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Statistical methods and complex Analysis

Complex variables - Differentiation.

Functions of complex variable, differentiation, Cauchy-Riemann equations, analytical functions, harmonic functions, elementary analytical functions, Conformal mappings.

Complex variables - Integration.

Contour integrals, Caunchy-Goursat theorem, Cauchy integral formula, Liouville's theorem, and Maximum-Modulus theorem, Taylor's series, zeros of analytic functions, singularities, Laurent's series, Residues, Residue theorem and evaluation of real integrals.

Textbook: J.W. Brown & R.V. Churchill
Complex variables & applications

Complex numbers:

Defination: Complex numbers can be defined as ordered pairs (x, y) of real numbers that can be interpreated as a points in the z or complex plane with rectangular coordinate x and y.

We denote the complex number (x, y) by z i.e $z = (x, y)$ where the numbers x & y are known as real and imaginary part of complex no. respectively.

$\therefore z = x + iy$ $\quad Re(z) = x$

$Im(z) = y$

Note: let $z = (x, y)$ be any complex no.

1) if $x = 0$, $y \neq 0$ then $z = iy$ & is called as purely imaginary no.

2) if $x \neq 0$, $y = 0$ then $z = x$ & is called as purely real no.

thus y is imaginary axis and x is real axis.

Equality of complex numbers:

Two complex numbers z_1 & z_2 are equal whenever they have the same real & imaginary part.

let $z_1 = (x_1, y_1)$ $\quad z_2 = (x_2, y_2)$

$z_1 = z_2$ if $x_1 = x_2$ & $y_1 = y_2$.

Thus it means that z_1 & z_2 corresponds to same point in the complex / z plane.

Addition and multiplication of two complex numbers:

Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$
then $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$
The addition of 2 complex no.'s is commutative

$$z_1 + z_2 = z_2 + z_1$$

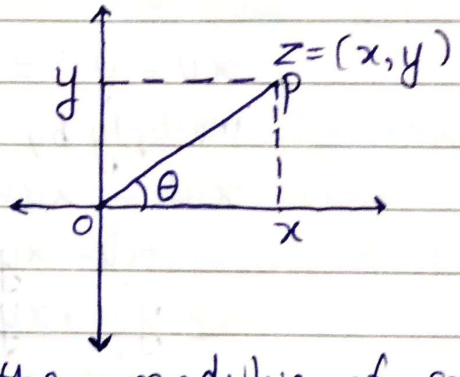
Let $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2,\ y_1 x_2 + y_2 x_2)$
Let $z_1 = (x_1 + iy_1)$ $z_2 = (x_2 + iy_2)$

$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= x_1 x_2 + iy_2 x_1 + iy_1 x_2 + i^2 y_1 y_2$$
$$= x_1 x_2 - y_1 y_2 + i(y_1 x_2 + y_2 x_1)$$

Argand's diagram:

Mathematician Argand's represented a complex no. i[n] a complex plane. which is called Argand's diagra[m]

Then representation of z is:



A complex no. $z = x + iy$ can be represented by a point p. whose coordinates are (x, y)
In fig. the distance OP is the modulus of complex no. z. and the angle θ made by z with +ve real axis is called argument of complex no. z

Additive identity:

Let $z = x + iy$ and $0 = 0 + i0$ then $z + 0 = z$
$\therefore$ 0 is called as additive identity of complex no.

Additive inverse:

let $z = (x, y)$ and $-z = (-x, -y)$ then $z + (-z) =$

$= (x + iy) + (-x - iy)$

$= (x - x) + (iy - iy)$

$z - z = 0$

Multiplicative identity:

let $z = (x, y)$ & $(1, 0) = 1 + 0i$

Now $z \cdot 1 = (x + iy)(1 + 0i) = (x, y)$

1 is multiplicative identity.

Multiplicative inverse:

For any non-zero ~~no.~~ complex no. $z = (x, y)$ there is a number z^{-1} such that $z \cdot z^{-1} = 1$

To find multiplicative inverse i.e $z^{-1} = (u, v)$

$z \cdot z^{-1} = (x, y) \cdot (u, v) = 1$

$\Rightarrow (x + iy) \cdot (u + iv) = 1$

$= (xu - yv, xv + yu) = (1, 0)$

$xu - yv = 1$ — ① $xv + yu = 0$ — ②

multiply by x multiply by y

$x^2u - xyv = x$ $xyv + y^2u = 0$

$x^2u - xyv = x$

$y^2u + xyv = 0$

$x^2u + y^2u = x$

$u(x^2 + y^2) = x$

$$\boxed{u = \frac{x}{x^2 + y^2}}$$

Put u in eq^n ②

$xv + y\left(\frac{x}{x^2 + y^2}\right) = 0$

There is one to one corresponding betn $\mathbb{R}^2$ and $\mathbb{C}$
(x,y) (x,y)

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$$xv = \frac{-xy}{x^2+y^2}$$

$$\boxed{v = \frac{-y}{x^2+y^2}}$$

$$\therefore z^{-1} = (u,v) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

where $z \neq (0,0)$
The inverse z^{-1} for complex no. $z=(0,0)$ is not defined.

Prove that the set of complex number $\mathbb{C}$ is a field:

Let $\mathbb{C} = \{z = x+iy \mid x,y \in \mathbb{R}\}$

1) If $z_1, z_2 \in \mathbb{C}$ then $z_1+z_2 \in \mathbb{C}$ → closed property
2) $z_1+z_2 = z_2+z_1$ → Commutative
3) $z_1+(z_2+z_3) = (z_1+z_2)+z_3$ → Associative.
4) $z+0 = z$ → Additive identity
5) $z \in \mathbb{C}$ $\exists$ $(-z) = (-x,-y)$
$z+(-z) = (-z)+z = 0$ → Additive inverse.

6) $z_1, z_2 \in \mathbb{C}$ then $z_1 \cdot z_2 \in \mathbb{C}$ → closed property
7) $z_1 \cdot z_2 = z_2 \cdot z_1$ → commutative
8) $z_1(z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$ → Associative
9) $z_1(z_2+z_3) = z_1z_2 + z_1z_3$ → Distributative
10) $z \cdot 1 = z$ → Multiplicative identity
11) For $z \neq 0$ $\exists$ an element $z^{-1} \in \mathbb{C}$
$z \cdot z^{-1} = z^{-1}z = 1$ → Multiplicative inverse

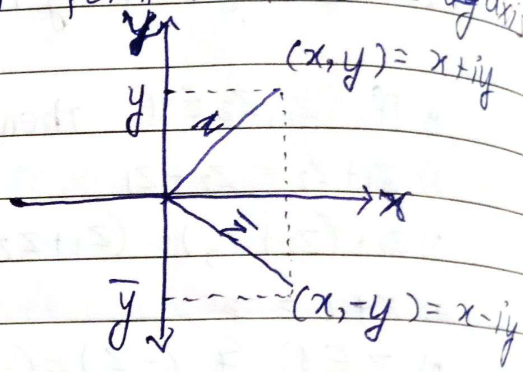
Conjugate of a complex no. z

If $z = x+iy = (x,y)$ is any complex no. then the conjugate of z is denoted by $\bar{z}$ and is defined as $\bar{z} = x-iy$

$$\Rightarrow \overline{(\bar{z})} = x+iy = z$$

Note: Replace i by $-i$ to find conjugate of any complex no. z and vice-versa.

Thus $\bar{z}$ is the reflection of point z about x axis as shown in fig.



Properties of conjugate of z.

1) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

in general:

$$\overline{z_1+z_2+z_3+\dots+z_n} = \bar{z}_1+\bar{z}_2+\bar{z}_3+\dots+\bar{z}_n$$

2) $\overline{z_1 \cdot z_2 \cdot z_3 \cdots z_n} = \bar{z}_1 \cdot \bar{z}_2 \cdot \bar{z}_3 \cdots \bar{z}_n$

3) $z \cdot \bar{z} = |z|^2$

4) $z+\bar{z} = 2\,\mathrm{Re}(z)$

5) $z-\bar{z} = 2\,\mathrm{Im}(z)$

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Modulus of complex no. z

Let $z = x+iy$ be any complex no. then modulus of z is the magnitude of vector z.

Therefore modulus of z is denoted by $|z|$

$$|z| = \sqrt{x^2+y^2}$$

Properties of mod z / |z|

1) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

in general:

$$|z_1 \cdot z_2 \cdot \ldots \cdot z_n| = |z_1| \cdot |z_2| \cdot \ldots \cdot |z_n|$$

2) $|z^2| = |z|^2$

in general:

$$|z^n| = |z|^n$$

3) Triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

in general:

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots |z_n|$$

4) $|z_1 - z_2| \geq |z_1| - |z_2|$

in general:

$$|z_1 - z_2 - \ldots - z_n| \geq |z_1| - |z_2| - \ldots - |z_n|$$

5) $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad z_2 \neq 0$$

Note: 1) If $z = x+iy$ then $Re(z) = x$ and $Im(z) = y$.

We know that $x^2 \geq 0$ and $y^2 \geq 0$

$$\Rightarrow x^2 \leq x^2+y^2$$

$$y^2 \leq x^2+y^2$$

Taking root on both side

$$|x| \leq \sqrt{x^2+y^2} = |z|$$

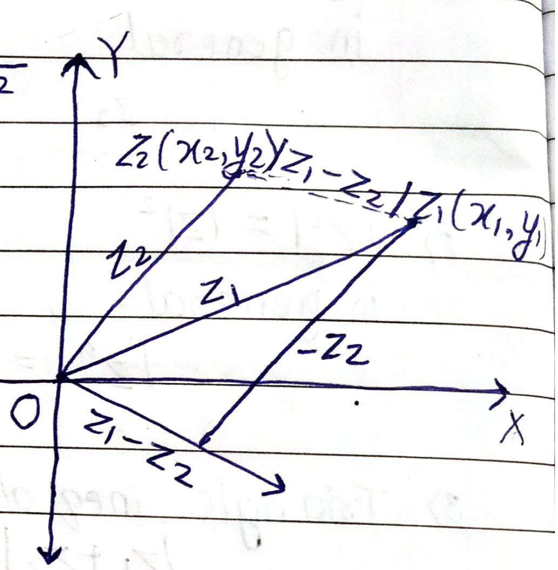
$$|y| \leq \sqrt{x^2+y^2} = |z|$$

we have: $|x| \leq |z|$

$|y| \leq |z|$

2) Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ then distance betn z_1 and z_2 is (x_1-x_2, y_1-y_2)

$$|z_1-z_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$



The complex no.'s z corresponding to the points lying on the circle with center z_0 and radius R satisfies the equation $|z-z_0| = R$ and we say that the set of all points is a circle of radius R which is centered at z_0

e.g: 1) The equation $|z-1+3i| = 2$ is representing the circle of radius 2. and centered as $(1,-3)$

3) We know that $|z|^2 = \cancel{\sqrt{x^2+y}}\ x^2+y^2$

$$= [Re(z)]^2 + [Im(z)]^2$$

$$\therefore (Re(z))^2 \leq |z|^2$$

$$(Im(z))^2 \leq |z|^2$$

$$Re(z) \leq |Re(z)| \leq |z|$$

$$Im(z) \leq |Im(z)| \leq |z|$$

Prove that $|z_1+z_2| \leq |z_1|+|z_2|$

Proof: In a triangle

let $|z_1+z_2|^2 = (z_1+z_2)\cdot\overline{(z_1+z_2)}$

$= (z_1+z_2)\cdot(\overline{z_1}+\overline{z_2})$

$= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}$

$= |z_1|^2 + \overline{\overline{z_1}\cdot z_2} + z_2\overline{z_1} + |z_2|^2$

$= |z_1|^2 + \overline{\overline{z_1}\cdot z_2} + \overline{z_1}\cdot z_2 + |z_2|^2$

$= |z_1|^2 + 2Re(\overline{z_1}\cdot z_2) + |z_2|^2$

$\leq |z_1|^2 + 2|\overline{z_1}||z_2| + |z_2|^2$ —— $x \leq |x| \leq |z|$

$\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2$ —— $|\overline{z_1}| = |z_1|$

$\leq (|z_1|+|z_2|)^2$

Taking root:

$|z_1+z_2| \leq |z_1|+|z_2|$

Properties arising from triangle inequality:

Q Prove that $|z_1+z_2| \geq ||z_1|-|z_2||$

$|z_1| = |(z_1+z_2) - z_2|$

$\leq |(z_1+z_2)+(-z_2)|$

$\leq |z_1+z_2| + |z_2|$

$|z_1|-|z_2| \leq |z_1+z_2|$

$|z_1+z_2| \geq |z_1|-|z_2|$ —— 1

By interchanging z_1 & z_2 we get:

$|z_1+z_2| \geq |z_2|-|z_1|$

$|z_1+z_2| \geq -(|z_1|-|z_2|)$ —— 2

from 1 & 2

we have:

A2 $|z_1+z_2| \geq ||z_1|-|z_2||$

J6 term

the

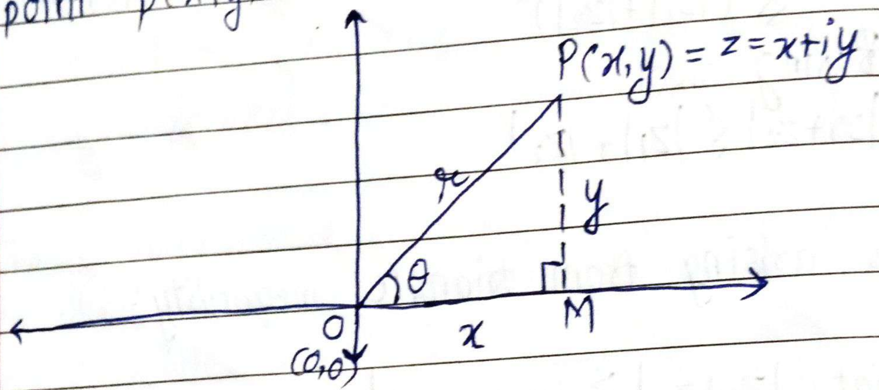
on left hand sign is smaller square which is ≥ 0

so it is true ∴ $\sqrt{2}|z| \geq |Re z| + |Im z|$

Representation of complex no. in polar form:

- We know that every complex no. z can be represented by a two directional vector. Suppose the magnitude of complex no. $z = x+iy$ is $|z| = r$ $= \sqrt{x^2+y^2}$

Suppose θ is the angle made by the vector z with +ve real axis, then polar coordinates of the point $p(x,y)$ are (r,θ) as shown in fig.



z-plane.

In ΔOPM,

$$\angle PMO = 90^\circ$$
$$\angle POM = \theta$$

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

$$\boxed{y = r\sin\theta} \qquad \boxed{x = r\cos\theta}$$

$$\therefore z = r\cos\theta + ir\sin\theta$$

$$\boxed{z = r[\cos\theta + i\sin\theta]} \rightarrow \text{polar form}$$

Let us denote:

$$\cos\theta + i\sin\theta = e^{i\theta}$$

$$\therefore \boxed{z = re^{i\theta}} \longrightarrow \text{exponential form.}$$

Argument of complex no. z:

The argument of complex no. z is the angle made by the vector z with +ve x-axis.

Let $z = x+iy = r(\cos\theta + i\sin\theta)$

where $\frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

where θ is called as an argument of z.
The quadrant containing the point (x,y) i.e the point corresponding to complex no. z must be specified.

We know that ~~cos2π~~ $\cos(\theta+2\pi) = \cos\theta$

$\sin(\theta+2\pi) = \sin\theta$

$\therefore z = r[\cos(\theta+2n\pi) + i\sin(\theta+2n\pi)]$

$\therefore$ arg. of $z = \theta + 2n\pi$, $n = 0, \pm1, ---$

Hence for every complex no. z, arg. of z has an infinite no. of values.

Note: The value of arg. z is unique for $-\pi < \underset{(\theta)}{\arg. z} \leq \pi$.
So this θ is called as principle argument of complex no. z.

We denote the principle arg. of z by Arg z.

e.g:1) Find the value of arg z and principal argument of z.

$z = 1+i$

$\mathrm{Arg}\, z = 45^\circ = \pi/4$

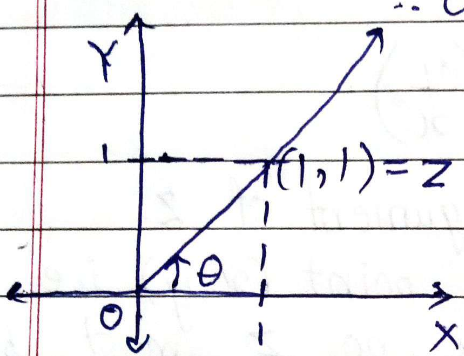
we have $x=1$, $y=1$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\theta = \tan^{-1}(1) = \pi/4$

As $-\pi < \pi/4 \leq \pi$

$\therefore \theta = \pi/4$ is a Principle arg. of z

$\mathrm{Arg}\, z = \pi/4$



$\arg z = \frac{\pi}{4} + 2n\pi$, $n = 0, \pm 1, \ldots$

2) $z = -1+i$

comparing with $z = x+iy$

$x = -1$, $y = 1$

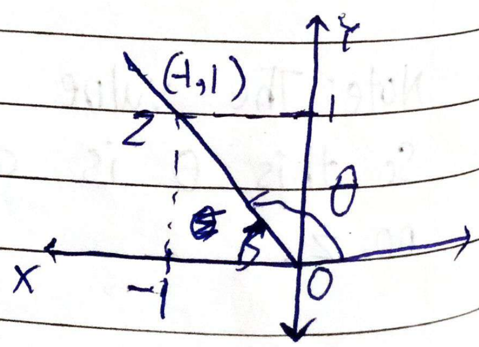
$\therefore \theta = \tan^{-1}\left(\frac{1}{-1}\right)$

The given point lies in 2^{nd} quadrant

So that $\arg z = \theta' + \pi/2$

where $\theta' = \tan^{-1}\left(\frac{y}{|x|}\right)$

$\therefore \theta' = \tan^{-1}(1/1) = \pi/4$



$\arg z = \theta' + \pi/2 = \pi/4 + \pi/2 = \frac{3\pi}{4}$

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As $-\pi < 3\pi/4 < \pi \Rightarrow \operatorname{Arg} z = 3\pi/4$

$$\arg z = 3\pi/4 + 2n\pi \quad , n = 0, \pm 1, ---$$

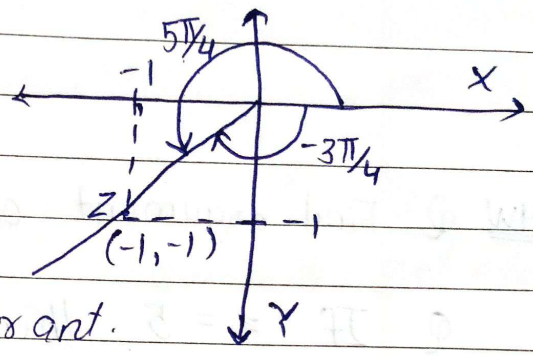
3) $z = -1 - i$

comparing with $z = x + iy$

$x = -1$, $y = -1$

$$\theta' = \tan^{-1}\left(\frac{|y|}{|x|}\right)$$

$$\theta' = \pi/4$$



The given point lies in 3^{rd} quadrant.

$$\therefore \arg z = \theta' + \pi$$
$$= \pi/4 + \pi$$
$$= 5\pi/4$$

$$\arg z = 5\pi/4 + 2n\pi \quad , n = 0, \pm 1, ---$$

if $n = -1$, then $\arg z = \frac{5\pi}{4} - 2\pi = -3\pi/4$

As, $-\pi < -3\pi/4 < \pi$

$$\therefore \operatorname{Arg} z = -3\pi/4$$

4) $z = 1 - i$

comparing with $z = x + iy$

$x = 1$, $y = -1$

$$\theta' = \tan^{-1}\left(\frac{|y|}{\cancel{x}}\right) = \pi/4$$

If we take z in polar form then $z \neq 0$ thus $\theta =$ not defined
we can't find argument for $z=0$

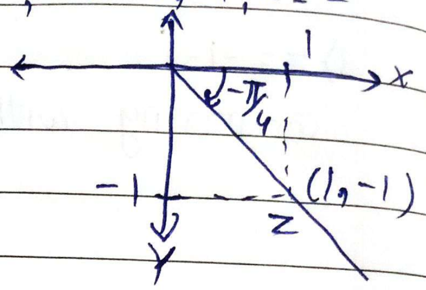
The given point lies in 4^{th} quadrant.

$\therefore \arg z = -\theta = -\pi/4$

$\arg z = -\pi/4 + 2n\pi$, $n = 0, \pm 1, \ldots$

As, $-\pi < -\pi/4 < \pi$

$\therefore \operatorname{Arg} z = -\pi/4$



HW Q Find argument of z where $z = \pm \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Q If $z = 5$ then principle argument is :
$z = -i$

Note: 1) The argument of purely real +ve complex no. is zero.

2) The argument of purely real -ve complex no is π.

3) The argument of purely imaginary +ve complex no. is $\pi/2$

4) The argument of purely imaginary -ve complex no. is $-\pi/2$

De-moivre's formula:

- Let $z = x+iy$ then polar form representation of z is
$z = r(\cos\theta + i\sin\theta)$, $r = |z|$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Let $z_1 = r_1 \cdot (\cos\theta_1 + i\sin\theta_1)$
$z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$

4 !
15 i
26 -1
37 $-i$

then $z_1 \cdot z_2 = r_1 \cdot r_2 [(\cos\theta_1 + i\sin\theta_1) \cdot (\cos\theta_2 + i\sin\theta_2)]$

$= r_1 \cdot r_2 [\cos\theta_1 \cdot \cos\theta_2 + i\cos\theta_1 \cdot \sin\theta_2 + i\sin\theta_1 \cdot \cos\theta_2 + i^2 \sin\theta_1 \sin\theta_2$

$= r_1 r_2 [\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2 + i(\cos\theta_1 \cdot \sin\theta_2 + \sin\theta_1 \cdot \cos\theta_2)]$

$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

where $|z_1 \cdot z_2| = r_1 \cdot r_2$

$\arg(z_1 \cdot z_2) = \theta_1 + \theta_2$

- Thus if we multiply any two complex no.'s then their moduli gets multiplied and arguments get added.

- In general if we have n complex no's as

$z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, - - -, $z_n = r_n e^{i\theta_n}$

then,

$z_1 \cdot z_2 \cdot z_3 \cdots z_n = (r_1 \cdot r_2 \cdots r_n) \cdot e^{i(\theta_1 + \theta_2 + \cdots + \theta_n)}$

$z_1 \cdot z_2 \cdot z_3 \cdots z_n = (r_1 \cdot r_2 \cdots r_n) \cdot [\cos(\theta_1 + \theta_2 + \cdots \theta_n) + i\sin(\theta_1 + \theta_2 \cdots$

If $z_1 = z_2 = \cdots = z_n$

where $z = r(\cos\theta + i\sin\theta)$

$\therefore z^n = r^n [\cos n\theta + i\sin n\theta]$ — (1)

Now: $z = r(\cos\theta + i\sin\theta)$

Taking n power on both side

$z^n = r^n (\cos\theta + i\sin\theta)^n$ — (2)

comparing eq^n (1) & (2) we get

$$\boxed{\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n}$$

This result is known as De-moivre's formula.

$$\boxed{(e^{i\theta})^n = e^{in\theta}}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$\frac{1}{z} = z^{-1} = \frac{1}{r} e^{-i\theta}$$

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Let $z_1 = -1$, $z_2 = i$ then find principal argument of $z_1 z_2$
$arg(z_1) = \pi$
$arg(z_2) = \pi/2$

$arg(z_1 \cdot z_2) = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$

Principal argument $z_1 \cdot z_2 = -\frac{\pi}{2}$

$z^n = r^n e^{in\theta}$

e.g.) Find the principal argument of $z = \frac{-2}{1+\sqrt{3}i}$

Let $z = \frac{-2}{1+\sqrt{3}i}$ comparing with $z = \frac{z_1}{z_2}$

Then, $z_1 = -2$, $z_2 = 1+\sqrt{3}i$

$arg\ z = arg\ z_1 - arg\ z_2$
$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

As $-\pi < \frac{2\pi}{3} < \pi$ $\therefore$ $Arg(z) = \frac{2\pi}{3} = Arg\left(\frac{-2}{1+\sqrt{3}i}\right)$

2) $z = (\sqrt{3} - i)^6$
By demoiver's thm
$z^n = r^n(\cos\theta + i\sin\theta)^n = r^n[\cos n\theta + i\sin n\theta]$
$z^n = r^n(e^{i\theta})^n = r^n e^{in\theta}$
Let $z = re^{i\theta} = \sqrt{3} - i$
so $arg.\ (\sqrt{3}-i)^6 = 6(arg(\sqrt{3}-i))$ — ①
Now, $arg(\sqrt{3}-i) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

~~$Arg(\sqrt{3}-i) = -\frac{\pi}{6}$~~

$-\pi < \theta \leq \pi$

from eqⁿ ①

$$\arg(\sqrt{3}-i)^6 = 6(-\pi/6) = -\pi$$

So ~~the arg~~ $-\pi$ is one of the arg. of $(\sqrt{3}-i)^6$

$\therefore -\pi + 2\pi = \pi$ is the principal arg. of $z = (\sqrt{3}-i)^6$

HW 3) Use the demoivre's formula to establish the following trigonometric identities:

a) $\cos 3\theta = \cos^3\theta - 3\cos\theta . \sin^2\theta$

b) $\sin 3\theta = 3\cos^2\theta . \sin\theta - \sin^3\theta$

Roots of complex no:

- If $z_0 = x_0 + iy_0 = r_0 e^{i\theta_0}$ is a complex no. then we assume $z = re^{i\theta}$ is the n^{th} root of z_0

Then now to find $z = re^{i\theta}$ such that $z^n = z_0$

$z = (z_0)^{1/n}$

$$r^n e^{in\theta} = r_0 \cdot e^{i\theta_0}$$

On comparing.

$$r^n = r_0 \qquad n\theta = \theta_0$$

$$r = \sqrt[n]{r_0} \qquad \theta = \frac{\theta_0}{n}$$

but we know that,

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}, \quad k = 0, \pm 1, ---$$

$$z = \sqrt[n]{r_0} \left(e^{i(\theta_0/n + 2k\pi/n)}\right)$$

This is the n^{th} root

$$z = \sqrt[n]{r} \exp\left(i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right) \text{ where } k = 0, \pm 1, --$$

$n = 2, 3, \cdots$

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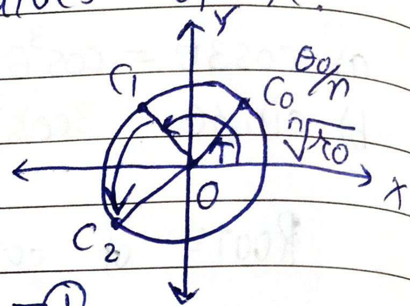
~~for~~ the n^{th} roots lie on the circle $|z| = \sqrt[n]{r_0}$ about the origin and are equally spaced by $2\pi/n$ radian

* Starting with the first argument $\frac{\theta_0}{n}$

Now all the distinct roots are obtained when $k = 0, 1, \ldots n$

and no further roots for another values of k.



Now we denote all the distinct roots by c_k where $k = 0, 1, \ldots n-1$

$$c_k = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, 2, \ldots n-1 \quad -(1)$$

Note: In expression (1) the value of θ_0 is taken as principal value of $\arg(z_0)$ $(-\pi < \theta_0 < \pi)$ and the root c_0 is called as principal root of z_0

$\sqrt{x^2+y^2}$

Find all the values of $z = (-8i)^{1/3}$ or find all the 3 cuberoots of $-8i$

Let $z = -8i$ $\therefore |z| = 8$ $\quad \sqrt[3]{8}\, \exp i[$

$\arg z = -\pi/2$

$\theta_0 = \operatorname{Arg} z = -\pi/2$ $\quad \left(\frac{y}{x}\right)$

$$\therefore c_k = (8)^{1/3} \cdot e^{i\left(\frac{-\pi/2}{3} + \frac{2k\pi}{3}\right)}, \quad k = 0, 1, 2$$

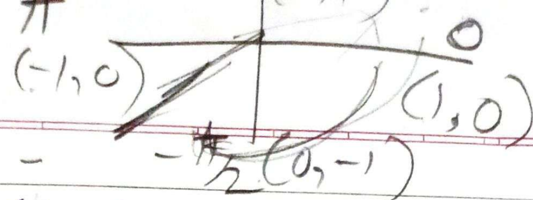
$= 2 \cdot e^{i(-\pi/6)}$ (for $k=0$)

$= 2 \cdot e^{i(-\pi/6 + 2\pi/3)}$ (for $k=1$)

$= 2 \cdot e^{i(-\pi/18)}$

$= 2 \cdot e^{i(-\pi/6 + 4\pi/3)}$ (for $k=2$)

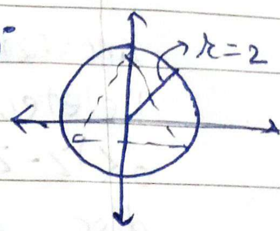
$= 2 \cdot e^{i(\pi/18)}$



for $k=0$;

$$c_0 = 2\cdot\exp(-i\pi/6) = 2[\cos\pi/6 - i\sin(\pi/6)]$$

$$c_0 = 2\left[\frac{\sqrt{3}}{2} - \frac{i}{2}\right] \Rightarrow \sqrt{3}-i$$



for $k=1$;

$$c_1 = 2\cdot\exp\left[i\left(\frac{-\pi+4\pi}{6}\right)\right] = 2\exp\left[i\frac{\pi}{2}\right]$$

$$= 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = 2i$$

$k=2$, $c_2 = 2\cdot\exp\left[i\left(-\frac{\pi}{6}+\frac{4\pi}{3}\right)\right] = 2\exp\left[i\left(\frac{-\pi+8\pi}{6}\right)\right]$

$$= 2\left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right] = 2\left[\frac{-1}{2}i + \frac{-\sqrt{3}}{2}\right] \quad \sqrt[n]{r}\left[\exp\left(\frac{\theta_0}{n}+\frac{2k\pi}{n}\right)\right]$$

$$= -\sqrt{3} - i$$

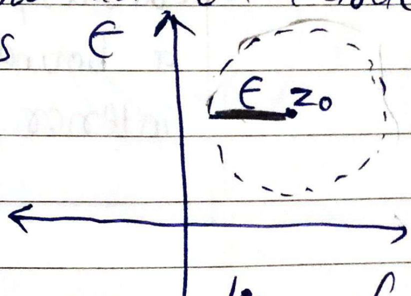
$$\tan^{-1}\left(\frac{|y|}{|x|}\right) \Rightarrow$$

Find $z = (\sqrt{3}+i)^{1/2}$

Regions in a complex plane:

- ϵ neighbourhood of given point z_0

An ϵ neighbourhood of point z_0 in complex plane is consist of all points z lying inside but not on a circle centered at z_0 and with a +ve radius ϵ

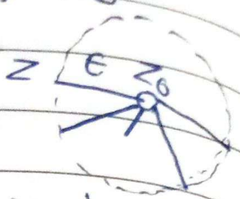


$$D_\epsilon(z_0) = \{z \,/\, |z-z_0| < \epsilon\}$$

An ϵ neighbourhood is also called as open disc of radius epsilon (ϵ)

→ If we exclude the point z_0 from open disc $|z-z_0| < \epsilon$ then it is called as deleted neighbourhood of z_0 or ~~punch~~ punctured disc at z_0

Punchered disc is written an $0 < |z - z_0| < \epsilon$

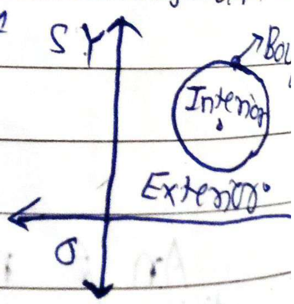


- ~~Interior points~~
Deleted neighbourhood of z_0 is $0 < |z - z_0| < \epsilon$ is consisting of points z in an open disc of z_0 except z_0

- Interior points :- A point z_0 said to be an interior point of set 'S', $S \subseteq C$ whenever a some neighbourhood z_0 contains only points in 'S'
(or) $S = \{ z \in C / |z| < 1 \}$
A point z_0 said to be an interior point of set S if there exist an ϵ neighbourhood of z_0 which contains only points of S

- Exterior points :- A point z_0 is said to be an exterior point of set S, $S \subseteq C$, if there exist a neighbourhood of z_0 that contains no points of S
i.e $D_\epsilon(z_0) \subseteq S^c$ then z_0 is called as exterior point of S.

boundary points : A point z_0 belong to C is said to be a boundary point of S, $S \subseteq C$, if it is neither an interior point nor an exterior point of S



e.g: Let the circle $|z| = 1$ is the boundary to the set $|z| < 1$ & $|z| \leq 1$

→ Open set - A set ~~none~~ none of certain boundary point
close set : contain all boundary point

Closed set:

- A set is closed it contains all of its boundary points is the closed set containing all of its boundary points together with all points of its boundary points & is denoted by $\overline{S}$

i.e: $\overline{S} = S \cup S'$

S' is collection of all boundary of S.

Note: 1) There are some sets which are neither open or closed e.g: punchered disc : $0 < |z| \leq 1$

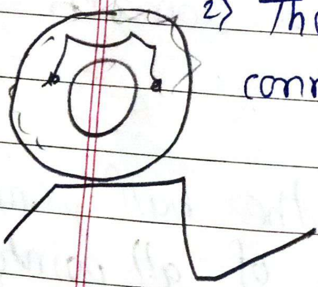
2) The set of complex no. C is both open & closed. since it has no boundary point.

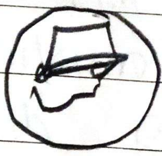
→ Connected set : A set is said to be connected set. A open set S is said to be a connected set if

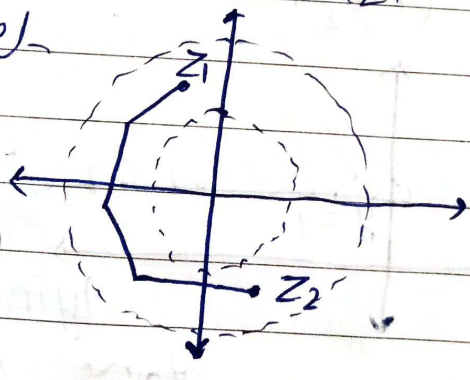
- ~~for~~ each pair of points z_1 & z_2 in it can be joined by a polygonal line which is consisting of ~~finite~~ finite no. of segments joined end to end & that lies entirely in S

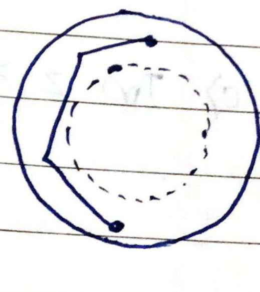
e.g: 1) The open set $|z| < 1$ is connected set & opened.

2) The annulus between $|z|$ i.e $1 < |z| < 2$ is also connected set.









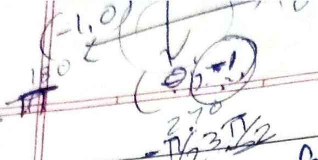
$y = f(x)$

Domain

- Domain: A non-empty open set that is connected is called as Domain.

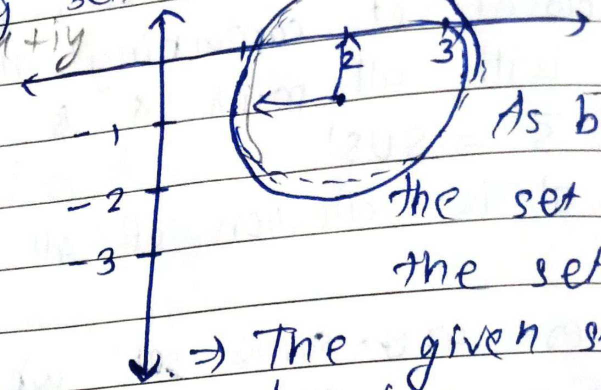
domain

- eg: Any neighbourhood is a domain



Sketch the following sets & determine which are domain

$z = x+iy$



a) $|z-(2+i)| \leq 1$

$|z-z_0| \leq r$

$z_0 = (2,-1)$

$r = 1$

As boundary of the set included in the set.

$\Rightarrow$ The given set is not a domain.

$|(x,y)-(2,-1)| \leq 1 \Rightarrow |(x-2, y-1)| \leq 1$

$\sqrt{(x-2)^2+(y+1)^2} \leq 1$

$(x-2)^2+(y-(-1))^2 \leq 1$

$z_0 = (2,-1)$

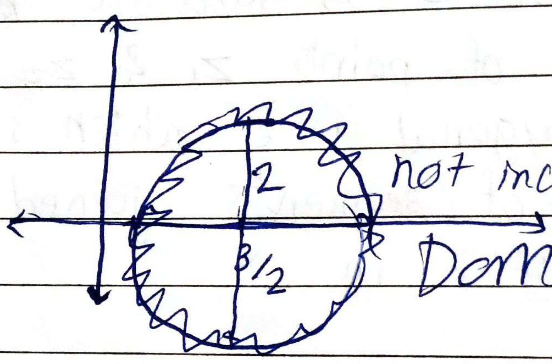
cen

$|z-z_0| =$

cen N

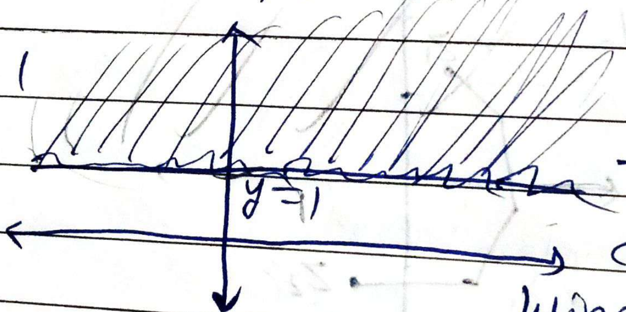
b) $|2z+3| > 4$

$|z+3/2| > 2$



It contains all points exterior ~~points~~ of the circle centered at $(+3/2, 0)$ thus it is a domain.

c) $\operatorname{Im} z > 1$



domain.

- This is the half plane consisting of all points lying above the horizontal ~~plane~~ line $y=1$ & thus domain

d) $\operatorname{Im} z = 1 \Rightarrow$ The set is simply horizontal line $y=1$ & it is not a open set thus it is not a domain

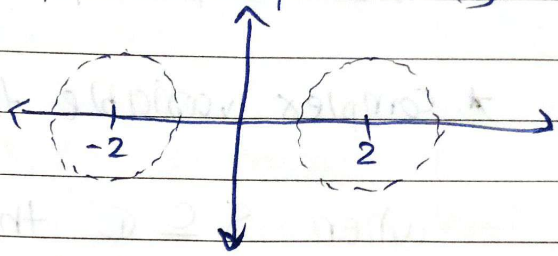
1) $0 \leq \arg z \leq \pi/4 \quad (z \neq 0)$

2) $|z-4| \geq |z|$

2) $0 \leq \arg \leq \pi/4$

Q Let the set $B = \{z \in \mathbb{C} \mid |z+2|<1$, or, $z \in \mathbb{C} \mid |z-2|<1\}$

→ So graphically the set B is:
is not a connected set



- Bounded set: A set S is bounded if every point of S lies inside a circle $|z|=r$ otherwise it is unbounded.

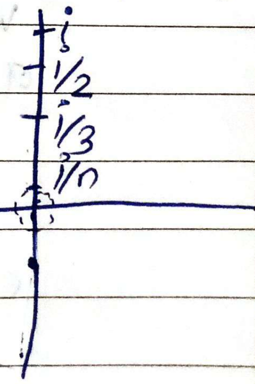
e.g $|z|<1$, $|z| \leq 1$ are bounded sets
~~Im~~ $Im(z) \geq 0$ is unbounded set.

Accumulation points: The point z_0 is said to be accumulation point of set S if each deleted neighbourhood of z_0 contains atleast one point of S

- If set S is closed then it contains each of its accumulation points.
(or) a set is closed if and only if it contains all of accumulation points.

Let $z_n = \frac{i}{n}$ $\{n=1,2,3,\ldots\}$ then $z_n = \{i, i/2, i/3, i/4, \ldots i/n\}$

The deleted neighbourhood of point i doesn't contain any point of set z_n
∴ point i is not an accumulation point for set z_n



Similarly all points of set z_n are not the accumulation point of z_n, but origin is the only accumulation point because every deleted neighbourhood of origin contains points of S

It can be possible that z_0 is accumulation point of S but $z_0 \notin S$.

* complex variable function

when $S \subseteq \mathbb{C}$ then:
A function f is defined on S is a rule that assigns to each z in S a complex no. w. The no. w is called as the value of function f at z and denoted as f(z)
$\therefore f(z) = w$ and the set S is called as domain of f.

e.g: $f(z) = \frac{1}{z}$, where $z \neq 0$

$f(z) = \omega$ $\qquad \omega = \frac{1}{z}$ $\qquad z = re^{i\theta}$

$\omega = u + iv$

$u + iv = \frac{1}{x+iy}$

The real no.'s u and v depends on real no. x & y
$\therefore$ f(z) can be expressed as ~~f(z)~~ a pair of real valued functions u and v.

as $f(z) = u(x,y) + iv(x,y)$
$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ $\longrightarrow$ polar form representation

e.g: if $f(z) = z^2$ then $\omega = z^2$
$(u+iv) = (x+iy)^2$
$u+iv = x^2 - y^2 + i2xy$
$u = x^2 - y^2$, $v = 2xy$

$\therefore u(x,y) = x^2 - y^2$ & $v(x,y) = 2xy$

for polar coordinates (r,θ):

$$f(re^{i\theta}) = (re^{i\theta})^2$$

$$u(r,\theta) + iv(r,\theta) = r^2 e^{i2\theta}$$

$$= r^2[\cos 2\theta + i\sin 2\theta]$$

$$= r^2\cos 2\theta + ir^2\sin 2\theta$$

$u(r,\theta) = r^2\cos 2\theta$, $v(r,\theta) = r^2\sin 2\theta$

If $v=0$, then the function is called as real valued function of complex variable.

* The polynomial of degree n.

If n is zero or positive integer and if $a_0, a_1, \ldots a_n$ are complex constants where $a_n \neq 0$. then the function $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots a_n z^n$. is a polynomial of degree n.

The domain of polynomial function is all $\mathbb{C}$.

* Rational functions

The quotient $\frac{P(z)}{Q(z)}$ is called a rational function for all points, where $Q(z) \neq 0$

Finding domains of functions

e.g. $f(z) = \frac{1}{z^2+1}$

Let $f(z) = \frac{1}{z^2+1}$, it is decimal when $z^2+1 \neq 0$

$z^2 \neq -1$

$z \neq \pm i$

Domain of $f(z) = \{z \in \mathbb{C} / z \neq \pm i\}$

$D = \{z \in \mathbb{C} / z \neq \pm i\}$

1) $f(z) = Arg\left(\frac{1}{z}\right)$ $D = \{ z \in \mathbb{C} / z \neq 0 \}$

2) $f(z) = \frac{z}{z+\bar{z}}$ $D = \{ z \in C / z+\bar{z} \neq 0 \}$
$D = \{ z \in C / Re z \neq 0 \}$

Mapping:
Let the function is form $\mathbb{C}$ to $\mathbb{C}$ defined by $f(z)$ where $z = x+iy$ and $w = u+iv$ is called as complex variable function.

If we thought a function f as a two separate planes for its domain and range the it is called as mapping (or) transformation.

Let f is function from $C \to C$ defined as $f(z)=$ where w is the image of z under f

Now similarly the inverse image of a point w is the set of all points z in domain of function that have w as their image.

The inverse image of point may just contain one point, many points or no points.

Types of mapping: 1) Translation mapping
2) Reflection mapping
3) Rotation mapping.

1) Translation mapping: Let $f(z) = w$ is the function defined on $\mathbb{C}$ ~~then~~ if the function $f(z)=$ then this function is called translation mapping

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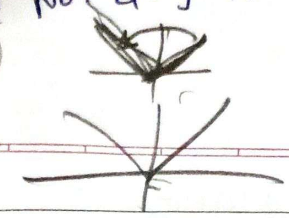
$re^{i\theta}$ $\frac{\pi}{2}$ $\frac{3\pi}{4}$



Translation
Rotational
Reflection

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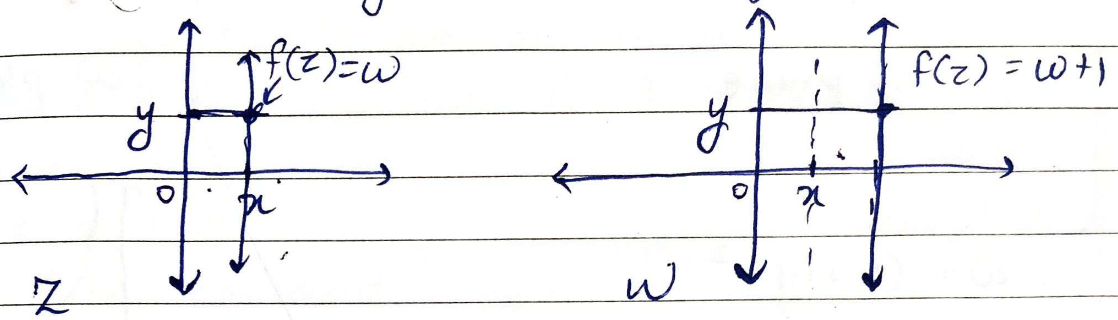
by a magnitude b.

e.g. Under the translation map the set in the w plane has same size, shape but it is translated or displaced by the magnitude b.

e.g: $w = z+1 = x+iy+1 = (x+1)+iy$.



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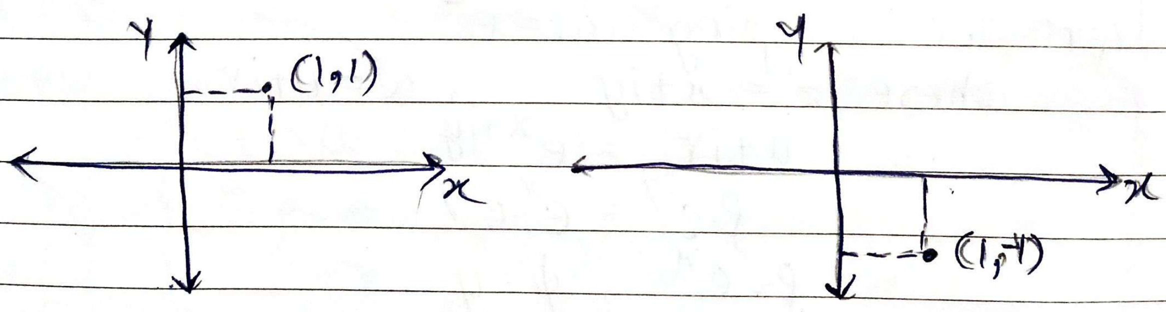
3. Reflection Map:

The mapping $w=\bar{z}$ is called as reflection map which transforms $z = x+iy$ to be $w = x-iy$.

i.e. each point z transforms into it's reflection about real axis.



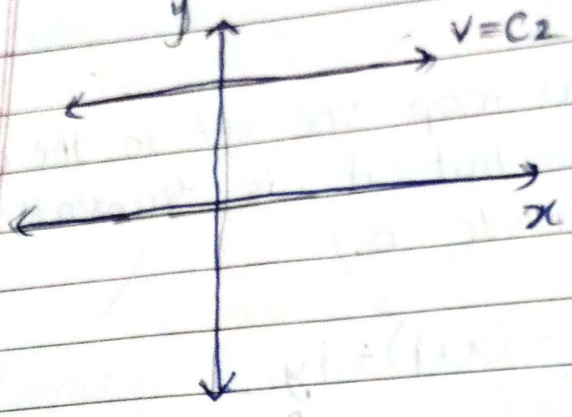
z-plane   w plane

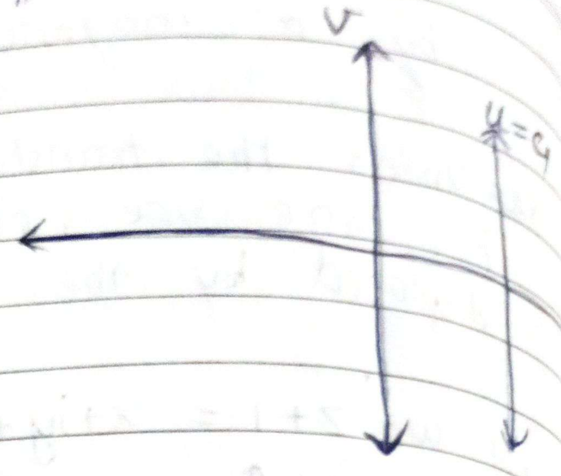
For eg.,

Let $w = z^2$ & $w = (x+iy)^2$ --- $z = x+iy$

Let $x^2-y^2 = c_1$ ---- $(c_1 > 0)$

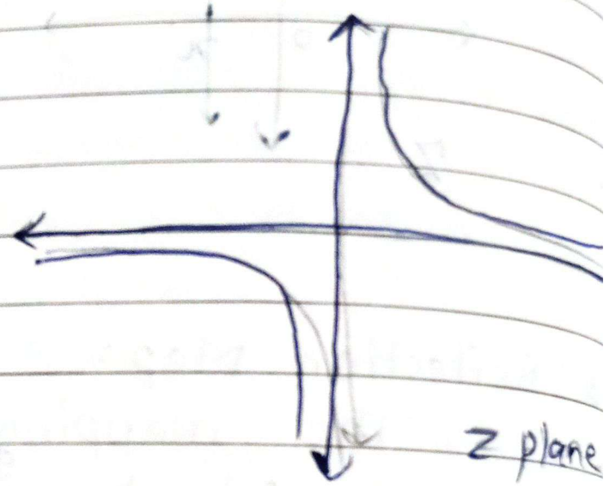
Now its easy to observe that the image of hyperbola $x^2-y^2 = c_1$ is mapped in one to one onto the line $u = c_1$ in uv plane





ω plane

ω-plane



$$\therefore \omega = (x+iy)^2$$
$$= \underbrace{x^2 - y^2}_{u} + i\underbrace{(2xy)}_{v}$$

z plane

Mappings by the exponential functions

Let the mapping $\omega = e^z$ $\qquad z = x+iy$

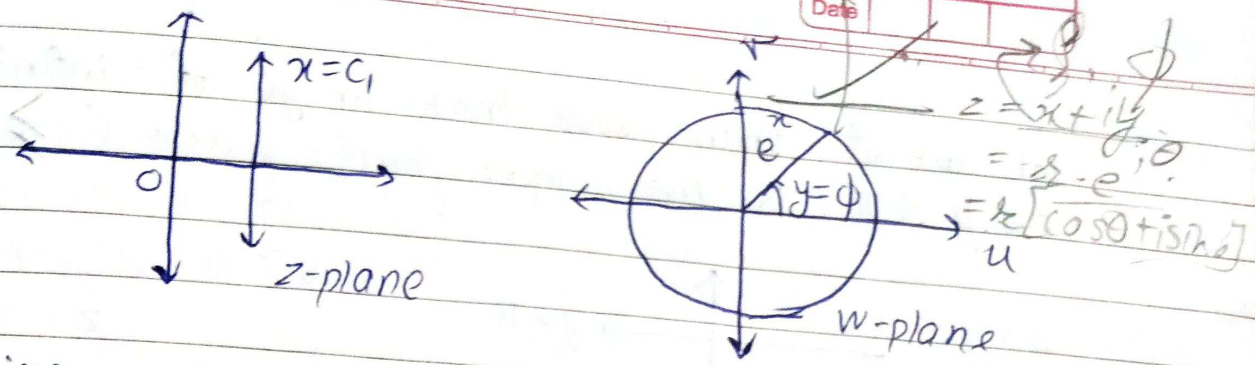
where $z = x+iy$, $\omega = u+iv$ $\qquad u+iy$

$$u+iv = e^{x+iy}$$
$$\rho \cdot e^{i\phi} = e^x \cdot e^{iy}$$
$$\rho = e^x, \quad \phi = y.$$

$$\rho = e^x$$
$$\phi = y$$

The image of a typical point $z(c_1, y)$ where x is constant and on a ~~...~~ vertical line where $x = c$ and has a polar coordinates $\rho = e^{c_1}$, $\phi = y$

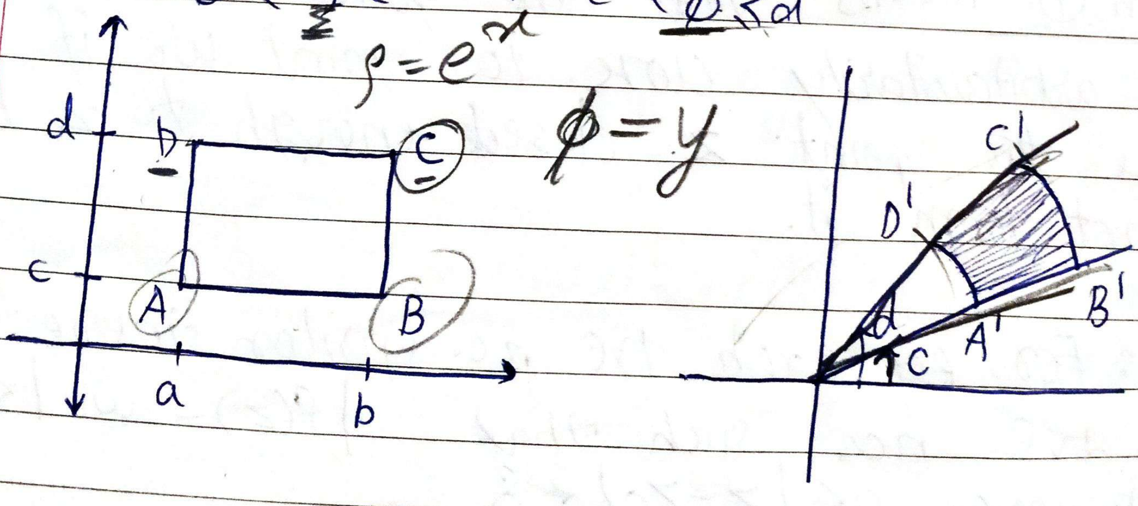
∴ The image



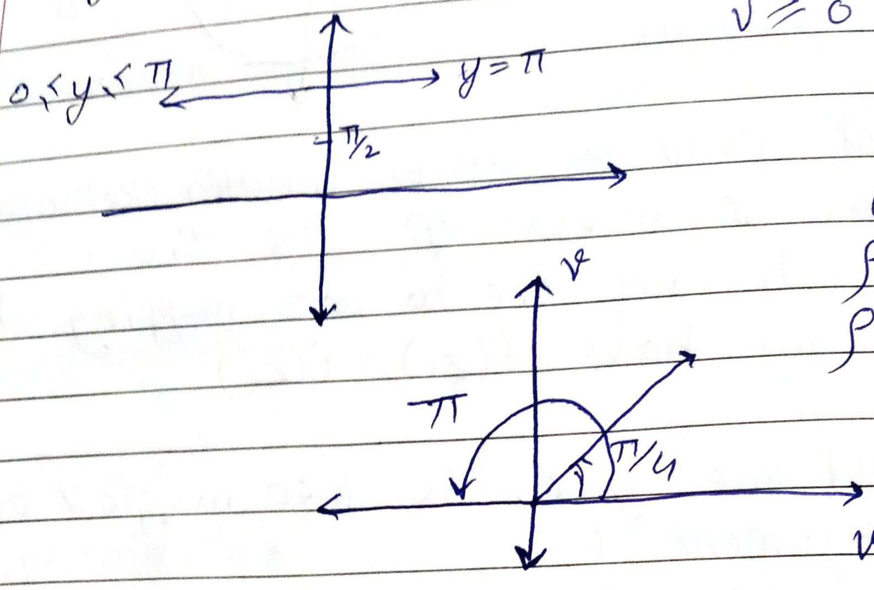
The image of point z moves counter clockwise around the circle as z moves up the line
This mapping is not one to one mapping because two $z_1 \neq z_2$ we have $f(z_1) = f(z_2)$

let a horizontal line $y = c_2$ is then mapped on a point which has argument $\rho = c_2$
$\therefore$ the image of point $z = (x, c_1)$ is ~~as shown~~ the point $w = (e^x, c_2)$ as shown in fig.
As the point z moves along the entire line from left to right its image moves outward on the infinite ray $\phi = c_2$
The vertical & horizontal lines in z plane mapped onto portion of circles & planes.

shows that the transformation $w = e^z$ maps the rectangle region. where $a \leq x \leq b$ & $c \leq y \leq d$ onto the region where $e^a \leq \rho \leq e^b$ & $c \leq \phi \leq d$



Let $w = e^z$ then prove that image of infinite strip $0 \le y \le \pi$ is the upper half plane ($v \ge 0$) in w-plane



$v \ge 0$ in w-plane

$z = x + iy$

$w = e^z = e^{(x+iy)}$

$u + iv = e^x . e^{iy}$

$\rho . e^{i\phi} = e^x . e^{iy}$

$\rho = e^x , \phi = y$

Limit for complex variable function:

Let a function f be defined at all points z in some deleted neighbourhood of z_0. Then the statement i.e the limit of $f(z)$ as z approaches to z_0 is a number w_0

Such that we write:

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{—— (1)}$$

Equation (1) means that the point $w = f(z)$ can be made arbitarilarily close to point w_0 if we choose the point z close enough to z_0 but distinct from it.

i.e: For each +ve no. epsilon there exist a +ve no. such that $|f(z) - w_0| < \epsilon$, whenever $0 < |z - z_0| < \delta$

Geometrically this defination says that for each ϵ neighbourhood $|w-w_0|<\epsilon$ of w_0 there is a deleted δ neighbourhood $0<|z-z_0|<\delta$ of z_0 such that every point z in it has the image w in the ϵ neighbourhood

Note: It is to be note that eventhough all points in neighbourhood $0<|z-z_0|<\delta$ are considered but their images need not fill up the entire neighbourhood $|w-w_0|<\epsilon$

Result: for a complex variable function if limit exists then it is unique

In other words. if $\lim_{z\to z_0} f(z)=w_0$ and $\lim_{z\to z_0} f(z)=w_1$ then $w_0=w_1$

→ Proof: Let funct^n $f(z)$ have $\lim_{z\to z_0} = w_0$ &

$\epsilon\ \delta$ $\lim_{z\to z_0} f(z)=w_1$

so by defination:

$$|f(z)-w_0|<\epsilon \text{ whenever } 0<|z-z_0|<\delta_0$$

$$\& \ |f(z)-w_1|<\epsilon \text{ whenever } 0<|z-z_0|<\delta_1$$

let δ be any +ve no. which is $\min\{\delta_0,\delta_1\}$ so that we can write $0<|z-z_0|<\delta$

Now consider

$$|w_1-w_0| = |[f(z)-w_0]-[f(z)-w_1]|$$

$$\leq |f(z)-w_0| + |f(z)-w_1|$$

$$< \epsilon + \epsilon$$

$$|w_1-w_0| < 2\epsilon$$

As $|w_1 - w_0|$ is a +ve constant +ve & ϵ can be chossen arbitary small so we have $|w_1 - w_0| = 0$

$\therefore w_1 = w_0$

let us show that if $f(z) = \frac{i\cdot\bar{z}}{2}$ defined on the region $|z| < 1$ ~~which~~ ~~st~~ then $\lim_{z \to 1} f(z) = \frac{i}{2}$

~~By definition~~:

$$\text{let } \left|f(z) - i/2\right| = \left|\frac{i\bar{z}}{2} - \frac{i}{2}\right| = \left|\frac{i(x - iy)}{2} - \frac{i}{2}\right|$$

$$= \left|\frac{(xi + y - i)\,i}{2\,i}\right|$$

$$= \left|\frac{-x + iy + 1}{2i}\right|$$

$$= \left|\frac{x - iy - 1}{-2i}\right|$$

$$= \frac{|x - iy - 1|}{|-2i|}$$

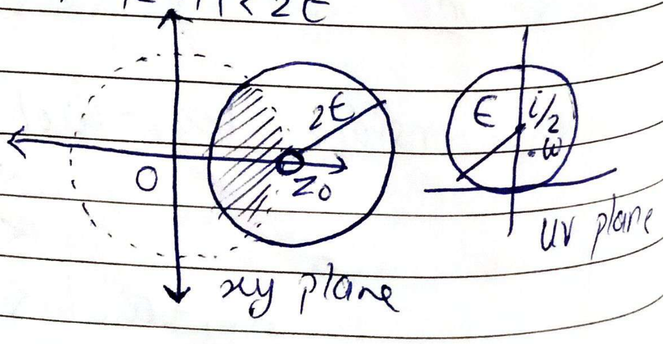
$$= \frac{|z - 1|}{2}$$

$$\left|f(z) - i/2\right| = \frac{|z-1|}{2}$$

$\left|f(z) - 1/2\right| < \epsilon$ whenever $0 < |z - 1| < 2\epsilon$

$\therefore \frac{|z-1|}{2} < \epsilon \quad \Rightarrow \quad |z - 1| < 2\epsilon$

By def: $\lim_{z \to 1} f(z) = i/2$



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Note: If $\lim_{z \to z_0} f(z) = w_0$ exists then the symbol $z \to z_0$

implies that z is allowed to approch in any arbitany manner not just from a particular direction

Show that the limit. $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist.

$f(z) = \frac{z}{\bar{z}}$, $z \neq 0$

→ let to show that $\lim_{z \to 0} f(z)$ does not exist.

we check the limit of $f(z)$
along x-axis
i.e $z \to (0,0)$
along x axis, i.e $(x,0) \to (0,0)$

$$\lim_{z \to 0} f(z) = \lim_{(x,0) \to (0,0)} \frac{z}{\bar{z}} = \lim_{x \to 0} \frac{x+i0}{x-i0} = \frac{x}{x} = 1 \quad \text{--- (1)}$$

along y axis, i.e $(0,y) \to (0,0)$

$$\lim_{z \to 0} f(z) = \lim_{(0,y) \to (0,0)} \frac{z}{\bar{z}} = \lim_{x \to 0} \frac{0+iy}{0-iy} = -1 \quad \text{--- (2)}$$

from (1) & (2) we can say it is contradiction to the uniqueness of value of limit of function.
$\therefore \lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist.

Now consider z approaching origin along y axis

HW. Show that $\lim_{z \to 2} \frac{z^2 - 4}{z - 2} = 4$

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we know that

$$|f(z) - w_0| < \epsilon \quad , \text{where} \quad 0 < |z-2| < \delta$$

considering: $f(z) = \frac{z^2-4}{z-2} = \frac{(z+2)(z-2)}{(z-2)}$

$$= z+2$$

$$\lim_{z \to 2} (z+2) = 4$$

i.e $|z+2-4| = |z-2| < \epsilon$

whenever $0 < |z-2| < \delta$ we choose $\delta = \epsilon$

(or)

Suppose $0 < |z-2| < \delta = \epsilon$

then $|z+2-4| < \epsilon$ $\quad \lim_{z \to 2} (z+2) = 4$

Theorems on limit

1) Suppose that $f(z) = u(x,y) + iv(x,y)$ and $z_0 = x_0 + iy_0$, $w_0 + u_0 + iv_0$ then $\lim_{z \to z_0} f(z) = w_0$ if and only if (iff)

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \quad \& \quad \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$$

2) Suppose that $\lim_{z \to z_0} f(z) = w_0$ & $\lim_{z \to z_0} F(z) = W_0$ then

$$\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0$$

and also $\lim_{z \to z_0} [f(z) \cdot F(z)] = w_0 \cdot W_0$

if $W_0 \neq 0$ then $\lim_{z \to z_0} \left[\frac{f(z)}{F(z)}\right] = \frac{w_0}{W_0}$

differentiable always continuous
continuous may or may not be differentiable

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Defination of continuity:

- ~~A function~~ let a function f is continuous at point z_0 if all three of the following conditions are satisfied.

1) $\lim_{z \to z_0} f(z)$ exists

2) $f(z_0)$ exists

3) $\lim_{z \to z_0} f(z) = f(z_0)$

$$f(z) = \begin{cases} \dfrac{z^2+1}{z-i}, & z \neq i \\ 2i, & z = i \end{cases}$$

~~lim~~ let $f(z) = \dfrac{z^2+1}{z-i}$, when $z \neq i$

$f(z) = 2i$, when $z = i$

let $\lim_{z \to i} f(z) = \lim_{z \to i} \dfrac{z^2-(-1)}{z-i}$

$= \lim_{z \to i} \dfrac{z^2-(i)^2}{z-i}$

$= \lim_{z \to i} \dfrac{(z+i)(z-i)}{z-i}$

$= 2i$

As all the three conditions are satisfied thus the given function is continuous.

$\lim_{z \to i} f(z) = f(i) = 2i$

$\therefore f(z)$ is continuous at $z = i$

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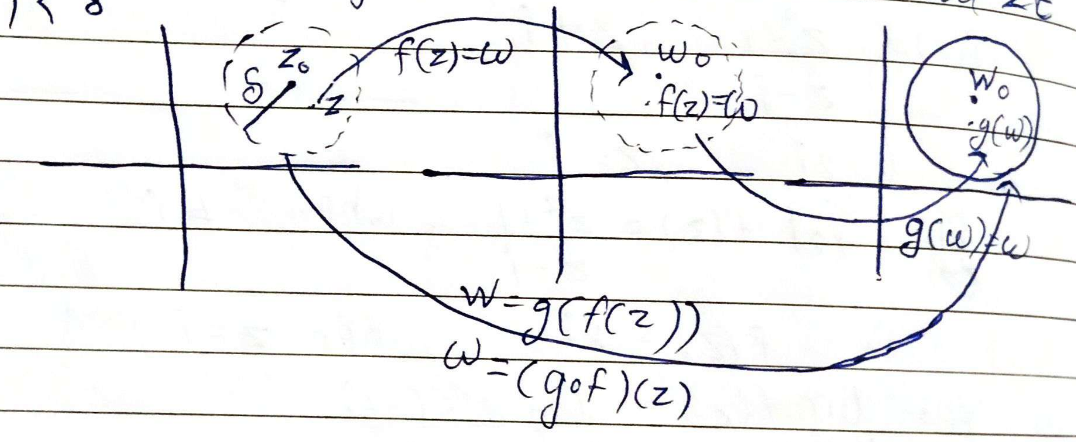
$$f(z) = \begin{cases} \dfrac{Re(z)^2}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Composition of a function f and g.

Let $w = f(z)$ be a function defined on neighbourhood $|z - z_0| < \delta$ and let $W = g(w)$ be a function whose domain of defination contains the image of neighbourhood $(|z - z_0| < \delta)$ under f then The composition $W = g(f(z))$ is defined for all $z \in |z - z_0| < \delta$



- Composition of any ~~fun~~ continuous functions is continuous.

- A function f(z) is continuous and non-zero at a point z_0 then $f(z) \neq 0$ throughout the domain

- If a function f is continuous throughout the region R i.e both closed & bounded then there exist a non negative real no. M such that $|f(z)| \leq M \quad \forall z \in R$

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Recall: let a function $f : U \subseteq \mathbb{R} \to \mathbb{R}$, is said to be differentiable at point $x_0 \in U$ if x_0 is an interior point of U and $\lim_{\substack{x \to x_0 \\ x \in U - \{x_0\}}} \frac{f(x) - f(x_0)}{x - x_0}$ exist

and we call this limit as derivative of f at point x_0 and we denote it as:

$$f'(x_0) = \lim_{\substack{x \to x_0 \\ x \in U - \{x_0\}}} \frac{f(x) - f(x_0)}{x - x_0}$$

complex differentiability:

let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is said to be a differentiable function at point z_0 if z_0 is an interior point of U and the ~~limit $z \to z_0$~~

$$\lim_{\substack{z \to z_0 \\ z \in U - \{z_0\}}} \frac{f(z) - f(z_0)}{z - z_0} \text{ exist}$$

and we call this limit as derivative of f at point z_0 & we denote it as:

$$f'(z_0) = \lim_{\substack{z \to z_0 \\ z \in U - \{z_0\}}} \frac{f(z) - f(z_0)}{z - z_0} = \frac{d f(z_0)}{dz}$$

Differentiability of complex variable function:

let $f(z)$ be a function on neighbourhood which contained in the domain of f then the derivative of function f at z_0 is a limit

$$f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z \cdot z_0} \quad \text{—} \; (1)$$

The fuction f is said to be differentiable if $f'(z_0)$ exist let's denote $f'(z_0)$ and we call it as ~~derivative~~ derivative of f at z_0

$$f'(z_0) = \frac{f(z+z_0)-f(z)}{\Delta z}$$

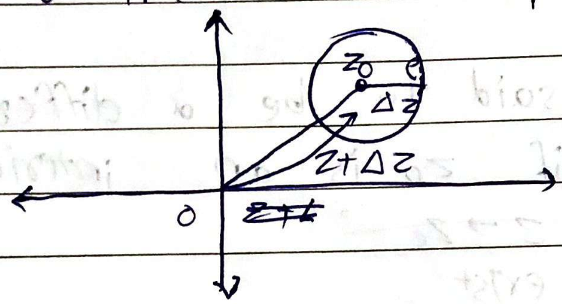
if $\Delta z = z - z_0$

$\therefore z = \Delta z + z_0$ — (2) $\qquad z \neq z_0$

where $\Delta z = z - z_0$

$z = \Delta$

Note 1: As f is defined through out the nbdh of z_0 then it is implised at $f(z+\Delta z)$ exist.



Note 2: while using eqn (2) we drop the subscript as introduced $\Delta w = f(z+\Delta z) - f(z)$ which denote the change in value of the function $f(w) = \Delta w$ corresponding to change Δ in the point at which the function is to be equated.

Note 3: we denote $f'(z_0)$ by $\frac{dw}{dz}$ also $f(z_0) = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$

Q discuss the differentiability of $f(z) = z^2$ in the complex plane.

→ so to check $f(z) = z^2$ is differentiable

let $f'(z) = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$

where $\Delta w = f(z+\Delta z) - f(z)$

$\Delta w = 2z \cdot \Delta z + \Delta z^2$

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$$\frac{\Delta w}{\Delta z} = 2z+\Delta z$$

$$f'(z) = 2z$$

As their is no condition on z so function is differentiable in everywhere in complex plane

If $f'(z) = \bar{z}$ discuss the differentiability of given funct^n:

→ Let $f'(z) = \bar{z}$

then $f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$

where: $\Delta w = f(z+\Delta z) = f(z)$

$$\Delta w = (\overline{z+\Delta z}) - \bar{z}$$

$$\frac{\Delta w}{\Delta z} = \frac{\bar{z}+\overline{\Delta z}-\bar{z}}{\Delta z}$$

$$\frac{\Delta w}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} \quad - (*)$$

Now to check the limit along x-axis where

$$\lim_{z\to 0} \frac{\Delta w}{\Delta z} = \lim_{(\Delta x,0)\to(0,0)} \frac{\Delta \omega}{\Delta z} = \lim_{\Delta x \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$\lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z} = \lim_{x\to 0} \frac{\Delta x}{\Delta x} = 1$$

is not differentiable at any point.

Q 1) $f(z) = |z|^2$

2) Show that $\lim_{z\to z_0} \mathrm{Re}\, z = \mathrm{Re}\, z_0$ use ϵ delta defn of limit

3) Show that: $\lim_{z\to z_0} \bar{z} = \bar{z_0}$

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$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ along x axis & along y axis will be equal

for only 1 pt. i.e origin.

$\Rightarrow$ given fun $f(z) = |z|^2$ is differentiable only at origin

Note :- 1) If a function is continuous in a domain then it is not necessary that it is differentiable in the same.

2) If a fun^n is differentiable at a point then it implies that the fun^n is also continuous at the same pt.

e.g: The fun^n $f(z) = |z|^2$

Differentiation formulas:

1) If c is any constant and f be a function then:

$$f'(z) = \frac{d}{dz} f(z) \text{ --- } ①$$

1) $\frac{d}{dz}(c) = 0$

2) $\frac{d}{dz}(z) = 1$

3) $\frac{d}{dz} f(cz) = c\frac{d}{dz} f(z) = cf'(z)$

4) $\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$

5) $\frac{d}{dz}[f(z) \cdot g(z)] = f(z) \cdot g'(z) + g(z) \cdot f'(z)$

6) $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)[f'(z)] - g(z) \cdot [f'(u)]}{g(z)^2}$

7) $\frac{d}{dz}(z^n) = nz^{n-1}$

8) If $f(z) = \omega$ and $g(\omega) = W$
then $\frac{d}{dz} g(\omega) = \frac{dW}{d\omega} \cdot \frac{d\omega}{dz} = \frac{dW}{d\omega} \cdot \frac{d\omega}{dz}$

Find derivative of $W = (2z^2+i)^5$

$\frac{d}{dz} z^n = 5(2z^2+i)^4 (4z)$

$= 20z(2z^2+i)^4$

Q $f(z) = \frac{(1+z^2)^4}{z^2}, z \neq 0$ $\quad f(z) = 1 + z^3 + \frac{1}{z} \; ; z \neq 0$

Q Show that $\lim_{z \to 0} \frac{\bar{z}^2}{z} = 0$ use ϵ-δ def^n. (find along path)

CR equations (cauchy-Riemann equations)

Let $f(z) = u(x,y) + iv(x,y)$ then the partial derivatives of u and v satisfies some special conditions by which we can arrive at a conclusion from which we can decide the differentiability of & we can ~~derive~~ calculate derivative at a particular point.

let $z_0 = x_0 + iy_0$ + $\Delta z = \Delta x + i\Delta y$.
then $\Delta w = f(z + \Delta z) - f(z_0)$

$= u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y) - [u(x_0,y_0) + iv(x_0,y_0)]$

$\Delta w = u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) + i[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)]$

Dividing with Δz on both side.

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$$\frac{\Delta w}{\Delta z} = \frac{[u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)] + i[v(x_0+\Delta x, y_0+\Delta y)-v(x_0,y_0)]}{\Delta z}$$

$$\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} = \lim_{\Delta z\to 0}\left(\frac{Re\,\Delta w}{\Delta z}\right) + i\lim_{\Delta z\to 0}\left(\frac{Im\,\Delta w}{\Delta z}\right) \quad \text{—} \star$$

So now, $\lim_{\Delta z\to 0}\left(\frac{Re\,\Delta w}{\Delta z}\right) = \lim_{\Delta z\to 0}\left[\frac{u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)}{\Delta z_0}\right]$

To calculate limit we choose $\Delta z \to 0$ along x axis i.e $\Delta z\to 0$ along the points $(\Delta x, 0)$

$$\Rightarrow \lim_{\Delta x\to 0}\left(\frac{Re\,\Delta w}{\Delta z}\right) = \lim_{\Delta x\to 0}\left[\frac{u(x_0+\Delta x, y_0)-u(x_0,y_0)}{\Delta x}\right]$$

$$\lim_{\Delta z\to 0}\left(\frac{Re\,\Delta w}{\Delta z}\right) = u_x(x_0,y_0) \quad \text{—} (1)$$

Similarly $\lim_{\Delta z\to 0}\left(\frac{Im\,\Delta w}{\Delta z}\right) = v_x(x_0,y_0)$ — (2)

To calculate the limit we approached origin along x axis

from eq^n (1) & (2) eq^n $\star$ becomes

$$\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} = u_x(x_0,y_0) + i v_x(x_0,y^0)$$

$$f'(z_0) = u_x(x_0,y_0) + i v_x(x_0,y_0) \quad \text{—} (3)$$

Similarly $\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} = \lim_{\Delta z\to 0}\left(\frac{Re\,\Delta w}{\Delta z}\right) + i\lim_{z\to 0}\left(\frac{\Delta w}{\Delta z}\right)$ along y axis

To check the limit of above equation along y axis i.e $\Delta z\to 0$ along the point $(0,\Delta y)$.

$$\lim_{\Delta y\to 0}\left(\frac{Re\,\Delta w}{\Delta z}\right) = \lim_{\Delta y\to 0}\left[\frac{u(x_0, y_0+\Delta y)-u(x_0,y_0)}{i\Delta y}\right]$$

$\because \Delta z = \Delta x + i\Delta y$ $\qquad 1/i = -i$

$$= -i\lim_{\Delta y\to 0}\left[\frac{u(x_0,y_0+\Delta y)-u(x_0,y_0)}{\Delta y}\right]$$

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$$\lim_{\Delta z \to 0}\left(\frac{\cancel{Re}\,\Delta w}{\Delta z}\right) = -i u_y(x_0, y_0) \quad \text{--- (4)}$$

$$\lim_{\Delta z \to 0}\left(\frac{Im\,\Delta w}{\Delta z}\right) = \lim_{\Delta y \to 0}\left[\frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)}{\Delta z}\right]$$

$$= \lim_{\Delta y \to 0}\left[\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i\Delta y}\right]$$

$$= \frac{1}{i}\lim_{\Delta y \to 0}\left[\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\Delta y}\right]$$

$$= \frac{1}{i} v_y(x_0, y_0) \quad \text{--- (5)}$$

from eq^n (4) & (5) eq^n (*) becomes

$$\lim_{\Delta z \to 0}\frac{\Delta w}{\Delta z} = -i u_y(x_0, y_0) + \frac{\cancel{(x)}}{i} v_y(x_0, y_0)$$

$$f'(z_0) = -i u_y(x_0, y_0) + v_y(x_0, y_0) \quad \text{--- (6)}$$

If function f is differentiable at z_0 then lim along any direction is same.

∴ from eq^n (3) & (6)

we get: $u_x(x_0, y_0) + i v_x(x_0, y_0) = v_y(x_0, y_0) - i u_y(x_0, y_0)$

$$u_x = v_y$$
$$u_y = -v_x$$

These are CR equations.

e.g: $f(z) = \cancel{|z|^2}\ z^2$

$f(z) = w = u + iv$

where $z^2 = x^2 - y^2 + i2xy$.

$u = x^2 - y^2$ $\qquad$ $v = 2xy$

$u_x = 2x$ $\qquad$ $v_x = 2y$

$u_y = -2y$ $\qquad$ $v_y = 2x$

∴ $u_x = v_y = 2x$ & $u_y = -v_x = -2y$

$f'(z) = u_x + i v_x$

$= 2x + i2y = 2(x+iy) = 2z$

(or) $f'(z) = v_y - i u_y = 2x - i(-2y)$

$= 2(x+iy) = 2z.$

Theorem: Suppose that $f(z) = u(x,y) + iv(x,y)$ and $f'(z)$ exist at a point $z_0 = x_0 + iy_0$ then the first order partial derivative of the components u & v must exist ~~and~~ at (x_0, y_0) & must satisfy CR equations at that point.

$$u_x = v_y \qquad u_y = -v_x.$$

And $f'(z_0)$ can be written as $u_x + iv_x$