Section A2: Sets (continued)

Recap and announcements

Proofs by contradiction

contrapositive

⋮

Sets

A collection of things specified by membership

$\emptyset$, $\mathbb{N}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, ...

$A \subset B \iff \forall x \, ((x \in A) \to (x \in B))$

New sets from old

Suppose that A and B are subsets of a universe U.

The **union** of A and B, denoted $A \cup B$, is the set

$$\{x \in U \mid (x \in A) \underset{\text{or}}{\vee} (x \in B)\}.$$

The **intersection** of A and B, denoted $A \cap B$, is the set

$$\{x \in U \mid (x \in A) \underset{\text{and}}{\wedge} (x \in B)\}.$$

The **difference** of B minus A, or B **without** A, denoted $B - A$ or $B \setminus A$, is the set

$$\{x \in U \mid (x \in B) \wedge (x \notin A)\}.$$

New sets from old

Suppose that A and B are subsets of a universe U.

The **complement** of A (in U), denoted A^c, is the set

$$\{x \in U \mid x \notin A\}$$

The complement of A cannot be understood unless the universe of discourse has been communicated.

The **symmetric difference** of A and B, denoted $A \triangle B$, is the set

$$\{x \in U \mid (x \in A) \oplus (x \in B)\}.$$

xor

Some examples

Suppose that the universe of discourse is the set $\overset{\mathcal{U}}{\overset{\shortparallel}{\mathbb{Z}}}$ and let

O be the set of odd integers
E be the set of even integers
P be the set of primes
C be the set of composite numbers.

A **composite number** is a positive integer that can be formed by multiplying two smaller positive integers.

Find simple expressions for: $O \cup E$, $O \cap E$, $E \cap P$, $O \cap P$, $P \cup C$, O^c, P^c, $E \triangle P$, $(O \triangle P) \cap \mathbb{Z}_{\geq 0}$

Some examples

$$\left.\begin{aligned} O \cup E &= \mathbb{Z} \\ O \cap E &= \phi \end{aligned}\right\} \text{even/odd theorem}$$

$$E \cap P = \{2\}$$

$$O \cap P = P \setminus \{2\}$$

$$P^c = \{z \in \mathbb{Z} \mid z \leq 1\} \cup C$$

even xor prime $\rightarrow$

$$E \triangle P = (E \cup P) \setminus \{2\}$$

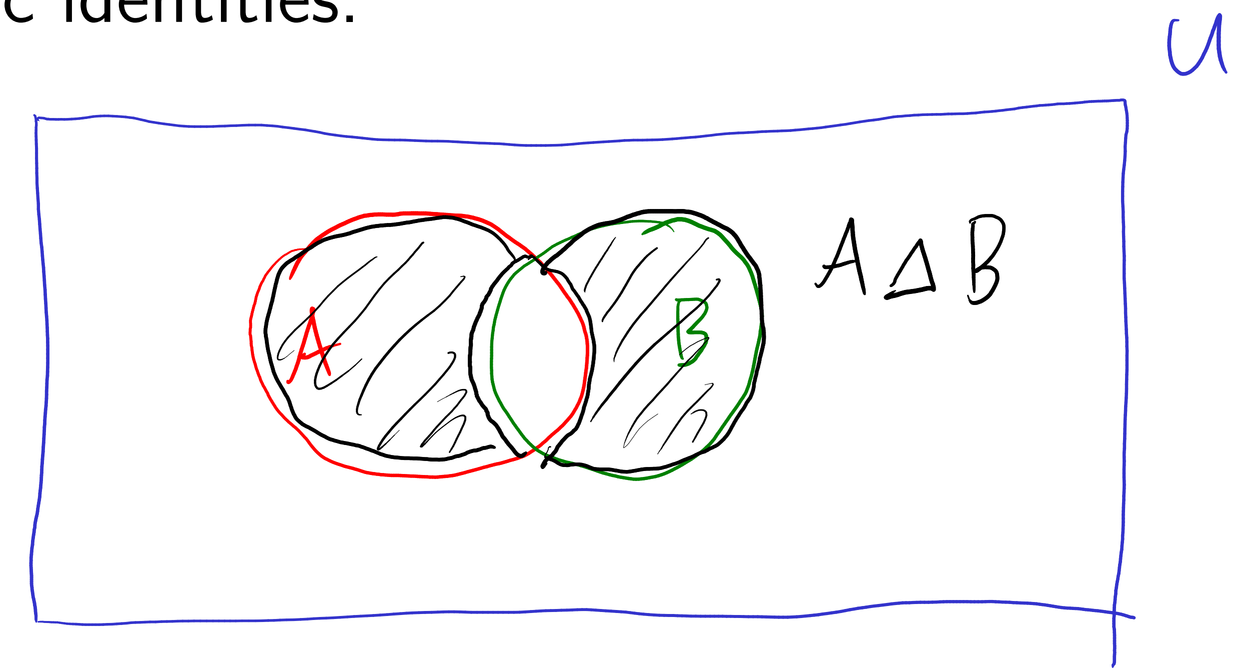
$$(O \triangle P) \cap \underset{\mathbb{N}}{\cancel{\mathbb{Z}}^+} = (O \cap C) \cup \{1, 2\}.$$

(odd xor prime) and positive

Using logic to prove set identities

An **identity** is a relationship that holds not matter which substitutions are made for the variables.

Since the set operations $\cup$, $\cap$, $\setminus$, $\subseteq$, ^{c} and $\triangle$ are defined using logical connectives, logical equivalences can be used to proved set theoretic identities.



An example

Let A, B and C be subsets of a universe U. Then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$p \wedge (q \vee r) \quad (p \wedge r) \vee (q \wedge r)$

Proof: Let $x \in U$.

$$\begin{aligned}
& x \in A \cap (B \cup C) & \\
\Leftrightarrow & (x \in A) \wedge (x \in B \cup C) & \text{Defn of } \cap \\
\Leftrightarrow & (x \in A) \wedge (x \in B \vee x \in C) & \text{Defn of } \cup \\
\Leftrightarrow & ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C)) & \text{Distr.} \\
\Leftrightarrow & (x \in A \cap B) \vee (x \in A \cap C) & \text{Defn of } \cap \\
\Leftrightarrow & x \in (A \cap B) \cup (A \cap C) & \text{Defn of } \cup \ \square
\end{aligned}$$

Note: we use $\Leftrightarrow$ between statemenst here, not $\equiv$. We shall reserve $\equiv$ for when we are working with statements and statement forms considering logic only.

An "element proof."

Let A, B and C be subsets of a universe U. Then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof:
We prove the set equality by two subset proofs. First we show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Let $x \in U$. Suppose that $x \in A \cap (B \cup C)$.

By definition of $\cap$, $x \in A$ and $x \in B \cup C$. By definition of $\cup$, $x \in B$ or $x \in C$. We consider cases.

Case $x \in B$: Since $x \in A$ and $x \in B$, $x \in A \cap B$. Hence $x \in (A \cap B) \cup (A \cap C)$.

Case $x \notin B$: Since $x \notin B$, $x \in C$. Since $x \in A$ and $x \in C$, $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$.

In all cases, $x \in (A \cap B) \cup (A \cap C)$.

Element proof (cont)

Now we show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Let $x \in U$. Suppose that $x \in (A \cap B) \cup (A \cap C)$.

By definition of $\cup$, $x \in A \cap B$ or $x \in A \cap C$. We consider cases.

Case $x \in A \cap B$: By definition of $\cap$, $x \in A$ and $x \in B$. Since $x \in B$, $x \in B \cup C$. Since $x \in A$ and $x \in B \cup C$, $x \in A \cap (B \cup C)$.

Case $x \notin A \cap B$: Since $x \notin A \cap B$, $x \in A \cap C$. By definition of $\cap$, $x \in A$ and $x \in C$. Since $x \in C$, $x \in B \cup C$. Since $x \in A$ and $x \in B \cup C$, $x \in A \cap (B \cup C)$.

In all cases, $x \in A \cap (B \cup C)$. $\square$

Another construction

For any set A, the power set of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

For example, if $A = \{1, 2, 3\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Q: If A has n elements, how many elements does $\mathcal{P}(A)$ have?

Another construction

For any set A, the power set of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

For example, if $A = \{1, 2, 3\}$, then

$$\mathcal{P}(A) \quad = \quad \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Q: If A has n elements, how many elements does $\mathcal{P}(A)$ have?

A: $\mathcal{P}(A)$ has 2^n elements... for reasons we will explain when we discuss counting techniques later in the course.

Cartesian products: Another way to make new sets from old

Order and multiplicity

In sets, there is no sense of the order in which elements appear and there is no idea of how many times an elements appears.

However, in many situations the order in which data appears is important, and the same data sometimes appears multiple times.

We now look at a construction that allows us to represent order and multiplicity.

Ordered n-tuples

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple** $(x_1, x_2, \ldots, x_n)$ consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair**, and an ordered 3-tuple is called an **ordered triple**.

Two ordered n-tuples are **equal** when their elements match up exactly in order. Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$$
$$\Leftrightarrow (x_1 = y_1) \wedge (x_2 = y_2) \wedge \cdots \wedge (x_n = y_n).$$

An ordered m-tuple and an ordered n-tuple cannot be equal if $m \neq n$.

Examples

$(a, b, c) \neq (b, c, a)$ because their first elements differ.

$(a, a, b, c) \neq (a, b, c)$ because one is an ordered 4-tuple and the other is an ordered triple.

The elements in ordered n-tuples do not need to be of the same type. For example, $(\text{cat}, \text{car}, 1, \$)$ is an ordered 4-tuple.

We are, however, usually interested in sets of ordered n-tuples where all of the elements in, say, the i-th position are of the "same type"...

Cartesian product

Given (not necessarily distinct) sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$, $\ldots$, $a_n \in A_n$.

$$\{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

This last expression may be read aloud as:

Cartesian product

Given (not necessarily distinct) sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$, $\ldots$, $a_n \in A_n$.

$$\{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

This last expression may be read aloud as: "the set of all ordered n-tuples with elements a_1, a_2, through, a_n such that a_1 comes from A_1, a_2 comes from A_2, through a_n comes from A_n."

A remark

$\{x \in U \mid p(x)\}$

The expression

$$A = \{(a_1, a_2, \ldots, a_n) \in A \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

does not appear to conform to the rules of set-builder notation we laid out in the last lecture because

- the domain part introduces variables but does not specify a domain for each;
- the "predicate" does not appear to be a single predicate.

We can fix the second concern easily by making a rule that in a predicate, each comma is read and understood as "and". It is usually better to use $\wedge$. ✓

What I have written is an entirely standard way to describe a Cartesian product, even though it seems like a poor use of set-builder notation.

Examples

Let

If A has n elements
B has m elements
Then $A \times B$ has $n \times m$ elements

$$A = \{\text{cat}, \text{dog}, \text{chicken}\}$$
$$B = \{\text{yes}, \text{no}\}$$
$$C = \{100, 300\}$$

Then

$$A \times B = \{(\text{cat}, \text{yes}), (\text{cat}, \text{no}), (\text{dog}, \text{yes}), (\text{dog}, \text{no}), (\text{chicken}, \text{yes}), (\text{chicken}, \text{no})\}.$$

and

$$C \times C = \{(100, 100), (100, 300), (300, 100), (300, 300)\}$$