}^{\infty} e^{-x^2}\left(\frac{x-1/\sqrt{2}}{-\sqrt{2}}\right)dx$$

$$\cancel{\left\{ x^2 = y \Rightarrow 2x\,dx = dy \;\&\; x = \sqrt{y} \right\}}$$

$$\cancel{\lambda_1 = -\frac{1}{2}\int_{0}^{\infty}\left(\sqrt{2}e^{-y}y - e^{-y}\right)dy = -\frac{1}{2}\left[\sqrt{2}\,\Gamma(2) - \Gamma(1)\right]}$$

$$\lambda_1 = -\frac{1}{2}\int_{-\infty}^{\infty}\left(\sqrt{2}\, e^{-x^2} x - e^{-x^2}\right) dx \qquad \because \int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\pi/a}$$

$$= -\frac{1}{2}\left[\left(\sqrt{2}\left(x\sqrt{\pi} - \int \sqrt{\pi}\, dx\right)\right) - \sqrt{\pi}\right] \xrightarrow{} 0 \qquad = \frac{\sqrt{\pi}}{2}$$

$$\lambda_2 = \int_{-\infty}^{\infty} e^{-x^2}\left(\frac{x - x_1}{x_2 - x_1}\right) dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\left(\sqrt{2}x + 1\right) dx = \frac{1}{2}\int_{-\infty}^{\infty}\left(\cancel{\sqrt{2}\, x e^{-x^2}} + e^{-x^2}\right) dx$$

$$= \frac{\sqrt{\pi}}{2}$$

$\longrightarrow$ 3 Point Technique:- $\quad H_3(x) = 4(2x^3 - 3x)$

Nodal points = $x_1 = 0, \quad x_2 = -\sqrt{3/2}, \quad x_3 = +\sqrt{3/2}$

Weights,

$$\lambda_1 = \int_{-\infty}^{\infty} e^{-x^2}\left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right) = \int_{-\infty}^{\infty} \frac{e^{-x^2}\left(x + \sqrt{3/2}\right)\left(x - \sqrt{3/2}\right)}{-3/2}$$

$$\lambda_1 = -\frac{2}{3}\int_{-\infty}^{\infty} e^{-x^2}\left(x^2 - 3/2\right) dx$$

$$= -\frac{2}{3}\left[\int_{-\infty}^{\infty} e^{-x^2} x^2 dx - \frac{3}{2}\int_{-\infty}^{\infty} e^{-x^2} dx\right]$$

$$y = x^2, \quad dy = 2x\,dx$$

$$= \left(-\frac{2}{3}\left[\frac{2}{2}\int_{0}^{\infty} e^{-y}\sqrt{y}\,dy - \frac{3}{2}\sqrt{\pi}\right]\right)$$

$$= \left[-\frac{2}{3}\Gamma(3/2) + \sqrt{\pi}\right] = \frac{2}{3}\sqrt{\pi} = 1.181$$

$$\lambda_2 = \int_{-\infty}^{\infty} e^{-x^2}\left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right) dx$$

$$= \cancel{\int_{-\infty}^{\infty} e^{-x^2}\left(\frac{2x}{-\sqrt{3}}\right.}$$

$$= \int_{-\infty}^{\infty} e^{-x^2}\left(\frac{\sqrt{2}x}{-\sqrt{3}}\right)\left(\frac{x - \sqrt{3/2}}{-\sqrt{6}}\right) dx$$

$$= +\frac{1}{3\sqrt{2}}\int_{-\infty}^{\infty}\left(\sqrt{2}e^{-x^2}x^2 - \sqrt{3}e^{-x^2}x\right) dx$$

this becomes zero

odd f^n

= using $y = x^2$

$$+\frac{1}{3\sqrt{2}}\left[\sqrt{2}\,\Gamma\left(\frac{3}{2}\right)\right] = \frac{\sqrt{\pi}}{6} = 0.295$$

λ_3 will be same as $x_3 = \sqrt{3/5}$

Q8) $\int_a^b f(x)dx \approx \int_a^b P_n(x)\,dx$

Error is because of $(n+1)^{th}$ term in the polynomial

$$E_n = \int_a^b \frac{\Delta^{n+1} f\, h^{n+1}[t(t-1)\ldots]\, h\,dt}{(n+1)!\, h^{n+1}}$$

$$E_n \leq \frac{f^{n+1}(\max)}{(n+1)!} \times C$$

Now if $f(x) = x^{n+1}$

$$C = \int_{-1}^{1} \{f(x)\,dx\} - \sum_{k=0}^{n} \lambda_k f(x_k)$$

→ One - point :- $x = 0$

$P(x) = 1 + x$, $f(x) = x^2$

$$C = \int_{-1}^{1} x^2 dx - \lambda_0 f(x_0) = \frac{2}{3} - 2f(0) = 2/3$$

$$E_0 \leq \frac{2}{3}\,\frac{f^2(\max)}{2!} = \frac{(f'')_{\max}}{3}$$

→ Two point : $n = 1$, $P_2(x) = 1 + x + x^2 + x^3$ $\quad f(x) = x^4$

$$C = \int_{-1}^{1} x^4 dx - \lambda_0 f(x_0) - \lambda_1 f(x_1)$$

$$= \frac{2}{5} - f\left(-\frac{1}{\sqrt{3}}\right) - f\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{5} - \frac{1}{9} - \frac{1}{9} = \frac{8}{45}$$

$$E_1 = \frac{8}{45} \frac{f^{4}_{max}}{4!}$$

→ Three point $(x=3)$ $\quad P_3(x) = 1 + x + x^2 + x^3 + x^4 + x^5$

$f(x) = x^6$

$$C = \int_{-1}^{1} x^6 dx - \lambda_0 f(x_0) - \lambda_1 f(x_1) - \lambda_2 f(x_2)$$

$$= \frac{2}{7} - \frac{5}{9} f\left(\frac{-\sqrt{3}}{\sqrt{5}}\right) - \frac{8}{9} f(0) - \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$= \frac{2}{7} - \frac{5}{9} \times \frac{27}{125} - \frac{8}{9} \times (0) - \frac{5}{9} \times \left(\frac{27}{125}\right)$$

$$= \frac{2}{7} - \frac{6}{25} = \frac{8}{175}$$

$$E_2 = \frac{8}{175} \frac{\left(f^{6}_{max}\right)}{6!}$$