Lecture 4 blackboard notes

Linear Independence (from here)

Consider a set of linear functions.

$$\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k = 0.$$

The vectors $x_1 \dots x_k$ are li if the only solution is the trivial one.

$$\lambda_1 = \lambda_2 = \lambda_3 \dots = \lambda_k = 0.$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{array}{ccc} \left[\begin{matrix} 1 & 3 & 0 \\ 0 & 0 & 2 \end{matrix}\right] \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 \end{matrix} \end{array}$$

free!

how we can verify if vectors are li.

$x_2 = 3x_1$

$A_1, A_2, A_3 \in M_{3\times 2}(\mathbb{R})$

$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$

$A_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 = 0.$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \\ 0 & \lambda_1 \end{pmatrix} + \begin{pmatrix} 0 & \lambda_2 \\ \lambda_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \lambda_3 \\ 0 & 0 \\ 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Downarrow$

$$\begin{pmatrix} \lambda_1 & \lambda_2 + \lambda_3 \\ \lambda_2 & 0 \\ 0 & \lambda_1 + \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proposition

Let $x_1 \dots x_k \in V$ be k l.i. vectors.

Let $u \in V$ s.t.

$$u \notin \text{Span}(x_1 \dots x_k)$$

Then $x_1, x_2 \dots, x_k, u$ are linearly independent.

Basis

A set of vectors $\{x_1 \dots x_\ell\}$ form a basis for V if:

i) $\{x_1 \dots x_\ell\}$ span the vector space V.

ii) they are linearly independent.

Example

$\mathbb{R}^3$: $\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{canonical basis}}$ confirm they are a basis

1) linear independence

$$\underbrace{\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}}$$

3 pivots : vectors are linearly independent

$\forall v \in \mathbb{R}^3$, $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\exists\ \lambda_1, \lambda_2, \lambda_3$

s.t. $v = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$.

$$\begin{cases} e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{cases}$$

We can write:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

So then $\lambda_1 = a$, $\lambda_2 = b$, $\lambda_3 = c$. | They exist!

Theorem

Let $x_1, x_2, \dots, x_k \in V$. Then the following statements are pairwise equivalent.

1) $x_1, x_2, x_3, \dots, x_k$ form a basis $\mathcal{B}$.

2) $x_1, x_2, x_3 \dots x_k$ form a maximal linear independent set in V, i.e. one cannot add another vector in V to the x_i's and still obtain a linearly independent set.

3) Any vector v in V can be <u>uniquely</u> written as:

$$v = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \dots + \lambda_k x_k$$

4) The vectors $x_1, x_2, \dots, x_k$ form a minimal spanning set of V, i.e. one cannot remove one of the x_i's and still have a spanning set of V.

Let's prove 2) $\rightarrow$ 3).

Proof.

$$v = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k =$$

$$= \mu_1 x_1 + \mu_2 x_2 + \dots + \mu_k x_k \Rightarrow$$

$$\Rightarrow 0 = (\lambda_1 - \mu_1) x_1 + \dots + (\lambda_k - \mu_k) x_k.$$

But $x_1, x_2, \dots, x_k$ are linearly independent (2).

$$\Rightarrow \lambda_i - \mu_i = 0 \Rightarrow \boxed{\lambda_i = \mu_i}$$

Theorem

Let V be a finite - dimensional vector Space. Then every basis has the same number of elements and we call this the dimension of V and write $\boxed{\dim(V)}$.

Furthermore, any subspace of V, U, is also finite - dimensional and we have:

$$\dim(U) \leq \dim(V) \quad \text{with}$$

$$\dim U = \dim V \iff \boxed{U = V}.$$

Determining a Basis

Vector Subspace $U \subseteq \mathbb{R}^5$.

U Spanned by:

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} \quad X_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{bmatrix} \quad \in \mathbb{R}^5$$

$$\Downarrow$$

$$[X_1, X_2, X_3, X_4]$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x_1, x_2, x_4$ are linearly independent (pivots).

Therefore, x_1, x_2, x_4 is a basis of U.

Coordinates of a vector

Let V be a vector space of dim n and B be an ordered basis of V,

$B = (b_1, b_2, \dots, b_n)$.

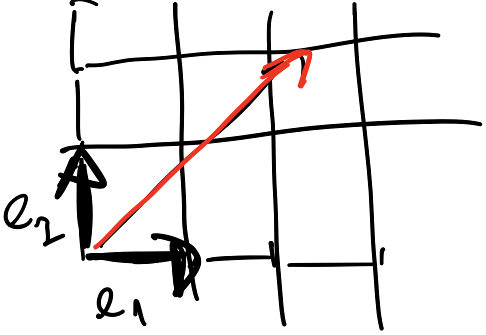
$\forall x \in V$, we have a unique representation

$$\boxed{x = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n}$$

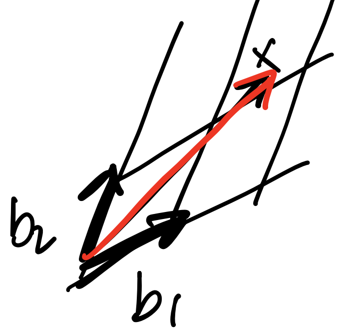
of x with respect to B.

Then $\alpha_1, \alpha_2 \dots, \alpha_n$ are the coordinates of x with respect to B. and:

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n$$ is the coordinate vector/ coordinate representation of x with respect to ordered basis b.



Cartesian coordinates



under different basis

Rank

The number of linearly independent columns of a matrix $A \in M_{m \times n}(\mathbb{R})$ is called the (column) rank of A and is denoted by $rk(A)$.

Gives us an idea of how much information is encoded in a matrix.

try to prove this

Properties

1) $rk(A) = rk(A^T)$ (rank of A also equals the number of linearly independent rows)

2) $\forall \; A \in M_{n \times n}(\mathbb{R})$, A is invertible iff $rk(A) = n$.

3) The basis of the subspace spanned by the columns (rows) of $A \in M_{m \times n}(\mathbb{R})$ can be found by Gaussian (GJ) elimination to identify the pivot columns.

4) $\forall A \in M_{m\times n}(\mathbb{R})$, $b \in \mathbb{R}^m$, $Ax = b$ can be solved iff $rk(A) = rk(A|b)$, where $A|b$ denotes the augmented matrix.

5) For $A \in M_{m\times n}(\mathbb{R})$ the subspace of solutions for $Ax=0$ possess dimension $n - rk(A)$.

6) A matrix has full rank if its rank equals the largest possible rank for a matrix of the same dimensions.

7) $rk(A) = \min(m, n)$ for fr

8) A matrix is said to be rank deficient if it doesn't have full rank.

Bringing it all together

$A \in R_{3\times4}(\mathbb{R})$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} a_{11}' & a_{12}' & a_{13}' & a_{14}' \\ 0 & a_{22}' & a_{23}' & a_{24}' \\ 0 & 0 & 0 & a_{34}' \end{pmatrix}$$

$c_1 \quad c_2 \quad c_3 \quad c_4$

free

3.

$C(A) = \text{Span}\{ c_1, c_2, c_3, c_4 \}$

$C(A)$ subspace of $\mathbb{R}^3 = \{ b \in \mathbb{R}^3 : Ax = b \text{ has solution} \}$

$\dim C(A) = \#$ of pivots $=$ rank of $A = 3$

$rk(A) = rk(A|b)$

$N(A) = \{ x \in \mathbb{R}^4 : Ax = 0 \}$

Subspace in $\mathbb{R}^4$

how many dimesions? 1 (number of free variables)

rank $= 3$.

$\dim N(A) = n - r = 4 - 3 = 1.$

The fundamental subspaces of A

$A \in M_{m \times n}(\mathbb{R})$.

Can also define $C(A^T)$: row space

$N(A^T) = \{ y \in \mathbb{R}^m : A^T y = 0 \}$

$\mathbb{R}^m$

$C(A)$ is a subspace of this

$\dim C(A) = r$

$\mathbb{R}^n$

$N(A)$

$\dim N(A) = n - r$.

One final trick

told you yesterday that multiplying row with λ $\equiv$ multiplying by a specific matrix $M_r(\lambda)$ that 's invertible.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

multiply row 1 by 5:

$$B = \begin{pmatrix} 5 & 10 \\ 3 & 4 \end{pmatrix}$$

Define $M_1(5) = \underbrace{5 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{mul}} = \underbrace{\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{identity matrix whose diagonal entry for } r^{\text{th}} \text{ row is } \lambda.}$

Then $M_1(5) \cdot A = B$

$$\left[\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 3 & 4 \end{pmatrix} \right.$$

$M_1(5)$ is invertible tho!

$$M_1^{-1}(5) = M_1\left(\frac{1}{5}\right)$$

$$\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1/5 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$A \rightarrow R$

$\underbrace{E_k E_{k-1} \dots E_1}_{\text{invertible}} A = R \Rightarrow$ So that's why we can say the solutions to $Ax = 0$ are the same as $Rx = 0$.

invertible elementary matrices associated with EROs!

linear mappings

Let V & W be 2 vector spaces.

A mapping $\Phi: V \to W$ is a linear mapping if:

$$\forall x, y \in V, \lambda, \psi \in \mathbb{R}:$$

$$\Phi(\lambda x + \psi y) = \lambda \Phi(x) + \psi \Phi(y)$$

This implies:

$$* \ \Phi(x+y) = \Phi(x) + \Phi(y).$$

$* \quad \Phi(\lambda x) = \lambda \Phi(x).$

$T(x) := Ax \quad , A \in M_{m \times n}(\mathbb{R})$

$T: \mathbb{R}^n \to \mathbb{R}^m$

$T(u+w) = A(u+w) = Au + Aw =$
$= T(u) + T(w).$

$T(\lambda u) = A \lambda u = \lambda A u = \lambda T(u).$

Lemma

$\Phi: V \longrightarrow W$

$\psi: W \to X.$

$\Phi \cdot \psi \to V \to X$ is also linear.

Proof.

Lemma

$\Phi : \underset{\sim}{V} \rightarrow \underset{\sim}{W}$.

$\Psi : \underset{\sim}{V} \rightarrow \underset{\sim}{W}$.

$\Phi + \Psi$ is also a linear mapping.

$\lambda \Phi$ is also a linear mapping.

$\lambda \in \mathbb{R}$.

V, W — $C = (c_1, c_2 \dots c_m)$ basis

$B = (b_1, b_2 \dots b_n)$ basis.

$\Phi : V \rightarrow W$.

$\Phi(b_j) = \alpha_{1j} c_1 + \alpha_{2j} \cdot c_2 + \dots \alpha_{mj} \cdot c_m =$

$$= \sum_{i=1}^{m} \alpha_{ij} c_i.$$

unique representation of $\phi(b_j)$ with respect to C.

$$\underbrace{A_\phi}_{\text{transformation matrix.}}(i,j) = \alpha_{ij}$$

<u>Proposition</u>

$\phi : V \to W.$

V, $x \in V$, $\hat{x}$ is the coordinate vector of x with respect to B (basis of V).

$\hat{\underline{y}}$ is the coordinate vector of $y = \phi(x) \in \underline{W}$ with respect to C.

then:

$$\boxed{\hat{y} = A_\phi \hat{x}.}$$