CS6462

Probabilistic and Explainable AI

Lesson 4

Probability Calculation as an Exercise of Counting

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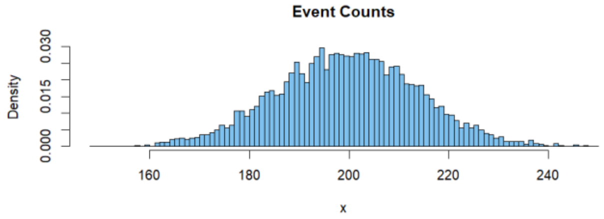
Probability and Counting



Definition:

- ***P(E)*** is the number of outcomes in ***E*** divided by the number of outcomes in ***S***
- ***P(E)*** = cardinality of ***E*** vs cardinality of ***S***

$$P\,(E) = \frac{n(E)}{n(S)}\quad,\ \text{iff } S \text{ is a finite set}$$

Problem:

- find the numerator and denominator
- combinatorics deals with counting

Combinatorics rules:

- permutation – the order does matter (*the order we put numbers in matters*)
- combination – the order does not matter

Counting Rules - Permutations

Definition:

- the number of permutations is number of ways of choosing ***E*** elements from ***S*** distinguishable elements where the order of choice matters

Permutations with repetition:

- *we use all the **S** elements every time (no reduction of the base **S**)*

  $\boldsymbol{P_{E,S} = n(S)^{n(E)}}$, ***n(S)*** – the number of options, ***n(E)*** – the size of combinations
- $\boldsymbol{P_{E,S}}$ - number of permutations (<u>not probability</u>)
- ***n(S)*** , ***n(E)*** – cardinality of ***S*** *and* ***E***

Example:

The combination to a safe is composed of 4 digits. There are 10 numbers to choose from 0,1,2,3,4,5,6,7,8, 9 and we need to pick up 4 of them. The number of permutations is:

$\boldsymbol{P_{S,E} = 10^4 = 10{,}000}$

Counting Rules – Permutations (cont.)

Permutations without repetition:

- choosing ***E*** elements from ***S*** distinguishable elements
- *we use all the **S** elements only the first time and consecutively reduce* the number of available choices every following time

$$P_{E,S} = \frac{n(S)!}{(n(S)-n(E))!}$$, use the factorial function ***!***

$$P_{r,n} = \frac{n!}{(n-r)!}$$

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$$

Example:

How many different ways are there that 3 pool balls could be arranged out of 16 balls.

$$P_{E,S} = \frac{16!}{(16-3)!} = \frac{20{,}922{,}789{,}888{,}000}{6{,}227{,}020{,}800} = 3{,}360$$

Counting Rules - Combinations

Definition:

- the number of combinations is the number of ways of choosing ***E*** elements from ***S*** distinguishable elements where the order of choice does not matter

Combinations with repetition:

$$C_{r,n} = \frac{(n+r-1)!}{r!\,(n-1)!}$$

- *we use all the **S** elements every time (no reduction of the base **S**)*

$$C_{E,S} = \frac{(n(S) + n(E) - 1)!}{n(E))! * (n(S) - 1)!}$$, ***n(S)*** – the number of options, ***n(E)*** – the size of combinations

Example:

There are five flavors of ice cream: *banana*, *chocolate*, *lemon*, *strawberry* and *vanilla* (***n(S) = 5***). We can have three scoops (***n(E) =3***). How many variations will there be?

$$C_{E,S} = \frac{(3+5-1)!}{3! * (5-1)!} = \frac{7!}{3!*4!} = \frac{5040}{144} = 35$$

Counting Rules – Combinations (cont.)

Combinations without repetition:

- we use *all the **S*** elements only the first time and consecutively reduce the number of available choices every following time

$$C_{E,S} = \frac{n(S)!}{n(E))! * (n(S)-n(E))!}$$, use the factorial function ***!***

$$C_{r,n} = \frac{n!}{r!(n-r)!}$$

Example:

How many different choices are there where 3 pool balls could be chosen out of 16.

$$C_{E,S} = \frac{16!}{3! * (16-3)!} = \frac{20,922,789,888,000}{37,362,124,800} = 560$$

Permutations/Combinations & Probability

Recall:

A probability ***P(E)*** is the number of permutations/combinations ***n(E)*** considered to be an event ***E*** divided by the total number of permutations/combinations ***n(S)***

$$P(E) = \frac{n(E)}{n(S)} \quad , \text{ iff } S \text{ is a finite set}$$

Rules to solve a probability problem using permutations/combinations:

- Set up a ratio to determine the probability.
- Determine whether the numerator and denominator require combinations, permutations, or a mix?
- Are these permutations/combinations with or without repetitions?
- Both types of permutations/combinations require you to identify the ***n(S)*** and ***n(E)*** to enter into the equations.

Permutations & Probability - Example

Example: ***X:*** *A* lottery game where 69 white balls are used by a machine to randomly select 5 of the balls. What is the probability of winning the game if the order does matter?

$\boldsymbol{S = \{1,\ldots,69\}}$

$\boldsymbol{E = \{x_1, x_2, x_3, x_4, x_5\}\ ,\ E \subseteq S\ ,\ E_w \subseteq E}$ (one wining combination $\boldsymbol{E_w}$)

<u>Example of computing probability using permutation without repetition</u>.

$$P\,(E_w) = \frac{n(\text{winning combinations})}{n(\text{combinations})} = \frac{n(E_w)}{P_{E,S}} = \frac{1}{P_{E,S}} = \frac{1}{1{,}348{,}621{,}560} = 7.4149786e\text{-}10$$

Recall how we compute $\boldsymbol{P_{E,S}}$ *for permutations without repetition*:

$$P_{E,S} = \frac{n(S)!}{(n(S)-n(E))!} = \frac{69!}{(69-5)!} = 1{,}348{,}621{,}560$$

Combinations & Probability - Example

Example: ***X:*** *A* lottery game where 69 white balls are used by a machine to randomly select 5 of the balls. What is the probability of winning the game if the order does not matter?

S = {1,…,69}

$E = \{x_1, x_2, x_3, x_4, x_5\}$, $E \subseteq S$, $E_w \subseteq E$ (one wining combination E_w)

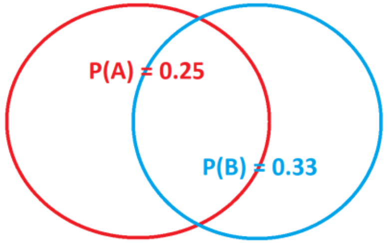
<u>Example of computing probability using combinations without repetition</u>.

$$P(E) = \frac{n(\text{winning combinations})}{n(\text{combinations})} = \frac{n(E_w)}{C_{E,S}} = \frac{1}{C_{E,S}} = \frac{1}{11{,}238{,}513} = 0.000000089$$

Recall how we compute $C_{E,S}$ *for combinations without repetition*:

$$C_{E,S} = \frac{n(S)!}{n(E))! * (n(S)-n(E))!} = \frac{69!}{5!*(69-5)!} = 11{,}238{,}513$$

Summary



Probability as an exercise of counting:

$$P\,(E) = \frac{n(E)}{n(S)}\quad,\ \text{iff } S \text{ is a finite set}$$

- *Problem to solve: find the numerator and denominator*
- *Combinatorics rules:*
  - permutation – the order does matter (*the order we put numbers in matters*)
  - combination – the order does not matter

- *Next Lesson* – Random Variables and Probability Distribution

Thank You!

Questions?