Question 1 In rôle-playing games, such as the original, *Dungeons and Dragons*, gamers use a variety of dice shapes, not just the standard cube. For example a 'd12' has the shape of a regular dodecahedron. A (fair) d12 has twelve faces numbered 1,2,...,12, and each face is equally likely to become the top face (the face that is read) when the die is 'thrown'.

Suppose gamer Alice does not have a d12 but does have a d6 (*i.e.* a regular cubic die with six faces labelled 1,2,...,6.) To simulate throwing a d12, Alice throws her d6 twice, resulting in a pair of values $(a, b) \in \{1, \ldots, 6\}^2$. She then combines the two values in some way to come up with a value in $\{1, \ldots, 12\}$.

(a) Write out the sample space S for this 'experiment'.

(b) One way to combine a and b would be to add them. But this would not accurately simulate a d12 throw because it would be impossible to score 1.
To avoid this problem Alice decides to use the formula: $v = \left\lceil \frac{ab}{3} \right\rceil$.
Write out the event $E \subseteq S$ coresponding to $v = 1$.

(c) What is the probability of event E above and in what way does it demonstrate that Alice's method also does not accurately simulate a d12 throw?

(d) How *could* Alice combine a and b to accurately simulate a d12 throw?
(Gamers actually do this, so you can Google an answer if you're desperate!)

Question 2 To evaluate the quality of a micro-finance program, some participants are selected, and their economic situation measured. In a given evaluation a sample of 100 participants, out of a total 10 000, is randomly selected. Every participant has the same probability of being selected.

Assume that 40 out of the 10 000 participants have actually seen their economic situation deteriorate. We seek the probability that at least one of these is in the sample.

(a) Describe a sample space S for this problem, and give a formula for $|S|$.

(b) Let E be the event "at least one of the participants in the random selection has seen her/his economic situation deteriorate". Give a formula for $|E^c|$.

(c) With the help of WolframAlpha, or some other electronic aid, calculate $\mathbb{P}(E)$.

Question 3 A *Binomial experiment* comprises a fixed number n of 'trials' where each trial has the same probability p of 'success'. The probability that a binomial experiment results in k successes is given by

$$\mathbb{P}(k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}.$$

For example, what is the probability of scoring two 6s from twelve throws of a standard die? Here $n = 12$, $p = \dfrac{1}{6}$ and $k = 2$, so the the probability is $\dbinom{12}{2}\left(\dfrac{1}{6}\right)^2\left(\dfrac{5}{6}\right)^{10}$.

(a) Use your calculator to find the probabilities of

(i) two 6s from twelve throws,

(ii) one 6 from twelve throws and

(iii) no sixes from twelve throws

(b) Calculate the probability of scoring *at least two 6s* from twelve throws.

(c) In practice Binomial probabilities are usually found from a book of tables or from on-line tables or calculators. Both density and (cumulative) distribution values are available. With the help of an on-line statistical calculator such as *Stat Trek* (`http://stattrek.com/online-calculator/binomial.aspx`) find the probability of scoring more than two but less than eight 6s from 30 throws of a standard die. Give all the values you obtained, what they represented, and how you used them to obtain your answer.

Question 4 The income and education level of each person on the electoral roll for Queanberra is recorded as a pair $(x, y) \in \{1, 2, 3\}^2$, where 1 stands for low, 2 for average, and 3 for high, *e.g.* $(2, 3)$ represents a highly educated person with average income.

Let S denote the set of all people on the Queanberra electoral roll, and define random variables $X, Y : S \to \{1, 2, 3\}$ by $X(s), Y(s)$ are the income and educational levels of person s. Let $p_{i,j} = \mathbb{P}(\{(X(s) = i) \wedge (Y(s) = j)\})$ for $1 \leq i, j \leq 3$. Assume that

$$(p_{i,j})_{1\leq i,j\leq 3} = \begin{pmatrix} 0.05 & 0.10 & 0.05 \\ 0.10 & 0.20 & 0.10 \\ 0.15 & 0.20 & 0.05 \end{pmatrix}.$$

(a) Explain what $p_{1,2}$ represents.

(b) Find the probability of the event AI: "the person has an average income"?

(c) Find the probability of the event AE: "the person has an average education level"?

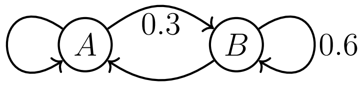
(d) Describe the event $AI \cup AE$ and find its probability.

(e) Are the events AI and AE independent?

(f) Are the random variables X and Y independent?

Question 5 A Markov process has two states A and B with transition graph below.

(a) Write in the two missing probabilities.



(b) Suppose the system is in state A initially. Use a tree diagram to find the probability that the system will be in state B after three steps.

(c) The transition matrix for this process is $T =$

(d) Use T to recalculate the probability found in (b).

Question 6 Let $T = \begin{bmatrix} 9/10 & 1/10 \\ 7/10 & 3/10 \end{bmatrix} = \begin{bmatrix} .9 & .1 \\ .7 & .3 \end{bmatrix}$.

(a) Use the 'Matrix Calculator' computer application `http://matrixcalc.org/en/` to calculate T^2, T^4, T^8 and T^{16} to 3dp accuracy. (You can progressively insert the results back into an input matrix, so there is no need to physically enter anything more than the four entries of T.)

(b) Based on (a) guess a steady state vector for the Markov process with transition matrix T.

(c) Use the transpose matrix T' to verify that your guess from (b) is correct. Do the calculation by hand.

Question 7

(a) By solving the relevant system of equations find the steady state vector for the Markov process with transition matrix $T = \begin{bmatrix} .7 & .3 \\ .2 & .8 \end{bmatrix}$. Do this by hand calculation, using matrix inverse.

(b) Using Matrix Reshish[1] to solve the relevant equations by Gauss-Jordan Elimination, find the steady state vector for the Markov process with transition matrix

$T = \begin{bmatrix} 2/5 & 0 & 3/5 \\ 3/5 & 2/5 & 0 \\ 1/5 & 1/2 & 3/10 \end{bmatrix}$. Use the 'Fractional' input style.

[1]`https://matrix.reshish.com/`

Question 8

(a) Carefully prove that the steady state vector for the Markov process with transition matrix $T = \begin{bmatrix} t_{1\,1} & t_{1\,2} \\ t_{2\,1} & t_{2\,2} \end{bmatrix}$ is $S = \dfrac{1}{t_{1\,2} + t_{2\,1}} \begin{bmatrix} t_{2\,1} \\ t_{1\,2} \end{bmatrix}$ (providing $t_{1\,2} + t_{2\,1} \neq 0$).

(b) What happens if $t_{1\,2} + t_{2\,1} = 0$?

(c) Check that the formula proved for (a) gives the correct result for Q2c and/or Q3a.

Question 9 The *Ehrenfest urns model* is used as part of an explanation of how gas diffusion works. Two urns A and B contain between them a fixed number of balls. At each time step one ball is selected at random (equal probabilities), removed from its current urn and placed into the other one. For the case of four balls the model has five states corresponding to A containing 0, 1, 2, 3 or 4 balls.

(a) Explain why the transition matrix for this Markov process is $T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

(b) Explain why, for any n, T^n will always contain some zeroes.
(Hint: Can the number of balls in A change by 1 in an even number of steps?)

(c) The property of T stated in (b) makes this Markov process *non-regular*. Non-regular Markov processes do not necessarily converge towards a steady state. Find the steady state vector for this process (use the computer as in 3(b)) and explain why this will never be reached starting from any *known* (*i.e.* probability 1) starting state.