A Brief History of Solving The Gravitational Three Body Problem

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The Three-Body Problem was theorized by Issac Newton in order to determine the long term stability of the Earth-Moon-Sun system. No general closed form solution exists for the problem, resulting in a system that is chaotic for most initial conditions. Modeling the system is useful in astrophysics to model the long term trajectories of stellar systems; trajectories of bodies in a three-body system are impossible to determine without knowing its initial conditions to infinite precision. This report aims to highlight the various numerical and analytical methods used to solve the problem and comment on their practicality.

1 Classical Mechanics

1.1 Issac Newton's Propositions

In Philosophæ Naturalis Principia Mathematica, Issac Newton qualitatively comments on the motion of a third body under the influence of the gravity of two interacting bodies, using the Sun, the Earth and the Moon as an example. He described a set of three differential equations to determine their acceleration. The proposition then describes the complexity of motion of three similar masses moving under the influence of gravity.

$$\begin{aligned} \ddot{r_1} &= -Gm_2\frac{r_1 - r_3}{|r_1 - r_3|^3} - Gm_3\frac{r_1 - r_2}{|r_1 - r_2|^3} \\ \ddot{r_2} &= -Gm_3\frac{r_2 - r_3}{|r_2 - r_3|^3} - Gm_1\frac{r_2 - r_1}{|r_2 - r_1|^3} \\ \ddot{r_3} &= -Gm_1\frac{r_3 - r_1}{|r_3 - r_1|^3} - Gm_2\frac{r_3 - r_2}{|r_3 - r_2|^3} \end{aligned} \tag{1}$$

1.2 Restricted Three-Body Problem

Under certain conditions, a system of three gravitationally interacting bodies is a largely non-chaotic system. In a "restricted" three-body problem, a small mass moves under the influence of large masses. Since the mass is negligible, its influence on the larger bodies can be neglected during calculations. For example, in the Alpha Centauri star system, Proxima Centauri orbits two larger stars (Alpha Centauri A and B).

2 Analytical Solutions

2.1 Lagrange's Family of Solutions

In 1772, Lagrange determined a family of solutions in which three masses are at an equilateral triangle at all times. Jupiter and the Greek and Trojan family of asteroids, for example, form an equilateral triangle and interact like a three-body system. Lagrange determined a family of five solutions - points that are viewed from reference frames that rotate with the masses - which are known as Lagrangian points. However, his family of solutions only apply to a circular restricted three-body problem which require very specific conditions in order to work, which is why Lagrange's method cannot be used as a general solution to the problem.

2.2 Karl Sundman's Approximation

In 1912, Karl Sundman proved the existence of an analytical solution to the three-body problem that corresponds to values of time $t^{1/3}$. His solution is a power series that was discovered while searching for singularities; in Sundman's three-body problem, this occurred when the angular momentum of the system vanishes, i.e., points at which binary or triple collisions occur. The set of nine differential equations that govern the three-body problem, which has 18 degrees of freedom, is restricted to 12 degrees of freedom by using the center of mass of two of the bodies as a frame of reference. Sundman's approximation converges for all real values of t, however the series converges far too slowly for practical purposes; astronomers estimate that if Sundman's calculations were used in astronomical observations, computations would involve $10^{8000000}$ terms.

3 Conclusion

Over the years, mathematicians proved that no general solution to the three-body problem can be found analytically. For a restricted three-body system, Henri Poincaré concluded that an infinite number of periodic solutions exist, but without knowing the initial conditions of a system to infinite precision, it is impossible to study its long-term behavior.

4 References

1. The Three Body Problem with Python
2. On Periodic Solutions of the three-body problem (Shijun Liao, Xiaoming Li)
3. Newton's Philosophae Naturalis Principia Mathematica
4. Sundman's General Solution of the Three Body Problem
5. A (Less Than Practical) Solution to the N-Body Problem
6. Exposition of Sundman's Regularization of The Three Body Problem