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COMP3670 Assignment - 1

Q1) Question 1

(1) a) To prove, all eigenvalues of A are non-zero

Proof

Let λ be a eigenvalue of A that is equal to 0

There exists a non 0 vector x such that $Ax = \lambda x \Rightarrow Ax = 0$ $(\because \lambda = 0)$

Mulp both side by A^{-1}

$$A^{-1}Ax = 0$$
$$x = 0$$

But this contradicts our statement of x is a non zero vector.

$\therefore$ Hence proved that all eigenvalues of A are non-zero ..

b) To prove, for any eigenvalue λ of A, λ^{-1} is an eigenvalue of A^{-1}.

Proof

Let x be a eigenvector for eigenvalue λ of A

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$Ax = \lambda x$

Mulp both side by A^{-1}

$x = A^{-1} \lambda x$

Mulp λ^{-1} on Bs

$\lambda^{-1} A x = A^{-1} x$

thus proves that λ^{-1} is an eigenvalue of A^{-1} with x as corresponding eigenvector.

(8) Given,

$Bx = \lambda x \longrightarrow (1)$

$p(n) = B^n x = \lambda^n x$

By mathematical Induction

$p(1)$: $B^1 x = \lambda^1 x$

$Bx = \lambda x$

$\therefore p(1)$ is true

Lets assume $p(k)$ is true.

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To prove,: $P(k+1)$ is true.

$B^{k} x = \lambda^k x$

Mlp by B on B.S.

$B^{k+1} x = \lambda^k B x$

WKT,

$Bx = \lambda x$ from $P(1)$.

$B^{k+1} x = \lambda^{k+1} x$

$\therefore$ $P(k+1)$ is true

$\therefore$ $P(n)$ is true for all $n \geq 1$

Question 2

① Given,
A has n distinct eigenvalues $\lambda_1 \dots \lambda_n$
and corresponding non-zero eigenvectors
$x_1, \dots x_n$

To prove,
$x_1 \dots x_n$ are linearly independent

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Proof

We prove using PMI

To prove $P(1)$ is true

WKT at $P(1)$ $c_1 x_1 = 0$,
here $c_1 = 0$ since x_1 is a non-zero vector.

Assume that $P(k)$ is true for $1 \leq k < n$,
therefore $x_1, x_2 \ldots x_k$ is linearly independent.

$\therefore c_1 x_1 + c_2 x_2 \ldots + c_k x_k = 0$

To prove, $P(k+1)$ is true.

$$c_1 x_1 + c_2 x_2 \ldots + c_k x_k + c_{k+1} x_{k+1} = 0 \rightarrow (1)$$

Mlp (1) with A.

$$c_1 A x_1 + c_2 A x_2 \ldots + c_k A x_k + c_{k+1} A x_{k+1} = 0$$

WKT, $A x_n = \lambda_n x_n$

$$c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 \ldots + c_k \lambda_k x_k + c_{k+1} \lambda_{k+1} x_{k+1} = 0$$

$\rightarrow$ (2)

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mlp ① with λ_{k+1}.

$$c_1\lambda_{k+1}x_1 + c_2\lambda_{k+1}x_2 \dots c_k\lambda_{k+1}x_k + c_{k+1}\lambda_{k+1}x_{k+1} = 0 \rightarrow ③$$

Sub ② - ③.

$$c_1(\lambda_1-\lambda_{k+1})x_1 + c_2(\lambda_2-\lambda_{k+1})x_2 \dots c_k(\lambda_k-\lambda_{k+1})x_k = 0$$

But wkt,

$$\lambda_1 - \lambda_{k+1} \neq 0$$

so,

$$c_1x_1 + c_2x_2 \dots c_kx_k = 0$$

Sub this in ①.

$$0 + c_{k+1}x_{k+1} = 0$$

Since x_{k+1} is a non zero vector

$$c_{k+1} = 0$$

∴ Hence proved by PMI

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⑤ Given: $B \in R^{n \times n}$

<u>To prove</u>: There can be atmost n distinct eigenvalues for B.

<u>Proof</u>

Let B have k eigenvalues $\lambda_1 \dots \lambda_k$ where $k > n$. Let $x_1 \dots x_k$ be the corresponding eigenvectors.

WKT,

eigenvectors are ~~for~~ linearly independent to eachother

$\therefore \{x_1 \dots x_k\}$ are linearly independent

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and form the basis of R^n

$\therefore$ Any linear combination of the basis $\{x_1 \dots x_k\} \in R^n$.

$B \in R^{n \times n}$ can have atmost n linearly independent columns

Let the columns of B span a subspace of dimension k.

But if $k > n$ then dimension of subspace is greater than number of columns of B. This is not possible.

$\therefore$ Our assumption is wrong

$\therefore$ There can be atmost n distinct eigenvalues for B for any matrix $B \in R^{n \times n}$

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Question 3

(1) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & -2 \\ 0 & -1 & -2 \end{bmatrix}$

Extend the matrix by rewriting first 2 columns at the end.

$$\begin{bmatrix} 1 & 2 & 4 & 1 & 2 \\ 2 & 3 & -2 & 2 & 3 \\ 0 & -1 & -2 & 0 & -1 \end{bmatrix}$$

According to same rule.

$$\det(A) = [(1)(3)(-2) + (2)(-2)(0) + (4)(2)(-1)] - [0(3)(4) + (-1)(-2)(1) + (-2)(2)(2)]$$

$$= [-6 - 8] - [2 - 8]$$

$$= -14 + 6$$

$$= -8$$

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(2) To prove $\det(A^T) = \det(A)$.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\det(A) = 1((3)(-2) - (-1)(-2)) - 2(-4-0) + 4(-2)$$

$$= -8 + 8 - 8$$

$$= -8 = RHS.$$

$$A^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 4 & -2 & -2 \end{bmatrix}$$

$$\det(A^T) = 1(-6-2) - 2(-4+4) + 0$$

$$= -8 = LHS$$

$\therefore$ LHS = RHS

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8 To prove,
$\det(I_n) = 1$

Proving by mathematical induction
Let us prove $P(1)$

Let $I_1 = [1]$

$\det(I_1) = 1$

$\therefore P(1)$ is true

~~To prove~~
Now, $P(1), P(2) \dots P(k)$ is true.

Prove $P(k+1)$ is true

Consider a matrix $I_{(k+1)(k+1)}$

$$I_{(k+1)(k+1)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ 0 & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & \cdots & 0 \\ \vdots & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & a_{(k+1)(k+1)} \end{bmatrix}$$

$$\det(I_{(k+1)(k+1)}) = 0 + 0 \dots (-1)^{k+1+k+1} I_{(k+1)(k+1)}$$

$$\times \det\left(\begin{bmatrix} a_{11} & \cdots & \cdots & 0 \\ \vdots & a_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & \cdots & a_{kk} \end{bmatrix}\right)$$

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Next, $\det(I_k) = 1$

for $(k+1) \times (k+1)$ matrix

$a_{(k+1)(k+1)} = 1$

$\therefore \det(I_{(k+1)(k+1)}) = 1 \times 1 = 1$

$\therefore P(k+1)$ is true

$\therefore \det(I_n) = 1$ where I_n is the $n \times n$ identity matrix

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(4) To prove,

$\det(A) = -\det(\sigma_{i,j}(A))$ for $i \neq j$ where $\sigma_{i,j}$ swaps the ith and jth row of input matrix.

Proof

LHS $= \det(A) = 8$

RHS $= -\det(\sigma_{ij}(A))$

$$\sigma_{ij}(A) = \sigma_{12}(A) = \begin{bmatrix} 2 & 3 & -2 \\ 1 & 2 & 4 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\det(\sigma_{12}(A)) = 2(-4+4) - 3(-2) - 2(-1)$$
$$= 6 + 2$$
$$= 8$$

$$\sigma_{23}(A) = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -2 \\ 2 & 3 & -2 \end{bmatrix}$$

$$\det(\sigma_{23}(A)) = 1(2+6) - 2(4) + 4(2)$$

$$= 8 - 8 + 8$$
$$= 8$$

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$$a_{13}(A) = \begin{bmatrix} 0 & -1 & -2 \\ 2 & 3 & -2 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\det(a_{13}(A)) = 0 + 1(8 + 2) - 2(4-3)$$

$$= 10 - 2$$

$$= 8$$

$\therefore$ RHS $= -\det(a_{ij}(A)) = 8$

$\therefore$ LHS = RHS

⑤ Proof

Proving by mathematical induction.

$P(n)$: Let the determinant of $n \times n$ upper triangular matrix be the product of its diagonal elements.

Proving $P(1)$

Consider 1×1 matrix $A = [a_{11}]$. Here a_{11} is the diagonal element since it is the only element.

$\det(A) = a_{11}$

$P(1)$ is true.

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Assume that P(k) is true

To prove, P(k+1) is true.
Consider (k+1) × (k+1) be a upper triangular matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(k+1)} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & a_{(k+1)(k+1)} \end{bmatrix}$$

$$\det(A) = 0 + 0 \cdots (-1)^{k+1+k+1}$$

$$= a_{k+1} \det\left(\begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & a_{22} & \vdots \\ \vdots & \ddots & \vdots \\ 0 & 0 \cdots & a_{kk} \end{bmatrix}\right)$$

WKT, det of upper triangular matrix is given as product of its diagonal elements.

$$\therefore \det(A) = a_{k+1} (a_{11} \times a_{12} \times \cdots a_{kk})$$
$$= a_{11} \times a_{12} \cdots a_{kk} \times a_{(k+1)(k+1)}$$

Hence P(k+1) is true
∴ det of upper triangular matrix is the product of its diagonals.

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Question 4

To prove,

$$tr\left((A+B)(A+B)^T\right) \leq 2 \cdot tr\left(AA^T + BB^T\right)$$

Proof

LHS = $tr\left((A+B)(A+B)^T\right)$

$tr\left((A+B)(A^T+B^T)\right)$ $\quad (\because (A+B)^T = A^T+B^T)$

$tr\left[AA^T + AB^T + BA^T + BB^T\right]$

$tr(AA^T) + tr(AB^T) + tr(BA^T) + tr(BB^T)$

using cyclic property

$tr(AA^T) + tr(BB^T) + 2tr(AB^T)$

$\therefore tr((A+B)(A+B)^T) = tr(AA^T) + tr(BB^T) + 2tr(AB^T) \rightarrow (1)$

But according to cauchy schwartz inequality

$tr(AB^T)^2 \leq tr(AA^T) + tr(BB^T)$

$tr(AB^T) \leq \sqrt{tr(AA)^T + tr(BB^T)}$

Sub in (1)

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Question 4:

mltp 2 on both side.

$$2\,tr(AB^T) \leq 2\sqrt{tr(AA^T)\,tr(BB^T)}$$

Add $\left[\sqrt{tr(AA^T)} - \sqrt{tr(BB^T)}\right]^2$ on RHS.

$$2tr(AB^T) \leq 2\sqrt{tr(AA^T)\,tr(BB^T)} + \left(\sqrt{tr(AA^T)} - \sqrt{tr(BB^T)}\right)^2$$

$$2tr(AB^T) \leq tr(AA^T) + tr(BB^T)$$

Add $tr(AA^T)$ and $tr(BB^T)$ on B-S

$$2tr(AB^T) + tr(AA^T) + tr(BB^T) \leq 2(tr(AA^T) + tr(BB^T)) \rightarrow ⑧.$$

Comp ① & ⑧.

$$tr((A+B)(A+B)^T) \leq 2tr(AA^T + BB^T)$$

Hence proved.

Question 5

$$A = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$$

(i) $A - \lambda I$

$$\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & -2 \\ 0 & 1-\alpha \end{bmatrix}$$

Finding determinant and eq to 0

$$(2-\alpha)(1-\alpha) - (-2)(0) = 0$$

$2-\alpha = 0$ or $1-\lambda = 0$

$\alpha = 2$ or $\alpha = 1$

$\therefore \alpha = 1, 2$ are eigenvalues of A.

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(2) $\lambda_1 = 1$, $\lambda_2 = 2$

Let x_1 be the eigenvector for $\lambda_1 = 1$

Let $x_1 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$(A - \lambda_1 I) x_1 = 0$

$\left(\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$

$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$

$a_1 - 2a_2 = 0$

$a_1 = 2a_2$

Let $a_2 = t$

$\therefore a_1 = 2t$

$\therefore x_1 = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}, t \in R$

$\therefore$ Eigenspace $E_{\lambda=1} = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

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Let $x_2 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$(A - \lambda_2 I)x_2 = 0$

$\left(\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$

$\begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$

$\begin{bmatrix} -2a_2 \\ -1a_2 \end{bmatrix} = 0$

~~Let $a_2 = t$.~~

$\cancel{x_2 = \begin{bmatrix} -2t \\ t \end{bmatrix} = 0}$

$\therefore a_2 = 0$

a_1 = any real value $t \in \mathbb{R}$

$\therefore x_2 = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ t \in \mathbb{R}$

$\therefore$ Eigenspace $E_{\lambda=2} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

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(8) $\{x_1, x_2\}$ are the eigenvectors of A

$$\left\{\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}2\\1\end{bmatrix}\right\}$$

Expressing this as a linear combination.

$$c_1x_1 + c_2x_2 = 0$$

$$c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1 + 2c_2\\c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$c_2 = 0$$
$$c_1 = 0$$

Since $c_1 = c_2 = 0$, the only solution is the trivial one and ~~hence~~ x_1 & x_2 could be extended as a linear combination $c_1x_1 + c_2x_2 = 0$ where the only solution is trivial one.

$\therefore$ x_1 & x_2 are linearly independent and hence spans R^2

(2) Given: $A = PDP^{-1}$

Construct P using set of eigenvector of A and we construct D as a diagonal matrix using eigenvalues of A

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$\therefore P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\therefore D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Now,

$LHS = A = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$

$RHS = PDP^{-1}$

$$P^{-1} = \frac{1}{|P|} \text{Adjoint}(P) = \frac{1}{-1}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix}$$

$LHS = RHS$

Hence verified.

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5) NKT,

$$A = PDP^{-1}$$

From the properties of eigenvalues, if λ is an eigenvalue of A, then λ^n will be eigenvalue of A^n.

$\therefore A^n = PD^nP^{-1}$

where, D^n is the diagonal matrix obtained by raising each eigenvalue to the power of n and putting it as diagonal element.

Question 6

Given

$$D = \{x_1, x_2 \dots, x_n\}$$

Projecting a vector x_i into a subspace of dimension M

$$z_n = v^T x$$

To find,
vector v that maximises the variance.

$$\vartheta = 1/n \sum^{n} (v^T x_i)^2$$

$$\vartheta = v^T C v$$

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where

$C = \sum_{i=1}^{n} x_i x_i^T$, covariance matrix.

To find v, we have to optimise using Lagrangian multiplier.

$$\mathcal{L}(x, \lambda) = f(x) - \lambda(g(x) - c)$$

$$\mathcal{L}(v, \lambda) = ? - \lambda(\|v_2\| - 1)$$

$$= v^T C v - \lambda v^T v - \lambda$$

diff wrt λ

$$\frac{\partial \mathcal{L}(v, \lambda)}{\partial \lambda} = 0 - v^T v - 1$$

Setting the derivative to zero

$$v^T v = 1 \longrightarrow (1)$$

diff wrt v

$$\frac{\partial \mathcal{L}(v, \lambda)}{\partial v} = 2Cv - 2\lambda v$$

setting derivative to zero

$$2Cv - 2\lambda v = 0$$

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$$Cv = \lambda v \longrightarrow (8)$$

from (1) & (8) wkt,

v is orthonormal

$Cv = \lambda v$ is in the form of characteristic equation of eigenvalues.

Therefore v can be described as the eigenvector of the covariance matrix 'C'

Hence the vector v that maximises the resulting variance $\nu = v^T C v$ subject to constraint $\|v\|_2 = 1$ is an eigenvector of the covariance matrix C.