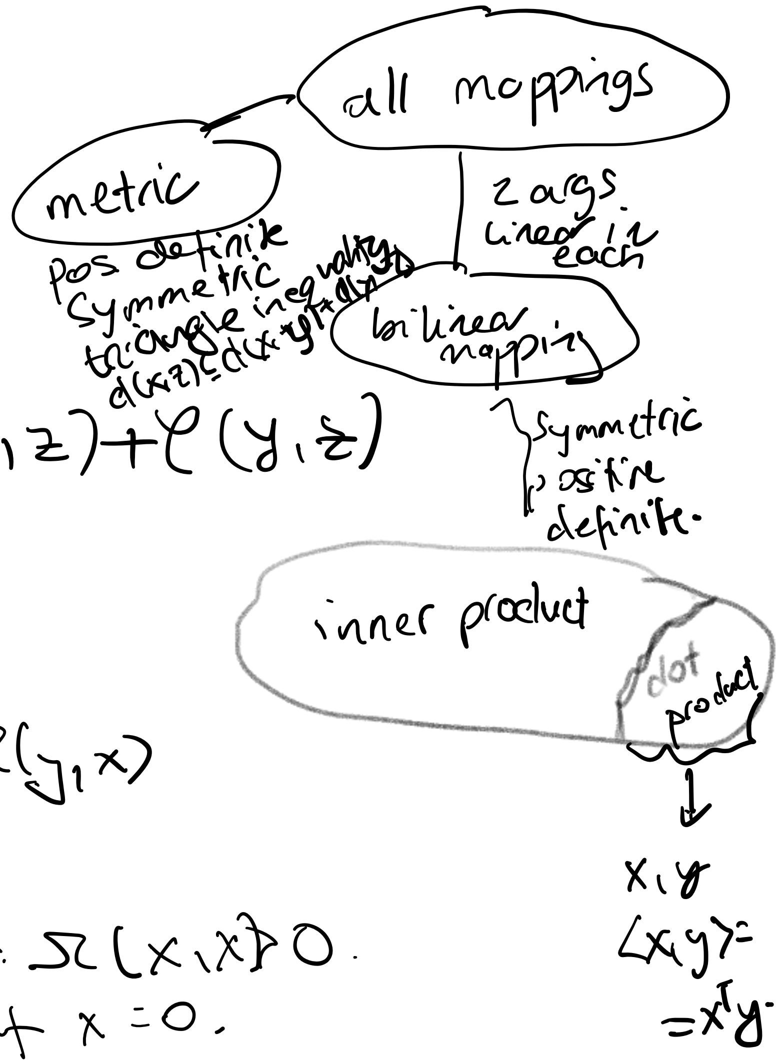
Lecture 5

Bilinear mapping

- 2 arguments
- linear in each



$x, y, z \in V$

$\lambda, \psi \in \mathbb{R}$

$$\Omega(\lambda x + \psi y, z) = \lambda \Omega(x, z) + \psi (y, z)$$

Inner product

$\Omega : V \times V \to \mathbb{R}$

Ω symmetric $\Omega(x, y) = \Omega(y, x)$

Ω positive definite

$$\forall x \in V \setminus \{0\} : \Omega(x, x) > 0.$$

$$\Omega(x, x) = 0 \text{ if } x = 0.$$

$$\Omega(0, 0) = 0.$$

Lemma

V

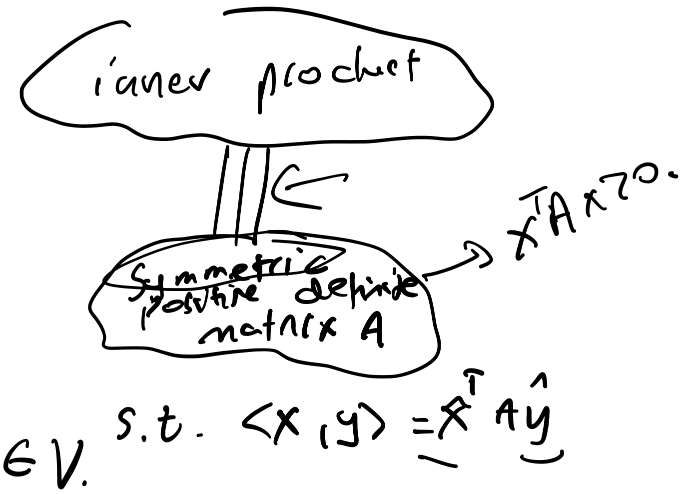
$B = (b_1 \dots b_n)$

$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$

is an inner product iff

$\exists$ symmetric p.d. matrix A

s.t. $\langle x, y \rangle = \hat{x}^T A \hat{y}$, $\forall x, y \in V$.



inner product — distance

$\|x-y\| = \sqrt{\langle x-y, x-y\rangle}$.

norm

$\|x\| = \sqrt{\langle x, x\rangle}$

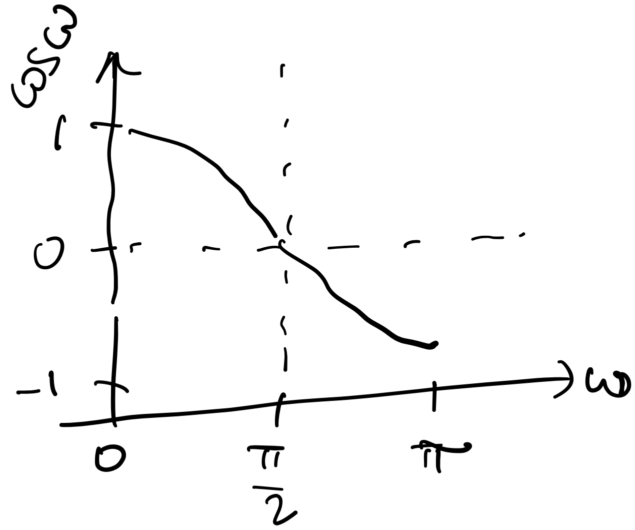
CS inequality

$|\langle x, y\rangle| \le \|x\| \, \|y\|$ $\quad \Big| \frac{1}{\|x\| \, \|y\|}$

$$-1 \le \frac{\langle x, y\rangle}{\|x\| \, \|y\|} \le 1$$

$\exists\, \omega \in [0, \pi]$ st

$$\cos\omega = \frac{\langle x, y\rangle}{\|x\| \, \|y\|}$$



$\langle x, y\rangle = 0 \Rightarrow x, y$ are orthogonal.

$\left.\begin{array}{l} \|x\| = 1 \\ \|y\| = 1 \end{array}\right\}$ orthonormal.

Square matrix A is orthogonal

$A^T A = I$.

$A^{-1} = A^T$

$\|Ax\|^2 = \|z\|^2$

orthogonal / orthonormal vectors

$\Downarrow$

<u>matrix</u>

$\Downarrow$

<u>basis</u>

n-dimensional vector space V

$\{b_1, b_2, \dots, b_n\}$

$\langle b_i, b_j \rangle = 0$ if $i \neq j$

$\langle b_i, b_i \rangle = 1$

Orthogonal subspaces $\equiv$ orthogonal complement

V : D-dimensional

U : M-dimensional

$U^{\perp} = \{ v \in V \mid \langle u, v \rangle = 0 \text{ for all } u \text{ in } U \}$

$\dim U^{\perp} = \underline{D - M}$.

$U^{\perp} \cap U = \{0\}$.

Why do we care about projections?

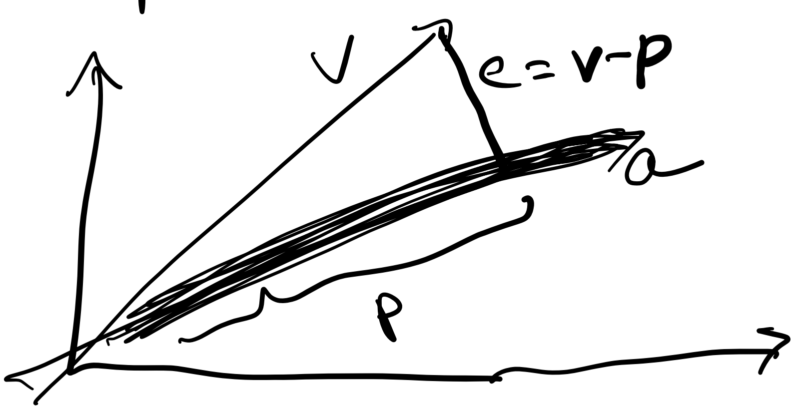
$Au = v$ may never have solutions

$2u_1 + u_2 = 5$

$2u_1 + u_2 = 11$

$v = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$ is not in the column space of matrix $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$.

$p \in C(A)$ $\quad \boxed{A\hat{x} = p}$



$a^T \cdot e = 0$

$a^T(v-p) = 0.$

$a^T v = a^T \underbrace{p}_{\lambda a}.$

$a^T v = \lambda a^T a$

$\lambda = \dfrac{a^T v}{a^T a}.$

$p = \lambda a$

$P_v = p$

$$P_{1\text{-dim}} = \frac{a a^T}{\|a\|^2}$$ (1 dimension)

$$P_{N\text{-dim}} = \frac{\underline{A(A^TA)^{-1}A^T}}{4\underline{As}.}$$ (N dimensional)

<u>GS.</u> Make A into a matrix with orthonormal columns.