/Chains

C3. Markov Processes

Notes originally prepared by Judy-anne Osborn.
Editing, expansion and additions by Malcolm Brooks.

This material is not covered in the textbook by Epp. Check books on Finite Mathematics or Discrete Mathematics in the Library, *e.g. Finite Mathematics* By Maki & Thompson Chapter 8

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- and the probable state after **many** time-steps.
- We often don't make a sharp distinction between **proportions** and **probabilities** as you will see in the examples.

  This works well for large samples but you may need to be careful with small samples.

Introductory example

adapted from 'Finite Mathematics', Maki & Thompson

A freelance computer network consultant, let's call her Cathy, is employed in weekly contracts. Each week she is either:

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(a) If she's employed this week, then next week she'll be employed with probability 0.8 and unemployed with probability 0.2.

(b) If she's unemployed this week, then next week she'll be employed with probability 0.6 and unemployed with probability 0.4.

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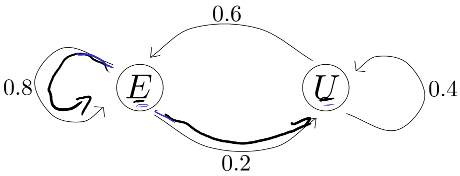
- A **Transition Diagram** encodes the transition probabilities:



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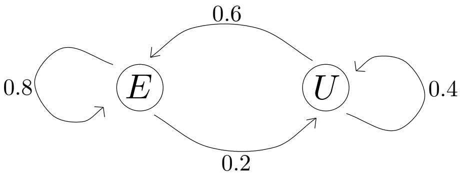

It is a property of a Markov Process that the probability of stepping from one state to another *only depends on the current state*.

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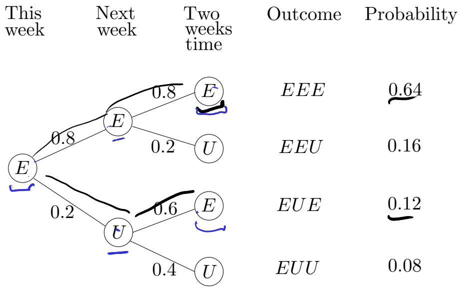

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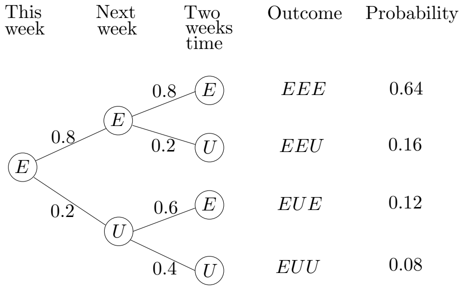

From the tree diagram, the probability that Cathy will be employed two weeks from now is

$$\Pr(\text{EEE or EUE}) = \Pr(\text{EEE}) + \Pr(\text{EUE}) = 0.64 + 0.12 = 0.76.$$

Two time-steps on the transition diagram

Starting employed, then employment after two weeks can be shown on the transition diagram as

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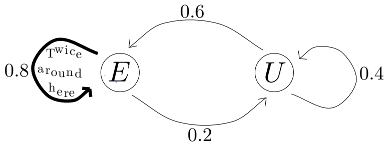

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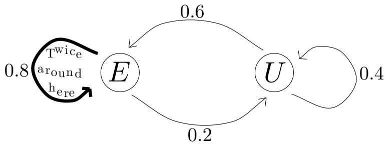
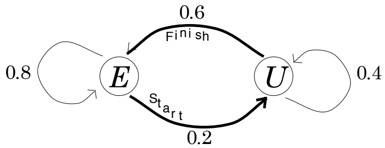
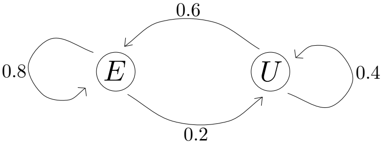


or



Transition Matrix

The information in Cathy's transition diagram



can be encoded in the **transition matrix**

$$T = \begin{array}{cc} & \begin{array}{cc} E & \quad U \end{array} \\ \begin{array}{c} E \\ U \end{array} & \begin{bmatrix} 0.8 \;+ & 0.2 \\ 0.6 \;+ & 0.4 \end{bmatrix} \begin{array}{l} = 1 \\ = 1 \end{array} \end{array}$$

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100% employed and 0% unemployed, or
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Transpose of the Transition Matrix

Recall that the transition (transfer) matrix is

$$T = \underbrace{\begin{array}{cc} & \begin{array}{cc} E & U \end{array} \\ \begin{array}{c} E \\ U \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \end{array}}.$$

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We will need

$$T' = \begin{array}{c c} & \begin{array}{cc} E & U \end{array} \\ \begin{array}{c} E \\ U \end{array} & \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \end{array}.$$

flipped diagonally

transposition

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This is the **transpose** of the transition matrix.
It is very important to remember that it is always the *transpose* of the transition matrix that is used in calculations.

Using Matrices and State Vectors

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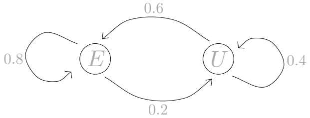
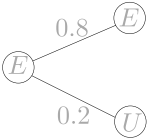
This can be expressed as:

$$\mathbf{x_1} = \mathbf{T}'\mathbf{x_0}$$

$$= \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$





Outcome	Prob.
EE	0.8
EU	0.2

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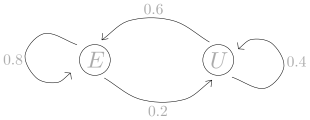


$$\mathbf{x_2} = \mathbf{T}'\mathbf{x_1}$$

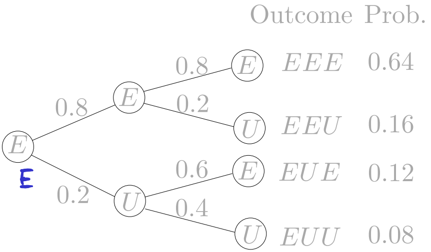
$$= \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.64 + 0.12 \\ 0.16 + 0.08 \end{bmatrix}$$

$$= \begin{bmatrix} 0.76 \\ 0.24 \end{bmatrix}$$

prob E & E

prob E & U



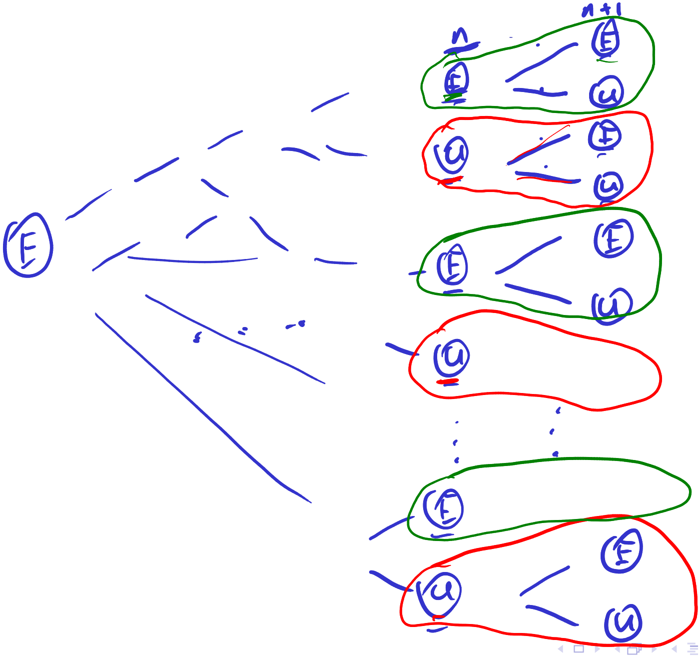
$$x_n = \begin{bmatrix} \mathbb{P}(\overbrace{E \to \to \cdots \to E}^{n+1}) \\ \mathbb{P}(\underbrace{E \to \cdots \to U}_{n+1}) \end{bmatrix}$$

$$T' = \begin{bmatrix} \mathbb{P}(E \to E) & \mathbb{P}(U \to E) \\ \mathbb{P}(E \to U) & \mathbb{P}(U \to U) \end{bmatrix}$$

$$T' x_n = \begin{bmatrix} \mathbb{P}(E \to E)\,\mathbb{P}(E \dots E) + \mathbb{P}(U \to E)\,\mathbb{P}(E \dots U) \\ \mathbb{P}(E \to U)\,\mathbb{P}(E \dots E) + \mathbb{P}(U \to U)\,\mathbb{P}(E \dots U) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{P}(\overbrace{E \dots EE}^{n+2}) + \mathbb{P}(E \dots UE) \\ \mathbb{P}(E \dots EU) + \mathbb{P}(E \dots UU) \end{bmatrix} = \begin{bmatrix} \mathbb{P}(\overbrace{E \dots E}^{n+2}) \\ \mathbb{P}(\overbrace{E \dots U}^{n+2}) \end{bmatrix}$$

$$= x_{n+1}$$


n
n+1
E
E
U
U
E
U
E
E
U
U
E
U
E
U

n time-steps

Continuing:
$$\begin{aligned}
\mathbf{x}_3 &= T'\mathbf{x}_2 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.76 \\ 0.24 \end{bmatrix} = \begin{bmatrix} 0.752 \\ 0.248 \end{bmatrix} \\
\mathbf{x}_4 &= T'\mathbf{x}_3 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.752 \\ 0.248 \end{bmatrix} = \begin{bmatrix} 0.7504 \\ 0.2496 \end{bmatrix} \\
\mathbf{x}_5 &= T'\mathbf{x}_4 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.7504 \\ 0.2496 \end{bmatrix} = \begin{bmatrix} 0.75008 \\ 0.24992 \end{bmatrix} \\
\vdots \quad &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots
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$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

Thus:

$$\mathbf{x}_1 = T'\mathbf{x}_0 \quad , \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_2 = T'\mathbf{x}_1 = T'T'\mathbf{x}_0 = (T')^2\mathbf{x}_0$$

$$\mathbf{x}_3 = T'\mathbf{x}_2 = T'T'T'\mathbf{x}_0 = (T')^3\mathbf{x}_0$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{x}_n = (T')^n\mathbf{x}_0$$

The n^{th} power of T'

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$$
\begin{aligned}
(T')^2 &= T'T' = \begin{bmatrix} 0.76 & 0.72 \\ 0.24 & 0.28 \end{bmatrix} \\
(T')^3 &= T'(T')^2 = \begin{bmatrix} 0.752 & 0.744 \\ 0.248 & 0.256 \end{bmatrix} \\
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&\vdots \qquad \vdots \qquad\qquad \vdots
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So: $(T')^n \simeq \begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix}$ for large values of n.

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Notice that the columns of this matrix are equal, and that

$$\begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}.$$

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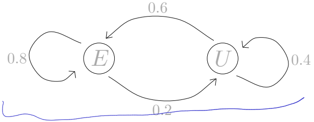
So, irrespective of the initial state, in the long term the state vector becomes approximately $\begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$. This means

No matter what, eventually Cathy will be employed 75% of the time.

The Steady State Vector

Cathy's employment situation can now be summed up by:



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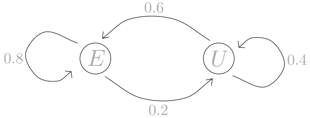

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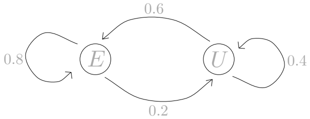

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$$(T')^n \underline{\mathbf{u}} \simeq \mathbf{v}$$

for any initial state vector $\mathbf{u}$.

The steady state vector is an eigenvector

The steady state vector has the property that multiplication by the transposed transition matrix does not change it, *e.g.* for Cathy:

$$T'\mathbf{v} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.60 + 0.15 \\ 0.15 + 0.10 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = \mathbf{v}.$$

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No change in time from v

More generally, for *any* transition matrix T we call *any* vector $\mathbf{v}$ for which

$$T'\mathbf{v} = \mathbf{v}$$

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Caution: A Markov process may not always reach a steady state!

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The steady state vector has the property that multiplication by the transposed transition matrix does not change it, *e.g.* for Cathy:

$$T'\mathbf{v} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.60 + 0.15 \\ 0.15 + 0.10 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = \mathbf{v}.$$

More generally, for *any* transition matrix T we call *any* vector $\mathbf{v}$ for which

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if $v \neq \vec{0}$

Note: The definition of $\mathbf{v}$ makes it a special case of an **eigenvector**. Courses in linear algebra cover more about eigenvectors and also numbers called **eigenvalues**.

A steady state vector has an associated eigenvalue of 1.

17 Announcement: Final exam is finalised, post to come.

Some Definitions

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A (discrete) **Markov** $\overset{\text{chain}}{\textbf{process}}$ is a system that, at each of a sequence of time steps, can be in exactly one of a finite number k of states, with the probability of the system being in any particular state at time step $n \geq 1$ being dependent only on

(i) its state at the $(n-1)$-th time step, and

(ii) a fixed stochastic matrix $T \in M_k(\overset{\mathbb{R}}{Q}_+)$ called the **transition matrix** of the process.

Some Definitions (continued)

The (i,j)-entry T_{ij} of the transition matrix T specifies the probability that the system will be in the j-th state at any time step $n \geq 1$, given that it was in the i-th state at time step $n-1$.

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for this particular system

A **steady state vector** $\mathbf{v}$ is any probability vector for which $T'\mathbf{v} = \mathbf{v}$.

May or may not exist.

Using the transition matrix

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&= \sum_{j=1}^{k} (T_{ji})(\text{prob. of being in state } j \text{ at step } n-1) = \sum_{j=1}^{k} T_{ji} q_j .
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Proofs of (ii) and (iii): These are simple corollaries to (i).

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- It does not depend on the state at any earlier time. In other words, it is a *first-order* (matrix) recurrence.
- Because of this, Markov processes are said to "have ~~no~~ one-step memory".

Finding steady state vectors

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Worked for Cathy's example, might not always work.

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- There are more direct methods of finding steady state vectors, and we demonstrate these in the next example.

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As a table: Probabilities of weather tomorrow are:

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	cloudy	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
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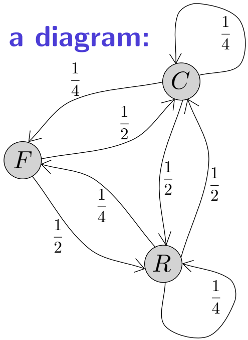

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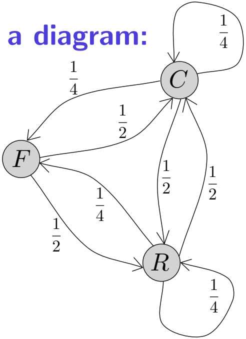

As a matrix:

$$
T = \begin{array}{c} \\ F \\ C \\ R \end{array}
\begin{array}{c}
\begin{array}{ccc} F & C & R \end{array} \\
\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}
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- Define $\mathbf{x}_n = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
- Then, according ot the Markov process theorem:

$$\begin{aligned} \mathbf{x}_{n+1} &= T'\mathbf{x}_n \\ &= \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (1/4)y + (1/4)z \\ (1/2)x + (1/4)y + (1/2)z \\ (1/2)x + (1/2)y + (1/4)z \end{bmatrix} \end{aligned}$$

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Day n		Day n+1	Probabilities	Summarized	
Fine (x)	0	Fine	0		
	1/2	Cloudy	$x/2$		
	1/2	Rain	$x/2$		
Cloudy (y)	1/4	Fine	$y/4$	Fine:	$0+y/4+z/4$
	1/4	Cloudy	$y/4$	Cloudy:	$x/2+y/4+z/2$
	1/2	Rain	$y/2$	Rain:	$x/2+y/2+z/4$
Rain (z)	1/4	Fine	$z/4$		
	1/2	Cloudy	$z/2$		
	1/4	Rain	$z/4$		

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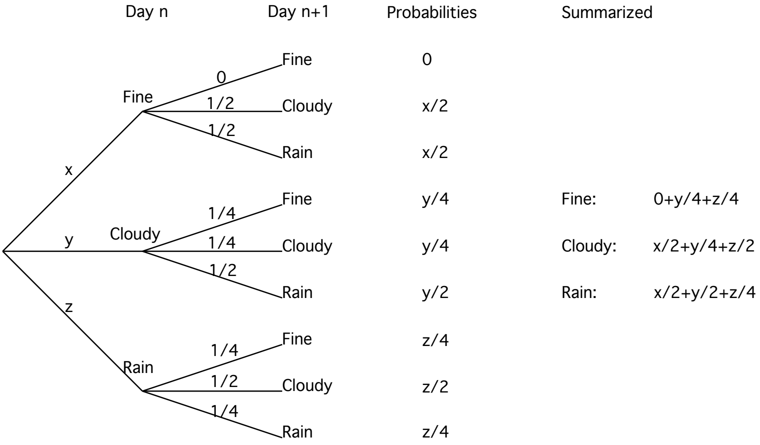

Yes, the state vector $\mathbf{x}_{n+1}$ and probability tree agree. ✓

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Using a computer to calculate the 7th power of the matrix, we get

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Perhaps decimals would be more illuminating?

Days 1 through 10 in Oz

Computer calculations give:

$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ .5 \\ .5 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} .250 \\ .375 \\ .375 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} .18750 \\ .40625 \\ .40625 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} .2031250 \\ .3984375 \\ .3984375 \end{bmatrix},$$

$$\mathbf{x}_5 = \begin{bmatrix} .199218750 \\ .400390625 \\ .400390625 \end{bmatrix}, \quad \mathbf{x}_6 = \begin{bmatrix} .1999511719 \\ .4000244141 \\ .4000244141 \end{bmatrix}, \quad \mathbf{x}_7 = \begin{bmatrix} .19995511719 \\ .4000244141 \\ .4000244141 \end{bmatrix},$$

$$\mathbf{x}_8 = \begin{bmatrix} .2000122070 \\ .3999938965 \\ .3999938965 \end{bmatrix}, \quad \mathbf{x}_9 = \begin{bmatrix} .1999969438 \\ .4000015260 \\ .4000015260 \end{bmatrix}, \quad \mathbf{x}_{10} = \begin{bmatrix} .2000007629 \\ .3999996185 \\ .3999996185 \end{bmatrix}.$$

A steady state for the weather in Oz

These values seem to be converging to a long-term steady state of

$$S = \begin{bmatrix} .2 \\ .4 \\ .4 \end{bmatrix},$$

i.e. a probability of 0.2 of fine weather, a probability of 0.4 of cloudy weather and a probability of 0.4 of rainy weather.

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+
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To check that this S really is a steady state vector, we calculate

$$T'S = \begin{bmatrix} 0 & 0.25 & 0.25 \\ 0.50 & 0.25 & 0.50 \\ 0.50 & 0.50 & 0.25 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.1 + 0.1 \\ 0.1 + 0.1 + 0.2 \\ 0.1 + 0.2 + 0.1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix} = S \checkmark$$

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i.e. a probability of 0.2 of fine weather, a probability of 0.4 of cloudy weather and a probability of 0.4 of rainy weather. As they must, these probabilities sum to 1.

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$$T'S = \begin{bmatrix} 0 & 0.25 & 0.25 \\ 0.50 & 0.25 & 0.50 \\ 0.50 & 0.50 & 0.25 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.1 + 0.1 \\ 0.1 + 0.1 + 0.2 \\ 0.1 + 0.2 + 0.1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix}.$$

Therefore $T'S = S$. ✓

Finding steady state vectors

We derive another way to find steady state vectors, illustrating with weather from Oz.

Assume that $S = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a steady state vector.

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(In other words we have reached a stage where the probabilities don't change from day to day any more.)

Notice that we can rearrange this equation in the form

$$T'S - S = 0.$$

nxn matrix

Remember to *think about what kinds of objects* are in this equation

3x1

$$T'S - S = 0$$

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$$T'S - S = 0$$

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We can rewrite the equation yet again in the form

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where I is the 3×3 identity matrix.

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Finally using a distributive law, we can re-write it as:

$$(T' - I)S = 0.$$

Methods for solving the matrix equation

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Since

$$T' - I = \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1/4 & 1/4 \\ 1/2 & -3/4 & 1/2 \\ 1/2 & 1/2 & -3/4 \end{bmatrix}$$

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our augmented matrix is

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Row reducing,

$$\left[\begin{array}{ccc|c} -1 & 1/4 & 1/4 & 0 \\ 1/2 & -3/4 & 1/2 & 0 \\ 1/2 & 1/2 & -3/4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1/4 & 1/4 & 0 \\ 1 & -3/2 & 1 & 0 \\ 1 & 1 & -3/2 & 0 \end{array}\right] \begin{array}{l} R_2' = 2R_2 \\ R_3' = 2R_3 \end{array}$$

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$$\sim \left[\begin{array}{ccc|c} -1 & 1/4 & 1/4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2' = (-4/5)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1/4 & -1/4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1' = -R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1' = R_1 + (1/4)R_2$$

$\uparrow$
This column
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Let $z = t$, $t \in \mathbb{R}$

So our original matrix equation is equivalent to

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leading to the solution
$$\begin{aligned} x &= (1/2)t \\ y &= t \\ z &= t. \end{aligned}$$

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– the same as we found before by exponentiating and guessing.

A short cut

A short cut to this process is to take the augmented matrix $[T' - I|0]$ as below,

$$\left[\begin{array}{ccc|c} -1 & 1/4 & 1/4 & 0 \\ 1/2 & -3/4 & 1/2 & 0 \\ 1/2 & 1/2 & -3/4 & 0 \end{array}\right]$$

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and solve this new system to directly obtain the unique solution for S.

After row-reducing the new system we find that

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Can you figure out *why* this short cut works?

Solving by Computer (using Reshish)

The system of equations

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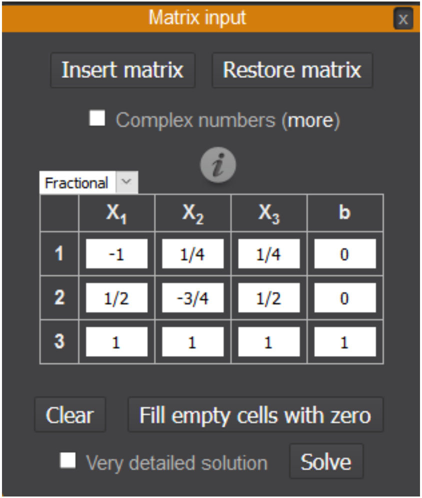

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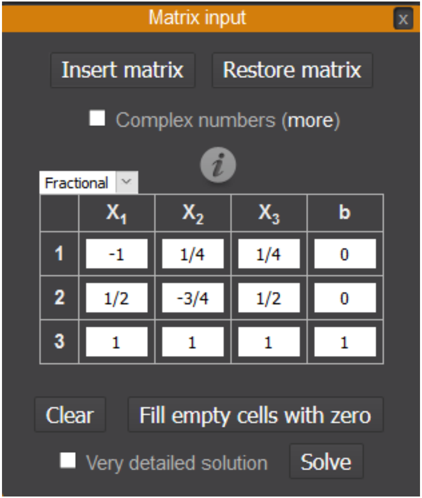

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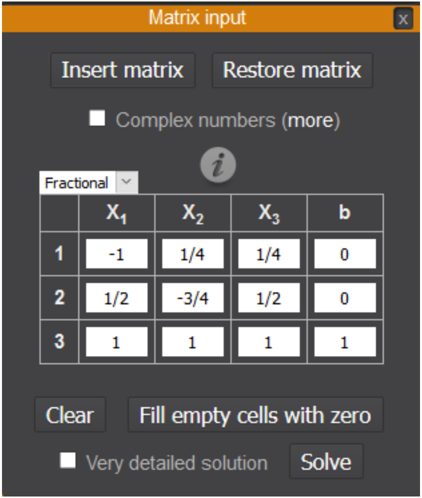
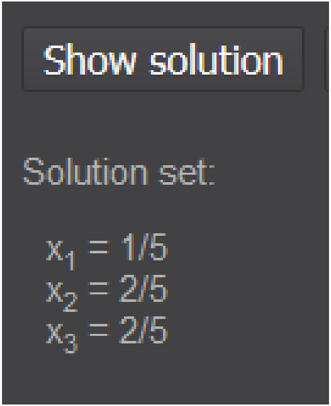


Here is how Reshish responds:



Back to the first example

We have seen that to find the steady state vector S for a Markov process with transition matrix T we need to solve the linear system that results from replacing the last equation in

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For Cathy's employment process we had

$$T = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

and, by a 'guess and check' method, we discovered that

$$S = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}.$$

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Because T is 2×2, and we have a formula for the inverse of a 2×2 matrix, we can find Cathy's steady state vector directly, without Gaussian elimination or computer. There are three steps:

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$$\left(\begin{bmatrix}0.8 & 0.6\\0.2 & 0.4\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\text{i.e. } \begin{bmatrix}-0.2 & 0.6\\0.2 & -0.6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$x+y \to \begin{bmatrix}1 & 1\end{bmatrix}$ $\quad \begin{bmatrix}1\end{bmatrix} \leftarrow = 1$

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$$\text{i.e.} \quad \begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. Replace the second equation by $x + y = 1$:

$Ax = b$

$x = A^{-1}b$

$$\underbrace{\begin{bmatrix} -0.2 & 0.6 \\ 1 & 1 \end{bmatrix}}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution by matrix inverse (conclusion)

3. Solve this system using matrix inverse:

$$\overset{x}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{A^{-1}}{\begin{bmatrix} -0.2 & 0.6 \\ 1 & 1 \end{bmatrix}^{-1}} \overset{b}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

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$$= \frac{1}{-0.8} \begin{bmatrix} -0.6 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 6/8 \\ 2/8 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

New Example — Colours of flowers

A species of flower (carnations say) has three colour varieties. The relevant genetics are as shown in the table:

Colour	Genotype
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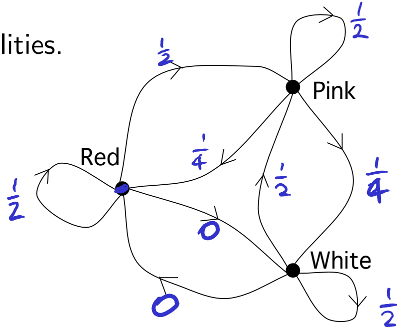

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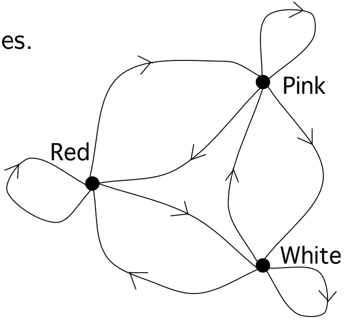
The transition matrix is

$$T = \begin{array}{c} \\ \text{Red} \\ \text{Pink} \\ \text{White} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Red} & \text{Pink} & \text{White} \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.5 & 0.5 \end{bmatrix} \end{array} .$$



Finding the steady state

(a) $[T' - I|0]$ is

$$\left[\begin{array}{ccc|c} -0.5 & 0.25 & 0 & 0 \\ 0.5 & -0.5 & 0.5 & 0 \\ 0 & 0.25 & -0.5 & 0 \end{array}\right]$$

(b) Replacing the bottom row with all 1's gives

$$\left[\begin{array}{ccc|c} -0.5 & 0.25 & 0 & 0 \\ 0.5 & -0.5 & 0.5 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

Finding the steady state (cont.)

(c) Row reduction gives

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$$\sim \left[\begin{array}{ccc|c} -1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0 \\ 0 & 1.5 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + R_1 \\ R_3' = R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{array}\right] \quad R_3' = R_3 + 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -0.5 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1/4 \end{array}\right] \quad R_3' = (1/4)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -0.5 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/4 \end{array}\right] \quad R_2' = R_2 + 2R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/4 \end{array}\right] \quad R_1' = R_1 + (1/2)R_2$$

yielding $S = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$.

Finding the steady state by computer

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Solution set:

$x_1 = 1/4$
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Hence there is a unique steady state vector of

$$S = \begin{array}{cl} \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix} & \begin{array}{l} \leftarrow \text{Red} \\ \leftarrow \text{pink} \\ \leftarrow \text{white} \end{array} \end{array}$$

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	X_1	X_2	X_3	b
1	-1/2	1/4	0	0
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Solution set:

$x_1 = 1/4$
$x_2 = 1/2$
$x_3 = 1/4$

Hence there is a unique steady state vector of

$$S = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

So the species has a steady state in which 25% of the flowers are coloured red, 50% pink, and 25% white.

Checking the answer

The steady state vector S must be an eigenvector of T'.

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Let's check:

$$T'S = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/8 + 1/8 \\ 1/8 + 1/4 + 1/8 \\ 1/8 + 1/8 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

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The steady state vector S must be an eigenvector of T'.
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$$T'S = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$
$$= \begin{bmatrix} 1/8 + 1/8 \\ 1/8 + 1/4 + 1/8 \\ 1/8 + 1/8 \end{bmatrix}$$
$$= \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

So yes, $T'S = S$. ✓

Will a Markov process always get to a steady state?

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Not necessarily!

Example: chemical compounds in transition

Consider a chemical compound whose molecule can exist in any one of five states, termed A, B, C, D and E.

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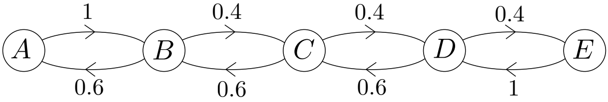
Each molecule frequently undergoes transitions from one state to another, always to an ‘adjacent’ state, according to the probabilities shown in the transition diagram.



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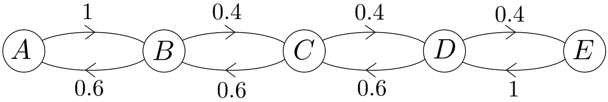

The transition matrix for this Markov Process is

$$T = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

A beaker full of chemical

- Now suppose we have a beaker full of this chemical. We expect the proportions of A, B, C, D, E to relate to transition probabilities.

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- To do a thorough analysis of all possible behaviours of this Markov Process, you need to study '*eigenvalues and eigenvectors*' – a reason to take a course or read a book on Linear Algebra.

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- What proportions of the compound will be in the various states?
- To do a thorough analysis of all possible behaviours of this Markov Process, you need to study '*eigenvalues and eigenvectors*' – a reason to take a course or read a book on Linear Algebra.
- But let's see what we can figure out without those tools.

Chemical example — investigating with a computer

Suppose the beaker only contains form 'A' to start with, *i.e.* $\mathbf{x}_0 = [1, 0, 0, 0, 0]'$. Then by computer to 6dp we find:

$$
\begin{aligned}
\mathbf{x}_{100} &= (T')^{100}\mathbf{x}_0 \\
&= \underline{[\overset{A}{0.415383}\,,\ 0.000000\,,\ \overset{C}{0.461538}\,,\ 0.000000\,,\ \overset{E}{0.123077}]'} \\
\mathbf{x}_{101} &= T'\mathbf{x}_{100} \\
&= [0.000000\,,\ \overset{B}{0.692308}\,,\ 0.000000\,,\ \overset{D}{0.307692}\,,\ 0.000000]'
\end{aligned}
$$

Chemical example — investigating with a computer

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&= [0.000000\,,\ 0.692308\,,\ 0.000000\,,\ 0.307692\,,\ 0.000000]'
\end{aligned}$$

and continuing in the same manner

$$\begin{aligned}
\mathbf{x}_{102} &= [0.415383\,,\ 0.000000\,,\ 0.461538\,,\ 0.000000\,,\ 0.123077]' \\
\mathbf{x}_{103} &= [0.000000\,,\ 0.692308\,,\ 0.000000\,,\ 0.307692\,,\ 0.000000]' \\
\mathbf{x}_{104} &= [0.415383\,,\ 0.000000\,,\ 0.461538\,,\ 0.000000\,,\ 0.123077]' \\
\mathbf{x}_{105} &= [0.000000\,,\ 0.692308\,,\ 0.000000\,,\ 0.307692\,,\ 0.000000]' \\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

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Suppose the beaker only contains form 'A' to start with, *i.e.* $\mathbf{x}_0 = [1, 0, 0, 0, 0]'$. Then by computer to 6dp we find:

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\mathbf{x}_{105} &= [0.000000\,,\ 0.692308\,,\ 0.000000\,,\ 0.307692\,,\ 0.000000]' \\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

It appears to alternate!

However starting with a beaker half full of A and half of B, *i.e.* $\mathbf{x}_0 = [0.5, 0.5, 0, 0, 0]'$, and again using formulae

$$\mathbf{x}_n = (T')^n \mathbf{x}_0 \quad \text{and} \quad \mathbf{x}_{n+1} = T' \mathbf{x}_n$$

repeatedly we get

$$\begin{aligned}
\mathbf{x}_{100} &= [0.207692\,,\ 0.346154\,,\ 0.230769\,,\ 0.153846\,,\ 0.061539]' \\
\mathbf{x}_{101} &= [0.207692\,,\ 0.346154\,,\ 0.230769\,,\ 0.153846\,,\ 0.061539]' \\
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&\ \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

This looks like a steady state!

$T's = s$

steady state

$(T')^n s_0 \to s$ for any s_0

limiting state

However starting with a beaker half full of A and half of B, *i.e.* $\mathbf{x}_0 = [0.5, 0.5, 0, 0, 0]'$, and again using formulae

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This looks like a steady state!

So this Markov Process is different to those we used to model
Catlugs employment, weather in Oz, and flower-colours because

eventual behaviour depends on where you start!

Steady state(s) for a beaker of chemical?

We can solve for the steady state to find out if it is unique.

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We need to solve

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for $S = [x_1, x_2, x_3, x_4, x_5]'$ subject to additional constraint

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1.$$

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We use the 'short cut' method:

(a) First construct $[T' - I|0]$.

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(a) First construct $[T' - I|0]$.

(b) Then replace the last row with all 1's.

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We use the 'short cut' method:

(a) First construct $[T' - I|0]$.
(b) Then replace the last row with all 1's.
(c) Then solve by Gaussian elimination or computer.

Steady state(s) for a beaker of chemical?

(a) $[T' - I \mid 0]$ is

$$\left[\begin{array}{ccccc|c} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & -1 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0.4 & -1 & 0 \end{array}\right]$$

Steady state(s) for a beaker of chemical?

(a) $[T' - I \mid 0]$ is

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(b) Replace the last row with all 1's

$$\left[\begin{array}{ccccc|c} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & -1 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & -1 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & -1 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & -1 & 1 & 0 \\ 0 & 1.6 & 1 & 1 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + R_1 \\ R_5' = R_5 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & -1 & 1 & 0 \\ 0 & 0 & 3.4 & 1 & 1 & 1 \end{array}\right] \quad R_3' = R_3 + R_2$$

$$\sim \left[\begin{array}{ccccc|c} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 1 & 0 \\ 0 & 0 & 0 & 6.1 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_4' = R_4 + R_3 \\ R_5' = R_5 + (8.5)R_3 \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 16.25 & 1 \end{array}\right] \quad R_5' = R_5 + (16.25)R_3$$

$$\sim \left[\begin{array}{ccccc|c} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4/65 \end{array}\right] \quad R_5' = (4/65)R_5$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2.5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4/65 \end{array}\right] \quad \begin{array}{l} R_1' = -R_1 \\ R_2' = (-5/2)R_2 \\ R_3' = (-5/2)R_3 \\ R_4' = (-5/2)R_4 \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 10/65 \\ 0 & 0 & 0 & 0 & 1 & 4/65 \end{array}\right] \quad R_4' = R_4 + (5/2)R_5$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 15/65 \\ 0 & 0 & 0 & 1 & 0 & 10/65 \\ 0 & 0 & 0 & 0 & 1 & 4/65 \end{array}\right] \quad R_3' = R_3 + (3/2)R_4$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 45/130 \\ 0 & 0 & 1 & 0 & 0 & 15/65 \\ 0 & 0 & 0 & 1 & 0 & 10/65 \\ 0 & 0 & 0 & 0 & 1 & 4/65 \end{array}\right] \quad R_2' = R_2 + (3/2)R_3$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 27/130 \\ 0 & 1 & 0 & 0 & 0 & 45/130 \\ 0 & 0 & 1 & 0 & 0 & 15/65 \\ 0 & 0 & 0 & 1 & 0 & 10/65 \\ 0 & 0 & 0 & 0 & 1 & 4/65 \end{array}\right] \quad R_1' = R_1 + (5/3)R_2$$

Steady state for a beaker of chemical - by computer

Decimal

	X_1	X_2	X_3	X_4	X_5	b
1	-1	0.6	0	0	0	0
2	1	-1	0.6	0	0	0
3	0	0.4	-1	0.4	0	0
4	0	0	0.4	-1	1	0
5	1	1	1	1	1	1

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4	0	0	0.4	-1	1	0
5	1	1	1	1	1	1

Solution set:

$x_1 = 27/130$
$x_2 = 9/26$
$x_3 = 3/13$
$x_4 = 2/13$
$x_5 = 4/65$

Steady state for a beaker of chemical - by computer

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$x_4 = 2/13$
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This confirms the unique steady-state solution found by row reduction on the previous slide:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underline{\begin{bmatrix} 27/130 \\ 45/130 \\ 15/65 \\ 10/65 \\ 4/65 \end{bmatrix}} = \begin{bmatrix} 0.2077 \\ 0.3462 \\ 0.2308 \\ 0.1538 \\ 0.0615 \end{bmatrix} .$$

Steady state for a beaker of chemical - by computer

Decimal

	X_1	X_2	X_3	X_4	X_5	b
1	-1	0.6	0	0	0	0
2	1	-1	0.6	0	0	0
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So the steady-state proportions of the five forms of the chemical are:

A: 20.77%, B: 34.62%,
C: 23.08%, D: 15.38%,
E: 6.15%.

A steady state for a beaker of chemical - conclusion

We found that **provided** the beaker reaches a steady-state, then proportions of the various forms of the chemical remain stable at

$$A : 20.77\%, \quad B : 34.62\%, \quad C : 23.08\%, \quad D : 15.38\%, \quad E : 6.15\%.$$

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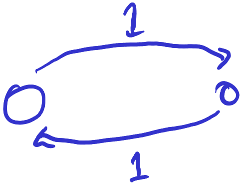
END OF SECTION C3

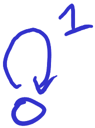


Unique stationary state, but not as a limit

$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, no limit state, i.e. $(T)^n S_0$ doesn't stabilise

stationary state $= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, all states are stationary



Lots of stationary states