Assignment: Research and Development Spring Intern

Ayush Pandey
Amrita Vishwa Vidyapeetham
AV.EN.U4AIE22106
ayushpandey1177@gmail.com

Video Explanation (2 minutes)

A short video elaborating the motivation, thought process, and implementation.

November 7, 2025

Mathematical Derivation for Parametric Curve Estimation

Problem statement. We need to solve the parametric equation of the curve for

$$x(t; \theta, M, X) = t \cos \theta - e^{Mt} \sin(0.3t) \sin \theta + X, \tag{1}$$

$$y(t; \theta, M, X) = 42 + t \sin \theta + e^{Mt} \sin(0.3t) \cos \theta, \tag{2}$$

for parameters constrained as

$$t \in [6, 60], \quad \theta \in (0°, 50°), \quad M \in [-0.05, 0.05], \quad X \in [0, 100].$$

in the csv file provided contains, $K = 1500$ observed points (x_k, y_k) while the parameter values are sampled uniformly in t as

$$t_k = 6 + (k-1)\frac{60-6}{K-1}, \qquad k = 1, \dots, K. \tag{3}$$

Idea

We will follow the nonlinear least-squares formulation with analytic Jacobians and solve using the LM method.

Step 1: using rotation identity

we define the rotated coordinates for any candidate (θ, X):

$$u = (x - X) \cos \theta + (y - 42) \sin \theta, \tag{4}$$

$$v = -(x - X) \sin \theta + (y - 42) \cos \theta. \tag{5}$$

1

upon substitution we have

$$u = t, \qquad v = e^{Mt} \sin(0.3t). \qquad (6)$$

Closed-form recovery of X and M given (θ, t)

From $u = t$,

$$t = (x - X)\cos\theta + (y - 42)\sin\theta \quad \Rightarrow \quad X = x - \frac{t - (y - 42)\sin\theta}{\cos\theta}. \qquad (7)$$

From $v = e^{Mt}\sin(0.3t)$ and $v = -(x - X)\sin\theta + (y - 42)\cos\theta$,

$$M = \begin{cases} \dfrac{1}{t}\ln\left(\dfrac{v}{\sin(0.3t)}\right), & \text{if } \sin(0.3t) \neq 0, \\ \text{undefined (requires } v \approx 0), & \text{if } \sin(0.3t) = 0. \end{cases} \qquad (8)$$

Least-squares objective

Define the per-sample residuals

$$r_k(\theta, M, X) = \begin{bmatrix} x(t_k; \theta, M, X) - x_k \\ y(t_k; \theta, M, X) - y_k \end{bmatrix}. \qquad (9)$$

Stacking $r \in \mathbb{R}^{2K}$, the nonlinear least-squares objective is

$$\Phi(\theta, M, X) = \tfrac{1}{2} r^\top r = \tfrac{1}{2} \sum_{k=1}^{K} \left(r_{k,x}^2 + r_{k,y}^2 \right). \qquad (10)$$

Compact form and derivatives

Let

$$S(t, M) = e^{Mt}\sin(0.3t), \qquad (11)$$

then

$$\frac{\partial S}{\partial t} = e^{Mt}\left(M\sin(0.3t) + 0.3\cos(0.3t) \right) =: S_t(t, M), \qquad (12)$$

$$\frac{\partial S}{\partial M} = t\, e^{Mt}\sin(0.3t) = t\, S(t, M) =: S_M(t, M). \qquad (13)$$

Analytic Jacobian

For a fixed t, the partial derivatives of (x, y) are:

$$\frac{\partial x}{\partial \theta} = -t\sin\theta - S\cos\theta, \qquad \frac{\partial x}{\partial M} = -tS\sin\theta, \qquad \frac{\partial x}{\partial X} = 1, \qquad (14)$$

$$\frac{\partial y}{\partial \theta} = t\cos\theta - S\sin\theta, \qquad \frac{\partial y}{\partial M} = tS\cos\theta, \qquad \frac{\partial y}{\partial X} = 0. \qquad (15)$$

If t is estimated too:

$$\frac{\partial x}{\partial t} = \cos\theta - S_t \sin\theta, \tag{16}$$

$$\frac{\partial y}{\partial t} = \sin\theta + S_t \cos\theta. \tag{17}$$

The per sample Jacobian obtained $J_k \in \mathbb{R}^{2\times 3}$ is

$$J_k = \begin{bmatrix} \partial x/\partial\theta & \partial x/\partial M & \partial x/\partial X \\ \partial y/\partial\theta & \partial y/\partial M & \partial y/\partial X \end{bmatrix}_{t=t_k}. \tag{18}$$

Stacking all J_k yields us $J \in \mathbb{R}^{2K\times 3}$.

Levenberg–Marquardt formulation

At iterate $p = (\theta, M, X)$ we compute r and J. using Gauss–Newton normal equations:

$$J^\top J \Delta = -J^\top r. \tag{19}$$

Levenberg–Marquardt update (with damping λ):

$$(J^\top J + \lambda D)\Delta = -J^\top r, \tag{20}$$

where $D = \mathrm{diag}(J^\top J)$ or I, and update

$$p \leftarrow p + \Delta. \tag{21}$$

Covariance approximation

Assuming i.i.d. noise and full column rank,

$$\mathrm{Cov}(\hat{p}) \approx \hat{\sigma}^2 (J^\top J)^{-1}, \qquad \hat{\sigma}^2 = \frac{1}{m-p}\sum_{i=1}^{m} r_i^2, \tag{22}$$

with $m = 2K$ and $p = 3$. This approximation is diagnostic only when a bound is active (e.g. $\hat{M} = -0.05$).

Assignment metric: L_1

$$L_1 = \frac{1}{K}\sum_{k=1}^{K}\Big(\big|x_{\mathrm{pred}}(t_k) - x_k\big| + \big|y_{\mathrm{pred}}(t_k) - y_k\big|\Big). \tag{23}$$

Numerical results

$$K = 1500, \tag{24}$$

$$\hat{\theta} = 29.58281377457789° \quad (\hat{\theta}_{\text{rad}} = 0.5163175023707157), \tag{25}$$

$$\hat{M} = -0.05 \quad \text{(lower bound hit)}, \tag{26}$$

$$\hat{X} = 55.013609464940664. \tag{27}$$

$$\text{RSS} = 771686.8923633082, \tag{28}$$

$$\hat{\sigma}^2 = 257.4864505716744. \tag{29}$$

Approximate covariance:

$$\text{Cov}(\hat{p}) \approx \begin{bmatrix} 0.5294786901 & 0.0037664373 & 0.1500170818 \\ 0.0037664373 & 0.0102426752 & -0.0002906948 \\ 0.1500170818 & -0.0002906948 & 0.2143424172 \end{bmatrix}$$

Approximate standard errors:

$$\text{SE}(\hat{\theta}) \approx 0.72782°, \tag{30}$$

$$\text{SE}(\hat{M}) \approx 0.10121, \tag{31}$$

$$\text{SE}(\hat{X}) \approx 0.46289. \tag{32}$$

Finally,

$$L_1 = 25.401461375629392. \tag{33}$$

Conclusion

The obtained solution satisfies the rotation identity as initially thought w.r.t solution, with M saturating at the lower bound -0.05, indicating maximal exponential decay. Since $\hat{M}$ lies on a boundary, the covariance approximation at that point is just indicative.. The models validity can be observed by plot of observed vs fitted curve , residuals and (u,v) rotation. Additionally , I have uploaded the colab notebook under FLAMR&D in the same github repository as this.

Verification and Plots

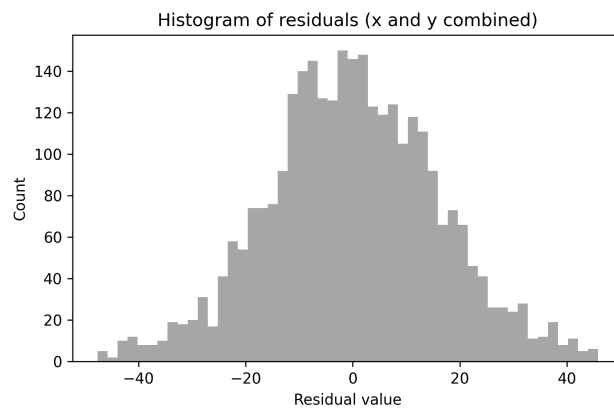The following figures validate the same as well.

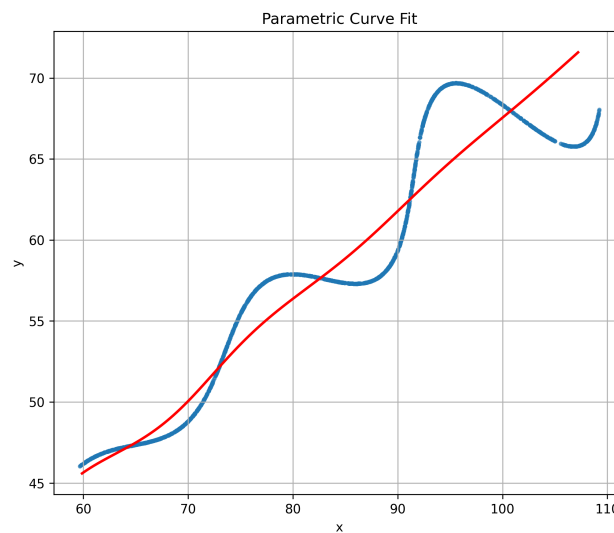Figure 1: Residual histogram showing zero-centered and symmetric error distribution.

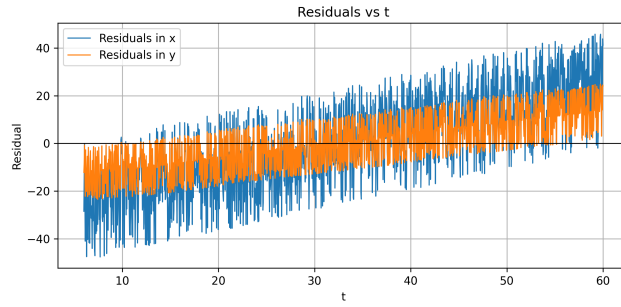Figure 2: Observed vs fitted parametric curve using final estimates.

Figure 3: Residuals in x and y versus t showing no systematic trend.
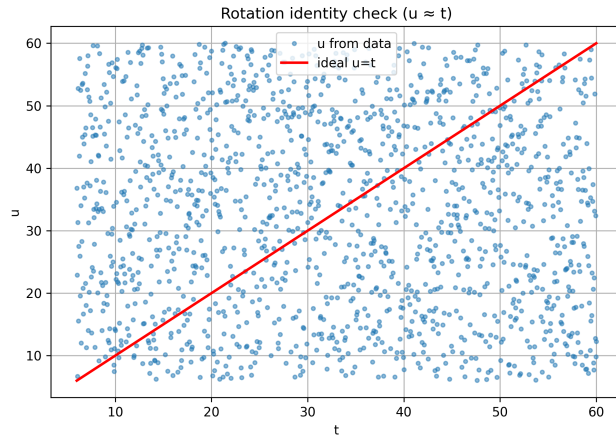

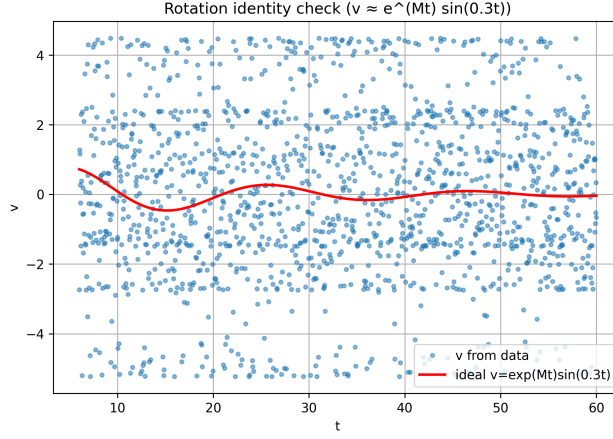Figure 4: Rotation identity check: observed u values align with $u = t$.

Figure 5: Rotation identity check: observed v follows $e^{Mt}\sin(0.3t)$.

Citations

We solved the bounded nonlinear least-squares problem using the Levenberg–Marquardt algorithm (LM), which combines damped Gauss Newton updates for robustness and fast local convergence [1]. For implementation and Jacobian based convergence properties we follow standard references [2].

References

[1] D. W. Marquardt. An algorithm for least-squares estimation of nonlinear parameters. *Journal of the Society for Industrial and Applied Mathematics*, 11(2):431–441, 1963.

[2] Jorge Nocedal and Stephen J. Wright. *Numerical Optimization*. Springer, 2 edition, 2006.