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COMP3670 ASSIGNMENT-2

Aryan Odugoudar u7689173

Exercise 1

(a) $U = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \right\}, \; x = \begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix}$

(i) Since U is spanned by the given vectors, it can be expressed as the linear combination of those vectors

$$U = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

To prove

$x \notin U$

i.e there exists no $c_1, c_2 \in \mathbb{R}$ for which

$$x = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$-c_1 + 2c_2 = 8$

$c_1 - 2c_2 = -8 \longrightarrow (1)$

$c_1 - c_2 = 1 \longrightarrow (2)$

$c_1 - 2c_2 = 16 \longrightarrow (3)$

(1) and (3) contradict each other

$\therefore$ No solution exists for c_1 and c_2

Hence $x \notin V$

(2) Since V is a span of $[-1\ 1\ 1]^T$ and $[2\ -1\ -2]^T$, we can say that

$$c_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad c_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

are the basis of vector space $V \subset R^3$.

Let $B = [v_1 \quad v_2]$

$$B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \\ 1 & -2 \end{bmatrix}$$

$$B^T = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$$

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$$B^T B = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix}$$

$$B^T x = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix}$$

NKT,

$B^T B x = B^T x$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -20 \end{bmatrix}$$

$$3x_1 - 5x_2 = 12 \longrightarrow (1)$$
$$-5x_1 + 9x_2 = -20 \longrightarrow (2).$$

(1) $\times 5$ (2) $\times 3$

$$\cancel{15}x_1 - 25x_2 = 60$$
$$-\cancel{15}x_1 + 27x_2 = -60$$

$$2x_2 = 0$$

$$\boxed{x_2 = 0}$$

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Sub λ_2 in (3)

$3\lambda_1 = 12$

$$\boxed{\lambda_1 = 4}$$

$$\therefore \lambda = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

v. $(\pi_U(x))$

$$\pi_{(U)} x = B\lambda$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix}$$

(3) WKT, if $\pi_U(x)$ is the projection of $x \in R^3$ onto $U \in R^3$, then

$\| x - \pi_U(x) \|$ is minimal

$$\therefore d(x, U) = \min_{y \in U} \| x - y \| = \| x - \pi_U(x) \|$$

$$x - \pi_U(x) = \begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix} - \begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 2 \end{bmatrix}$$

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$$\|x - \pi_u(x)\| = \sqrt{12^2 + 0^2 + 12^2}$$
$$= \sqrt{288}$$
$$= 16.97$$

(2) $A = \begin{bmatrix} -1 & 2 \\ 1 & -1 \\ 1 & -2 \end{bmatrix}$

Let

$c_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and $c_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ represent the columns of A

Let v_1 and v_2 represent the columns of B such that Gram Schmidt orthogonalisation on A gives B with orthogonal columns

$A = [c_1 \quad c_2]$ $\quad B = [v_1 \quad v_2]$

WKT, using Gram Schmidt orthogonalization

$v_1 = c_1$

$v_2 = c_2 - \text{proj}(c_2, v_1)$, where $\text{proj}(c_2, v_1)$ is the projection of c_2 onto v_1

$$\text{proj}(c_2, v_1) = \frac{\langle c_2, v_1 \rangle v_1}{\langle v_1, v_1 \rangle}$$

$$= \frac{\langle c_2, v_1 \rangle v_1}{\|v_1\|^2}$$

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$$\frac{C_2^T v_1}{\|v_1\|^2}$$

$$= \frac{\begin{bmatrix} 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}{(-1)^2 + 1^2 + 1^2}$$

$$= -\frac{5}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5/3 \\ -5/3 \\ 5/3 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ -5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ -1/3 \end{bmatrix}$$

normalizing v_1

$$\|v_1\| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$$

Orthonormal $v_1 = \frac{1}{\sqrt{3}} v_1 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

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Normalizing V_2

$$\|V_2\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2}$$

$$= \sqrt{6}/3$$

Orthonormal $V_2 = \frac{3}{\sqrt{6}} V_2 = \begin{bmatrix} \cancel{\sqrt{6}/9} \\ \cancel{-2\sqrt{6}/9} \\ \cancel{-\sqrt{6}/9} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$

$\therefore$ Required Vector $B = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$

(5) $f(\theta) = \|x - Q\theta\|^2 + \lambda \|\theta\|^2$

To minimize the equation we sets its gradient to zero.

$$\nabla f(\theta) = \nabla \left\{ \begin{array}{l} (x - Q\theta)^T (x - Q\theta) \\ + \lambda \theta^T . \theta \end{array} \right\} = 0$$

Differentiate wrt θ

$$= \nabla \left\{ (x^T - \theta^T Q^T)(x - Q\theta) + \lambda \theta^T \theta \right\}$$

$$= \nabla \left\{ x^T x - x^T Q\theta - \theta^T Q^T x + \theta^T Q^T Q \theta + \lambda \theta^T \theta \right\}$$

$$= \frac{d(x^T x)}{d\theta} - \frac{d(x^T Q\theta)}{d\theta} - \frac{d(\theta^T Q^T x)}{d\theta} + \frac{d(\theta^T Q^T Q\theta)}{d\theta} + \frac{\lambda\, d(\theta^T \theta)}{d\theta}$$

Solving the differentiation independently

$$\frac{d(x^T x)}{d\theta} = 0$$

$$\frac{d(x^T Q\theta)}{d\theta} = \frac{d(\theta^T Q^T x)}{d\theta} = Q^T x$$

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$$\frac{d(\theta^T Q^T x)}{d\theta} = (Q^T x)$$

$$\frac{d(\theta^T Q^T Q \theta)}{d\theta} = (Q^T Q \theta)^T + \theta^T Q^T Q$$
$$= 2\theta^T Q^T Q$$

$$\frac{d(\theta^T \theta)}{d\theta} = \theta + \theta = 2\theta$$

Substituting the solution

$$2\theta^T Q^T Q - 2Q x^T + 2\theta$$

WKT,

$$\left(2x^T Q = 2Q^T n\right), \left(2\theta^T Q^T Q = 2Q^T Q\theta\right)$$

$$2\theta^T Q^T Q - 2Q^T x + 2\theta\lambda$$

To minimize $f(\theta)$, we need to equate $\nabla f(\theta) = 0$

$$\therefore 2\theta Q^T Q - 2Q^T x + 2\theta\lambda = 0$$

$$\theta = (Q^T Q + \lambda I)^{-1} Q^T x$$

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(b) To compute θ for $B = \begin{bmatrix} -4/\sqrt{3} & 4/\sqrt{6} \\ 4/\sqrt{3} & -2/\sqrt{6} \\ 4/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$ $\lambda = 10$

Substituting B & λ on

$$(B^T B + \lambda I)^{-1} B^T x$$

$$B^T B = \begin{bmatrix} -4/\sqrt{3} & 4/\sqrt{3} & 4/\sqrt{3} \\ 4/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} -4/\sqrt{3} & 4/\sqrt{6} \\ 4/\sqrt{3} & -2/\sqrt{6} \\ 4/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$$

$$B^T B = \begin{bmatrix} 1 & -4/\sqrt{18} \\ -4/\sqrt{18} & 1 \end{bmatrix}$$

$$B^T B + \lambda I = \begin{bmatrix} 11 & -4/\sqrt{18} \\ -4/\sqrt{18} & 11 \end{bmatrix}$$

$$(B^T B + \lambda I)^{-1} = \begin{bmatrix} 99/1081 & 6\sqrt{2}/1081 \\ 6\sqrt{2}/1081 & 99/1081 \end{bmatrix}$$

$$(B^T B + \lambda I)^{-1} \cdot B^T = \begin{bmatrix} -31\sqrt{3}/1081 & 29\sqrt{3}/1081 & 31\sqrt{3}/1081 \\ 29\sqrt{3}/1081\sqrt{2} & -31\sqrt{6}/1081 & -29\sqrt{3}/1081\sqrt{2} \end{bmatrix}$$

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Therfore for a given vector $x \in \mathbb{R}^3$ we can find θ as the vector that minimises

$$\theta = \begin{bmatrix} \frac{-31\sqrt{3}}{1081} & \frac{29\sqrt{3}}{1081} & \frac{31\sqrt{3}}{1081} \\ \frac{29\sqrt{3}}{1081\sqrt{2}} & \frac{-31\sqrt{6}}{1081} & \frac{-29\sqrt{3}}{1081\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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Exercise 2

(1) $\frac{\partial x^T ABC x}{\partial x}$

WKT,

$$\frac{\partial x^T B x}{\partial x} = x^T (B + B^T)$$

So,

$$\frac{\partial x^T ABC x}{\partial x} = x^T (ABC + (ABC)^T)$$

$$= \boxed{x^T (ABC + C^T B^T A^T)}$$

(2) $\frac{\partial (Bx+b)^T C(Dx+d)}{\partial x}$

using multiplication rule of derivatives

$$\frac{\partial (Bx+b)^T}{\partial x} [C(Dx+d)] + (Bx+b)^T \frac{\partial (C(Dx+d))}{\partial x}$$

$$\left[\frac{\partial (Bx)^T}{\partial x} + \frac{\partial b^T}{\partial x}\right] C(Dx+d) + (Bx+b)^T \left[\frac{\partial CDx}{\partial x} + \frac{\partial Cd}{\partial x}\right]$$

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$$\left[\frac{\partial x^T B^T}{\partial x} + 0\right] C (Dx + d) + (Bx + b)^T \left[CD + 0 \right]$$

According to identity of matrix gradient

$$\frac{\partial x^T a}{\partial x} = a^T$$

$$= (B^T)^T C(Dx + d) + (Bx + b)^T CD$$

$$= B((Dx + CD) + (Bx + b)^T CD$$

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③ $$\frac{\partial tr(x^2)}{\partial x} = \begin{bmatrix} \frac{\partial tr(x^2)}{\partial x_{11}} & \frac{\partial tr(x^2)}{\partial x_{12}} & \cdots & \frac{\partial tr(x^2)}{\partial x_{1n}} \\ \frac{\partial tr(x^2)}{\partial x_{21}} & \frac{\partial tr(x^2)}{\partial x_{22}} & \cdots & \frac{\partial tr(x^2)}{\partial x_{2n}} \\ \vdots & & & \vdots \\ \frac{\partial tr(x^2)}{\partial x_{n1}} & \cdots & & \frac{\partial tr(x^2)}{\partial x_{nn}} \end{bmatrix}$$

Assuming that X is a $n \times n$ matrix and since $X^2 = X X$

$$\frac{\partial tr(X^2)}{\partial X} = \begin{bmatrix} \frac{\partial \sum_{i=1}^{n} (X^2)_{ii}}{\partial x_{11}} & \cdots & \frac{\partial \sum_{i=1}^{n} (X^2)_{ii}}{\partial x_{1n}} \\ \vdots & & \vdots \\ \frac{\partial \sum_{i=1}^{n} (X^2)_{ii}}{\partial x_{n1}} & & \frac{\partial \sum_{i=1}^{n} (X^2)_{ii}}{\partial x_{nn}} \end{bmatrix}$$

Considering a general case of row i and column j

$$\frac{\partial \sum_{i=1}^{n} (X^2)_{ii}}{\partial x_{ij}}$$

Numerator contains sum of diagonal elements of X^2

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if $i \neq j$, $\dfrac{\partial \sum_{i=1}^{n} (X^2)_{ii}}{\partial X_{ij}} = 0$ (∵ there is no relevant term in the numerator & derivatives of a constant is 0)

if $i = j$, $\dfrac{\partial \sum_{i=1}^{n} (X^2)_{ii}}{\partial X_{ii}} = \dfrac{\partial \sum_{i=1}^{n} (X^2)_{ii}}{\partial X_{ii}} = \dfrac{\partial \sum_{j=1}^{n} (X^2)_{jj}}{\partial X_{jj}}$

$$= 2\,tr(X)$$

$$\therefore \frac{\partial\, tr(X^2)}{\partial x} = 2\,tr(X)$$

Exercise 3

① $f(x) = \left(\sum_{i=1}^{n} x_i^p \right)^{1/p} \quad p<1, \; p \neq 0$

On Expanding

$f(x) = \left(x_1^p + x_2^p + x_3^p \ldots x_n^p \right)^{1/p}$

$\frac{\partial f}{\partial x_i} = \frac{1}{p} \left(x_1^p + x_2^p + x_3^p \ldots x_n^p \right)^{1/p - 1} p x_i^{p-1}$

$= \left(x_1^p + x_2^p + x_3^p \ldots x_n^p \right)^{1-p/p} x_i^{p-1}$

Let $\left(x_1^p + x_2^p \ldots + x_n^p \right) = A$

$\frac{\partial f}{\partial x_i} = A^{(1-p)/p} x_i^{p-1}$

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Computing $\frac{\partial^2 f}{\partial x_i \partial x_j}$

If $i \neq j$, $\frac{\partial^2 f}{\partial x_i \partial x_j} = 0$ (∵ ~~non~~ for derivative wrt x_j, non x_j terms are constants and derivative of constant is 0)

If $i = j$, $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_i^2}$

$$= (p-1) x_i^{p-2} A^{(1-p)/p} + x_i^{p-1} \frac{(1-p)}{p} A^{(1-2p)/p} \, p x_i^{p-1}$$

$$= (p-1)\left[x_i^{p-2} A^{(1-p)/p} - x_i^{2p-2} A^{(1-2p)/p}\right]$$

(3) To prove: H is negative semidefinite, hence f is concave since it has a convex domain

Proof:

WKT, Hessian matrix H is a symmetric matrix

For a symmetric matrix to be negative semidefinite all of its principal minors are non negative.

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The Hessian matrix for the above function is

$$H_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ (p-1)\left[x_i^{p-2} A^{(1-p)/p} - x_i^{2p-2} A^{(1-2p)/p}\right], & \text{if } i=j \end{cases}$$

If $n=1$, then H will be 1×1 matrix.
The leading principal minor is same as

$$(p-1)\left[x_i^{p-2} A^{(1-p)/p} - x_i^{2p-2} A^{(1-2p)/p}\right] = \det(H_{1\times 1})$$

Since it is given that $p<1$ and the convex domain $\text{dom}(f) = R^n_{++}$, i.e. it has strictly elementwise positive vector

$\det(H_{1\times 1}) \geq 0$, i.e. non negative.

If $n=2$ then H will be a 2×2 matrix.
The leading principal minor is given as

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

$$\det(H_{2\times 2}) = \left[(p-1)\left[x_i^{p-2} A^{(1-p)/p} - x_i^{2p-2} A^{(1-2p)/p}\right]\right]^2$$

Since square of any $c \in R$ cannot be negative
$\det(H_{2\times 2}) \geq 0$ i.e. non negative

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Similarly for all $n \in N$, the leading principal minors of H will be non negative

$\therefore$ H is a negative semidefinite

And we know that if Hessian H is negative semidefinite, then the function is locally concave.

$\therefore$ f is locally concave since the domain is convex

Hence Proved.

(5) RHS

$$\text{diag}(x^{p-2}) = \begin{bmatrix} x_1^{p-2} & 0 & \cdots & 0 \\ 0 & x_2^{p-2} & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & x_n^{p-2} \end{bmatrix}$$

$$f(x)^p = \text{diag}(x)^{p-2} = \begin{bmatrix} f(x)^p x_1^{p-2} & 0 & \cdots & 0 \\ 0 & f(x)^p x_2^{p-2} & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & f(x)^p x_n^{p-2} \end{bmatrix}$$

$$x^{(p-1)} x^{(p-1)T} = \begin{bmatrix} x_1^{(p-1)} \\ \vdots \\ x_n^{(p-1)} \end{bmatrix} \times \begin{bmatrix} x_1^{(p-1)} & x_2^{(p-1)} & \cdots & x_n^{(p-1)} \end{bmatrix}$$

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$$= \begin{bmatrix} x_1^{2(p-1)} & (x_1 x_2)^{(p-1)} & \cdots & (x_1 x_n)^{(p-1)} \\ \vdots & & & \vdots \\ \vdots & & \ddots & \vdots \\ (x_n x_1)^{(p-1)} & (x_n x_2)^{(p-1)} & \cdots & x_n^{2(p-1)} \end{bmatrix}$$

$||^{y}$ the non-diagonal entry at row 'i' & column 'j' is given as:

$$(1-p)\, f(x)^{1-2p} \left\{ x_i^{(p-1)} x_j^{(p-1)} \right\}$$

which is of the form : $\frac{\partial f}{\partial x_i x_j}$

$\therefore$ we can rewrite it as

$$\begin{Bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \partial^2 f / \partial x_1 \partial x_n \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & & \partial^2 f / \partial x_n^2 \end{Bmatrix}$$

Therfore we conclude that

$$(1-p)\, f(x)^{1-2p} \left[x^{p-1} x^{p-1^T} - f(x)^p \cdot \text{diag}(x^{p-2}) \right]$$

is equivalent to the hessian matrix of $f(x)$

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Exercise 4

$$F(q(\theta)) = \int d\theta\, q(\theta) \log \frac{q(\theta)}{p(\theta)\, p(y|\theta, x)}$$

$$= \int d\theta\, q(\theta) [\log q(\theta) - \log p(\theta) - \log p(y|\theta, x)]$$

$$p(\theta) = \mathcal{N}(\theta; 0, 1) = \frac{1}{2\pi} e^{-\theta^2/2}$$

$$p(y|\theta, x) = \mathcal{N}(y; \theta x, \sigma_n^2) = \frac{1}{2\pi\sigma_n^2} e^{-\frac{(y-\theta x)^2}{2\sigma_n^2}}$$

$$q(\theta) = \mathcal{N}(\theta; \mu, \sigma^2) = \frac{1}{2\pi\sigma^2} e^{-\frac{(\theta-\mu)^2}{2\sigma^2}}$$

① $$F = \frac{\theta}{2\pi\sigma^2} e^{-\frac{(\theta-\mu)^2}{2\sigma^2}} \Big[\log \frac{1}{2\pi\sigma^2} e^{-\frac{(\theta-\mu)^2}{2\sigma^2}} - \log \frac{1}{2\pi} e^{-\theta^2/2}$$

$$- \log \frac{1}{2\pi\sigma_n^2} e^{-(y-\theta x)^2/2\sigma_n^2} \Big]$$

$$F = \frac{\theta e^{-\frac{(\theta-\mu)^2}{2\sigma^2}}}{2\pi\sigma^2} \log \left[\frac{2\pi\sigma_n^2 e^{-\frac{(\theta-\mu)^2}{2\sigma^2}}}{\sigma^2 e^{-\frac{\theta^2\sigma_n^2 - y^2 - \theta^2 x^2 + 2y\theta x}{2\sigma_n^2}}} \right]$$

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(8) $$\frac{\partial F}{\partial \mu} = \left(\log\left[\frac{2\pi a_n^2 e^{-\frac{(\theta-\mu)^2}{2a^2}}}{a^2 e^{-\frac{\theta^2 a_n^2 - y^2 - \theta^2 x^2 + 2y\theta x}{2a_n^2}}}\right] \frac{\theta}{2\pi a^2} e^{-\frac{(\theta-\mu)^2}{2a^2}} \times \frac{(-1)(2)(\theta-\mu)(-1)}{2a^2} \right)$$

$$+ \left(\frac{\theta}{2\pi a^2} e^{-\frac{(\theta-\mu)^2}{2a^2}} \cdot \frac{a^2 e^{-\frac{\theta^2 a_n^2 - y^2 - \theta^2 x^2 + 2y\theta x}{2a_n^2}}}{2\pi a_n^2 e^{-\frac{(\theta-\mu)^2}{2a^2}}} \times \frac{2\pi a_n^2 e^{\frac{(\theta-\mu)^2}{2a^2}} \cdot (-1)(2)(\theta-\mu)(-1)}{a^2 (2a^2) \exp\left(\frac{-\theta^2 a_n^2 - y^2 - \theta^2 x^2 + 2y\theta x}{2a_n^2}\right)} \right)$$

$$= \frac{\theta \exp\left(\frac{-(\theta-\mu)^2}{2a^2}\right)(\theta-\mu)}{2\pi a^4} \left[\log\left(\frac{2\pi a_n^2 e^{-\frac{(\theta-\mu)^2}{a^2}}}{a^2 \exp\left(\frac{\theta^2 a_n^2 - y^2 - \theta^2 x^2 + 2y\theta x}{2a_n^2}\right)} \right) + 1 \right]$$

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$$\frac{\partial F}{\partial a} = \left[\log\left[\frac{2\pi a_n^2 \exp\left(\frac{-(\theta-\mu)^2}{2a^2}\right)}{a^2 \exp\left(\frac{-\theta^2 a_n^2 - y^2 - \theta^2 x^2 + 2y\theta x}{2a_n^2}\right)}\right]\right]$$

$$\times \left[\left[\left(2\pi a^2 \theta \exp\left(\frac{-(\theta-\mu)^2}{2a^2}\right)\left(\frac{(-1)(\theta-\mu)^2(-2)}{2a^3}\right)\right.\right.\right.$$

$$\left.\left. - \left(\theta \cancel{\mu} \cancel{\pi} a \exp\left(\frac{-(\theta-\mu)^2}{2a^2}\right)\right]\right] \Big/ (2\pi a^2)^2 \Big]$$

$$+ \left[\frac{\theta \exp\left(\frac{-(\theta-\mu)^2}{2a^2}\right)}{2\pi a^2} \cdot \frac{a^2 \exp\left(\frac{-\theta^2 a_n^2 - y^2 - \theta^2 x^2 + 2y\theta x}{2a_n^2}\right)}{2\pi a_n^2 \exp\left(\frac{-(\theta-\mu)^2}{2a^2}\right)}\right.$$

$$\times \left(a^2 \exp\left(\frac{-\theta^2 a^2_n - y^2 - \theta^2 x^2 + 2y\theta x}{2a_n^2}\right) 2\pi a_n^2 \exp\left(\frac{-(\theta-y)^2}{2a^2}\right)\right.$$

$$(-1)\left(\frac{\theta-\mu)^2}{2a^2}\right)\frac{(-2)}{a^3} - 2\pi a_n^2 \exp\left(\frac{-(\theta-\mu)^2}{2a^2}\right)$$

$$\frac{\left. 2a \exp\left(\frac{-\theta^2 a_n^2 - y^2 - \theta^2 x^2 + 2y\theta x}{2a_n^2}\right)\right)}{a^2 \exp\left(\frac{-\theta^2 a_n^2 - y^2 - \theta^2 x^2 + 2y\theta x}{2a_n^2}\right)}$$

$$\therefore \frac{\partial F}{\partial a} = \frac{\log\left[\frac{2\pi a_n^2 \exp\left(\frac{-(\theta-\mu)^2}{2a^2}\right)}{a^2 \exp\left(-\theta^2 a_n^2 - y^2 - \theta^2 x^2 + 2y\theta x\right.}\right] \theta e^{\frac{-(\theta-\mu)^2}{2a^2}} \left((\theta-\mu)^2 - 2a^2\right)}{2\pi a^3}$$

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$$+ \frac{\theta \exp\left(\frac{-(\theta-\mu)^2}{2a^2}\right)\left[\frac{(\theta-\mu)^2}{a^3} - 2a\right]}{2\pi a^4}$$

(2) $\frac{\partial F}{\partial u} = 0$

$$\theta \exp\left(\frac{-(\theta-u)^2}{2a^2}\right)(\theta-u)\left[1+\log\left(\frac{2\pi a_n^2 \, e^{-\frac{(\theta-u)^2}{a^2}}}{a^2 \exp\left(\frac{\theta^2 a_n^2 - y^2 - \theta^2 u^2 + 2xy\theta}{2a_n^2}\right)}\right)\right]$$

$$= 0$$

$$\ln(2\pi a_n^2) + \ln\left(\exp \frac{-(\theta-\mu)^2}{2a^2}\right) - \ln(a^2)$$

$$- \ln\left(\exp\left(- \frac{\theta^2 a_n^2 - y^2 - \theta^2 x^2 + 2y\theta x}{2a_n^2}\right)\right) = -1$$

$$\therefore \mu(\mu - 2\theta) = 2a^2 + 2a^2\ln(2\pi a_n^2) - 2a^2\ln(a^2)$$

$$- \theta^2 + \frac{\theta^2 a_n^2 a^2 + y^2 a^2 + \theta^2 x^2 a^2 - 2y\theta x a^2}{a_n^2}$$

$$\frac{\partial F}{\partial a} = 0$$

$$\Rightarrow \frac{\log\left[\dfrac{2\pi a_n^2 \exp\left(\dfrac{(\theta-\mu)^2}{2a^2}\right)}{a^2 \exp\left(\dfrac{-\theta^2 a_n^2 - y^2 - \theta^2 x^2 + 2y\theta x}{2a_n^2}\right)}\right] \theta \exp\left(\dfrac{-(\theta-\mu)^2}{2a^2}\right)\left[(\theta-\mu)^2 - 2a^2\right]}{2\pi a^5}$$

$$\left[(\theta-\mu)^2 - 2a^2\right] a^2 = \left[(\theta-\mu)^2 - 2a^2\right] \times \frac{1}{\ln\left[\dfrac{2\pi a_n^2 \exp\left(\dfrac{-(\theta-\mu)^2}{2a^2}\right)}{a^2 \exp\left(\dfrac{-\theta^2 a_n^2 - y^2 - \theta^2 x^2 + 2y\theta x}{2a_n^2}\right)}\right]}$$