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Satellite orbits in Levi-Civita space

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Abstract

In this paper we consider satellite orbits in central force field with quadratic drag using two formalisms. The first using polar coordinates in which the satellite angular momentum plays a dominant role. The second is in Levi-Civita coordinates in which the energy plays a central role. We then merge these two formalisms by introducing polar coordinates in Levi-Civita space and derive a new equation for satellite orbits which unifies these two paradigms. In this equation energy and angular momentum appear on equal footing and thus characterize the orbit by its two invariants. Using this formalism we show that equatorial orbits around oblate spheroids can be expressed analytically in terms of Elliptic functions. In the second part of the paper we derive in Levi-Civita coordinates a linearized equation for the relative motion of two spacecrafts whose trajectories are in the same plane. We carry out also a numerical verification of these equations.

Keywords: Satellite orbits; Invariants; Levi-Civita space; Oblate body

1. Introduction

The motion of a particle in central force field is a classical problem that has been addressed in many books on classical mechanics (Goldstein, 1981). In principle the problem requires only the application of Newton's second law $\mathbf{F} = m\mathbf{a}$ where $\mathbf{F}$ is the force, m is the particle mass and $\mathbf{a}$ is the acceleration. The resulting orbit can be computed by numerical integration or in some cases analytically e.g when $\mathbf{F}$ is the gravitational force due to a point mass. When no drag is present the system is conservative and the motion is characterized by two invariants, the energy E and the angular momentum J. It follows then that an equation which represents the orbit in terms of these two invariants is desirable from both a fundamental and practical point of view. However to our best knowledge such a representation has not appeared in the literature so far.

One particular current application where accurate orbit determination is of critical importance is related to satellites and spacecraft trajectories. The computation of these trajectories has been the subject of numerous monographs (King-Hele and Merson, 1958; King-Hele, 1987; Lawden, 1963; Prussing and Conway, 1993; Vallado, 1997) and research papers (Condurache and Martinusi, 2012; Gim and Alfriend, 2003; Junkins and Turner, 1979; Kechichian, 1998) (to name a few). In particular the effect of the Earth oblateness (Humi and Carter, 2008; Humi, 2012; Schweighart and Sedwick, 2002) and drag forces on satellite orbits (Carter and Humi, 2008; Humi and Carter, 2007; Kustaanheimo and Stiefel, 1965). Furthermore some recent publications discussed the effects of orbit eccentricity and drag variability on the determination of a satellite orbit (Fumenti et al., 2013; Humi, 2010, 2012, 2013).

Since satellite orbits in a central force field are in a plane it is natural to use polar representation for the orbit equations. In this formalism the angular momentum of the

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satellite emerges as a key variable for the derivation of the equations of motion. However in 1920 [6,23] Levi-Civita introduced another two dimensional formalism whose primary objective was to regularize the equations of motion near collision which is advantageous from computational point of view. To this end a coordinate transformation was defined and in the resulting equations of motion, the particle energy emerges as the major quantity. A seminal generalization of Levi-Civita coordinates to three dimensions was made by Kustaanheimo and Steifel (KS-coordinates) (Hernandez and Akella, 2016, 2014; Kustaanheimo, 1964; Kustaanheimo and Stiefel, 1965; Stiefel and Scheifele, 1971; Woollands et al., 2015). These contributions regularized the singularity that exists when two celestial bodies approach each other. Furthermore a host of new applications of this formalism emerged especially when a problem has to considered in three dimensions.

Related to the problem of satellite trajectory determination is the issue of relative motion of two satellites in orbit and the rendezvous problem (Carter and Humi, 1987, 2002; Clohessy and Wiltshire, 1960; Lane and Axelrad, 2007; Melton, 2000; Tschauner and Hempel, 1965). This problem was considered in several settings. In particular the rendezvous problem in the non-central force field of an oblate body was addressed in Humi and Carter (2008). However as far as we know this problem was not addressed even in two dimensions using Levi-Civita coordinates which mitigate to some extent the singular nature of these equations.

Our primary objective in this paper is to introduce polar coordinates in Levi-Civita plane and derive a *new equation which characterizes the motion of a particle in a central force field in terms of its two invariants energy and angular momentum*. These invariants appear on equal footings in this equation. Thus this new equation unifies the two paradigms mentioned above (angular momentum vs. energy) and might find applications in classical mechanics and satellite theory.

Our secondary objective in this paper is to recast in Levi-Civita coordinates the relative motion of two satellites orbiting in the same plane thereby mitigating the singular nature of these equation when these satellites are in close proximity.

Within this framework we consider also the dissipative effects of quadratic drag on the angular momentum and energy of a satellite and their impact on its orbits. This problem was addressed in various context by several authors (Carter and Humi, 2002; Celletti et al., 2011; Humi and Carter, 2008; Humi, 2010; Mavraganis and Michalakis, 1994) in the past. In this paper we address the impact that these dissipative effects have on the invariants that characterize the orbit.

The plan of the paper is as follows: In Section 2 we review the general theory for satellite trajectories in a central force field and derive the orbit equation under the action of quadratic drag. Section 3 provides a short review of Levi-Civita formalism. In Section 4 we introduce polar coordinates in Levi-Civita plane and derive the orbit equation in these coordinates. In Section 5 we derive a linearized equation for the relative motion of two satellites in the same plane. Section 6 carries out some numerical simulations to verify the accuracy of the formulas that were derived in Sections 4 and 5. We end up with some conclusions in Section 7.

2. Angular momentum representation of the orbit

In this section we review the derivation of the orbit equation for a satellite in a central force field with quadratic drag. In this derivation the satellite angular momentum plays a central role.

The general equation for the orbit under these assumptions is,

$$\ddot{\mathbf{R}} = -f(R)\mathbf{R} - g(\alpha, R)(\dot{\mathbf{R}} \cdot \dot{\mathbf{R}})^{1/2}\dot{\mathbf{R}}, \quad R = |\mathbf{R}| \tag{2.1}$$

In this equation $\mathbf{R}$ is the radius vector of the satellite from the center of attraction and α is a parameter that lumps together the drag constant and the atmospheric density proportionality constants. Differentiation with respect to time is denoted by a dot. We assume that f and g are differentiable on the domain of R under consideration.

Taking the vector product of (2.1) with $\mathbf{R}$ on the left and introducing the angular momentum vector

$$\mathbf{J} = \mathbf{R} \times \dot{\mathbf{R}} \tag{2.2}$$

we obtain

$$\dot{\mathbf{J}} = -g(\alpha, R)(\dot{\mathbf{R}} \cdot \dot{\mathbf{R}})^{1/2}\mathbf{J} \tag{2.3}$$

It follows from this equation that $\dot{\mathbf{J}}$ is always parallel to $\mathbf{J}$ and therefore $\mathbf{J}$ has a fixed direction. As a consequence the motion is in a fixed plane which we take, without loss of generality, to be the $x - y$ plane. Introducing polar coordinates R and θ in this plane the angular momentum vector can be written as

$$\mathbf{J} = R^2\dot{\theta}\mathbf{e}_J \tag{2.4}$$

where $\mathbf{e}_J$ is a unit vector in the direction of $\mathbf{J}$. Hence (2.3) can be rewritten as

$$\frac{\dot{J}}{J} = -g(\alpha, R)(\dot{\mathbf{R}} \cdot \dot{\mathbf{R}})^{1/2} \tag{2.5}$$

where $J = |\mathbf{J}|$. Substituting this result in (2.1) and dividing by J yields

$$\frac{d}{dt}\left(\frac{\dot{\mathbf{R}}}{J}\right) + \frac{f(R)\mathbf{R}}{J} = 0 \tag{2.6}$$

i.e

$$\dot{\mathbf{R}} = -J\int \frac{f(R)\mathbf{R}}{J}\,dt \tag{2.7}$$

In polar coordinates (2.1) becomes

$$R\ddot{\theta} + 2\dot{R}\dot{\theta} = -g(\alpha, R)(\dot{R}^2 + R^2\dot{\theta}^2)^{1/2} R\dot{\theta} \tag{2.8}$$

$$\ddot{R} - R\dot{\theta}^2 = -f(R)R - g(\alpha, R)(\dot{R}^2 + R^2\dot{\theta}^2)^{1/2} \dot{R} \tag{2.9}$$

Dividing the first equation by $R\dot{\theta}$ and integrating we obtain

$$J = R^2\dot{\theta} = h \exp\left(-\int g(\alpha, R)(\dot{R}^2 + R^2\dot{\theta}^2)^{1/2} dt\right) \tag{2.10}$$

where h is an integration constant.

Using (2.4) and (2.5) to change the independent variable from t to θ we obtain after some algebra the orbit equation

$$\frac{R''}{R} - 2\left(\frac{R'}{R}\right)^2 + \frac{f(R)R^4}{J^2(\theta)} = 1 \tag{2.11}$$

where primes denote differentiation with respect to θ.

Eq. (2.11) is the orbit equation for the motion of the satellite in the "angular momentum representation".

When there exist a potential function V so that $\nabla V = f(R)\mathbf{R}$ (2.1) becomes

$$\ddot{\mathbf{R}} = -\nabla V - g(\alpha, R)(\dot{\mathbf{R}} \cdot \dot{\mathbf{R}})^{1/2} \dot{\mathbf{R}} \tag{2.12}$$

Taking the scalar product of this equation by $\dot{\mathbf{R}}$ we obtain

$$(\ddot{\mathbf{R}}, \dot{\mathbf{R}}) = -(\nabla V, \dot{\mathbf{R}}) - g(\alpha, R)(\dot{\mathbf{R}} \cdot \dot{\mathbf{R}})^{3/2}. \tag{2.13}$$

In this equation the scalar product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ is $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$. Eq. (2.13) can be rewritten as

$$\frac{dE}{dt} = -g(\alpha, R)(\dot{\mathbf{R}} \cdot \dot{\mathbf{R}})^{3/2}. \tag{2.14}$$

where E is the particle energy

$$E = V + \frac{(\dot{\mathbf{R}}, \dot{\mathbf{R}})}{2}$$

Eq. (2.14) gives the rate of the particle energy decay due to the dissipative effects of the drag force.

3. Energy representation of the orbit

Energy is an invariant which characterizes satellite motion when no drag is present. To take advantage of this Levi-Civita (1920) introduced a two dimensional formalism (which was later generalized to three dimensions (Kustaanheimo, 1964; Kustaanheimo and Stiefel, 1965)) in which energy plays a central role (and has additional advantages when collisions are present).

3.1. Levi-Civita formalism

There are in the literature excellent expositions of Levi-Civita formalism (Kustaanheimo and Stiefel, 1965; Levi-Civita, 1920; Stiefel and Scheifele, 1971). Here we present a short overview of this formalism for completeness.

To begin with introduce a "fictitious time" s which is defined by the relation

$$\frac{d}{ds} = R\frac{d}{dt}. \tag{3.1}$$

It is then easy to see that for $\mathbf{R} = (x, y)$

$$\ddot{\mathbf{R}} = \frac{1}{R^2}\mathbf{R}'' - \frac{1}{R^3}R'\mathbf{R}'. \tag{3.2}$$

where primes denote (now) differentiation with respect to s. Furthermore the velocity v satisfies,

$$v^2 = (\dot{\mathbf{R}}, \dot{\mathbf{R}}) = \frac{1}{R^2}(\mathbf{R}', \mathbf{R}') \tag{3.3}$$

Using (3.2) and (3.3) the equation of motion (2.1) becomes

$$\frac{1}{R^2}\mathbf{R}'' - \frac{1}{R^3}R'\mathbf{R}' = -f(R)\mathbf{R} - \frac{g(\alpha, R)}{R^2}(\mathbf{R}', \mathbf{R}')^{1/2}\mathbf{R}' \tag{3.4}$$

For a particle of unit mass whose equation of motion is (2.1) where $f(R)\mathbf{R} = \nabla V$ and $g(\alpha, R) = 0$ (no drag) the energy E is conserved and

$$E = V + \frac{v^2}{2} = V + \frac{1}{2R^2}(\mathbf{R}', \mathbf{R}'). \tag{3.5}$$

When this particle is in the gravitational field of a point mass M,

$$f(R) = \frac{\mu}{R^3}, \quad V = -\frac{\mu}{R} \tag{3.6}$$

and the expression for energy becomes

$$E = -\frac{\mu}{R} + \frac{1}{2R^2}(\mathbf{R}', \mathbf{R}'), \tag{3.7}$$

where $\mu = GM$ and G is the gravitational constant. When the drag force is included the rate of change in E in this coordinate system can be obtained from (2.14) by a change of variables

$$\frac{dE}{ds} = -\frac{g(\alpha, R)}{R^2}(\mathbf{R}', \mathbf{R}')^{3/2} \tag{3.8}$$

Next define a transformation from the (x, y) coordinate system to a new one (u_1, u_2) (Levi-Civita coordinates) which is defined by the following relations,

$$x = u_1^2 - u_2^2, \quad y = 2u_1u_2. \tag{3.9}$$

(We shall refer to this as the U-plane).

Introducing the scalar product of any two vectors $\mathbf{w_1}, \mathbf{w_2}$ as

$$(\mathbf{w_1}, \mathbf{w_2}) = \mathbf{w_1}^T\mathbf{w_2},$$

it follows that $R = (\mathbf{u}, \mathbf{u}) = |\mathbf{u}|^2$ where $\mathbf{u} = (u_1, u_2)$. Next we introduce Levi-Civita matrix,

$$L(\mathbf{u}) = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}.$$

Observe that the transpose L^T and the inverse L^{-1} of this matrix satisfy the following relationships

$$L^T(\mathbf{u})L(\mathbf{u}) = R\mathbf{I}, \quad L^{-1}(\mathbf{u}) = \frac{1}{R}L^T(\mathbf{u}) \tag{3.10}$$

where **I** is the unit two dimensional matrix. Moreover for any two vectors $\mathbf{u}, \mathbf{v}$ we have the following identities

$$L(\mathbf{u})\mathbf{v} = L(\mathbf{v})\mathbf{u}, \quad (\mathbf{u},\mathbf{u})L(\mathbf{v})\mathbf{v} + (\mathbf{v},\mathbf{v})L(\mathbf{u})\mathbf{u} = 2(\mathbf{u},\mathbf{v})L(\mathbf{u})\mathbf{v} \tag{3.11}$$

and

$$L(\mathbf{u})' = L(\mathbf{u}') \tag{3.12}$$

It is then easy to see that for **R**

$$\mathbf{R} = L(\mathbf{u})\mathbf{u}, \quad \mathbf{R}' = 2L(\mathbf{u})\mathbf{u}', \quad \mathbf{R}'' = 2L(\mathbf{u})\mathbf{u}'' + 2L(\mathbf{u})'\mathbf{u}' = 2L(\mathbf{u})\mathbf{u}'' + 2L(\mathbf{u}')\mathbf{u}' \tag{3.13}$$

To convert (3.4) to an equation in U space we use (3.13) and (3.11) with the vectors $\mathbf{u}, \mathbf{u}'$. For the left hand side of (3.4) we then have

$$\frac{1}{R^2}\mathbf{R}'' - \frac{1}{R^3}R'\mathbf{R}' = \frac{1}{(\mathbf{u},\mathbf{u})^2}[2L(\mathbf{u})\mathbf{u}'' + 2L(\mathbf{u}')\mathbf{u}'] - \frac{4}{(\mathbf{u},\mathbf{u})^3} \times (\mathbf{u},\mathbf{u}')L(\mathbf{u})\mathbf{u}'$$

However from (3.11) with $\mathbf{v} = \mathbf{u}'$ we have

$$(\mathbf{u},\mathbf{u})L(\mathbf{u}')\mathbf{u}' = 2(\mathbf{u},\mathbf{u}')L(\mathbf{u})\mathbf{u}' - (\mathbf{u}',\mathbf{u}')L(\mathbf{u})\mathbf{u}$$

This leads after some algebra to

$$\mathbf{u}'' - \frac{(\mathbf{u}',\mathbf{u}')}{(\mathbf{u},\mathbf{u})}\mathbf{u} = \frac{(\mathbf{u},\mathbf{u})^2}{2}\left\{-f(R)\mathbf{u} - 4g(\alpha,R)(\mathbf{u},\mathbf{u})^{-3/2}(\mathbf{u}',\mathbf{u}')^{1/2}\mathbf{u}'\right\} \tag{3.14}$$

where R has to be replaced by $(\mathbf{u},\mathbf{u})$. When $f(R)$ is given by (3.6) this equation becomes

$$\mathbf{u}'' + \frac{1}{2}\frac{\mu - 2(\mathbf{u}',\mathbf{u}')}{(\mathbf{u},\mathbf{u})}\mathbf{u} = -2g(\alpha,R)(\mathbf{u},\mathbf{u})^{1/2}(\mathbf{u}',\mathbf{u}')^{1/2}\mathbf{u}' \tag{3.15}$$

When no drag forces are present the particle energy Eq. (3.7) becomes

$$E = -\frac{\mu - 2(\mathbf{u}',\mathbf{u}')}{(\mathbf{u},\mathbf{u})}. \tag{3.16}$$

Using this expression for E in (3.15) we have

$$\mathbf{u}'' - \frac{1}{2}E\mathbf{u} = -2g(\alpha,R)(\mathbf{u},\mathbf{u})^{1/2}(\mathbf{u}',\mathbf{u}')^{1/2}\mathbf{u}'. \tag{3.17}$$

However observe that due to the dissipative nature of the drag force, E is not constant under the present settings.

4. Polar representation of the orbit equation

Polar coordinates representation of satellite orbit equations in the $x - y$ plane offers several advantages over their Cartesian counterparts. Motivated by this observation we develop in this section a new representation of (3.14) using polar and Levi-Civita coordinates.

4.1. Polar geometry in the U-plane

We introduce polar coordinates (u, ϕ) in the U-plane in a manner similar to polar coordinates in the $x - y$ plane.

$$u = (\mathbf{u},\mathbf{u})^{1/2}, \quad \phi = tan^{-1}\frac{u_2}{u_1}. \tag{4.1}$$

The relationship between these variables and (R, θ) is given by

$$R = (\mathbf{u},\mathbf{u}) = u^2, \quad \theta = 2\phi. \tag{4.2}$$

Furthermore in parallel to the definitions of the radial and tangential unit vectors in $x - y$ plane

$$\mathbf{e}_r = (\cos\theta, \sin\theta), \quad \mathbf{e}_\theta = (-\sin\theta, \cos\theta), \tag{4.3}$$

we define in the U-plane

$$\mathbf{e}_u = (\cos\phi, \sin\phi), \quad \mathbf{e}_\phi = (-\sin\phi, \cos\phi). \tag{4.4}$$

Using (4.2) we find that

$$\mathbf{e}_r = \cos\phi\,\mathbf{e}_u + \sin\phi\,\mathbf{e}_\phi, \quad \mathbf{e}_\theta = -\sin\phi\,\mathbf{e}_u + \cos\phi\,\mathbf{e}_\phi. \tag{4.5}$$

Since $\mathbf{u} = u\mathbf{e}_u$ we have the following formulas for the derivatives of $\mathbf{u}$,

$$\mathbf{u}' = u'\mathbf{e}_u + u\phi'\mathbf{e}_\phi, \quad \mathbf{u}'' = (u'' - u(\phi')^2)\mathbf{e}_u + (u\phi'' + 2u'\phi')\mathbf{e}_\phi. \tag{4.6}$$

4.2. Derivation of the new orbit equation

Using (4.6) the orbit Eq. (3.14) becomes

$$[u'' - u(\phi')^2]\mathbf{e}_u + [u\phi'' + 2u'\phi']\mathbf{e}_\phi - \frac{(\mathbf{u}',\mathbf{u}')}{u}\mathbf{e}_u = \frac{(\mathbf{u},\mathbf{u})^2}{2}\left\{-f(R)u\mathbf{e}_u - 4g(\alpha,R)(\mathbf{u},\mathbf{u})^{-3/2}(\mathbf{u}',\mathbf{u}')^{1/2}[u'\mathbf{e}_u + u\phi'\mathbf{e}_\phi]\right\} \tag{4.7}$$

This yields the following two equations for the tangential and radial components

$$u\phi'' + 2u'\phi' = -2g(\alpha,R)u^2(\mathbf{u}',\mathbf{u}')^{1/2}\phi', \tag{4.8}$$

$$u'' - u(\phi')^2 - \frac{(\mathbf{u}',\mathbf{u}')}{u} = -\frac{f(R)u^5}{2} - 2g(\alpha,R)u(\mathbf{u}',\mathbf{u}')^{1/2}u', \tag{4.9}$$

where

$$(\mathbf{u}',\mathbf{u}') = (u')^2 + u^2(\phi')^2. \tag{4.10}$$

Multiplying (4.8) by u, dividing by $u^2\phi'$ and integrating we obtain

$$L = u^2\phi' = h_1\exp\left(-2\int g(\alpha,u)u(\mathbf{u}',\mathbf{u}')^{1/2}ds\right). \tag{4.11}$$

Hence

$$\frac{d}{ds} = \frac{L}{u^2}\frac{d}{d\phi} \tag{4.12}$$

We note that L defined in (4.11) is equal to the angular momentum J up to a constant. In fact using (4.2) we have

$$L = u^2\phi' = u^2 R\frac{d\phi}{dt} = \frac{R^2\dot{\theta}}{2}$$

Using (4.12) to change the variable from s to ϕ in (4.9) we obtain after a long algebra the following orbit equation,

$$\frac{1}{u}\frac{d^2u}{d\phi^2} - \frac{2}{u^2}\left(\frac{du}{d\phi}\right)^2 - \frac{u^2(\mathbf{u}',\mathbf{u}')}{L^2} = 1 - \frac{f(u)u^8}{2L^2}. \tag{4.13}$$

In this equation there is no explicit dependence on the drag force but it is present implicitly in the expression for L.

When $f(R)$ is given by (3.6) we obtain

$$\frac{1}{u}\frac{d^2u}{d\phi^2} - \frac{2}{u^2}\left(\frac{du}{d\phi}\right)^2 + \frac{u^2[\mu - 2(\mathbf{u}',\mathbf{u}')]}{2L^2} = 1 \tag{4.14}$$

Using (3.16) this equation becomes

$$\frac{1}{u}\frac{d^2u}{d\phi^2} - \frac{2}{u^2}\left(\frac{du}{d\phi}\right)^2 - \frac{Eu^4}{2L^2} = 1 \tag{4.15}$$

This equation represent the central result of this paper. The satellite orbit is represented in terms of it two invariants, energy and angular momentum. The equation "unifies" the angular momentum and energy formalisms given by (2.11) and (3.17) respectively.

Eq. (4.15) can be simplified by the transformation $u = 1/v$ which leads to

$$\frac{d^2v}{d\phi^2} + \frac{E}{2L^2v^3} = -v \tag{4.16}$$

When no drag forces are present the solution to (4.15) (or (4.16)) can be written as

$$R = u^2 = \frac{L\sqrt{2(e^2-1)}}{\sqrt{E}(1 + e\cos(2\phi))} \tag{4.17}$$

When there are no drag forces E, L are constants and Eq. (4.16) may be referred to as the "Energy-Angular-M omentum" orbit equation in Levi-Civita coordinates. In absence of dissipative forces this equation characterizes the orbit by its generic invariants.

We believe that this equation is important from a theoretical and practical points of view. It will facilitate satellite mission planning and control due to the fact that measurements of energy and angular-momentum can be made accurately. As a result corrections to satellite orbits can be executed with high precision.

It might be possible to argue that (3.17) has no (explicit) singularity near the origin while (4.16) does. However, in general, this is not a valid argument since, satellites and spacecraft trajectories are (usually) far from the force center. Furthermore when the trajectory is close to the center then the value of E in (3.17) will increase (in absolute value) without bound while the ratio of $\frac{E}{L^2}$ in (4.15) remains bounded. Observe also that the singularity at $v = 0$ in (4.16) correspond to $u = \infty$ and hence has no practical consequence since satellites do not reach infinity far from the central body.

Similarly when the force term in the primitive equations of motion (2.8) and (2.9) contain higher negative powers of R (e.g near an oblate body or non-spherical asteroids) these equations become highly singular. However in (4.15) this singularity is mitigated by the fact that the ratio of $\frac{E}{L^2}$ remains bounded and the solution retains its accuracy. This constitutes another advantage of (4.15) over the primitive equations of motion. This is another illustration of the "regularization power" of Levi-Civita formalism which remains valid even when two celestial bodies are close to collision.

4.3. Equatorial orbits around oblate body

The gravitational potential around an oblate body is approximated by

$$V \approx -\frac{\mu}{R}\left[1 - \frac{R_0^2J_2}{R^2}P_2(\cos\phi)\right] \tag{4.18}$$

In this equation J_2 is a constant, ϕ is the colatitude angle, R_0 is the body radius at the equator and P_2 is the second order Legendre polynomial. At the equator $\phi = \pi/2$ and we have

$$f(R)\mathbf{R} = \nabla V = \frac{\mu}{R^3}\left(1 + \frac{3R_0^2J_2}{2R^2}\right)\mathbf{R}. \tag{4.19}$$

Hence using (4.2)

$$f(u) = \frac{\mu}{u^6}\left(1 + \frac{3R_0^2J_2}{2u^4}\right). \tag{4.20}$$

Using this expression for $f(u)$ in (4.13) we obtain

$$\frac{1}{u}\frac{d^2u}{d\phi^2} - \frac{2}{u^2}\left(\frac{du}{d\phi}\right)^2 - \frac{u^2(\mathbf{u}',\mathbf{u}')}{L^2} = 1 - \frac{\mu u^2\left(1 + \frac{3R_0^2J_2}{2u^4}\right)}{2L^2} \tag{4.21}$$

To derive an equation similar to (4.15) we note that the expression for the energy E (see (3.5) in this case is

$$E = -\frac{\mu}{R}\left(1 + \frac{R_0^2J_2}{2R^2}\right) + \frac{(\mathbf{R}',\mathbf{R}')}{2R^2}, \tag{4.22}$$

or equivalently (using(3.13))

$$E = -\frac{\mu}{u^2}\left(1 + \frac{R_0^2J_2}{2u^4}\right) + 2\frac{(\mathbf{u}',\mathbf{u}')}{u^2}. \tag{4.23}$$

Using this expression for E in (4.21) yields

$$\frac{1}{u}\frac{d^2u}{d\phi^2} - \frac{2}{u^2}\left(\frac{du}{d\phi}\right)^2 - \frac{u^4E}{2L^2} = 1 - \frac{\mu R_0^2J_2}{2L^2u^2} \tag{4.24}$$

When there is no drag E and L are constants and it is possible to obtain an implicit solution for $u(\phi)$. To this end we make the transformation $u(\phi) = 1/v(\phi)$ to obtain the following equation

$$v^3 \frac{d^2 v}{d\phi^2} - Bv^6 + v^4 + A = 0 \tag{4.25}$$

where $A = \frac{E}{2L^2}$ and $B = \frac{\mu R_0^2 J_2}{2L^2}$. The solution of (4.25) is

$$\phi + C_2 = \int^{v(\phi)} \frac{2sds}{\sqrt{2Bs^6 - 4s^4 + 4C_1 s^2 + 4A}} = \int^{v(\phi)} \frac{dw}{\sqrt{2Bw^3 - 4w^2 + 4C_1 w + 4A}} \tag{4.26}$$

where C_1, C_2 are integration constants and $w = s^2$. The last integral has an exact analytical solution in terms of Elliptic functions.

5. Relative motion of satellites

In this section we derive the equations for the relative motion of two satellites in orbit in a central force field (Condurache and Martinusi, 2012). The goal is derive equations which are less singular when the two satellites are close to each other. This endeavor is carried out in the spirit of the original contribution by Levi-Civita for the regularization of the equations of motion of two celestial bodies near collision (Levi-Civita, 1920). In fact the equations of relative motion formulated in term of s might hold but impossible to solve in terms of time t.

5.1. Linearized equations in physical space

If the positions of the two satellites are denoted by $\mathbf{R_1}, \mathbf{R_2}$ then their respective equations of motion in a (conservative) central force field are

$$\ddot{\mathbf{R}}_1 = -\nabla V(\mathbf{R}_1), \quad \ddot{\mathbf{R}}_2 = -\nabla V(\mathbf{R}_2) \tag{5.1}$$

where the dots represent differentiation with respect to time. The relative position of the second satellite with respect to the first is $\mathbf{w} = \mathbf{R}_2 - \mathbf{R}_1$. Using (5.1) we then have,

$$\ddot{\mathbf{w}} = \nabla V(\mathbf{R}_1) - \nabla V(\mathbf{R}_2). \tag{5.2}$$

Assuming that $|\mathbf{w}| \ll R_1$ we can approximate

$$\nabla V(\mathbf{R}_1) - \nabla V(\mathbf{R}_2) = \nabla[V(\mathbf{R}_1) - V(\mathbf{R}_1 + \mathbf{w})]$$

by a first order Taylor polynomial in $\mathbf{w}$. This leads to the following linear relative motion of the second satellite with respect to first in the inertial coordinate system attached to the central body center,

$$\ddot{\mathbf{w}} = -\nabla(\nabla V(\mathbf{R}_1) \cdot \mathbf{w}) \tag{5.3}$$

In particular if the motion is around a spherical body where V is given by (3.6) we have

$$\ddot{\mathbf{w}} = -\frac{\mu}{R_1^3}\mathbf{w} + \frac{3\mu(\mathbf{R_1} \cdot \mathbf{w})}{R_1^5}\mathbf{R_1} = \mathbf{F}. \tag{5.4}$$

In a coordinate system rotating with the first satellite the relative-motion Eq. (5.3) becomes(Goldstein, 1981; Carter and Humi, 1987)

$$\ddot{\mathbf{w}} + 2\mathbf{\Omega} \times \dot{\mathbf{w}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{w}) + \dot{\mathbf{\Omega}} \times \mathbf{w} = \mathbf{F}. \tag{5.5}$$

where $\mathbf{\Omega}$ is the orbital angular velocity of the first satellite. The reduction of this formula to a system of ordinary differential equations for the motion of two satellites around an oblate body was carried in Humi and Carter (2008). When the motion of the two satellites is in the equatorial plane of the oblate body $\mathbf{F}$ contains terms of the form R_1^{-5}.

We now consider this equation in the special case where the two satellites are in the same $x-y$ plane. In this case

$$\mathbf{\Omega} = (0, 0, \dot{\theta}), \quad \mathbf{w} = (w_1, w_2, 0). \tag{5.6}$$

We have

$$\mathbf{\Omega} \times \dot{\mathbf{w}} = \dot{\theta}(-\dot{w}_2, \dot{w}_1, 0)^T, \quad \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{w}) = -\dot{\theta}^2(w_1, w_2, 0)^T, \quad \dot{\mathbf{\Omega}} \times \mathbf{w} = \ddot{\theta}(-w_2, w_1, 0)^T$$

(In the following we suppress the third component of the vectors).

5.2. Relative equation of motion in Levi-Civita coordinates

We now introduce Levi-Civita transformation

$$w_1 = v_1^2 - v_2^2, \quad w_2 = 2v_1v_2, \quad r^2 = w_1^2 + w_2^2 = (\mathbf{v}, \mathbf{v})^2, \quad \frac{d}{ds} = r\frac{d}{dt}. \tag{5.7}$$

Due to the appearance of the vector $(-w_2, w_1)$ in the equation of motion (5.5) we introduce

$$\overline{L}(\mathbf{u}) = \begin{pmatrix} -u_2 & -u_1 \\ u_1 & -u_2 \end{pmatrix}.$$

We then have for a generic vector $\mathbf{u}$

$$\overline{L}(\mathbf{u})\mathbf{u} = (-u_2, u_1)^T. \tag{5.8}$$

Also for $\mathbf{v}$

$$\overline{L}(\mathbf{v})\mathbf{v} = (-w_2, w_1)^T$$

The matrix $\overline{L}(\mathbf{u})$ has the following properties

$$\overline{L}(\mathbf{u})^T\overline{L}(\mathbf{u}) = (\mathbf{u}, \mathbf{u})I, \quad L(\mathbf{u})^T\overline{L}(\mathbf{u}) = (\mathbf{u}, \mathbf{u})\Gamma, \quad L^{-1}(\mathbf{u})\overline{L}(\mathbf{u}) = \frac{1}{(\mathbf{u}, \mathbf{u})}L(\mathbf{u})^T\overline{L}(\mathbf{u}) = \Gamma, \tag{5.9}$$

where

$$\Gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Observe also that

$$\begin{pmatrix} -w_2' \\ w_1' \end{pmatrix} = 2\overline{L}(v)\mathbf{v}'.$$

Using this data (5.5) for the relative motion can be written as

$$\frac{1}{r^3}[2(\mathbf{v},\mathbf{v})L(\mathbf{v})\mathbf{v}'' - 2(\mathbf{v}',\mathbf{v}')L(\mathbf{v})\mathbf{v}]$$
$$+\frac{4\dot{\theta}}{r}\overline{L}(\mathbf{v})\mathbf{v}' - \dot{\theta}^2 L(\mathbf{v})\mathbf{v} + \ddot{\theta}\overline{L}(\mathbf{v})\mathbf{v} = \mathbf{F}$$

After some algebra similar to the one in Section 4 we obtain the following representation of Eq. (5.5)

$$\mathbf{v}'' - \left[\frac{(\mathbf{v}',\mathbf{v}')}{(\mathbf{v},\mathbf{v})} + \frac{r^2\dot{\theta}^2}{2}\right]\mathbf{v} + 2r\dot{\theta}\Gamma\mathbf{v}' + \frac{r^2\ddot{\theta}}{2}\Gamma\mathbf{v} = \frac{r^2L^T(v)\mathbf{F}}{2}. \tag{5.10}$$

Using

$$\dot{\theta} = \frac{\theta'}{r}, \quad \ddot{\theta} = \frac{1}{r^2}\left[\theta'' - \frac{2}{r}(\mathbf{v},\mathbf{v}')\theta'\right].$$

Eq. (5.10) becomes

$$\mathbf{v}'' - \left\{\frac{(\mathbf{v}',\mathbf{v}')}{(\mathbf{v},\mathbf{v})} + \frac{(\theta')^2}{2} - \frac{1}{2}\left[\theta'' - \frac{2(\mathbf{v},\mathbf{v}')}{(\mathbf{v},\mathbf{v})}\theta'\right]\Gamma\right\}\mathbf{v}$$
$$+2(\theta)'\Gamma\mathbf{v}' = r^2\frac{L^T(v)\mathbf{F}}{2}. \tag{5.11}$$

In particular when $\mathbf{F}$ is given by (5.4) the right hand side of this equation becomes

$$r^2\frac{L^T(\mathbf{v})\mathbf{F}}{2} = -\frac{\mu r^3}{2R_1^3}\mathbf{v} + \frac{3\mu r^2}{2R_1^5}(\mathbf{R_1}\cdot\mathbf{w})L^T(\mathbf{v})\mathbf{R_1} \tag{5.12}$$

We observe that in this formulation $\mathbf{F}$ contains (effectively) only powers of R_1^{-3} and the singularity in the expression of the force in (5.5) has been reduced. Thus in the long run (over many orbits) (5.12) will reduce the cumulative floating point error in the computation of the orbit. It is therefore superior to (5.5) from a numerical point of view.

6. Numerical verification

6.1. Motion of a satellite in an exponential atmosphere

A common model for the earth atmosphere density ρ with height $R - R_0$ is

$$\rho = C_1 \exp\left(\frac{R_0 - R}{H}\right) \tag{6.1}$$

where C_1, R_0, H are constants. For this atmospheric model

$$g(\alpha, R) = \alpha \exp\left(\frac{R_0 - R}{H}\right) = \alpha \exp\left(\frac{u_0^2 - u^2}{H}\right) \tag{6.2}$$

where the constant C_1 was lumped with the drag coefficient α. Eq. (3.8) for the energy becomes

$$\frac{dE}{ds} = -\frac{8\alpha}{u}(\mathbf{u}'\cdot\mathbf{u}')^{3/2}\exp\left(\frac{u_0^2 - u^2}{H}\right). \tag{6.3}$$

Similarly for L we have

$$\frac{1}{L}\frac{dL}{ds} = -2\alpha u(\mathbf{u}'\cdot\mathbf{u}')^{1/2}\exp\left(\frac{u_0^2 - u^2}{H}\right). \tag{6.4}$$

Using (4.12) to change variables from s to ϕ in (6.3), (6.4) yields,

$$\frac{dE}{d\phi} = -\frac{8\alpha L^2}{u^5}\left(\frac{d\mathbf{u}}{d\phi}, \frac{d\mathbf{u}}{d\phi}\right)^{3/2}\exp\left(\frac{u_0^2 - u^2}{H}\right) \tag{6.5}$$

$$\frac{dL}{d\phi} = -2\alpha u L\left(\frac{d\mathbf{u}}{d\phi}, \frac{d\mathbf{u}}{d\phi}\right)^{1/2}\exp\left(\frac{u_0^2 - u^2}{H}\right) \tag{6.6}$$

Using the fact that $\mathbf{u} = u\mathbf{e}_u$ and (4.4), (6.5) and (6.6) become

$$\frac{dE}{d\phi} = -\frac{8\alpha L^2}{u^5}\left[u^2 + \left(\frac{du}{d\phi}\right)^2\right]^{3/2}\exp\left(\frac{u_0^2 - u^2}{H}\right) \tag{6.7}$$

$$\frac{dL}{d\phi} = -2\alpha u L\left[u^2 + \left(\frac{du}{d\phi}\right)^2\right]^{1/2}\exp\left(\frac{u_0^2 - u^2}{H}\right) \tag{6.8}$$

The system of Eqs. (4.15), (6.7), (6.8) comprise of three equations in three unknowns and can be solved by standard numerical methods. Fig. 1 is a plot of the solution of this system with $R_0 = 7000$ km, $\dot{\theta}_0 = \sqrt{\frac{\mu}{R_0^3}}$ (initial circular orbit), $\alpha = 3.10^{-10}, H = 88.667$ and step error of 10^{-12}. On the same figure we plotted also the numerical solution of the system (2.8) and (2.9) with the same parameters but with very small time step (and stiff equations integrator). We note that these two curves are almost indistinguishable. Fig. 2 displays the difference between these curves which over ten periods remains less than 0.75 m. This difference is most probably due to the cumulative error in the numerical integration. However when an adaptive step size is used for the computation of the same orbits we obtain Fig. 3. This figure demonstrates clearly that while the solution using the system (4.15), (6.7), (6.8) (red line) remains

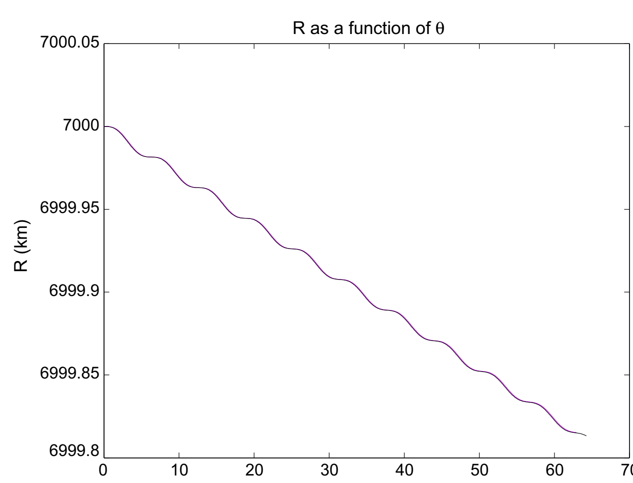

Fig. 1. Illustrative trajectory for a satellite orbit using Eq. (4.15) (red line) which is indistinguishable from the one obtained from (2.8) and (2.9) with extremely small time-step. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

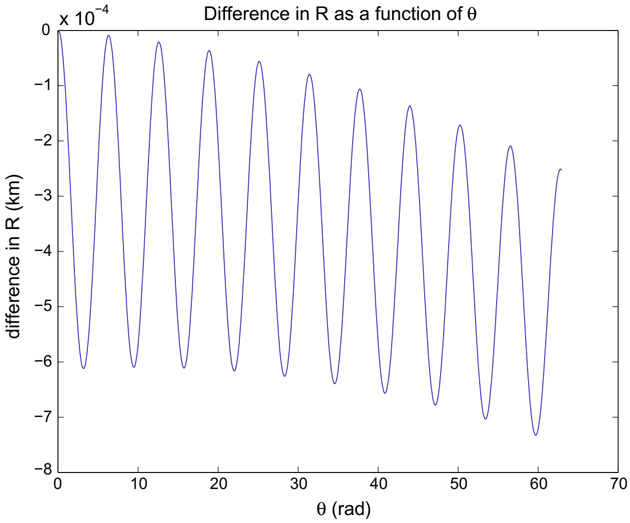

Fig. 2. Difference between the trajectories in Fig. 1.

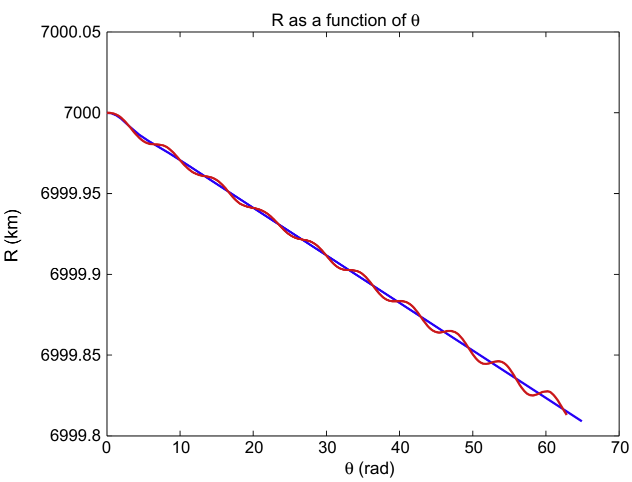

Fig. 3. Adaptive integration of (4.15) (red line) and (2.8) and (2.9) (blue line). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

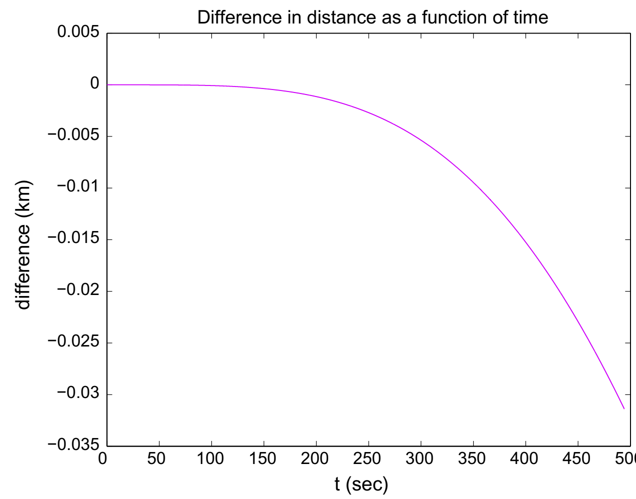

Fig. 4. Difference between the analytic and numerical value of the distance between two satellites.

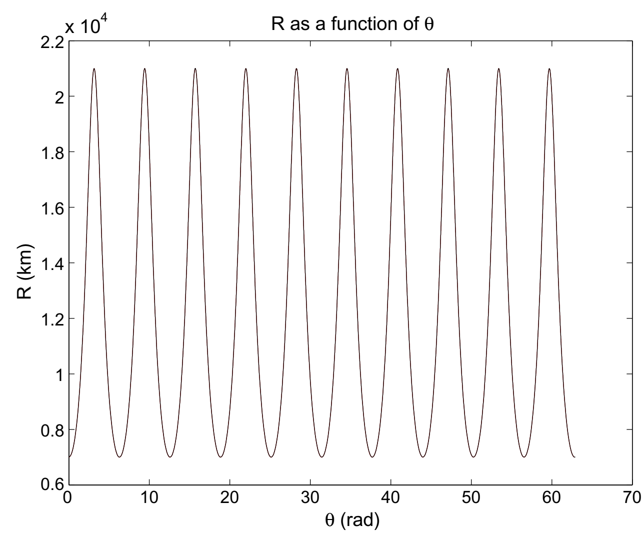

Fig. 5. Illustrative trajectory for a satellite orbit with eccentricity 0.5 using Eq. (4.15) which is essentially indistinguishable from the one obtained from (2.8) and (2.9) with extremely small time-step.

effectively unchanged (and therefore represents the correct solution for the trajectory) the solution derived from Eqs. (2.8) and (2.9) deviates from this solution after few periods and the difference becomes larger as time goes by.

In addition it should be observed, that when the central force field contains high orders of negative powers in R (e.g around an oblate body) Eqs. (2.8) and (2.9) (even without the presence of drag) will yield in the long run results which deviate significantly from the actual satellite orbit due to the singularities present in these equations. These singularities are mitigated in Eq. (4.15) and therefore the numerical results obtained from this equation will be superior to those obtained from (2.8) and (2.9).

We performed also a simulation with the same parameters used to obtain Fig. 1 but with eccentricity $e = 0.5$. Fig. 5 demonstrates that the two solutions are almost indistinguishable also in this case.

6.2. Relative motion of satellites

To verify numerically the formula for the relative motion of satellites which was derived in the previous section we considered two satellites in circular orbit whose positions at time $t = 0$ (in polar coordinates) are $(R_1, 0)$ and $(R_2, 0)$ with $R_1 = 7000$ km and $R_2 = 6999$ km, respectively. The angular velocities of these satellites respectively are

$$\omega_i = \dot{\theta}_i = \sqrt{\frac{\mu}{R_i^3}}, \quad i = 1, 2$$

Hence their distance d at time t satisfies

$$d^2 = R_1^2 + R_2^2 - 2R_1R_2\cos(\omega_1 - \omega_2)t$$

Fig. 4 is a plot of the difference between this analytical expression for the distance and the numerical value obtained from the linearized formula (5.10) as a function of time. This figure demonstrates that the difference between the analytical expression and the numerical value of the linearized formula remains less than 32 m for the duration of the simulation. We observe that the difference remains small (less than 1 m) up to $t \approx 250$ s. However for $t > 250$ s this difference grows sharply due to the accumulation of integration errors.

7. Conclusion

The representation of a physical system in terms of it (generic) invariants is a major perennial problem. Such a representation can lead to insights and practical applications that can not be acquired by other means. In this paper we developed such a representation for the motion of a satellite (or particle) in a central force field. Furthermore we derived equations for the impact of dissipative effects (e.g drag) on these invariants and hence on the trajectory of the particle.

In the current literature there exist two "disjoint" formulations of satellite trajectories. One in terms of its energy invariant and the other in terms of it angular momentum. This paper unifies these two representation into one. We compared numerically this new representation with other formulations to compute the trajectory of a satellite and found that this new formula provides (in many instances) a superior numerical solution.

In the second part of the paper we derived a formula for the relative motion of two satellites moving in the same plane. This formula can be generalized to the case where drag effects have to be taken into account and the orbits of the satellites are not in the same plane using the KS-formalism.

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