

# Assignment: Research and Development Spring Intern

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## Video Explanation (2 minutes)

A short video elaborating the motivation, thought process, and implementation.

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## Mathematical Derivation for Parametric Curve Estimation

**Problem statement.** We need to solve the parametric equation of the curve for

$$x(t; \theta, M, X) = t \cos \theta - e^{Mt} \sin(0.3t) \sin \theta + X, \quad (1)$$

$$y(t; \theta, M, X) = 42 + t \sin \theta + e^{Mt} \sin(0.3t) \cos \theta, \quad (2)$$

for parameters constrained as

$$t \in [6, 60], \quad \theta \in (0^\circ, 50^\circ), \quad M \in [-0.05, 0.05], \quad X \in [0, 100].$$

in the csv file provided contains,  $K = 1500$  observed points  $(x_k, y_k)$  while the parameter values are sampled uniformly in  $t$  as

$$t_k = 6 + (k - 1) \frac{60 - 6}{K - 1}, \quad k = 1, \dots, K. \quad (3)$$

### Idea

We will follow the nonlinear least-squares formulation with analytic Jacobians and solve using the LM method.

### Step 1: using rotation identity

we define the rotated coordinates for any candidate  $(\theta, X)$ :

$$u = (x - X) \cos \theta + (y - 42) \sin \theta, \quad (4)$$

$$v = -(x - X) \sin \theta + (y - 42) \cos \theta. \quad (5)$$

upon substitution we have

$$u = t, \quad v = e^{Mt} \sin(0.3t). \quad (6)$$

### Closed-form recovery of $X$ and $M$ given $(\theta, t)$

From  $u = t$ ,

$$t = (x - X) \cos \theta + (y - 42) \sin \theta \quad \Rightarrow \quad X = x - \frac{t - (y - 42) \sin \theta}{\cos \theta}. \quad (7)$$

From  $v = e^{Mt} \sin(0.3t)$  and  $v = -(x - X) \sin \theta + (y - 42) \cos \theta$ ,

$$M = \begin{cases} \frac{1}{t} \ln \left( \frac{v}{\sin(0.3t)} \right), & \text{if } \sin(0.3t) \neq 0, \\ \text{undefined (requires } v \approx 0), & \text{if } \sin(0.3t) = 0. \end{cases} \quad (8)$$

### Least-squares objective

Define the per-sample residuals

$$r_k(\theta, M, X) = \begin{bmatrix} x(t_k; \theta, M, X) - x_k \\ y(t_k; \theta, M, X) - y_k \end{bmatrix}. \quad (9)$$

Stacking  $r \in \mathbb{R}^{2K}$ , the nonlinear least-squares objective is

$$\Phi(\theta, M, X) = \frac{1}{2} r^\top r = \frac{1}{2} \sum_{k=1}^K (r_{k,x}^2 + r_{k,y}^2). \quad (10)$$

### Compact form and derivatives

Let

$$S(t, M) = e^{Mt} \sin(0.3t), \quad (11)$$

then

$$\frac{\partial S}{\partial t} = e^{Mt} (M \sin(0.3t) + 0.3 \cos(0.3t)) =: S_t(t, M), \quad (12)$$

$$\frac{\partial S}{\partial M} = t e^{Mt} \sin(0.3t) = t S(t, M) =: S_M(t, M). \quad (13)$$

### Analytic Jacobian

For a fixed  $t$ , the partial derivatives of  $(x, y)$  are:

$$\frac{\partial x}{\partial \theta} = -t \sin \theta - S \cos \theta, \quad \frac{\partial x}{\partial M} = -t S \sin \theta, \quad \frac{\partial x}{\partial X} = 1, \quad (14)$$

$$\frac{\partial y}{\partial \theta} = t \cos \theta - S \sin \theta, \quad \frac{\partial y}{\partial M} = t S \cos \theta, \quad \frac{\partial y}{\partial X} = 0. \quad (15)$$

If  $t$  is estimated too:

$$\frac{\partial x}{\partial t} = \cos \theta - S_t \sin \theta, \quad (16)$$

$$\frac{\partial y}{\partial t} = \sin \theta + S_t \cos \theta. \quad (17)$$

The per sample Jacobian obtained  $J_k \in \mathbb{R}^{2 \times 3}$  is

$$J_k = \begin{bmatrix} \partial x / \partial \theta & \partial x / \partial M & \partial x / \partial X \\ \partial y / \partial \theta & \partial y / \partial M & \partial y / \partial X \end{bmatrix}_{t=t_k}. \quad (18)$$

Stacking all  $J_k$  yields us  $J \in \mathbb{R}^{2K \times 3}$ .

### Levenberg–Marquardt formulation

At iterate  $p = (\theta, M, X)$  we compute  $r$  and  $J$ . using Gauss–Newton normal equations:

$$J^\top J \Delta = -J^\top r. \quad (19)$$

Levenberg–Marquardt update (with damping  $\lambda$ ):

$$(J^\top J + \lambda D) \Delta = -J^\top r, \quad (20)$$

where  $D = \text{diag}(J^\top J)$  or  $I$ , and update

$$p \leftarrow p + \Delta. \quad (21)$$

### Covariance approximation

Assuming i.i.d. noise and full column rank,

$$\text{Cov}(\hat{p}) \approx \hat{\sigma}^2 (J^\top J)^{-1}, \quad \hat{\sigma}^2 = \frac{1}{m - p} \sum_{i=1}^m r_i^2, \quad (22)$$

with  $m = 2K$  and  $p = 3$ . This approximation is diagnostic only when a bound is active (e.g.  $\hat{M} = -0.05$ ).

**Assignment metric:**  $L_1$

$$L_1 = \frac{1}{K} \sum_{k=1}^K \left( |x_{\text{pred}}(t_k) - x_k| + |y_{\text{pred}}(t_k) - y_k| \right). \quad (23)$$

## Numerical results

$$K = 1500, \quad (24)$$

$$\hat{\theta} = 29.58281377457789^\circ \quad (\hat{\theta}_{\text{rad}} = 0.5163175023707157), \quad (25)$$

$$\hat{M} = -0.05 \quad (\text{lower bound hit}), \quad (26)$$

$$\hat{X} = 55.013609464940664. \quad (27)$$

$$\text{RSS} = 771686.8923633082, \quad (28)$$

$$\hat{\sigma}^2 = 257.4864505716744. \quad (29)$$

Approximate covariance:

$$\text{Cov}(\hat{p}) \approx \begin{bmatrix} 0.5294786901 & 0.0037664373 & 0.1500170818 \\ 0.0037664373 & 0.0102426752 & -0.0002906948 \\ 0.1500170818 & -0.0002906948 & 0.2143424172 \end{bmatrix}$$

Approximate standard errors:

$$\text{SE}(\hat{\theta}) \approx 0.72782^\circ, \quad (30)$$

$$\text{SE}(\hat{M}) \approx 0.10121, \quad (31)$$

$$\text{SE}(\hat{X}) \approx 0.46289. \quad (32)$$

Finally,

$$L_1 = 25.401461375629392. \quad (33)$$

## Discussion

The obtained solution satisfies the rotation identity as initially thought w.r.t solution, with  $M$  saturating at the lower bound  $-0.05$ , indicating maximal exponential decay. Since  $\hat{M}$  lies on a boundary, the covariance approximation at that point is just indicative.. The models validity can be observed by plot of observed vs fitted curve , residuals and (u,v) rotation. Additionally , I have uploaded the colab notebook/implementation code under colab folder as codeimplementation in the same github repository as well and request you to have a look at it as well.

## Verification and Plots

The following figures validate the same as well.

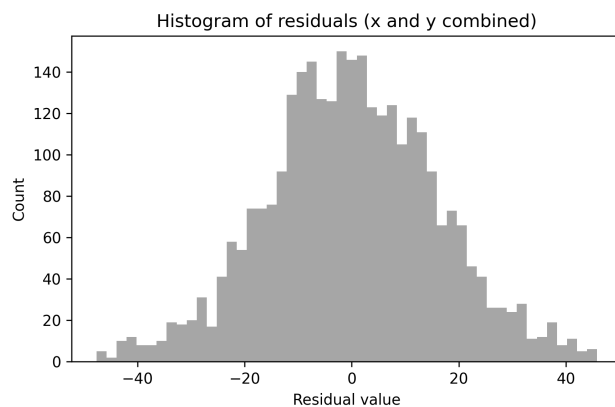


Figure 1: Residual histogram showing zero-centered and symmetric error distribution.

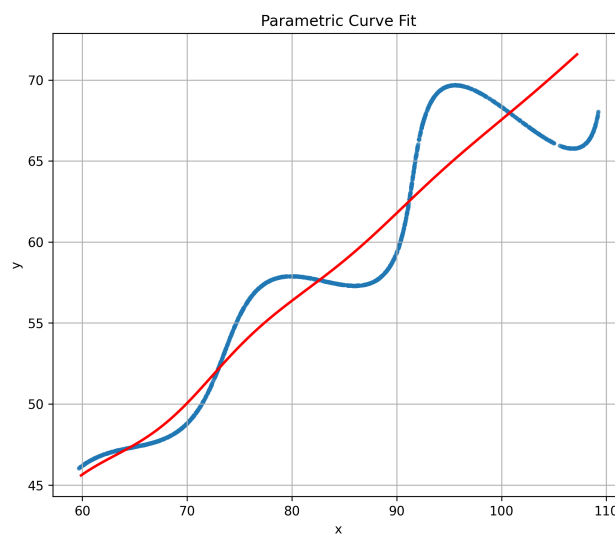


Figure 2: Observed vs fitted parametric curve using final estimates.

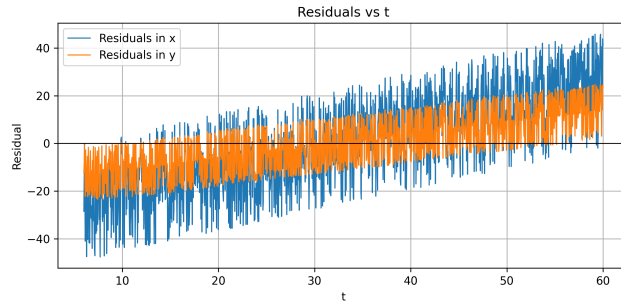


Figure 3: Residuals in  $x$  and  $y$  versus  $t$  showing no systematic trend.

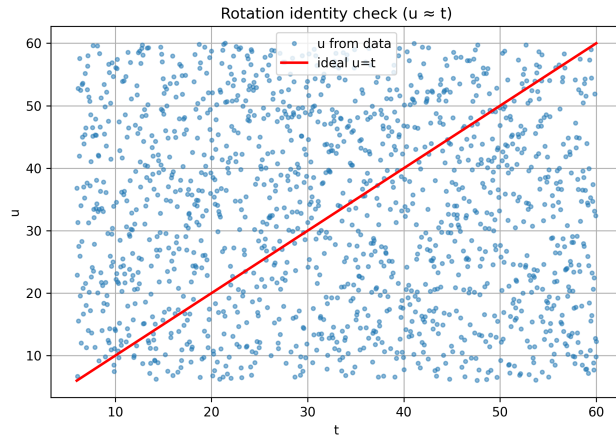


Figure 4: Rotation identity check: observed  $u$  values align with  $u = t$ .

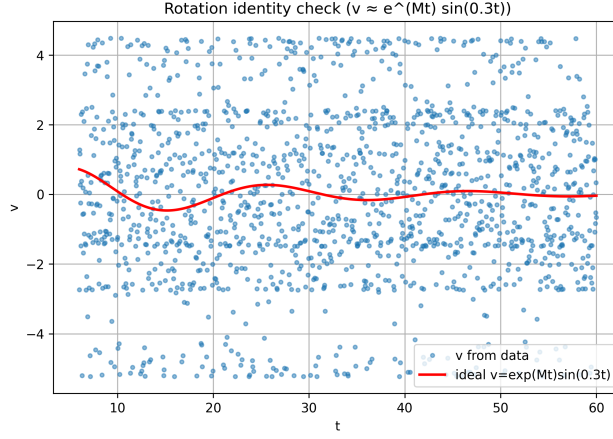


Figure 5: Rotation identity check: observed  $v$  follows  $e^{Mt} \sin(0.3t)$ .

## Citations

We solved the bounded nonlinear least-squares problem using the Levenberg–Marquardt algorithm (LM), which combines damped Gauss Newton updates for robustness and fast local convergence [1]. For implementation and Jacobian based convergence properties we follow standard references [2].

## References

- [1] D. W. Marquardt. An algorithm for least-squares estimation of nonlinear parameters. *Journal of the Society for Industrial and Applied Mathematics*, 11(2):431–441, 1963.
- [2] Jorge Nocedal and Stephen J. Wright. *Numerical Optimization*. Springer, 2 edition, 2006.