

# **MATH1005/MATH6005**

## **Discrete Mathematical Models**

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Semester 2, 2023

# **Preface to the course**

# An acknowledgment of country

We acknowledge and celebrate the First Australians on whose traditional lands we live, work and study. We pay our respects to the elders past and present. In particular, we acknowledge the Ngunnawal and Ngambri people, the Traditional Owners of the land upon which the University's Acton campus is located.

# Credits

This course has been developed by a number of mathematicians over more than a decade. Contributing authors include:

- Judy-anne Osborne
- Pierre Portal
- Malcolm Brooks
- Adam Piggott

The mathematics in this course has been developed over millennia. Credit for discovery will be given only occasionally.

# Learning Outcomes

Upon successful completion of MATH1005, students will have the knowledge and skills to:

1. Recall, invent, interpret examples of motivation for mathematical constructs used in discrete mathematics as models of processes in the world.
2. Recognise, define, explain and use terminology and notation from discrete mathematics.
3. Identify the logical structure of a statement, and then identify the logical structure of an argument that may be used to prove or disprove the statement.
4. Perform mathematical calculations in discrete mathematics using methods presented in the course.
5. Write simple proofs/construct explicit counterexamples for statements relating to discrete mathematics topics covered in the course.

# Learning Outcomes

Upon successful completion of MATH6005, in addition to the knowledge and skills attained by students completing MATH1005, students will have the knowledge and skills to:

6. Use their deep knowledge and understanding of the material presented in the course to formulate responses to complex concrete and abstract problems.
7. Communicate their understanding and skills in discrete mathematics with colleagues and non-experts and apply their knowledge in an occupational situation.

# Reference materials

The lecture notes for the course are intended to be self-contained. Notes will be available through the course Wattle page immediately after the lecture, if not before. For most students, these notes, the workshop materials and the web will be sufficient.

If you would like an alternative presentation of the material, our optional text is Susanna S. Epp. *Discrete Mathematics with Applications: Metric Version*. Cengage Learning, 5th edition, 2019. ISBN: 9780357114087.

- An ebook copy of the text is available through the ANU library.
- From the publisher: “Students who wish to purchase their own eBook copies they can do so through <https://au.cengage.com/> Use the coupon code WOW10 to receive a discount at checkout.”

# Assessment

## MATH1005 Assessment:

- workshop quizzes (best 8 out of 10, worth 16% total),
- workshop participation (best 8 out of 10, worth 16% total),
- weekly assignments (best 8 out of 10, worth 16% total),
- a final exam (52% with a 40% hurdle).

## MATH6005 Assessment:

- workshop quizzes (best 8 out of 10, worth 16% total),
- four graduate assignments (8% each),
- a final exam (52% with a 40% hurdle).



# Other notes

- You must check the Wattle page, particularly the announcements, often.
- All students should sign up for a workshop through MyTimetable.

# Learning design

**Lectures** (3 hours per week): The discussion progresses and there is something new to wrestle with, but you probably won't understand it just from listening.

**Processing the lectures** (~3 hours per week) You spend time with the new material, unpack definitions, work examples from class yourself, try other examples.

**Workshops** (2 hours per week) A short quiz, then new examples to try with your classmates and demonstrators to assist. This is the most useful time of the week provided you have prepared for it.

**Assignments** (~2 hours per week) Give you a chance to measure your skills and understanding. They are NOT designed to communicate everything you need to take out of the lectures.

**Exam** Summative assessment items in which you demonstrate the extent to which you have obtained the learning outcomes.

# Teaching feedback

- Course representatives
- Anonymous ongoing feedback survey

# **Why study discrete mathematical models?**

# What are discrete mathematical models?

**Discrete mathematical models** are abstract representations of processes and objects, the steps or units of which can be indexed by the positive integers. In particular, we avoid continua (like the open interval  $(0, 1)$ ).

**Discrete mathematics** is the study of discrete mathematical models.

# Sudoku?

Q: Is completing a Sudoku puzzle an exercise in discrete mathematics?

# Example: Which webpages are the most interesting?

Problem: A search on the internet can produce an overwhelming number of “hits.” We need an effective and efficient method to order the hits from a web search so that the most interesting pages appear first.

Google’s PageRank algorithm is an excellent solution to this problem that has made its inventors very wealthy. The algorithm combines a simple discrete mathematical model of the internet with some linear algebra discovered circa 1910. We will learn this algorithm in our course.

# Computing and discrete mathematical models

Discrete mathematics and computer science go hand-in-hand because ...

Q: Describe an abstract model of the memory of a computer. That is, how do you think about memory?

Q: Do you think about computers working in continuous time or discrete time (“clock cycles”)?

Q: Can the interval  $(0, 1)$  be modeled inside a computer?



# Discrete mathematics and computer science

In 1974, Knuth wrote:

“Discrete mathematics, especially combinatorial theory, has been given an added boost by the rise of computer science, in addition to all the other fields in which discrete mathematics is currently being extensively applied.”

Donald E. Knuth. [Computer science and its relation to mathematics](#). *Amer. Math. Monthly*, 81:323-343, 1974.

# Discrete mathematics and computer scientists

If you're interested, you should check out the backgrounds of some [A. M. Turing Award](#) laureates.

- [Donald Knuth](#) (1974)
- [Andrew Chi-Chih Yao](#) (2000)
- [Barbara Liskov](#) (2008)
- [Shafi Goldwasser](#) (2012)
- [David Patterson](#) (2017)

You'll find they've all had a lot of exposure to (discrete) mathematics.

# Some optional activities

- (Optional) Read the first 4 sections (pp. 323-329) of Knuth's paper referenced on slide #17 (the paper is available on Wattle), and (even more optionally) take notes.
- (Optional) Play Sudoku at [Sudoku.com](http://Sudoku.com). Ask yourself the following questions: Can you describe your strategy? Can you guarantee that your strategy will work every time? Is your strategy "the best"? Are these questions examples of discrete mathematics?

# Section A1: Logic

# Statements: The basic unit of logic

A **statement** is a sentence that is true or false but not both.

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- Australia is in the Southern Hemisphere.
- Tropical storms spin clockwise in the Southern Hemisphere.
- Canberra is in New Zealand.
- $3 > 2$

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- Canberra is in New Zealand.
- $3 > 2$

Some sentences that are not statements:

- How are you going? (question)
- Canberra is a better city than Sydney. (ambiguous)
- Wake up! (instruction/advice)
- This statement is false. (neither true nor false; self-referencing)

# Compound statements

**Logical connectives** such as 'and', 'or', 'not', 'if-then', 'if and only if' may be used to glue statements together to make new statements.

**Compound statements** are statements built from other statements using logical connectives.



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Example: Australia is in the Southern Hemisphere and tropical storms spin clockwise in the Southern Hemisphere.

Example: If 2 divides 4238 and 3 divides 4238, then 6 divides 4238.

# Compound statements

Sometimes the way a compound statement is built from other statements is less clear, because our everyday use of language does not make the logical structure obvious. For example, the word 'not' sometimes causes confusion because of its placement.

# Examples of compound statements

Q: In each of the following examples, identify clearly the way the statement is built from other statements using logical connectives:

- Australia is in the Southern hemisphere and tropical storms spin clockwise in the Southern Hemisphere.
- Australia is in the Southern Hemisphere and Canberra is not in New Zealand.
- I don't like either the one or the other.

# Compound statements

Takeway: What makes for nice spoken or written language, may not necessary make the logical structure of a compound statement most clear. Worse, the everyday use of language can be ambiguous.

Mathematicians and computer scientists must agree upon the meaning of logical connectives, and must be proficient at recognising the way that compound statements are built from simpler statements, so that we can communicate effectively (human-to-human, and human-to-machine).

# How to introduce a symbol for a statement

When you first learned algebra, you learned that it was useful to use variables to make abstract statements about relationships between numbers. In logic, we sometime save time and space, and clarify the logical structure of a compound statement, by introducing symbols as names for statements.

For example, we write

$$p : \text{Australia is in the Southern Hemisphere}$$

to mean that the symbol  $p$  now represents the statement “Australia is in the Southern Hemisphere.”

# Statement variables and statement forms

We can also use variables to represent arbitrary statements. When a variable represents an arbitrary statement, it is called a **statement variable**.

We have symbols for various logical connectives (such as  $\neg$  for 'not',  $\wedge$  for 'and',  $\vee$  for 'or'). We will define these properly in a moment.

An expression built using statement variables, parentheses and logical connectives is called a **statement form** if the expression becomes a statement when actual statements are substituted for the component statement variables.

# Examples of statement forms

Example: Let  $p$  and  $q$  be statement variables.

The expression  $q \vee \neg(p \wedge q)$  is a statement form.

The expression  $p \neg \wedge q$  is not a statement form (just like  $x + \times y$  is not an algebraic expression).



# About statement forms

The value of the algebraic expression  $x \times y + y$  depends on the values taken by  $x$  and  $y$ . You need to understand the multiplication and addition operations, and the order of precedence among operations, to figure out the value of the expression.

Analogously, the **truth value** of the statement form  $q \vee \neg(p \wedge q)$  depends on the truth values taken by  $p$  and  $q$ . You need to understand the logical connectives  $\neg$ ,  $\wedge$ ,  $\vee$  and the order of precedence among them to figure out the truth value of the statement form.

# Truth tables

We are now ready to define logical connectives. We shall do so using truth tables. A **truth table** records the truth value taken by a statement form for each possible combination of truth values taken by the statement variables appearing in the statement form.

# A logical connective: NOT

If  $p$  is a statement variable, the **negation** of  $p$  is read “not  $p$ ”, denoted  $\neg p$ , and defined by the following truth table:

$p$	$\neg p$
$T$	$F$
$F$	$T$

# A logical connective: **AND**

If  $p$  and  $q$  are statement variables, the **conjunction** of  $p$  and  $q$  is read “ $p$  and  $q$ ”, denoted  $p \wedge q$ , and defined by the following truth table:

$p$	$q$	$p \wedge q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$F$
$F$	$F$	$F$

# A logical connective: OR

If  $p$  and  $q$  are statement variables, the **disjunction** of  $p$  and  $q$  is read “ $p$  or  $q$ ”, denoted  $p \vee q$ , and defined by the following truth table:

$p$	$q$	$p \vee q$
$T$	$T$	$T$
$T$	$F$	$T$
$F$	$T$	$T$
$F$	$F$	$F$

# A logical connective: XOR

If  $p$  and  $q$  are statement variables, the **exclusive disjunction** of  $p$  and  $q$  is read “ $p$  or  $q$  but not both”, denoted  $p \oplus q$ , and defined by the following truth table:

$p$	$q$	$p \oplus q$
$T$	$T$	$F$
$T$	$F$	$T$
$F$	$T$	$T$
$F$	$F$	$F$

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$T$	$T$	$F$
$T$	$F$	$T$
$F$	$T$	$T$
$F$	$F$	$F$

**Note:** Grammatically, the words ‘and’ & ‘but’ are both conjunctions, so are both interpreted as AND in logic. You could read  $p \oplus q$  as “ $p$  or  $q$  and not both.”

# A logical connective: IMPLIES

If  $p$  and  $q$  are statement variables, the **conditional** of  $q$  by  $p$  is read “if  $p$  then  $q$ ” or “ $p$  implies  $q$ ”, denoted  $p \rightarrow q$ , and defined by the following truth table:

$p$	$q$	$p \rightarrow q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$



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$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$

There is extra vocabulary associated with conditional. In the statement form  $p \rightarrow q$ , we call  $p$  the **hypothesis** (or antecedent) and  $q$  the **conclusion** (or consequent).

# Understanding IMPLIES

Pay careful attention to the truth table for a conditional. On first sight, some find the last two lines surprising. Spend some time making sure you know the details of the definition.

$p$	$q$	$p \rightarrow q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
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$p$	$q$	$p \rightarrow q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$

It sometimes helps to use language to emphasize *why* a conditional is true. When the hypothesis in a conditional is false, we say that the conditional is **vacuously true**.

# A logical connective: IFF

If  $p$  and  $q$  are statement variables, the **biconditional** of  $p$  and  $q$  is read “ $p$  if and only if  $q$ ”, denoted  $p \leftrightarrow q$ , and defined by the following truth table:

$p$	$q$	$p \leftrightarrow q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$F$
$F$	$F$	$T$

# Order of precedence among connectives

When evaluating a statement form:

- Obey parentheses over any precedence rule
- Evaluate  $\neg$  first
- Evaluate  $\wedge$ ,  $\vee$  and  $\oplus$  second. When two or more are present, parentheses may be needed
- Evaluate  $\rightarrow$  and  $\leftrightarrow$  third. When both are present, parentheses may be needed.

When writing statement forms, you must use parentheses to avoid ambiguity. For example, you should write  $p \wedge (q \vee r)$  rather than  $p \wedge q \vee r$ .

# Example: A truth table for a statement form

Q: Write a truth table for  $(p \wedge q) \vee (\neg p \wedge \neg r)$ .

First we note that:

- Our table needs a row for each combination of truth values among the statement variables. Since there are three statement variables in the statement form, our table will need  $2^3 = 8$  rows.
- We need to obey the order of precedence among logical connectives as we evaluate the truth value of the statement form.

We shall present two ways to lay out our work.

# Method 1: Extra columns

$p$	$q$	$r$	$p \wedge q$	$\neg p$	$\neg r$	$\neg p \wedge \neg r$	$(p \wedge q) \vee (\neg p \wedge \neg r)$
$T$	$T$	$T$	$T$	$F$	$F$	$F$	$T$
$T$	$T$	$F$	$T$	$F$	$T$	$F$	$T$
$T$	$F$	$T$	$F$	$F$	$F$	$F$	$F$
$T$	$F$	$F$	$F$	$F$	$T$	$F$	$F$
$F$	$T$	$T$	$F$	$T$	$F$	$F$	$F$
$F$	$T$	$F$	$F$	$T$	$T$	$T$	$T$
$F$	$F$	$T$	$F$	$T$	$F$	$F$	$F$
$F$	$F$	$F$	$F$	$T$	$T$	$T$	$T$

An advantage of this method is that you leave a record of your working. A disadvantage is that it can take a lot of room on the page.

# Method 2: Align T/F under connectives

$p$	$q$	$r$	$(p \wedge q) \vee (\neg p \wedge \neg r)$
$T$	$T$	$T$	
$T$	$T$	$F$	
$T$	$F$	$T$	
$T$	$F$	$F$	
$F$	$T$	$T$	
$F$	$T$	$F$	
$F$	$F$	$T$	
$F$	$F$	$F$	

- Be careful because your work leaves less of a record of how you did it.
- Bold, circle or highlight the column with the final result.



# Statements and statement forms

Each statement is an instance of one or more statement forms.

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EXAMPLE: Consider the statement “ $\sqrt{2}$  is negative or non-negative.” The statement is an instance of the statement form  $p \vee q$  with

$p : \sqrt{2}$  is negative

$q : \sqrt{2}$  is not negative.

The statement is also an instance of the statement form  $p \vee \neg p$ .

# Another example

EXAMPLE: Consider the statement “Tropical storms spin clockwise in the Southern hemisphere.” The statement is an instance of the statement form

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EXAMPLE: Consider the statement “Tropical storms spin clockwise in the Southern hemisphere.” The statement is an instance of the statement form  $p \rightarrow q$  with

$p$  : We are in the Southern Hemisphere

$q$  : Tropical storms spin clockwise.

Do you agree that the following statements mean the same thing?

- Tropical storms spin clockwise in the Southern hemisphere
- Tropical storms spin clockwise if we are in the Southern hemisphere
- If we are in the Southern Hemisphere then tropical storms spin clockwise.

# Tautologies

A statement form that is true regardless of the truth values of its variables is called a **tautology**. A statement whose form is a tautology is also called a **tautology**.

We can show that a statement form is a tautology by making a truth table. We can show that a statement is a tautology by making a truth table for a corresponding statement form.

Please note that a statement may be true even though it is not a tautology.

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Please note that a statement may be true even though it is not a tautology.

EXAMPLE: The statement “6 is an even integer” is true, but it is not a tautology. The statement “6 is even or 6 is not even” is true and it is a tautology.

# Showing a statement form is a tautology

Example: The statement form  $p \vee \neg p$  is a tautology, as shown by:

$p$	$\neg p$	$p \vee \neg p$
$T$	$F$	$T$
$F$	$T$	$T$

# Showing a statement form is a tautology

Example: The statement form  $p \vee \neg p$  is a tautology, as shown by:

$p$	$\neg p$	$p \vee \neg p$
$T$	$F$	$T$
$F$	$T$	$T$

Example: The statement “ $\sqrt{2}$  is negative or non-negative” is a tautology because it is an instance of the statement form  $p \vee \neg p$  with

$p : \sqrt{2} \text{ is negative,}$

and the statement form  $p \vee \neg p$  is a tautology.



# Contradiction

A statement form that is false regardless of the truth values of its variables is called a **contradiction**. A statement whose form is a contradiction is also called a **contradiction**.

We can show that a statement form is a contradiction by making a truth table. We can show that a statement is a contradiction by making a truth table for the corresponding statement form.

# Logical equivalence

When two statement forms  $f, g$  have identical truth tables we say they are **logically equivalent** and write  $f \equiv g$ .

(ALTERNATIVE DEFINITION: When  $f, g$  are statement forms and  $f \leftrightarrow g$  is a tautology, we say that  $f$  and  $g$  are **logically equivalent** and write  $f \equiv g$ . )

Knowing some logical equivalences allows us to replace complicated statement forms by simpler, logically equivalent, statement forms.

Some well-known logical equivalences are shown on the next slide.

# From page 49 of your optional text

## Theorem 2.1.1 Logical Equivalences

Given any statement variables  $p$ ,  $q$ , and  $r$ , a tautology  $\mathbf{t}$  and a contradiction  $\mathbf{c}$ , the following logical equivalences hold.

1. Commutative laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Identity laws:	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. Negation laws:	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. Double negative law:	$\sim(\sim p) \equiv p$	
7. Idempotent laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. Universal bound laws:	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. De Morgan's laws:	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. Negations of $\mathbf{t}$ and $\mathbf{c}$ :	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

Note: The text uses  $\sim$  for negation instead of  $\neg$ .

# Example: An equivalent form of XOR

Q: Justify the earlier claim that  $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$ .

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Q: Justify the earlier claim that  $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$ .

$p$	$q$	$p \oplus q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
$T$	$T$					
$T$	$F$					
$F$	$T$					
$F$	$F$					

# Example: Another equivalent form for XOR

Q: Show that  $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

# Example: Another equivalent form for XOR

Q: Show that  $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

We compute a truth table:

$p$	$q$	$p \oplus q$	$\neg p$	$\neg q$	$p \wedge \neg q$	$\neg p \wedge q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
$T$	$T$						
$T$	$F$						
$F$	$T$						
$F$	$F$						

# Suggested activities

- Ensure you know how to watch videos and download slides from the ECHO360 system. Do this by watching Lecture 0 and downloading the slides for Lecture 1.
- Master the vocabulary introduced in this lecture and practice making truth tables for complex statement forms.
- Find the ebook of our optional text in the ANU library catalogue and see what you think
- Look at the materials, including the practice problems, in the "Practice problems" and "Week 1" section on Wattle.