

Announcements:

Next Monday public holiday.

3 Monday 6005 workshops

3 replacements + make-up workshop Friday afternoon

Attend any of the four options.

steady state:
 $T'S = S$

Recap:

Markov processes.

Steady states vs. long-run probabilities



has unique steady state.
Doesn't stabilize



Every state is stationary



unique steady state
Any starting state tends to steady state

D1. Graph Theory

Notes originally prepared by Judy-anne Osborn.

Editing, expansion and additions by Malcolm Brooks.

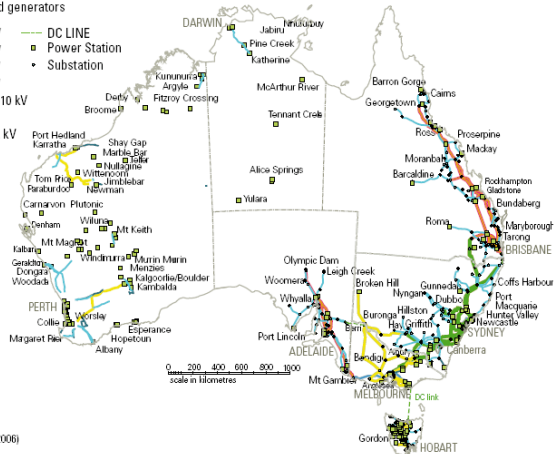
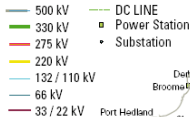
Text Reference (Epp) 3ed: Chapter 11
 4ed: Chapter 10
 5ed: Chapter 10

These references may not *completely* cover everything in this section, but they do have most it. They also contain a few items we do not cover.

Real-world phenomena often
modeled with graphs:

Australian Power Transmission Network

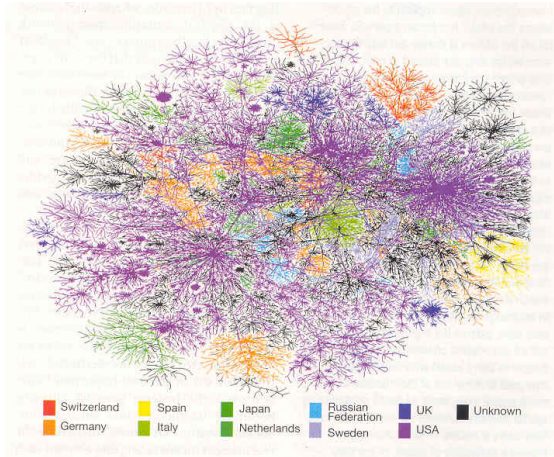
Transmission lines and generators



Locations are indicative only.

Sources: NEMMCO, ESAA (2006)

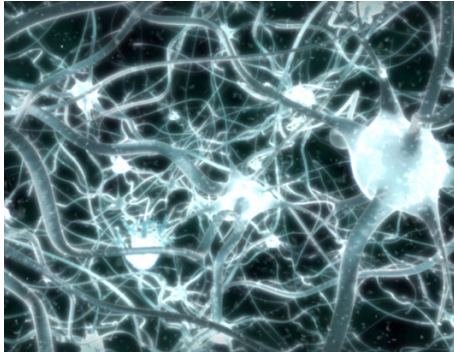
Complex Network Example: Internet



(William R. Cheswick)

18 Sept 2009 ©David J Hill The Australian National University Networked Decision

Brain Network



from documentary, 'Inside the living body'
<http://abcnews.go.com/2020/popup?id=3560899>

Graphs

A **graph** G is a collection of **vertices** and **edges**.

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- Each edge is specified by a pair of vertices.
(The two vertices could be the same.)

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Graphs

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- The set of vertices of G is denoted $V(G)$.
- The (multi)set¹ of edges of G is denoted $E(G)$.
- Each edge is specified by a pair of vertices.
(The two vertices could be the same.)
- In this course we only consider **finite** graphs;
i.e. $V(G)$ and $E(G)$ are both finite sets.

¹As explained in Section C1, a 'multiset' is just like a set except that it may contain the same element more than once.

Diagrams of Graphs

To draw a graph we use:

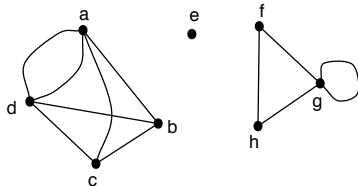
- dots/circles for vertices
- lines for edges

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- lines for edges

Example: The graph G

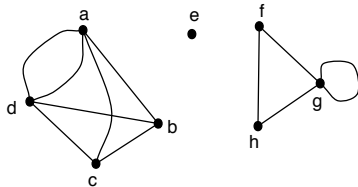


Diagrams of Graphs

To draw a graph we use:

- dots/circles for vertices
- lines for edges

Example: The graph G



has vertex set $V(G) = \{a, b, c, d, e, f, g, h\}$

and edge multiset $E(G) =$

$\{\{a,b\}, \{a,c\}, \{a,d\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{f,g\}, \{f,h\}, \{g,g\}, \{g,h\}\}.$

A table of edges

- The same graph as a table of labelled edges:

Edge	Endpoints
e_1	$\{a, b\}$
e_2	$\{a, c\}$
e_3	$\{a, d\}$
e_4	$\{a, d\}$
e_5	$\{b, c\}$
e_6	$\{b, d\}$

Edge	Endpoints
e_7	$\{c, d\}$
e_8	$\{f, g\}$
e_9	$\{f, h\}$
e_{10}	$\{g, g\}$
e_{11}	$\{g, h\}$

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e_7	$\{c, d\}$
e_8	$\{f, g\}$
e_9	$\{f, h\}$
e_{10}	$\{g, g\}$
e_{11}	$\{g, h\}$

- Notice that e_3 is distinct from e_4 even though both edges have the same endpoints.


A vertex adjacency listing

- The same graph as a vertex adjacency listing:

Vertex	Adjacent to:
<u>a</u>	<i>b, c, <u>d</u>, <u>d</u></i>
<i>b</i>	<i>a, c, d</i>
<i>c</i>	<i>a, b, d</i>
<u>d</u>	<i><u>a</u>, <u>a</u>, b, c</i>
<i>e</i>	
<i>f</i>	<i>g, h</i>
<i>g</i>	<i>f, g, h</i>
<i>h</i>	<i>f, g</i>

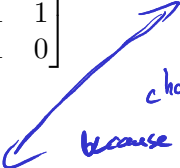
An adjacency matrix

- An adjacency matrix for the same graph






	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>
<i>a</i>	0	1	1	<u>2</u>	0	0	0	0
<i>b</i>	1	0	1	1	0	0	0	0
<i>c</i>	1	1	0	1	0	0	0	0
<i>d</i>	<u>2</u>	1	1	0	0	0	0	0
<i>e</i>	0	0	0	0	0	0	0	0
<i>f</i>	0	0	0	0	0	0	1	1
<i>g</i>	0	0	0	0	0	1	1	1
<i>h</i>	0	0	0	0	0	1	1	0

no
change,
because edges
are undirected.



An adjacency matrix

- An adjacency matrix for the same graph

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>
<i>a</i>	0	1	1	2	0	0	0	0
<i>b</i>	1	0	1	1	0	0	0	0
<i>c</i>	1	1	0	1	0	0	0	0
<i>d</i>	2	1	1	0	0	0	0	0
<i>e</i>	0	0	0	0	0	0	0	0
<i>f</i>	0	0	0	0	0	0	1	1
<i>g</i>	0	0	0	0	0	1	1	1
<i>h</i>	0	0	0	0	0	1	1	0

- The $(i, j)^{\text{th}}$ entry is the number of edges between vertices i and j .

An adjacency matrix

- An adjacency matrix for the same graph

	a	b	c	d	e	f	g	h
a	0	1	1	2	0	0	0	0
b	1	0	1	1	0	0	0	0
c	1	1	0	1	0	0	0	0
d	2	1	1	0	0	0	0	0
e	0	0	0	0	0	0	0	0
f	0	0	0	0	0	0	1	1
g	0	0	0	0	0	1	1	1
h	0	0	0	0	0	1	1	0

- The $(i, j)^{\text{th}}$ entry is the number of edges between vertices i and j .
- Thus $a_{i,j}$ is number of ways that i is adjacent to j .

Some Graph Terminology

- An edge connects its **endpoints**.

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Some Graph Terminology

- An edge connects its **endpoints**.
- An edge with both endpoints the same is called a **loop**.
- Two edges may connect the same pair of endpoints, in which case they are said to be **parallel**.
- Two vertices are **adjacent** if they are connected by an edge; two edges are **adjacent** if they share an endpoint.

Some Graph Terminology

- An edge is **incident on** its endpoints.

Some Graph Terminology

- An edge is **incident on** its endpoints.
- A vertex with no incident edges is **isolated**.

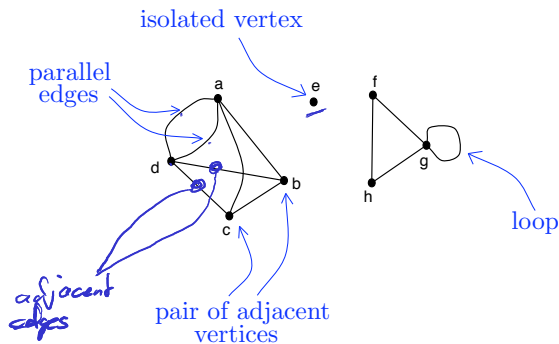
Some Graph Terminology

- An edge is **incident on** its endpoints.
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- A graph with no vertices (hence no edges) is **empty**.

Some Graph Terminology

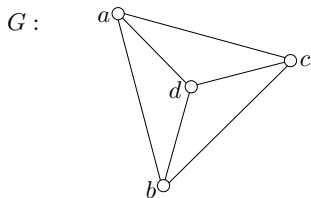
- An edge is **incident on** its endpoints.
- A vertex with no incident edges is **isolated**.
- A graph with no vertices (hence no edges) is **empty**.
- The **order** of a graph, G , is the number of vertices in it, i.e. $|V(G)|$.
(A graph of order '0' is empty.)

Some graph concepts illustrated



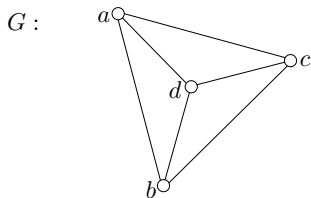
Another example

- Tetrahedron Graph

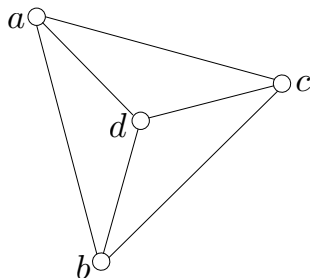


Another example

- Tetrahedron Graph



- $V(G) = \{a, b, c, d\}$
- $E(G) = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$



- Tetrahedron Graph:

Adjacency listing:

Vertex	Adjacent to:
a	b, c, d
b	a, c, d
c	a, b, d
d	a, b, c

Adjacency matrix:

$$\begin{array}{c}
 a \quad b \quad c \quad d \\
 \begin{array}{c}
 a \\
 b \\
 c \\
 d
 \end{array}
 \begin{bmatrix}
 0 & 1 & 1 & 1 \\
 1 & 0 & 1 & 1 \\
 1 & 1 & 0 & 1 \\
 1 & 1 & 1 & 0
 \end{bmatrix}
 \end{array}$$

More about graph diagrams

- Position, length, curvedness and orientation in a graph diagram do not matter for the graph represented.

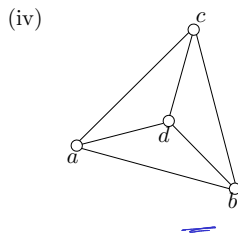
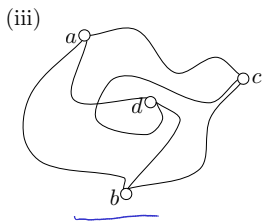
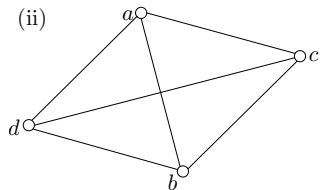
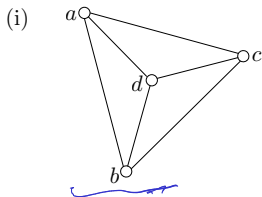
More about graph diagrams

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More about graph diagrams

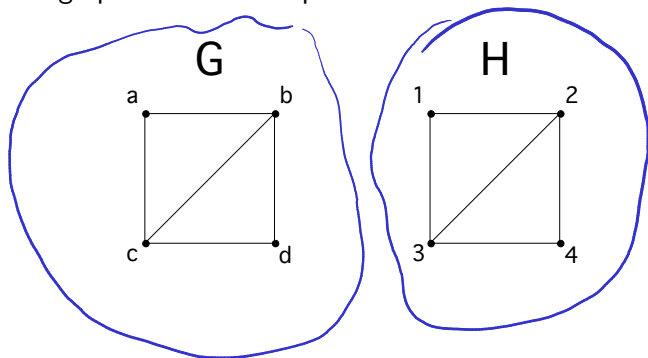
- Position, length, curvedness and orientation in a graph diagram do not matter for the graph represented.
- The only things which matter are that precisely those vertices in $V(G)$ are shown and precisely those edges in $E(G)$ are shown.
- For instance, the following diagrams all represent the same graph.

Four diagrams of the same graph



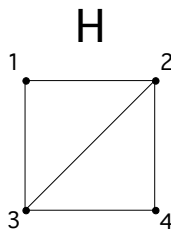
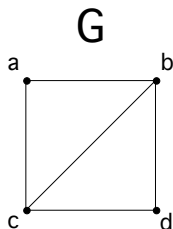
Isomorphic Graphs

Consider graphs G and H as pictured:



Isomorphic Graphs

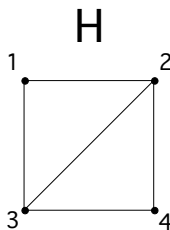
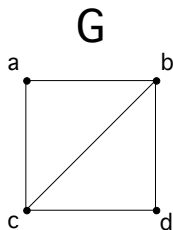
Consider graphs G and H as pictured:



- They are different graphs because their vertex labels are different.

Isomorphic Graphs

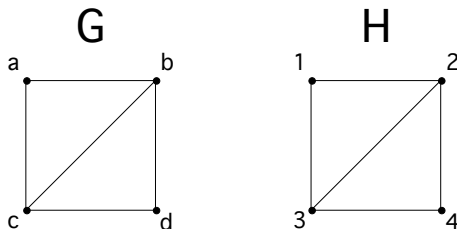
Consider graphs G and H as pictured:



- They are different graphs because their vertex labels are different.
- But they are the same in some sense.

Isomorphic Graphs

Consider graphs G and H as pictured:



- They are different graphs because their vertex labels are different.
- But they are the same in some sense.
- Formally these graphs are called 'isomorphic'.

Isomorphisms

An **isomorphism** between two graphs G_1 and G_2 is a bijection

$$f : \underline{V(G_1)} \rightarrow \underline{V(G_2)}$$

such that:

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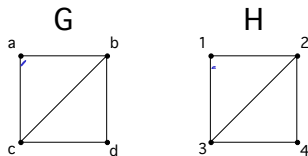
$\{u, v\}$ is an edge in $E(G_1)$

if and only if

$\{f(u), f(v)\}$ is an edge in $E(G_2)$

with multiplicity

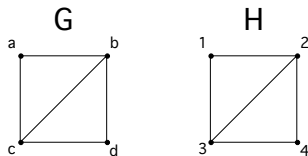
Isomorphisms



An example of an isomorphism between G and H is the mapping

$$f : V(G) \rightarrow V(H)$$

Isomorphisms

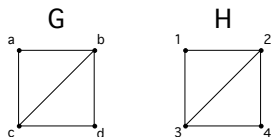


An example of an isomorphism between G and H is the mapping

$$f : V(G) \rightarrow V(H)$$

$$\begin{array}{lcl} a & \mapsto & 1 \\ b & \mapsto & 2 \\ c & \mapsto & 3 \\ d & \mapsto & 4 \end{array}$$

Isomorphisms



The mapping 'preserves' edges:

$$\{a, b\} \mapsto \{1, 2\} \quad \checkmark$$

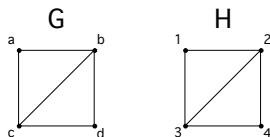
$$\{a, c\} \mapsto \{1, 3\} \quad \checkmark$$

$$\{b, c\} \mapsto \{2, 3\} \quad \checkmark$$

$$\{b, d\} \mapsto \{2, 4\} \quad \checkmark$$

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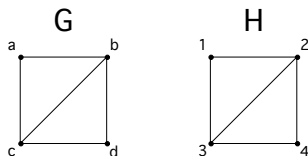
$$\{b, d\} \mapsto \{2, 4\} \quad \checkmark$$

$$\{c, d\} \mapsto \{3, 4\} \quad \checkmark$$

....and non-edges:

$$\{a, d\} \mapsto \{1, 4\} \quad \checkmark$$

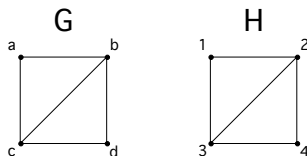
Isomorphisms



A different example of an isomorphism between G and H is the mapping

$$g : V(G) \rightarrow V(H)$$

Isomorphisms



A different example of an isomorphism between G and H is the mapping

$$g : V(G) \rightarrow V(H)$$

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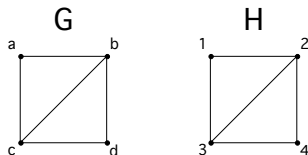
Isomorphic Graphs

If there exists an isomorphism between two graphs then the graphs are said to be **isomorphic**.

Isomorphic Graphs

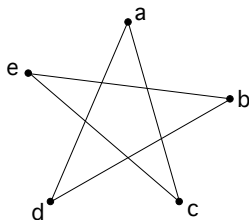
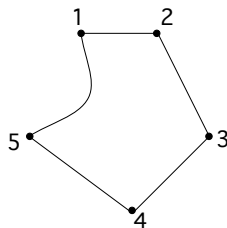
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Example: Graphs G and H are isomorphic.



Isomorphic Graphs

- The following graphs pictured are isomorphic.

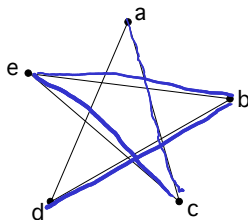
 G_1  G_2 

Isomorphic Graphs

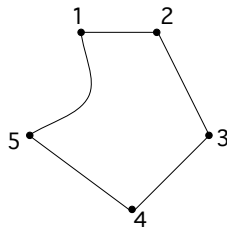
$a \mapsto 1$
 $c \mapsto 2$
 $e \mapsto 3$
 $b \mapsto 4$
 $d \mapsto 5$

- The following graphs pictured are isomorphic.

G_1



G_2



- Can you specify an explicit isomorphism between them?

Directed Graphs

Digraphs

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- A **directed graph** (or **digraph**) is the same as a graph except that edges are *ordered* pairs of endpoints.
- Each edge has an **initial vertex** and a **final vertex**. *terminal*

Digraphs

Digraphs were introduced in Section A3 in order to represent some relations diagrammatically. Here we look at digraphs in general.

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Digraphs

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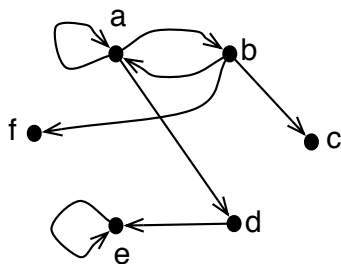
- A **directed graph** (or **digraph**) is the same as a graph except that edges are *ordered* pairs of endpoints.
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- The edges of a digraph are sometimes called **arcs**.

Digraphs

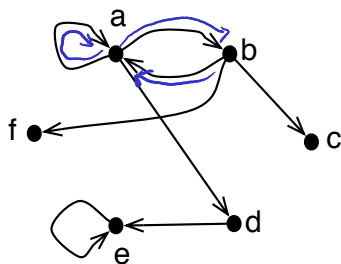
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- Each edge has an **initial vertex** and a **final vertex**.
- Loops are still allowed.
- The edges of a digraph are sometimes called **arcs**.
- In a diagram of a digraph, the direction of an arc is given by an arrow.

Example



Example

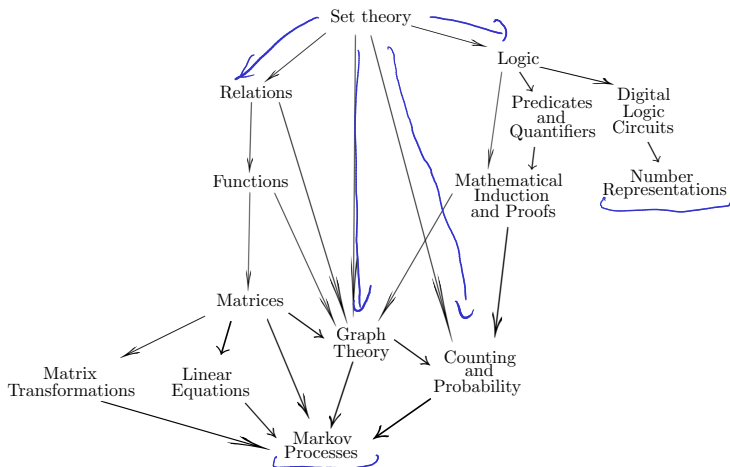


The vertex set and edge set for this graph are:

$$V(G) = \{a, b, c, d, e, f\}$$

$$E(G) = \{(a,a), (a,b), (a,d), (b,a), (b,c), (b,f), (d,e), (e,e)\}$$

An application: Recording Information Dependencies



An arrow from A to B means that B depends upon A in some way.

Foodwebs

- An application of digraphs in ecology is in describing a *foodweb*.

Foodwebs

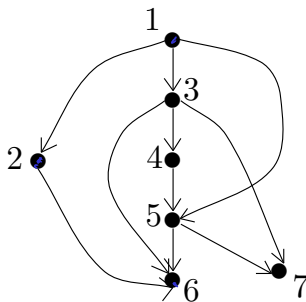
- An application of digraphs in ecology is in describing a *foodweb*.
- The next slide shows a foodweb developed by Parsons and LeBrasseur, as adapted by Cohen, pertaining to the following species in the Strait of Georgia, British Columbia.

KEY SPECIES

1. Juvenile pink salmon
2. P. Minutus
3. Calanus and Euphausiid Burcillia
4. Euphausiid Eggs
5. Euphausiids
6. Chaetoceros Socialis and Debilis
7. Mu-Flagellates

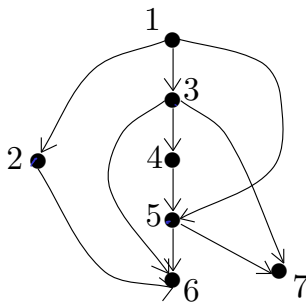
Example

An arrow from i to j
means ' i eats j ' :



Example

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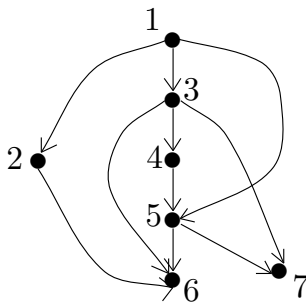


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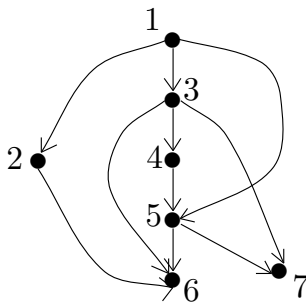


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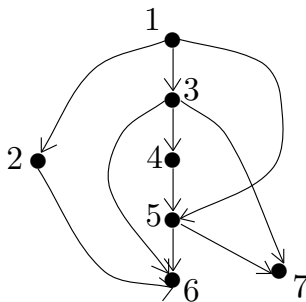
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We shall use the first convention unless stated otherwise.

Niche Overlap Graphs

- An **application of graphs** in ecology is in describing commonalities (or competition) between species in a *Niche Overlap Graph*.

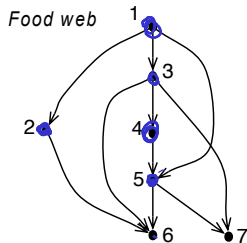
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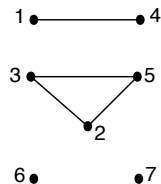
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- The following niche overlap graph is constructed from the Food Web data of the previous example.

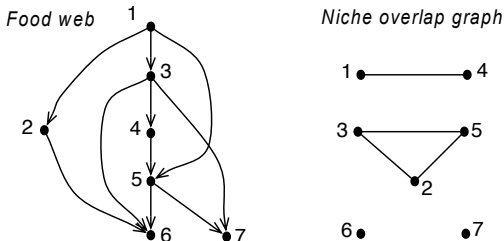
Food Webs and Niche Overlap Graphs



Niche overlap graph



Food Webs and Niche Overlap Graphs



- For example:
species 1 and 4 compete for food (species 5), so are connected by an edge in the niche overlap graph.

Types of Graphs and Digraphs

Sometimes it is useful to restrict our attention to two types of graphs and digraphs, called:

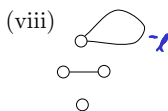
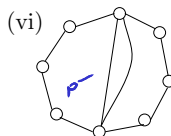
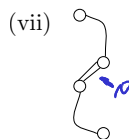
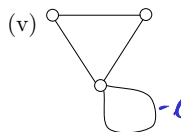
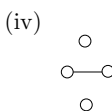
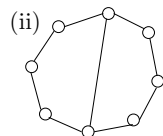
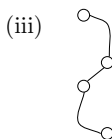
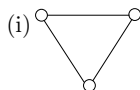
- **Simple Graphs** and
- **Simple Digraphs**

Simple Graphs

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Some simple Graphs

Some non-simple Graphs

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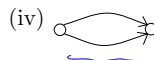
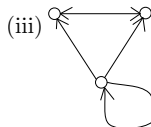
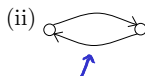
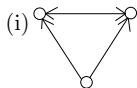
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- For some, what we would call a graph with parallel edges, is a *multi-graph*.

Simple Digraphs

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Simple Digraphs

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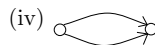
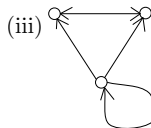
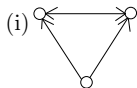
Some simple Digraphs

Some non-simple Digraphs

Don't allow duplicate
ordered pairs in
our edge set

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Some simple Digraphs

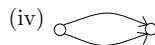
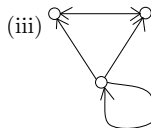
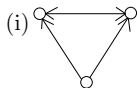
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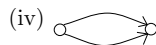
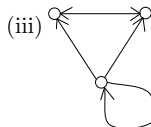
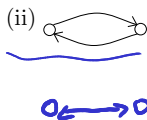
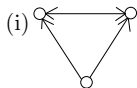
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- It is not okay to have (a, b) twice - those would be parallel.
- We sometimes draw a single edge with an arrow at each end to indicate a pair of edges (a, b) and (b, a) , instead of drawing two distinct lines.

Special simple graphs I: Complete Graphs

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A complete graph on n vertices is denoted by K_n .

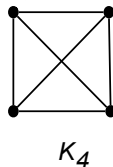
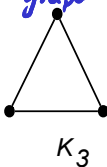
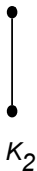
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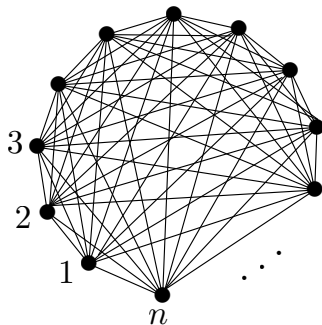
If G, H have n vertices and are complete, then any bijection $f: G \rightarrow H$ is going to preserve edges, so is a graph isomorphism.

Examples:



How many edges in K_n ?

How many edges are there in K_n , the complete graph of order n ?

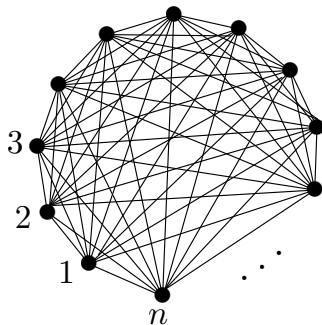


$$\binom{n}{2}$$

since we need an edge for every pair of vertices.

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The answer is $\boxed{\binom{n}{2} = \frac{n(n-1)}{2}}$. *Why?*

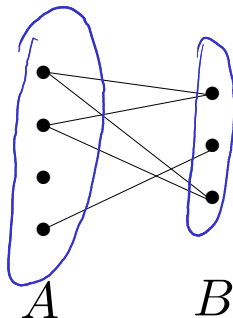
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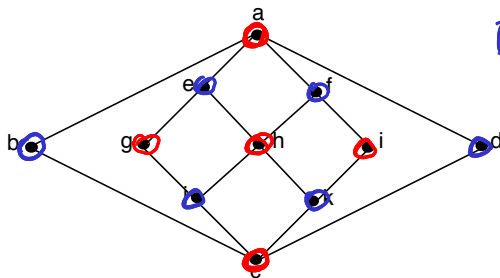
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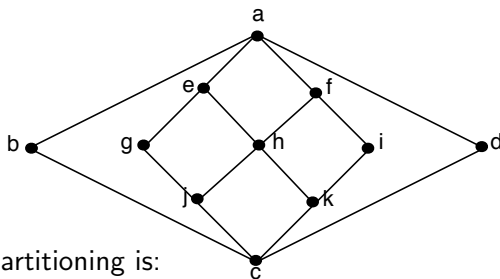
$A =$ ~~red scribble~~

$B =$ ~~blue scribble~~

A larger example of a bipartite graph

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The vertex partitioning is:

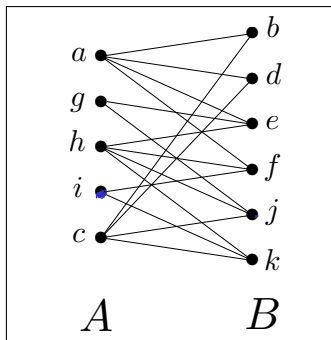
$$A = \{a, g, h, i, c\}$$

$$B = \{b, d, e, f, j, k\}$$

Larger example continued

*planar graphs
are all 4-partite
& colourable*

This is the same graph, redrawn.



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If A has m vertices and B has n vertices, the complete bipartite graph on A and B is denoted by $K_{m,n}$.

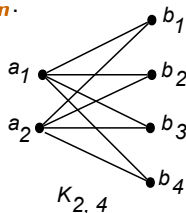
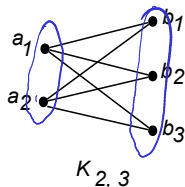
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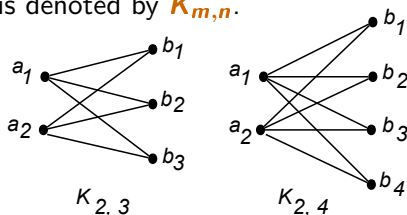
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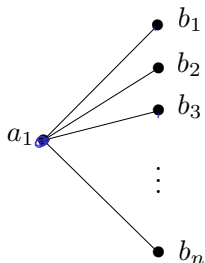
Examples:



Caution: The name “Complete Bipartite Graph” is misleading. Except for $K_{1,1}$, such graphs are **not complete graphs**. The adjective ‘complete’ qualifies ‘bipartite’, not ‘graph’.

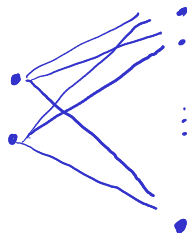
How many edges in $K_{1,n}$?

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- \vdots
- How many edges in $K_{m,n}$? ma



Subgraphs

- A **subgraph**, S , of a graph G , is a graph whose vertices are a subset of $V(G)$ and whose edges are a subset of $E(G)$, i.e.

$$\underline{V(S)} \subseteq V(G)$$

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G is a subgraph
of G

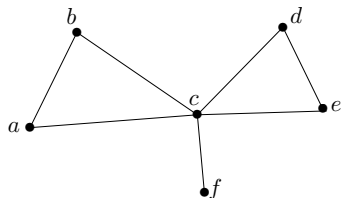
- Since S is a graph, if the edge

$$\{a, b\} \in E(S),$$

we require its endpoints to be in $V(S)$, i.e. $a, b \in V(S)$.

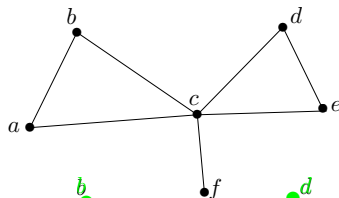
A subgraph example:

Let G be the graph:

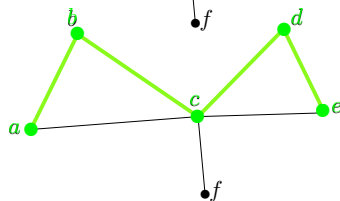


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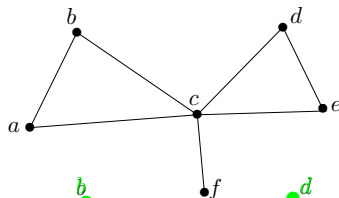


Select some edges and vertices:

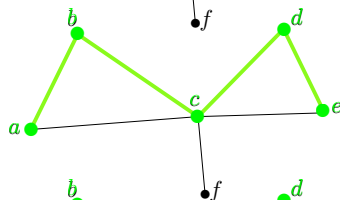


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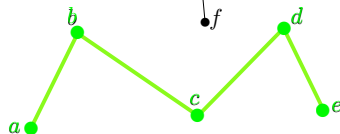
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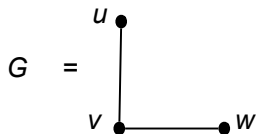


Now S is a subgraph of G :



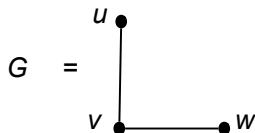
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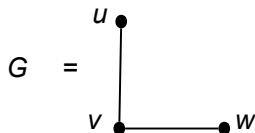
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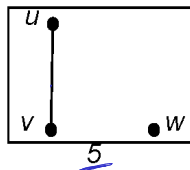
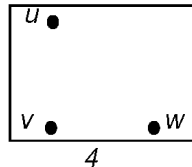
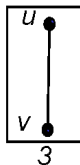
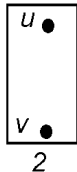
The following five graphs (numbered 1-5) are each subgraphs of G :

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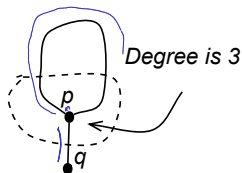
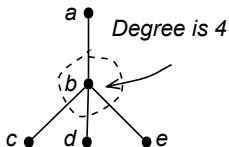


Degree of a vertex

- The **degree** of a vertex is the number of edges incident on it (but with each loop counted twice — once for each 'end').

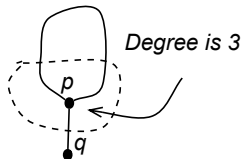
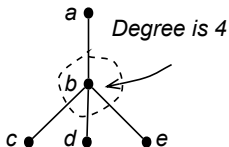
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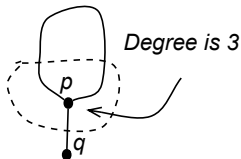
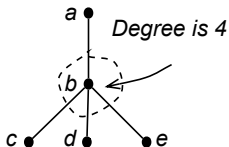
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- Let v be a vertex and let k be its degree. We write $\deg(v)=k$.

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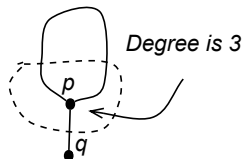
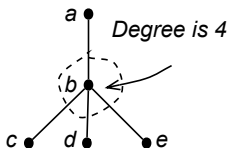
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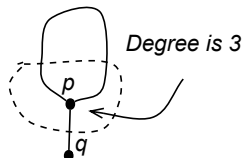
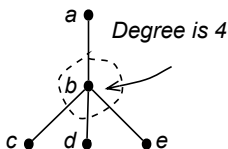


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We shall always use the ‘adding two’ version.

Total degree of a graph

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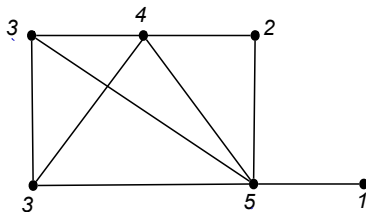
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Example: In this graph the degree of each vertex is shown.



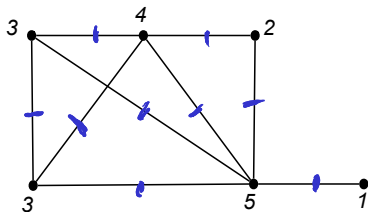
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Example: In this graph the degree of each vertex is shown.



edges = 9

The total degree of the graph is $3 + 4 + 2 + 3 + 5 + 1 = \underline{18}$.

The Handshake Theorem

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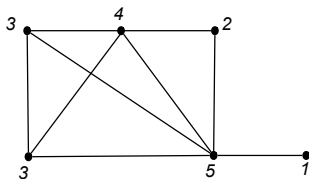
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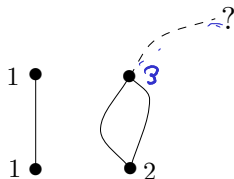
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How does this corollary follow from the handshake theorem?

Example:

- The set $\{1, 1, 2, 3\}$ cannot possibly be the set of degrees of the vertices of some graph.
- However we try, we always end up with an edge that doesn't have a vertex to connect to:



A useful abbreviation

We often abbreviate

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Note that

- for graphs, ab means *the same* as ba , since $\{a, b\} = \{b, a\}$; but
- for digraphs, ab is *different* from ba since $(a, b) \neq (b, a)$.

Announcements

- Grand assignment C due next Monday
- Q3: You should do external research (should look familiar).

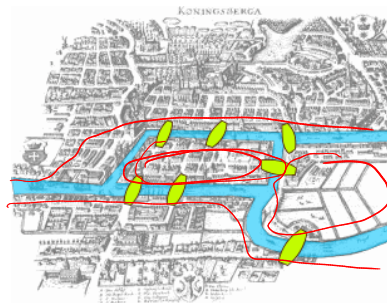
Walks on Graphs

The bridges of Königsberg

- The city of Königsberg in Prussia was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges.

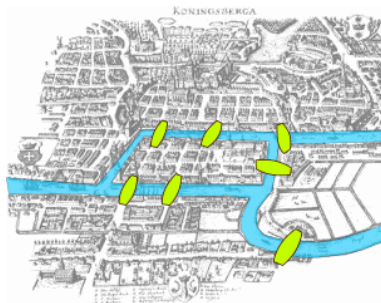
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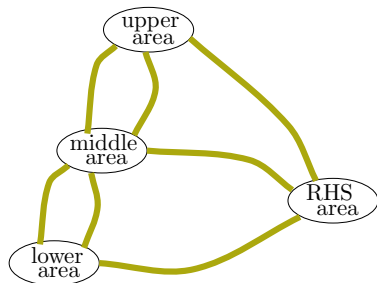
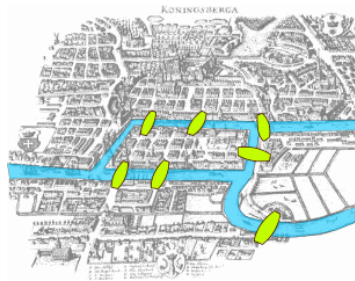


Adapted from:

http://en.wikipedia.org/wiki/Bridges_of_Konigsberg

The bridges of Königsberg

Leonard Euler realized that the task can be modeled as a problem in graph theory, which he invented for the purpose.



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Walks

- A **walk** in a graph is a sequence of vertices alternating with edges:

$$v_0, e_1, v_1, e_2, v_2, e_3, \dots, e_n, v_n$$

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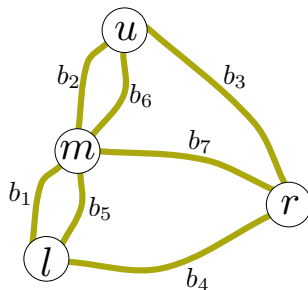
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- A **trivial walk**, say v_0 , contains no edges; hence has length 0.

Walks on the Königsberg Graph

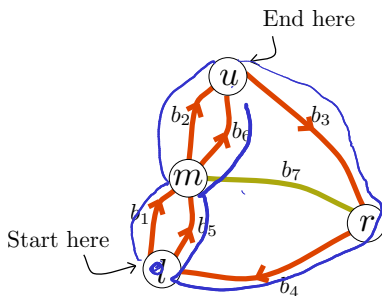
The question becomes, 'Is there a walk on the Königsberg Graph which traverses each edge exactly once?'



Walks on the Königsberg Graph

Try starting in the lower part of town, going via bridge b_1 to the middle island, then via bridge b_2 to the upper part, then via bridge b_3 to the right-most part; continuing as in the listed walk:

$$lb_1mb_2ub_3rb_4lb_5mb_6u$$

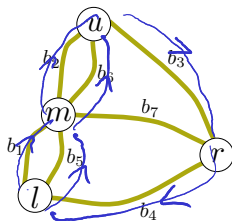


Walks on the Königsberg Graph

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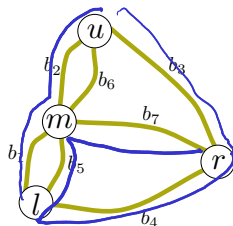
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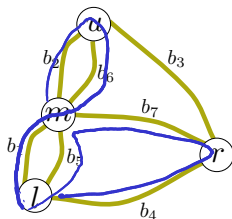
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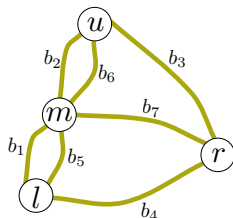
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Impossibility of Königsberg Bridge Walk

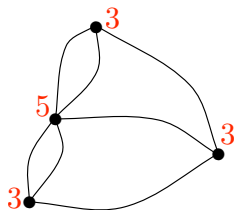
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- *How?* Think about the degrees of the vertices:



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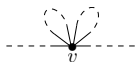
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Even degree
vertex
in a walk

OR



Odd degree
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OR

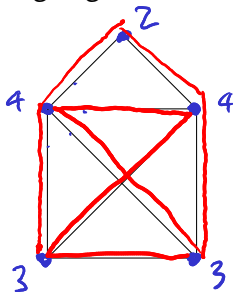


Odd degree
vertex
in a walk

A related puzzle

You may have come across the following puzzle:

Can you draw the following 'house' diagram without taking your pen off the paper or overwriting edges?

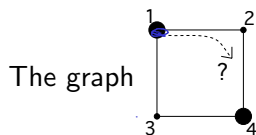


Can you?

Yes, start and end at an odd degree vertex.

Walks, paths and circuits

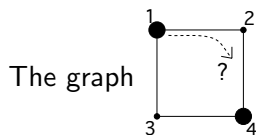
Counting Walks with Adjacency Matrices



has adjacency matrix

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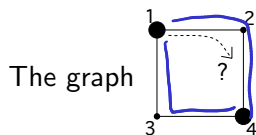


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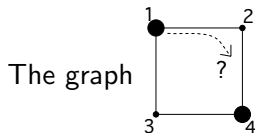
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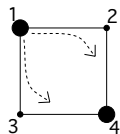
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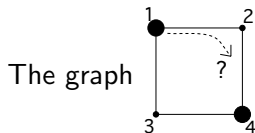
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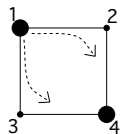
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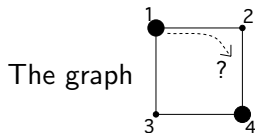
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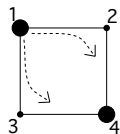
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For any graph, the number of ways to walk from vertex i to vertex j in t steps is given in terms of its adjacency matrix M by the $(i,j)^{\text{th}}$ entry of M^t .

Some special kinds of walks

Some properties that a walk on a graph may, or may not, possess are:

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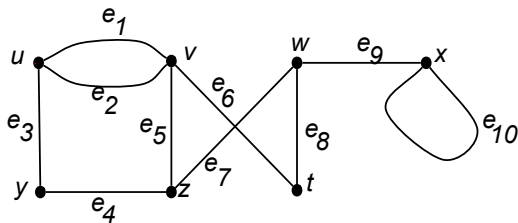
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We will now look at each of these potential properties in turn, with examples using the graph G below.



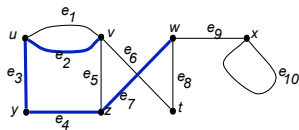
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Examples:



The walk

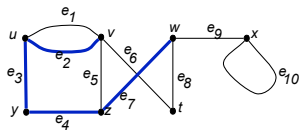
$v \mathbf{e}_2 u \mathbf{e}_3 y \mathbf{e}_4 z \mathbf{e}_7 w$

has length 4 and is **not** closed
because $v = v_0 \neq v_n = v_4 = w$.

Closed walks

A walk $v_0, e_1, v_1, e_2, \dots, e_n, v_n$ is called a **closed** when $v_0 = v_n$.

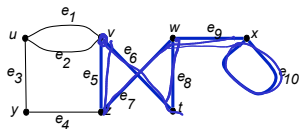
Examples:



The walk

$v \mathbf{e_2} u \mathbf{e_3} y \mathbf{e_4} z \mathbf{e_7} w$

has length 4 and is **not** closed
because $v = v_0 \neq v_n = v_4 = w$.



The walk

$v \mathbf{e_6} t \mathbf{e_8} w \mathbf{e_9} x \mathbf{e_{10}} x \mathbf{e_9} w \mathbf{e_7} z \mathbf{e_5} v$

has length 7 and **is** closed
because $v = v_0 = v_n = v_7 = v$.

Paths

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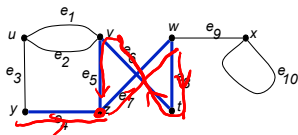
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The walk

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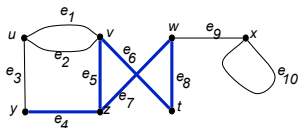
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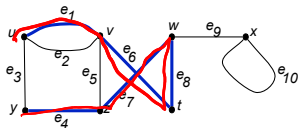
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The walk

$y \mathbf{e_4} z \mathbf{e_7} w \mathbf{e_8} t \mathbf{e_6} v \mathbf{e_1} u$

has length 5 and is a **simple path** because the six vertices are all different.

Circuits

A **circuit** is a closed path.

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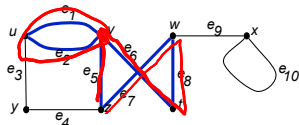
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Examples:



The walk

$z \mathbf{e_7} w \mathbf{e_8} t \mathbf{e_6} v \mathbf{e_1} u \mathbf{e_2} v \mathbf{e_5} z$

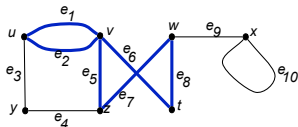
has length 6 and is a **circuit** as it is closed with all different edges, but is **not simple** because $v = v_3 = v_5$.

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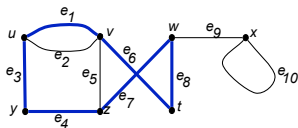
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The walk

$z \mathbf{e_7} w \mathbf{e_8} t \mathbf{e_6} v \mathbf{e_1} u \mathbf{e_3} y \mathbf{e_4} z$

has length 6 and is a **simple circuit** as it is closed without repeated vertices except the first and last $z = v_0 = v_6$.

Walks on digraphs

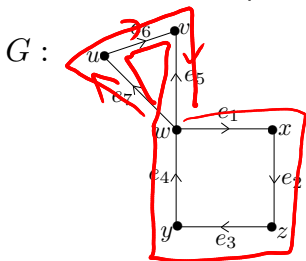
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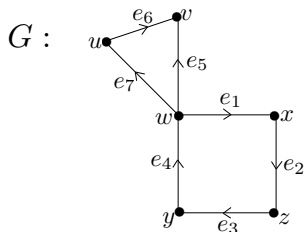


- $e_1 e_2 e_3 e_4 e_7 e_6$ is a directed walk in G
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Doesn't follow the direction of the edges.

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- Note that for a simple graph or digraph (with no loops and no parallel edges), **any** walk is uniquely determined by its sequence of vertices.

Connected Graphs

- A graph is **connected** if every pair of vertices can be connected by a walk (and therefore by a path).

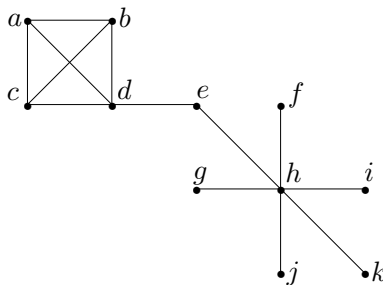
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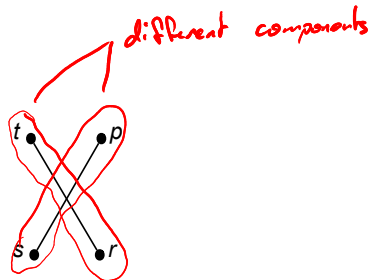
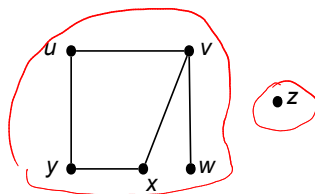
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Here is an example of a connected graph:

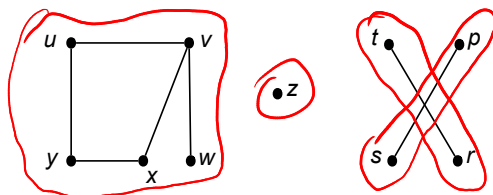


Example



The graph above is **not connected** and has **4 components**:

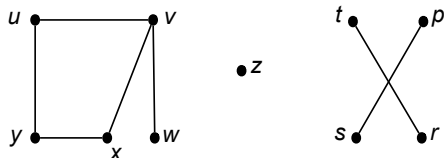
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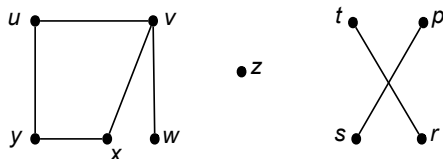
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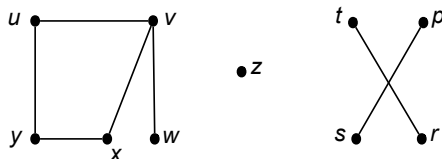
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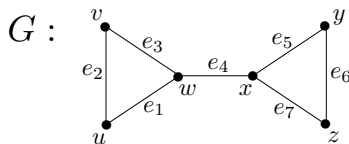
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Bridges and Cut Vertices

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Bridges and Cut Vertices

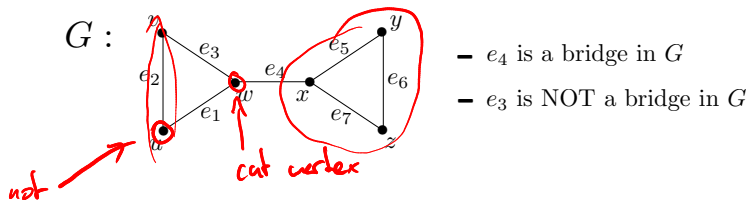
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- e_4 is a bridge in G
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Bridges and Cut Vertices

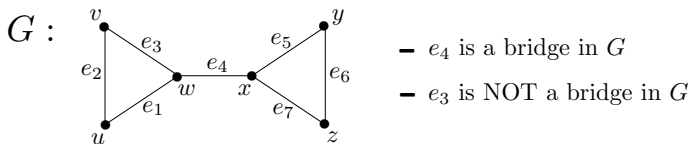
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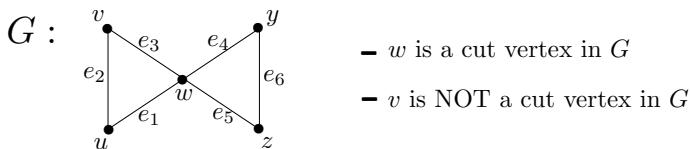
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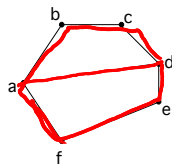
Euler Paths and Circuits

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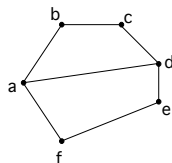
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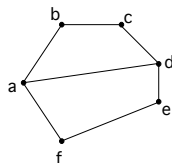


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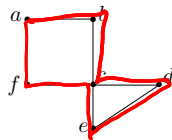
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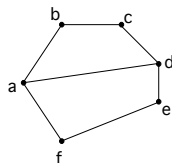
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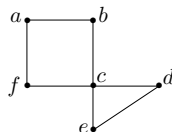
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Note: By convention, an Euler path must be open, *i.e* not a circuit.

When do Euler Paths and Circuits exist?

- **Theorem:** A connected graph has an Euler circuit if and only if each of its vertices has even degree.

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The algorithm easily adapts to this case.

Issues regarding finding an Euler Circuit in a graph G

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- Always leave an edge to return to the start vertex as the last step.

Fleury's Algorithm for finding Euler Circuits

Let G be a connected graph with all vertices of even degree.

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The Algorithm

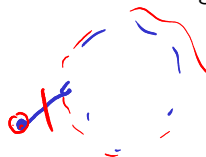
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1. Pick any vertex of G as a starting point.
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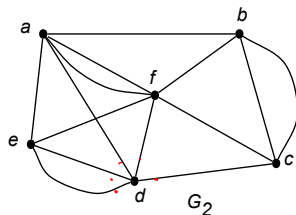
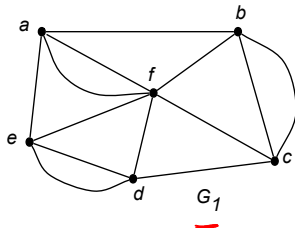
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5. Repeat steps 2 - 4 until all edges have been traversed, and you are back to the starting vertex.

4.5 If you've removed all edges from a single vertex, delete it

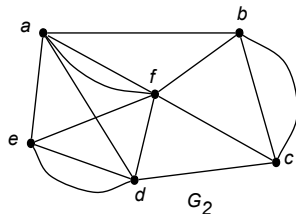
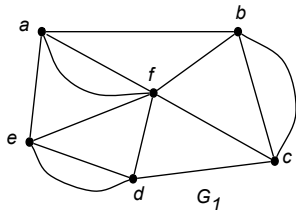
Candidate Graphs for Fleury's Algorithm

The graph G_1 below satisfies the criterion that all vertices have even degree, so it contains an Euler circuit and Fleury's algorithm can be used to find that circuit.



Candidate Graphs for Fleury's Algorithm

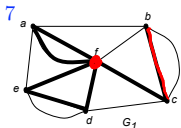
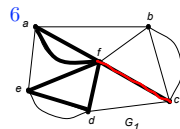
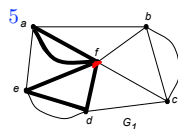
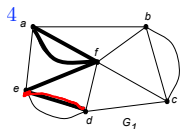
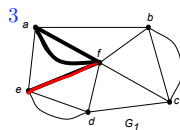
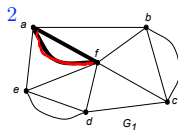
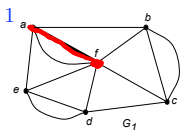
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The graph G_2 has two vertices of odd degree. Fleury's algorithm can be modified to find an Euler path in this graph. The only modification needed is that the first vertex must be one of the vertices of odd degree.

Fleury's Algorithm example in pictures

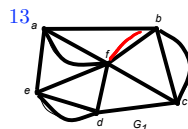
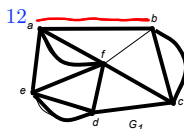
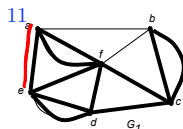
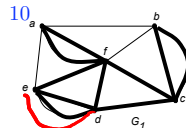
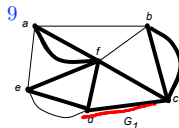
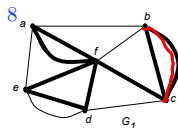
Start at f (just because we feel like it!)



Notice that from here
we MAY NOT step to f
but either a or c is allowed

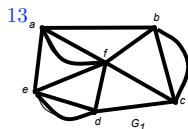
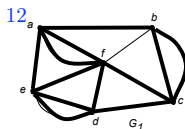
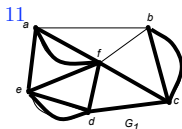
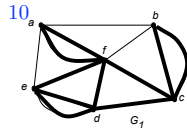
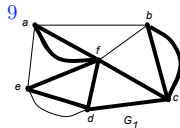
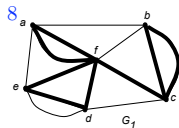
Fleury's Algorithm example in pictures

At step 9 we're forced to go to d ; and then all steps are forced.



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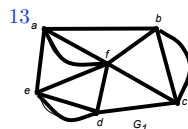
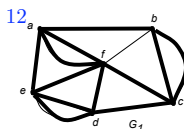
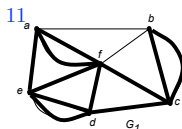
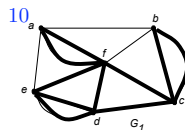
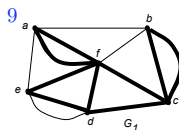
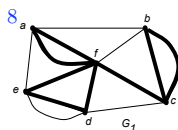
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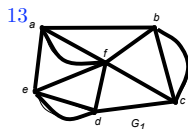
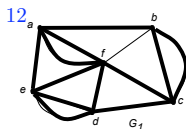
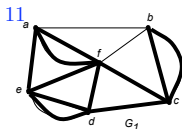
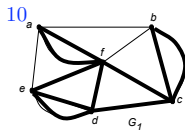
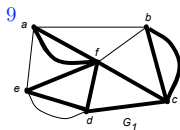


The final path is $fafedfcbcd eabf$.

The path is now closed, providing the Euler circuit we sought.

8

G_1



(As mentioned earlier, we do not call this path an Euler *path*, because, by convention, Euler paths are not closed.)

Fleury's Algorithm example discussion

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- In any implementation, we never have to back-track, so the algorithm is quite fast; as are some other algorithms to solve this problem.
- By contrast, the following problem – that of finding a *Hamilton* path or circuit – has no known 'fast' algorithm.

Hamilton Paths and Circuits

- A **Hamilton path** for a graph is a *simple* path which passes through every vertex.

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- A graph is called **Hamiltonian** if and only if it possesses a Hamilton circuit.

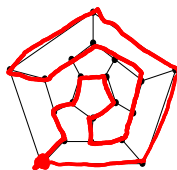
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Note: By convention, a Hamilton path must be open, *i.e* not a circuit.

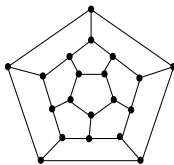
Graphs and Hamilton Paths / Circuits

- This graph has a Hamilton Circuit. *Can you find one?*

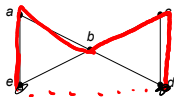


Graphs and Hamilton Paths / Circuits

- This graph has a Hamilton Circuit. *Can you find one?*



- This graph has no Hamilton circuit. *Why not?*



Types of Walks on Graphs – Summary

For a walk $v_0, e_1, v_1, e_2, \dots, e_n, v_n$ on a graph G :

Properties					
Name:	closed	—	Euler	simple	Hamilton
Description:	—	no repeated edges	uses all edges	no repeated vertices	uses all vertices
Requirement:	$v_0 = v_n$	$i \neq j \implies e_i \neq e_j$	$\forall e \in E(G) \exists i e_i = e$	$i \neq j \implies v_i \neq v_j$ ($v_0 = v_n$ OK)	$\forall v \in V(G) \exists i v_i = v$
path		✓			
simple path		✓		✓	
Euler path		✓	✓		
Hamilton path		✓		✓	✓
closed walk	✓				
circuit	✓	✓			
simple circuit	✓	✓		✓	
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Graph with tick in all boxes?

 path must be a ~~simple~~ loop

path		✓			
simple path		✓		✓	
Euler path		✓	✓		
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closed walk	✓				
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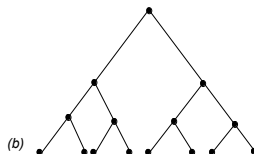
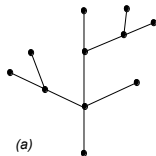
Trees

Trees

- A **tree** is a connected graph with no circuits other than the trivial ones.

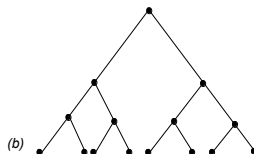
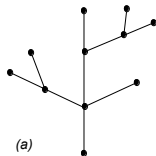
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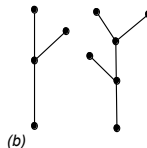
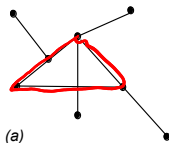


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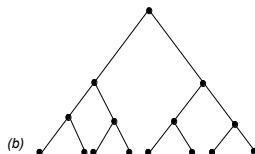
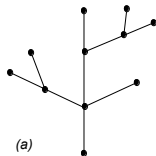
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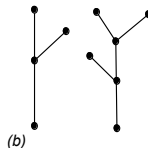
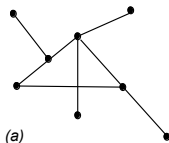
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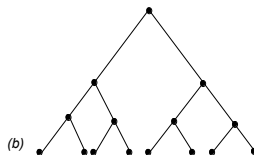
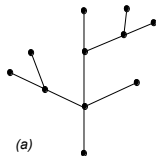
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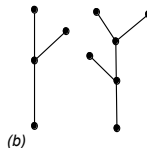
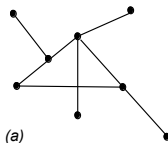
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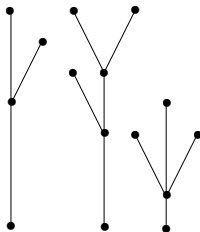
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Trees as Models

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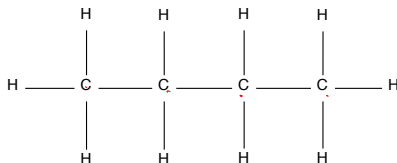
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- In the representation of such molecules we use C to represent a carbon atom and H to represent a hydrogen atom. These will be used instead of dots for the vertices.

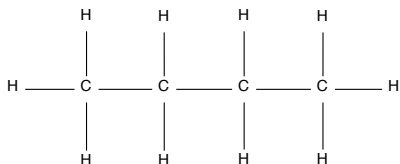
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- Butane:

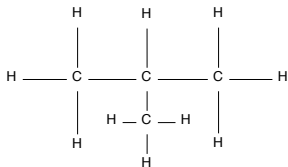


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- Isobutane:



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- Butane and Isobutane graphs each have 4 carbon atoms and 10 hydrogen atoms, but the configuration of the atoms is different.

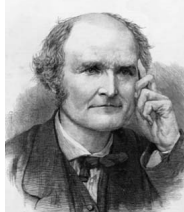
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- They have the same chemical formula, C_4H_{10} , but different chemical bonds and are called **isomers**.
- **Saturated hydrocarbon** molecules contain the maximum number of hydrogen atoms for a given number of carbon atoms.

Some History of Trees

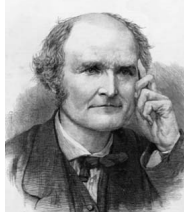


Arthur Cayley 1821 - 1895

Cayley Graph

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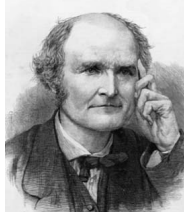
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- You will explore a proof of this formula in Workshops next week.

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Theorem: Let T be a graph with n vertices. The following statements are logically equivalent:

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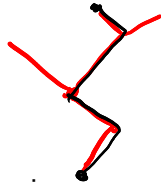
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- (i) T is a tree (it is connected and has no non-trivial circuits).
- (ii) T has no simple circuits and $n - 1$ edges. —
- (iii) T is connected and has $n - 1$ edges. ← Easy for computer to check.
- (iv) T is connected and every edge is a bridge.
- (v) Any two vertices of T are connected by exactly one simple path.
- (vi) T contains no non-trivial circuits, but the addition of any new edge (connecting an existing pair of vertices) creates a simple circuit.

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- The previous Theorem collects together several different ways of characterizing a tree. In different contexts, each of the different descriptions are useful.

Proving and Using the Theorem



- The previous Theorem collects together several different ways of characterizing a tree. In different contexts, each of the different descriptions are useful.
- To prove the Theorem, we should show that any statement in the list is derivable from any other statement. One way to do this is to show the chain of implications:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i).$$

Proving the Theorem

- *Can you prove that $(i) \Rightarrow (j)$ for any of the statements in the Tree-Characterising Theorem?*

Proving the Theorem

- *Can you prove that $(i) \Rightarrow (j)$ for any of the statements in the Tree-Characterising Theorem?*
- Some results that you might find helpful are ...

Rule

Lemma A: Any tree that has more than one vertex has at least one vertex of degree 1.

Proving the Theorem

- *Can you prove that $(i) \Rightarrow (j)$ for any of the statements in the Tree-Characterising Theorem?*
- Some results that you might find helpful are ...

Lemma A: Any tree that has more than one vertex has at least one vertex of degree 1.

Lemma B: If G is any connected graph, C is a non-trivial circuit in G , and one of the edges of C is removed, then the subgraph that remains is connected.



Applications of graphs, paths and trees

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- For instance, the problem of determining whether a message can be sent between two computers using intermediate links can be studied with a graph model.
- Problems of efficiently planning routes for mail delivery, garbage pickup, diagnostics in computer networks, and so on, can be solved using models that involve graphs.
- A subgraph of a connected graph that provides a unique path between any two vertices is a spanning tree. These trees have applications in many fields, including engineering.

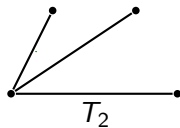
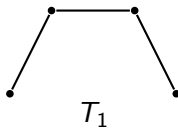
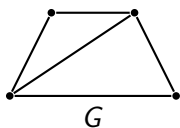
Spanning trees

- **Definition:** A **spanning tree** for a graph G is a subgraph of G which is a tree and contains all the vertices of G .

-
- spanning tree

Spanning trees

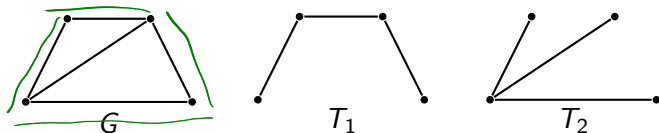
- **Definition:** A **spanning tree** for a graph G is a subgraph of G which is a tree and contains all the vertices of G .
- Spanning trees need not be unique. That is, a given graph can have a number of different spanning trees.
- Example: Two distinct spanning trees, T_1 and T_2 , for a graph G are shown below:



Same # vertices
 \Rightarrow same # of edges

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How many more spanning trees can you find for G ?

Which graphs have spanning trees?

Theorem:

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Each time you encounter a circuit, remove a random edge

2. Any 2 spanning trees for a graph have the same number of edges.

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Why?

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Why?

Hint: See (ii) or (iii) of the tree characterisation theorem.

A method to make a spanning tree

Let G be a connected graph with n vertices.

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The Algorithm

1. Initialize \mathcal{T} to be the vertices of G but no edges.

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4. Repeat steps (2) and (3) until T has $n - 1$ edges, then stop.

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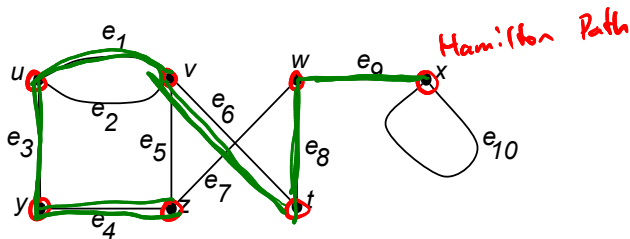
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For computer implementation it is necessary to **also**:

- at step 1 initialize a 'pool of potential edges' P to $E(G)$,
- at step 2 ensure the picked edge e comes from P and
- after step 3 remove e from P (whether it contributes to T or not).

Building a spanning tree: example

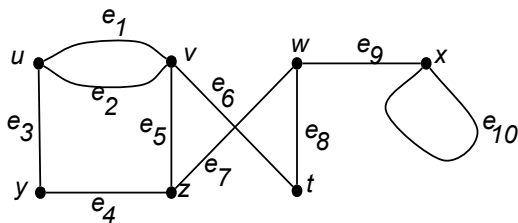
To demonstrate the spanning tree algorithm we will use a graph we have seen before:



No Hamilton path

Building a spanning tree: example

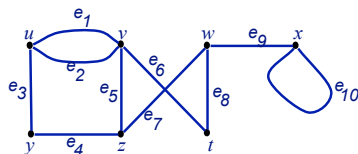
To demonstrate the spanning tree algorithm we will use a graph we have seen before:



Initialize T to

- vertex set $V(T) = \{u, v, w, x, y, z, t\}$,
- edge set $E(T) = \{\}$:

Building a spanning tree for


 $E(T) = \{ \}$
 $T :$
 u
•

 v
•

 w
•

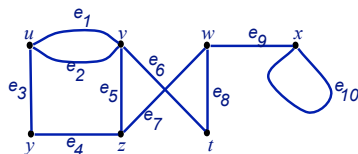
 x
•

 y
•

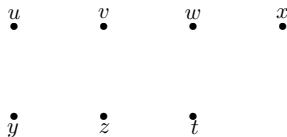
 z
•

 t
•

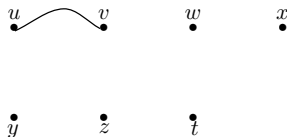
Building a spanning tree for



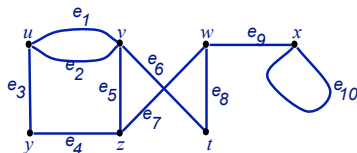
$$E(T) = \{\}: \quad T :$$



$$E(T) = \{e_1\} \quad T :$$

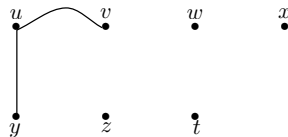


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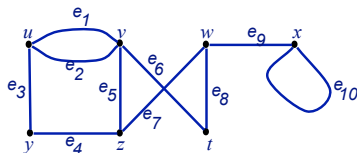


$$E(T) = \{e_1, e_3\}:$$

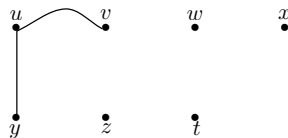
$T:$



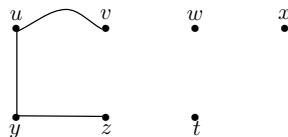
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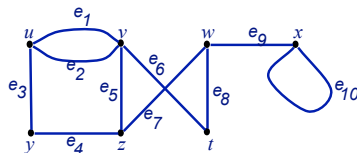
$$E(T) = \{e_1, e_3\}:$$

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$$E(T) = \{e_1, e_3, e_4\}$$

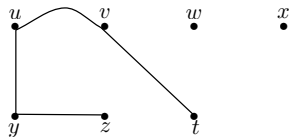
 $T:$


Building a spanning tree for

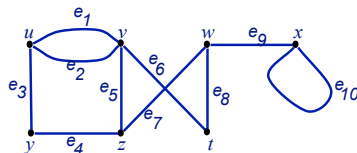


$$E(T) = \{e_1, e_3, e_4, e_6\}:$$

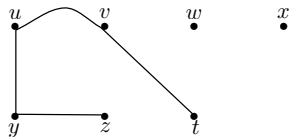
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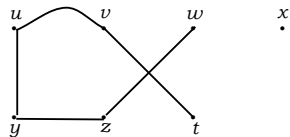
Building a spanning tree for



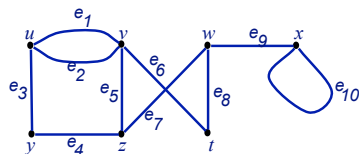
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$$E(T) = \{e_1, e_3, e_4, e_6, e_7\}$$

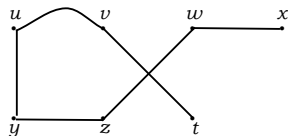
 $T :$


Completed spanning tree for

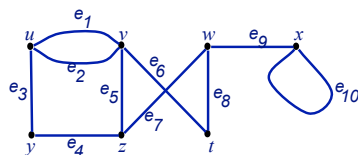


$$E(T) = \{e_1, e_3, e_4, e_6, e_7, e_9\}:$$

$T :$

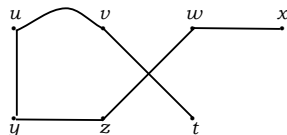


Completed spanning tree for



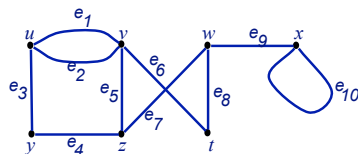
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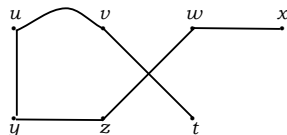
Our tree now has $7 - 1 = 6$ edges, so we are done (as you can see).

Completed spanning tree for



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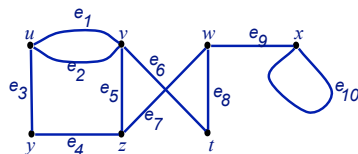
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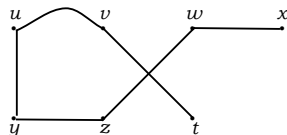
As it happens, this tree has no branching and so contains a path of length 6. That just results from our order of choosing edges.

Completed spanning tree for



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T :

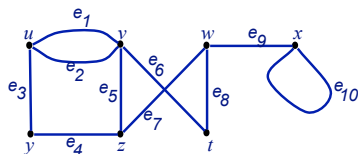


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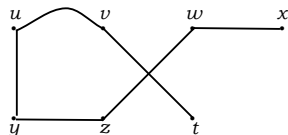
Can you make a spanning tree in which the longest path has length 4?

Completed spanning tree for



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T :

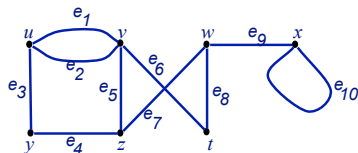
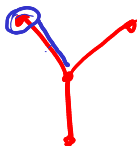


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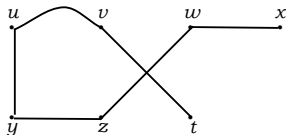
Can you make a spanning tree in which the longest path has length 4? Length 3?

Completed spanning tree for



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T :



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Can you make a spanning tree in which the longest path has length 4? Length 3?