D2. Weighted Graphs

Notes by Malcolm Brooks,
partly inspired by notes of Pierre Portal.

Text Reference (Epp) 3ed: Chapter 11
4ed: Chapter 10
5ed: Chapter 10

Some of the work in this section is not covered in our text by Epp.
I have based some examples on ones from:
Kolman, Busby & Ross *Discrete Mathematical Structures*
Johnsonbaugh *Discrete Mathematics*

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    - **The internet:** Vertices are internet nodes; edges are all direct connections between nodes; weights are times (in milliseconds) for a packet to travel across a connection.

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We will also look at a different kind of problem on a weighted **directed** graph: Maximal Flow. Details later.

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4. Repeat steps 2 and 3 until T has $n - 1$ edges.

Example: Applying Kruskal's algorithm

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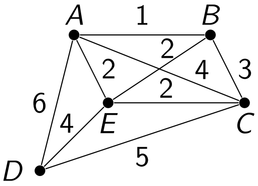

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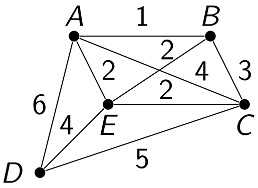

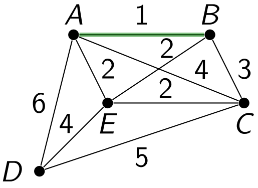

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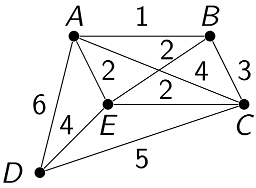

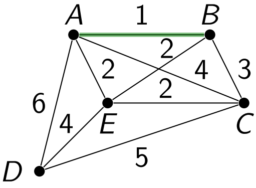

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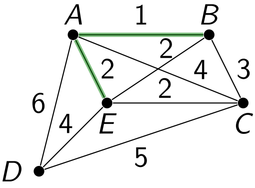

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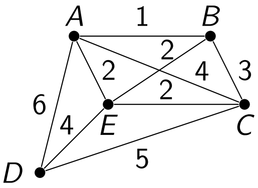

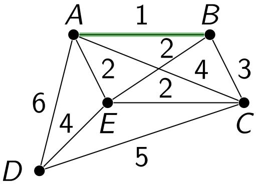

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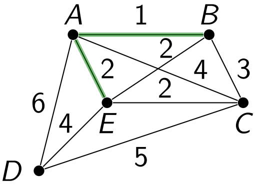

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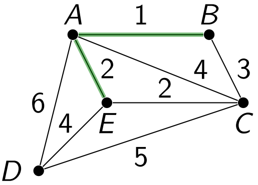

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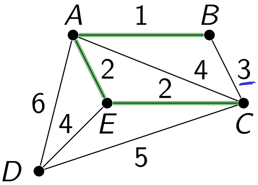
Example: Applying Kruskal's algorithm (cont.)

$\Longrightarrow$ 

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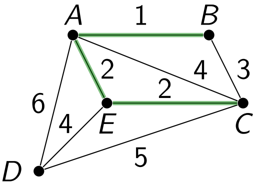
$\Longrightarrow$ 

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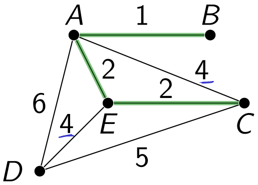
$\Longrightarrow$ 

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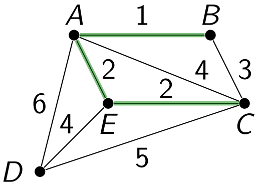

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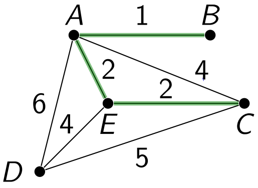

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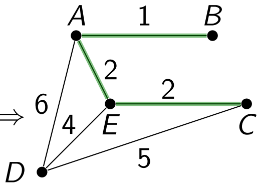

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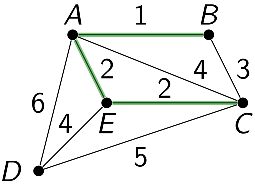

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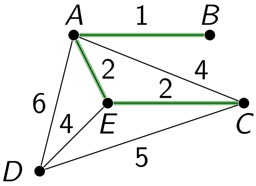

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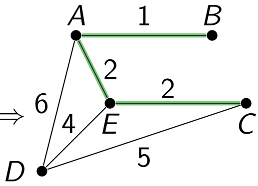

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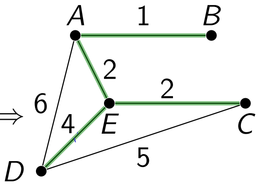

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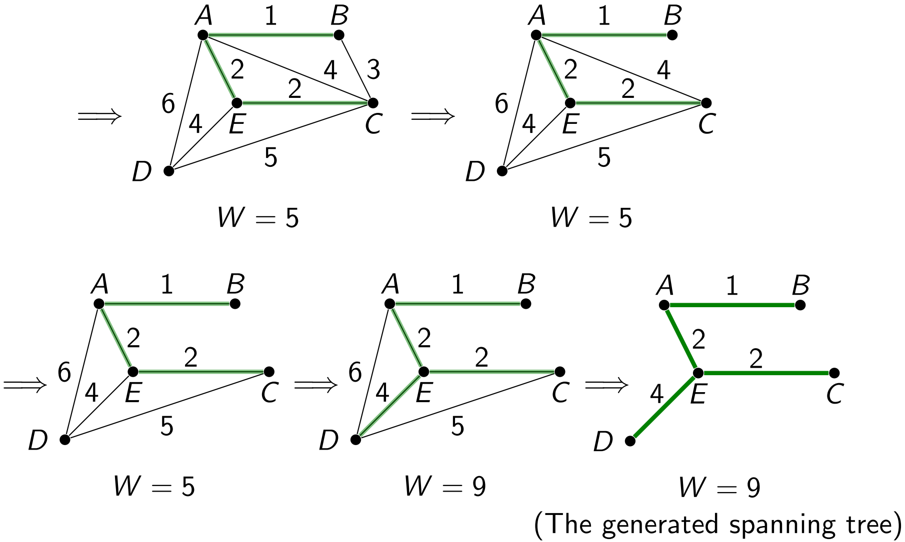

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- Kruskal's algorithm always succeeds! (Non-obvious theorem omitted)

  That is, it always finds a minimal spanning tree, given any weighted connected (finite) graph.

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- complete
  Model: Graph K_n with towns as vertices and edges weighted by the the inter-town distances.

  Find a Hamilton circuit of minimum possible total weight.

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6. Add weight$(L(n), L(1))$ to W. Append $L(1)$ to L as $L(n+1)$.

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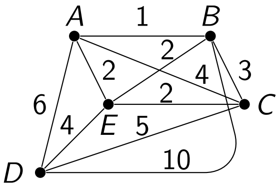
$P = NP$?

- In fact, no efficient successful algorithm for the travelling salesman problem is known at this time. Finding one, or proving that none exists, is a major outstanding problem in mathematics.

Example: Applying the Nearest Neighbour algorithm

Find a minimal Hamilton circuit (tour) for this weighted graph:

Note: This graph is as for the minimal spanning tree example but with the addition of an edge BD to make it complete.



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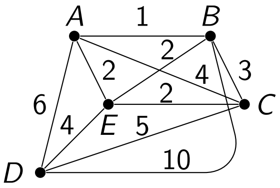

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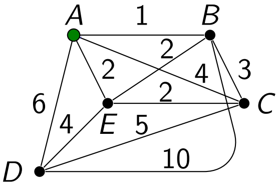

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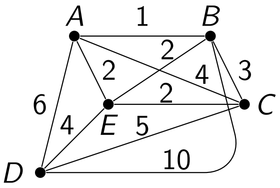

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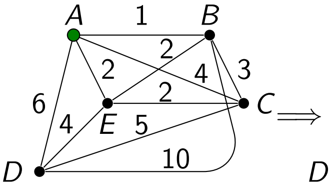

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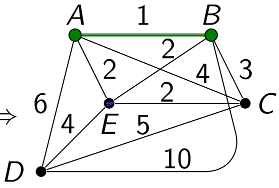

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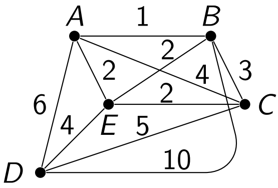

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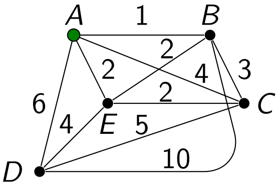
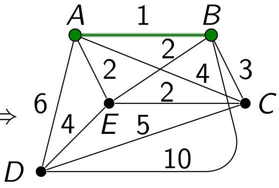


$L(1) = A,\ W = 0$

$\Longrightarrow$



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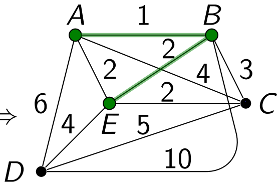

$L(3) = E,\ W = 3$

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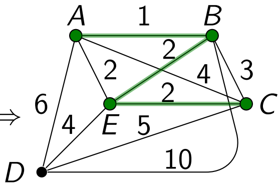
$\Longrightarrow$ 

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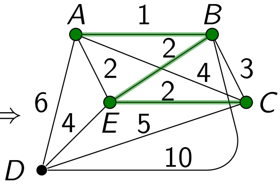

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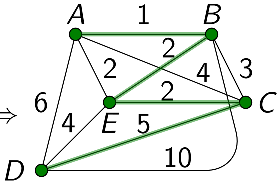
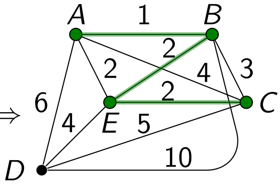
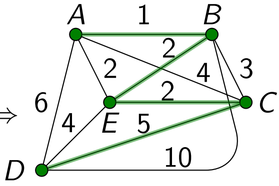
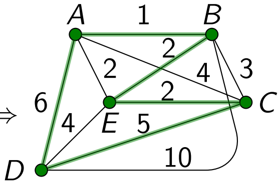


$L(5) = D, \;\; W = 10$

Example: Applying the Nearest Neighbour algorithm (cont.)

$\Longrightarrow$ 

$\Longrightarrow$ 

$\Longrightarrow$ 

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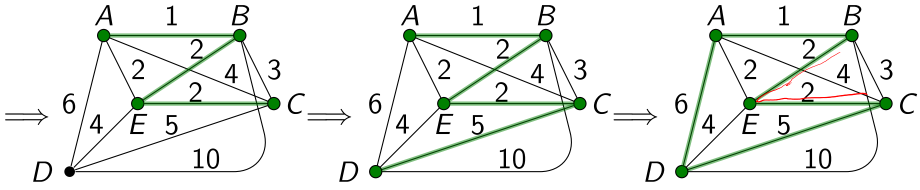

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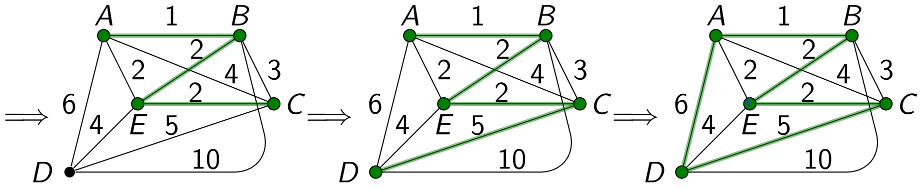

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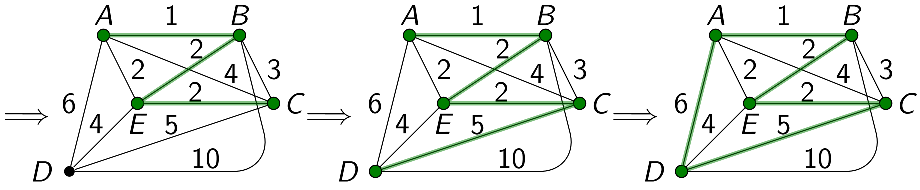

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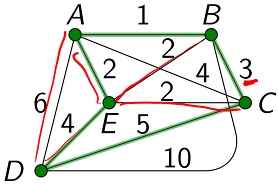

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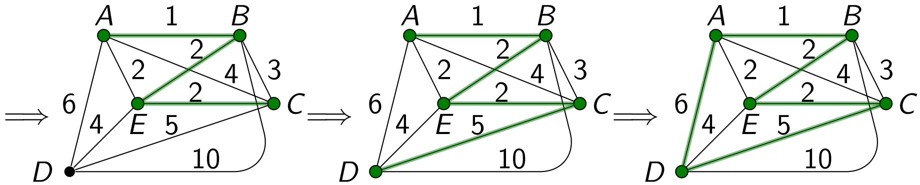

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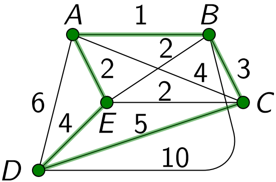

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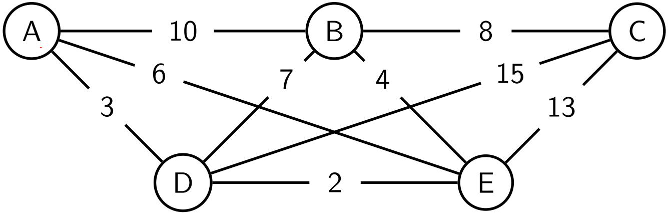
Note that Nearest Neighbour may generate this tour if we start at D instead of A. Then $L(2) = E$ and it just depends on the choice for $L(3)$.

Shortest Path — Introduction

Consider this weighted graph:



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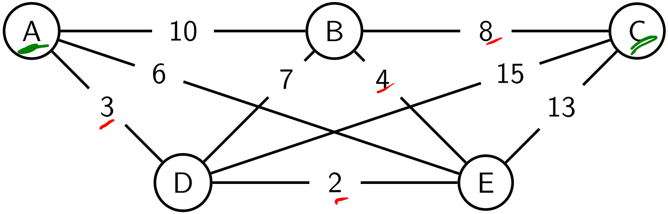

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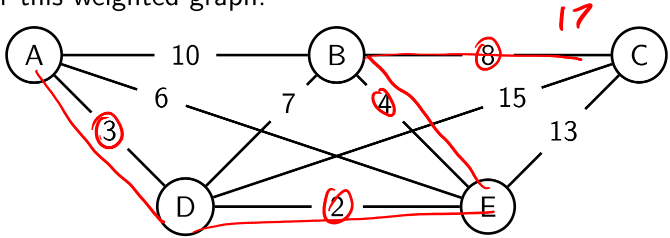

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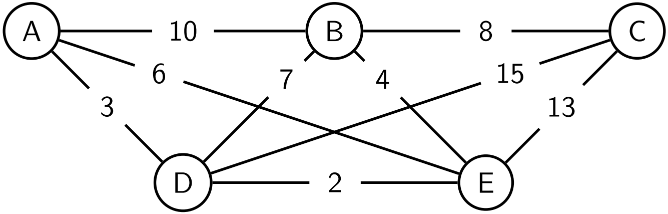

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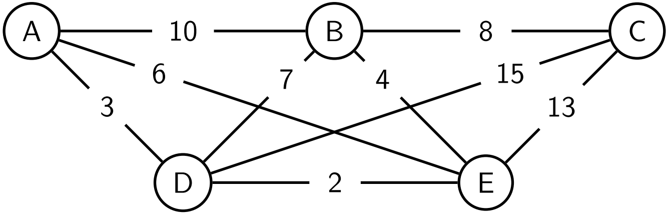

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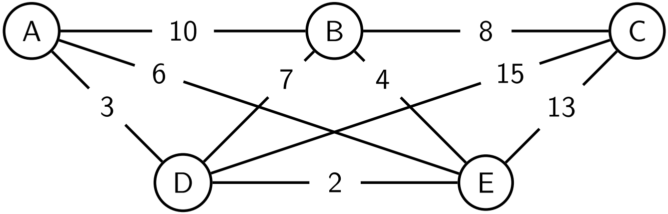

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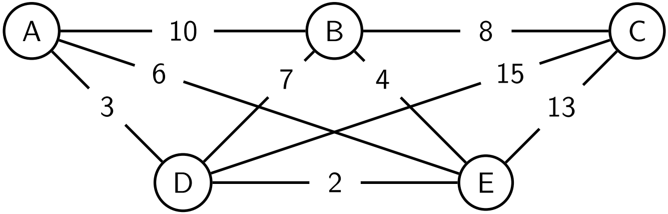

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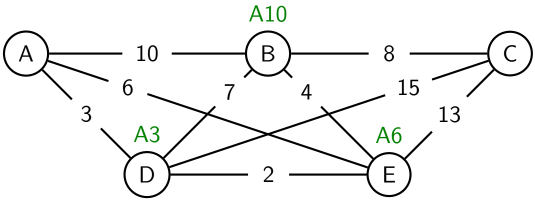
Dijkstra's Algorithm



Edsger Dijkstra 1930 - 2002

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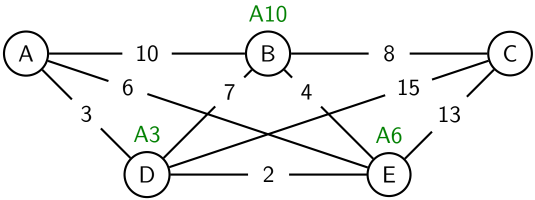

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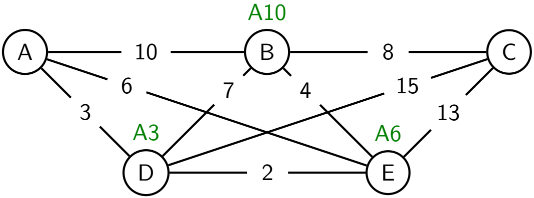

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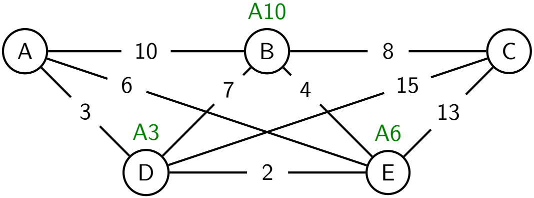

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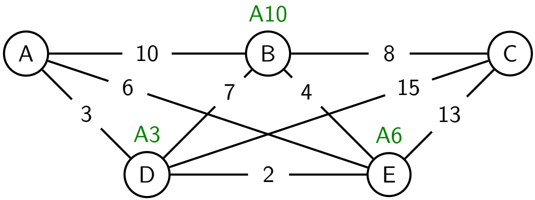

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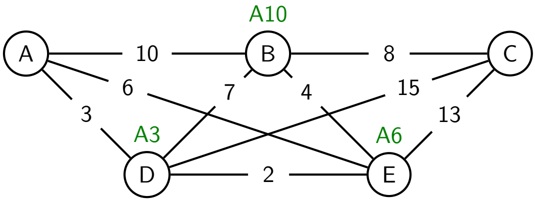

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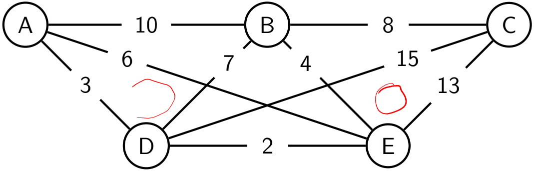
Dijkstra's algorithm has some similarities to Kruskal's algorithm for finding a minimum spanning tree but is a little more complicated. So I will delay setting out the algorithm formally and launch straight in to demonstrating its use on the above example.

Example 1 — Slide 1

We seek a minimal weight path from A to C in the graph below.



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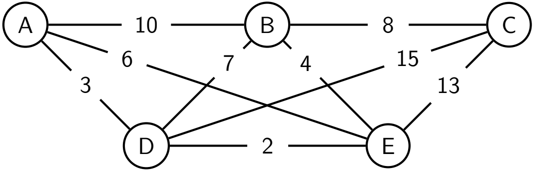

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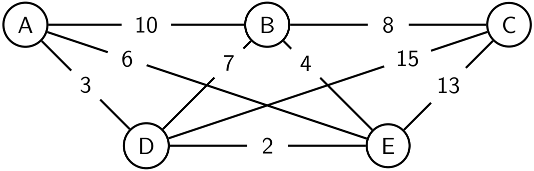

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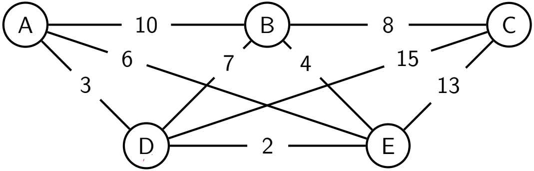

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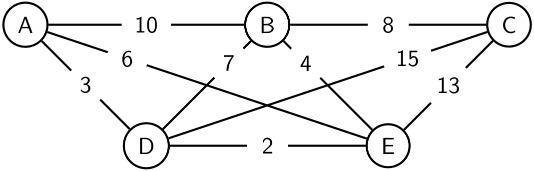

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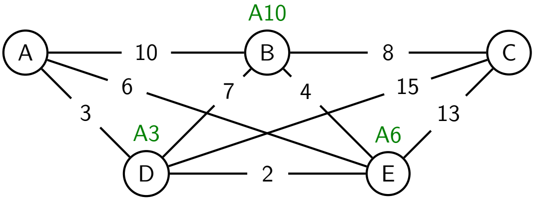

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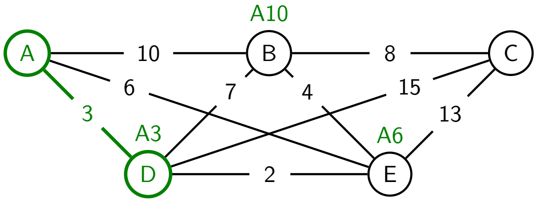

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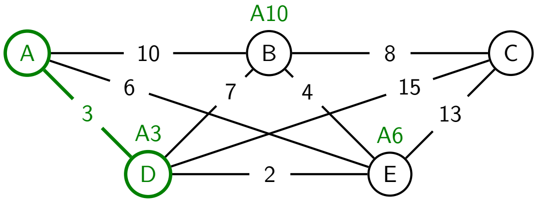

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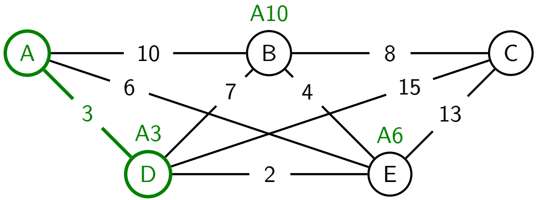

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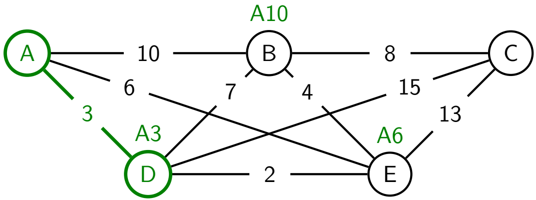

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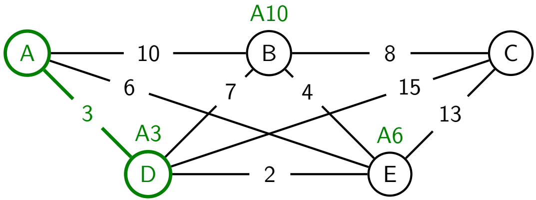

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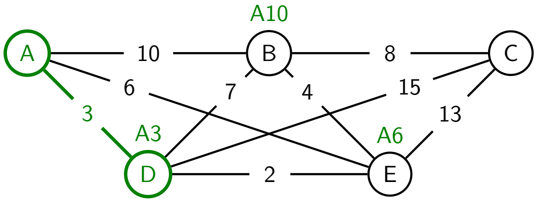

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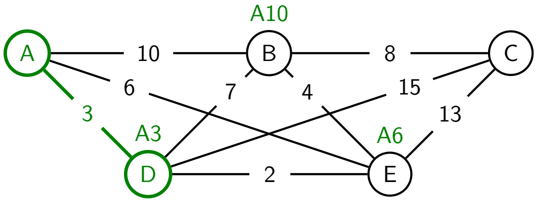

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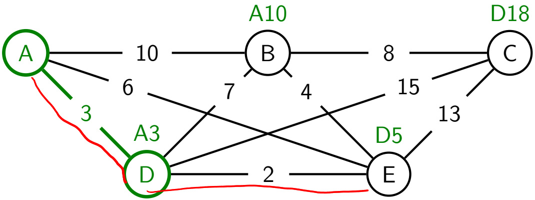
So we leave the A10 above B as it is.

The annotated graph now looks like this:



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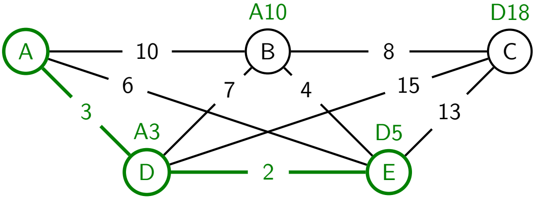

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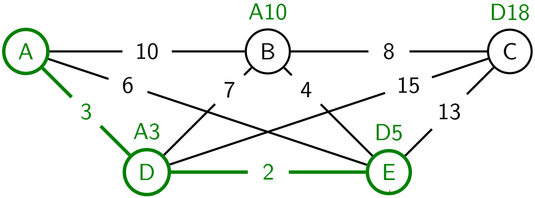

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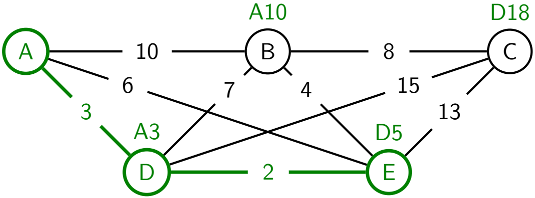

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We now repeat the process applied to the previous current vertex D.

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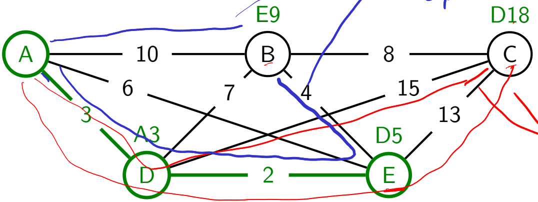

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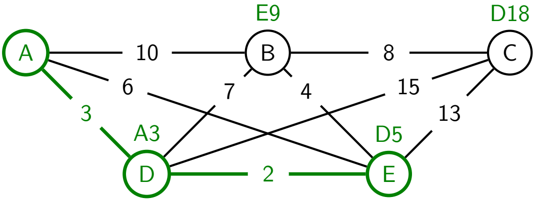
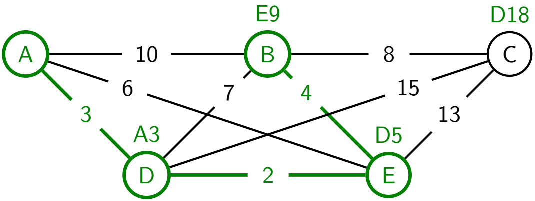
> Vertex C does not need updating since $5 + 13 = 18$, and C is already labelled 18.

The updated graph now looks like this:



The lowest fringe value is now 9 on B, so we lock in B and its lead-in edge EB (next slide).

Example 1 — Slide 6



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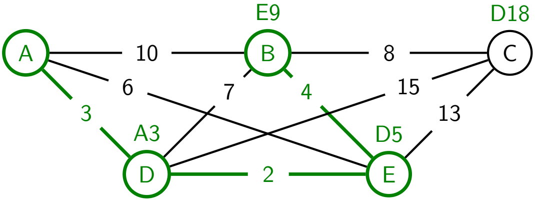
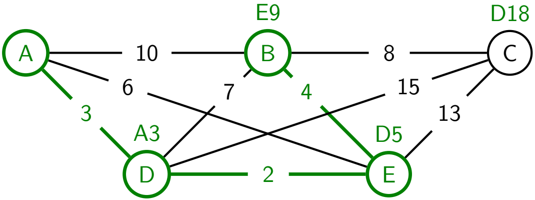


The new current vertex is the just locked-in B.

Example 1 — Slide 6



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There is only one vertex adjacent to B that has not already been locked in, namely C.

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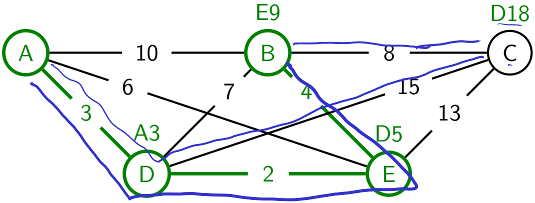

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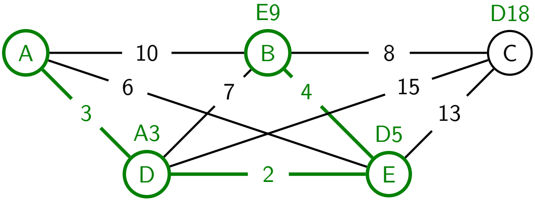

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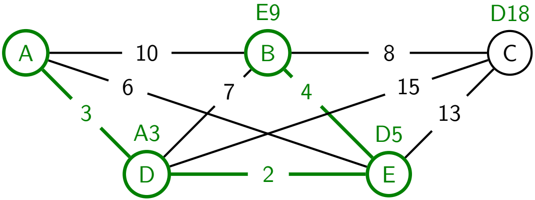

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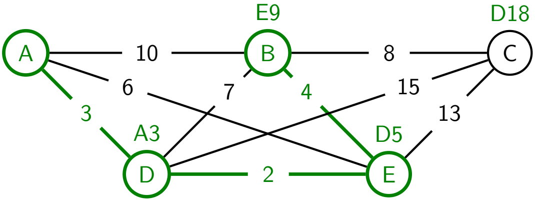

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We have now locked in the minimal distance 17 into our 'target' vertex C, so we can stop.

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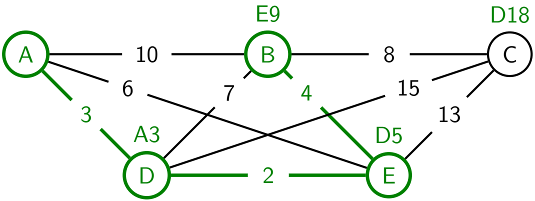

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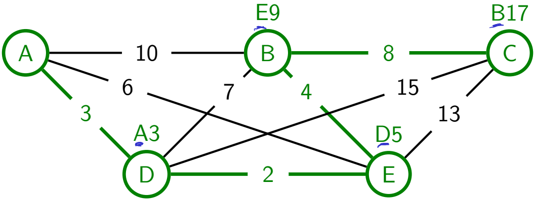
Example 1 — Slide 7; Results and Comments



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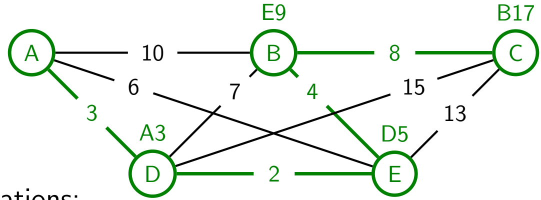

Some Observations:

- Besides the shortest path from A to C, the solution provides the shortest path to all the vertices along that path. For this example that happens to be the entire vertex set.

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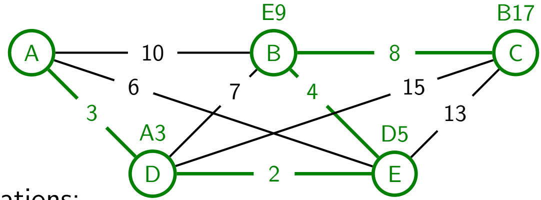

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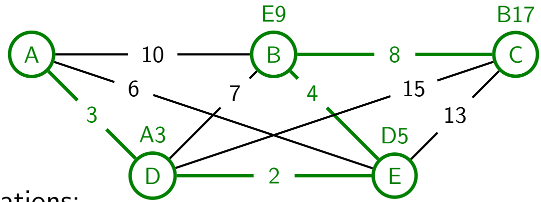

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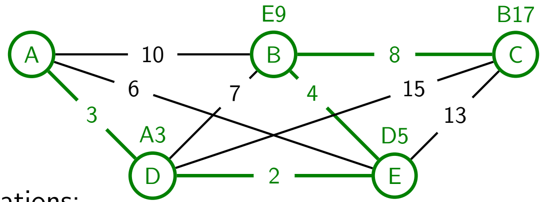

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- With all vertices locked, the solution provides a spanning tree for the graph.

Dijkstra's Algorithm — A Formal Description

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   remembering best-to-date vertex to pass through before v.
3. Set $L(\mathrm{A}) = 0$. Set 'current vertex' c to A.

While $c \neq \mathrm{Z}$:

4. For each vertex v adjacent to c but not in T:
   If v is unmarked (*i.e.* $M(v) = \text{blank}$)
   or if $L(v) > \underline{L(c)} + \mathrm{dist}(\{c, v\})$
   set $\underline{M(v) = c}$, $L(v) = L(c) + \mathrm{dist}(\{c, v\})$.

i.e. best to date path is via c

Dijkstra's Algorithm — A Formal Description (cont.)

5. From all marked $v \in G \setminus T$ (*i.e.* $M(v) \neq \text{blank}$ and $v \notin T$)
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This completes the formal description of Dijkstra's shortest path algorithm.

Note: we add vertices + edges anywhere along the tree, not just off the current vertex.

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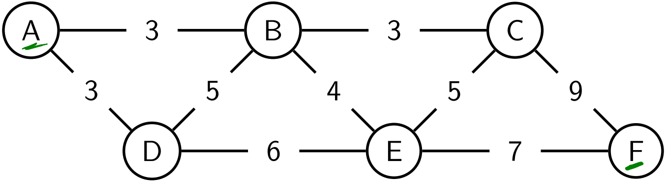
Next a second example, but this time with less commentary.

Example 2 – Slide 1

Find the shortest path for A to F:



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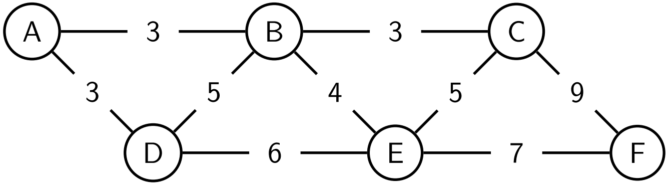

First annotate the vertices adjacent to the start vertex A:

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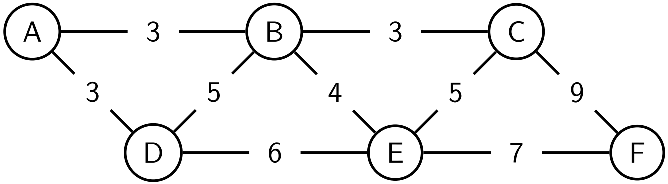

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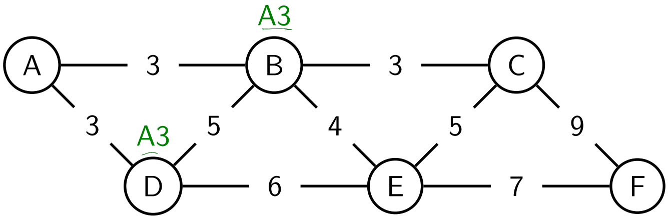

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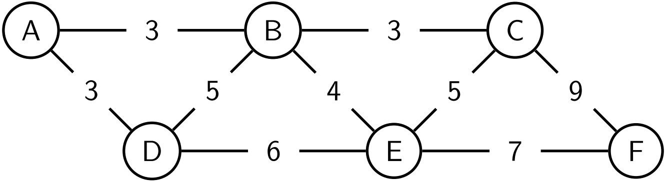

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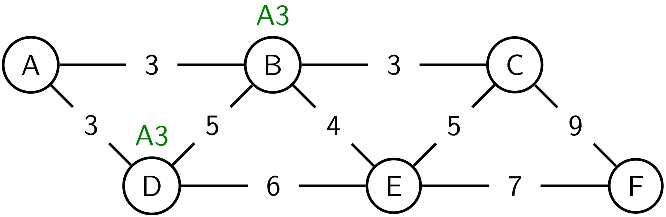
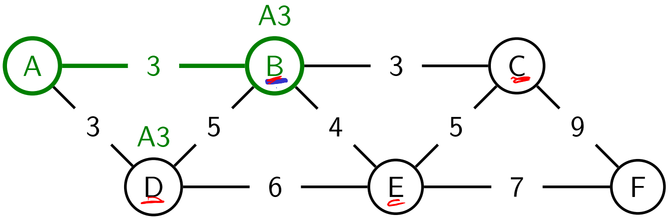


Vertices B and D have equal lowest label; let's lock in B:

Example 2 – Slide 2



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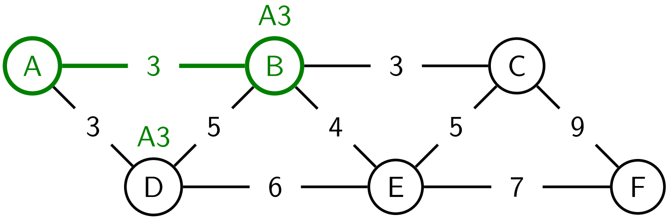

Current vertex is now B. Fringe vertices will be C,D,E.

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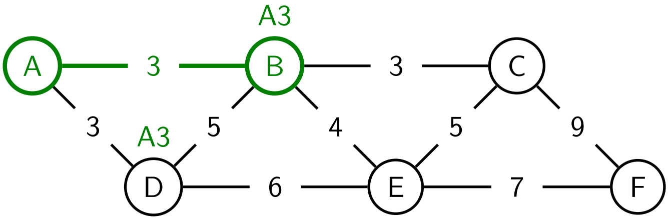

Current vertex is now B. Fringe vertices will be C,D,E.
Annotations required for C and E but D's does not need updating.

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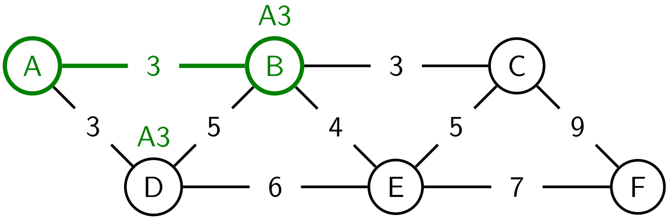

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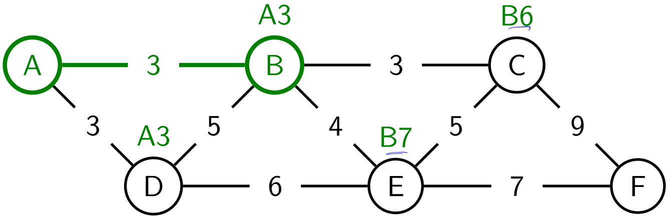

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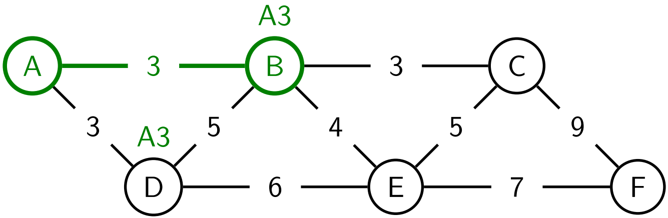

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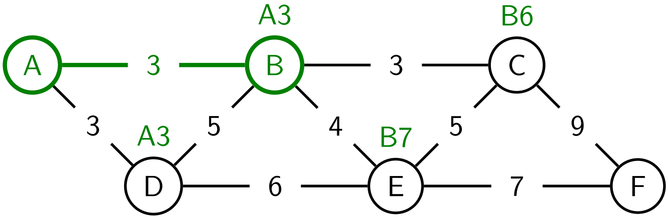
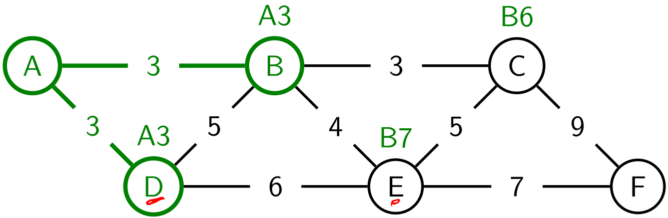


D now has lowest label so needs locking in next:

Example 2 – Slide 3



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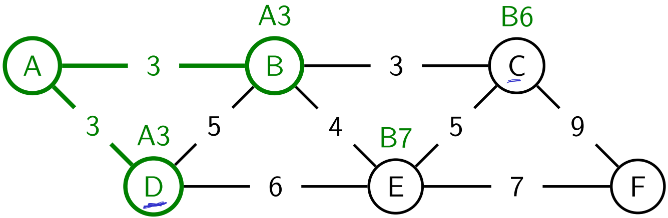
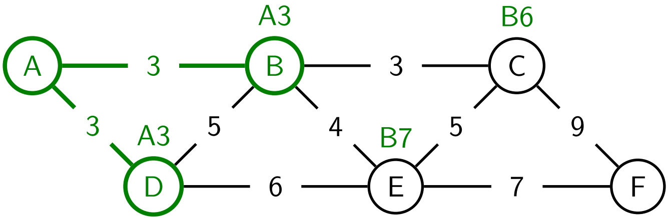


Current vertex is now D. Its only un-locked neighbour is E but no updating is required.

Example 2 – Slide 3



Current vertex is now D. Its only un-locked neighbour is E but no updating is required. The fringe vertices are C and E. Vertex C has lower label value so it is next to be locked:

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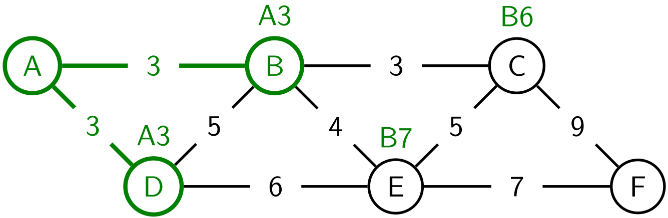

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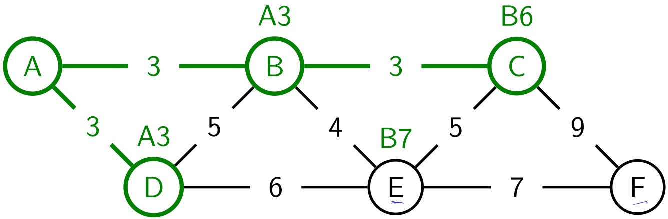

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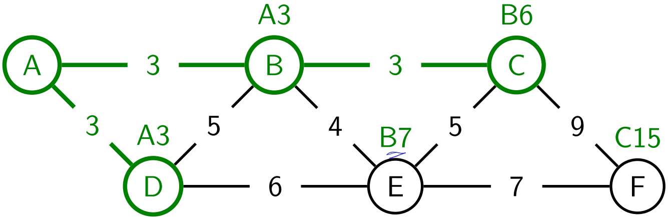

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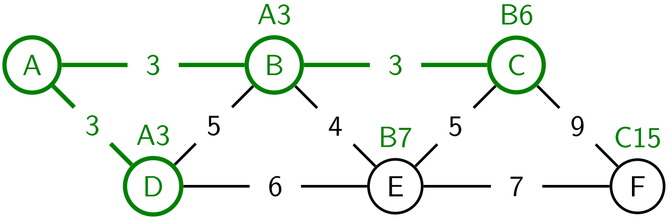
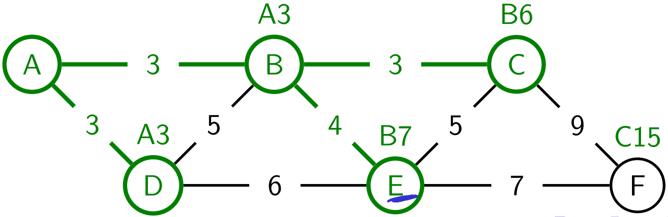


Of the two fringe vertices, E has the lower label value so is locked in. Its lead-in vertex is marked as B, so edge BE is also locked in.



Example 2 – Slide 5

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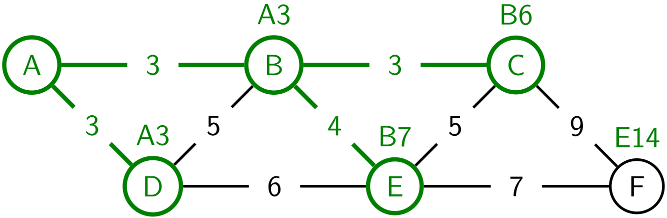

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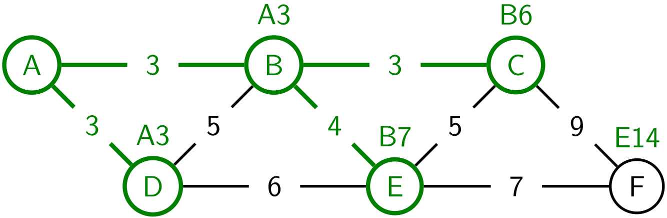
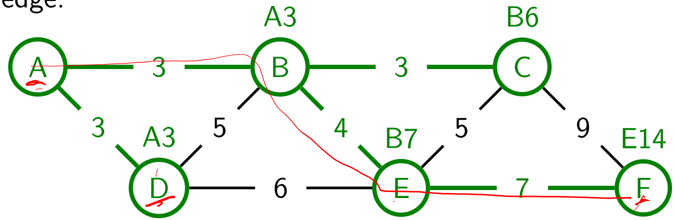


Now F is the only fringe vertex, so F is locked in, together with its lead-in edge.



Example 2 — Slide 6; Results and Comment

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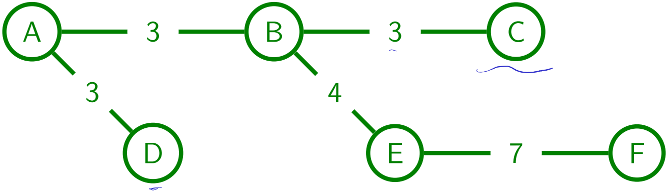

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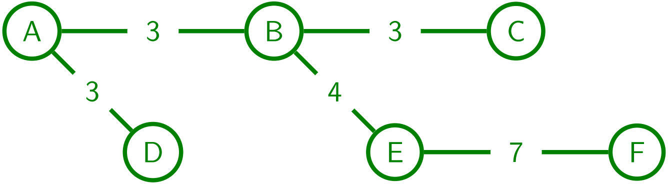

As it happens, this is a minimal spanning tree. However, in general a spanning tree produced by Dijkstra's algorithm will not be minimial.

Transport networks

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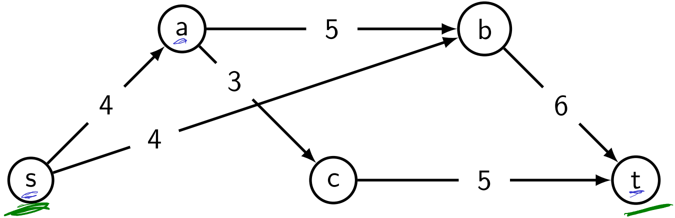

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e.g. pipe diameter
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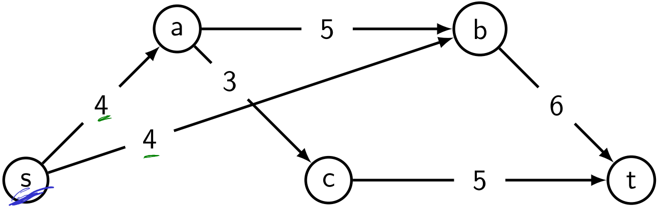

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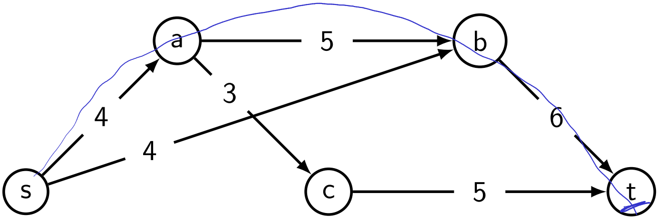

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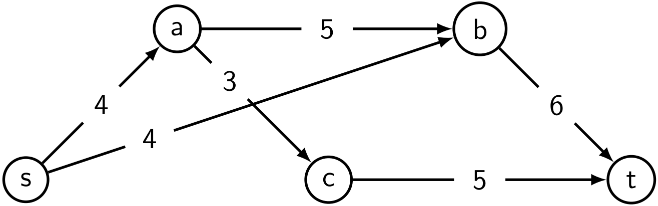

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For any flow F the constrains on the network imply that the total flow out of the source must equal the total flow into the target:

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At stage i, flow F_i is constructed as $F_i = F_{i-1} + f_i$, where the incremental flow f_i is based on a constant $k_i \in \mathbb{Q}^+$ and a simple path p_i from s to t:

$$f_i(e) = \begin{cases} k_i & \text{for every edge } e \text{ on the path } p_i \\ 0 & \text{for evey other edge } e. \end{cases}$$

Spare capacity

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Given a transport network D with capacity and flow functions C, F, we know that $F(e) \leq C(e)$ for every edge $e \in E(D)$. I will call the non-negative difference $S(e) = C(e) - F(e)$ the **spare capacity** of the edge e. (Some authors use the term "excess capacity".)

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To depict a flow I will follow the capacity value $C(e)$ on each (directed) edge e with the flow value $F(e)$ for that edge. For example

$$\text{x} \xrightarrow{\underset{C}{5},\underset{F}{3}} \text{y}$$

represents a flow of 3 in the edge from x to y, with spare capacity 2.

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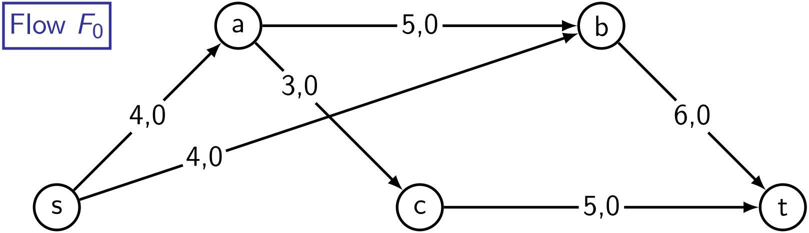

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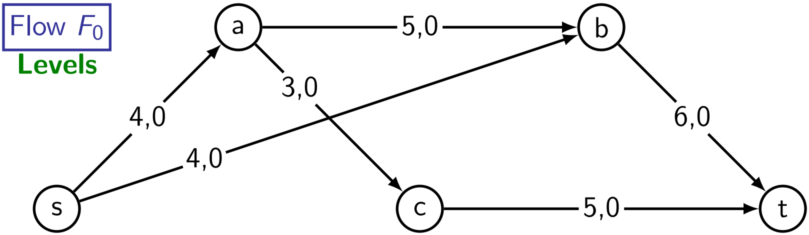

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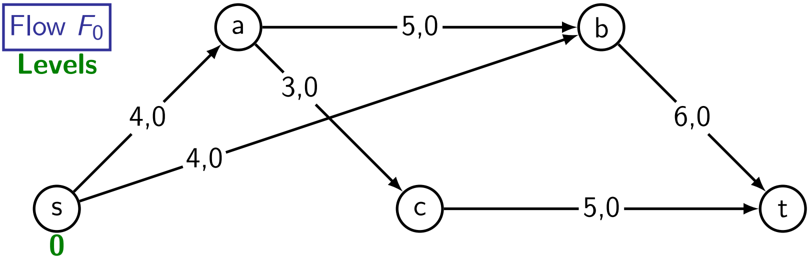

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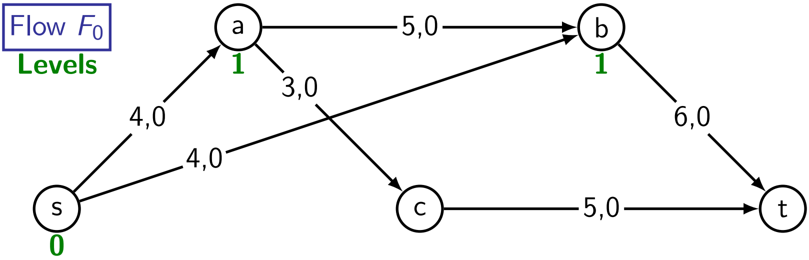

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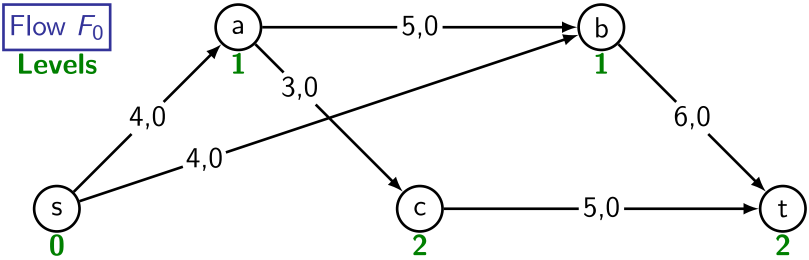

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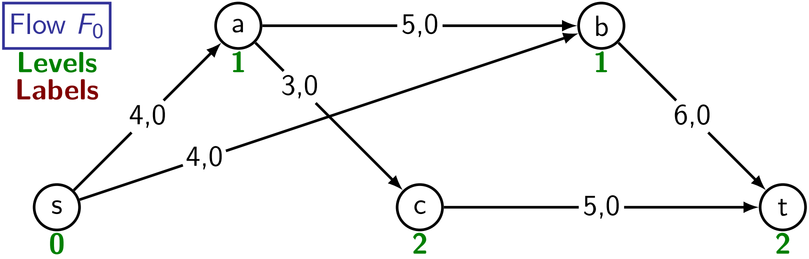

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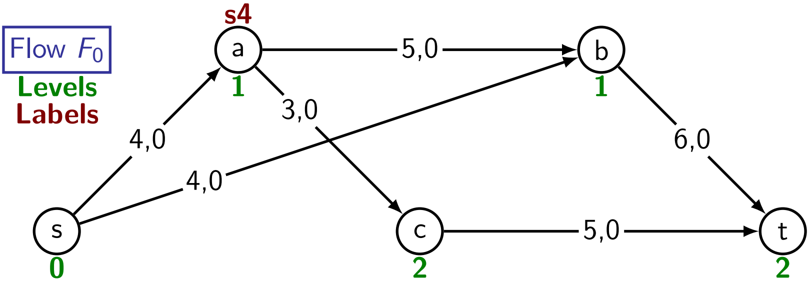

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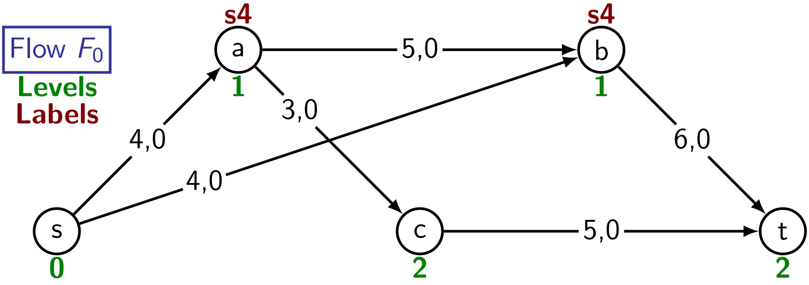

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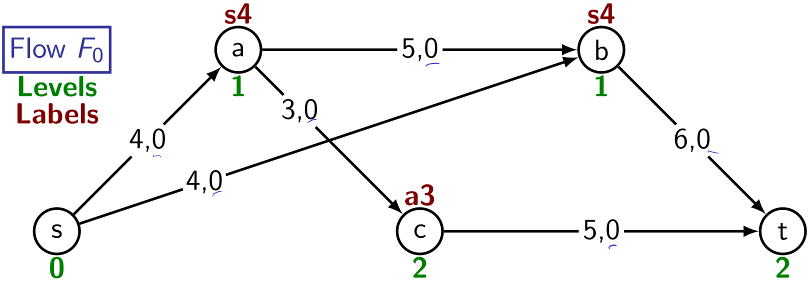

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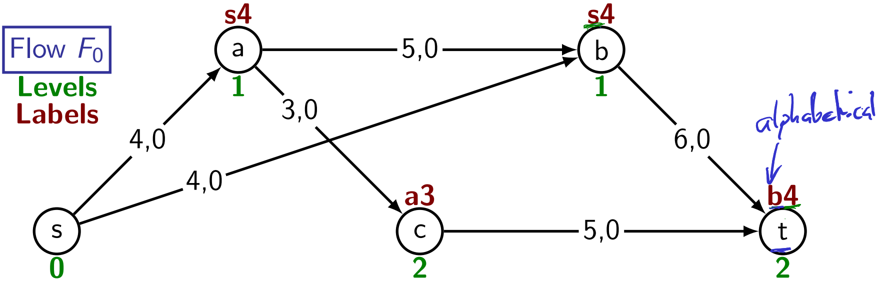

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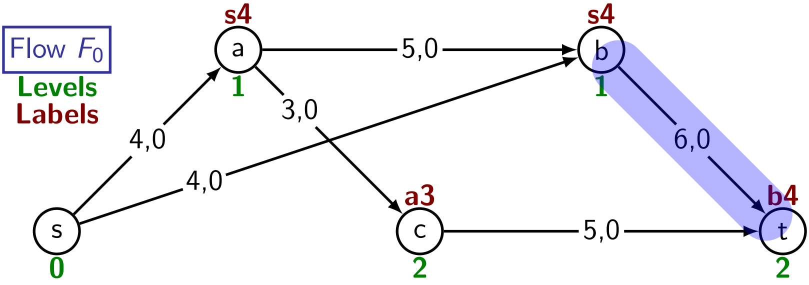

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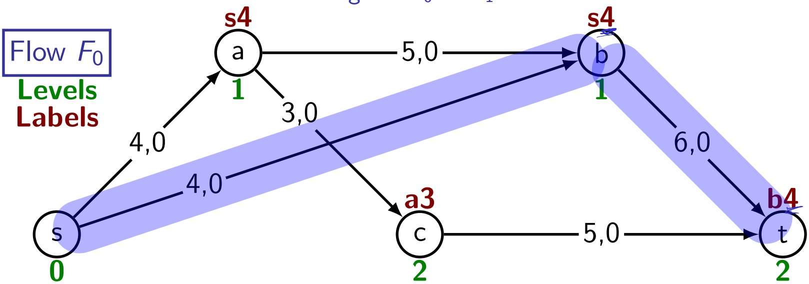

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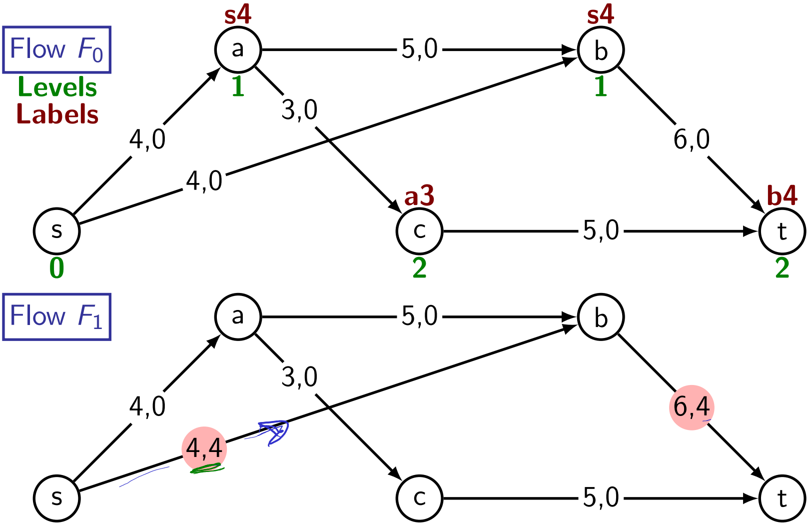

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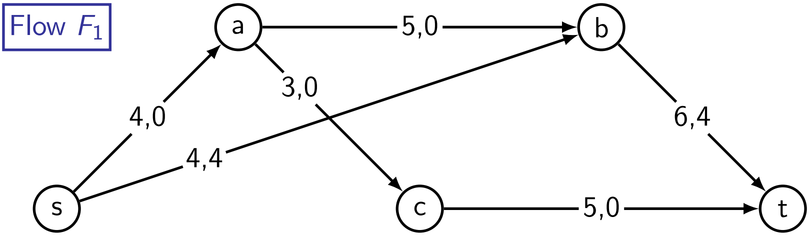

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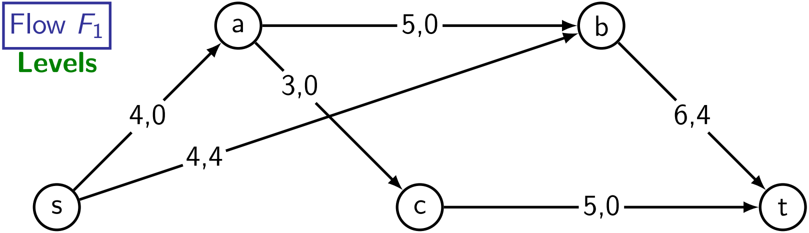

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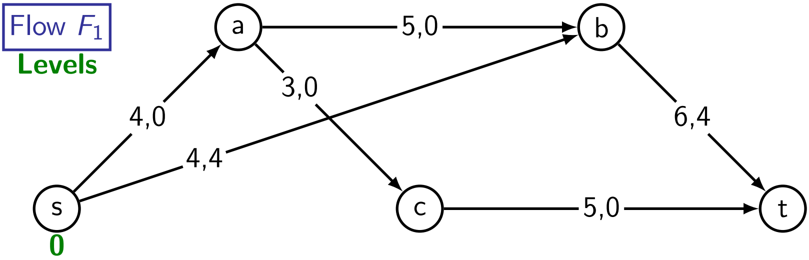

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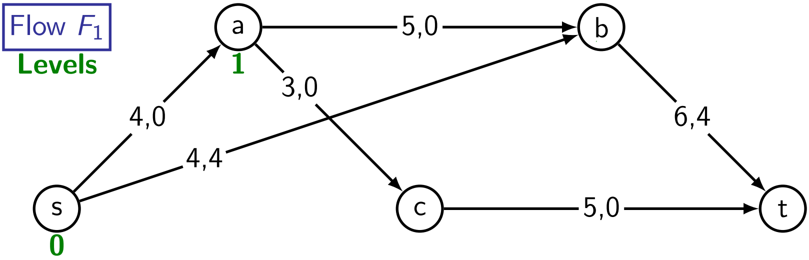

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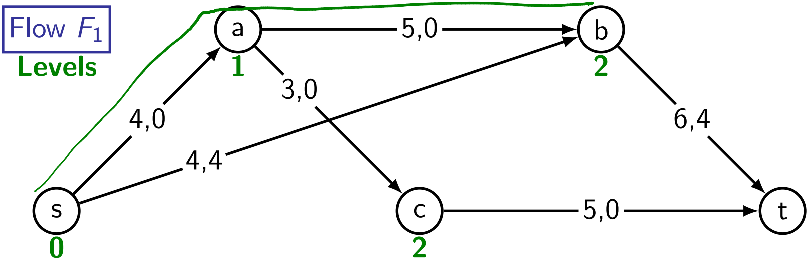

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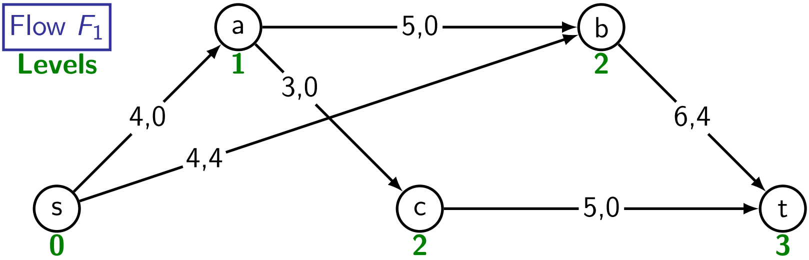

Vertex labelling algorithm, Example 1

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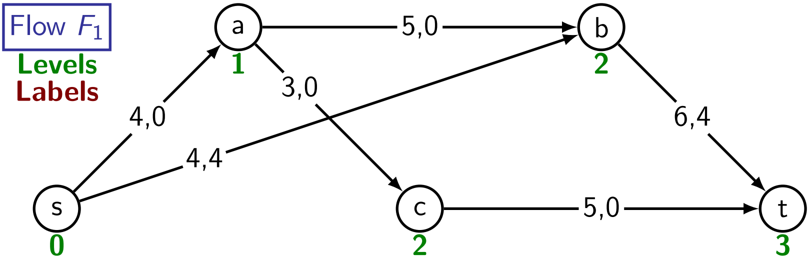

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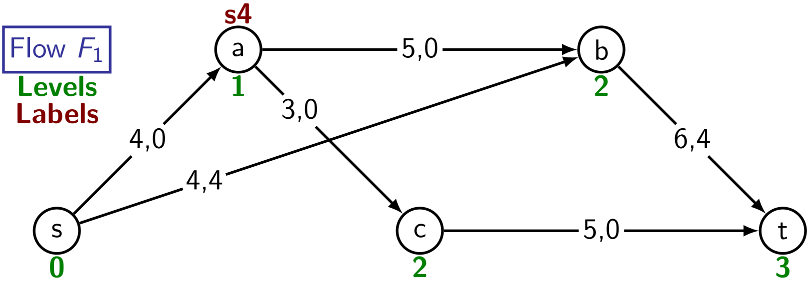

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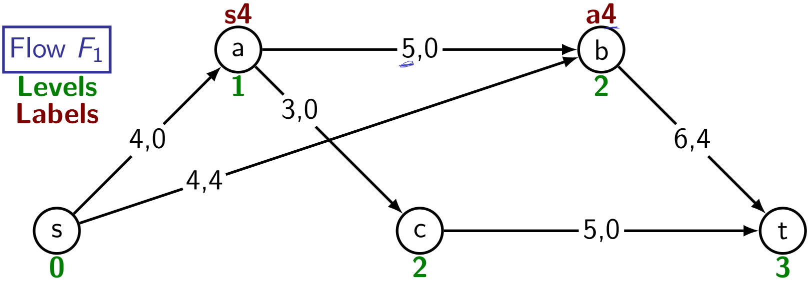

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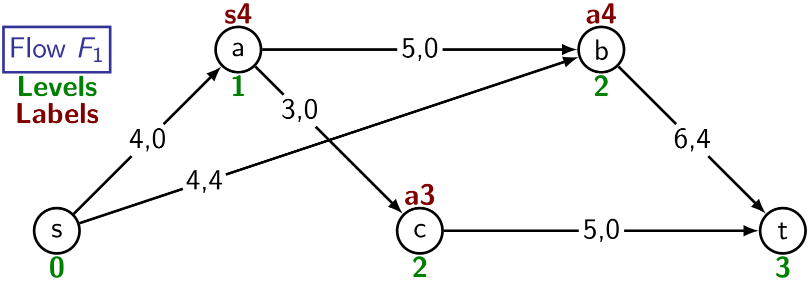

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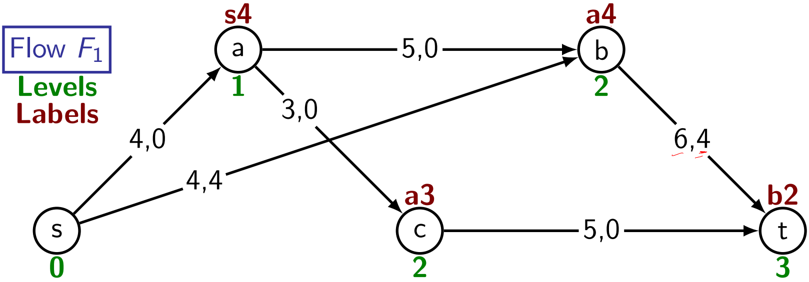

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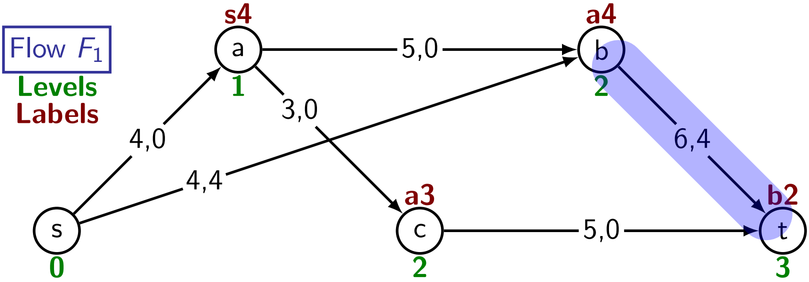

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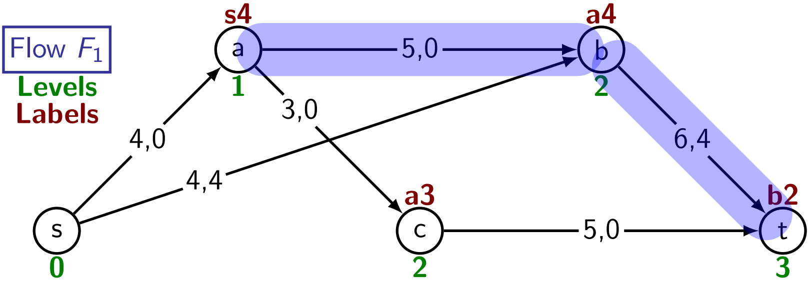

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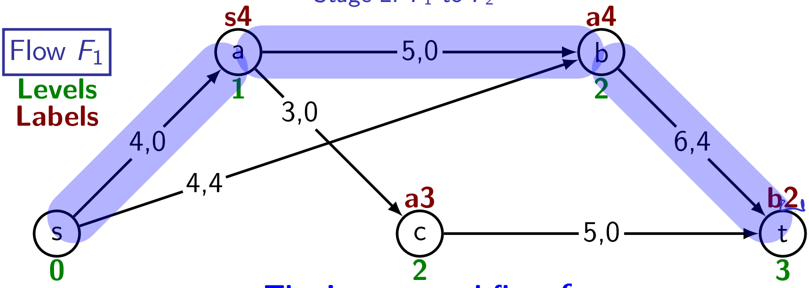


The incremental flow f_2

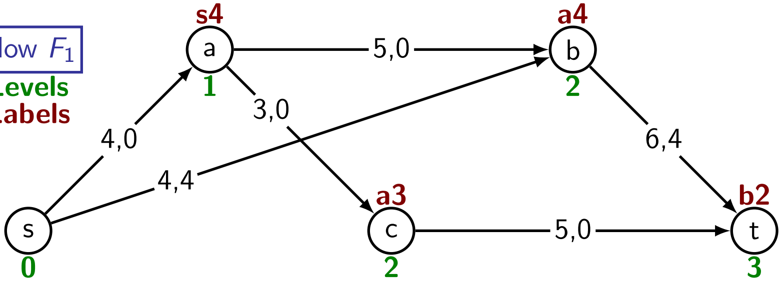
Vertex labelling algorithm, Example 1

Stage 2: F_1 to F_2

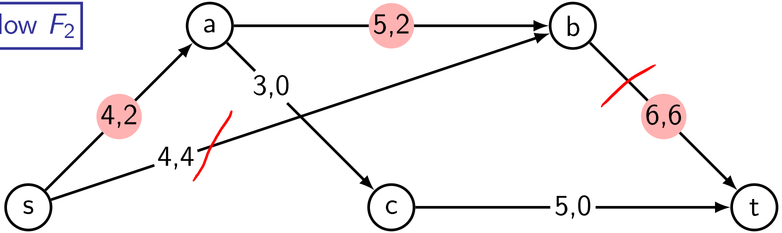
Flow F_1

Levels

Labels

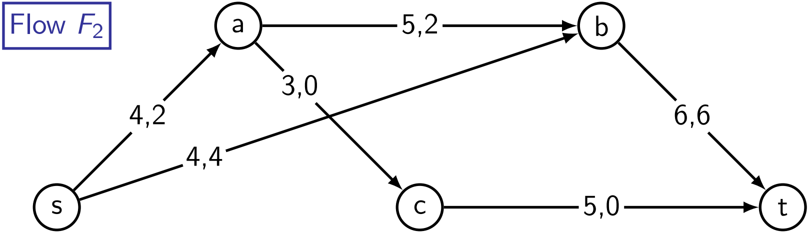


Flow F_2



Vertex labelling algorithm, Example 1

Stage 3: F_2 to F_3



Vertex labelling algorithm, Example 1

Stage 3: F_2 to F_3

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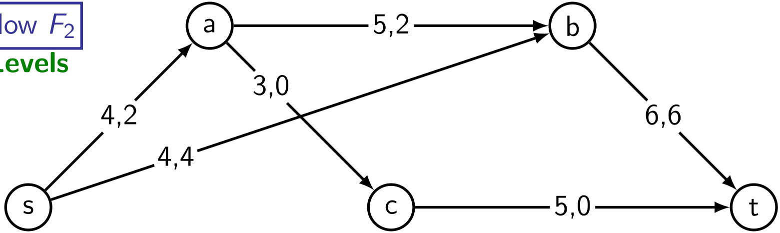
Levels



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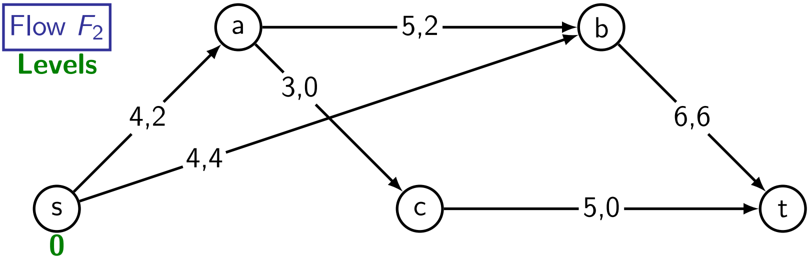

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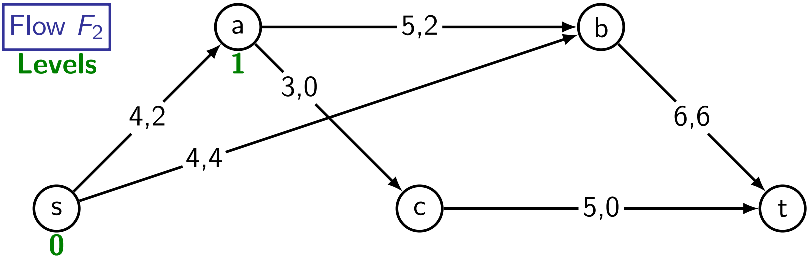

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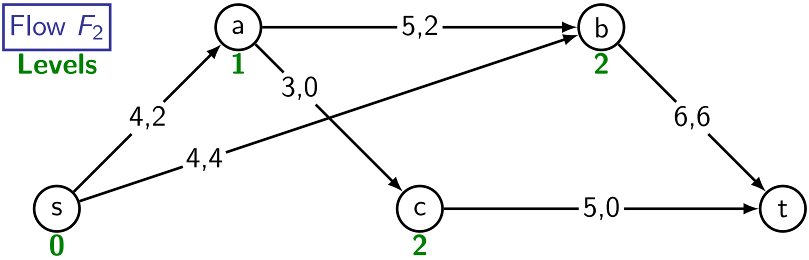

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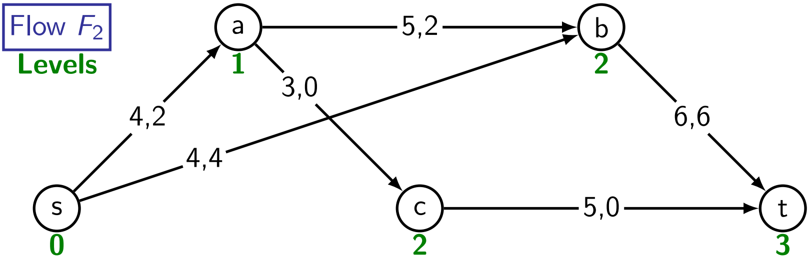

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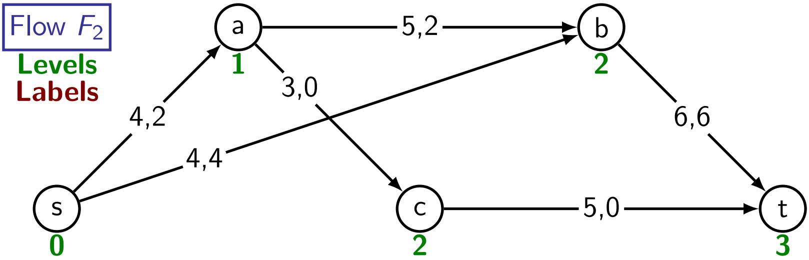

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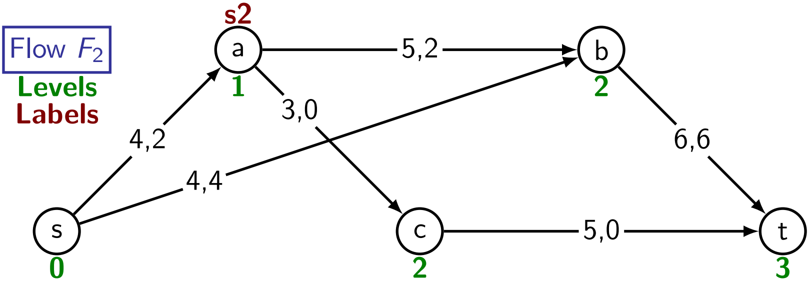

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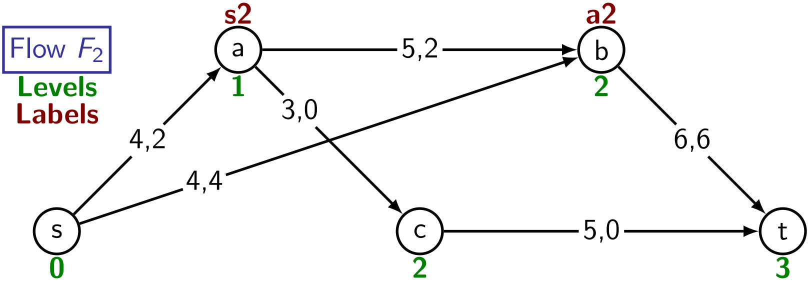

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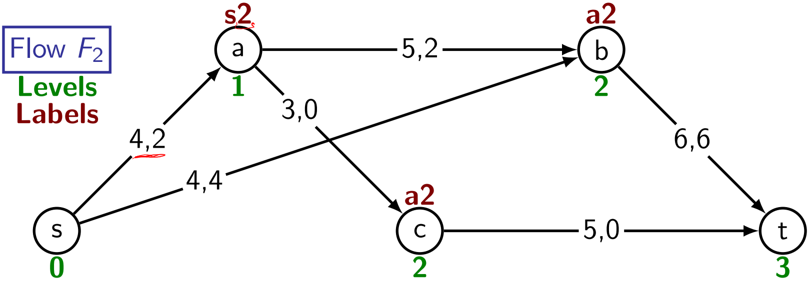

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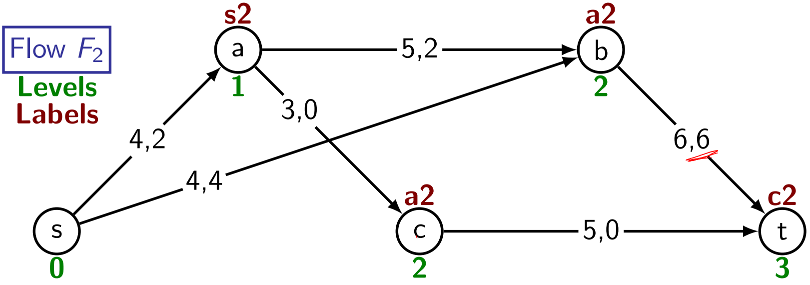

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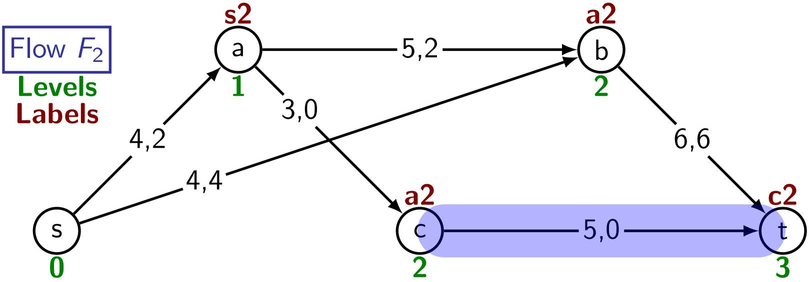

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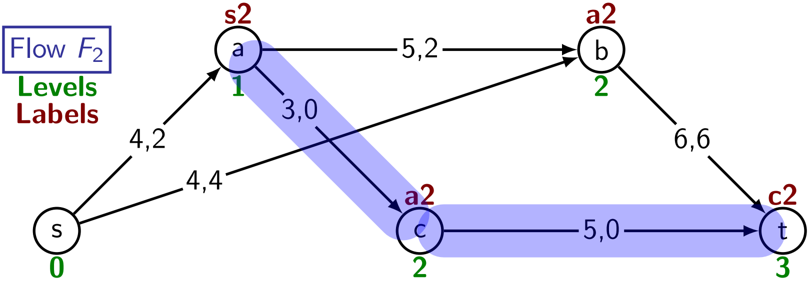

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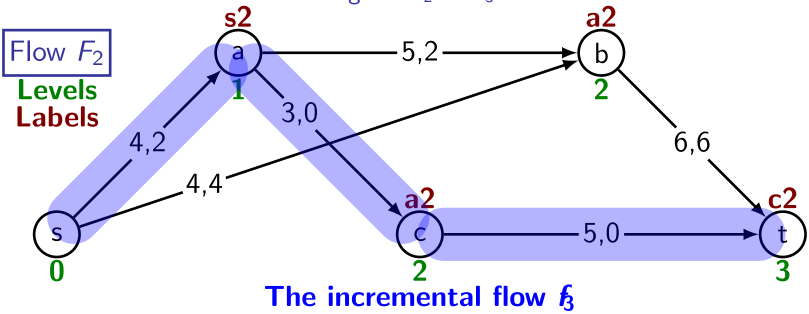

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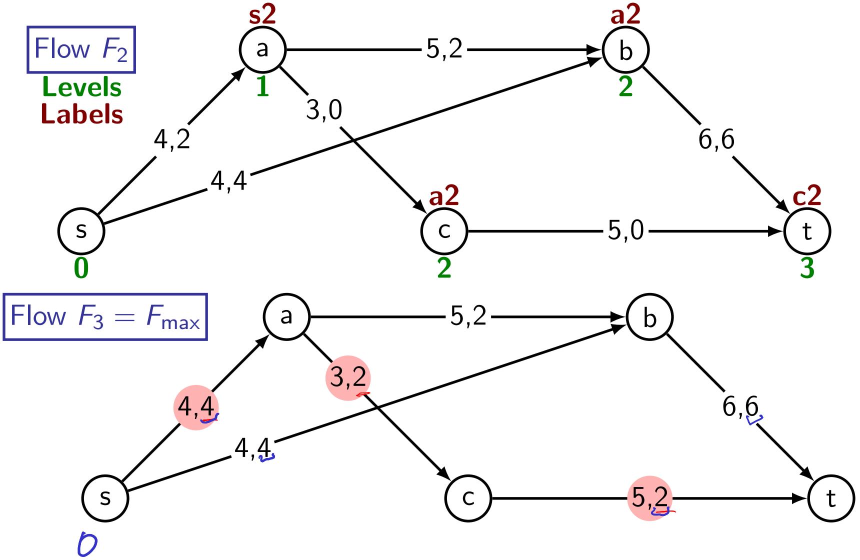

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3. Next slide....

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5. Let p_i be the path $\mathrm{u}_0\mathrm{u}_1 \ldots \mathrm{u}_n$ where $\mathrm{u}_n = \mathrm{t}$ and for $0 < j \leq n$ u_j has label $\mathrm{u}_{j-1}k_j$.

   Define f_i to be the incremental flow on p_i with flow value k_n.

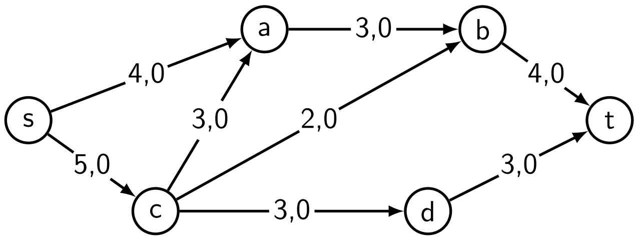
the amount which made it to the target

End of Method

Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

$\boxed{\text{Flow } F_0}$



Vertex labelling algorithm, Example 2

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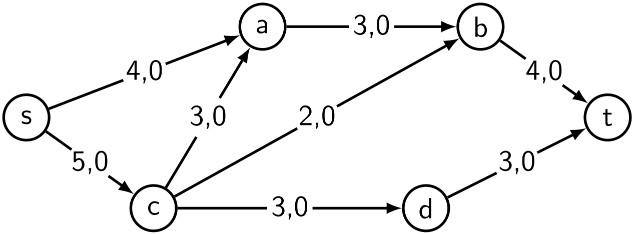
$\boxed{\text{Flow } F_0}$

Levels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

Flow F_0

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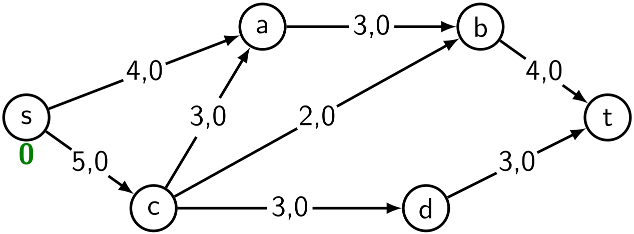

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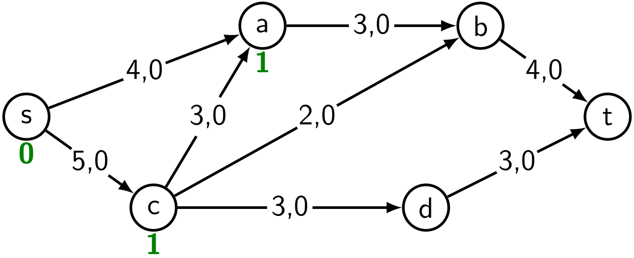

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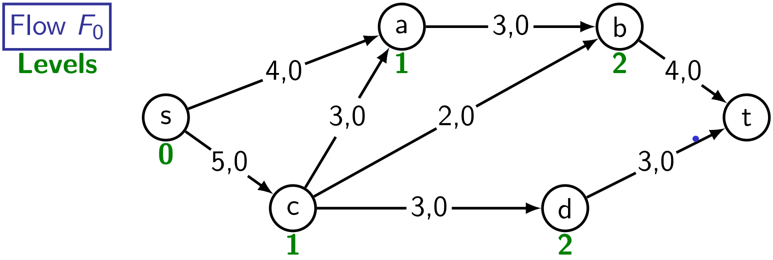

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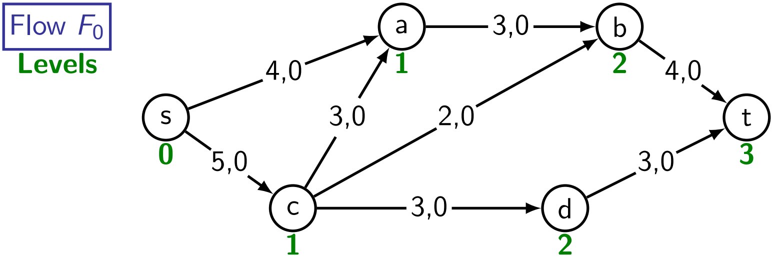

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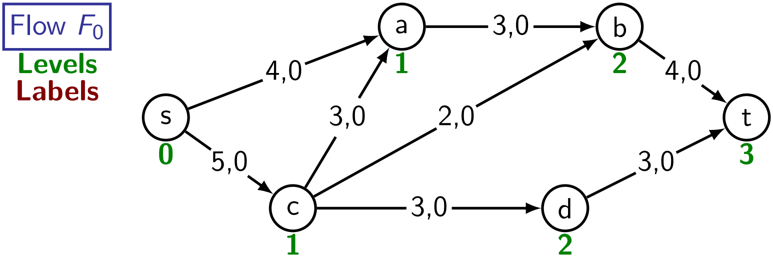

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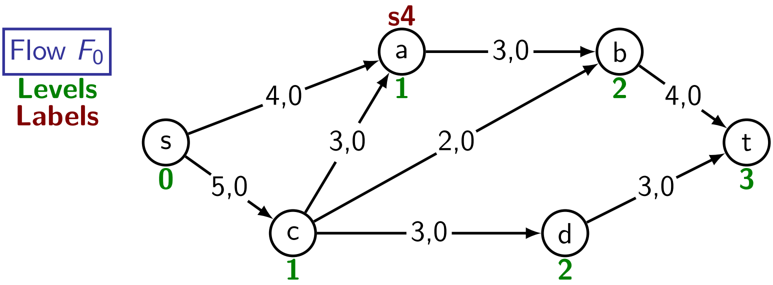

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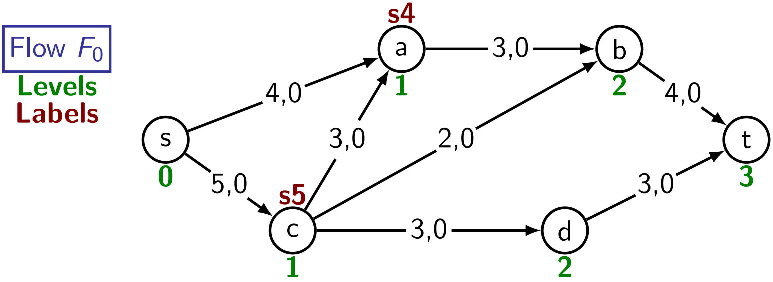

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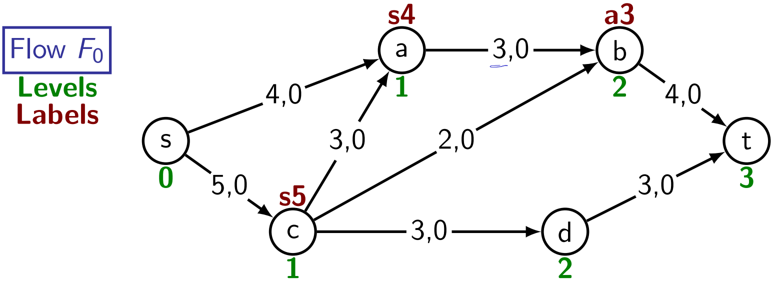

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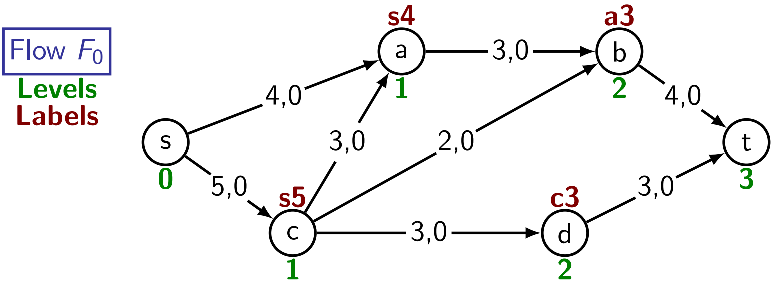

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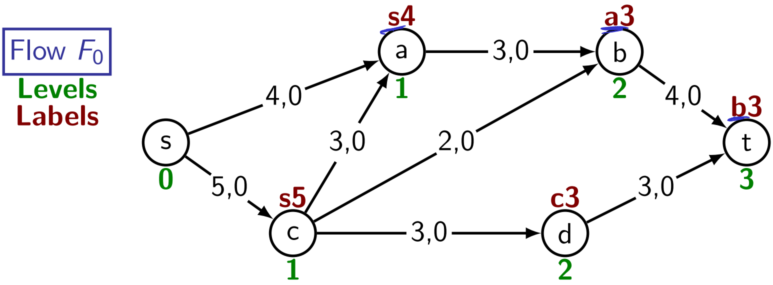

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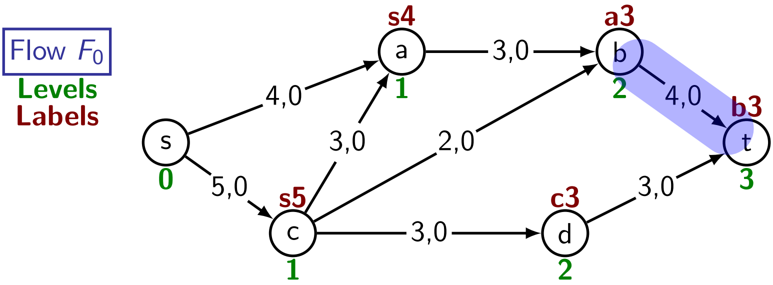

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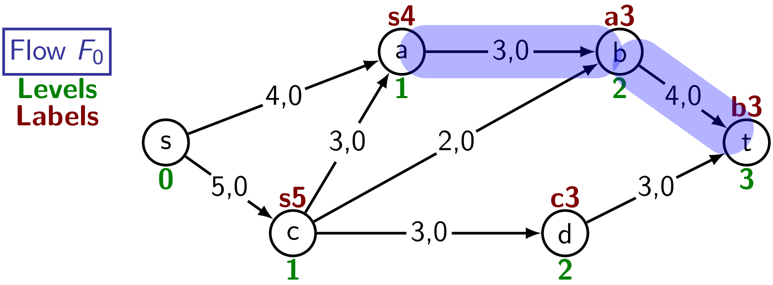

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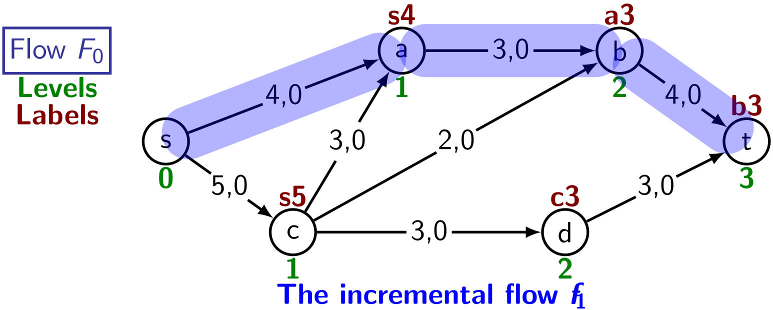

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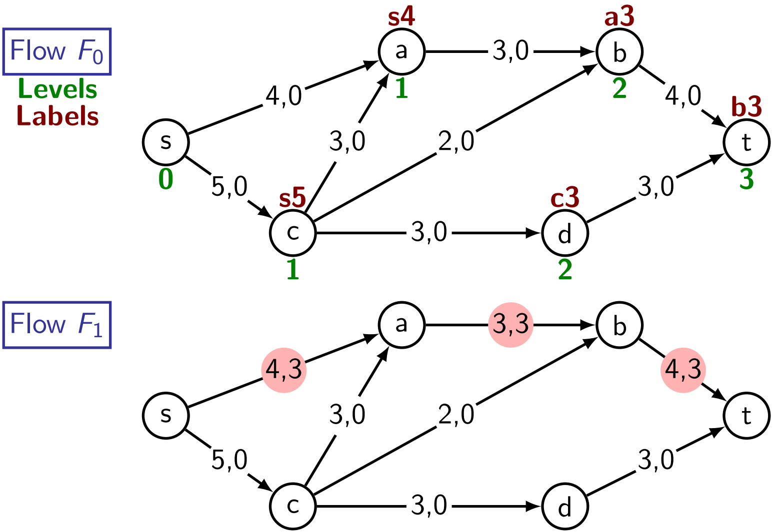

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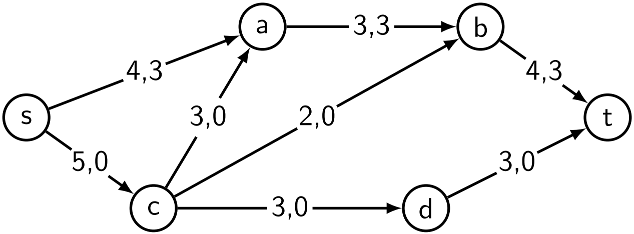
Stage 2: F_1 to F_2

Flow F_1



Vertex labelling algorithm, Example 2

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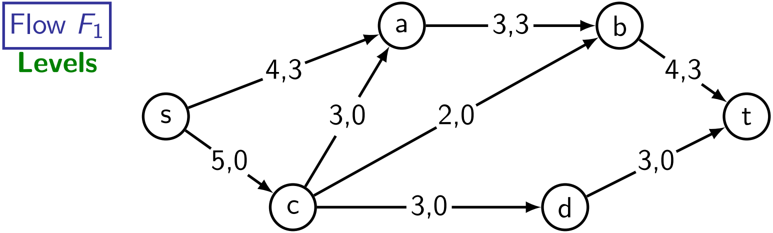
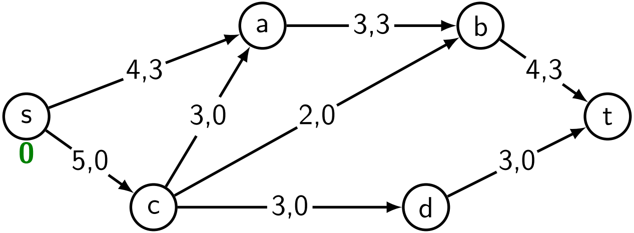


Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

Flow F_1

Levels



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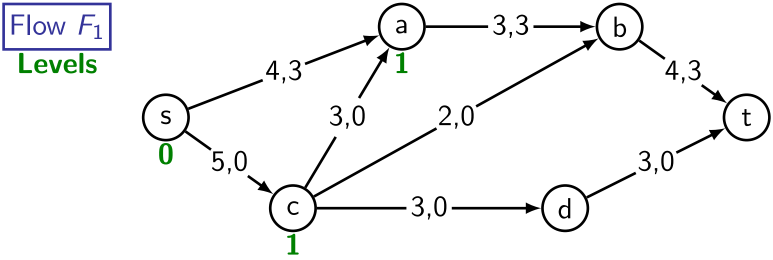

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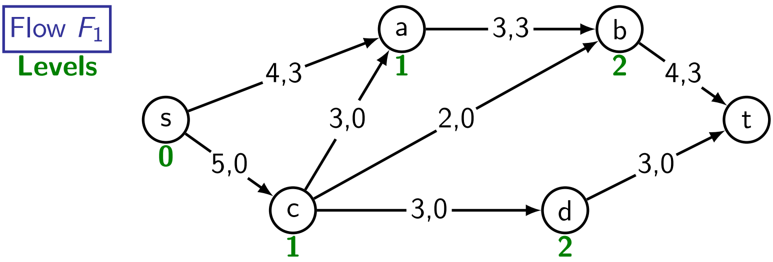

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$\boxed{\text{Flow } F_1}$

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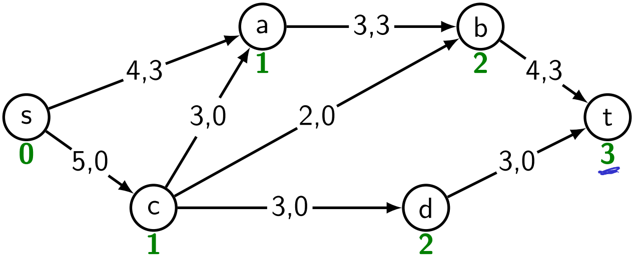

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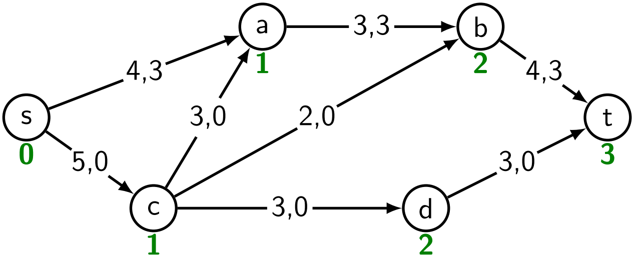
Flow F_1

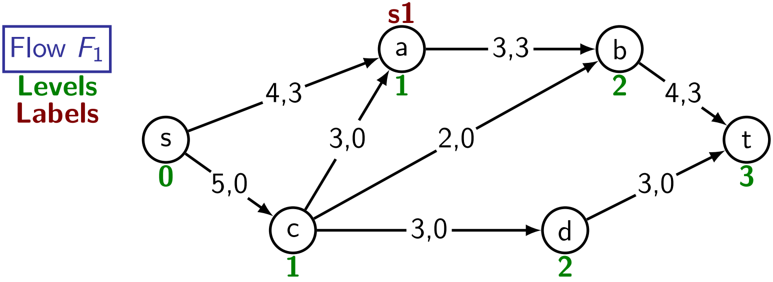
Levels

Labels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2



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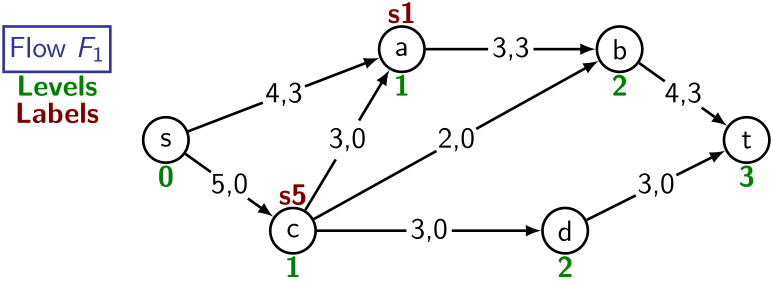

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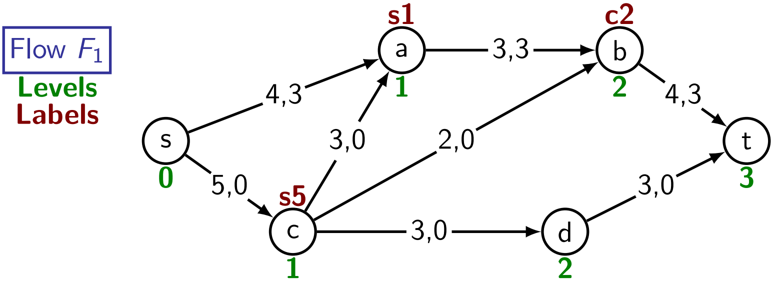

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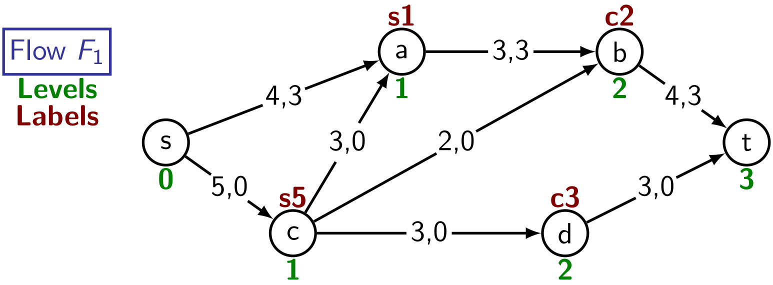

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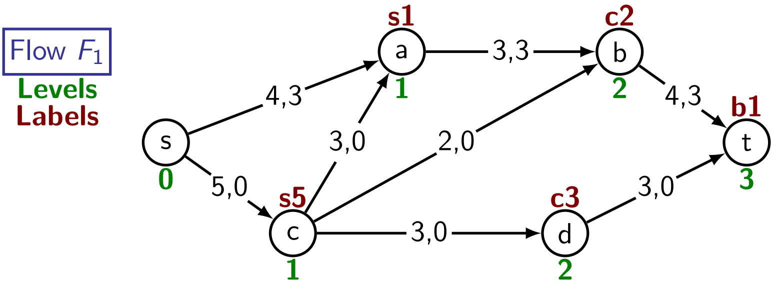

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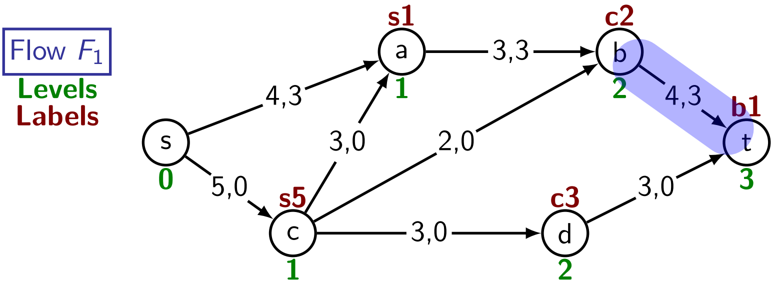

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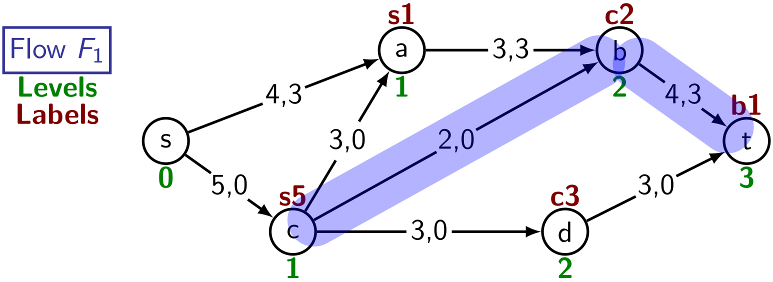

Vertex labelling algorithm, Example 2

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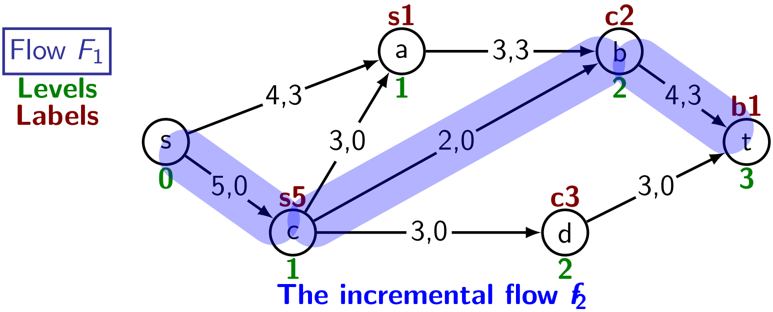

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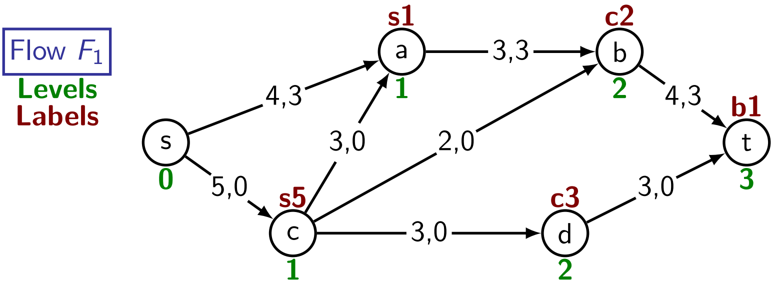
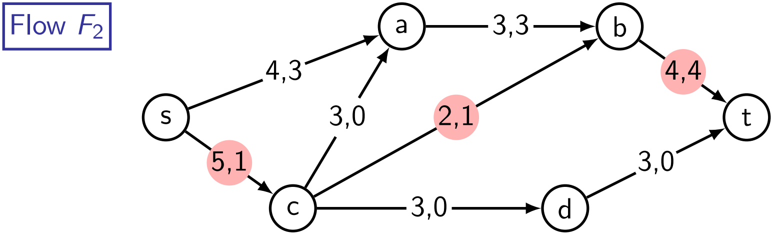
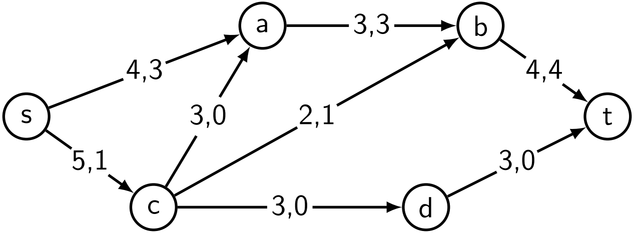




Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

Flow F_2

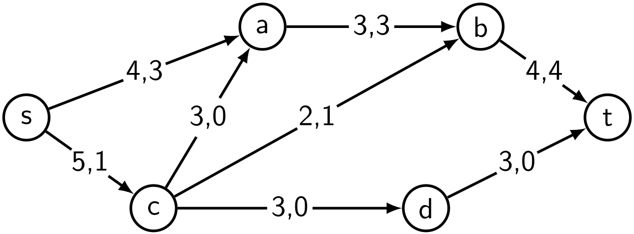


Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

$\boxed{\text{Flow } F_2}$

Levels



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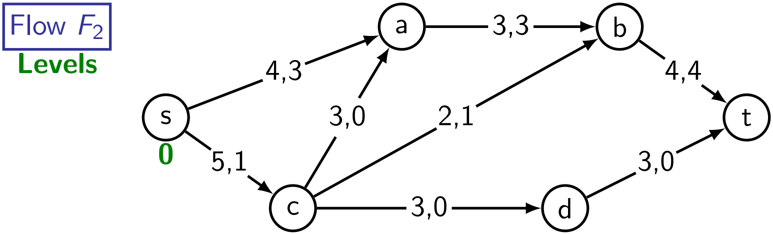

Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

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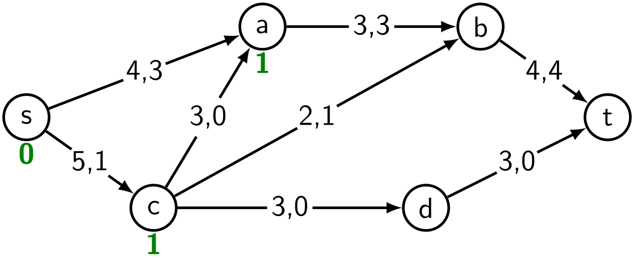
Levels



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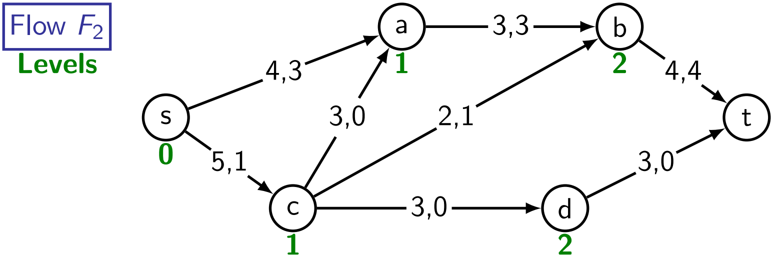

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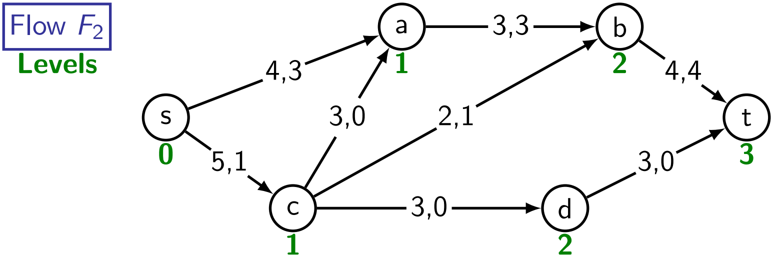

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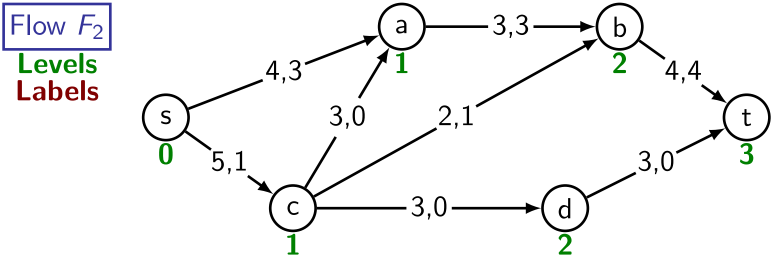

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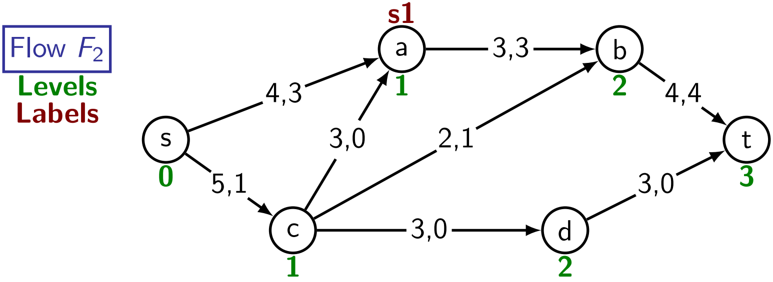

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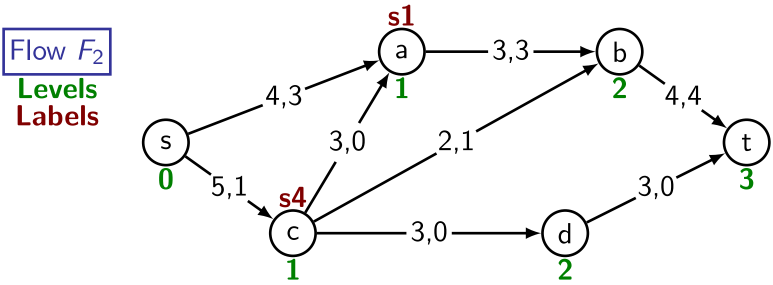

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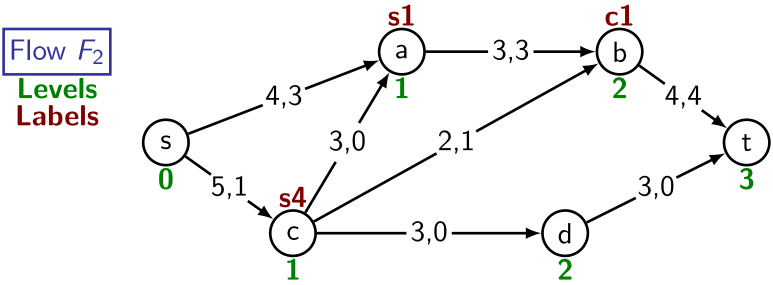

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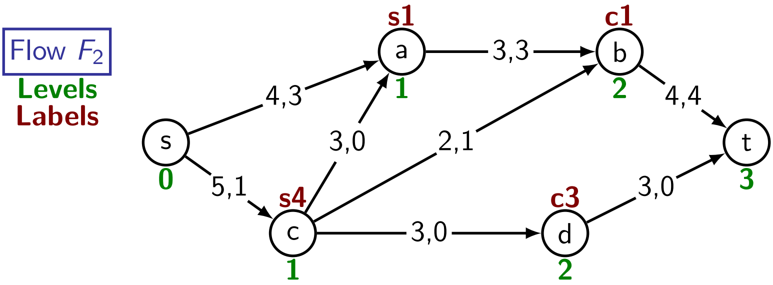

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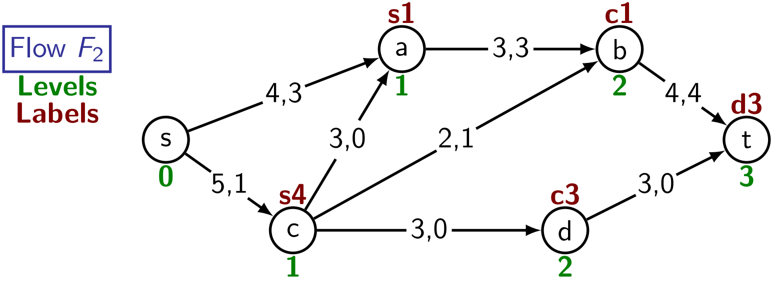

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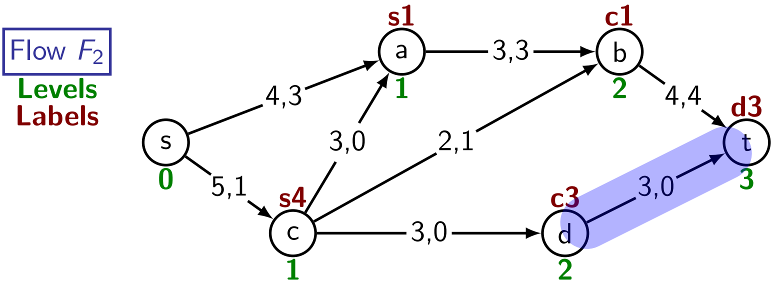

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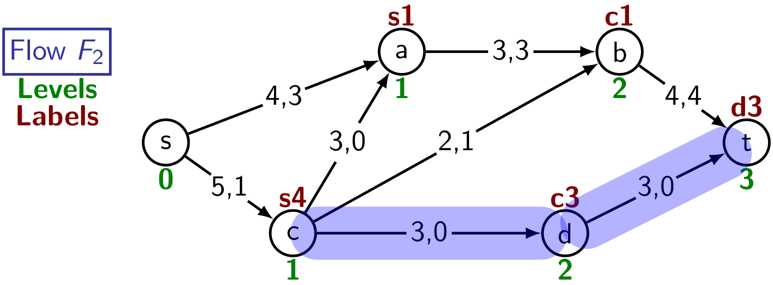

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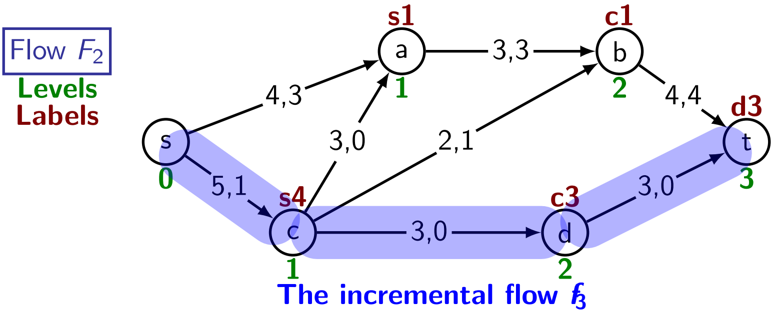

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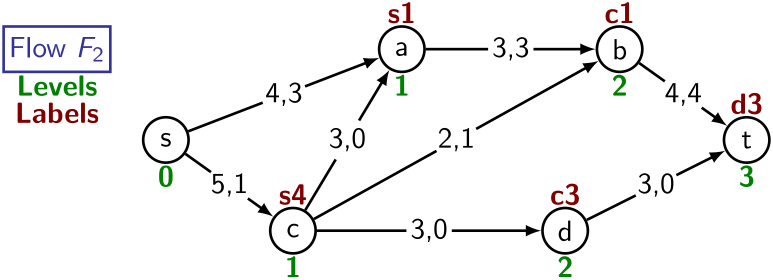
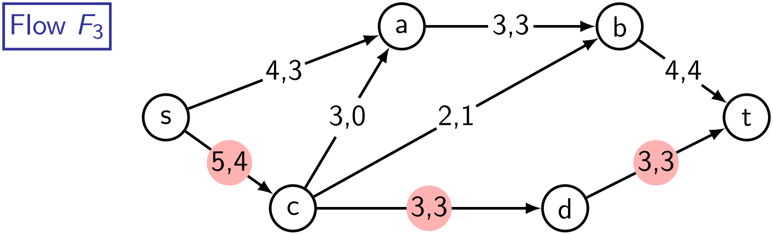
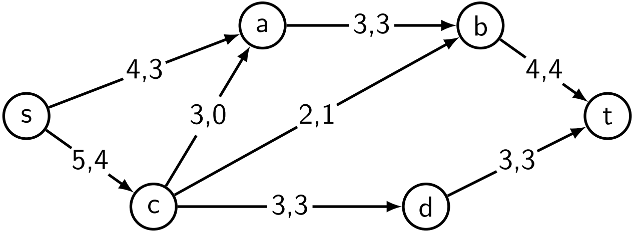




Vertex labelling algorithm, Example 2

Stage 4: F_3 is F_{max}

Flow F_3



Vertex labelling algorithm, Example 2

Stage 4: F_3 is $F_{\max}$

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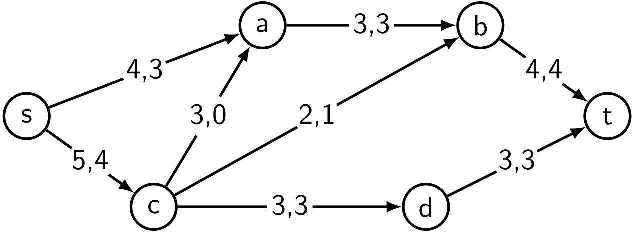
Levels



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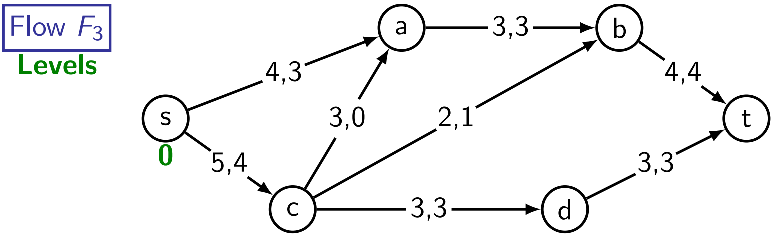

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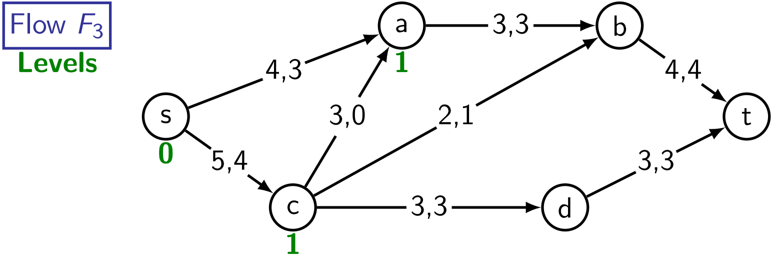

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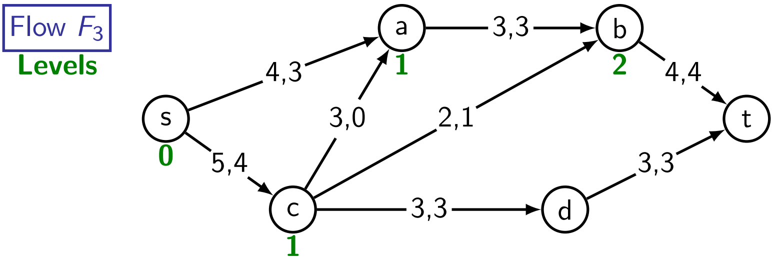

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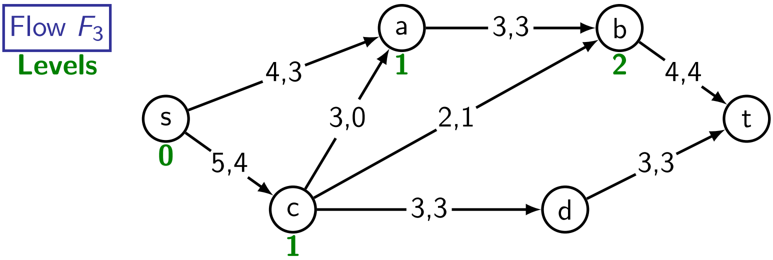

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Vertex labelling algorithm, Example 2

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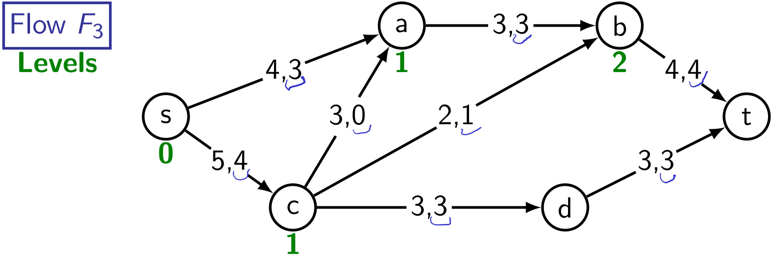

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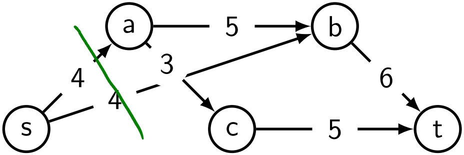
So the algorithm terminates with $F_{\max} = F_3$.

Cuts

Consider again Example 1 at right.



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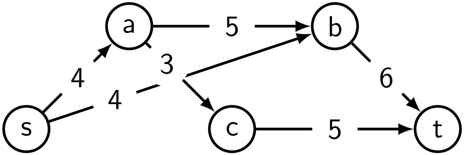

Since the total capacity of edges leading from the source is $4 + 4 = 8$ it is clear that the maximum flow value cannot be more than 8.

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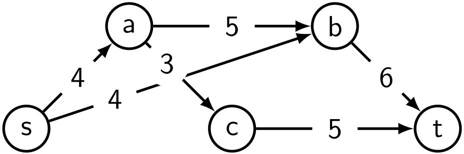

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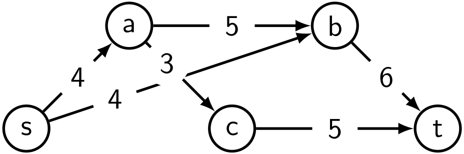

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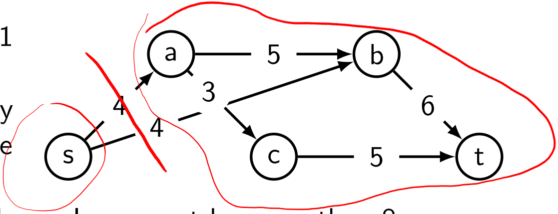

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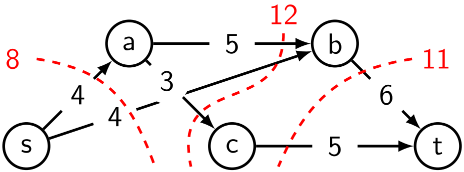

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Some examples, with values, are shown on the digraph above.

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Max flow min cut theorem: For any transport network:

$$\boxed{\textbf{maximum flow value} \quad = \quad \textbf{minimum cut capacity}}$$

Doesn't actually tell you how to construct F_{max}

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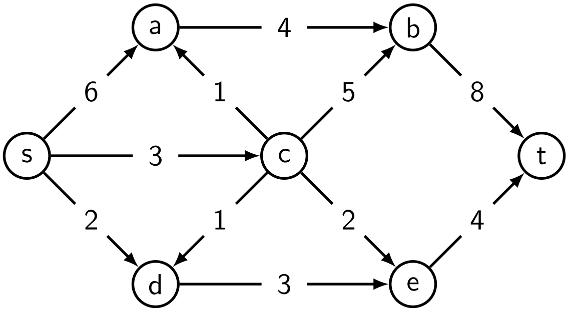
Though highly plausible, this theorem is little tricky to prove, and the proof will be omitted, as will the proof that the vertex labelling algorithm always finds a maximum flow.

Max flow min cut: Class example 1

What is the maximum flow value for this transport network?



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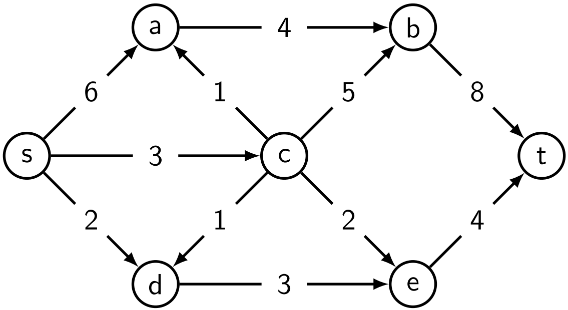

Answer:

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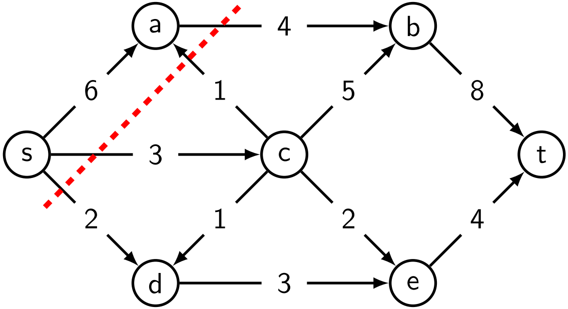

Answer: After some searching we find the minimum cut shown, for which $S = \{s, a\}$ and $T = \{b, c, d, e, t\}$.

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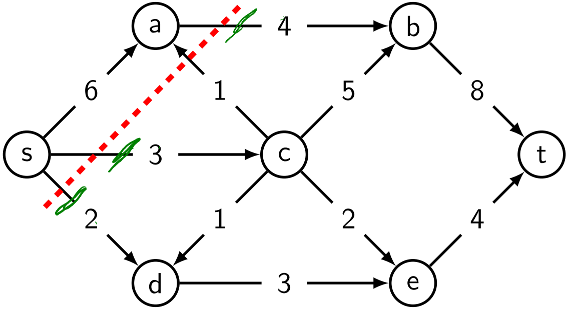

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The capacity of this cut is $4+3+2=9$, so this is the maximum flow.

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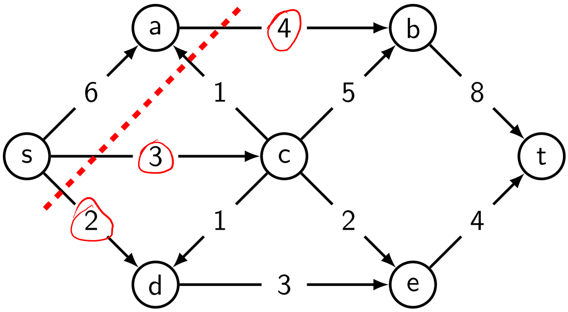

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Note: Edge (c,a) is not in the cut since it's in the wrong direction.

Max flow min cut: Class example 2

from Kolman, Busby & Ross

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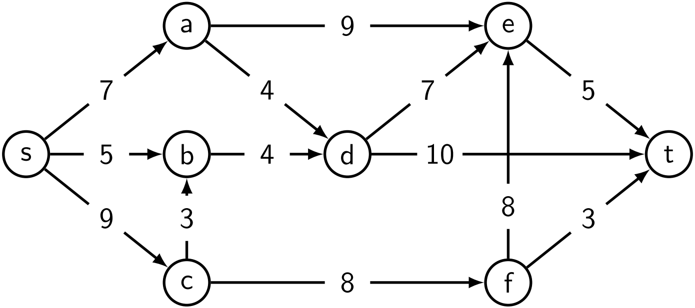

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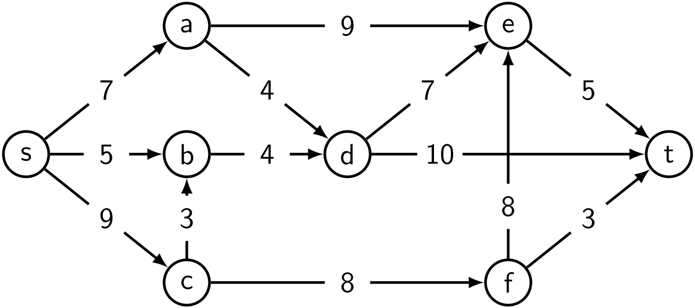

Hint:

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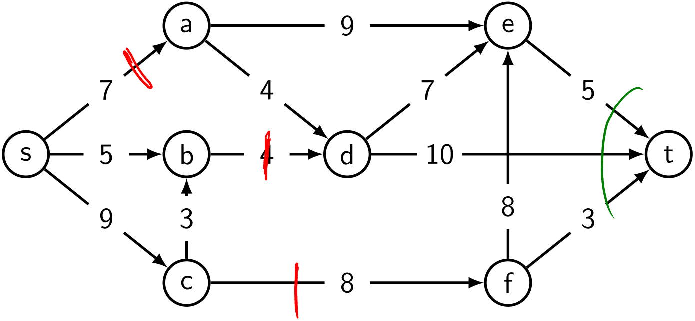

Hint: This time the minimum cut cannot be drawn as a single line!

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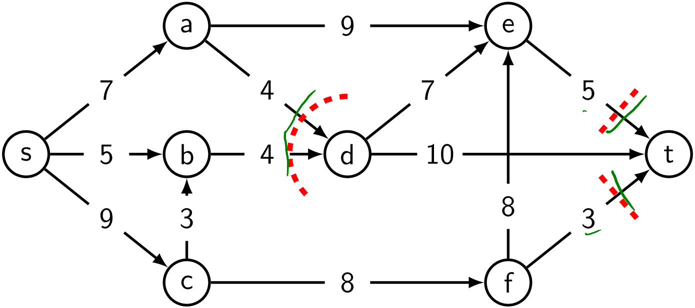

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Answer: Use all the edges from $S = \{s, a, b, c, e, f\}$ to $T = \{d, t\}$.

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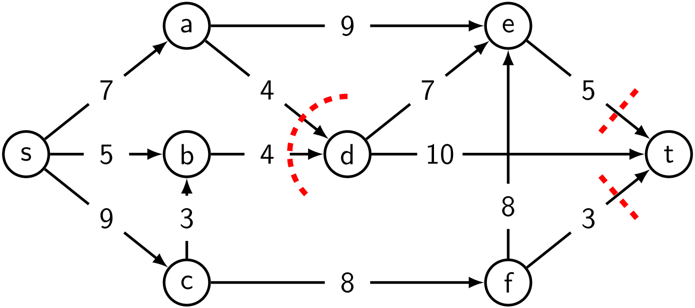

Hint: This time the minimum cut cannot be drawn as a single line!
Answer: Use all the edges from $S = \{s, a, b, c, e, f\}$ to $T = \{d, t\}$.
The capacity of this cut is $4+4+5+3=16$, so this is the max flow.

Introduction to "virtual flows"

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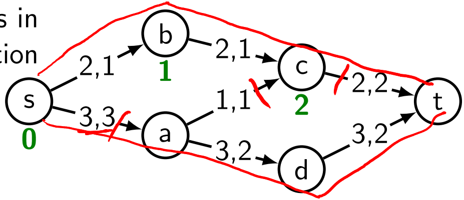

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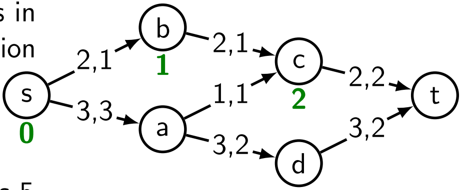

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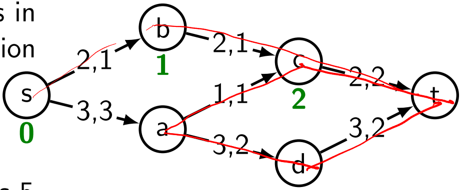

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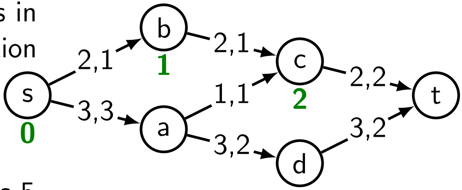

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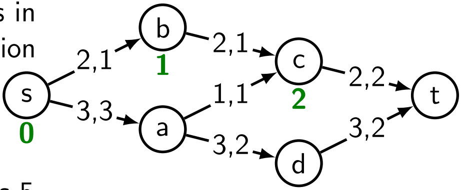

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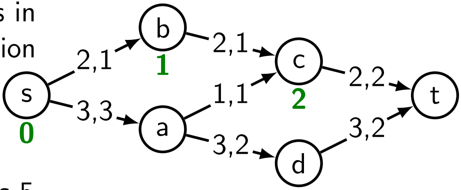

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First, the definition and an explanation of how the algorithm is modified.

The complete vertex labelling algorithm

Let (u,v) be a (directed) edge in a transport network D, and suppose there is currently a flow of $f > 0$ along this edge. The vertex labelling algorithm can reduce this flow to $g < f$ by introducing a **virtual flow** of $f - g$ in the opposite direction, *i.e.* from v to u.

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For vertices u,v of D, where D has capacity and flow functions C, F:

how much can be added

$$\boxed{S((\mathrm{u,v})) = \begin{cases} C((\mathrm{u,v})) - F((\mathrm{u,v})) & \text{if } (\mathrm{u,v}) \in E(D) \\ F((\mathrm{v,u})) & \text{if } (\mathrm{v,u}) \in E(D) \\ 0 & \text{otherwise} \end{cases}}$$

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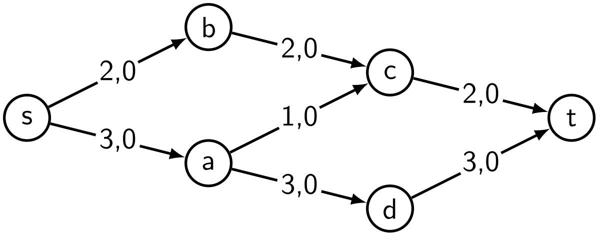
When $(\mathrm{v,u}) \in E(D)$, $S((\mathrm{u,v}))$ is called a **virtual capacity**.

Vertex labelling algorithm, Example 3

Stage 1: F_0 to F_1

$\boxed{\text{Flow } F_0}$



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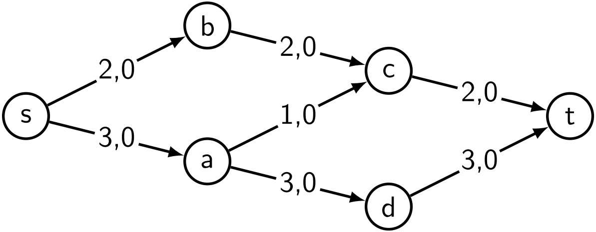
Levels



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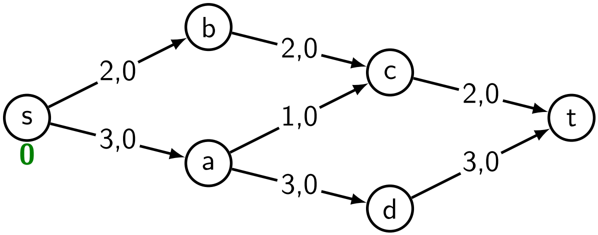

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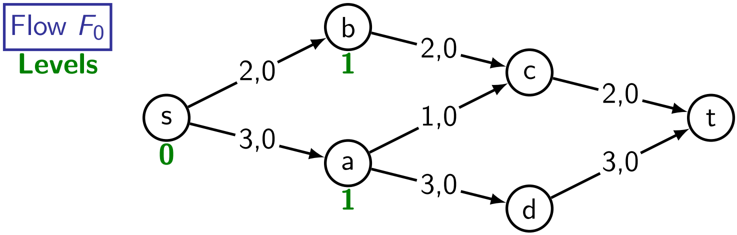

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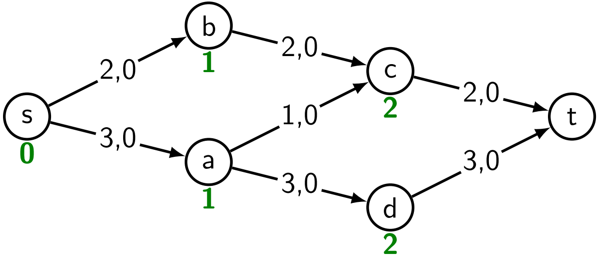

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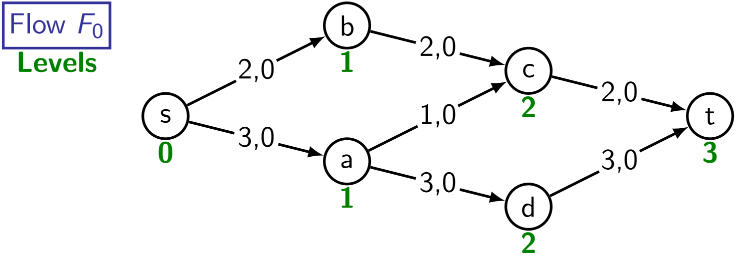

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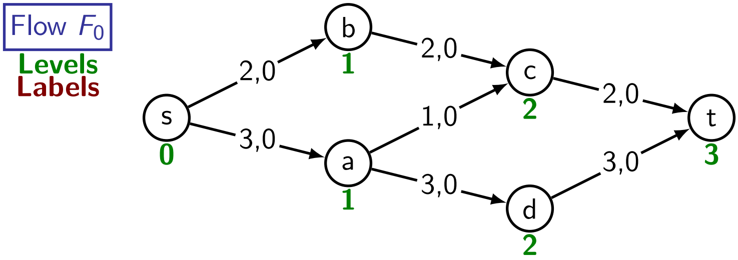

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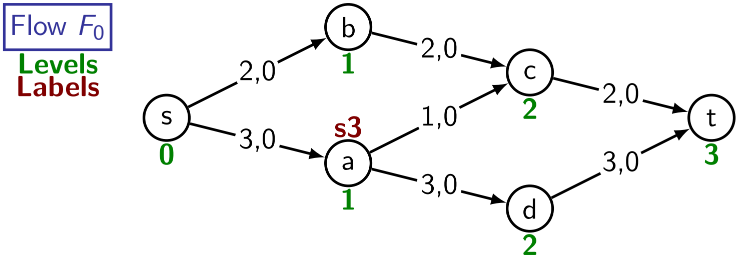

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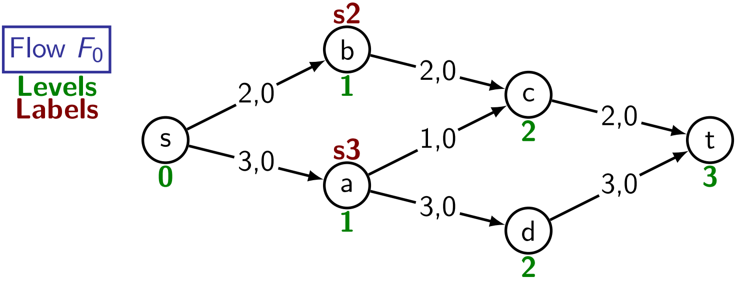

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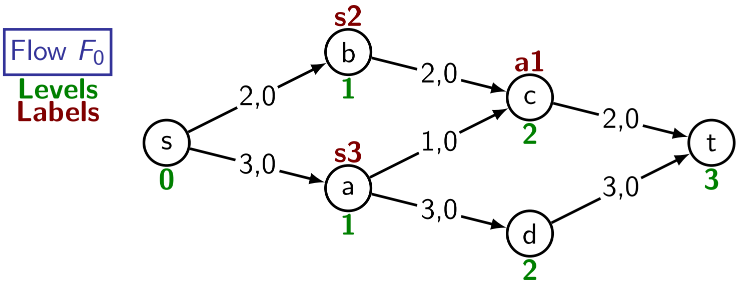

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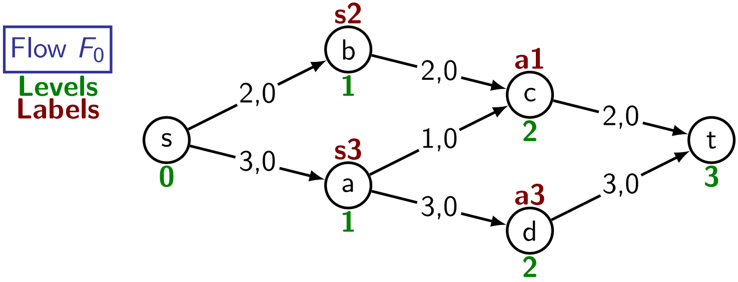

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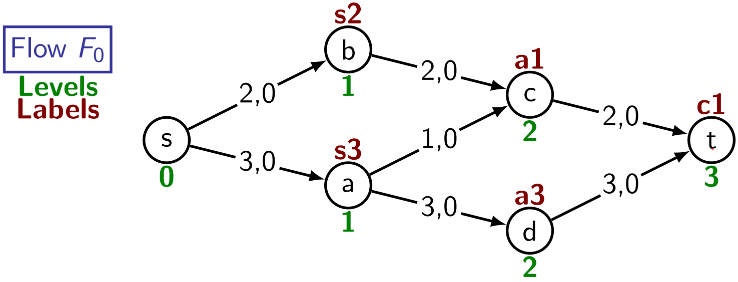

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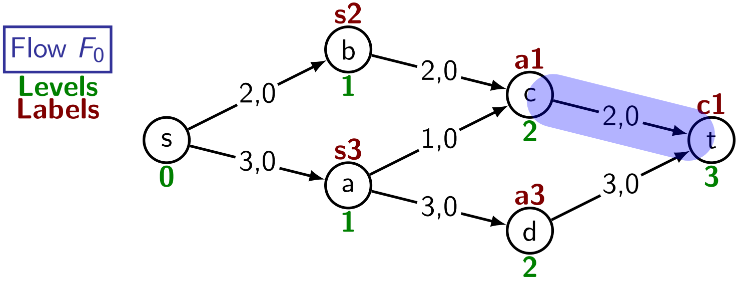

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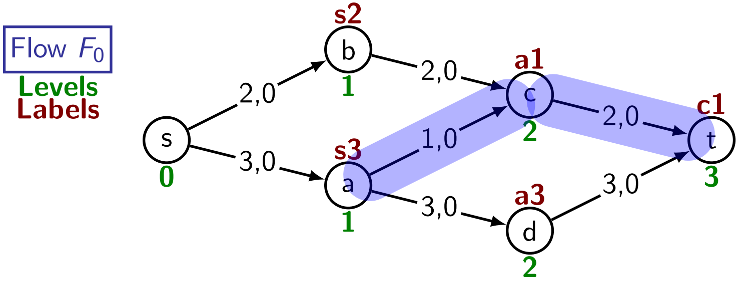

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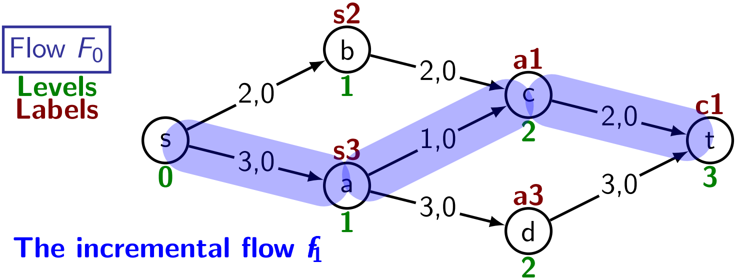


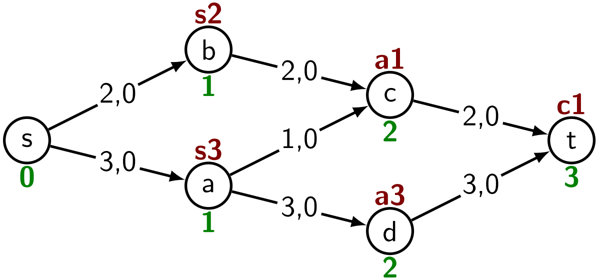
The incremental flow f_1

Vertex labelling algorithm, Example 3

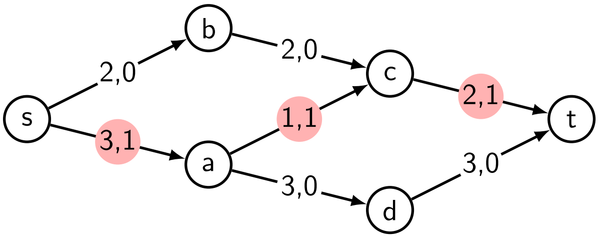
Stage 1: F_0 to F_1

Flow F_0

Levels
Labels



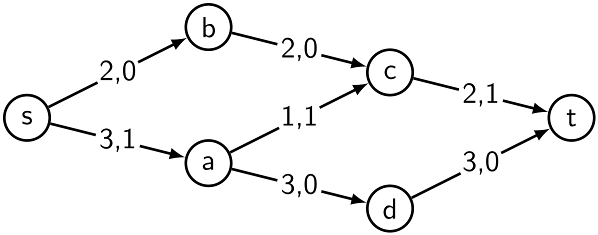
Flow F_1



Vertex labelling algorithm, Example 3

Stage 2: F_1 to F_2

Flow F_1

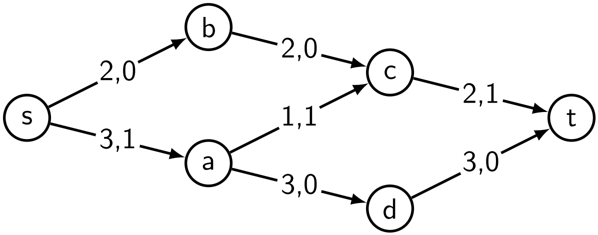


Vertex labelling algorithm, Example 3

Stage 2: F_1 to F_2

$\boxed{\text{Flow } F_1}$

Levels



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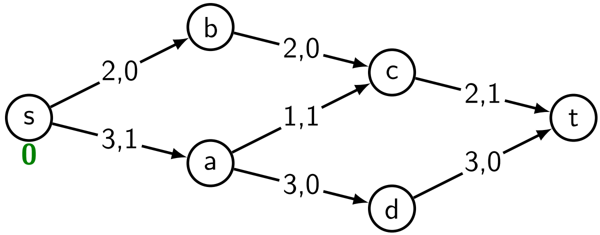

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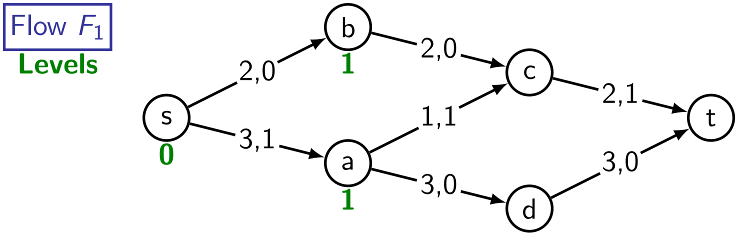

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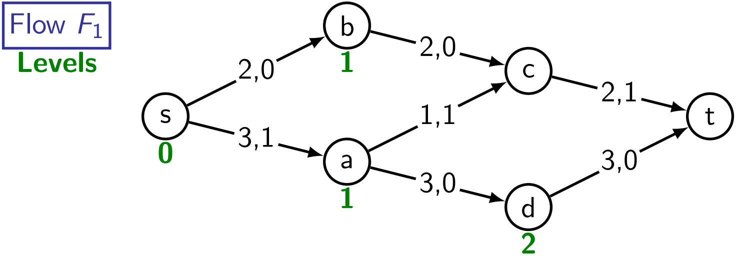

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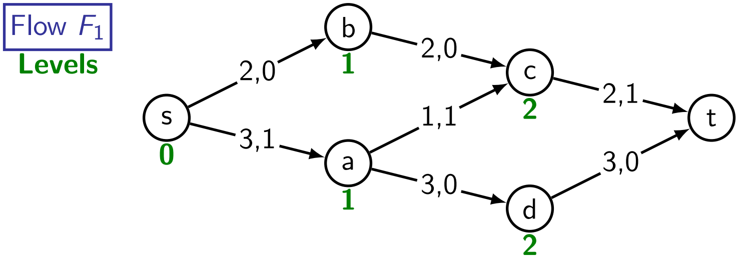

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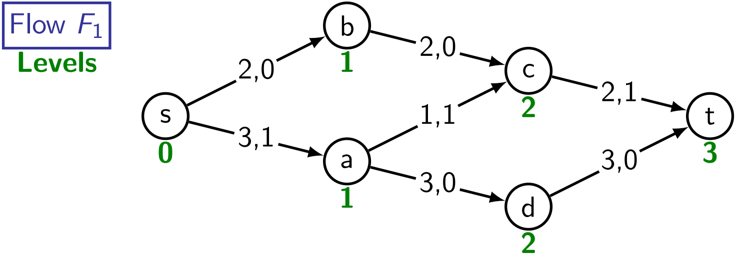

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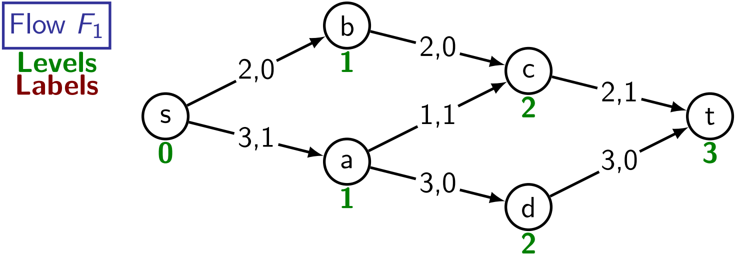

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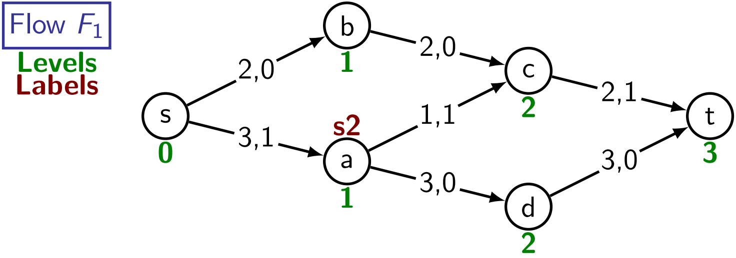

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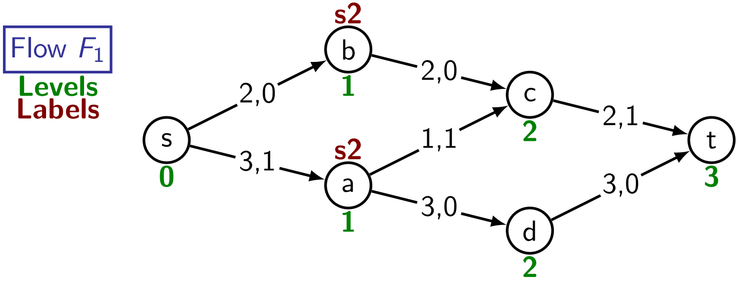

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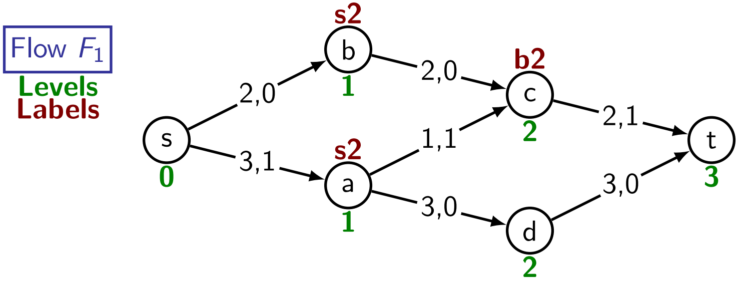

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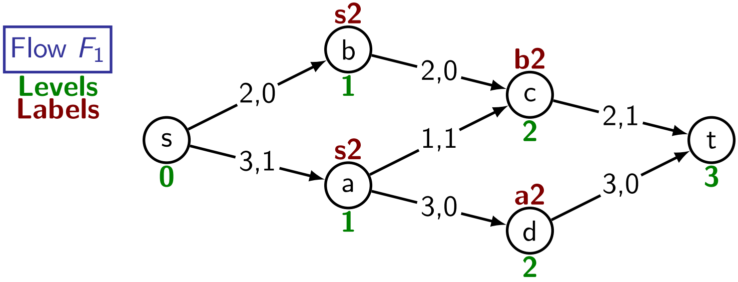

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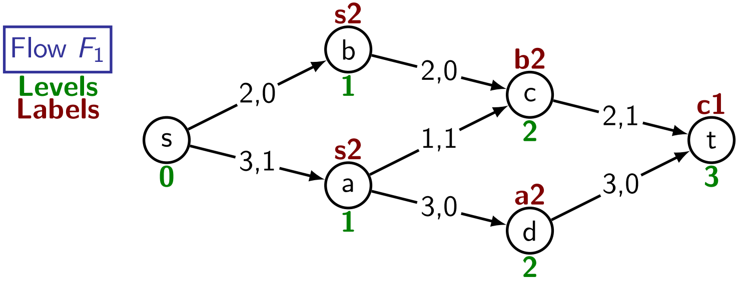

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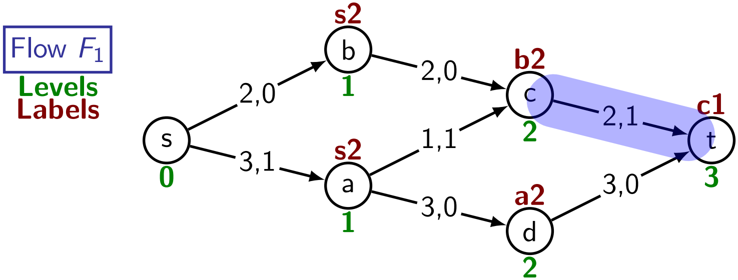

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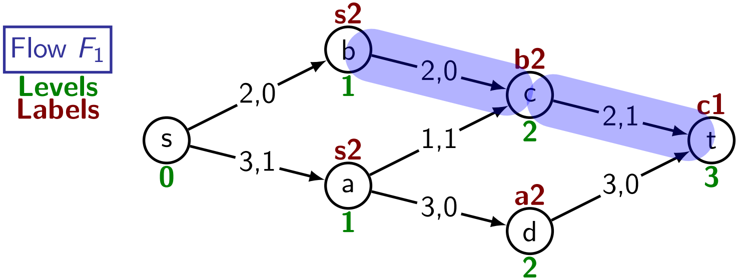
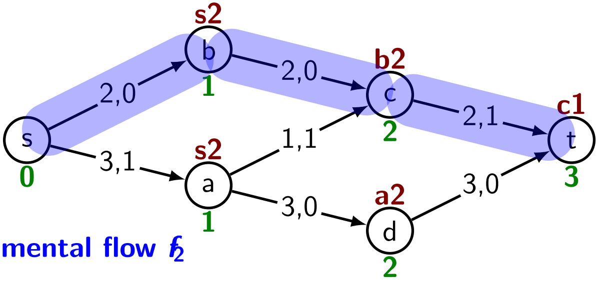


Vertex labelling algorithm, Example 3

Stage 2: F_1 to F_2

Flow F_1

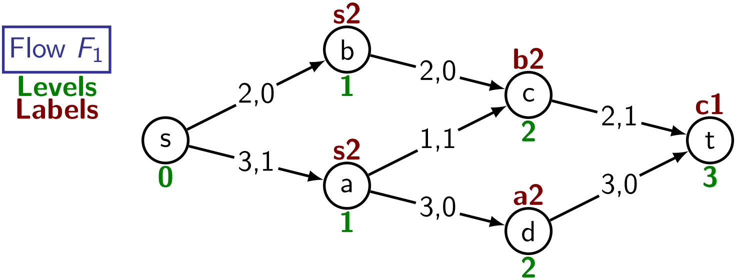
Levels
Labels

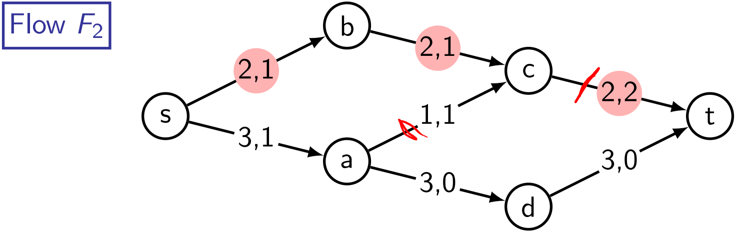


The incremental flow f_2

Vertex labelling algorithm, Example 3

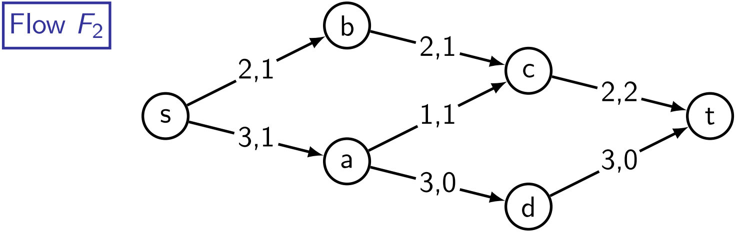
Stage 2: F_1 to F_2





Vertex labelling algorithm, Example 3

Stage 3: F_2 to F_3



Vertex labelling algorithm, Example 3

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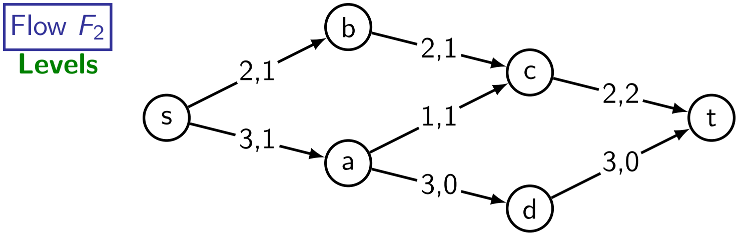

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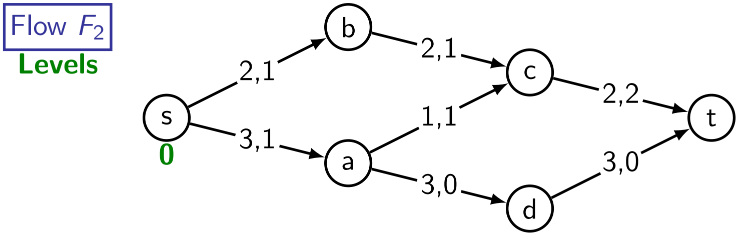

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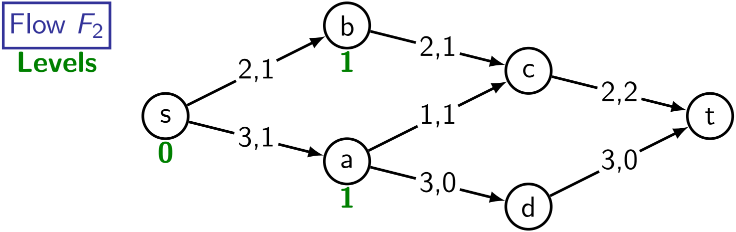
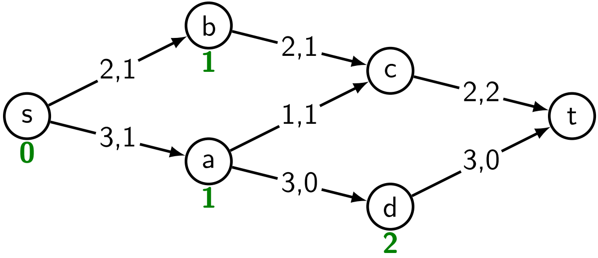


Vertex labelling algorithm, Example 3

Stage 3: F_2 to F_3

Flow F_2

Levels



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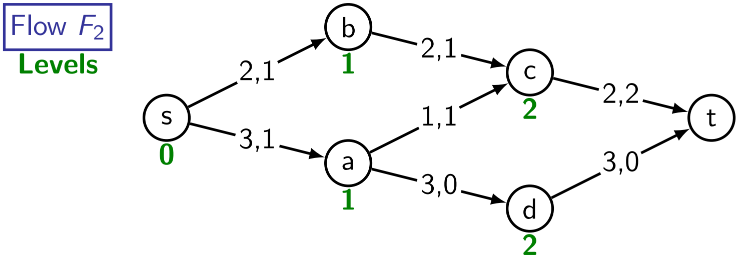

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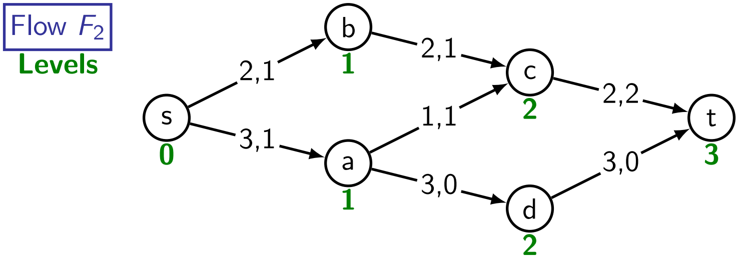

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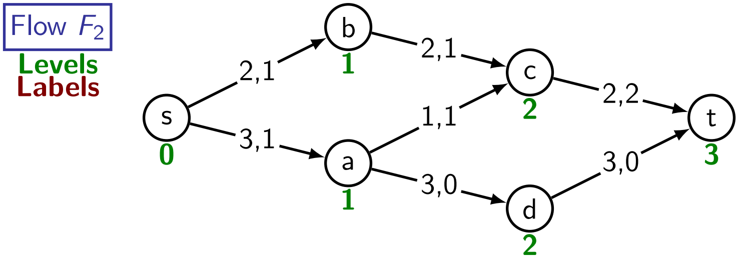

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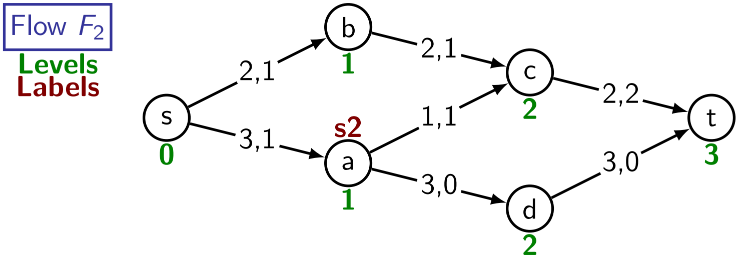
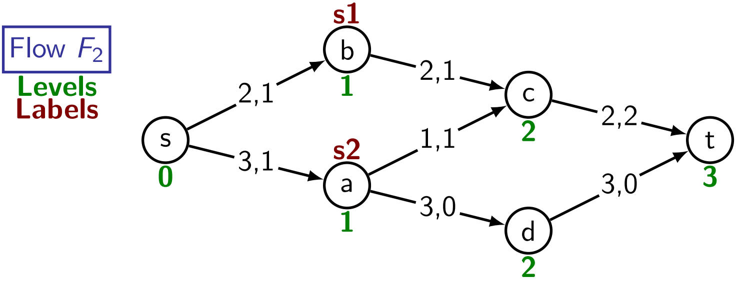


Vertex labelling algorithm, Example 3

Stage 3: F_2 to F_3



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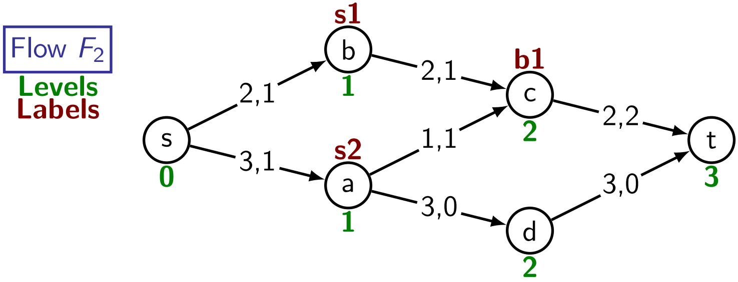

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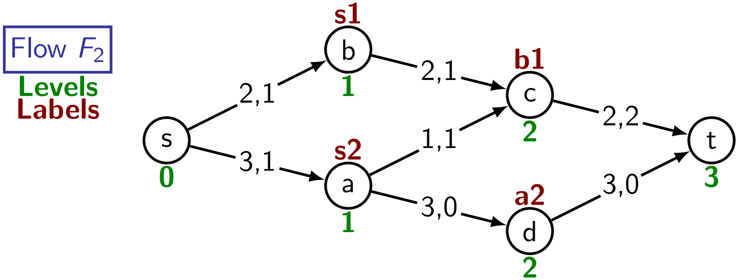

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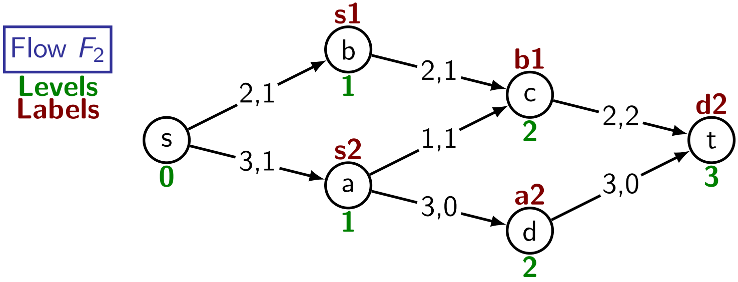

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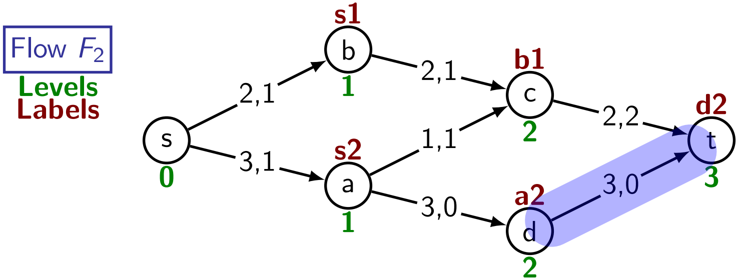

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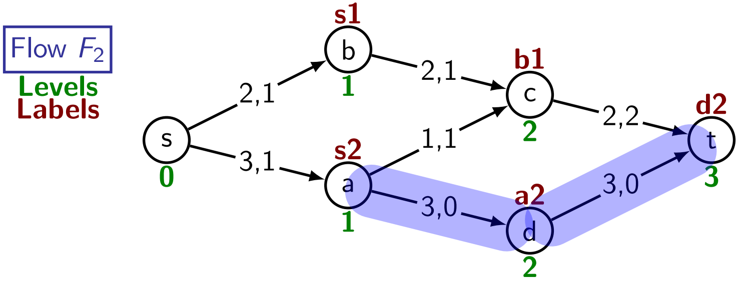

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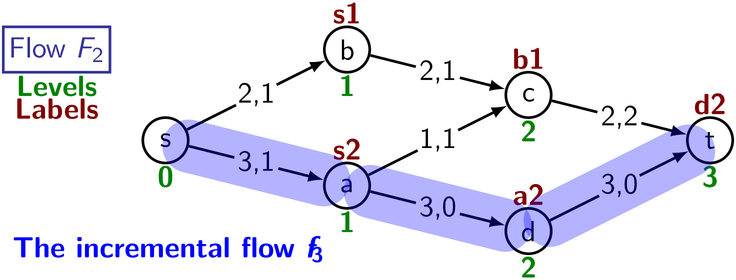

The incremental flow f_3

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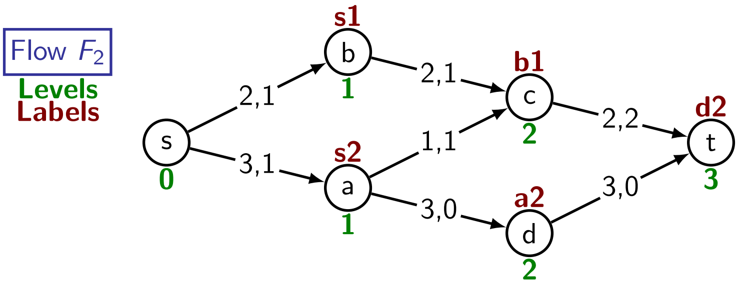
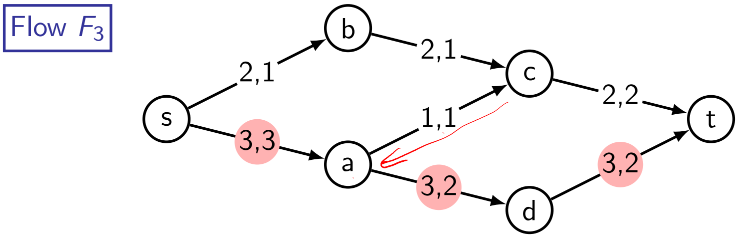
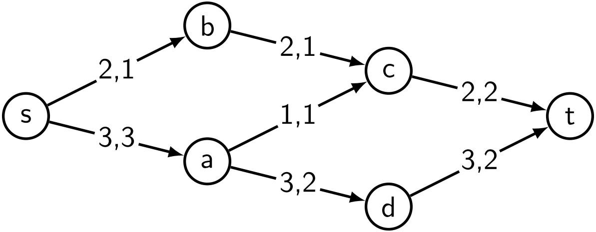




Vertex labelling algorithm, Example 3

Stage 4: F_3 to F_4

Flow F_3



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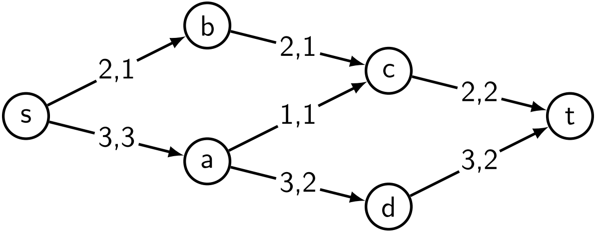

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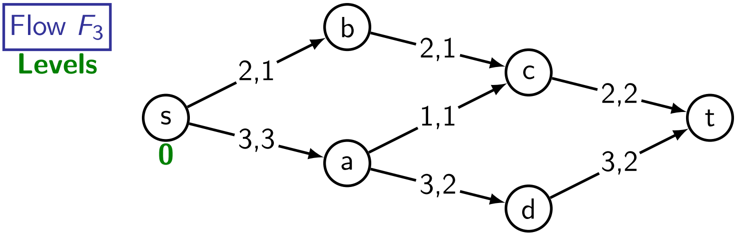

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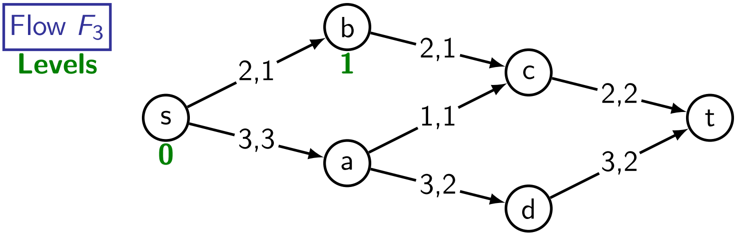

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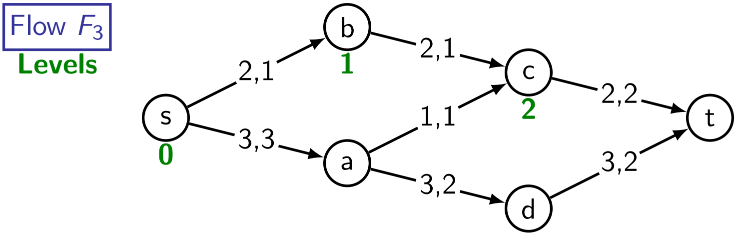

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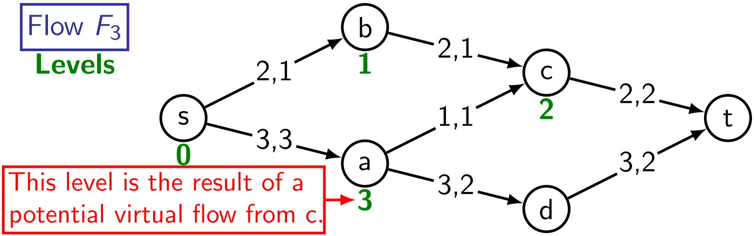

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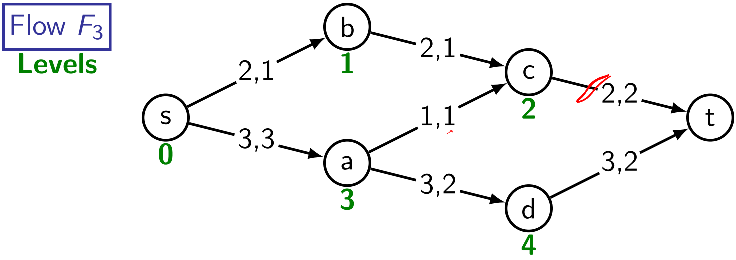

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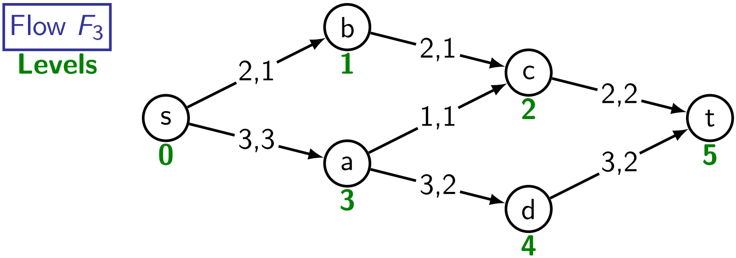

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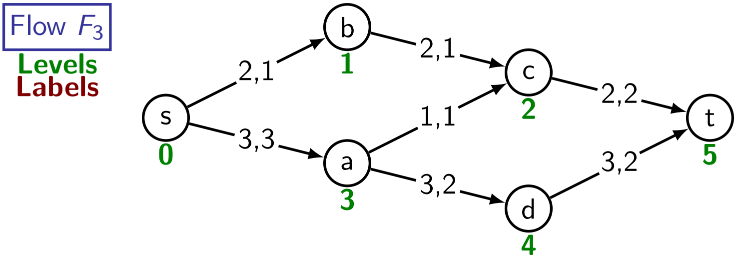

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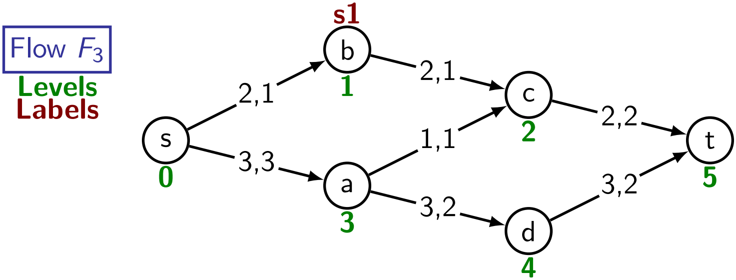

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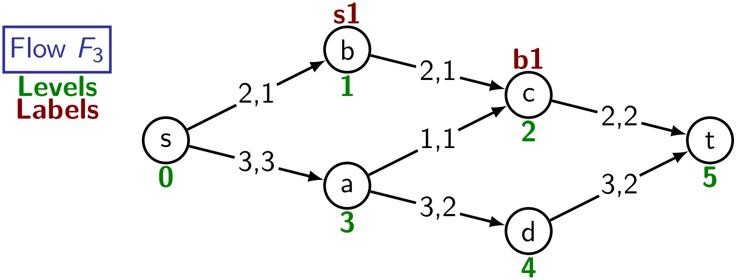

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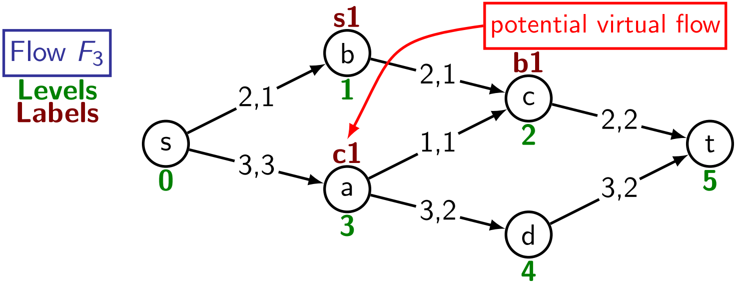

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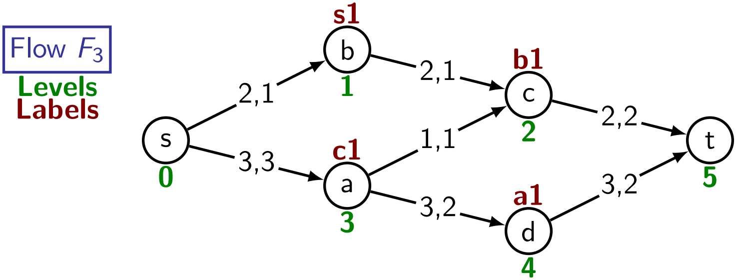

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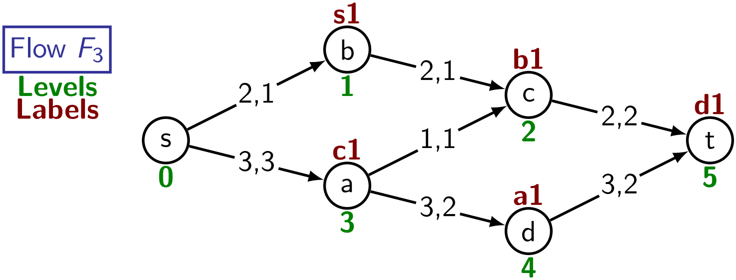

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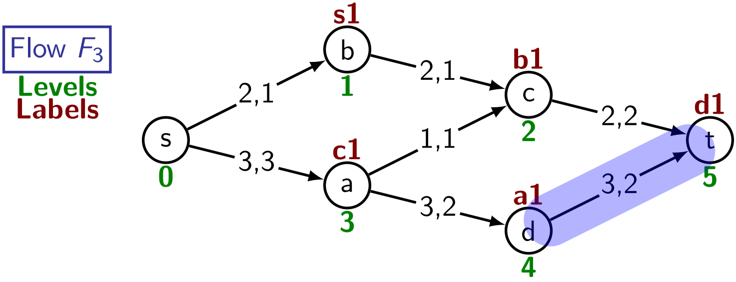

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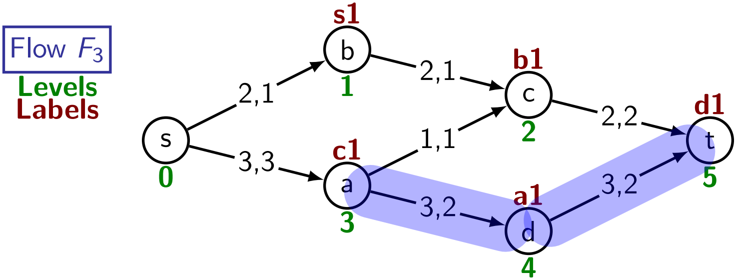

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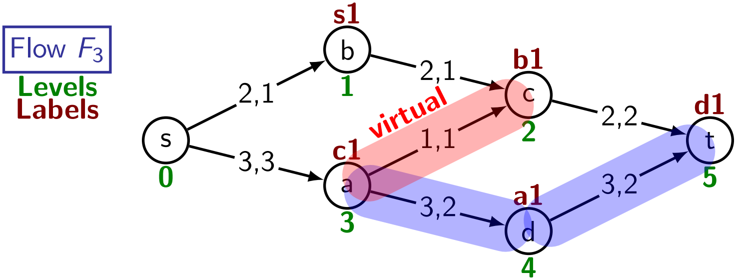

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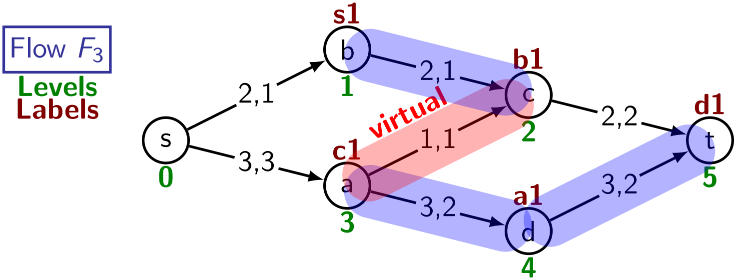

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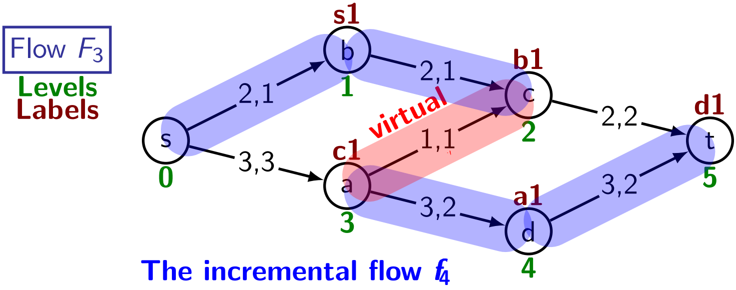

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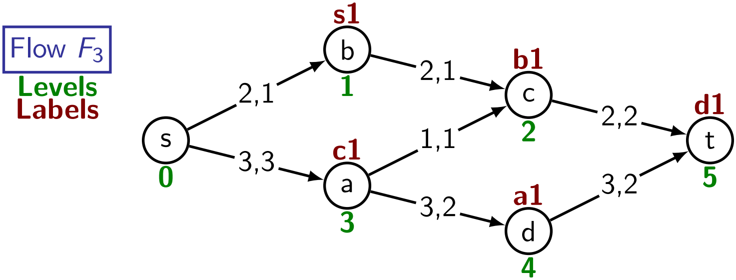

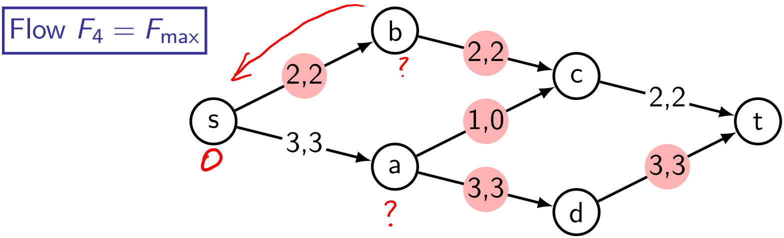

A matching problem

from Johnsonbaugh

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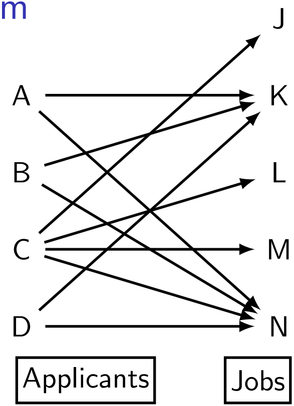

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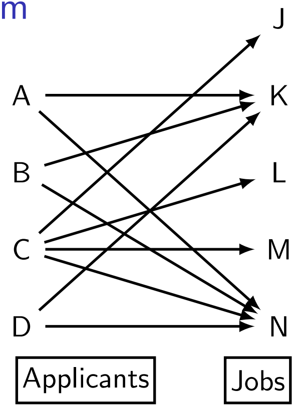

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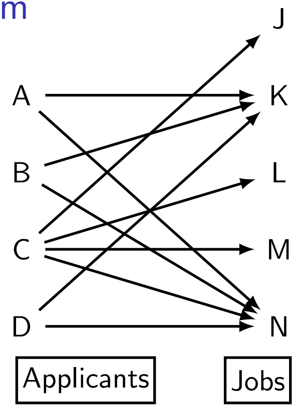

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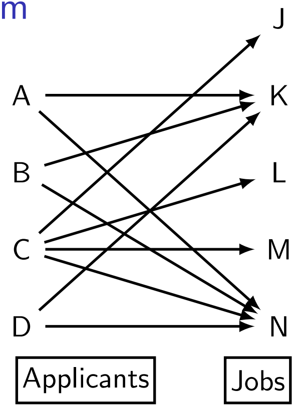

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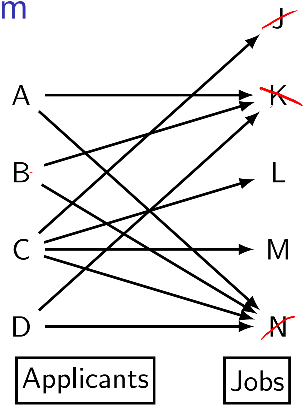

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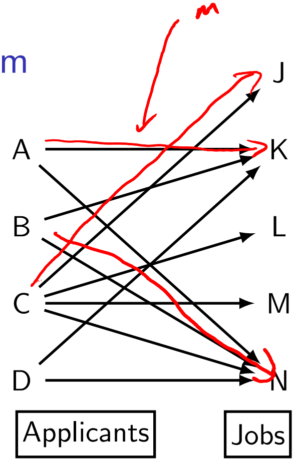

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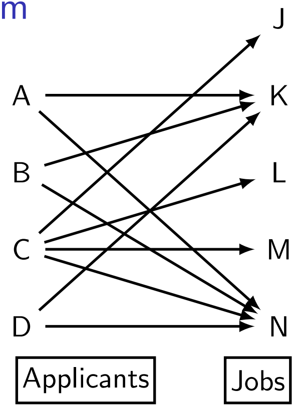

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This is a **injective** (one-to-one) function $f : S' \to T$ with domain $S' \subseteq S$ as large as possible subject to m being an injective subset of R.

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A solution to the max flow problem provides the matching:

$$\boxed{m = \{(x, y) \in S \times T \ : \ F_{\max}((x, y)) = 1\}.}$$

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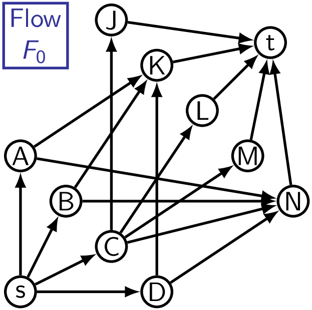

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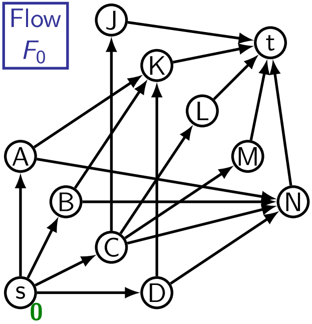

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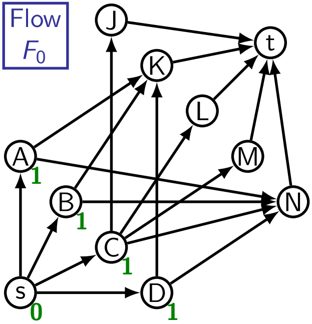

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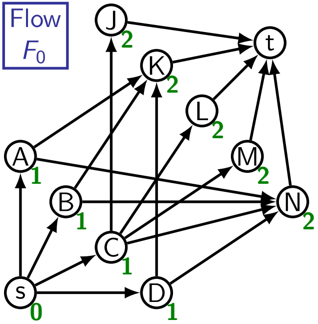

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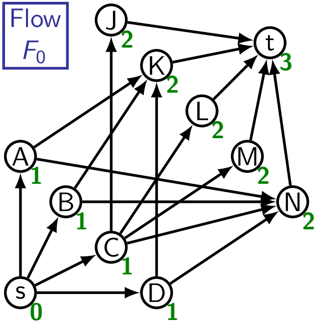

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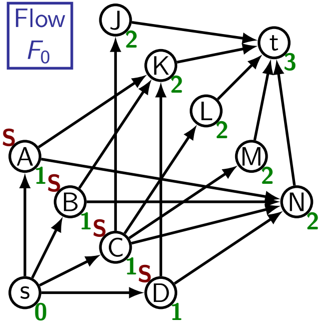

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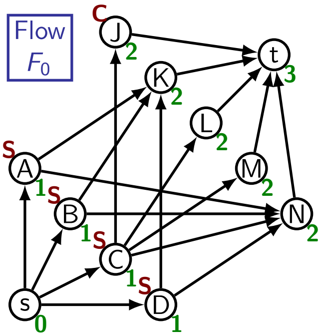

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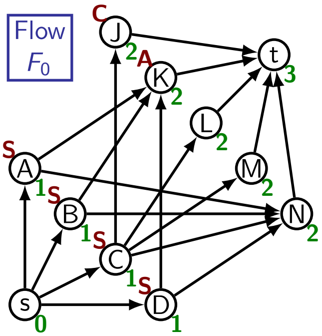
Here is how vertex labelling would be used on the Johnsonbaugh problem. This is just for demonstration, since, as we've seen, it's easily solved by eye.

With all edge capacities 1, edge flows are either 0 or 1. Notation will be simplified:

- Edge capacities not marked.
- (Non-zero) Edge flows marked by edge thickening.
- Potential flow values not indicated on labels.



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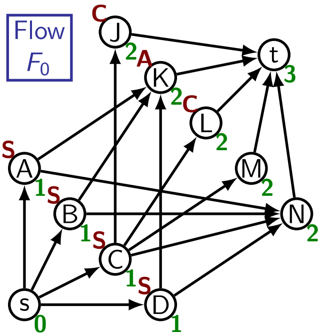

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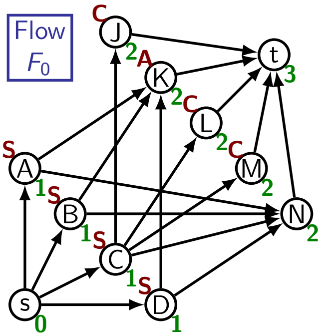

Vertex labelling for matching; Example 1

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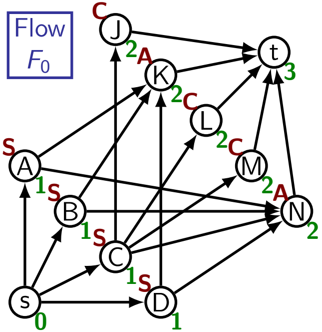

Vertex labelling for matching; Example 1

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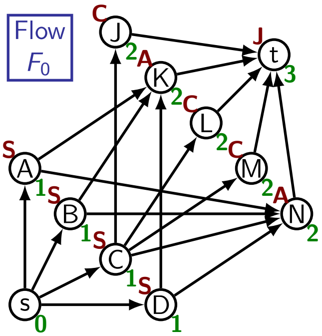

Vertex labelling for matching; Example 1

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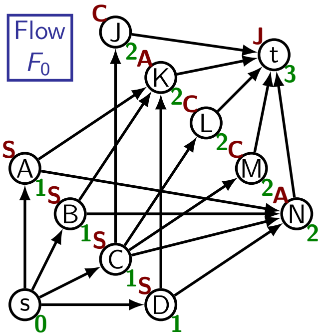

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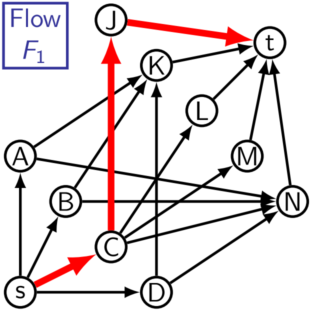

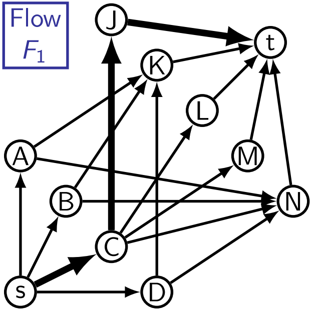
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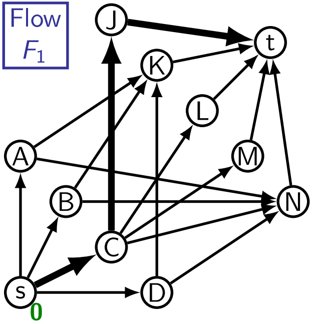
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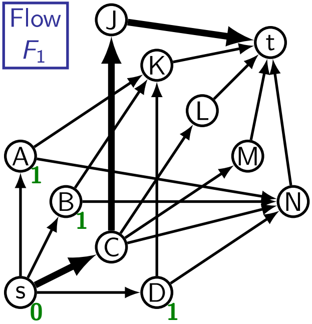
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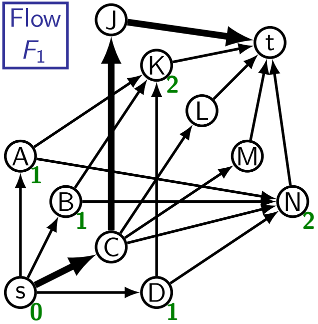

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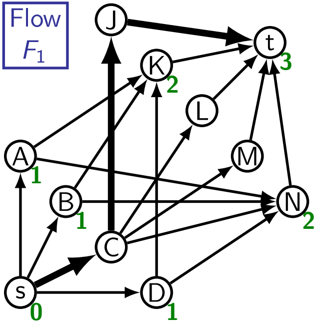
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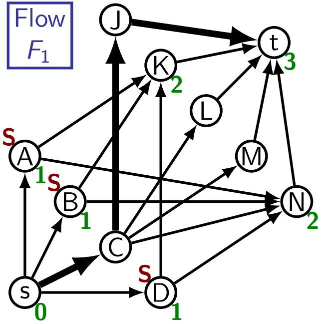

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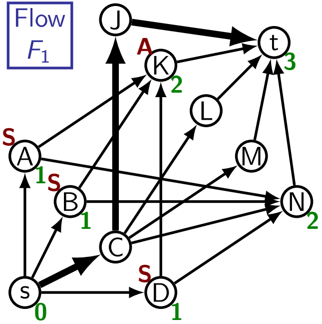

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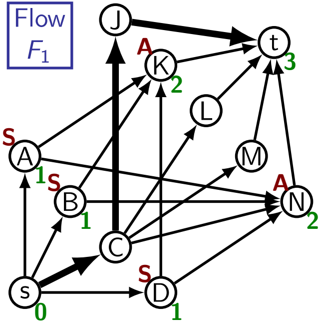
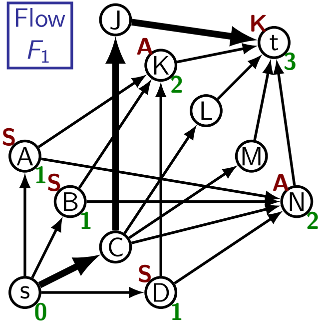
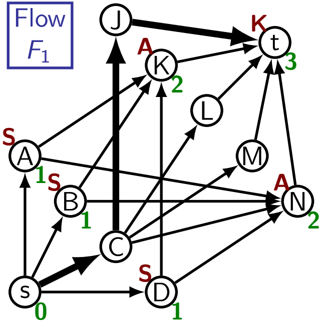


Flow F_1





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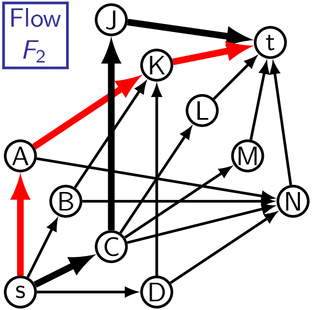

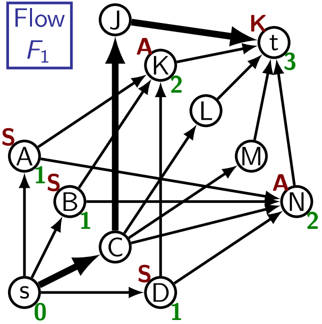

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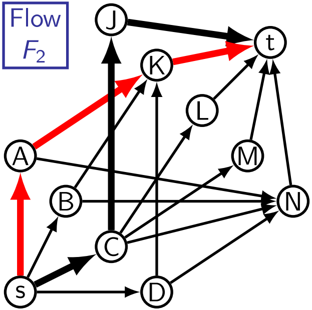
$\Rightarrow$



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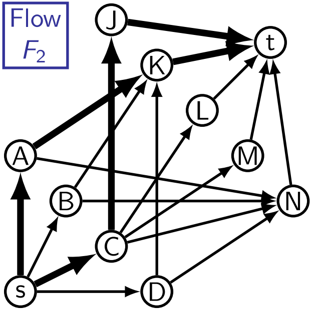

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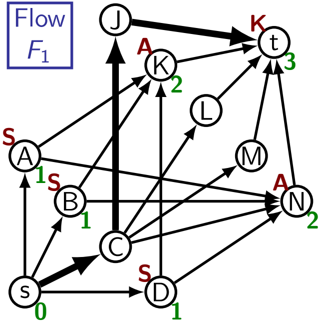
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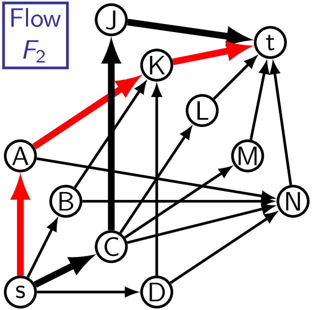
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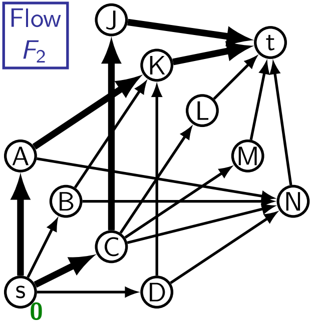
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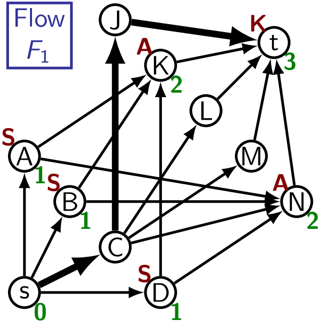
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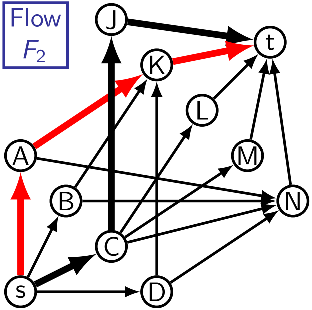
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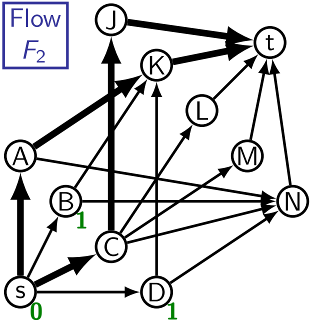
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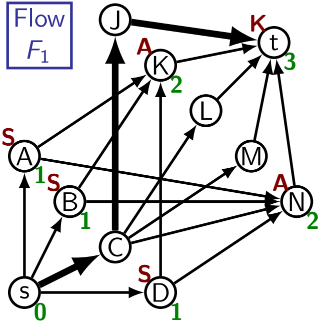
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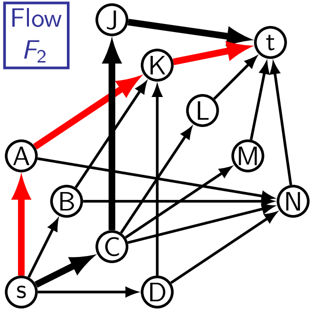
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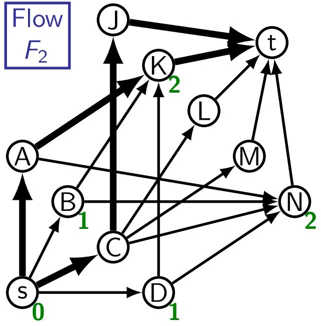
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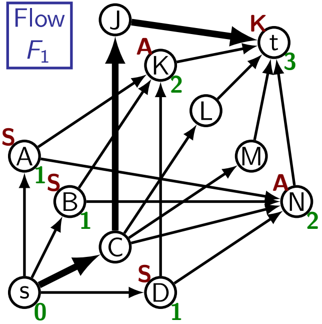
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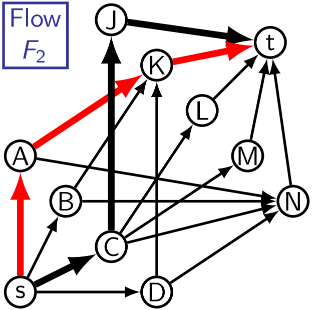
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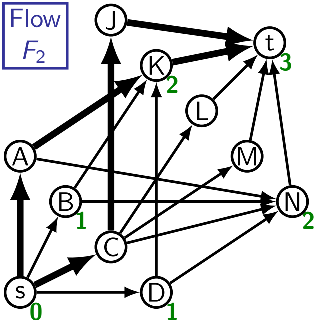
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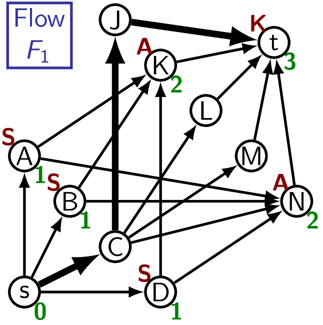
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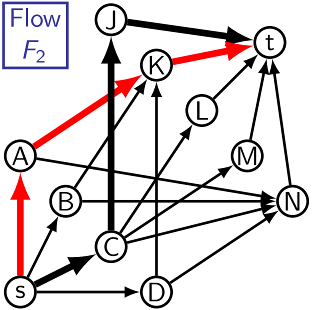
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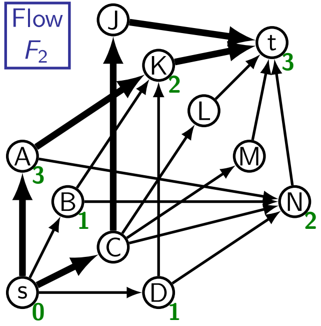
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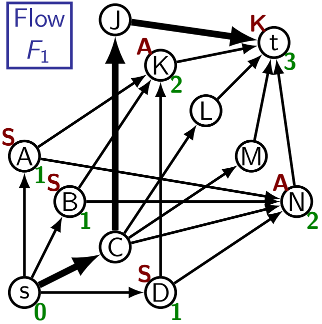
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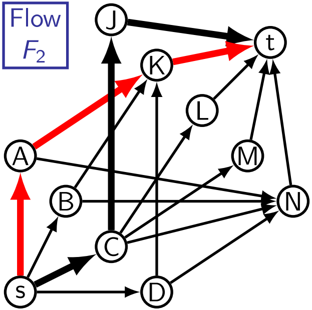
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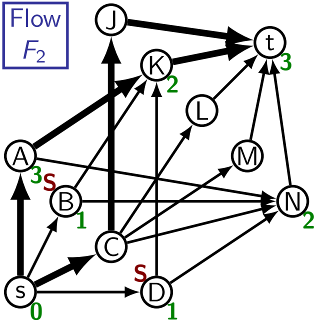
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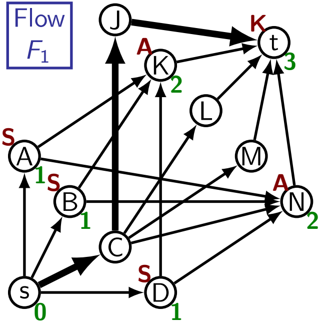
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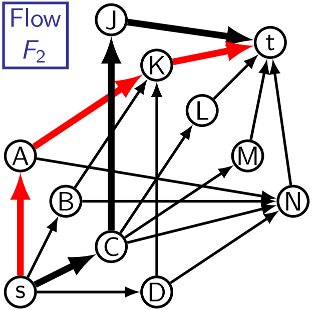
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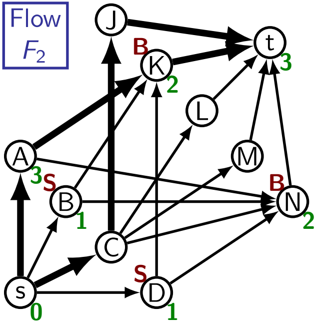
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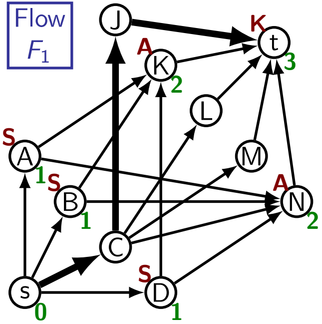
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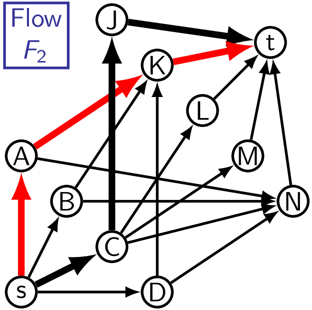
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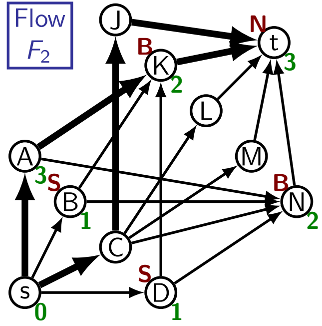
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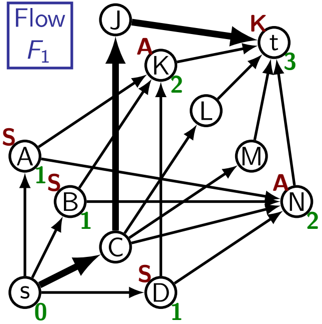
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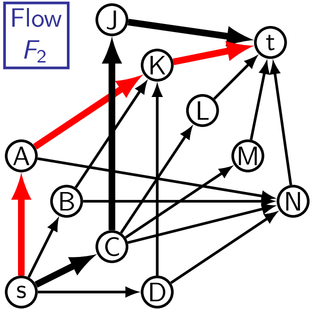
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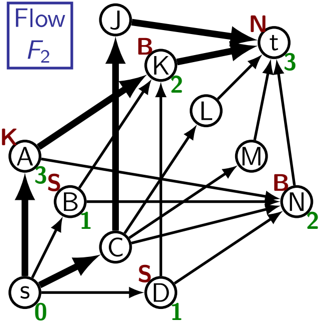
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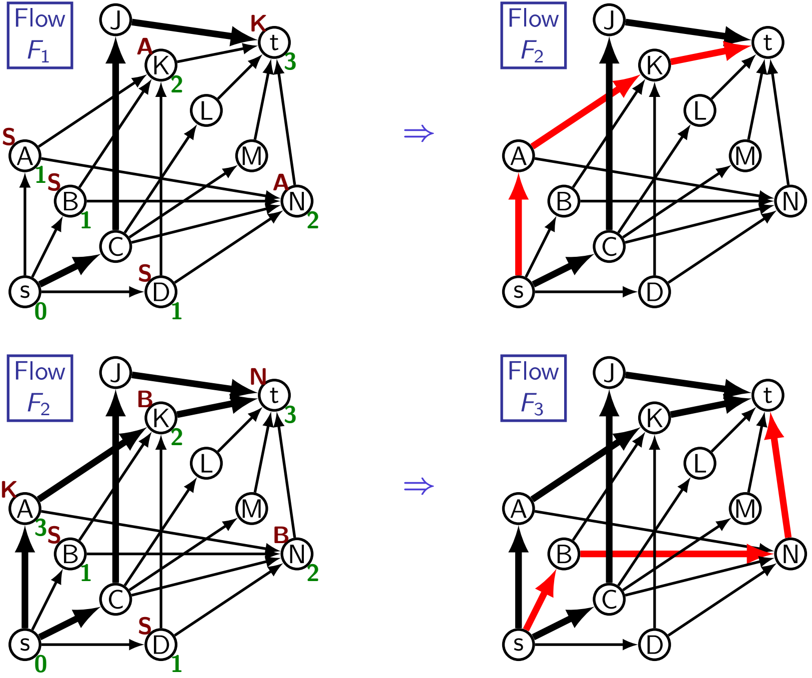
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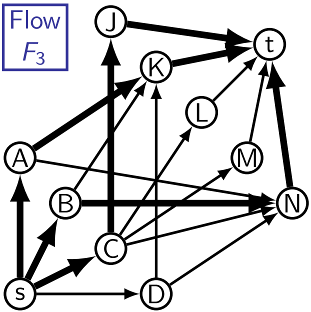
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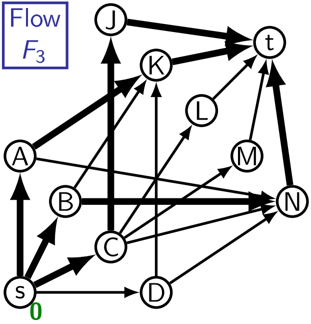
Flow
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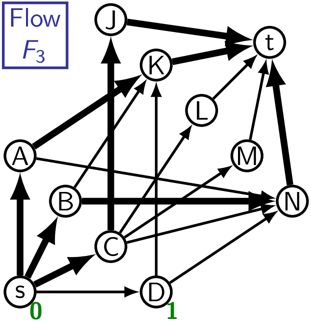
Flow
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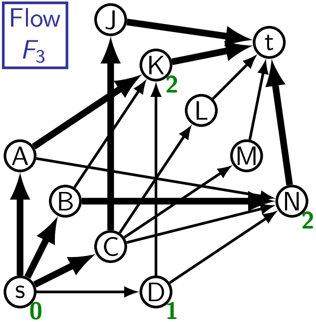
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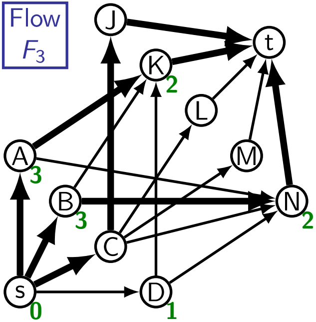
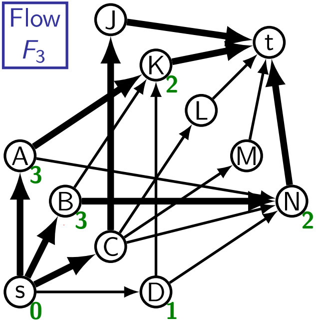
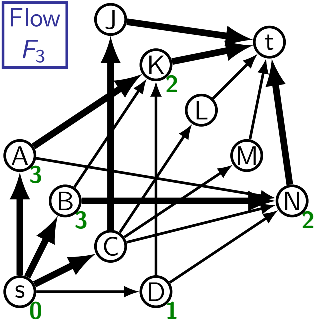
Flow
F_3





$\Rightarrow$

Even using virtual capacities to reach vertices of level 3, it isn't possible to assign a level to t, so the algorithm terminates.



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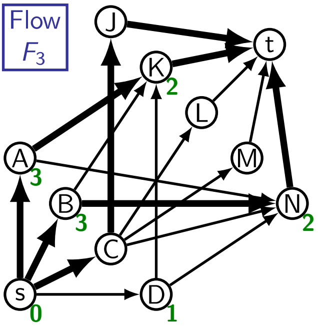
The matchings are indicated by the edges between $\{A, B, C, D\}$ and $\{J, K, L, M, N\}$ that have flow 1 (the thick edges).



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So the matching is

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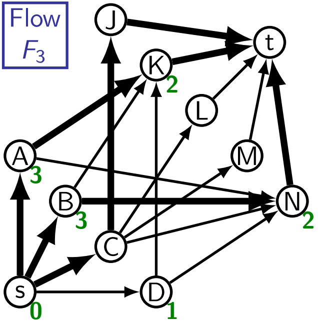

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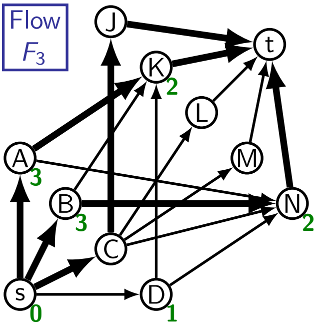
As remarked earlier, the above solution is very easy to get without the use of the vertex labelling algorithm. We have just matched the first member of T with the first 'available' member of S, then the next member of T with the next 'available' member of S and so on.



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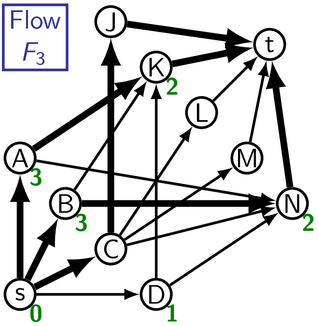

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The final example shows how it does this in the simplest possible case.

Vertex labelling for matching; Example 2

Find a (maximal) matching function m
for the relation $R = \{(a, p), (a, q), (b, p)\}$.

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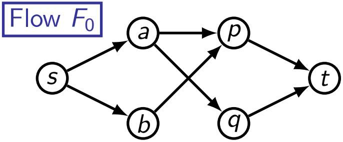

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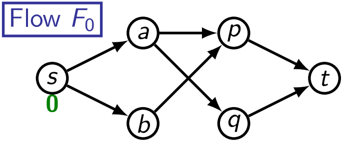

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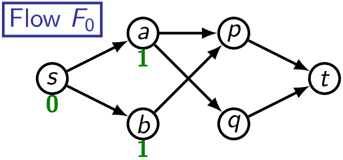

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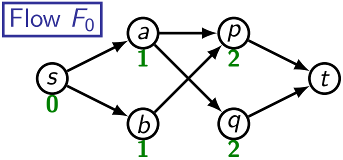

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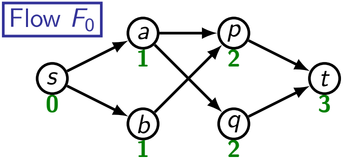

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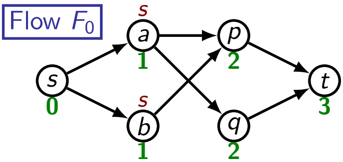

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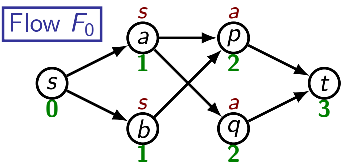

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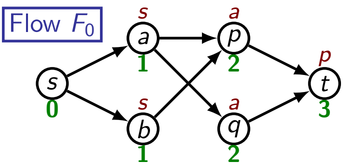

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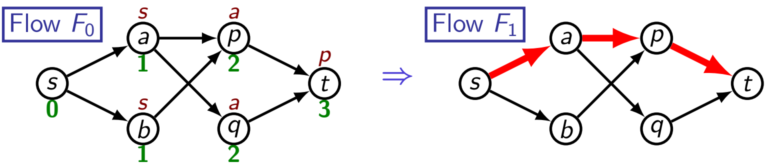

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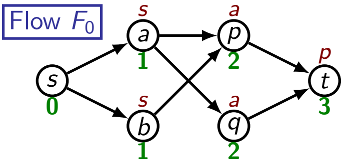
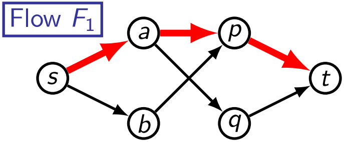
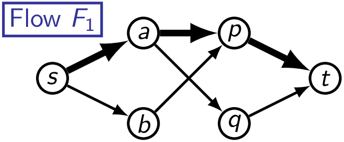


$\Rightarrow$





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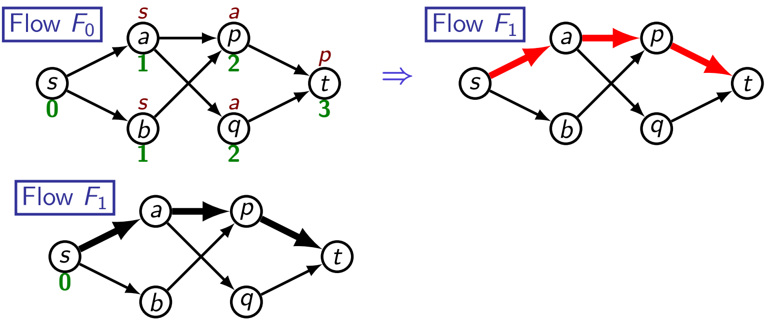

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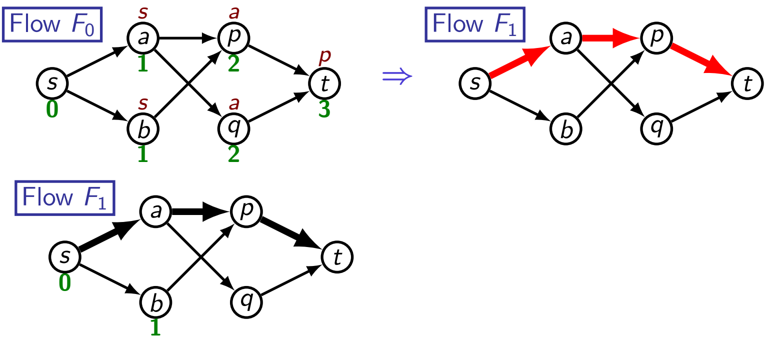

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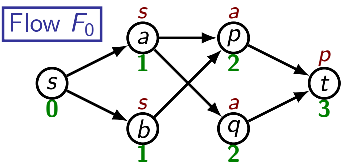

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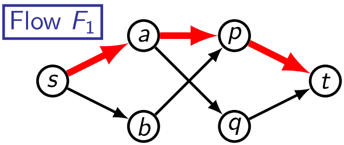

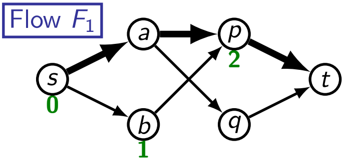

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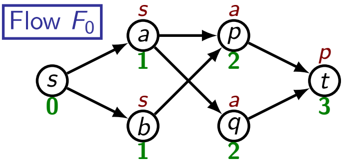

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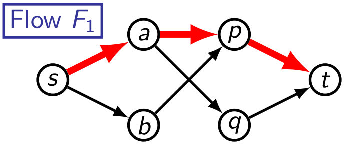

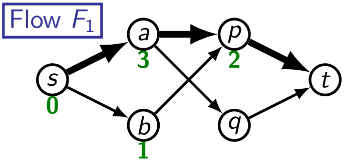

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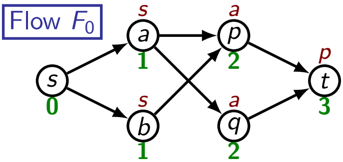

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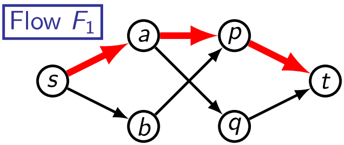

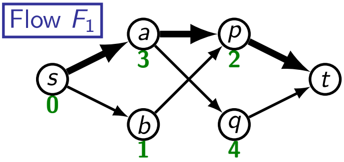

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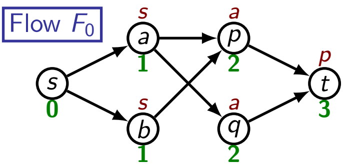

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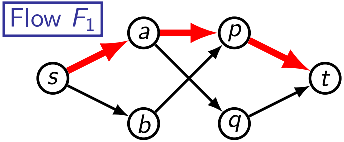

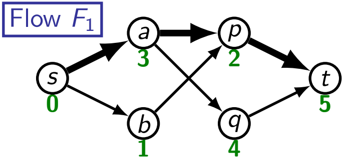

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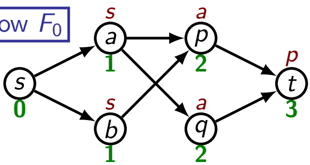
Flow F_0



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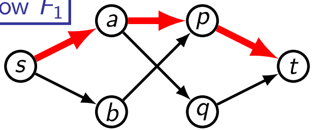
Flow F_1



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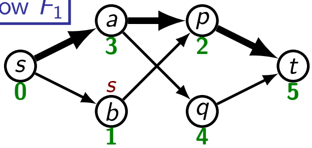

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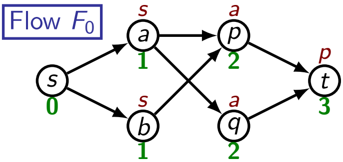

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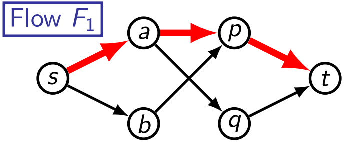

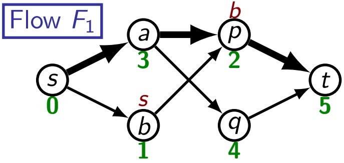

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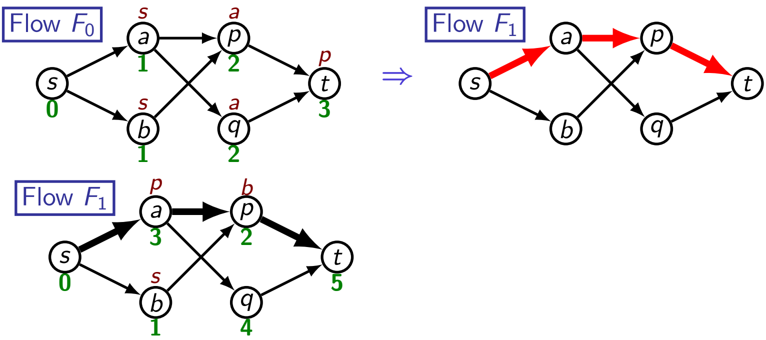

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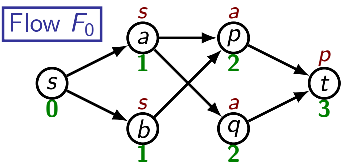
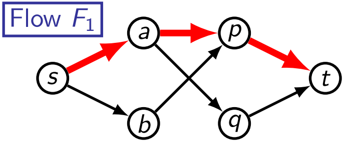
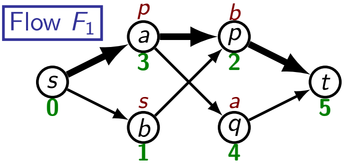


$\Rightarrow$

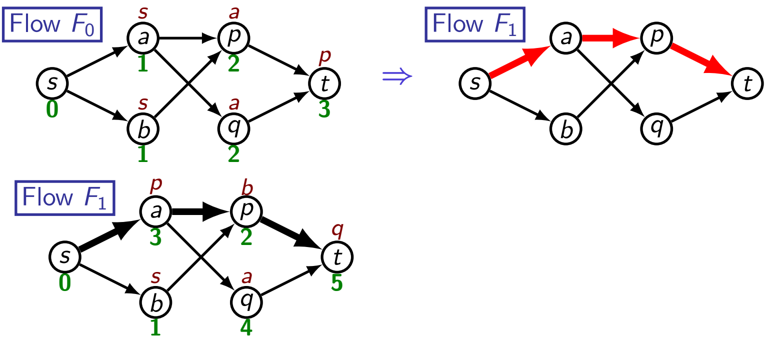




Vertex labelling for matching; Example 2

Find a (maximal) matching function m
for the relation $R = \{(a,p),(a,q),(b,p)\}$.

Obviously the only answer is $m = \{(a,q),(b,p)\}$, but the vertex labelling algorithm will first match a with p. However it will then use a virtual flow to undo this and match a with q, allowing b to be matched with p instead. Here are the diagrams:



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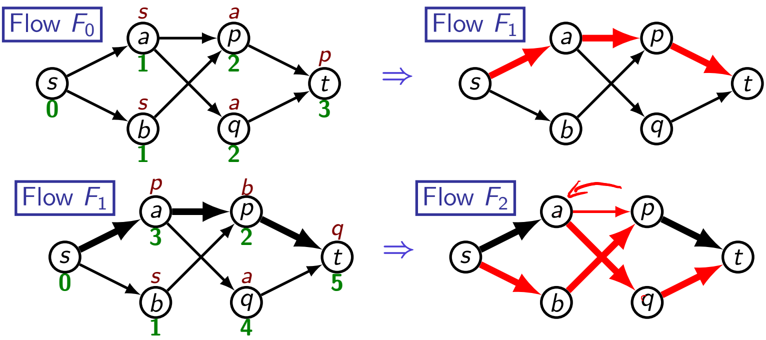

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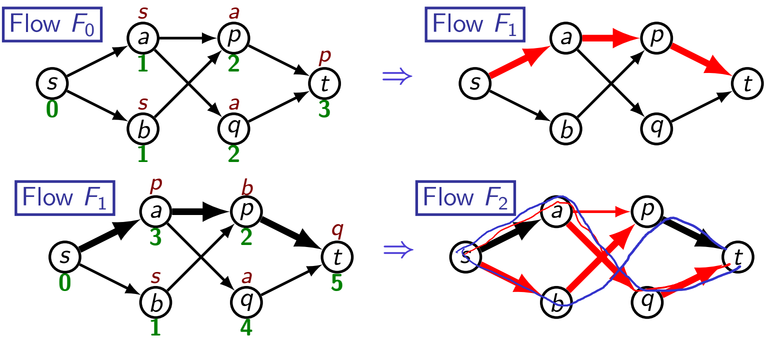

END OF SECTION D2