Question 1

	Location		Approximate air distance (000km) to:			
City	degrees North of Equator	degrees East of Gr'nwich	Auckland	Chennai	Marrakesh	Winnipeg
Auckland	-39	175	0	11	19	13
Chennai	13	80	11	0	9	13
Marrakesh	32	-8	19	9	0	7
Winnipeg	50	-97 ($\equiv 263$)	13	13	7	0

Relations R_1, R_2 and R_3 on the set of four cities tabulated above are defined below. Draw a directed graph for each relation (use arrows between 'A', 'C', 'M' and 'W') and say whether or not the relation is a function. If not a function, say why not.

(a) $xR_1y \Leftrightarrow x$ is more than 10 000km from y.

(b) $xR_2y \Leftrightarrow x$ is South of y.

(c) $xR_3y \Leftrightarrow x$ is between 80° and 100° East of y.

Question 2

Functions $a, b, c, d \ : \mathbb{N} \to \mathbb{N}$ are defined by the rules below. In each case decide whether the function is injective (one-to-one), surjective (onto), neither or both (bijective). Justify your answers.

$$a(n) = \begin{cases} 2n & \text{if } n \text{ is odd} \\ 3n & \text{if } n \text{ is even} \end{cases} \qquad b(n) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases} \qquad c(n) = n^2 \qquad d(n) = \lfloor \sqrt{n} \rfloor$$

For d, $\lfloor x \rfloor$ denotes the 'floor' of x, the greatest integer not greater than x. E.g. $\lfloor \sqrt{7} \rfloor = 2$.

Question 3

Let $A = \{x, y\}$ and let $B = \{4, 7, 9\}$.

1. Use set roster notation to list all possible functions from A to B. Give each of your functions a different name.
2. Use set roster notation to list all possible functions from B to A. Give each of your functions a different name.
3. Which, if any, of your functions from part 1 are injective?
4. Which, if any, of your functions from part 1 are surjective?
5. Which, if any, of your functions from part 2 are injective?
6. Which, if any, of your functions from part 2 are surjective?

Question 4 Prove or disprove each of the following statements:

1. There exists a bijection from $\mathbb{N}$ to $\mathbb{Z}$.
2. Every function from $\mathbb{Z}$ to $\mathbb{N}$ is surjective.
3. For any nonempty set A, any function from A to A is a bijection.
4. For any nonempty set A, there exists a bijection from A to A.

Question 5 Let $x = 11010_2,\ y = 10111_2,\ s = x + y,\ d = x - y,\ p = xy.$

(a) Calculate s, d and p directly in binary. Keep the answers in binary.

(b) Convert x, y to hexadecimal and recalculate s, d and p directly in hexadecimal.

(c) Finally convert x, y to decimal, recalculate s, d and p (in decimal) and convert the answers to hexadecimal and thence to binary.
Use the results to check your answers to (a) and (b).

Question 6 This question is about the toggle-plus-one method as it relates to the storage of integers in computer words. It also shows how this method avoids the need for separate subtraction circuits.

As demonstrated in lectures, for a binary word W, toggle-plus-one means:

toggle:	replace every 1 by 0, every 0 by 1, then
add one:	treating W as a binary number, add 1.
	Ignore any carry beyond the length of the word.

For $l \in \mathbb{N}$ let $S_l = \{n \in \mathbb{Z} : -2^{l-1} \leq n < 2^{l-1}\}$. Then S_l is the set of all integers that can be stored in words of length l bits, using the standard computer representation. The rules governing the storage of an integer n in a word W may be summarised as:

Rule 1:	n is negative if and only the left-most bit of W is 1
Rule 2:	$-n$ is stored as the word obtained from W by toggle-plus-one.
	This is true even when n is negative.
Rule 3:	If the left-most bit of W is 0 than $n = W_2$.
	i.e. n is retrieved by treating W as a binary number.

In computer arithmetic, subtraction uses negation and addition: $x - y = x + (-y)$.
In the addition, any carry beyond the length of the word is ignored.
As an example, take $x = 11010_2,\ y = 10111_2$ and use 8-bit computer arithmetic on:

(a) $x - y$ (b) $y - x$ (c) $-x - y$

Check your results by expressing x, y and your answers in decimal.

Question 7 Let A, B, C be nonempty sets and let $f : A \to B$ and $g : B \to C$ be functions. Prove or disprove each of the following statements:

1. If f and g are injective, then $g \circ f$ is injective.
2. If f is injective and g is surjective, then $g \circ f$ is surjective.

Question 8 Let $Q = \mathbb{Z} \times (\mathbb{Z} \setminus 0)$. Let $\sim$ be the relation on Q defined by

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc.$$

1. Use set roster notation to describe the set
$$F_{(2,3)} = \{(x, y) \in Q \mid (x, y) \sim (2, 3)\}$$
2. Use set roster notation to describe the set
$$F_{(-3,6)} = \{(x, y) \in Q \mid (x, y) \sim (-3, 6)\}$$
3. You have encountered the relation $\sim$ before, although you probably did not use the notation of relations. In what context have you seen this relation?