**Question 1** Recall that a natural number $n$ is *prime* provided that $n > 1$ and $n$ has no positive divisors other than 1 and itself (e.g. 17 is prime but 18 is not). For universal set $U = \{n \in \mathbb{N} : n \leq 15\}$ define sets $P$ and $H$ as follows:

$$P = \{n \in U : n \text{ is prime}\} \qquad H = \{n \in U : n > 10\}.$$

How many members have each of the following sets? Show your enumeration/calculation.

(a) $P^c$ (b) $P \cap H$ (c) $P \cup H$

(d) $P \backslash H$ (e) $P \triangle H$ (f) $\mathcal{P}(P)$ [the power set of $P$].

*First note that* $P = \{2, 3, 5, 7, 11, 13\}$ *has* 6 *members.*

*(a)* $15 - 6 = \boxed{9}$.

*(b)* $P \cap H = \{11, 13\}$ *has* $\boxed{2}$ *members.*

*(c)* $P \cup H = \{2, 3, 5, 7, 11, 12, 13, 14, 15\}$ *has* $4 + 5 = \boxed{9}$ *members.*

*(d)* $P \backslash H = \{2, 3, 5, 7\}$ *has* $6 - 2 = \boxed{4}$ *members.*

*(e)* $P \triangle H = \{2, 3, 5, 7, 12, 14, 15\}$ *has* $4 + 3 = \boxed{7}$ *members.*

*(f)* $\mathcal{P}(P)$ *has* $2^6 = \boxed{64}$ *members.*

**Question 2** Recall that a natural number $n$ is *prime* provided that $n > 1$ and $n$ has no positive divisors other than 1 and itself (e.g. 17 is prime but 18 is not) and that $n$ is *composite* if it is product of two smaller natural numbers (e.g. 18 is composite because $18 = 6 \times 3$). Define sets $U$, $P$ and $C$ as follows:

$$U = \{n \in \mathbb{N} : n \leq 15\} \quad P = \{n \in U : n \text{ is prime}\} \quad C = \{n \in U : n \text{ is composite}\}$$

Decide the truth or falsity of each of the following. Beware of deliberate traps! Briefly justify your answers.

(a) $\{1, 2, 3\} \subseteq P$ (b) $\{8, 9, 10\} \in C$ (c) $P \cap C = \emptyset$

(d) $\{11, 12\} \in P \times C$ (e) $P \subset P \times C$ (f) $\{P, C\}$ is a partition of U

*(a)* $\boxed{\textit{False}}$ $1 \notin P$.

*(b)* $\boxed{\textit{False}}$ *The members of $C$ are numbers, not sets.*
[*What is true is that* $\{8, 9, 10\} \subseteq C$.]

*(c)* $\boxed{\textit{True}}$ *No number is both prime and composite.*

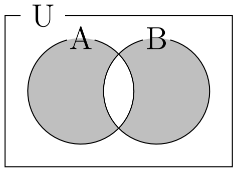
*(d)* $\boxed{\textit{False}}$ *The members of $P \times C$ are ordered pairs, not doubleton sets.*
[*What is true is that* $(11, 12) \in P \times C$.]

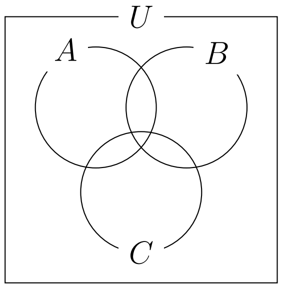
*(e)* $\boxed{\textit{False}}$ *The members of $P$ are not ordered pairs, so are not members of $P \times C$.*

*(f)* $\boxed{\textit{False}}$ *1 is neither prime nor composite, so $1 \notin P \cup C$. Hence $P \cup C \neq U$. But $P \cup C$ needs to be $U$, This is one of the two requirements for $\{P, C\}$ to be a pratitiion of $U$.*
[*The other requirement is that $P \cap C = \emptyset$, and this holds, as claimed for (c).*]

**Question 3** A **Venn diagram** is a graphical reprsentation of sets, subsets and elements. Inside a rectangle for the universal set are one or more circles (or shapes) representing sets. Dots inside the circles (or shapes) represent elements. A subset of the universe, and its relationship to the sets represented by circles, can be indicated by shading various regions on the Venn diagram.

For example, this Venn diagram represents the symmetric difference of $A$ and $B$.




(a) Using copies of the skeleton Venn diagram at right, draw five diagrams, one for each of the following:

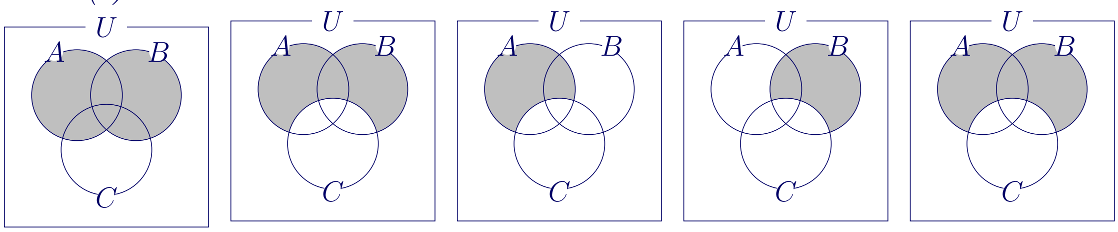
i) $A \cup B$ ii) $(A \cup B)\backslash C$

iii) $A\backslash C$ iv) $B\backslash C$ v) $(A\backslash C) \cup (B\backslash C)$

(b) Based on your answers to (a) decide whether $(A \cup B)\backslash C = (A\backslash C) \cup (B\backslash C)$.

(c) Use an element proof (and a logical equivalence) to prove your answer to (b).

*(a)* *(i)* *(ii)* *(iii)* *(iv)* *(v)*



$A \cup B$ $(A \cup B)\backslash C$ $A\backslash C$ $B\backslash C$ $(A\backslash C) \cup (B\backslash C)$

*(b) The second and fifth diagrams are the same, so the corresponding sets should be the same. So we decide that* $\boxed{\textit{the equation is correct}}$.

*(c) We prove that* $\forall x \in U\ (x \in (A \cup B) \setminus C) \Leftrightarrow (x \in (A \setminus C) \cup (B \setminus C))$.
*Let* $x \in U$. *Then*

$x \in (A \cup B)\backslash C$

$\Leftrightarrow x \in (A \cup B)\ \wedge\ \neg(x \in C)$ *(def. of set diff.)*

$\Leftrightarrow [(x \in A) \vee (x \in B)]\ \wedge\ \neg(x \in C)$ *(def. of union)*

$\Leftrightarrow [(x \in A) \wedge \neg(x \in C)] \vee [(x \in B) \wedge \neg(x \in C)]$ (***logical equivalence***)

$\Leftrightarrow (x \in A\backslash C) \vee (x \in B\backslash C)$ *(def. of set diff.)*

$x \in (A\backslash C) \cup (B\backslash C)$ *(def. of union).*

## Question 4

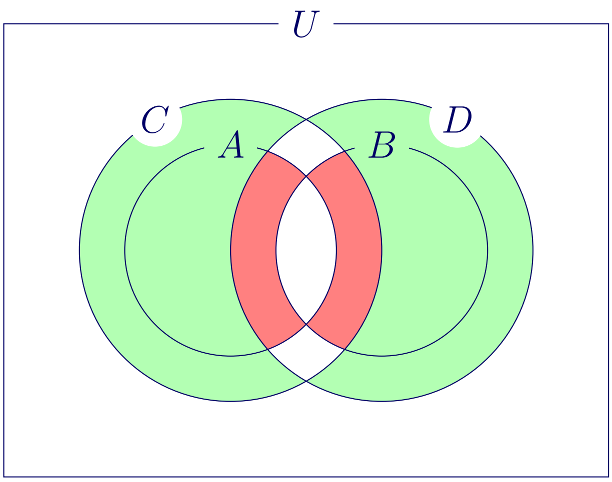
(a) Draw a Venn diagram showing four sets $A, B, C, D$ in the most general configuration for which $A \subseteq C$ and $B \subseteq D$.

(b) By referring to your answer to (a), decide on the truth or falsity of the claim that, for all sets $A, B, C, D$,

$$[(A \subseteq C) \wedge (B \subseteq D)] \Rightarrow [(A \triangle B) \subseteq (C \triangle D)].$$

(c) [Challenge] Prove your answer to (b).

*(a)*




*(b)* *Shown in red on the diagram are sections of* $A \triangle B$ *which are not part of* $C \triangle D$ *(shown green).*

*Thus even though* $(A \subseteq C) \wedge (B \subseteq D)$ *is true,* $(A \triangle B) \subseteq (C \triangle D)$ *is false.*

$\boxed{\textit{So the claim is false.}}$

*(c)* *An extreme counterexample occurs when* $A \neq B$ *but* $C = D$*. In this case* $(A \triangle B) \neq \emptyset$ *but* $(C \triangle D) = \emptyset$ *and so* $(A \triangle B)$ *cannot be a subset of* $(C \triangle D)$*.*

*To ensure that* $A$ *is a subset of* $C$ *and* $B$ *is a subset of* $D$ *we could take:*

$$\boxed{a = \{a\},\ B = \{b\},\ a \neq b,\ C = D = \{a, b\}}$$

## Question 5

For any integers $x$ and $y$, we say that $x$ **divides** $y$ if there exists an integer $k$ such that $y = kx$.

For each integer $x$, we define $M_x = \{z \in \mathbb{Z} \mid x \text{ divides } z\}$.

(a) Describe $M_5$ using set-roster notation.

(b) What is another name for $M_2$?

(c) What is another name for $\mathbb{Z} \setminus M_2$?

(d) Prove the following statement: $M_3 \neq M_5$.

(e) Prove or disprove the following statement: $M_7 \cap M_5 = \emptyset$

*(a)* $M_5 = \{\ldots, -5, 0, 5, 10, 15, \ldots\}$.

*(b)* $M_2$ *is the set of even integers.*

*(c)* $\mathbb{Z} \setminus M_2$ *is the set of odd integers.*

*(d) Since* $3 \in M_3$, *but* $3 \notin M_5$, $M_3 \neq M_5$.

*(e) The statement is false. Since* $35 = 5 \times 7$, *we have* $35 \in M_7$ *and* $35 \in M_5$. *Since* $35 \in M_7 \cap M_5$, $M_7 \cap M_5 \neq \emptyset$.

## Question 6

In this problem we use the definitions and notation of Question 5. Recall that the notation $A \subsetneq B$ reads "$A$ is a subset of $B$ and $A$ is not equal to $B$."

Determine which of the following statements, if any, are true. Give a brief reason in each case.

(a) $M_7 \subseteq M_{14}$

(b) $M_{14} \subseteq M_7$

(c) $M_7 \subsetneq M_{14}$

(d) $M_{14} \subsetneq M_7$

*(a) Statement (a) is false. For example,* $7 \in M_7$ *but* $7 \notin M_{14}$.

*(b) Statement (b) is true. Every multiple of 14 is a multiple of 7.*

*(c) Statement (c) is false. For example,* $7 \in M_7$ *but* $7 \notin M_{14}$.

*(d) Statement (d) is true. This follows from our answers to (i) and (ii).*

## Question 7

Let $S = \{z \in \mathbb{Z} \mid -5 \leq z \leq 5\}$. Let

$E$ denote the set of even integers,
$O$ denote the set of odd integers.

Determine whether or not each of the following sets is a partition of $S$. Justify your answer in each case.

(a) $\mathcal{P}_1 = \{\{-5, 1, 4, 5\}, \{-4, -3\}, \{-2, 0\}, \{-1, 2, 3\}\}$

(b) $\mathcal{P}_2 = \{\mathbb{N} \cap S, \mathbb{N} \setminus S\}$

(c) $\mathcal{P}_3 = \{E \cap S, O \cap S, \{0\}\}$

(d) $\mathcal{P}_4 = \{\mathbb{N} \cap S, S \setminus \mathbb{N}\}$

*(a) $\mathcal{P}_1$ is a partition of $S$. It is comprised of subsets of $S$, each element of $S$ appears is exactly one set in $\mathcal{P}_1$, and no set in $\mathcal{P}_1$ is empty.*

*(b) $\mathcal{P}_2$ is not a partition of $S$. Since $6 \in \mathbb{N} \setminus S$, and $6 \notin S$, the sets in $\mathcal{P}_2$ are not all subsets of $\mathcal{P}_2$.*

*(c) $\mathcal{P}_3$ is not a partition of $S$. Since $0 \in E \cap S$ and $0 \in \{0\}$, the sets in $\mathcal{P}_3$ are not pairwise disjoint.*

*(d) Since $\{\mathbb{N} \cap S, S \setminus \mathbb{N}\} = \{\{1, 2, 3, 4, 5\}, \{-5, -4, -3, -2, -1, 0\}\}$, $\mathcal{P}_4$ is a partition of $S$.*

## Question 8

Consider the following statement: No integers $x$ and $y$ exist for which $6x + 24y = 13$.

(a) Clearly identify the logical structure, including appropriate quantification, of the statement under consideration.

(b) Use a proof by contradiction to prove the statement under consideration.

*(a) The statement under consideration may be represented by*

$$\forall x \in \mathbb{Z} \ \ \forall y \in \mathbb{Z} \ \ (6x + 24y \neq 13)$$

*(b) We prove the statement using a proof by contradiction.*
*Suppose that there exist integers $x$ and $y$ such that*

$$6x + 24y = 13.$$

*Dividing throughout by 6 yields*

$$x + 4y = \frac{13}{6}. \tag{1}$$

*Since $x$ and $y$ are integers, the left-hand side of Equation (1) is an integer. But the right-hand side of Equation (1) is not an integer, and therefore Equation (1) cannot be a true statement. Our supposition allowed us to deduce a false statement, and must therefore be false.*