CS6462

Probabilistic and Explainable AI

Lesson 1

Probability of Events

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Probability

Definition

- mathematics concerned with numerical descriptions of how likely an event is to occur
- degree of belief of an experiment ***X*** (or phenomena) where all possible outcomes are known (denoted as sample space ***S***)
- any subset of ***S*** is called an event ***E***
- event ***E*** *occurs* if the outcome of experiment ***X*** is contained in ***E***

Example

X = tossing a die

S = {1, 2, 3, 4, 5, 6}

E = {2, 4, 6} - an event "the number is even"

Probability (cont.)

Example 2

X = tossing a coin twice

S = {hh, ht, th, tt}

E = {hh, ht} event “the first toss results in a Head”

Example 3

X = tossing a die twice

S = {f(i, j) : i, j = 1, 2, …. , 6} contains 36 elements

E = {f(i,j): f+j=10} event “the sum of the results of the two tosses is equal to 10”

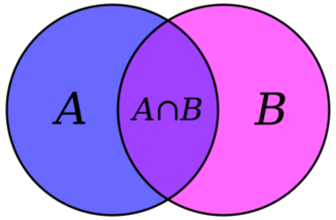
Example 4

X = choosing a point from the interval [0; 1]

S ⊆ Q rational numbers***,*** ***s*** ∈ ***S*** and ***0 ≤ s ≤ 1***

E = {1/3}

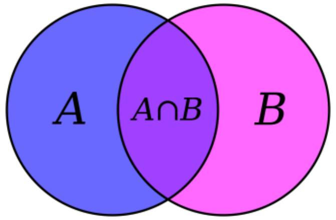
Events and Set Theory



Set Theory provides the *notation* to describe and manipulate events:

- $E \subseteq S$, $F \subseteq S$ events E and F are *subsets* of the sample space S
- E^c complement of E - the set of all outcomes not in **E**
- $E \cap F$ intersection of E and F - the set of all outcomes in both E and F
- $E \cup F$ union of E and F - the set of all outcomes in E or in F or in both E and F
- $E \subseteq F$ event E is subset of F
- $E \cap F = \emptyset$ events E and F are *mutually exclusive* (disjoint)
- union and intersection of more than two events:
  $\cup_{i=1}^{n} E_i$, $\cup_{i=1}^{\infty} E_i$, $\cap_{i=1}^{n} E_i$, $\cap_{i=1}^{\infty} E_i$

Events and Set Theory (cont.)



Commutativity:

- $E \cup F = F \cup E$, $E \cap F = F \cap E$

Associativity:

- $(E \cup F) \cup G = E \cup (F \cup G)$, $(E \cap F) \cap G = E \cap (F \cap G)$

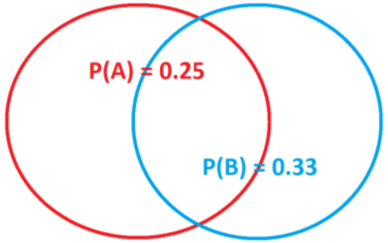
Distributivity:

- $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$, $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$

Morgan's Laws:

- $(E \cup F)^c = E^c \cap F^c$
- $(E \cap F)^c = E^c \cup F^c$

Probability of Events – Properties



Properties of probability ***P*** with respect to sample space ***S***:

- *notion*: probability ***P*** of an event ***E***

  $P(E)$

- *scaling:* measures uncertainty on a scale from 0 to 1, with 1 representing certainty

  $P(S) = 1,\ 0 \leq P(E)\ \forall\ E \in S$

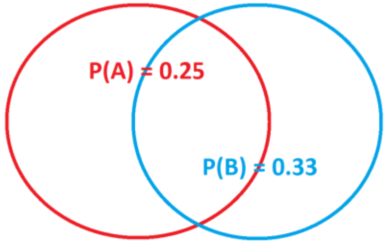
- *additivity: imposes* order on the assignment of probabilities

  $P(E \cup F) = P(E) + P(F)$, iff $E \cap F = \emptyset$

  for any sequence of events $E_1, E_2, \ldots, E_n$ which are mutually exclusive

  $P(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i)$

Probability of Events – Properties (cont.)



- *complementarity:*

  $P(E) + P(E^c) = 1 \ \forall\ E \in S$

- *general additivity:* additivity for non-mutually exclusive events

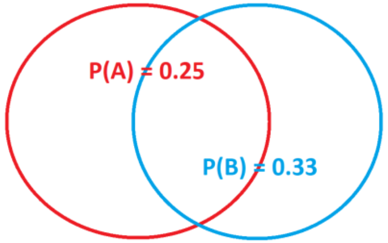
  $P(E \cup F) = P(E) + P(F) - P(E \cap F) \ \forall\ E,F \in S$

- *log odds:* measures the *odds of success* - probability of success/probability of failure

  $LogOdds(E) = \ln(P(E)/P(E^c))$

  $LogOdds(E) = \ln(P(E)/(1 - P(E))$

Probability of Events – Examples



*Example 1: **X** = Tossing a fair coin*

S = {h, t}

P(S) = 1, P(h) = P(t) = ½

*Example 2: **X** = Tossing a fair die*

S = {1, 2, 3, 4, 5, 6}

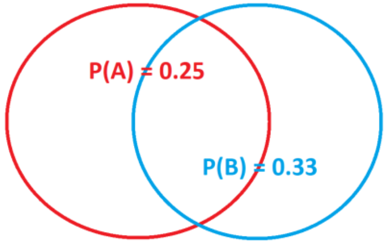
P(S) = 1, P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6

*Example 3: **X** = Tossing a fair coin twice.*

S = {hh, ht, th, tt}

P(S) = 1, P(hh) = P(ht) = P(th) = P(tt) = 1/4

Probability of Events – Examples (cont.)



Example 4: $\boldsymbol{X}$ *= Tossing a die -* $\boldsymbol{P(E \cup F)}$ **?**

- $S = \{1, 2, 3, 4, 5, 6\}$
- $P(S) = 1, P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$
- $E = \{1, 3, 6\}, F = \{3, 4, 6\}$
- $E = \{1\} \cup \{3\} \cup \{6\}, F = \{3\} \cup \{4\} \cup \{6\}$
- $E \cap F = \{3, 6\} = \{3\} \cup \{6\}$
- recall *additivity rule:* $P(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i)$

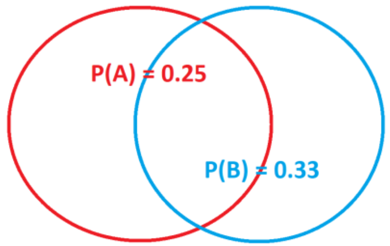
  $P(E) = P(1) + P(3) + P(6) = 3/6,$

  $P(F) = P(3) + P(4) + P(6) = 3/6$

  $P(E \cap F) = P(3) + P(6) = 2/6$
- recall *general additivity rule:* $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

  $P(E \cup F) = 3/6 + 3/6 - 2/6 = 4/6$

Probability of Events – Examples (cont.)



Example 5: ***X*** *= 80% chance of rain*

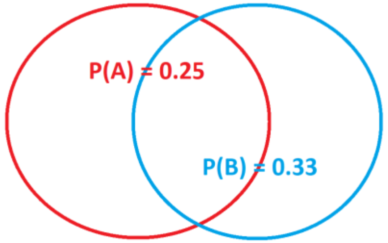
- ***S = [0,1]***
- ***E – chance of rain***
- ***P(E) = 0.8 odds of success 80%***
- ***P(E^c^) = 0.2 odds of failure 20%***
- recall *log odds rule:* ***LogOdds(E)*** **= ln(*P*(*E*)/*P*(*E^c^*))**

  LogOdds(E) **= ln(*P*(*E*)/*P*(*E^c^*)) = ln (0.8/0.2) = 1.38629436**

  the odds ratio (the probability of success/probability of failure) = 4

  the log odds = 1.38629436

Summary



Probability of Events

- *Probability* – mathematics concerned with numerical descriptions of how likely an event is to occur
- *Events* – outcomes ***E*** of experiment ***X*** with known set of possible outcomes ***S***

  event ***E*** *occurs* if the outcome of experiment ***X*** is contained in ***E***
- *Set Theory* provides the *notation* to describe and manipulate events:
  - notion ***P(E)***
  - properties (rules) – scaling, additivity, complementarity, general additivity
  - log odds – another way to express probability

- *Next Lesson* – Conditional Probability

Thank You!

Questions?