WW - DML

$B : D \times M$

$b : D \times 1 \quad M = 1$

$z_n = b^T . x_n \qquad \tilde{x}_n = b . z_n$

$= b . (b^T . \cancel{x}_n)$

Reconstruction loss

$$L(z, b) = \frac{1}{N} \sum_{n=1}^{N} \| \tilde{x}_n - x_n \|_2^2$$

$$= \frac{1}{N} \sum_{n=1}^{N} \| b b^T x_n - x_n \|_2^2$$

$$= \frac{1}{N} \sum_{n=1}^{N} \left(x_n^T x_n - 2 x_n^T b b^T x_n + x_n^T b \underbrace{b^T b}_{1} b^T x_n \right)$$

$$= \frac{1}{N} \sum_{n=1}^{N} \left(x_n^T x_n - x_n^T b b^T x_n \right)$$

$$= \underbrace{\frac{1}{N} \sum_{n=1}^{N} x_n^T x_n} - b^T \left(\frac{1}{N} \sum_{n} x_n x_n^T \right) b$$

$$\underbrace{L}_{\text{reconstruction loss}} = C - \underbrace{b^T \cdot S\, b}_{\text{variance of } z = b^T x}$$

minimize $\equiv$ maximize

$\|b\|_2^2 = 1$?, why variance $= b^T S b$?

$z_1 = b_1^T \cdot x$

$z_{11} = b_1^T x_1$

$\vdots$

$z_{1N} = b_1^T \cdot x_N$

$$\mathrm{Var}(z_1) = \frac{1}{N} \sum_n z_{1n}^2$$

$$= \frac{1}{N} \sum_n (b_1^T \cdot x_n)^2$$

$$= \frac{1}{N} \sum b_1^T x_n \cdot x_n^T b_1$$

$$= b_1^T \left[\frac{1}{N} \sum_n x_n x_n^T \right] b_1$$

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$$= \frac{1}{N} \sum b_1^T x_n \cdot x_n^T b_1$$

$$= b_1^T \left[\frac{1}{N} \sum_n x_n x_n^T \right] b_1$$

S

$$\boxed{V = b_1^T S b_1}$$

$b_1 \nearrow$

$$\max_{b_1} V$$

$$\boxed{\|b_1\|_2^2 = 1}$$ equality constraint

$$\max_{b_1} V + \alpha \cdot \left(\|b_1\|_2^2 - 1\right) = F$$

$$\frac{dF}{db_1} = 0 \qquad \frac{dF}{d\alpha} = 0$$

Q6, A4

$\rightarrow$ b_1 eigen vector of S

α eigen value of S

$$J = \frac{1}{N} \sum_n \| x_n - \tilde{x}_n \|_2^2$$

$$\tilde{x}_n = B \cdot z_n = \sum_{m=1}^{M} z_{mn} \cdot b_m$$

$\{\{z_{mn}\}_{m=1}^{M}\}_{n=1}^{N}$?

Given B

$$\frac{dJ}{dz_{in}} = \frac{dJ}{d\tilde{x}_n} \cdot \frac{d\tilde{x}_n}{dz_{in}}$$

$$= \left[\frac{1}{N}(-2)(x_n - \tilde{x}_n)\right]^T \cdot b_i$$

$$= -\frac{2}{N} \cdot \left[x_n^T b_i - \tilde{x}_n^T \cdot b_i \right]$$

$$= -\frac{2}{N} \cdot \left[x_n^T b_i - \left[\sum_{m=1}^{M} z_{mn} b_m\right]^T b_i \right]$$

$$= -\frac{2}{N} \left[x_n^T b_i - \sum_{m=1}^{M} \left(z_{mn} \cdot b_m^T \cdot b_i \right) \right]$$

0 if $m \neq i$

1 if $m = i$

$$= -\frac{2}{N}\left[x_n^T b_i - z_{in}\right] = 0$$

$$\boxed{z_{in} = x_n^T b_i = b_i^T x_n}$$

z_{in} = coordinate of the orthogonal proj onto a one-d subspace spanned by b_i

$$x_n = \sum_{m=1}^{D} z_{mn} b_m$$

$$M < D$$

$$\tilde{x}_n = \sum_{m=1}^{M} z_{mn} b_m$$

$$x_n - \tilde{x}_n = \sum_{m=M+1}^{D} \underline{z_{mn}} b_m$$

$$\hookrightarrow (b_m^T \cdot x_n)$$

$$\left(= \sum_{m=M+1}^{D} (b_m^T \cdot x_n) b_m \right)^2$$

$$J = \frac{1}{N} \sum_n \sum_{\underline{m}} (b_m^T x_n \underline{b_m} \underline{b_m^T} x_n^T b_m)$$

$$= \frac{1}{N} \sum_n \sum_m b_m^T \cdot x_n x_n^T \cdot b_m$$

$$= \sum_{m=M+1}^{D} b_m^T \cdot S \cdot b_m$$

$$S = \frac{1}{n} X X^T$$

$D \times D$

$X : D \times N$

$N \times D$

$$S v = \lambda v$$

$$\frac{1}{n} X X^T . v = \lambda v$$

of dim

D : no. of dimensions

$$\underbrace{\frac{1}{n} \underbrace{X^T X}\ }\ \underbrace{X^T v}_{c} = X^T \lambda v \quad M$$

$$X \to Z \to \tilde{X}$$

$$\bar{S} . c = \lambda . c \quad R^D \quad R^M \quad R^D$$

eigenvalue of $\bar{S} = N \times N$ matrix

$n < D$

$$\bar{S} = \frac{1}{n} X^T X$$

$$S = \frac{1}{n} X X^T$$

$\to$ same eigenvalues

$\lambda \quad \bar{S} \rightarrow$ eigenvector of S ?

$$\bar{S} c = \lambda c$$

$$\frac{1}{N} X^T X c = \lambda c$$

$$\underbrace{\frac{1}{N} \cancel{X} X^T}_{S} \underbrace{X c}_{v} = \cancel{X} \lambda c$$

$$S \, v = \lambda . v$$

$$\boxed{v = X . c}$$

$$\boxed{\begin{array}{ll} \underline{N < D} & \lambda, c \text{ eigen value + vec of } \bar{S} \\ & v = X . c \end{array}}$$

$$\boxed{\begin{array}{ll} N > D & \lambda, v : \text{ eigen values + vectors of } S \end{array}}$$