Assignment: Research and Development Spring Intern

Ayush Pandey
Amrita Vishwa Vidyapeetham
AV.EN.U4AIE22106
ayushpandey1177@gmail.com

Video Explanation (2 minutes)
A short video elaborating the motivation, thought process, and implementation.

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Mathematical Derivation for Parametric Curve Estimation

Problem statement. We need to solve the parametric equation of the curve for

$$x(t; \theta, M, X) = t \cos \theta - e^{Mt} \sin(0.3t) \sin \theta + X, \tag{1}$$

$$y(t; \theta, M, X) = 42 + t \sin \theta + e^{Mt} \sin(0.3t) \cos \theta, \tag{2}$$

for parameters constrained as

$$t \in [6, 60], \quad \theta \in (0°, 50°), \quad M \in [-0.05, 0.05], \quad X \in [0, 100].$$

in the csv file provided contains, $K = 1500$ observed points (x_k, y_k) while the parameter values are sampled uniformly in t as

$$t_k = 6 + (k - 1)\frac{60 - 6}{K - 1}, \qquad k = 1, \dots, K. \tag{3}$$

Idea
We will follow the nonlinear least-squares formulation with analytic Jacobians and solve using the LM method.

Step 1: using rotation identity

we define the rotated coordinates for any candidate (θ, X):

$$u = (x - X) \cos \theta + (y - 42) \sin \theta, \tag{4}$$

$$v = -(x - X) \sin \theta + (y - 42) \cos \theta. \tag{5}$$

upon substitution we have

$$u = t, \qquad v = e^{Mt} \sin(0.3t). \tag{6}$$

Closed-form recovery of X and M given (θ, t)

From $u = t$,

$$t = (x - X) \cos \theta + (y - 42) \sin \theta \quad \Rightarrow \quad X = x - \frac{t - (y - 42) \sin \theta}{\cos \theta}. \tag{7}$$

From $v = e^{Mt} \sin(0.3t)$ and $v = -(x - X) \sin \theta + (y - 42) \cos \theta$,

$$M = \begin{cases} \dfrac{1}{t} \ln\left(\dfrac{v}{\sin(0.3t)}\right), & \text{if } \sin(0.3t) \neq 0, \\ \text{undefined (requires } v \approx 0), & \text{if } \sin(0.3t) = 0. \end{cases} \tag{8}$$

Least-squares objective

Define the per-sample residuals

$$r_k(\theta, M, X) = \begin{bmatrix} x(t_k; \theta, M, X) - x_k \\ y(t_k; \theta, M, X) - y_k \end{bmatrix}. \tag{9}$$

Stacking $r \in \mathbb{R}^{2K}$, the nonlinear least-squares objective is

$$\Phi(\theta, M, X) = \tfrac{1}{2} r^\top r = \tfrac{1}{2} \sum_{k=1}^{K} \left(r_{k,x}^2 + r_{k,y}^2 \right). \tag{10}$$

Compact form and derivatives

Let

$$S(t, M) = e^{Mt} \sin(0.3t), \tag{11}$$

then

$$\frac{\partial S}{\partial t} = e^{Mt} \left(M \sin(0.3t) + 0.3 \cos(0.3t) \right) =: S_t(t, M), \tag{12}$$

$$\frac{\partial S}{\partial M} = t\, e^{Mt} \sin(0.3t) = t\, S(t, M) =: S_M(t, M). \tag{13}$$

Analytic Jacobian

For a fixed t, the partial derivatives of (x, y) are:

$$\frac{\partial x}{\partial \theta} = -t \sin \theta - S \cos \theta, \qquad \frac{\partial x}{\partial M} = -tS \sin \theta, \qquad \frac{\partial x}{\partial X} = 1, \tag{14}$$

$$\frac{\partial y}{\partial \theta} = t \cos \theta - S \sin \theta, \qquad \frac{\partial y}{\partial M} = tS \cos \theta, \qquad \frac{\partial y}{\partial X} = 0. \tag{15}$$

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If t is estimated too:

$$\frac{\partial x}{\partial t} = \cos\theta - S_t \sin\theta, \tag{16}$$

$$\frac{\partial y}{\partial t} = \sin\theta + S_t \cos\theta. \tag{17}$$

The per sample Jacobian obtained $J_k \in \mathbb{R}^{2\times3}$ is

$$J_k = \begin{bmatrix} \partial x/\partial\theta & \partial x/\partial M & \partial x/\partial X \\ \partial y/\partial\theta & \partial y/\partial M & \partial y/\partial X \end{bmatrix}_{t=t_k}. \tag{18}$$

Stacking all J_k yields us $J \in \mathbb{R}^{2K\times3}$.

Levenberg–Marquardt formulation

At iterate $p = (\theta, M, X)$ we compute r and J. using Gauss–Newton normal equations:

$$J^\top J \Delta = -J^\top r. \tag{19}$$

Levenberg–Marquardt update (with damping λ):

$$(J^\top J + \lambda D)\Delta = -J^\top r, \tag{20}$$

where $D = \text{diag}(J^\top J)$ or I, and update

$$p \leftarrow p + \Delta. \tag{21}$$

Covariance approximation

Assuming i.i.d. noise and full column rank,

$$\text{Cov}(\hat{p}) \approx \hat{\sigma}^2 (J^\top J)^{-1}, \qquad \hat{\sigma}^2 = \frac{1}{m-p} \sum_{i=1}^{m} r_i^2, \tag{22}$$

with $m = 2K$ and $p = 3$. This approximation is diagnostic only when a bound is active (e.g. $\hat{M} = -0.05$).

Assignment metric: L_1

$$L_1 = \frac{1}{K} \sum_{k=1}^{K} \left(\left| x_{\text{pred}}(t_k) - x_k \right| + \left| y_{\text{pred}}(t_k) - y_k \right| \right). \tag{23}$$

Numerical results

$$K = 1500, \tag{24}$$

$$\hat{\theta} = 29.58281377457789° \quad (\hat{\theta}_{\text{rad}} = 0.5163175023707157), \tag{25}$$

$$\hat{M} = -0.05 \quad \text{(lower bound hit)}, \tag{26}$$

$$\hat{X} = 55.013609464940664. \tag{27}$$

$$\text{RSS} = 771686.8923633082, \tag{28}$$

$$\hat{\sigma}^2 = 257.4864505716744. \tag{29}$$

Approximate covariance:

$$\text{Cov}(\hat{p}) \approx \begin{bmatrix} 0.5294786901 & 0.0037664373 & 0.1500170818 \\ 0.0037664373 & 0.0102426752 & -0.0002906948 \\ 0.1500170818 & -0.0002906948 & 0.2143424172 \end{bmatrix}$$

Approximate standard errors:

$$\text{SE}(\hat{\theta}) \approx 0.72782°, \tag{30}$$

$$\text{SE}(\hat{M}) \approx 0.10121, \tag{31}$$

$$\text{SE}(\hat{X}) \approx 0.46289. \tag{32}$$

Finally,

$$L_1 = 25.401461375629392. \tag{33}$$

Discussion

The obtained solution satisfies the rotation identity as initially thought w.r.t solution, with M saturating at the lower bound -0.05, indicating maximal exponential decay. Since $\hat{M}$ lies on a boundary, the covariance approximation at that point is just indicative.. The models validity can be observed by plot of observed vs fitted curve , residuals and (u,v) rotation. Additionally , I have uploaded the colab notebook/implementation code under colab folder as codeimplementaion in the same github repository as well and request you to have a look at it as well.

Verification and Plots

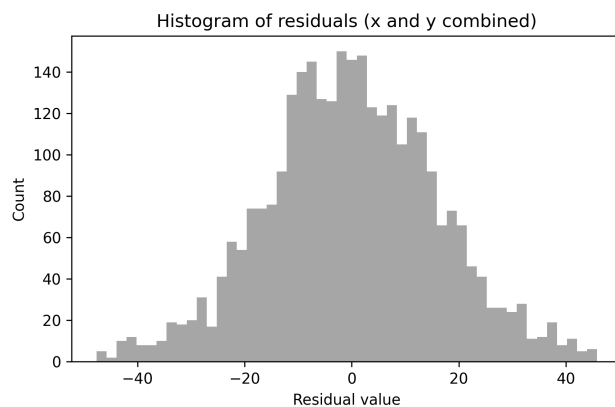The following figures validate the same as well.

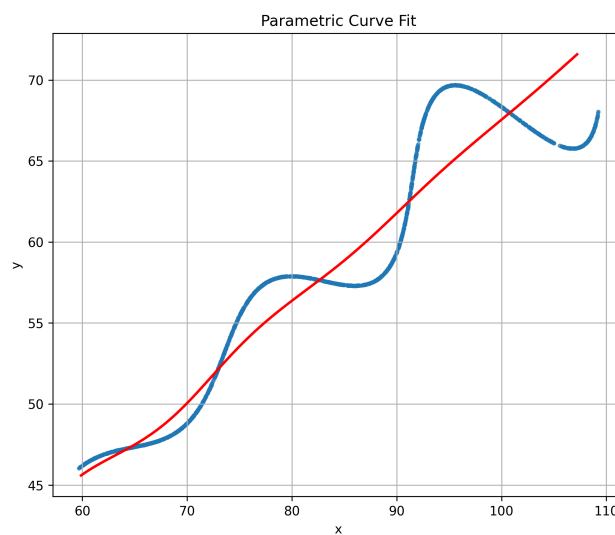Figure 1: Residual histogram showing zero-centered and symmetric error distribution.

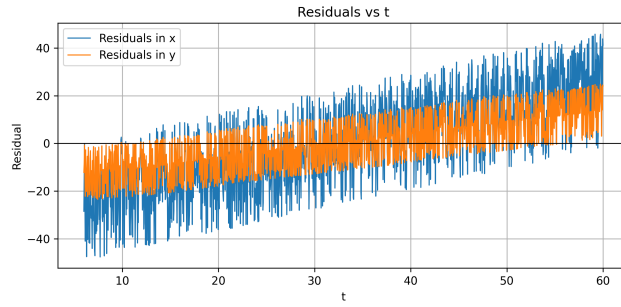Figure 2: Observed vs fitted parametric curve using final estimates.

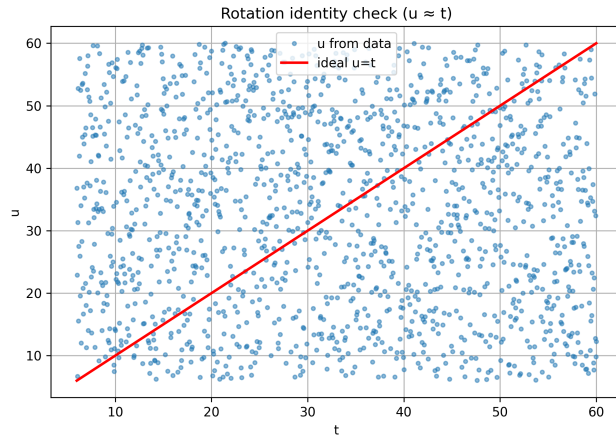Figure 3: Residuals in x and y versus t showing no systematic trend.

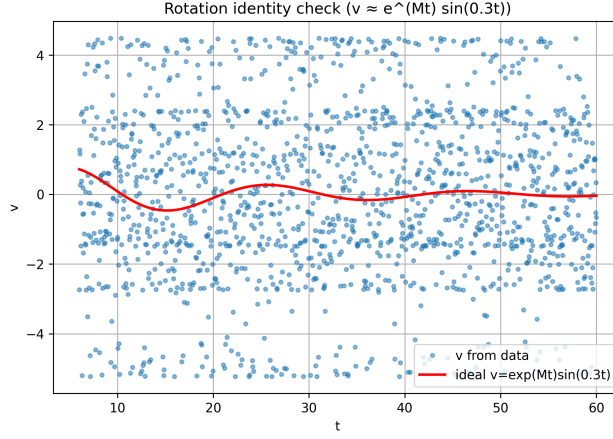Figure 4: Rotation identity check: observed u values align with $u = t$.

Figure 5: Rotation identity check: observed v follows $e^{Mt}\sin(0.3t)$.

Citations

We solved the bounded nonlinear least-squares problem using the Levenberg–Marquardt algorithm (LM), which combines damped Gauss Newton updates for robustness and fast local convergence [1]. For implementation and Jacobian based convergence properties we follow standard references [2].

References

[1] D. W. Marquardt. An algorithm for least-squares estimation of nonlinear parameters. *Journal of the Society for Industrial and Applied Mathematics*, 11(2):431–441, 1963.

[2] Jorge Nocedal and Stephen J. Wright. *Numerical Optimization*. Springer, 2 edition, 2006.