CS6462

Probabilistic and Explainable AI

Lesson 13
Bayesian Generalized Linear Models
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Exponential Family Form

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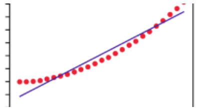


Exponential Family of Distribution

Probability distribution in Regression models :

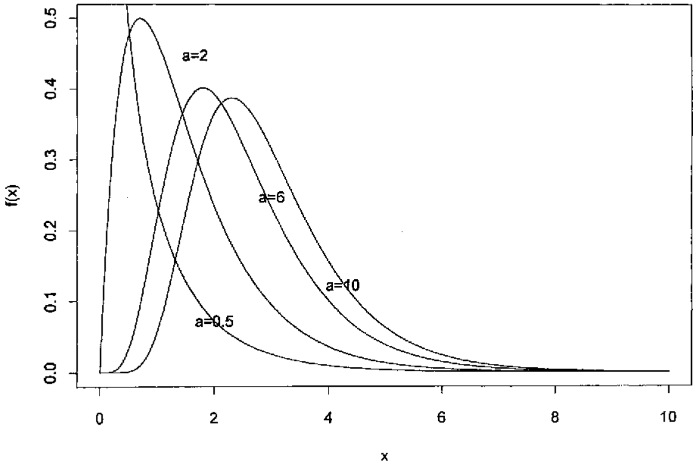
- usually, a normal distribution model of dependable variable
- dependable variable can follow different probability distributions: exponential, gamma, normal, etc.

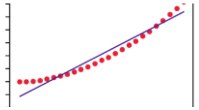
Exponential Family of Distribution (EFD): general exponential form or canonical form

- distribution model employed by GLM
- unifies linear and nonlinear regression models
- assumes that the distribution of the dependable variable is an exponential distribution

EFD models:

- Gaussian
- Multinomial
- Bernoulli
- Binomial
- Poisson





Exponential Family of Distribution (cont.)

Mathematical Definition of EFD: formal canonical model

- *meta-exponential model:* different distribution models need to be generalized via a meta-model
- random sample $\mathrm{X} = \{\mathrm{X}_1, \mathrm{X}_2, \dots, \mathrm{X}_\mathrm{n}\}$ with assumed probability distribution $\mathrm{P(X)}$ that depends on some unknown parameter θ
- $f(X_i; \theta)$ – the probability density (or mass) function of each $X_i \in X$ with θ has an exponential form

Exponential meta-models for $f(X_i; \theta)$: general exponential forms (canonical forms)

- version 1:

$$f(X_i; \theta) = \exp[\mathrm{a}(\mathrm{X_i}) * \mathrm{b}(\theta) + \mathrm{c}(\theta) + \mathrm{d}(\mathrm{X_i})]$$

$$f(X_i; \theta) = e^{\mathrm{a}(\mathrm{x_i}) * \mathrm{b}(\theta) + \mathrm{c}(\theta) + \mathrm{d}(\mathrm{x_i})}$$

$\mathrm{a}(\mathrm{X_i})$, $\mathrm{b}(\theta)$, $\mathrm{c}(\theta)$, $\mathrm{d}(\mathrm{X_i})$ - arbitrary functions

- version 2:

$$f(X_i; \theta, \varphi) = \exp\left\{\frac{\mathrm{X_i} * \theta - \mathrm{b}(\theta)}{\varphi} + \mathrm{c}(\mathrm{X_i}; \varphi)\right\}$$

φ – scale parameter
$\mathrm{b}(\theta)$, $\mathrm{c}(\mathrm{X_i}; \varphi)$ - arbitrary functions

Exponential Form of Poisson Distribution

Probability distribution model:

- version 1: $f(X_i; \theta) = e^{a(x_i)*b(\theta) + c(\theta) + d(x_i)}$

- $\boldsymbol{\lambda}$ is a shape parameter that indicates the average number of events in the given time interval

$$P(X_i = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \text{ where } x = \{0, 1, 2, ..., \infty\}$$

$$f(x; \lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \text{ where } x = \{0, 1, 2, ..., \infty\}$$

$$a = e^{\log(a)}$$

$$a^x = e^{x \log(a)}$$

$$\log(ab) = \log(a) + \log(b)$$

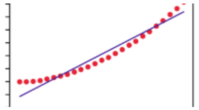
$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$$

Determine whether the Poisson model is a member of EFD:

1) write the Poisson probability mass function in the form of the meta-exponential model
2) prove that the Exponential Form of Poisson Distribution does not depend on λ parameter

$$f(x; \lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} = e^{-\lambda} e^{x*\ln(\lambda)} e^{-\ln(x!)} = e^{x*\ln(\lambda) - \lambda - \ln(x!)} \quad \text{(EFPD)}$$

Exponential Form of Poisson Distribution (cont.)



Exponential Form of Poisson Distribution:

$$f(x;\lambda) = \frac{\mathrm{e}^{-\lambda}\lambda^{\mathrm{x}}}{\mathrm{x}!} = e^{-\lambda}e^{x*\ln(\lambda)}e^{-\ln(x!)} = e^{x*\ln(\lambda)-\lambda-\ln(x!)}$$

$$f(x;\theta) = e^{\mathrm{a}(x)*\mathrm{b}(\theta) + \mathrm{c}(\theta) + \mathrm{d}(x)}$$

$$f(x;\lambda) = e^{x*\ln(\lambda)-\lambda-\ln(x!)}$$

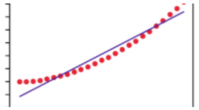
Factor mapping:

- $\theta = \lambda$
- $a(x) = x$
- $b(\theta) = ln(\lambda)$
- $c(\theta) = -\lambda$
- $d(x) = -ln(x!)$

Canonical link: $\mathrm{b}(\theta)$ could be modeled as a linear function of the explanatory variables

$$log(\lambda) = \beta_0 + \beta_1 * x_1 + \beta_2 * x_2 +, \cdots, + \beta_n * x_n$$

Exponential Form of Bernoulli Distribution



Probability distribution model: $f(x;\theta) = e^{\mathrm{a}(x)*\mathrm{b}(\theta) + \mathrm{c}(\theta) + \mathrm{d}(x)}$

- defined on a binary (0 or 1) random variable using parameter p (probability)

$f(x;p) = p^x(1-p)^{1-x}$ where $\mathrm{x} = \{0, 1\}$

Determine whether the Bernoulli model is a member of EFD:

$$f(x;p) = \exp\{ln(p^x(1-p)^{1-x})\} = \exp\{x * ln(p) + (1-x)ln(1-p)\}$$

$$f(x;p) = \exp\{x * ln(p) + ln(1-p) + \ ln(1-p)^{-x}\}$$

$$f(x;p) = e^{x*ln(p)+ln(1-p)+\ ln(1-p)^{-x}}$$

Factor mapping:

- $\theta = p$
- $a(x) = x$
- $b(\theta) = ln(p)$
- $c(\theta) = ln(1-p)$
- $d(x) = ln(1-p)^{-x}$

Exponential Form of Normal Distribution

Probability distribution model: $f(x;\theta) = e^{\mathrm{a}(x)*\mathrm{b}(\theta)\,+\,\mathrm{c}(\theta)\,+\,\mathrm{d}(x)}$

- continuous probability distribution
- μ is the mean (or expectation) of the distribution, σ is the standard deviation of the distribution

$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}}$ where $\mathrm{x} = \{-\infty ,\ldots, +\infty\}$, $\mu = \{-\infty ,\ldots, +\infty\}$

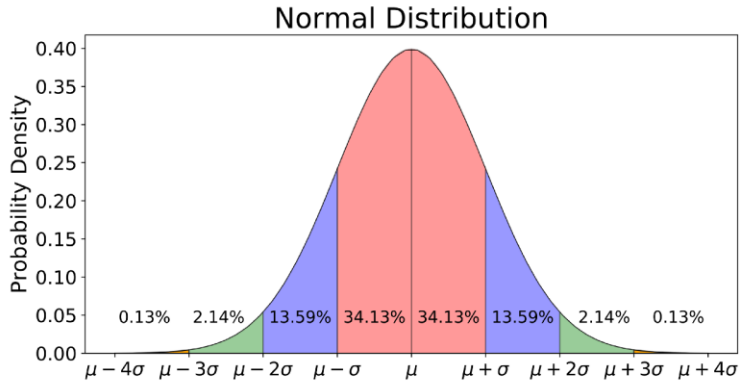
Determine whether the Normal model is a member of EFD:

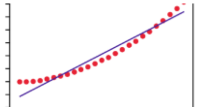
$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}} = exp\left\{x * \left(\frac{\mu}{\sigma^2}\right) + \left(-\frac{\mu}{2\sigma^2} - \frac{1}{2} * ln(2\pi\sigma^2)\right) - \frac{x^2}{2\sigma^2}\right\}$$

$$f(x;\mu,\sigma) = e^{x*\left(\frac{\mu}{\sigma^2}\right)+\left(-\frac{\mu}{2\sigma^2}-\frac{1}{2}*ln(2\pi\sigma^2)\right) - \frac{x^2}{2\sigma^2}}$$

Factor mapping:

- $\theta = \mu, \sigma$
- $a(x) = x$
- $b(\theta) = \frac{\mu}{\sigma^2}$
- $c(\theta) = -\frac{\mu}{2\sigma^2} - \frac{1}{2} * ln(2\pi\sigma^2)$
- $d(x) = -\frac{x^2}{2\sigma^2}$





Summary

Bayesian Generalized Linear Models – Exponential Family Form

Exponential Family of Distribution (EFD):

- distribution model employed by GLM
- unifies linear and nonlinear regression models
- assumes that the distribution of the dependable variable is an exponential distribution
- various canonical forms

$$f(X_i;\theta) = e^{\mathrm{a}(\mathrm{x_i}) * \mathrm{b}(\theta) + \mathrm{c}(\theta) + \mathrm{d}(\mathrm{x_i})}$$

EFD models:

- Poisson
- Bernoulli
- Normal

Next Lesson:

- Bayesian Generalized Linear Models - Generalized Linear Model Theory

Thank You!

Questions?