MATH1005/6005 Revision Lecture 1

For ~~today~~: $\boxed{\text{"Proofs"}}$

Preliminary: $\boxed{\text{arguments}}$

$$\Big[\underbrace{p_1, \ldots, p_n}_{\text{premises}} \;\therefore\; q\Big]$$

conclusion

$$\left.\begin{array}{c} p_1 \\ \vdots \\ p_n \end{array}\right\} $$
$$\overline{\quad q \quad}$$

What are $p_1, \dots, p_n, q$?

statements or statement forms

i.e. placeholders for statements

An argument is valid if

$p_1 \wedge \dots \wedge p_n \to q$ is a tautology (i.e. "$\to$" is true for any inputs into the statement forms.

What does an argument <u>not</u> do?

Does <u>not</u> tell us that the premises are true.

All it says is something about truth tables of statement forms.

Arguments are just the structure of a proof.

Some valid arguments:

Direct argument

$$\begin{array}{c} p \\ p \to q \\ \hline q \end{array}$$

$$\begin{array}{c} p \to q \\ \neg q \\ \hline \neg p \end{array}$$

Division into cases

$$\begin{array}{c} p \vee q \\ p \to r \\ q \to r \\ \hline r \end{array}$$

$\boxed{\text{Proofs:}}$

function evaluated at $x=2$ vs. function of x

Proof vs. Argument

Structure as an argument, but the statement forms are replaced by actual statements

Might need to check each statement is actually true.

Structure built out of statement forms

Valid if premises can be combined to produce the conclusion

Example

Direct argument

$$\begin{array}{c} p \\ (p \to q) \\ \hline q \end{array}$$

Isn't true upon substitution

But what if

$p =$ Is human

$q =$ has seven legs

Note: $p \to q$ is false for this particular p, q.

To prove something, you need to both:

- have a <u>valid argument structure</u>
- Check that all the premises are <u>actually</u> true

How to actually prove things*

*Involving $\to$

1. Figure out if you actually believe the statement

2. Identify the assumptions and conclusion

$$[\underbrace{p_1, \dots, p_n}\ \therefore r]$$

3.* Try to decide on a valid argument structure
Might require new premises

$$[p_1, \dots, p_n, \underbrace{q_1, \dots, q_m}_{\text{new premises}}, r]$$

4. Verify any added premises

*: Creative process, might require trial and error, easier with experience

Example:

$\forall n \ p(n)$ where $p(n)$ = "$f(n)$ is divisible by 6"

Theorem: For any $n \in \mathbb{Z}$,

$$f(n) := n^3 + 3n^2 + 2n$$

is divisible by 6.

For this course

$\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{Z}_{\geq} = \{0, 1, 2, 3, \dots\}$

$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$

Do we believe the statement? $f(0) = 0$, $f(1) = 6$, $f(-1) = 0$, $f(2) = 24$

all divisible by 6

Assumption(s): $n \in \mathbb{Z}$ Conclusion: $f(n)$ is divisible by 6

Structure: Direct / contrapositive / ~~counter~~example / contradiction

contrapositive: assume $f(n)$ not divisible by 6 ⋮ Find $n \notin \mathbb{Z}$

counterexample: since we believe the statement, $\exists n \ \neg p(x)$

contradiction: tricky

$\neg(p \to r) \to c$

column of "false" in a truth table

What do the different structures look like?

Direct:

Suppose ($n \in \mathbb{Z}$) p

$\vdots$

So ($f(n)$ is divisible by 6) r

$\boxed{\text{Contrapositive:}}$

Suppose ($f(n)$ is not divisible by 6) $\neg r$

$\vdots$

So ($n \notin \mathbb{Z}$.) $\neg p$

$$\frac{\begin{matrix} p \\ ??? \\ \vdots \end{matrix}}{r}$$

How do we fill in the gap?

Some (direct) options

"$\underbrace{p \to r}$"

"$p \to p_1 \vee \dots \vee p_n$"

$p_1 \to r$

$\vdots$

$p_n \to r$

e.g. $n \in \mathbb{Z} \to (n \text{ even}) \vee (n \text{ odd})$

n odd $\to r$

n even $\to r$

Let's go with the first option (since I already know it works).

So our proof is:

p

Suppose $n \in \mathbb{Z}$.

$p \to r$

Then $f(n) = n^3 + 3n^2 + 2n$

$= n(n^2 + 3n + 2)$

$= n(n+1)(n+2)$

So $f(n)$ is the product of three consecutive integers. At least one will be even, and at least one will be divisible by 3.

Creative, and difficult to teach

So $f(n)$ is divisible by 6.

r

Other sorts of statements and proof strategies

For any statement, the first thing to do is

Statement form	Prove	Disprove
$\forall x \ p(x)$	Assume x is fixed but arbitrary, argue $p(x)$ is true or show $\nexists x \ \neg p(x)$ by contradiction	• $\neg(\exists x \ p(x))$ • $\equiv \forall x \ \neg p(x)$
$\exists x \ p(x)$	• Find an example • Show $\forall x \ \neg p(x)$ must be false by contradiction	$\neg(\forall x \ p(x))$ $\equiv \exists x \ \neg p(x)$
$p \to q$	already covered	$\neg(p \to q) \equiv p \wedge \neg q$ So come up with an example with p and $\neg q$
$p \leftrightarrow q$	$p \to q$ and $q \to p$	Counter example of $p \to q$ or $q \to p$

$\forall x \; p(x)$ example

Statement:

$\overset{p}{(\forall z \in \mathbb{Z})}, \; \underset{r}{(z^2 + 3z + 1 \text{ is odd})}$

This statement is true

Try $\cancel{p \to r}$

Try
$$\begin{cases} p \to e \lor o \\ e \to r \\ o \to r \end{cases}$$

$\left(\begin{array}{l} e: \; z \text{ is even} \\ o: \; z \text{ is odd} \end{array}\right)$

Proof:

Suppose $z \in \mathbb{Z}$

Then z is even or odd (by even/odd theorem).

If z is even, $\exists k \in \mathbb{Z}$ with $z = 2k$

So
$$\begin{aligned} z^2 + 3z + 1 &= 4k^2 + 6k + 1 \\ &= 2(2k^2 + 3k) + 1 \quad \text{odd} \end{aligned}$$

If z is odd, $\exists k \in \mathbb{Z}$ with $z = 2k+1$

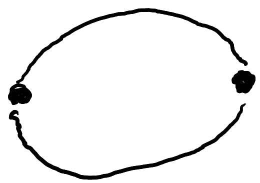
So
$$\begin{aligned} z^2 + 3z + 1 &= (2k+1)^2 + 3(2k+1) + 1 \\ &= 4k^2 + 4k + 1 + 6k + 3 + 1 \\ &= 4k^2 + 10k + 5 \\ &= 2(2k^2 + 5k + 2) + 1 \quad \text{odd} \end{aligned}$$

So $z^2 + 3z + 1$ is odd.

$\exists x \quad p(x)$ example

Statement:
There exists a graph with at least two vertices, which has a circuit which is both a Hamiltonian circuit and an Euler circuit.

Can prove by giving an example.



Has a circuit which is Euler and Hamiltonian, and it has at least two vertices.

Disproving $\exists$, $\forall$?

Disprove $\exists x \ p(x)$ $\iff$ Prove $\forall x \ \neg p(x)$

" $\forall x \ p(x)$ $\iff$ " $\exists x \ \underline{\neg p(x)}$

Disprove → example:

$\neg(p \to q) \equiv \underbrace{p} \wedge \neg \underbrace{q}$

Statement:

Let $a, b, c \in \mathbb{N}$. (If c divides ab)^{p}, then (c divides a)^{q_1} or$^{\vee}$ (c divides b.)^{q_2}

$q = q_1 \vee q_2$

$\neg(p \to (q_1 \vee q_2)) \equiv p \wedge \neg(q_1 \vee q_2)$

$\equiv p \wedge \neg q_1 \wedge \neg q_2$

Counter example:

$a = 4, b = 3, c = 6$

$2 \quad 2 \quad 4$

$p \wedge (c \text{ is prime}) \to q_1 \vee q_2$

is actually <u>true</u>.

General tips:

Rational Numbers and IEEE half-precision floating point

What are rational numbers?

At a more abstract level:

$\mathbb{Q}$

How can we represent rational numbers?

Idea 1:

Idea 2: Pick a base $b \in \{2,3,4,\dots\}$, and try write

$$q = a_N b^N + a_{N-1} b^{N-1} + \dots \quad , \quad a_N \in \{0, \dots, b-1\}$$

$$\| $$

$$(a_N a_{N-1} \dots a_0 . a_{-1} \dots)_b$$

Idea 3:

$$q = (-1)^s \, m \, b^n \,, \qquad s \in \{0,1\}, \quad n \in \mathbb{Z}, \quad 1 \le m < b,$$

combined with

$$m = (a_0 . a_{-1} a_{-2} \dots)_b$$

IEEE half-precision:

If we take $b = 2$ (i.e. what computers naturally handle)

we can optimise a bit further:

m will <u>always</u> be of the form:

So:

We can package everything into a bit-string as follows:

$$\underbrace{*}_{s}\quad \underbrace{*\ *\ *\ *\ *}_{n+15}\quad \underbrace{*\ *\ *\ *\ *\ *\ *\ *\ *\ *}_{m=(1.(\dots))_2}$$

(or with a different number of bits for exponent / mantissa or different offset)

Example

Suppose x is stored in half-precision floating point as

$$1\;0\;1\;1\;1\;1\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0$$

What is x?

Example

Suppose $y = (101100)_2$. Represent y in half-precision floating point.