Lecture 3 blackboard notes

We had:

$$
\begin{aligned}
2x_1 + 3x_2 - 2x_3 + 5x_4 &= 1 \\
x_1 + 2x_2 - x_3 + 3x_4 &= 2 \\
-x_1 - 2x_2 + x_3 - x_4 &= 4
\end{aligned}
$$

$$\Downarrow$$

$$
\left[\begin{array}{cccc|c}
2 & 3 & -2 & 5 & 1 \\
1 & 2 & -1 & 3 & 2 \\
-1 & -2 & 1 & -1 & 4
\end{array}\right]
$$

$$\Downarrow \text{ EROs}$$

$$
\left[\begin{array}{cccc|c}
\boxed{1} & 2 & -1 & 3 & 2 \\
0 & \boxed{-1} & 0 & -1 & -3 \\
0 & 0 & 0 & \boxed{2} & 6
\end{array}\right]
$$

pivot columns $\Rightarrow x_1, x_2, x_4$ pivot vars

x_3 free variables

* to get $Ax = b$

set $x_3 = 0$.

Back-substitution:

$2x_4 = 6 \Rightarrow x_4 = \frac{6}{2} = 3$ ✓

$-x_2 - x_4 = -3 \Rightarrow x_2 = 0$

$$x_1 + \underbrace{2x_2}_{0} \underbrace{- x_3}_{=0} + \underbrace{3x_4}_{9} = 2 \Rightarrow \boxed{x_1 = -7}$$

One solution : $\begin{bmatrix} -7 \\ 0 \\ 0 \\ 3 \end{bmatrix}$

* Solve $Ax = 0$.

$N(A)$ or $null(A) = \{x : Ax = 0\}$.

$\boxed{x_3 = 1}$

by back substitution. $\Rightarrow \begin{cases} x_2 = 0 \\ x_4 = 0 \\ x_1 = 1 \end{cases}$

Solution $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $A\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0 \quad | \cdot \lambda \quad , \lambda \in \mathbb{R}$

$A \cdot \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \lambda \cdot 0 = 0$, $\lambda \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is a solution

$$N(A) = \left\{ \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \lambda \in \mathbb{R} \right\}$$

* General solution

$$\left\{ x \in \mathbb{R}^4 : x = \begin{bmatrix} -7 \\ 0 \\ 0 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \lambda \in \mathbb{R} \right\}.$$

$M_{m \times n}(\mathbb{R})$

Exercise

Prove that given $A \in \mathbb{R}^{m\times n}$ $(m<n)$, then $Ax=0$ has infinitely many solutions.

Proof:

System $Ax=0$ has at least one solution 0.

$A\cdot 0 = 0.$

ERO: elementary row operations.

$$A \xrightarrow{EROs} R$$

$\exists$ At most m pivots in R.

$n>m \Rightarrow \exists\ n-m \geq 1$ free variables. $\Rightarrow$

$\Rightarrow \exists x^*$ s.t x^* is solution $\Rightarrow \lambda x^*$ are solutions (∞ many).

$\lambda \in \mathbb{R}$

$\therefore$ q.e.d.

Gauss-Jordan (GJ) method / Gaussian elimination

$$[A \mid I_n] \xrightarrow{EROs} [I_n \mid \underbrace{B}_{B=A^{-1}}]$$

$A \in M_{n\times n}(\mathbb{R})$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right) \longrightarrow$$

$$\xrightarrow[A_{23}(-2)]{A_{21}(-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right)$$

Define $A_{rs}(\lambda)$: add λ times row r to row s $(r \neq s)$

$M_r(\lambda)$: multiply row r by λ

P_{rs} : swap row r with row s.

$$\Big\Downarrow \begin{array}{l} A_{31}(-1) \\ A_{32}(2) \\ M_3(-1) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & +1 & -1 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right)$$

$$\underbrace{}_{I_n} \qquad \underbrace{\phantom{A^{-1}}}_{A^{-1}}$$

$$C = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 9 \\ 4 & 12 & 17 \end{pmatrix} \xrightarrow[\substack{A_{23}(-1) \\ A_{12}(-2) \\ A_{13}(-4)}]{\text{EROs}} \underbrace{\begin{pmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$$

can't be reduced to I_3, has <u>no inverse</u>!

$\Downarrow$

also looking at rows:
row 3 = row2 + 2·row1, linearly dependent.

Lemma $A \in M_{m \times n}(\mathbb{R})$

$Ax = b$

i) Let $B \in M_{m \times m}(\mathbb{R})$ be invertible.

$(BA)x = Bb$ has the same solutions as $Ax = b$. (proof at home)

ii) A is invertible. Then $Ax = b$ has a unique solution $\underline{x = A^{-1}b}$.

$$\underbrace{E_k \cdot E_{k-1} E_{k-2} \cdots E_1}_{\text{EROs}} A = R.$$

B.

$BAx = Bb$.

Theorem

$A \in M_{n \times n}(\mathbb{R})$ is a square matrix.

Then all the statements are pairwise equivalent.

i) RREF of A is the identity matrix

$$\begin{pmatrix} 1 & \cdots & 0 \\ 0 & 1 \cdots & 0 \\ 0 & \cdots & 1 \end{pmatrix}$$

ii) the only solution to $Ax = 0$ is the trivial Solution $x=0$.

MOORE-PENROSE INVERSE

$A \in M_{m\times n}(\mathbb{R})$

G is a pseudo-inverse for A if:

$AGA = A$

$GAG = G$

$(AG)^T = AG$

$(GA)^T = GA$.

Under mild conditions for A:

$Ax = b$.

$A^T A x = A^T b$

$x = \underbrace{(A^T A)^{-1} A^T}_{\text{pseudo-inverse}} b$.

pseudo-inverse (Moore-Penrose pseudo inverse)

VECTOR SPACES

Defn

A (real-valued) vector space V is a non-empty set V together with two operations.

$$+ : V \otimes V \to V \qquad \begin{bmatrix} 4 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

$$\cdot : R \otimes V \to V \qquad 5 \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 20 \\ 35 \end{bmatrix}$$

(i) addition: $\forall x, y \in V$, then $+$ allows us to get $x + y$.

(ii) Scalar multiplication: $\forall x \in V$, then we can get λx

CLOSURE AXIOMS

i) If $x, y \in V$, then $x + y \in V$.

ii) If $x \in V, \lambda \in R$, then $\lambda x \in V$.

AXIOMS FOR ADDITION

i) associativity

$(x+y) + z = x + (y+z)$, $\forall x, y, z \in V$.

ii) existence of additive identity (neutral element)

$\exists$ vector $v = 0 \in V$ s.t.

$x + 0 = 0 + x = x$, $\forall x \in V$.

iii) existence of additive inverse

$\forall x \in V \ \exists \ -x \in V$ s.t. $x + (-x) = 0$

iv) commutative

$\forall x, y \in V \quad x + y = y + x$

$(V, +)$ is an Abelian group.

AXIOMS FOR SCALAR MULTIPLICATION

i) associativity

$\lambda, \varphi \in \mathbb{R}, \quad x \in V$

$(\lambda \varphi) x = \lambda (\varphi x)$

ii) distributivity

$(\lambda + \varphi) x = \lambda x + \varphi x$

$\lambda \underbrace{(x+y)}_{\in V} = \lambda x + \lambda y.$

iii) $0 x = 0$

iv) $\exists$ of a neutral element

$1 \cdot x = x \cdot 1 = x \quad \forall x \in V.$

Examples

1) $\mathbb{R}^n$: YES
2) $M_{m\times n}(\mathbb{R})$: YES

$M_{3\times 2}(\mathbb{R})$

$$A_1 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{pmatrix}$$

$$\lambda A_1 = \begin{pmatrix} \lambda a_1 & \lambda a_2 \\ \lambda a_3 & \lambda a_4 \\ \lambda a_5 & \lambda a_6 \end{pmatrix}$$

$$A_1 + A_2 = \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \\ a_5+b_5 & a_6+b_6 \end{pmatrix}$$

$A_3 \in M_{3\times 2}(\mathbb{R})$

$(A_1 + A_2) + A_3 = A_1 + (A_2 + A_3)$

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad A_1 + 0 = 0 + A_1 = A_1$$

$-A_1 : A_1 + (-A_1) = 0$

$A_1 + A_2 = A_2 + A_1$

$\cancel{A_1}\ A_2 \neq A_2 A_1$ not the same operation!

$(\lambda \varphi) A_1 = \lambda(\varphi A_1)$.

Vector Subspaces

Proposition (Subspace Criterion)

A non-empty subset $\mathcal{U}$ ($\mathcal{U} \subseteq \mathcal{V}$) of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if it satisfies:

i) Closure under addition

$\forall x, y \in \mathcal{U}$, $x + y \in \mathcal{U}$.

ii) closure under scalar multiplication

$\forall x \in \mathcal{U}$, $\lambda \in \mathbb{R}$, then $\lambda x \in \mathcal{U}$.

iii) existence of an origin vector $0 \in \mathcal{U}$.

$Ax = 0$

$N(A) = \{ x : Ax = 0 \}$.

$A \cdot 0 = 0$?

$x_1 : Ax_1 = 0$.

$x_2 : Ax_2 = 0$.

$A(x_1 + x_2) = 0$

$x_1 : Ax_1 = 0 \quad | \cdot \lambda$

$A \lambda x_1 = \lambda \cdot 0 = 0.$

(λx_1) is also a solution.

$Ax = b \quad (b \neq 0)$

Linear combination

Let $x_1, \dots, x_k \in V$

A linear combination is a vector

$$v = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k = \sum_{i=1}^{k} \lambda_i x_i, \quad \lambda_i \in \mathbb{R}$$

$$\mathrm{Span}(x_1, \dots x_k) = \{ \lambda_1 x_1 + \lambda_2 x_2 \dots + \lambda_k x_k, \; \lambda_1, \dots \lambda_k \in \mathbb{R} \}.$$

Lemma

Let $x_1 \dots x_k$ be k vectors in V.

Then $\mathrm{Span}(x_1 \dots x_k)$ is a vector subspace of V.

Lemma

$x_1 \dots x_k$ vectors in V. x_1 is a linear combination of the others.

$$x_1 = \psi_1 x_2 + \psi_2 x_3 + \dots \psi_{k-1} x_k.$$

Then:

$$\text{Span}(x_1 \dots x_k) = \text{Span}(x_2, \dots x_k).$$

Linear Independence

Consider a set of linear functions.

$$\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k = 0.$$

The vectors $x_1 \dots x_k$ are li if the only solution is the trivial one.

$$\lambda_1 = \lambda_2 = \lambda_3 \dots = \lambda_k = 0.$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{ccc} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ \uparrow \quad \uparrow \quad \uparrow \\ x_1 \quad x_2 \quad x_3 \end{array}$$

$$x_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

free

$$x_2 = 3x_1$$

$A_1, A_2, A_3 \in M_{3\times 2}(\mathbb{R})$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 = 0.$$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \\ 0 & \lambda_1 \end{pmatrix} + \begin{pmatrix} 0 & \lambda_2 \\ \lambda_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \lambda_3 \\ 0 & 0 \\ 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} \lambda_1 & \lambda_2 + \lambda_3 \\ \lambda_2 & 0 \\ 0 & \lambda_1 + \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proposition

Let $x_1 \dots x_k \in V$ be k L.I. vectors.

Let $u \in V$ s.t.

$$u \notin \text{Span}(x_1 \dots x_k$$

Then $x_1, x_2 \dots, x_k, u$ are linearly independent.

Basis

A set of vectors $\{x_1 \dots x_\ell\}$ form a basis for V if,

i) if $\{x_1 \dots x_\ell\}$ span the vector space V.

ii) they are linearly independent.