SUBJECT: | DATE:

COMP3670 Assignment 1

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① a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

$\therefore$ $A = A^T$. So A is symmetric

b) $A^2 = A \times A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1+4+9 & 2+8+15 & 3+10+18 \\ 2+8+15 & 4+16+25 & 6+20+30 \\ 3+10+18 & 6+20+30 & 9+25+36 \end{bmatrix}$

$A^2 = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$

SUBJECT: DATE:

$$(A^2)^T = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$\therefore$ $A^2 = (A^2)^T$

$\therefore$ It is symmetric

c) Yes, because of the properties of matrix multiplication and transpose.

Proof

Say $A = A^T \longrightarrow ①$

Now, consider,

$(A^2)^T$

$= (A \times A)^T$

$= A^T \times A^T$

$= A \times A$ (from ①)

$= A^2$

$\therefore$ For any symmetric matrix A, A^2 is also symmetric

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d) Since A is a square matrix

$f(A) = a_0 A^0 + a_1 A^1 + a_2 A^2 + \dots a_n A^n$

$g(A) = b_0 A^0 + b_1 A^1 + b_2 A^2 + \dots b_n A^n$

$f(A) \cdot g(A) = (a_0 A^0 + a_1 A^1 + \dots a_n A^n) + (b_0 A^0 + b_1 A^1 + \dots b_n A^n)$

It can be represented as

$$\sum_{j=0}^{n} \sum_{i=0}^{n} a_i b_j A^{i+j} \longrightarrow (1)$$

Similarly

$$g(A) f(A) = \sum_{j=0}^{n} \sum_{i=0}^{n} b_i a_j A^{i+j} \longrightarrow (2)$$

Since we know that,

(1) and (2) are equal.

$\therefore f(A) g(A) = g(A) f(A)$

Hence they commute.

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SUBJECT: DATE:

e) $X = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$

$$X \otimes X = \begin{bmatrix} 8\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} & 1\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} & 6\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \\ 3\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} & 5\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} & 7\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \\ 4\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} & 9\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} & 2\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 64 & 8 & 48 & 8 & 1 & 6 & 48 & 6 & 36 \\ 24 & 40 & 56 & 3 & 5 & 7 & 18 & 30 & 42 \\ 32 & 72 & 16 & 4 & 9 & 2 & 24 & 54 & 12 \\ 24 & 3 & 18 & 40 & 5 & 30 & 56 & 7 & 42 \\ 9 & 15 & 21 & 15 & 25 & 35 & 21 & 35 & 49 \\ 12 & 27 & 6 & 20 & 45 & 10 & 28 & 63 & 14 \\ 32 & 4 & 24 & 72 & 9 & 54 & 16 & 2 & 12 \\ 12 & 20 & 28 & 27 & 45 & 63 & 6 & 10 & 14 \\ 16 & 36 & 8 & 36 & 81 & 18 & 8 & 18 & 4 \end{bmatrix}$$

Row sums = column sums = diagonal sum = counter diagonal sum = 225

$\therefore$ $X \otimes X$ is a magic square.

SUBJECT: DATE:

7) Since the matrix is a magic square.
∴ for given column i of $a(n \times n)$ matrix

$$\sum_{j=0}^{n} a_{ij} = \text{constant } (c)$$

Their sum is equivalent accross rows, columns and diagonals.

∴ for any column i, row k.

$$c = \sum_{j=0}^{n} a_{ij} = \sum_{m=0}^{n} a_{km} = \sum_{x=0}^{n} a_{xx} = \sum_{y=0}^{n} a_{y\,n-y} \quad (1)$$

→ Lets consider a magic square matrix X of shape $(n \times n)$

We should prove that $X \otimes X$ is also a magic square

$$X \otimes X = \begin{bmatrix} a_{00}X & a_{01}X & \cdots & a_{0n}X \\ a_{10}X & a_{11}X & \cdots & a_{1n}X \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0}X & a_{n1}X & & a_{nn}X \end{bmatrix}$$

where a is an arbitrary constant

This can be expanded as

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$$
\begin{bmatrix}
\begin{bmatrix}
a_{00}a_{00} & \cdots & a_{00}a_{00} \\
a_{00}a_{10} & \ddots & \vdots \\
\vdots & & \vdots \\
a_{00}a_{n0} & & a_{00}a_{nn}
\end{bmatrix}
&
\begin{bmatrix}
a_{0n}a_{00} & \cdots & a_{0n}a_{0n} \\
\vdots & \ddots & \vdots \\
a_{0n}a_{n0} & & a_{0n}a_{nn}
\end{bmatrix}
\\
\begin{bmatrix}
a_{n0}a_{00} & \cdots & a_{n0}a_{0n} \\
a_{n0}a_{10} & \ddots & \vdots \\
\vdots & & \vdots \\
a_{n0}a_{n0} & & a_{n0}a_{nn}
\end{bmatrix}
&
\begin{bmatrix}
a_{nn}a_{00} & \cdots & a_{nn}a_{0n} \\
\vdots & \ddots & \vdots \\
a_{nn}a_{00} & & a_{nn}a_{nn}
\end{bmatrix}
\end{bmatrix}
$$

SUBJECT:

DATE:

from this we can compute the column of x as

$$\sum_{j=0}^{n} a_{jx} \sum_{i=0}^{n} a_{ix} = c^2 \longrightarrow (2)$$

where c is sum along the column of X

$||^{ly}$

sum along a row of x is

$$\sum_{j=0}^{n} a_{xj} \sum_{i=0}^{n} a_{xi} = a^2 \longrightarrow (3)$$

where a is sum along a row of x

$||^{ly}$ sum along the diagonal of x is

$$\sum_{j=0}^{n} a_{jj} \sum_{i=0}^{n} a_{ii} = b^2 \longrightarrow (4)$$

where b is sum along a diagonal of x

$||^{ly}$
$$\sum_{j=0}^{n} a_{j(n-j)} \sum_{i=0}^{n} a_{i(n-i)} = d^2 \longrightarrow (5)$$

where d is sum along counter diagonal of x

SUBJECT: DATE:

Since X is a magic square

$\therefore \quad c = a = b = d$

$\therefore \quad c^2 = a^2 = b^2 = d^2$

$\therefore$ If X is a magic square then $X \otimes X$ is also a magic square.

Ex.

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Sum of rows, diagonals, columns $= 2$

$$X \otimes X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Here sum of rows, diagonals, columns $= 4$

$\therefore$ $X \otimes X$ gives a magic square

SUBJECT: DATE:

9) Let $x = \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}$ $y = \begin{bmatrix} y_1 & y_3 \\ y_2 & y_4 \end{bmatrix}$

$$x \otimes y \otimes x = \begin{bmatrix} x_1y_1x_1 & x_1y_1x_3 & x_3y_1x_1 & x_3y_3x_3 \\ x_1y_1x_2 & x_1y_1x_3 & x_3y_1x_2 & x_3y_3x_3 \\ x_1y_2x_1 & x_1y_2x_4 & x_3y_4x_1 & x_3y_4x_1 \\ x_1y_2x_2 & x_1y_2x_4 & x_3y_4x_2 & x_3y_4x_2 \\ & & & \\ x_2y_1x_1 & x_2y_1x_3 & x_4y_3x_1 & x_4y_3x_3 \\ x_2y_2x_2 & x_2y_1x_3 & x_4y_3x_1 & x_4y_3x_3 \\ x_2y_2x_1 & x_2y_2x_4 & x_4y_4x_2 & x_4y_4x_4 \\ x_2y_1x_2 & x_2y_2x_4 & x_4y_4x_2 & x_1y_4x_4 \end{bmatrix}$$

$$y \otimes x \otimes y = \begin{bmatrix} y_1x_1y_1 & y_1x_1y_1 & y_3x_3y_1 & y_3x_3y_3 \\ y_1x_1y_2 & y_1x_1y_2 & y_3x_3y_2 & y_3x_3y_4 \\ y_1x_2y_1 & y_1x_2y_1 & y_3x_4y_1 & y_3x_4y_3 \\ y_1x_2y_2 & y_1x_2y_2 & y_3x_4y_2 & y_3x_4y_4 \\ & & & \\ y_2x_1y_1 & y_2x_1y_1 & y_4x_3y_1 & y_4x_3y_3 \\ y_2x_1y_2 & y_2x_1y_2 & y_4x_3y_2 & y_4x_3y_4 \\ y_2x_2y_1 & y_2x_2y_1 & y_4x_4y_1 & y_4x_4y_3 \\ y_2x_2y_2 & y_2x_2y_2 & y_4x_4y_2 & y_4x_4y_4 \end{bmatrix}$$

for $x \otimes y \otimes x = y \otimes x \otimes x$

$x_1y_1x_1 = y_1x_1y_1 \Rightarrow x_1 = y_1$

$x_2y_2x_1 = y_2x_1y_1 \Rightarrow x_2 = y_2$

$x_3y_3x_3 = y_3x_3y_3 \Rightarrow x_3 = y_3$

$x_4y_3x_3 = y_4x_3y_3 \Rightarrow x_4 = y_4$

SUBJECT: DATE:

This means that

$$x = y$$

$$\therefore\ x \otimes y \otimes x = y \otimes x \otimes y$$

SUBJECT: | DATE:

Exercise 2

a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 6 & 2 & -5 \\ 2 & 1 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 6 & 2 & -5 & -2 \\ 2 & 1 & -1 & 3 \end{array}\right] \xrightarrow[R_3=(R_3-2R_1)]{R_2=(R_2-3R_1)(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 2 & 11 \\ 0 & 1 & -3 & -7 \end{array}\right]$$

$$\Big\downarrow R_3 = (R_3 - R_2)(-1/5)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/5 \\ 0 & 1 & 0 & 19/5 \\ 0 & 0 & 1 & 18/5 \end{array}\right] \xleftarrow[R_2=R_2-2R_3]{R_1=R_1-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 1 & 18/5 \end{array}\right]$$

$\therefore$ for $Ax = b$

$$x = \begin{bmatrix} 7/5 \\ 19/5 \\ 18/5 \end{bmatrix}$$

b) $A = \begin{bmatrix} 4 & 3 & 2 & 2 & -2 \\ 0 & 1 & 2 & 2 & 6 \\ 3 & 2 & 1 & 1 & -3 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 5 \\ 23 \\ -2 \\ 16 \end{bmatrix}$

$$\left[\begin{array}{ccccc|c} 4 & 3 & 2 & 2 & -2 & 5 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 3 & 2 & 1 & 1 & -3 & -2 \\ -1 & 0 & 1 & 1 & 1 & 16 \end{array}\right]$$

$\Big\downarrow$ $R_1 = R_1 - R_3$

$R_3 = R_3 + 3R_4$ | $R_4 = R_4 + R_1$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 0 & 2 & 4 & 4 & 0 & 46 \\ 0 & 1 & 2 & 2 & 2 & 23 \end{array}\right]$$

$\Big\downarrow$ $R_3 = R_3 - 2R_2$

$R_4 = R_4 - R_2$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 0 & 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \end{array}\right]$$

$R_4 = 4R_4 + R_3$ $\downarrow$ $R_3 = (R_3)(-1)$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

SUBJECT: DATE:

x_3 and x_4 are free variables

$$x_5 = 0$$

$$x_2 = 23 - 2x_3 - 2x_4$$

$$x_1 + 23 - 2x_3 - 2x_4 + x_3 + x_4 = 7$$

$$x_1 = -16 + x_3 + x_4$$

$$x = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -16 \\ 23 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

SUBJECT: DATE:

Exercise 3

a) WKT,

$$AA^{-1} = I$$

Taking Transpose on both side.

$$(AA^{-1})^T = (I)^T$$

$$(A^T)(A^{-1})^T = I$$

$$(A^{-1})^T = (A^T)^{-1} \quad (\because AI = A)$$

Hence proved.

SUBJECT: DATE:

b) $\begin{bmatrix} 1 & 1 & b \\ 1 & a & c \\ 1 & 1 & 1 \end{bmatrix}$

For the inverse to exist, the determinant should not be equal to 0

$$\det = 1(a-c) - 1(1-c) + b(1-a) \neq 0$$

$$a - \cancel{c} - 1 + \cancel{c} + b - ab \neq 0$$

$$a + b - ab \neq 1$$

Any values of a, b and c that satisfies the above equation are true.

SUBJECT: DATE:

i) The rank of the matrix is the maximum number of linearly independent columns or rows in the matrix.

Consider the column spaces of A as Col(A). The column space is a subspace spanned by the columns of A.
Similarly,
consider the row space of A^T as Row (A^T). The row space is the subspace spanned by rows of A^T.

We can observe that Col(A) is equivalent to Row (A^T) and vice versa.

Since Col(A) and Row (A^T) are equivalent, they have the same dimensions.

$\therefore \quad r(A) = r(A^T)$

Hence proved.

SUBJECT: DATE:

Exercise 4

a) i)

$A = \{(x_1, x_2, x_3 \dots x_n) \in R^n : x_1 \geq 0, x_2 \geq 0 \dots x_n \geq 0\}$

→ $A \subseteq R$

→ $A \neq \phi$ and $0 \in A$.
Thus zero vector is a set in A.

→ $(x_1, x_2, x_3 \dots x_n) \in A$ and $(y_1, y_2, y_3 \dots y_n) \in A$
and their sum $(x_1 + y_1, x_2 + y_2, \dots x_n + y_n)$
are also non negative integers.
Thus the sum is in set A.

→ If $(x_1, x_2, \dots x_n) \in A$ and λ is a negative real number, then $x_i \lambda$ is a negative number
Thus the product is not in set A.

∴ A is not a subspace of R^n because it fails the property of ~~does~~ closed under scalar multiplication.

ii) b) $B = \{(x_1, x_2, x_3 \dots x_n) \in R^n :$ atleast one x_i is irrational$\}$

→ $B \subseteq R$

SUBJECT: DATE:

→ $B \neq 0$ and $0 \in B$

Thus zero vector is in set ~~B~~ B

→ If $(x_1, x_2 \dots x_n) \in B$ and $(y_1, y_2 \dots y_n) \in B$ then their sums $(x_1+y_1, x_2+y_2 \dots x_n+y_n)$ may or maynot have an irrational component

Thus sum may not be in set B

∴ B is not a subspace of B because it fails closed under ~~scalar~~ Addition property

iii) $C = \{(x_1, x_2 \dots x_n) \in R^n : \sum_{i=1}^{n} (-1)^{i+1} x_i \geq 0\}$

→ $C \subseteq R$

→ $C \neq 0$ and $0 \in C$

Thus 0 vector is in set C

→ If $(x_1, x_2 \dots x_n) \in C$ and λ is a negative real number then the even element of x_i will be +ve, which fails to satisfy the condition.

∴ C is not a subspace of R^n because it fails to satisfy closed under scalar multiplication.

SUBJECT: DATE:

iv) The solution set for the equation $Ax=b$ can be in 2 scenarios.

* when $b=0$, homogeneous equation $Ax=0$ There is a subspace because

a) Zero vector is part of solution

b) For x_1 and x_2 in the eqn $Ax=0$

$$A(x_1+x_2) = Ax_1 + Ax_2 = 0+0=0$$

$\therefore$ It is closed under addition

c) For x in eqn $Ax=0$, and $\lambda \in R$

$$A(\lambda \cdot x) = \lambda(A(x)) = \lambda \cdot 0 = 0$$

$\therefore$ It is closed under multiplication.

* when $b \neq 0$, non homogeneous equation such that $Ax=b$, $b\neq 0$.

a) Zero vector is not ~~a~~ necessarily included in D.

b) For x_1 & x_2 in the eqn $Ax=b$

$$A(x_1+x_2) = Ax_1 + Ax_2 = b+b = 2b \neq b$$

$\therefore$ It is not closed under addition

$\therefore$ It fails the condition.

$\therefore$ It is not a subspace.

5) Let $v_1, v_2 \in W^{\perp}$

Then,

$$\langle v_1, w\rangle = 0 \quad \text{and} \quad \langle v_2, w\rangle = 0$$

Now,

$$\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$$
$$= 0 + 0$$
$$= 0$$

$\therefore$ $v_1 + v_2 \in W^{\perp}$ (Closed under vector addition)

Let α be a scalar and $v \in W^{\perp}$

Then $\langle v, w\rangle = 0$

Now,

$$\langle \alpha v, w\rangle = \alpha \langle v, w\rangle$$
$$= 0$$

$\therefore$ $\alpha v \in W^{\perp}$ (Closed under scalar multiplication)

$\therefore$ It satisfies all the conditions.

$\therefore$ $W^{\perp}$ is a subspace of V.

SUBJECT:

Exercise 5

Given,

Linear Transformation

$T: V \longrightarrow W$

Image: $Im(T) = \{w \in W \mid \exists v \in V$ such that $w = T(v)\}$

Kernel: $Ker(T) = \{v \in V \mid T(v) = 0\}$

a) Show that $T(0) = 0$

Let $0 = 0 - 0$

$\therefore T(0) = T(0-0)$

WKT, $T(u+w) = T(u) + T(w)$

$T(0) = T(0-0)$

$= T(0) - T(0)$

$= 0 - 0$

$\therefore T(0) = 0$

b) Given,

LHS: $T(c_1v_1 + \cdots + c_nv_n)$

WKT, by linear mapping

$T(u+w) = T(u) + T(w)$

SUBJECT: DATE:

Since $c_i v_i$ is a vector, we can say
$$c_i v_i = w_i$$

$$T(c_1 v_1 + \dots + c_n v_n) = T(w_1 + w_2 + \dots + w_n)$$

$$= T(w_1) + T(w_2) + \dots + T(w_n)$$
$$= T(c_1 v_1) + T(c_2 w_2) + \dots + T(c_n v_3)$$

From linear mapping wkt,
$$T(\lambda w) = \lambda T(w)$$

$$c_1 T(v_1) + c_2 T(v_2) \dots + c_n T(v_n) = RHS$$

$\therefore$ $LHS = RHS$

Hence Proved.

c) To prove $Im(T)$ ~~sts~~ is a subspace of W

WKT,
$$0 \in Im(T)$$
since $T(0) = 0$

$\therefore$ 0 vector exists

& Closure under vector addition

Let w_1 and $w_2 \in Im(T)$, then
$$w_1 = T(v_1) \text{ and } w_2 = T(v_2)$$

then $(w_1 + w_2) = T(v_1) + T(v_2)$

SUBJECT: | DATE:

$(w_1 + w_2) = T(v_1 + v_2)$

$\therefore (w_1 + w_2) \in Im(T)$

* Closed under multiplication

Let, $w \in Im(T)$, then $w = T(v)$

Let, $\lambda \in \mathbb{R}$

then, $\lambda w = \lambda \cdot T(v)$

WKT,

$\lambda w = T(\lambda v)$

$\therefore \lambda w \in Im(T)$

Since $Im(T)$ satisfies all the condition

It is a vector subspace of W

To prove $ker(T)$ is a subspace of V

* Zero ~~matrix~~ vector

WKT,

$T(0) = 0$

$\therefore 0 \in ker(T)$

* Closed under addition

Let v_1 & $v_2 \in ker(T)$ then,

$T(v_1) = 0$ and $T(v_2) = 0$

then, $T(v_1) + T(v_2) = 0 + 0$

$\Rightarrow T(v_1 + v_2) = 0$

$\therefore (v_1 + v_2) \in ker(T)$

SUBJECT: DATE:

Closure under multiplication.

Let $v \in \ker(T)$ and $\lambda \in R$

$T(v) = 0$

$\lambda T(v) = \lambda 0$

$T(\lambda v) = 0$

$\therefore \lambda v \in \ker(T)$

Since $\ker(T)$ satisfies all the condition

It is a vector subspace of V

d) Given,

$\dim(\mathrm{Im}(T)) = 3$

$\cancel{\dim(\mathrm{Im}(T))} =$

$\dim(\ker(T)) = 2$

By rank-nullity theorem we know,

for $T: V \to W$

$\dim(V) = \dim(\ker(T)) + \dim(\mathrm{Img}(T))$

$= 2 + 3$

$= 5$

$\therefore$ we can define linear maping as

$T: R^5 \to R^2$

Ex: $T(x_1, x_2, x_3, x_4, x_5)$

$= (x_1 + x_2, x_3 + x_4, x_5 + x_1)$

SUBJECT: | DATE:

The basis is

$$\left[\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right]$$

$\therefore \dim(\text{Img}(T)) = 3$

Basis of the kernel is

$$\ker(T) = \text{span}\left[\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\\-1\end{bmatrix}\right]$$

$\therefore \dim(\ker(T)) = 2$

e) For a transformation to be injective
$T(n_1) = T(n_2)$ then $n_1 = n_2$
i.e $T(n_1) - T(n_2) = 0$
$T(n_1 - n_2) = 0$

For a transformation to be injective, the kernal of the transformation should only contain zero vector.

SUBJECT: DATE:

For the given transformation $T(x) = Ax$ the equation should have zero vector.

To prove this,

$\det(A) = 0$

$\det(A) = 1(1-c) - a(1-c) + b(1-c) = 0$

$= (1-c)(1-a) + b(0) = 0$

$= (1-c)(1-a) = 0$

$\therefore$ For the linear transformation to be injective $a = 1$ or $c = 1$ and b can be any value.

SUBJECT: DATE:

Exercise 6

a) Let V be a vector space of finite dimension over field F

Let $\langle \cdot,\cdot \rangle$ be inner product on V
Then $\langle \cdot,\cdot \rangle : V \times V \to F$ is a function.

We claim that $\langle \cdot,\cdot \rangle$ is in bilinear form

$$\langle ax_1 + bx_2, y\rangle = \langle ax_1, y\rangle + \langle bx_2, y\rangle \quad (\because \text{linearity})$$
$$= a\langle x_1, y\rangle + b\langle x_2, y\rangle \quad (\because \text{by property of inner product})$$

Also, since they are symmetric

$$\langle y, ax_1 + bx_2\rangle = a\langle y, x_1\rangle + b\langle y, x_2\rangle$$

$\therefore$ It satisfies the 2 condition of bilinear maping

$\therefore$ $\langle \cdot,\cdot \rangle$ is bilinear.

b) To show that R preserves the inner product, we have to show that for all vectors x and y in R^2, $x^T y = (Rx)^T (Ry)$.

SUBJECT: DATE:

Given,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We have to show that $x^T y = (Rx)^T (Ry)$

$$Rx = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Rx = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}$$

$$Ry = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Ry = \begin{bmatrix} y_1\cos\theta - y_2\sin\theta \\ y_1\sin\theta + y_2\cos\theta \end{bmatrix}$$

$$x^T y = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\boxed{x^T y = x_1 y_1 + x_2 y_2} \rightarrow (1)$$

$$(Rx)^T (Ry) = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta & x_1\sin\theta + x_2\cos\theta \end{bmatrix} \begin{bmatrix} y_1\cos\theta - y_2\sin\theta \\ y_1\sin\theta + y_2\cos\theta \end{bmatrix}$$

$$= (x_1\cos\theta - x_2\sin\theta)(y_1\cos\theta - y_2\sin\theta) + (x_1\sin\theta + x_2\cos\theta)(y_1\sin\theta + y_2\cos\theta)$$

SUBJECT: DATE:

$$x_1 y_1 \cos^2\theta - x_1 y_2 \cancel{\sin\theta} \cos\theta - x_2 y_1 \cancel{\sin\theta} \cos\theta$$
$$+ x_2 y_2 \sin^2\theta + x_1 y_1 \sin^2\theta + x_1 y_2 \cancel{\sin\theta} \cos\theta$$
$$+ x_2 y_1 \cancel{\sin\theta} \cos\theta + x_2 y_2 \cos^2\theta$$

$$x_1 y_1 (\cos^2\theta + \sin^2\theta) + x_2 y_2 (\sin^2\theta + \cos^2\theta)$$

$$\boxed{x_1 y_1 + x_2 y_2} \longrightarrow (2).$$

Comparing (1) and (2).

$$\boxed{x^T y = (Rx)^T (Ry)}$$

c) To find D' such that

$$x^T D y = (Rx)^T D' (Ry)$$

Let $D' = R D R^T$

$\therefore$

$$x^T D y = x^T R^T R D R^T R y$$

$$= x^T I D I y$$

$$x^T D y = x^T D y$$

SUBJECT: DATE:

$\therefore$

$$D' = RDR^T = \begin{bmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos\theta - \operatorname{sen}\theta & \cos\theta - 3\operatorname{sen}\theta \\ 2\operatorname{sen}\theta + \cos\theta & \operatorname{sen}\theta + 3\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{bmatrix}$$

$$\cancel{= \left[(2\cos\theta - \operatorname{sen}\theta)\cos\theta + (\cos\theta - 3\operatorname{sen}\theta)(-\operatorname{sen}\theta) \right.}$$

$$= \begin{bmatrix} (2\cos^2\theta - \operatorname{sen}\theta\cos\theta) + (-\cos\theta\operatorname{sen}\theta + 3\operatorname{sen}^2\theta) & (2\cos\theta\operatorname{sen}\theta - \operatorname{sen}^2\theta) + (\cos^2\theta - 3\operatorname{sen}\theta\cdot\cos\theta) \\ (2\operatorname{sen}\theta\cdot\cos\theta + \cos^2\theta) + (-\operatorname{sen}^2\theta - 3\operatorname{sen}\theta\cdot\cos\theta) & (2\operatorname{sen}^2\theta + \cos\theta\cdot\operatorname{sen}\theta) + (\operatorname{sen}\theta\cos\theta + 3\cos^2\theta) \end{bmatrix}$$

$$D' = \begin{bmatrix} 2\cos^2\theta + 3\operatorname{sen}^2\theta - 2\operatorname{sen}\theta\cos\theta & \cos^2\theta - \operatorname{sen}^2\theta - \operatorname{sen}\theta\cos\theta \\ \cos^2\theta - \operatorname{sen}^2\theta - \operatorname{sen}\theta\cos\theta & 2\operatorname{sen}^2\theta + 3\cos^2\theta + 2\operatorname{sen}\theta\cdot\cos\theta \end{bmatrix}$$

$$D' = \begin{bmatrix} 2 + \operatorname{sen}^2\theta - \operatorname{sen}(2\theta) & \cos 2\theta - \operatorname{sen}\theta\cdot\cos\theta \\ \cos 2\theta - \operatorname{sen}\theta\cos\theta & 2 + \cos^2\theta + \operatorname{sen} 2\theta \end{bmatrix}$$

d) for $\theta = \frac{\pi}{4}$

$$D' = \begin{bmatrix} 2 + \operatorname{sen}^2(\pi/4) - \operatorname{sen}(\pi/2) & \cos(\pi/2) - \operatorname{sen}(\pi/4)\cdot\cos(\pi/4) \\ \cos(\pi/2) - \operatorname{sen}(\pi/4)\cos(\pi/4) & 2 + \cos^2(\pi/4) + \operatorname{sen}(\pi/2) \end{bmatrix}$$

SUBJECT: DATE:

$$D' = \begin{bmatrix} 2 + 1/2 - 1 & 0 - (\sqrt{2}/2)(\sqrt{2}/2) \\ 0 - (\sqrt{2}/2)(\sqrt{2}/2) & 2 + 1/2 + 1 \end{bmatrix}$$

$$D' = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

e) Angle between 2 vectors is

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \, \|y\|}$$

Since the inner product is defined by D

$$\langle x, y \rangle = x^T D y$$

$$\|x\| = \sqrt{x^T D x}$$

$$\|y\| = \sqrt{y^T D y}$$

So Angle is,

$$\cos\theta = \frac{[1, 1] \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}}{\sqrt{u^T D u} \, \sqrt{v^T D v}}$$

SUBJECT: DATE:

$$\cos\theta = \frac{\begin{bmatrix} 2+1 & 1+3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}}{\sqrt{\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \cdot \sqrt{\begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}}}$$

$$= \frac{[6-4]}{\sqrt{3+4}\ \sqrt{6+1}}$$

$$= \frac{2}{\sqrt{7}\sqrt{7}}$$

$$\cos\theta = \frac{2}{7}$$

$$\theta = \cos^{-1}(2/7)$$

$$\boxed{\theta = 1.281 \text{ radians}}$$

& Angle between Ru and Rv is

$$\cos\theta = \frac{\langle Ru, Rv \rangle}{\|Ru\|\ \|Rv\|}$$

$$= \frac{(Ru)^T D' (Rv)}{\sqrt{(Ru)^T D' (Ru)} \cdot \sqrt{(Rv)^T D' (Rv)}}$$

$$= \frac{u^T R^T R D R^T R v}{\sqrt{u^T R^T R D R^T R u} \cdot \sqrt{v^T R^T R D R^T R v}} \qquad (\because D' = RDR^T)$$

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$$\cos\theta = \frac{u^T R v}{\sqrt{u^T R u}\,\sqrt{v^T R v}}$$

Here we can see that this is equivalent to inner product of u and v

$$\therefore \cos\theta = \frac{2}{\sqrt{7}\sqrt{7}} = \frac{2}{7}$$

$$\theta = \cos^{-1}(2/7)$$

$$\boxed{\theta = 1.281 \text{ radians}}$$

$$\begin{bmatrix} \begin{bmatrix} a_{00}a_{00} & \cdots & a_{00}a_{00} \\ a_{00}a_{10} & \ddots & \vdots \\ \vdots & & \vdots \\ a_{00}a_{n0} & & a_{00}a_{nn} \end{bmatrix} & \begin{bmatrix} a_{0n}a_{00} & \cdots & a_{0n}a_{0n} \\ \vdots & \ddots & \vdots \\ a_{0n}a_{n0} & & a_{0n}a_{nn} \end{bmatrix} \\ \begin{bmatrix} a_{n0}a_{00} & \cdots & a_{n0}a_{0n} \\ a_{n0}a_{10} & & \vdots \\ \vdots & \ddots & \\ a_{n0}a_{n0} & & a_{n0}a_{nn} \end{bmatrix} & \begin{bmatrix} a_{nn}a_{00} & \cdots & a_{nn}a_{0n} \\ \vdots & \ddots & \vdots \\ a_{nn}a_{00} & & a_{nn}a_{nn} \end{bmatrix} \end{bmatrix}$$

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from this we can compute the column of x as

$$\sum_{j=0}^{n} a_{jx} \sum_{i=0}^{n} a_{ix} = c^2 \longrightarrow (2)$$

where c is sum along the column of X

lly

sum along a row of x is

$$\sum_{j=0}^{n} a_{xj} \sum_{i=0}^{n} a_{xi} = a^2 \longrightarrow (3)$$

where a is sum along a row of x

lly sum along the diagonal of x is

$$\sum_{j=0}^{n} a_{jj} \sum_{i=0}^{n} a_{ii} = b^2 \longrightarrow (4)$$

where b is sum along a diagonal of x

lly

$$\sum_{j=0}^{n} a_{j(n-j)} \sum_{i=0}^{n} a_{i(n-i)} = d^2 \longrightarrow (5)$$

where d is sum along counter diagonal of x

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Exercise 7

a) Given,

Inner product $\langle \cdot, \cdot \rangle : V \times V \to R$

x and y are non zero vectors

To prove:

x and y are orthogonal

x and y are linearly independent

Proof

2 vectors are orthogonal if $x \cdot y = 0$

let $x = \langle a, b \rangle$

$y = \langle c, d \rangle$

$x \cdot y = ac + bd = 0$

as a, b, c, d are non zero.

$\boxed{ac = -bd}$

$\therefore$ The vectors are orthogonal.

Let $\alpha x + \beta y = 0$ for some scalar α and β

$\langle \alpha x + \beta y, x \rangle = \langle 0, x \rangle = 0$

$\alpha \langle x, x \rangle + \beta \langle y, x \rangle = 0$

$\alpha \|x\|^2 + \beta \times 0 = 0$

Since $x \neq 0$

$\boxed{\alpha = 0}$

SUBJECT: DATE:

Similarly,

$\langle \alpha x + \beta y, y \rangle = \langle 0, y \rangle = 0$

$\beta \|y^2\| = 0$

Since $y \neq 0$

$\boxed{\beta = 0}$

$\therefore$ x and y are linearly independent

b) ~~#@@~~

Given

$p(x) = 3x^2 - 1$

$q(x) = 2x + 1$

Interval $= [0, 1]$

To find: If the function is orthogonal or not.

Proof

For function to be orthogonal, the inner product space $= 0$

$$\int_0^1 (3x^2 - 1)(2x + 1)\, dx$$

$$\int_0^1 (6x^3 + 3x^2 - 2x - 1)\, dx$$

SUBJECT: DATE:

$$\left[\frac{6x^4}{4} + \frac{3x^3}{3} - \frac{2x^2}{2} - x\right]_0^1$$

$$\left[\frac{3x^4}{2} + x^3 - x^2 - x\right]_0^1$$

$$\frac{3}{2} + 1 - 1 - 1$$

$$\frac{3}{2} - 1$$

$$\frac{1}{2} \neq 0$$

$\therefore$ They are not orthogonal.