

D2. Weighted Graphs

Notes by Malcolm Brooks,
partly inspired by notes of Pierre Portal.

Text Reference (Epp) 3ed: Chapter 11
 4ed: Chapter 10
 5ed: Chapter 10

Some of the work in this section is not covered in our text by Epp.
I have based some examples on ones from:
Kolman, Busby & Ross *Discrete Mathematical Structures*
Johnsonbaugh *Discrete Mathematics*

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 - **The internet:** Vertices are internet nodes; edges are all direct connections between nodes; weights are times (in milliseconds) for a packet to travel across a connection.

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We will also look at a different kind of problem on a weighted **directed** graph: **Maximal Flow**. Details later.

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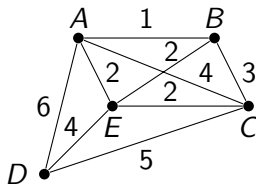
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 4. Repeat steps 2 and 3 until T has $n - 1$ edges.

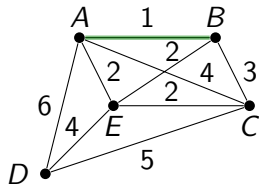
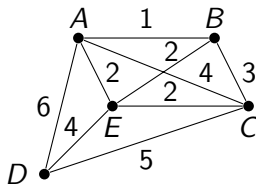
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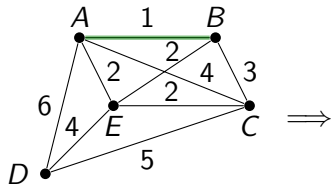
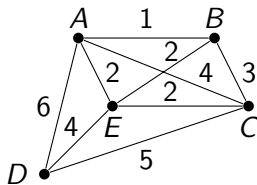
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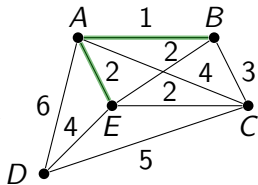
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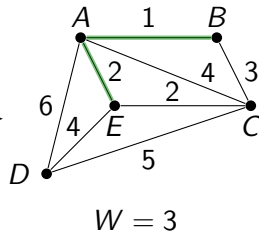
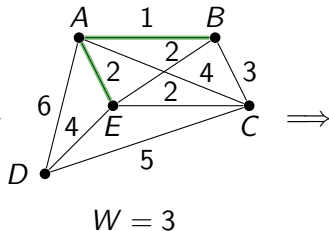
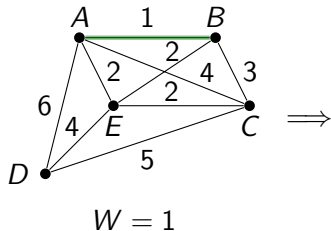
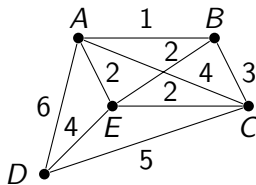
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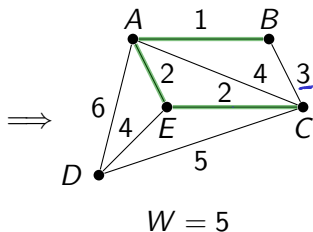
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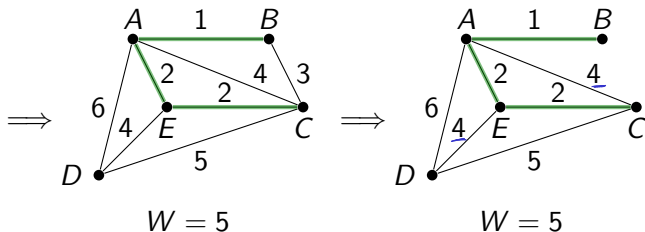
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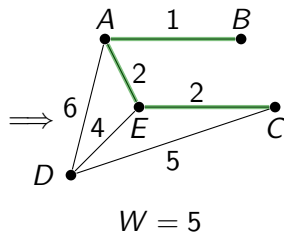
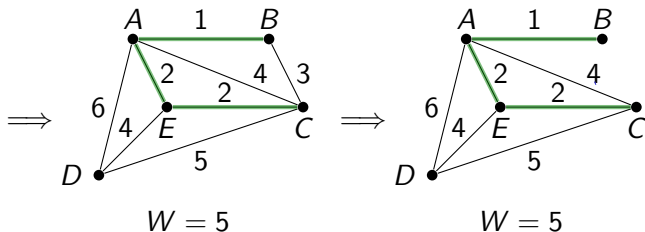
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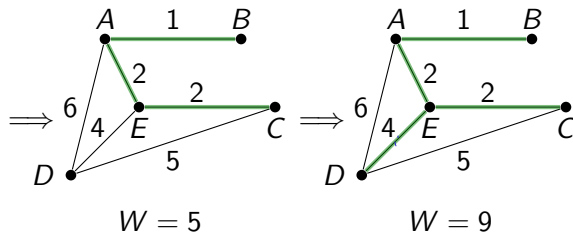
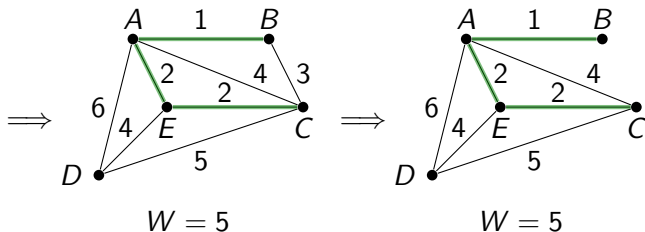
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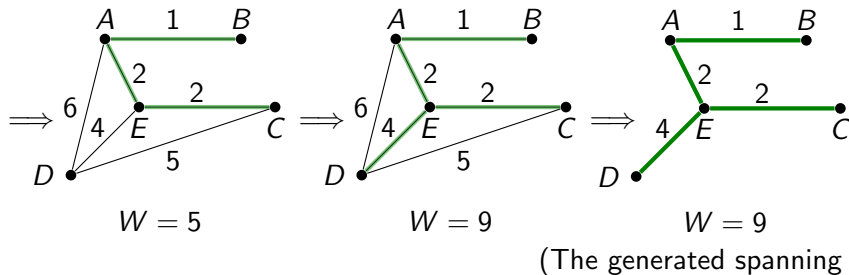
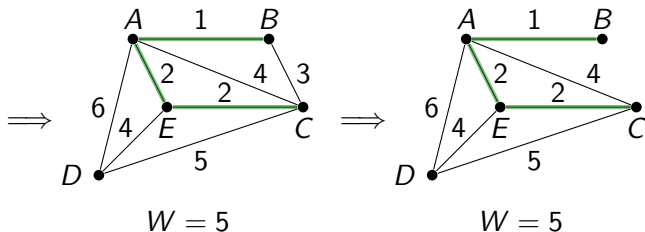
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- Kruskal's algorithm always succeeds! (Non-obvious theorem omitted)

That is, it always finds a minimal spanning tree, given any weighted connected (finite) graph.

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- Given: a table of distances between every pair of towns.
- **Model:** ^{complete} Graph K_n with towns as vertices and edges weighted by the the inter-town distances.

Find a Hamilton circuit of minimum possible total weight.

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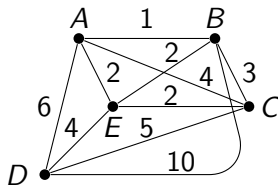
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- Greed doesn't always pay !!
- In fact, no efficient successful algorithm for the travelling salesman problem is known at this time. Finding one, or proving that none exists, is a major outstanding problem in mathematics.

$P = NP ?$

Example: Applying the Nearest Neighbour algorithm

Find a minimal Hamilton circuit (tour) for this weighted graph:

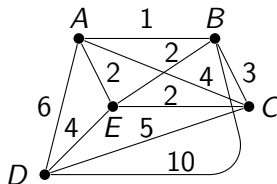
Note: This graph is as for the minimal spanning tree example but with the addition of an edge BD to make it complete.



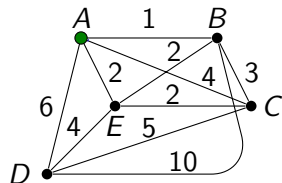
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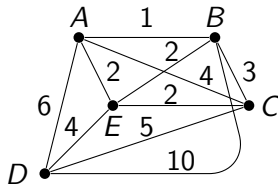


$$L(1) = A, W = 0$$

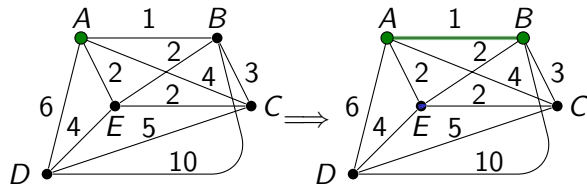
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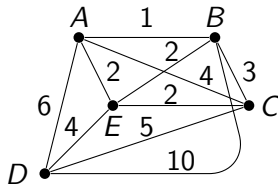
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$$L(2) = B, W = 1$$

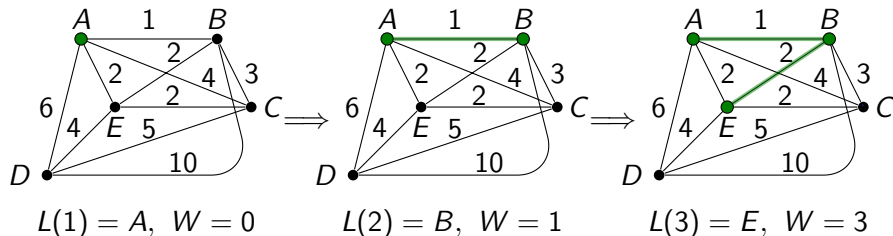
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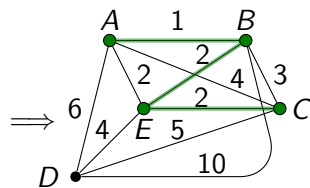
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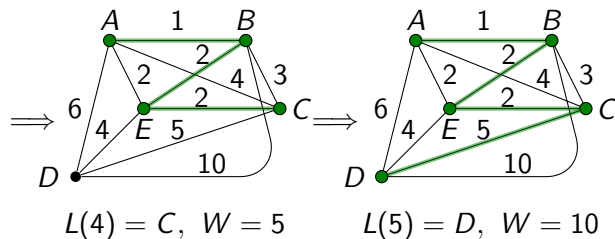


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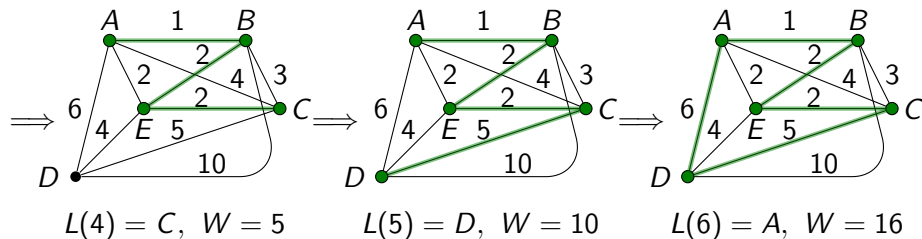


$$L(4) = C, W = 5$$

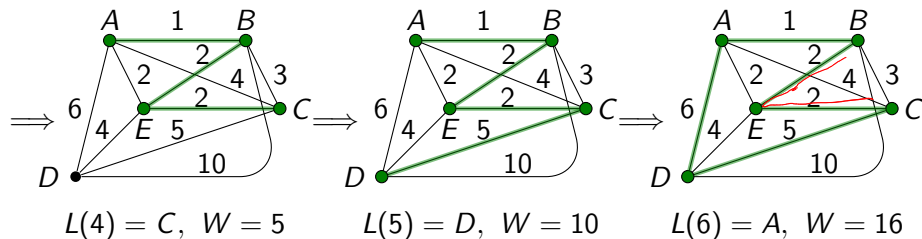
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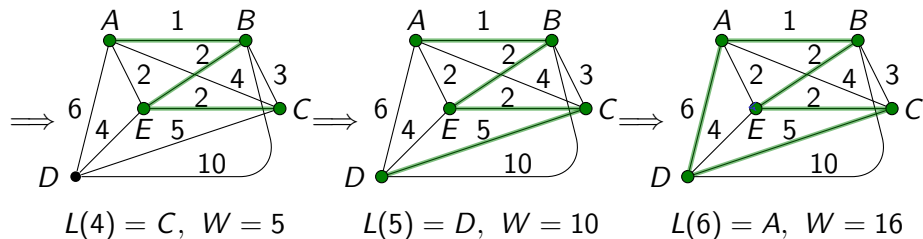


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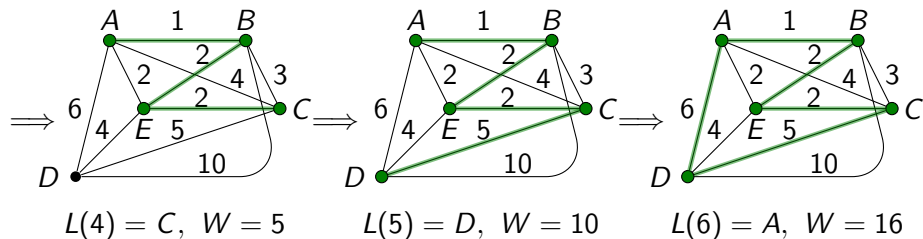
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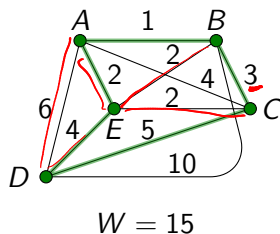


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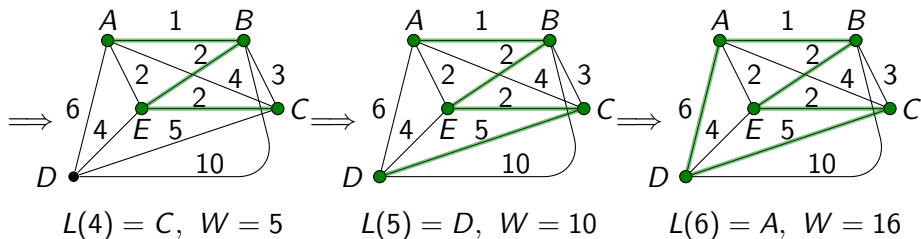
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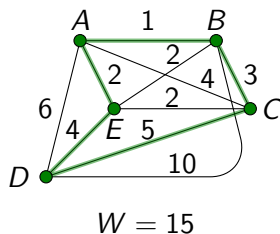
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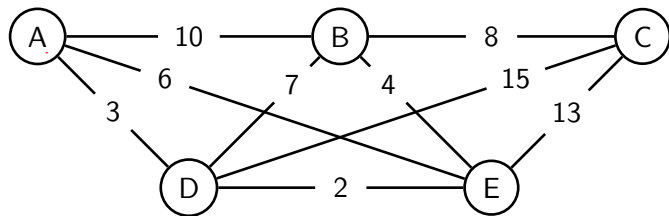
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Note that Nearest Neighbour may generate this tour if we start at D instead of A . Then $L(2) = E$ and it just depends on the choice for $L(3)$.



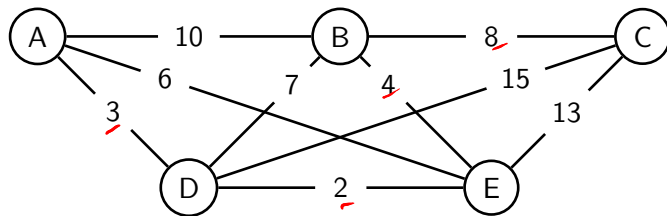
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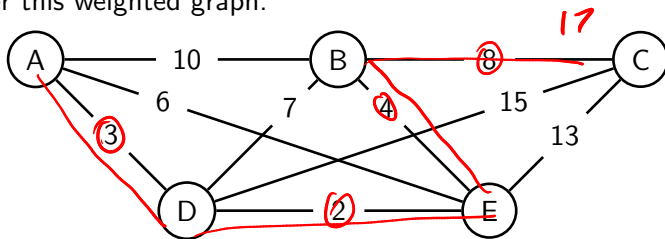
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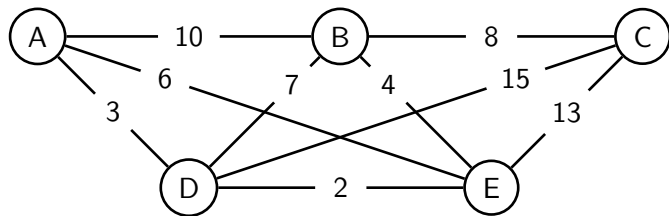


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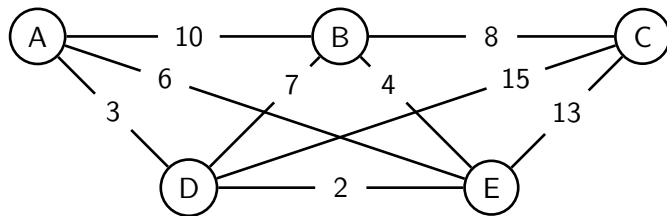
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For a small graph like this you can soon find a shortest path just by trying many alternatives (there are 10 or so simple A→C paths here).

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Consider this weighted graph:



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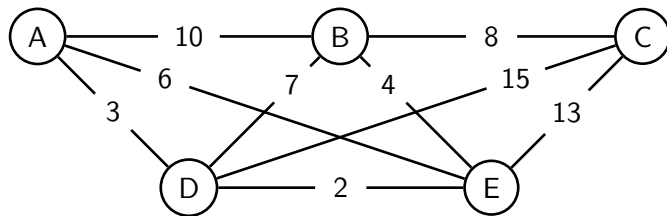
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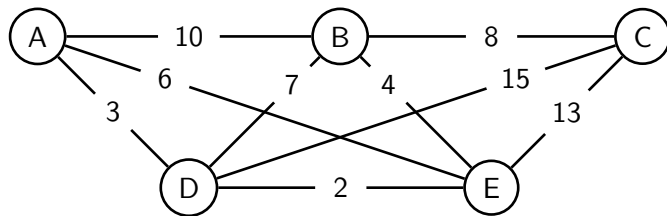
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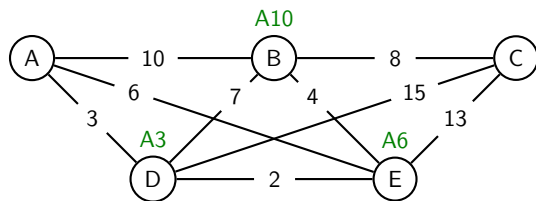
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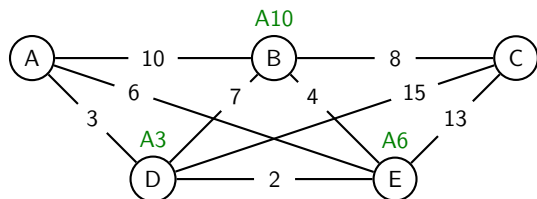
For large graphs this approach is not practical. We need an *algorithm*.

Dijkstra's Algorithm



Edsger Dijkstra 1930 - 2002

Dijkstra's Algorithm

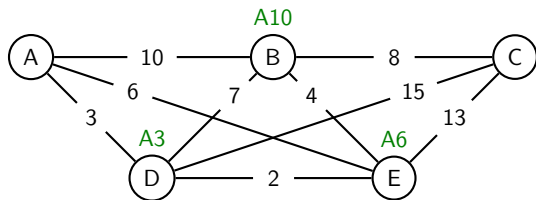


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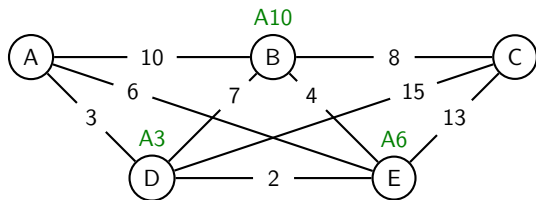


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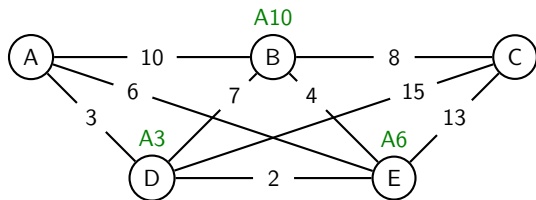


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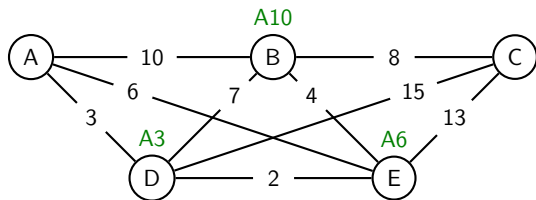
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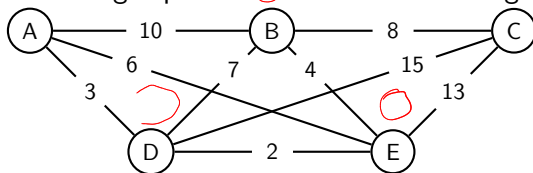
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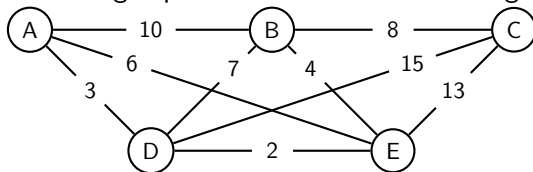
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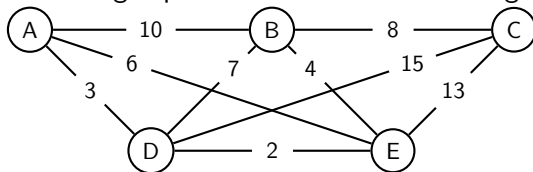
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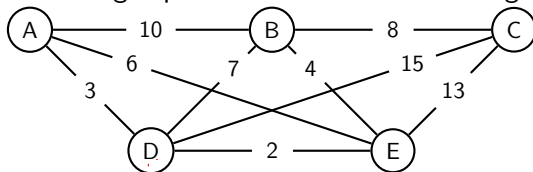
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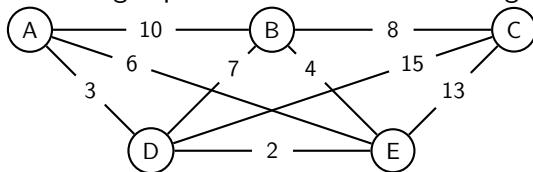


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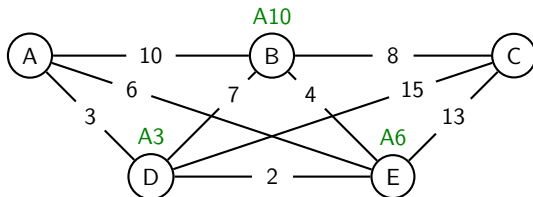
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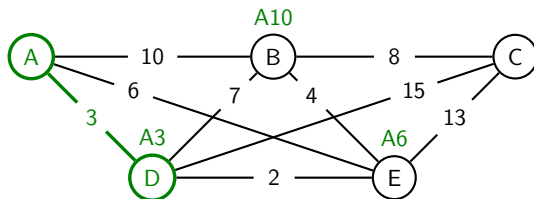


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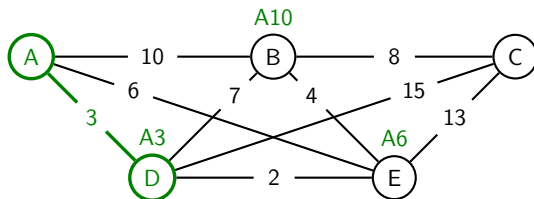
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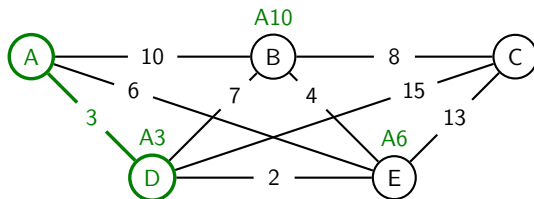
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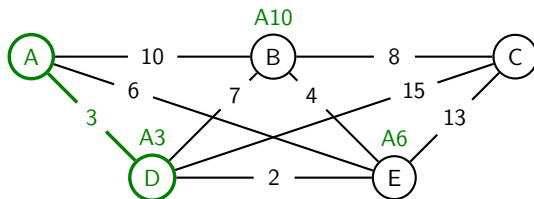


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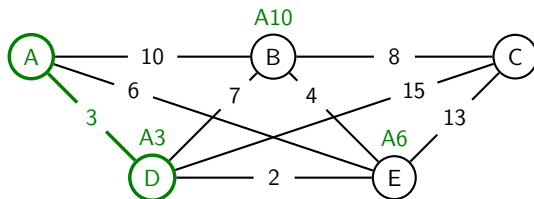
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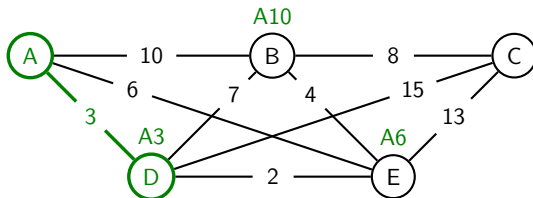
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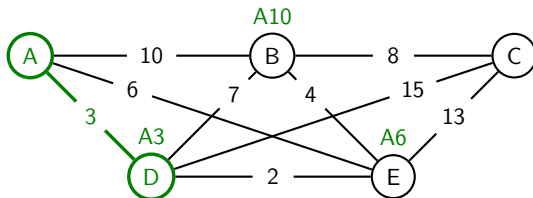
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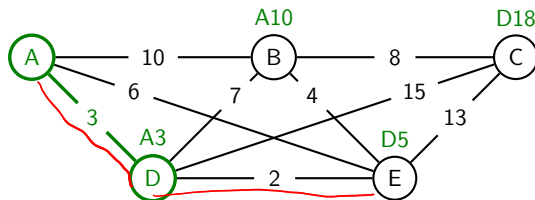
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So we leave the **A10** above B as it is.

The annotated graph now looks like this:



Example 1 — Slide 4

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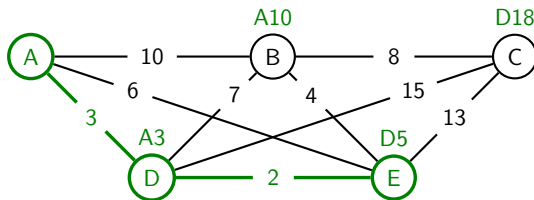
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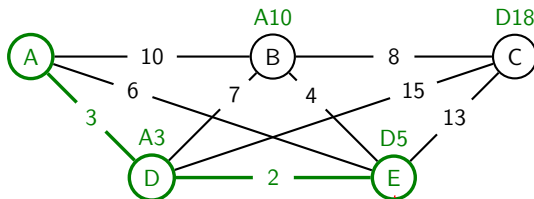
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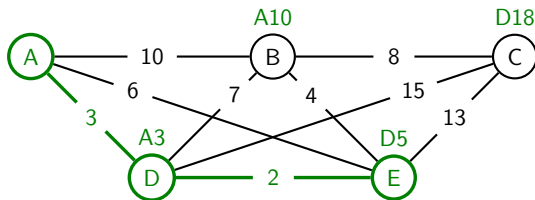
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We now repeat the process applied to the previous current vertex D.

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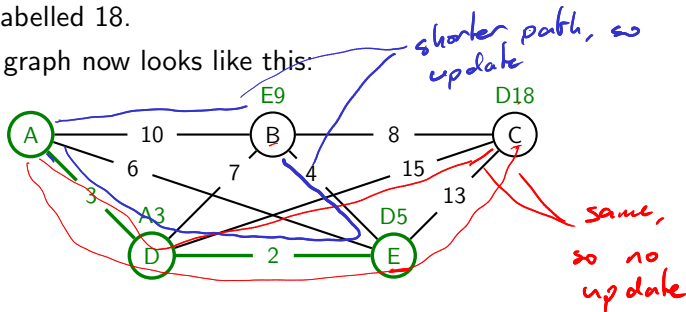
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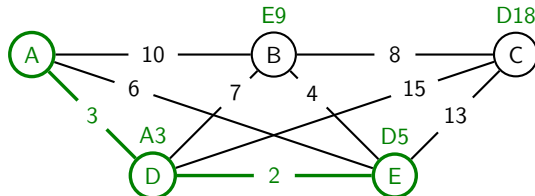
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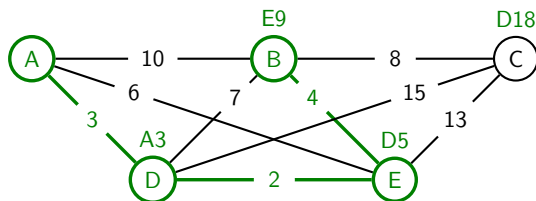
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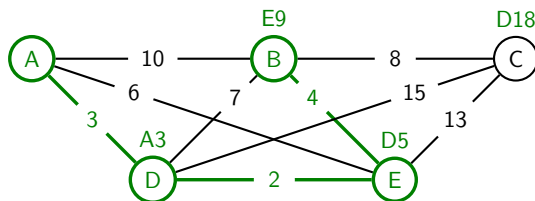


The lowest fringe value is now 9 on B, so we lock in B and its lead-in edge EB (next slide).

Example 1 — Slide 6

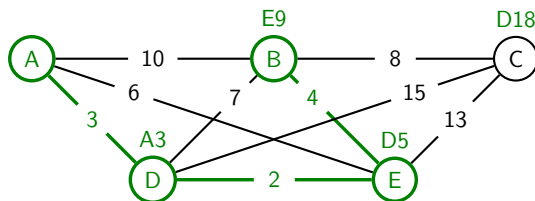


Example 1 — Slide 6



The new current vertex is the just locked-in B.

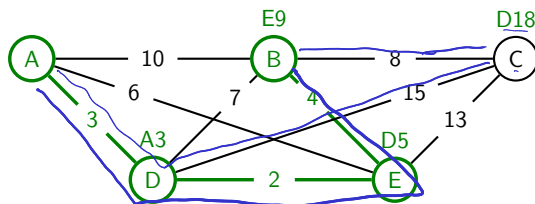
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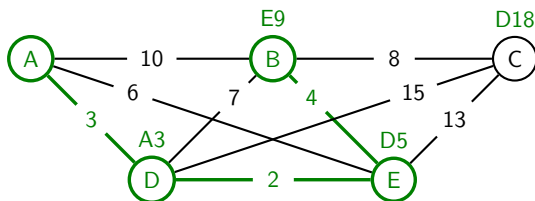
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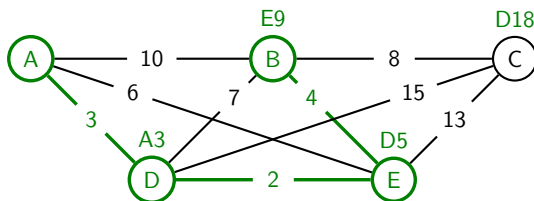


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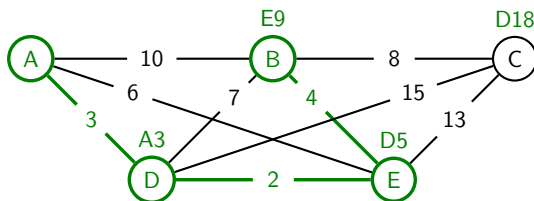
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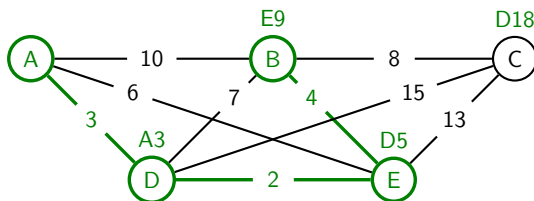
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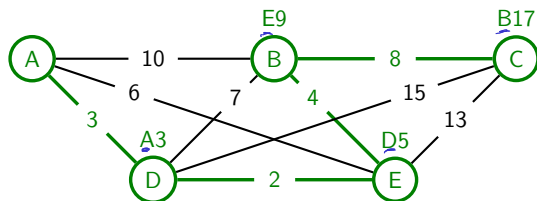
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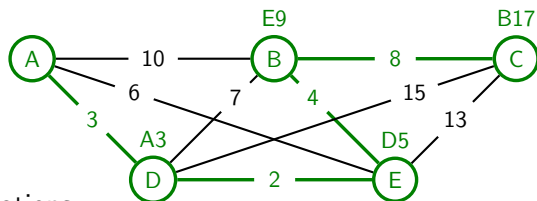
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Example 1 — Slide 7; Results and Comments



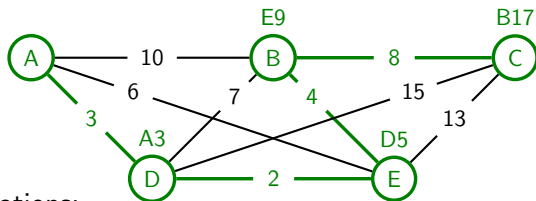
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Some Observations:

- Besides the shortest path from A to C, the solution provides the shortest path to all the vertices along that path. For this example that happens to be the entire vertex set.

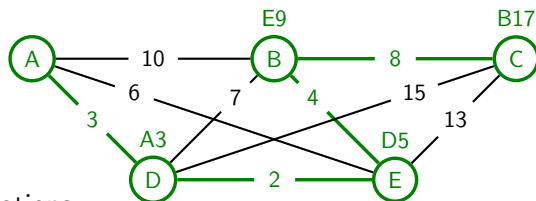
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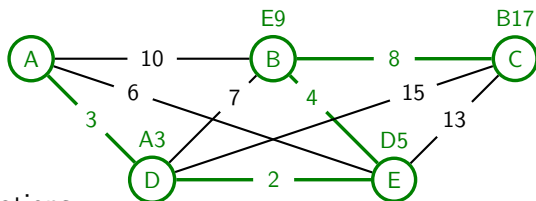
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- With all vertices locked, the solution provides a spanning tree for the graph.

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 (b) Distance function $\text{dist}: E(G) \rightarrow \mathbb{Q}^+$.

- Output:** (a) Tree T containing A and Z as vertices.
 T is a subgraph of G .

The unique path $A \rightarrow Z$ in T has minimal total distance of all paths $A \rightarrow Z$ in G .

- (b) 'Labelling' $L: V(T) \rightarrow \mathbb{Q}_+$; $L(v) = \min.\text{dist}(A \rightarrow v)$.

- Method:** 1. Initialize the tree T : Set $V(T) = \{A\}$, $E(T) = \emptyset$.
 2. Initialize a 'Marking' function $M: \underline{V(G)} \rightarrow \underline{V(G)} \cup \{\underline{\text{blank}}\}$:
 Set $M(v) = \text{blank}$ for all $v \in G$.

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While $c \neq Z$:

4. For each vertex v adjacent to c but not in T :

If v is unmarked (i.e. $M(v) = \text{blank}$)

or if $L(v) > L(c) + \text{dist}(\{c, v\})$

set $M(v) = c$, $L(v) = L(c) + \text{dist}(\{c, v\})$.

i.e. best path is to date via c

Dijkstra's Algorithm — A Formal Description (cont.)

5. From all marked $v \in G \setminus T$ (i.e. $M(v) \neq \text{blank}$ and $v \notin T$) (such v are said to be 'on the fringe') select one, say w , with minimal $L(v)$.

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This completes the formal description of Dijkstra's shortest path algorithm.

Note: we add vertices + edges anywhere along the tree, not just off the current vertex.

Dijkstra's Algorithm — A Formal Description (cont.)

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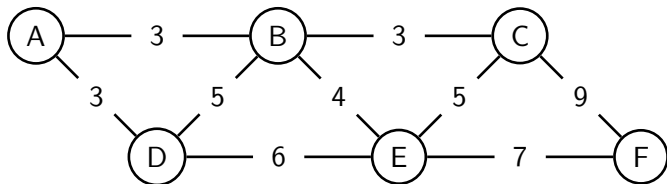
End of Method

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Next a second example, but this time with less commentary.

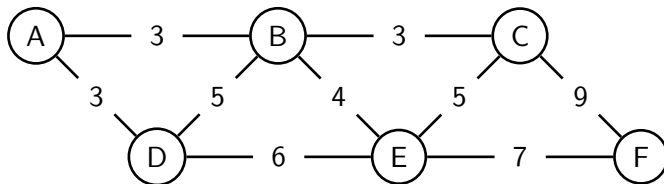
Example 2 – Slide 1

Find the shortest path for A to F:



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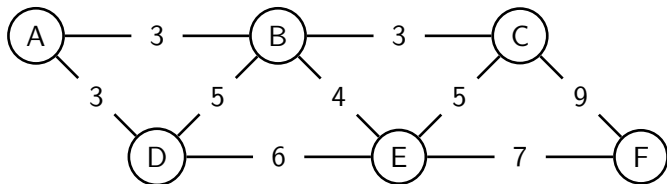
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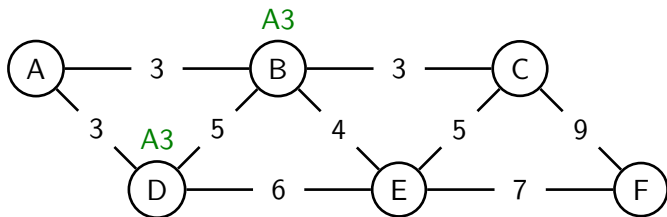
First annotate the vertices adjacent to the start vertex A:

Example 2 – Slide 1

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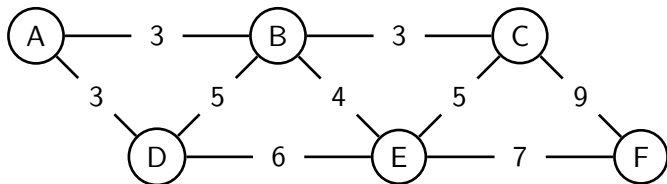


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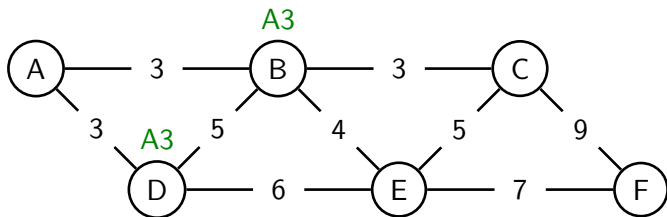


Example 2 – Slide 1

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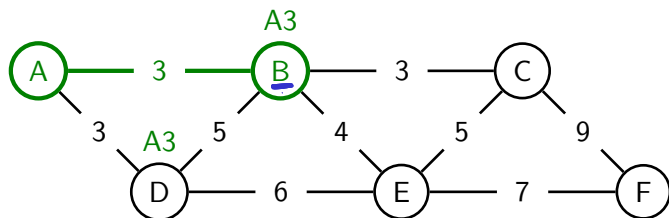


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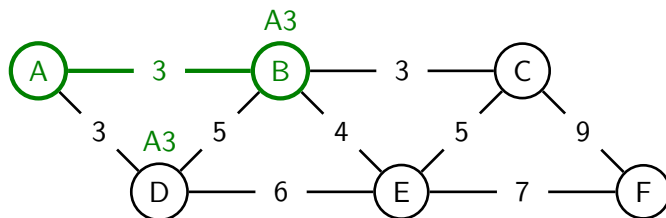


Vertices B and D have equal lowest label; let's lock in B:

Example 2 – Slide 2

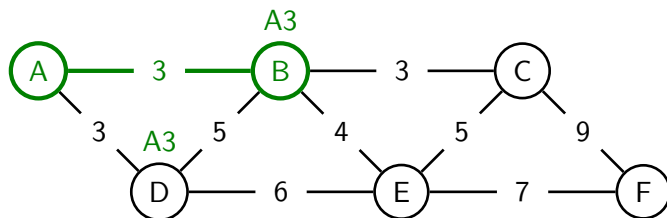


Example 2 – Slide 2



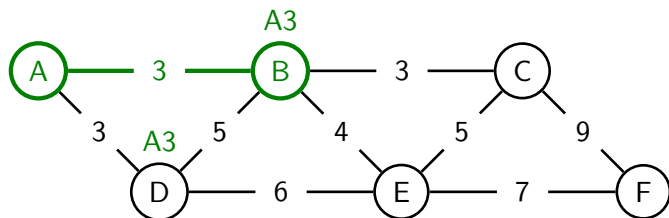
Current vertex is now B. Fringe vertices will be C,D,E.

Example 2 – Slide 2

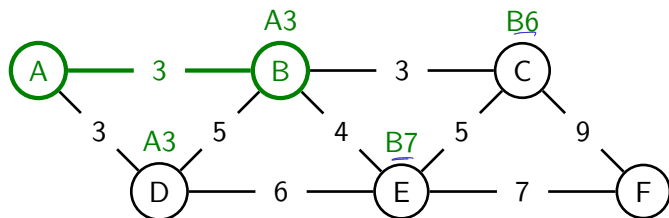


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Annotations required for C and E but D's does not need updating.

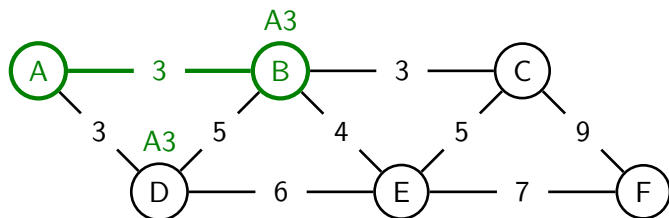
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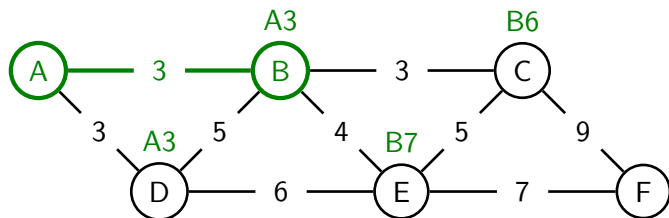
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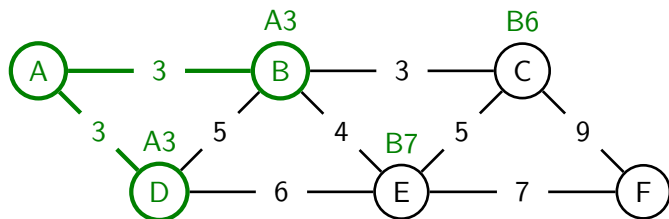


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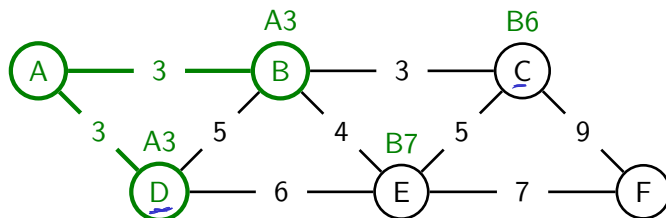


D now has lowest label so needs locking in next:

Example 2 – Slide 3

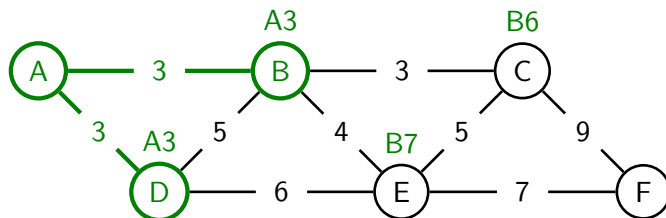


Example 2 – Slide 3



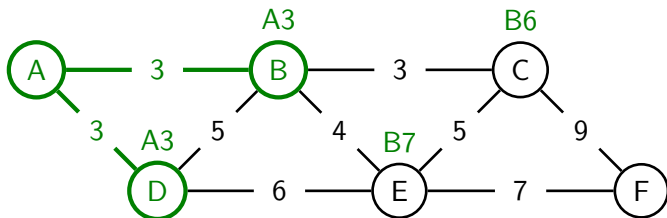
Current vertex is now D. Its only un-locked neighbour is E but no updating is required.

Example 2 – Slide 3

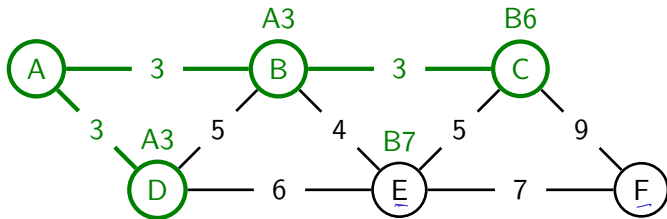


Current vertex is now D. Its only un-locked neighbour is E but no updating is required. The fringe vertices are C and E. Vertex C has lower label value so it is next to be locked:

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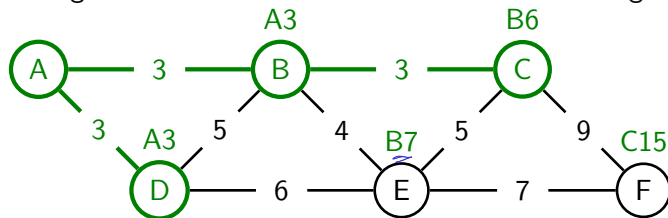


Example 2 – Slide 4

Of the two un-locked vertices adjacent to C, E is marked but does not need updating while F is unmarked and so needs annotating:

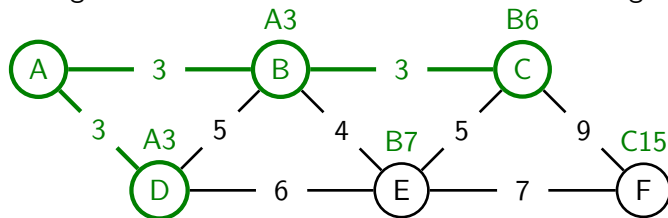
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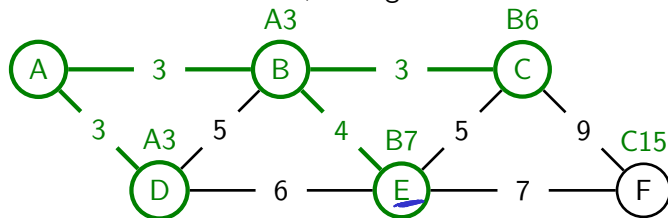


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Of the two fringe vertices, E has the lower label value so is locked in. Its lead-in vertex is marked as B, so edge BE is also locked in.

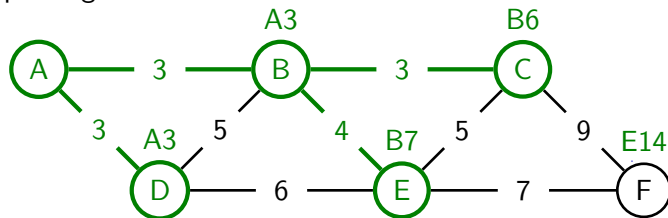


Example 2 – Slide 5

The new current vertex E has only one un-locked neighbour, F, and F needs updating since $7+7 < 15$:

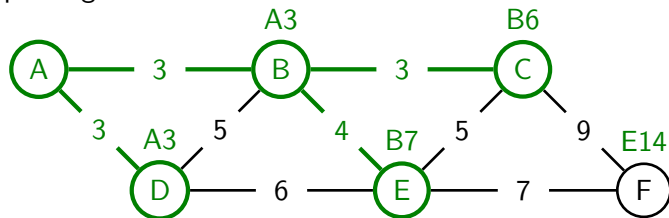
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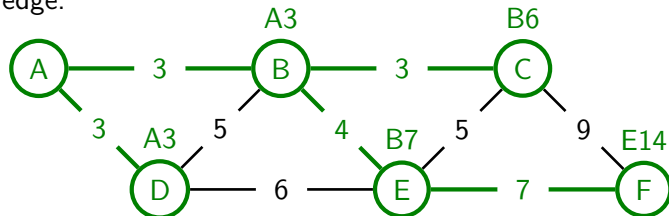


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Now F is the only fringe vertex, so F is locked in, together with its lead-in edge.



Example 2 — Slide 6; Results and Comment

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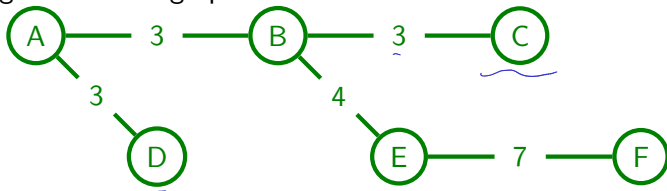
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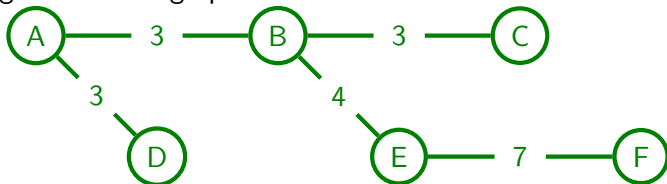


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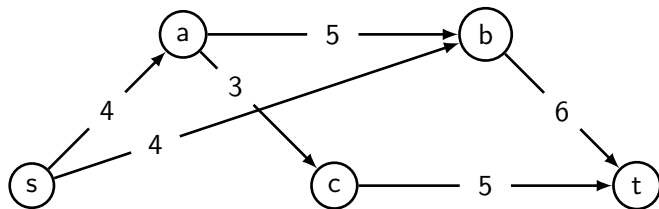
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As it happens, this is a minimal spanning tree. However, in general a spanning tree produced by Dijkstra's algorithm will not be minimal.

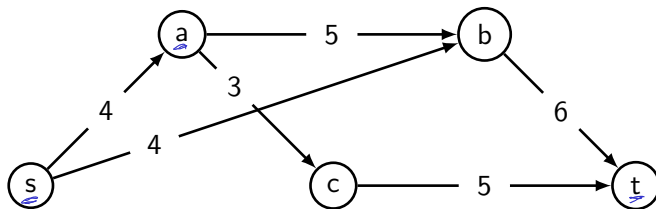
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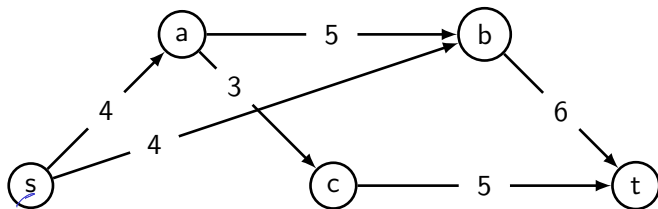
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*eg. pipe diameter
number of lanes
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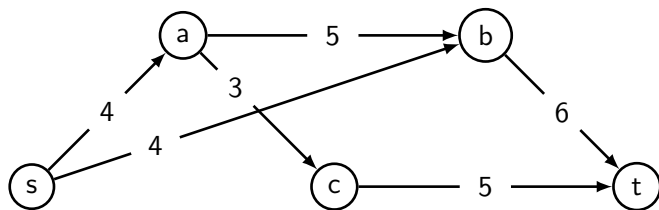


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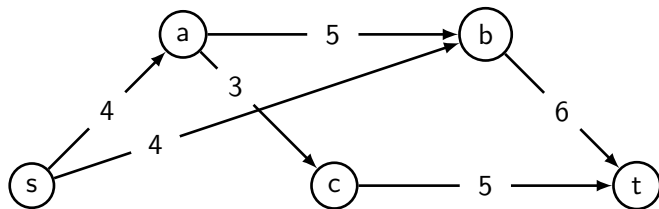


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- Every edge lies on some simple (directed) path from s to t.

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- (1) Flow cannot exceed capacity. $[\forall e \in E(D) \ F(e) \leq C(e).]$
- (2) In each edge, flow direction = edge direction.
- (3) Total flow into a node equals total flow out, except for nodes s, t .
 $[\forall v \in V(D) \setminus \{s, t\} \ \sum_{e \in v_{\text{in}}} F(e) = \sum_{e \in v_{\text{out}}} F(e), \text{ where } v_{\text{in}}, v_{\text{out}} \text{ are the sets of edges coming in to, and out of, } v, \text{ respectively.}]$

Finding a maximum flow

For any flow F the constraints on the network imply that the total flow out of the source must equal the total flow into the target:

$$\sum_{e \in s_{\text{out}}} F(e) = \sum_{e \in t_{\text{in}}} F(e) = v(F) \text{ say.}$$

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This common total $v(F)$ is the **flow value** and any flow achieving the maximum possible flow value is called a **maximum flow**, F_{max} . (There may be several flows that achieve the maximum.)

I will demonstrate a vertex labelling algorithm for finding an F_{max} . The algorithm starts with the **zero flow** F_0 , where $\forall e \in D \ F_0(e) = 0$. Then, over a number of stages, expanded flows F_1, F_2, \dots are obtained until F_{max} is reached.

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At stage i , flow F_i is constructed as $F_i = F_{i-1} + f_i$, where the incremental flow f_i is based on a constant $k_i \in \mathbb{Q}^+$ and a simple path p_i from s to t :

$$f_i(e) = \begin{cases} k_i & \text{for every edge } e \text{ on the path } p_i \\ 0 & \text{for every other edge } e. \end{cases}$$

Spare capacity

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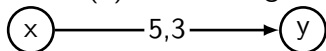
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To depict a flow I will follow the capacity value $C(e)$ on each (directed) edge e with the flow value $F(e)$ for that edge. For example



represents a flow of 3 in the edge from x to y , with spare capacity 2.

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At each stage of the vertex labelling algorithm levels and labels are associated afresh with the vertices of the network.

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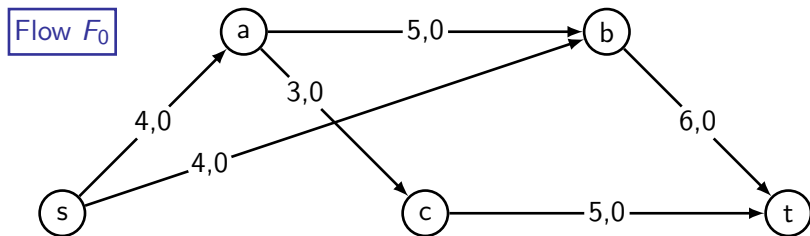
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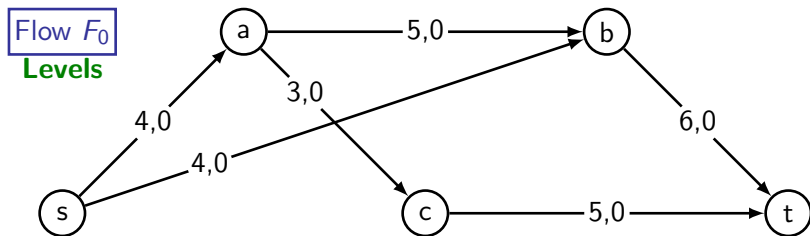
Vertex labelling algorithm, Example 1

Stage 1: F_0 to F_1



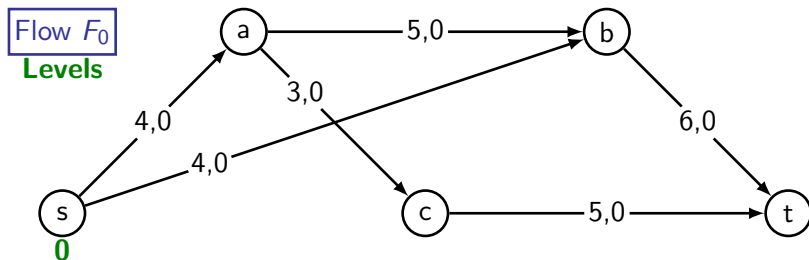
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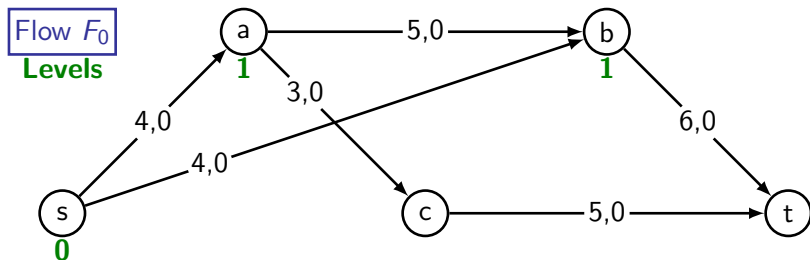
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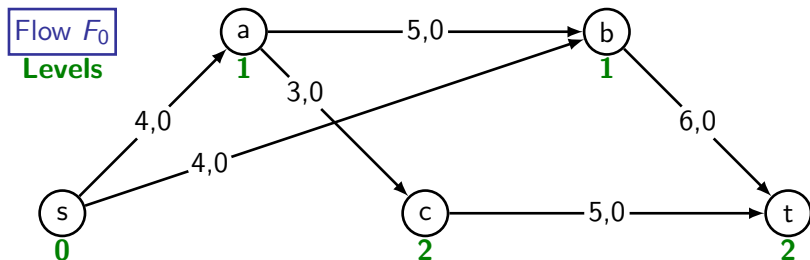
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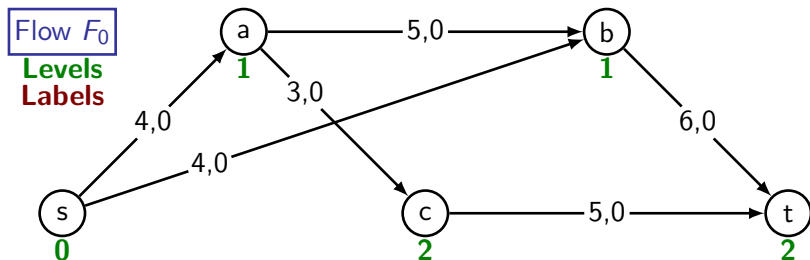
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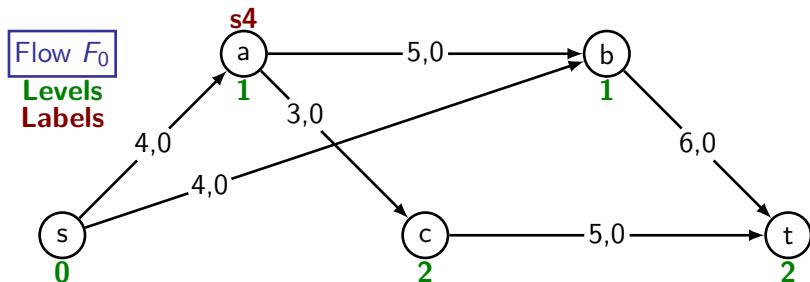
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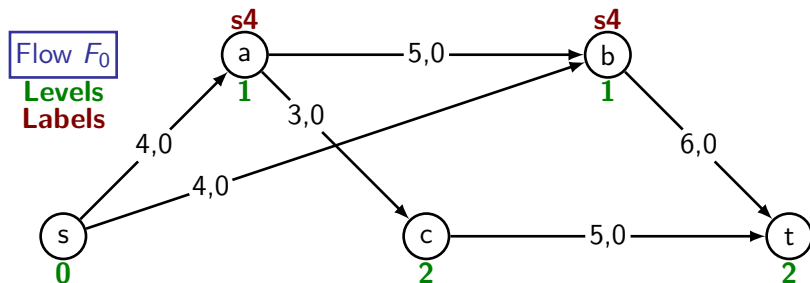
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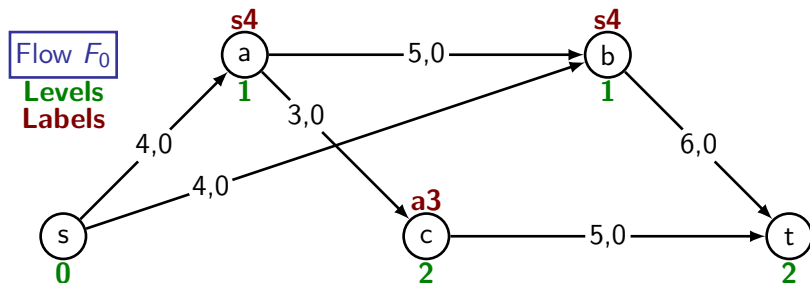
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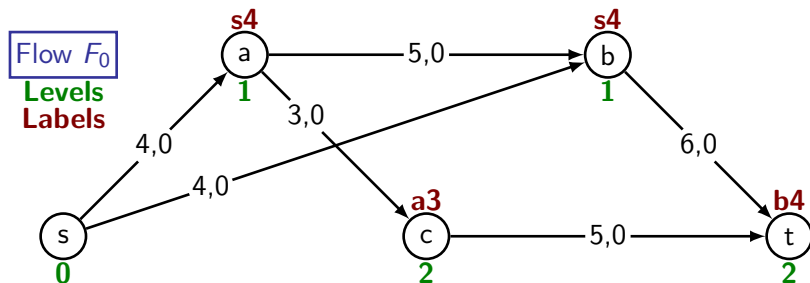
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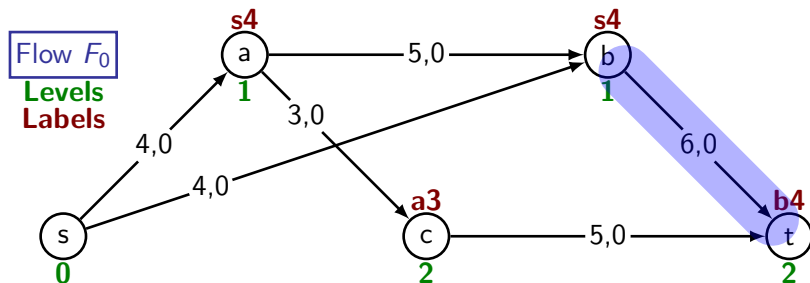
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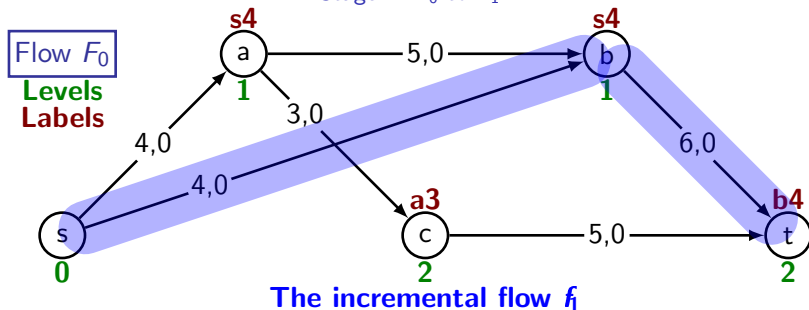
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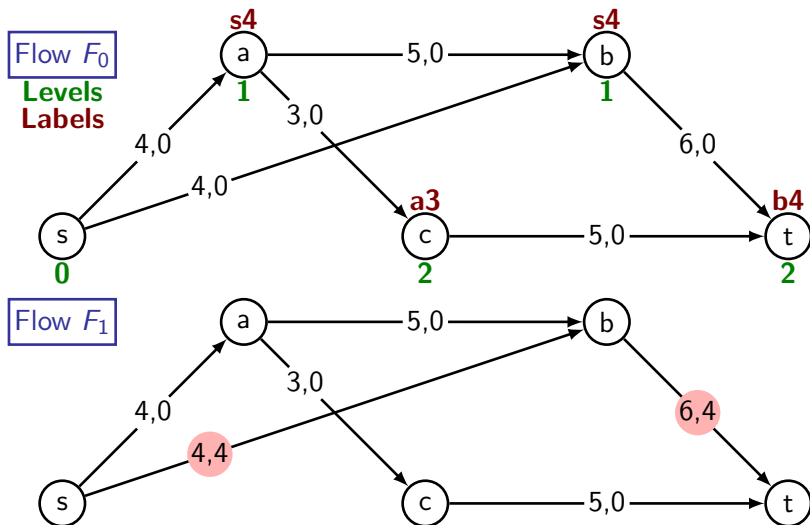
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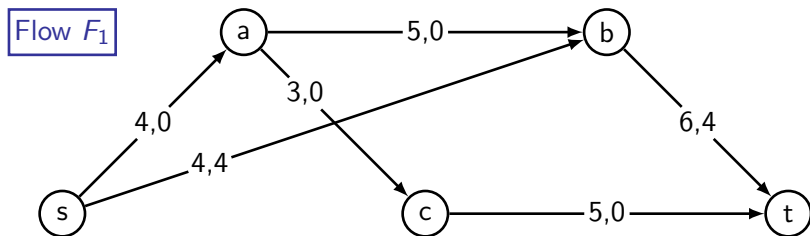
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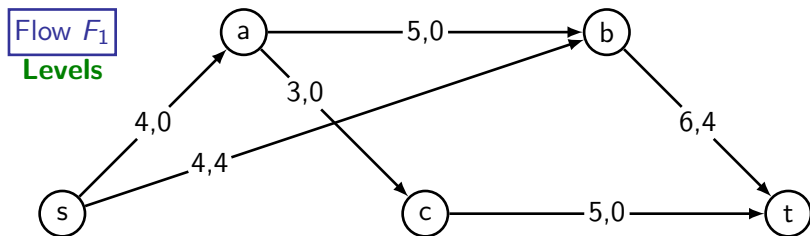
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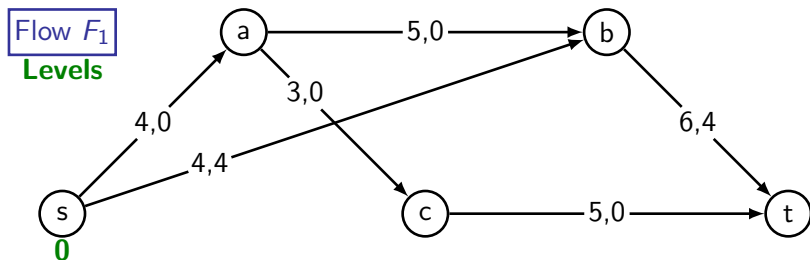
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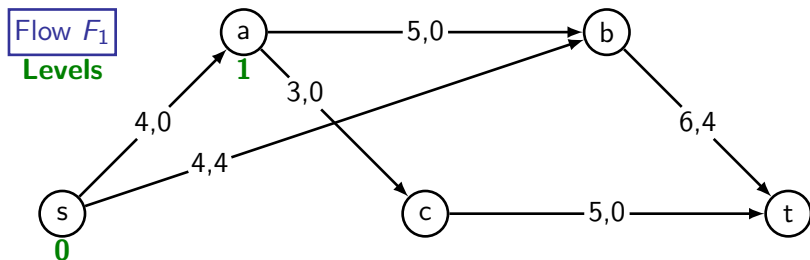
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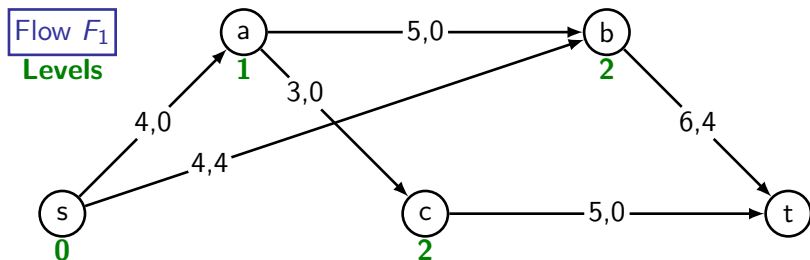
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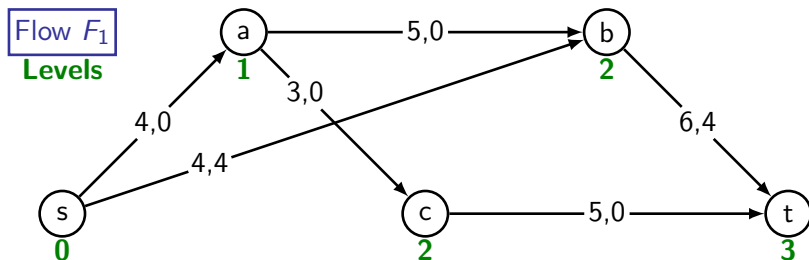
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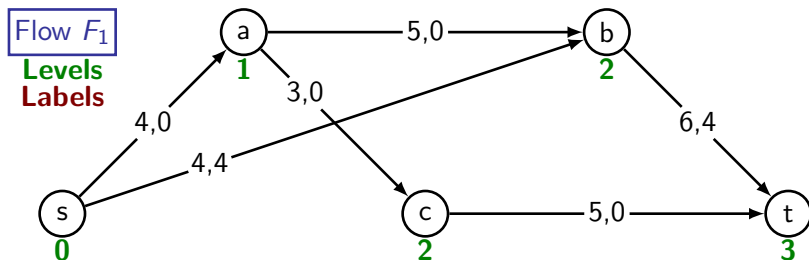
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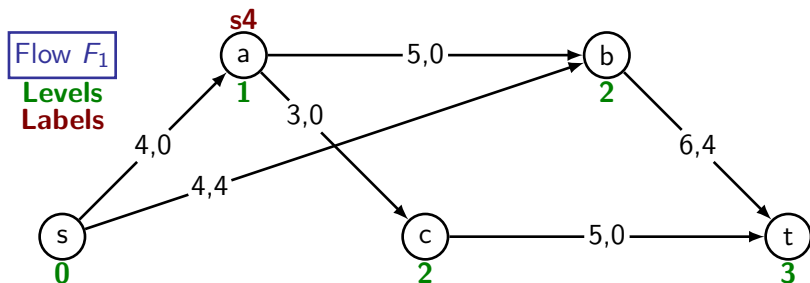
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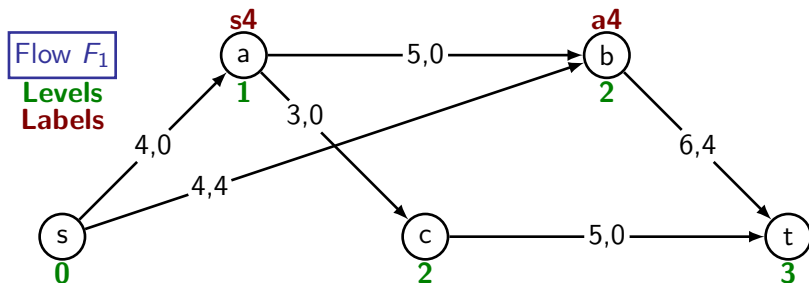
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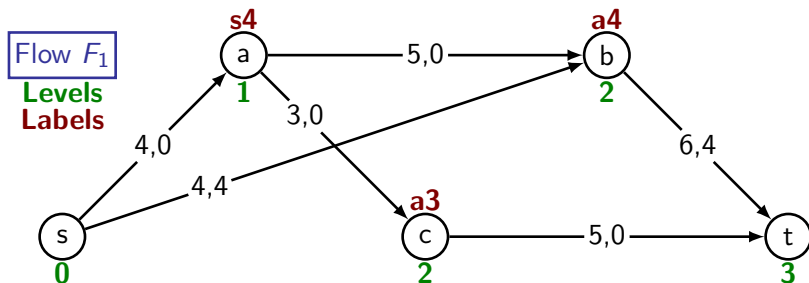
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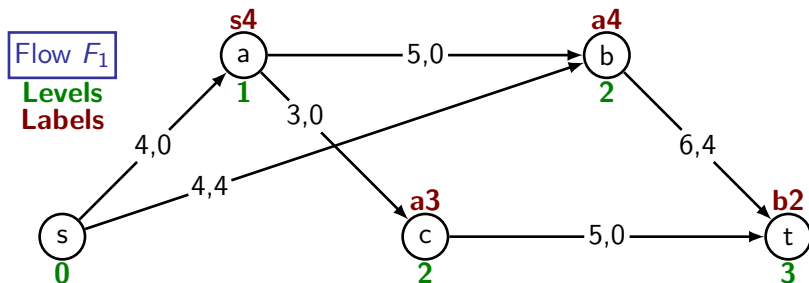
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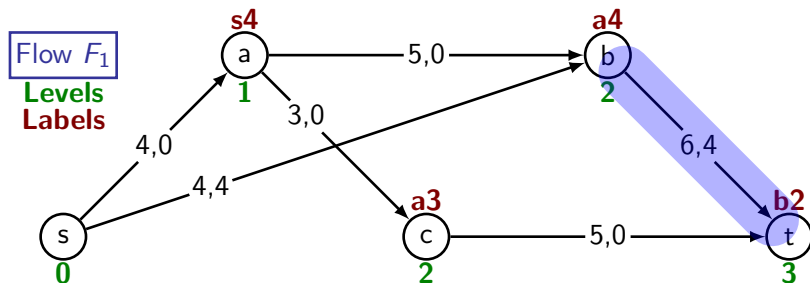
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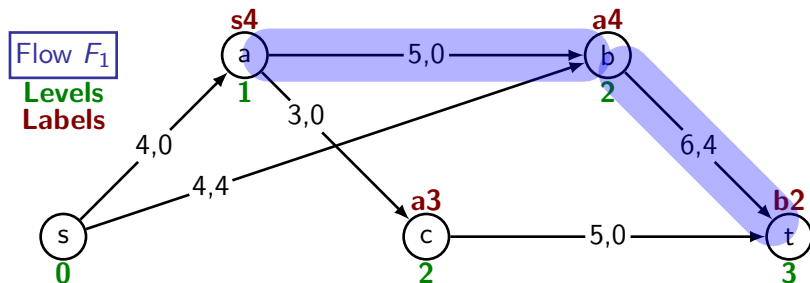
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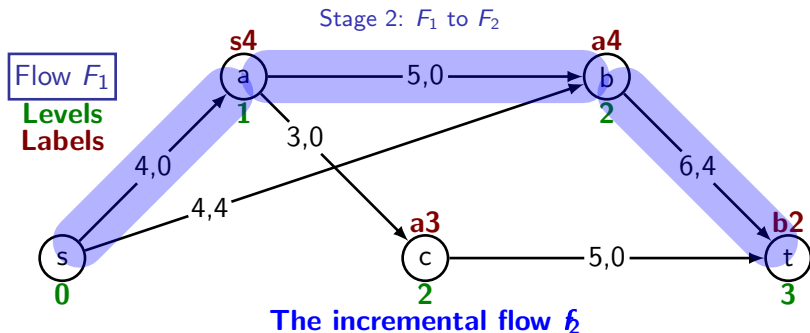


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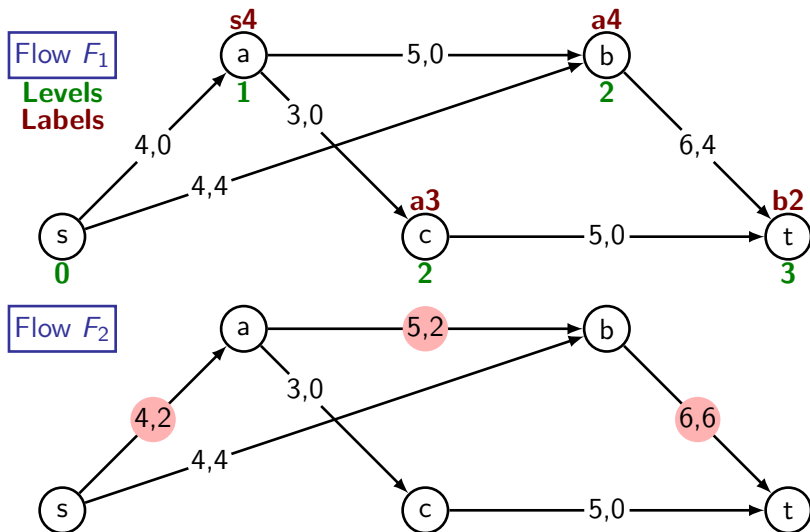


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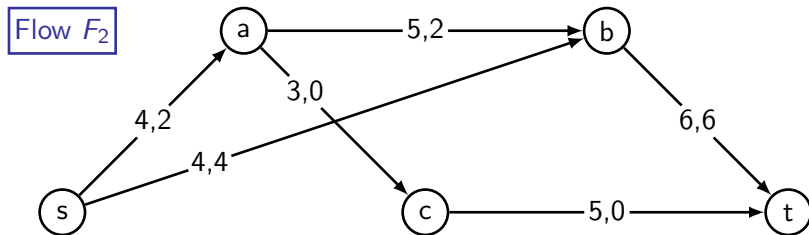
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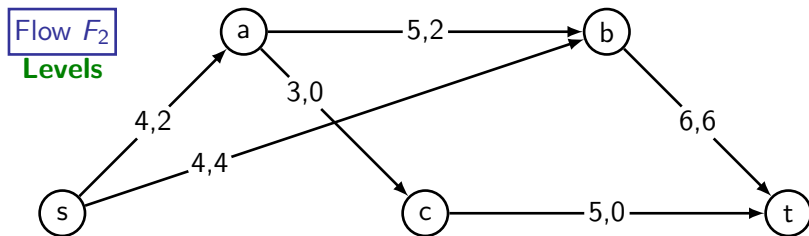
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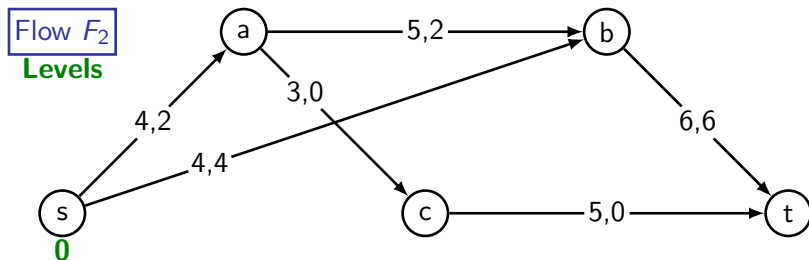
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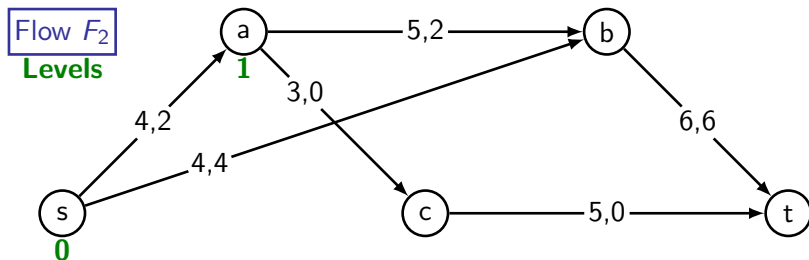
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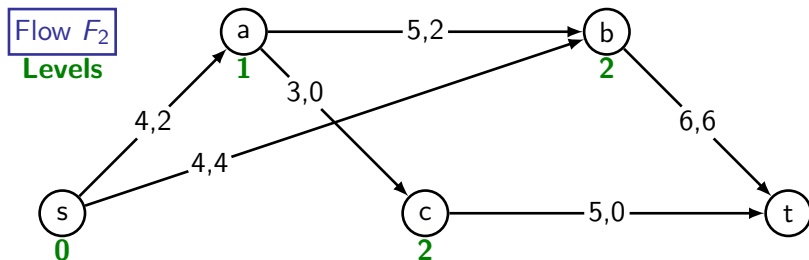
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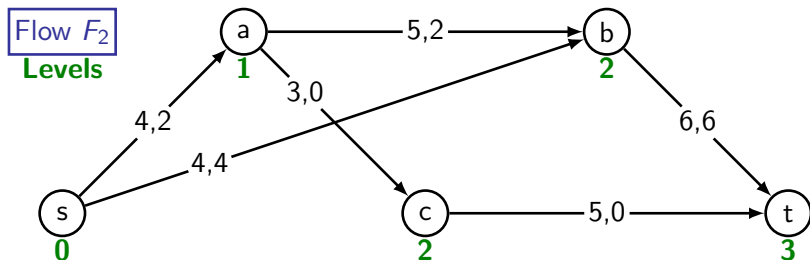
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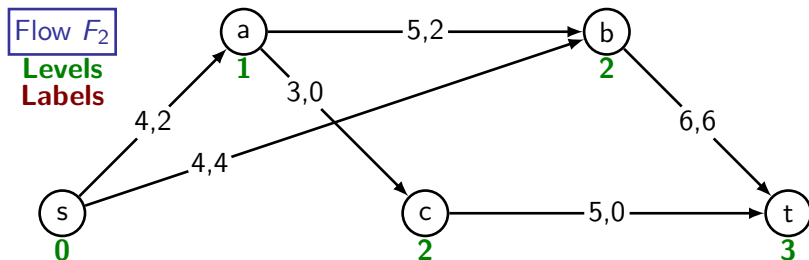
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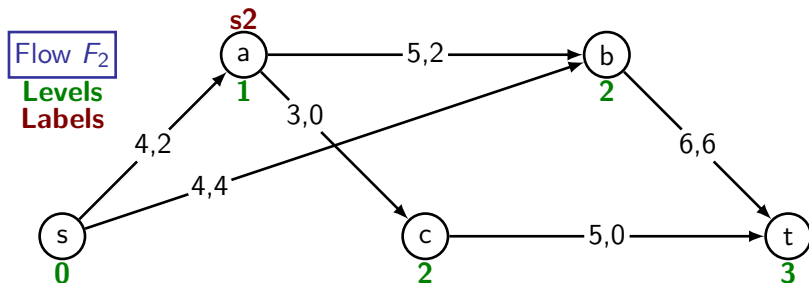
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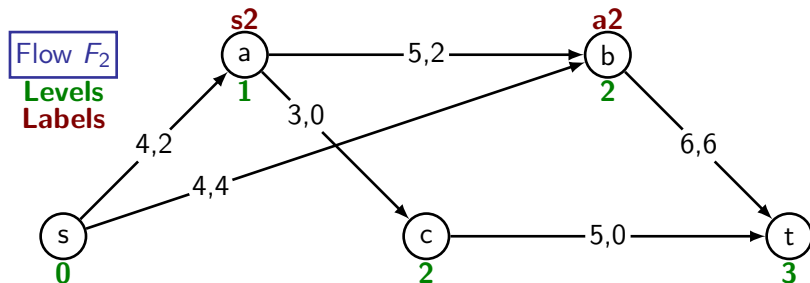
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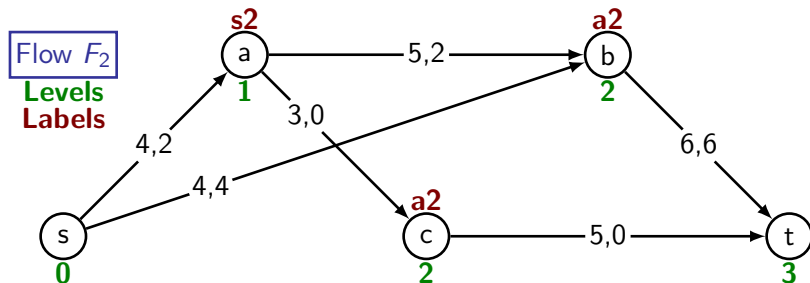
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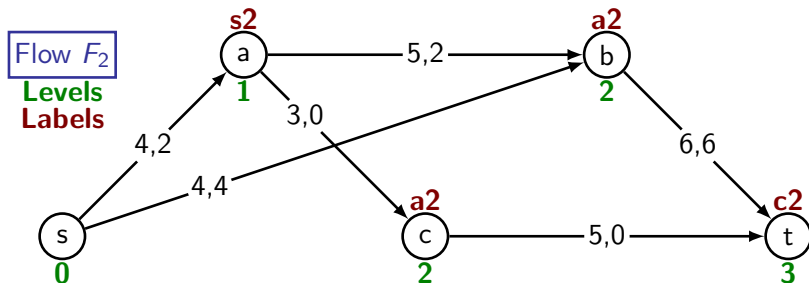
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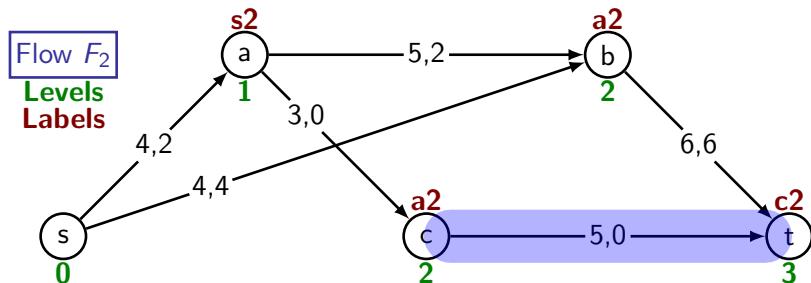
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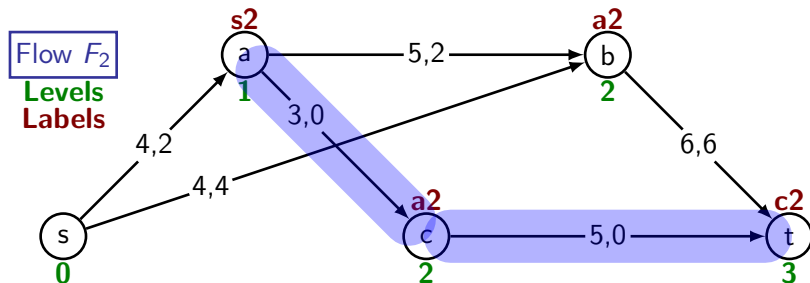
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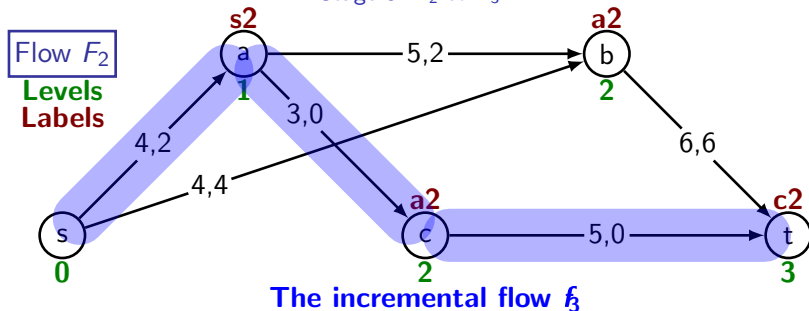
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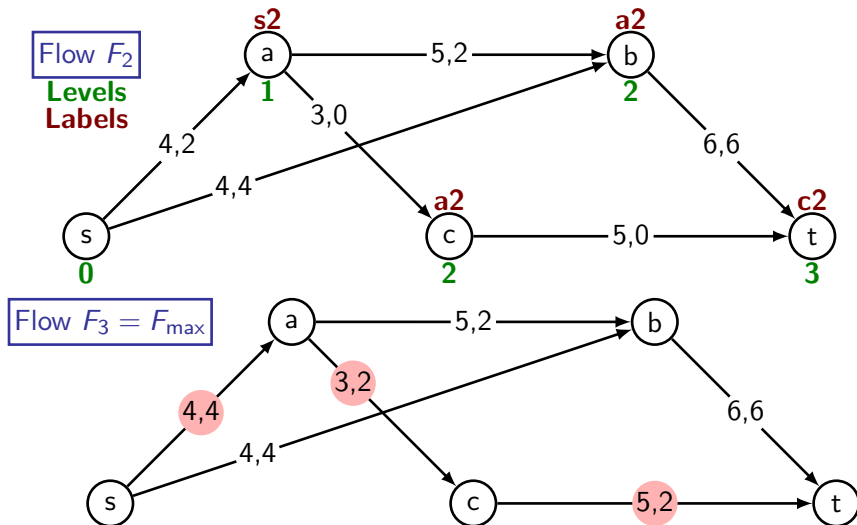
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3. Next slide....

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 - k_v is the minimum of $S((u,v))$ and the value part of u 's label.

3. If t is assigned a level, stage i will succeed, so continue.
If not, then stage i fails, so return above to define F_{\max} and terminate.
4. Mark up labels for F_{i-1} as follows until t is labelled:
 - (a) Label each level 1 vertex v with sk_v , where $k_v = S((s,v))$.
 - (b) If t has level 2 or more now work through the level 2 vertices in alphabetical order, labelling each vertex v with uk_u , where
 - u is the alphabetically earliest level 1 vertex with $(u,v) \in E(D)$ and $S((u,v)) > 0$,
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 - (c) If t has level 3 or more now work through the level 3 vertices in a similar manner and so on.

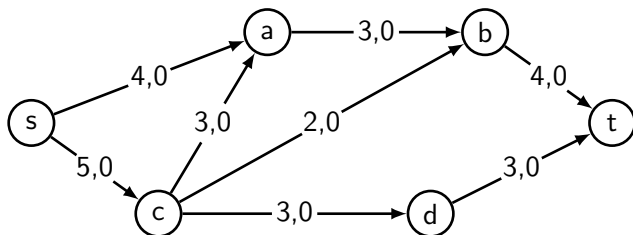
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 - (c) If t has level 3 or more now work through the level 3 vertices in a similar manner and so on.
5. Let p_i be the path $u_0 u_1 \dots u_n$ where $u_n = t$ and for $0 < j \leq n$ u_j has label $u_{j-1} k_j$.
Define f_i to be the incremental flow on p_i with flow value k_n .

End of Method

Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

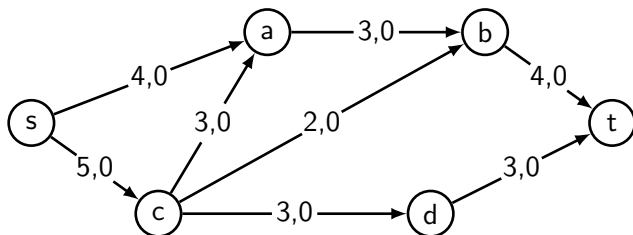
Flow F_0



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

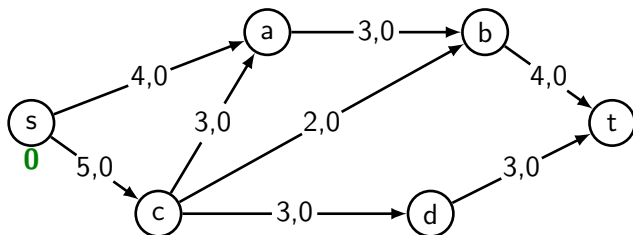
Flow F_0
Levels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

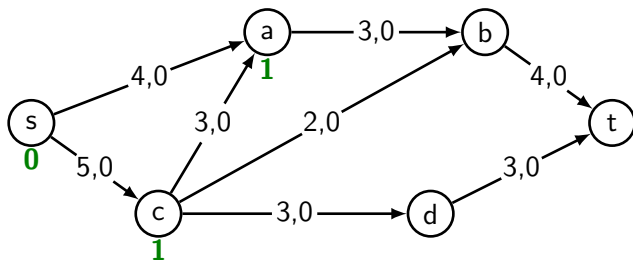
Flow F_0
Levels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

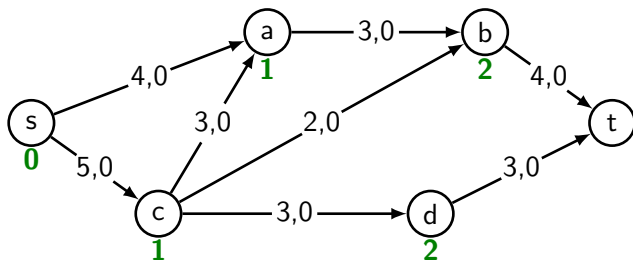
Flow F_0
Levels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

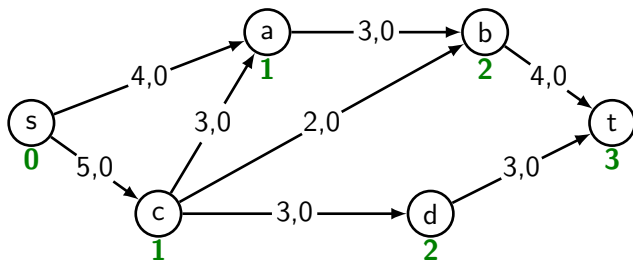
Flow F_0
Levels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

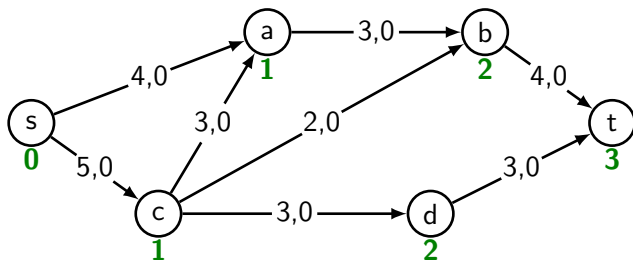
Flow F_0
Levels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

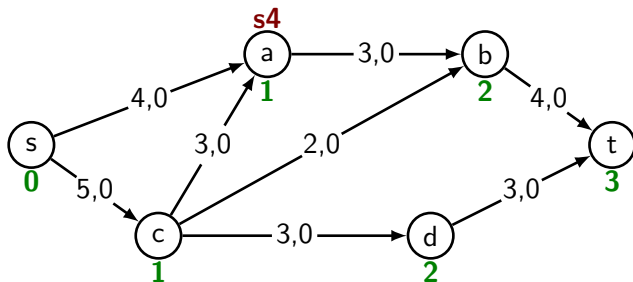
Flow F_0
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

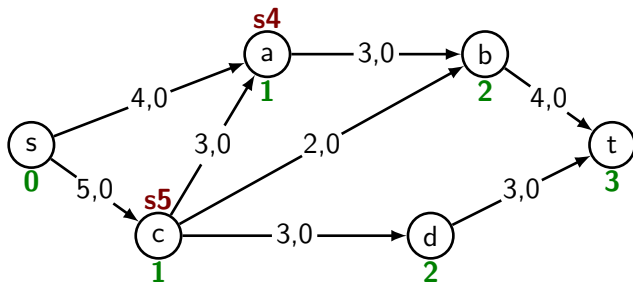
Flow F_0
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

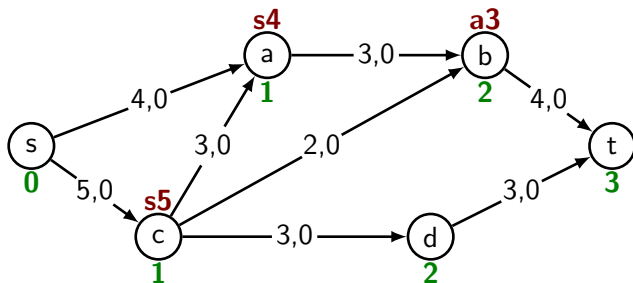
Flow F_0
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

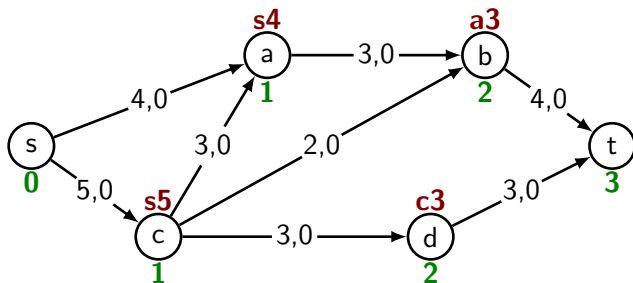
Flow F_0
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

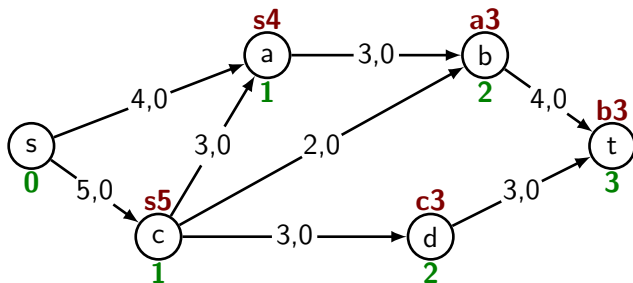
Flow F_0
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

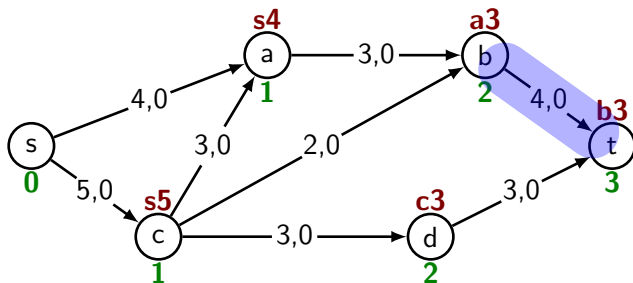
Flow F_0
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

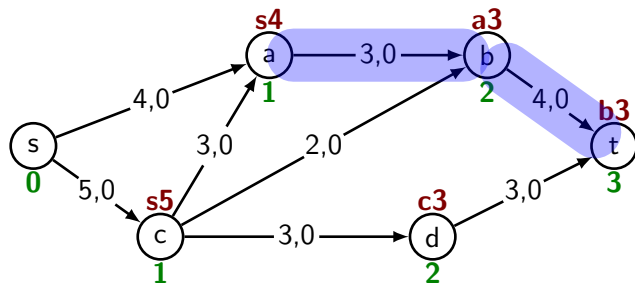
Flow F_0
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

Flow F_0
Levels
Labels

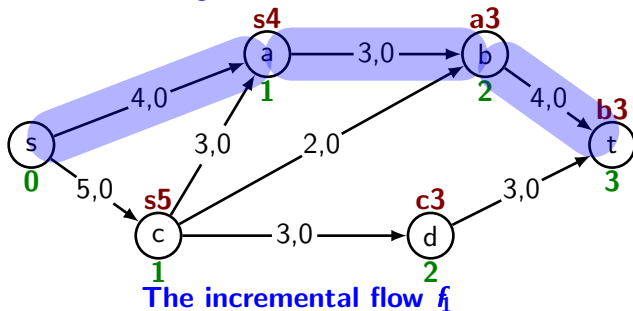


Vertex labelling algorithm, Example 2

Stage 1: F_0 to F_1

Flow F_0

Levels
Labels

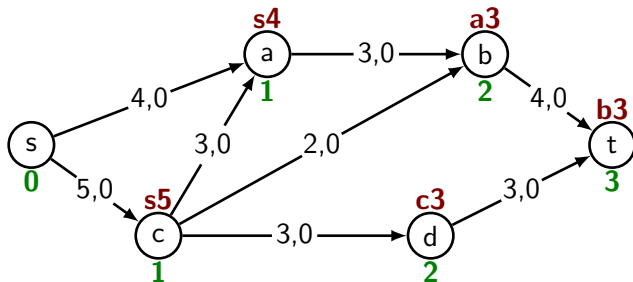


Vertex labelling algorithm, Example 2

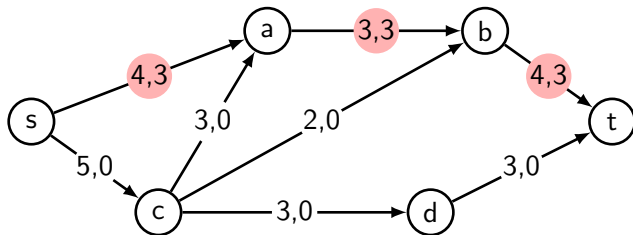
Stage 1: F_0 to F_1

Flow F_0

Levels
Labels



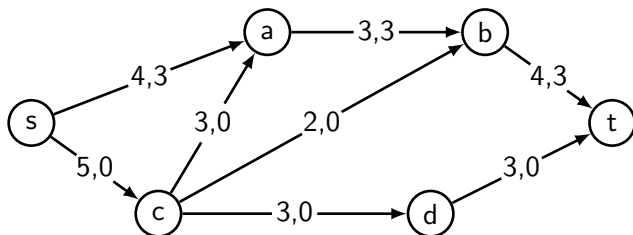
Flow F_1



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

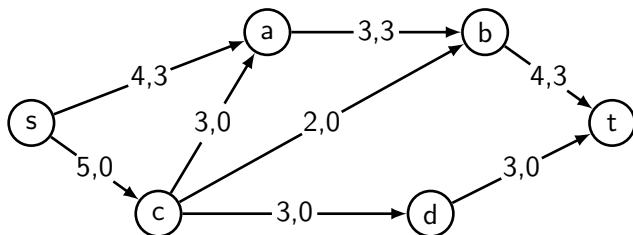
Flow F_1



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

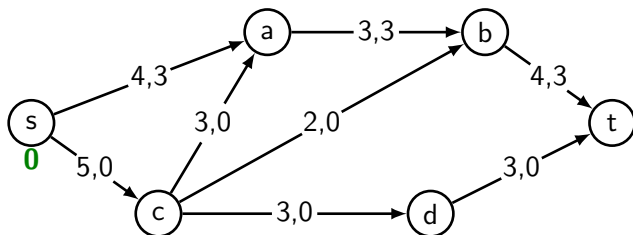
Flow F_1
Levels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

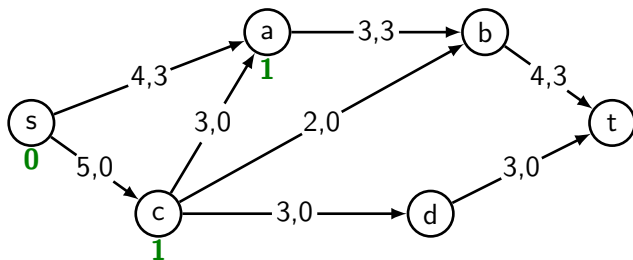
Flow F_1
Levels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

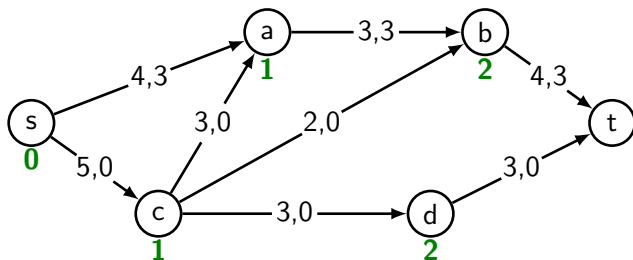
Flow F_1
Levels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

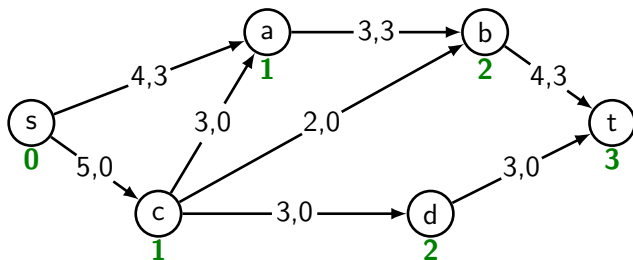
Flow F_1
Levels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

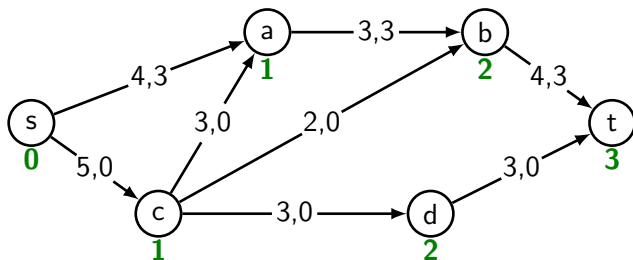
Flow F_1
Levels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

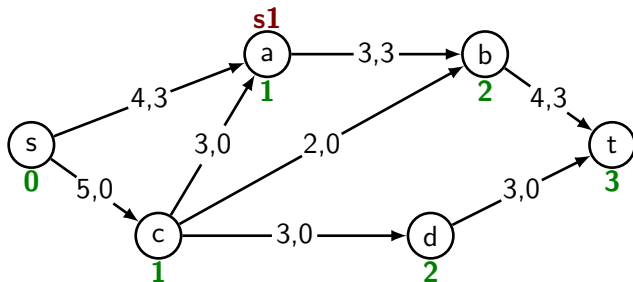
Flow F_1
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

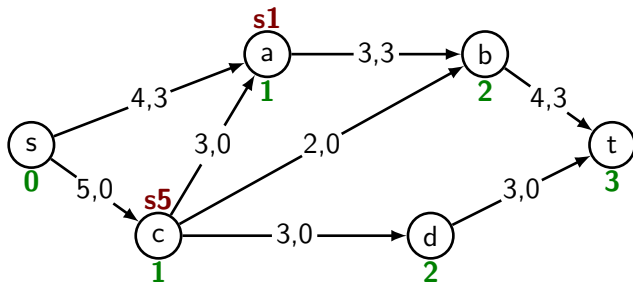
Flow F_1
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

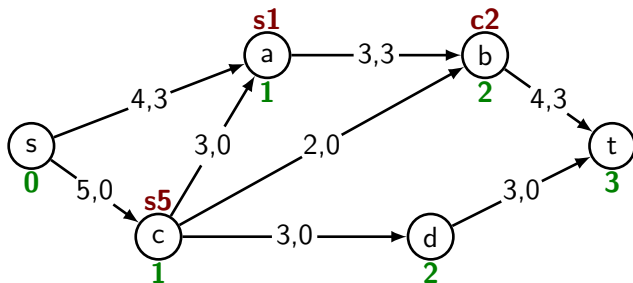
Flow F_1
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

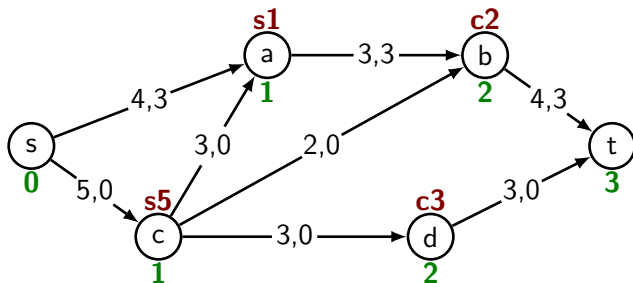
Flow F_1
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

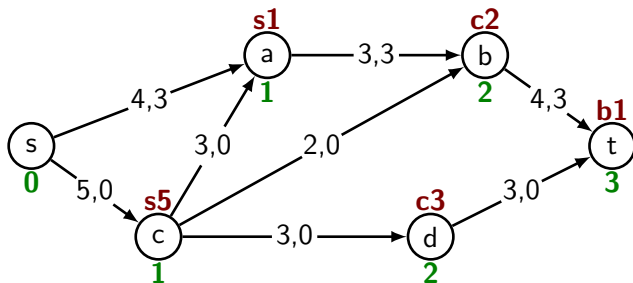
Flow F_1
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

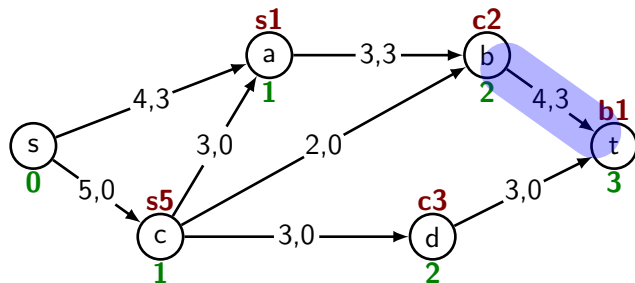
Flow F_1
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

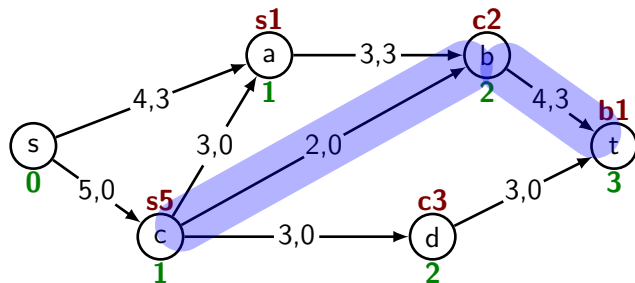
Flow F_1
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

Flow F_1
Levels
Labels

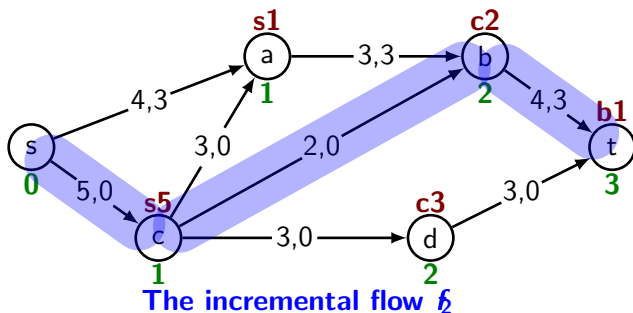


Vertex labelling algorithm, Example 2

Stage 2: F_1 to F_2

Flow F_1

Levels
Labels

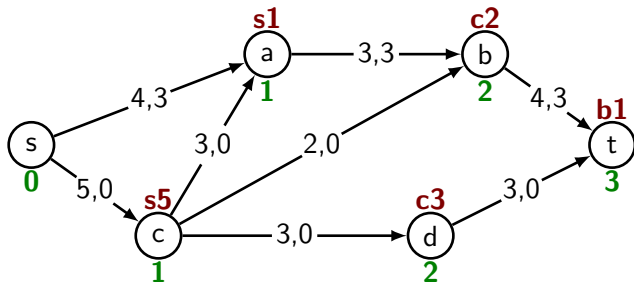


Vertex labelling algorithm, Example 2

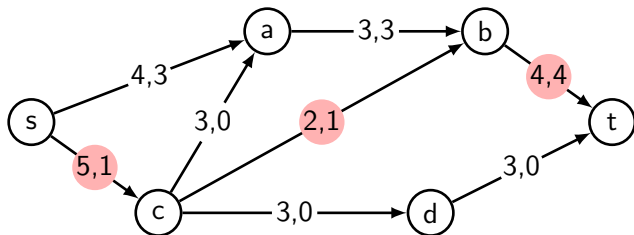
Stage 2: F_1 to F_2

Flow F_1

Levels
Labels



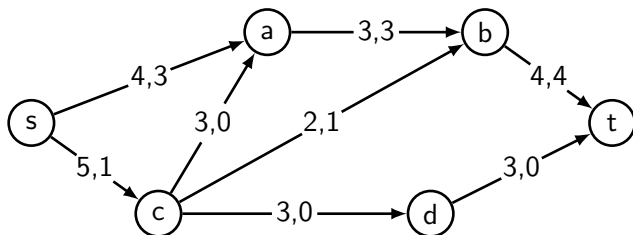
Flow F_2



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

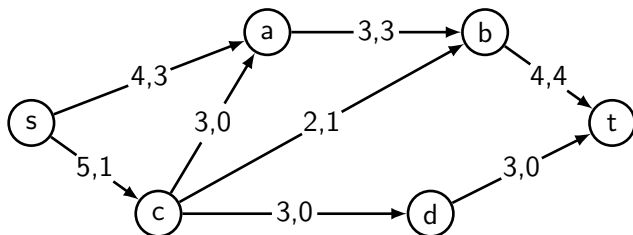
Flow F_2



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

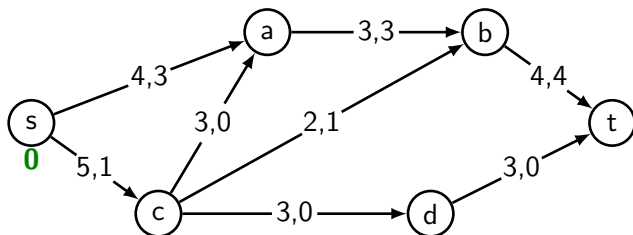
Flow F_2
Levels



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

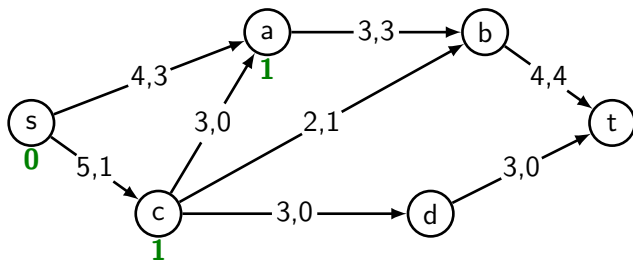
Flow F_2
Levels



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

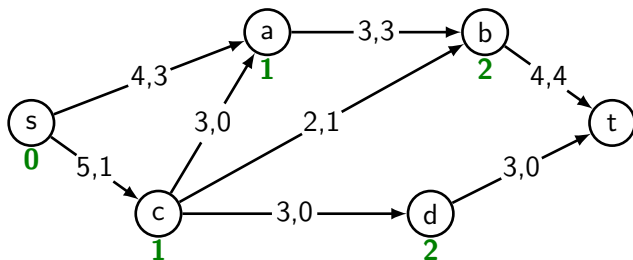
Flow F_2
Levels



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

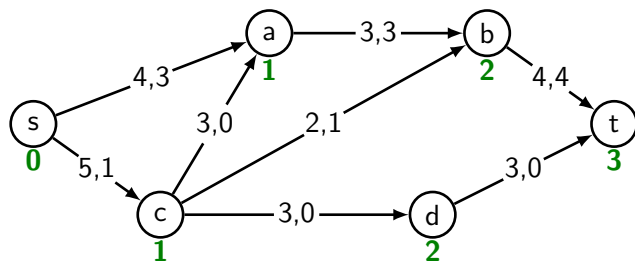
Flow F_2
Levels



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

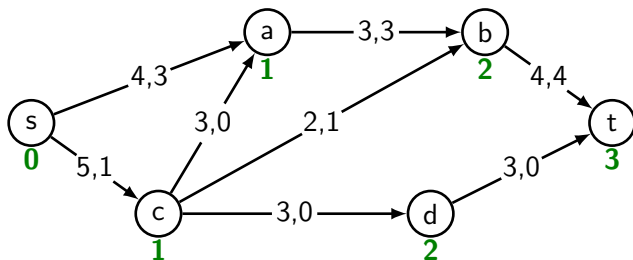
Flow F_2
Levels



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

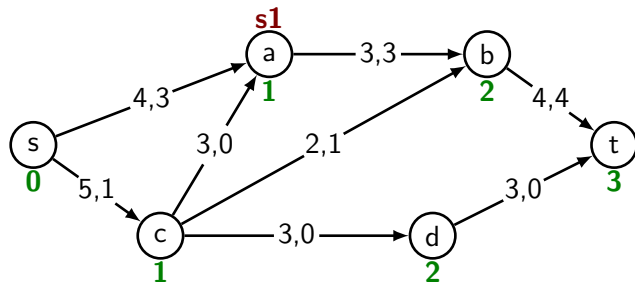
Flow F_2
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

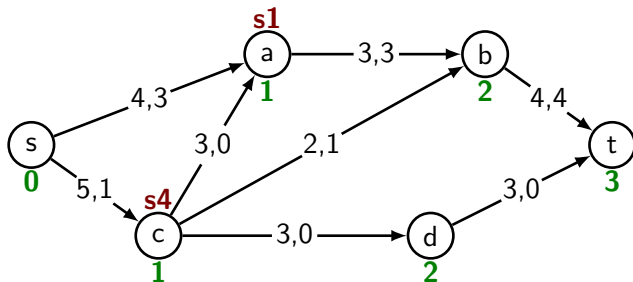
Flow F_2
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

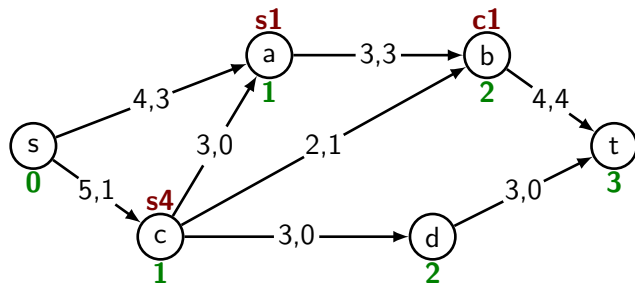
Flow F_2
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

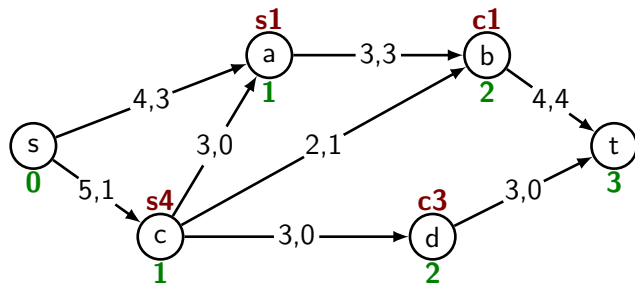
Flow F_2
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

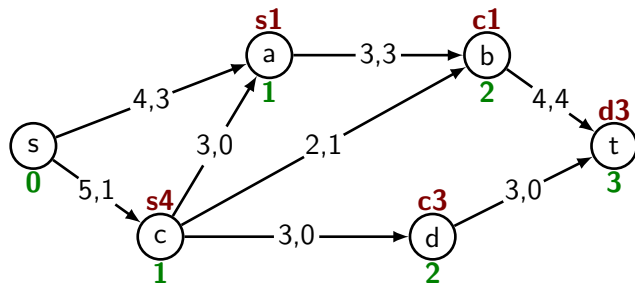
Flow F_2
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

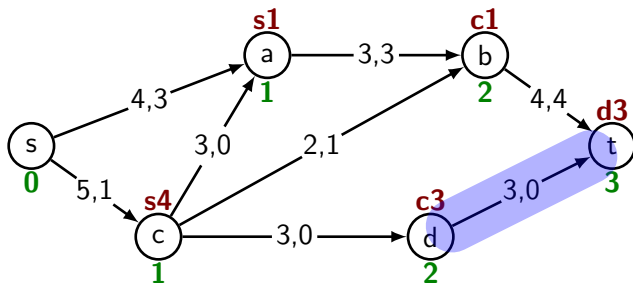
Flow F_2
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

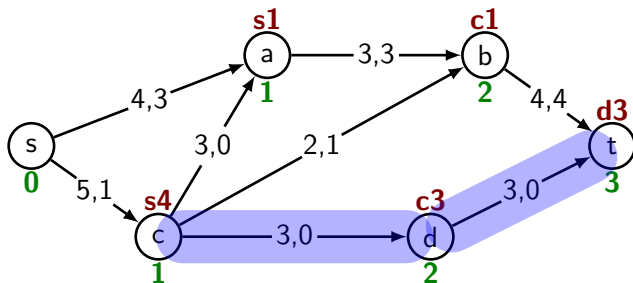
Flow F_2
Levels
Labels



Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

Flow F_2
Levels
Labels

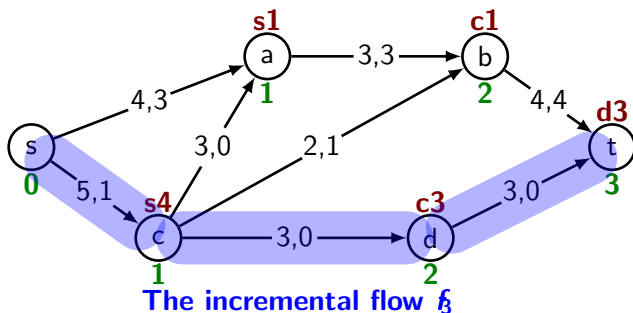


Vertex labelling algorithm, Example 2

Stage 3: F_2 to F_3

Flow F_2

Levels
Labels

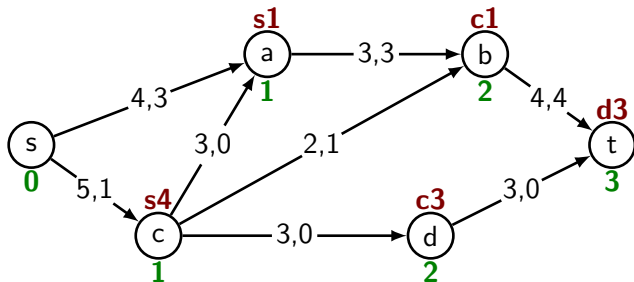


Vertex labelling algorithm, Example 2

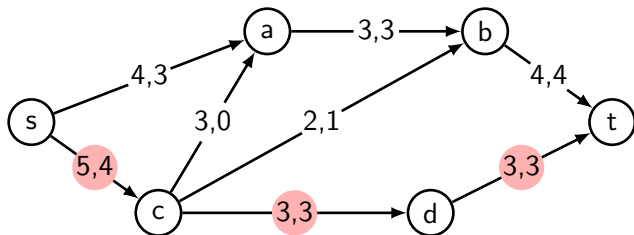
Stage 3: F_2 to F_3

Flow F_2

Levels
Labels



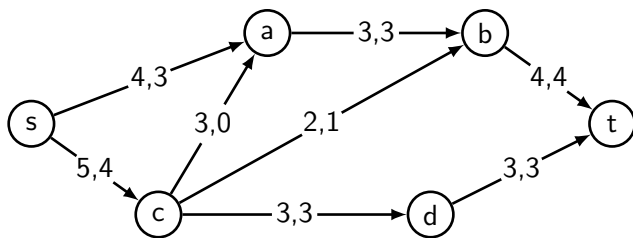
Flow F_3



Vertex labelling algorithm, Example 2

Stage 4: F_3 is F_{\max}

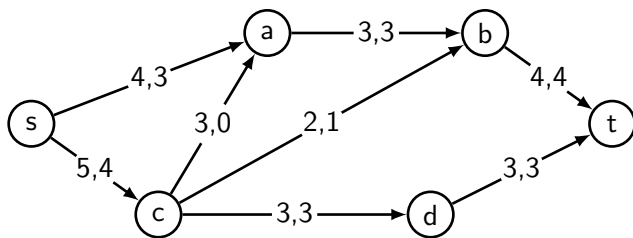
Flow F_3



Vertex labelling algorithm, Example 2

Stage 4: F_3 is F_{\max}

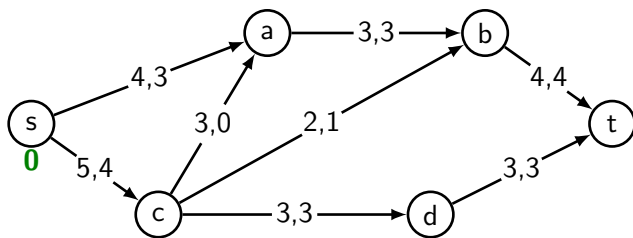
Flow F_3
Levels



Vertex labelling algorithm, Example 2

Stage 4: F_3 is F_{\max}

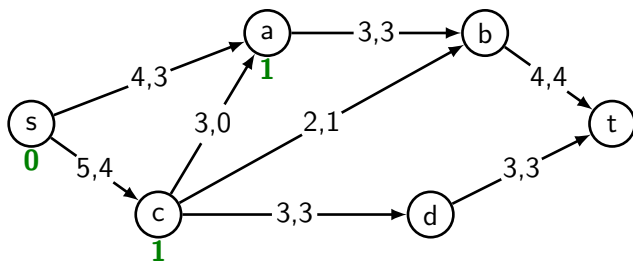
Flow F_3
Levels



Vertex labelling algorithm, Example 2

Stage 4: F_3 is F_{\max}

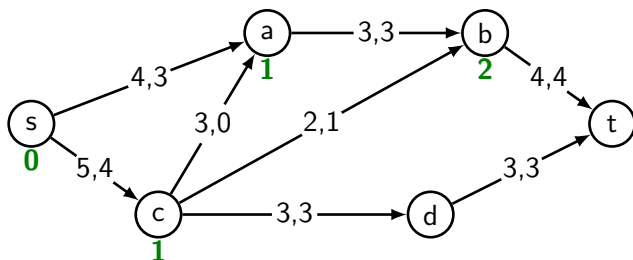
Flow F_3
Levels



Vertex labelling algorithm, Example 2

Stage 4: F_3 is F_{\max}

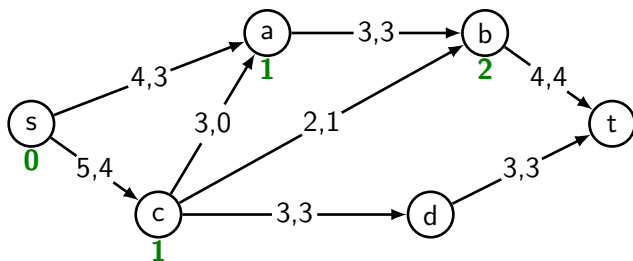
Flow F_3
Levels



Vertex labelling algorithm, Example 2

Stage 4: F_3 is F_{\max}

Flow F_3
Levels

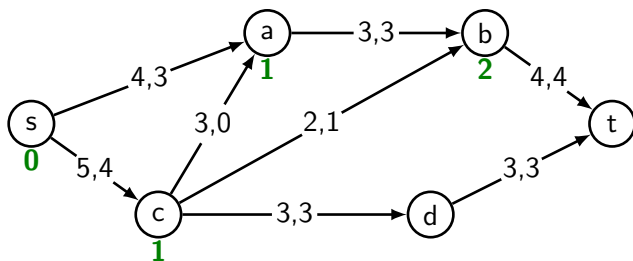


No level can be assigned to t !

Vertex labelling algorithm, Example 2

Stage 4: F_3 is F_{\max}

Flow F_3
Levels

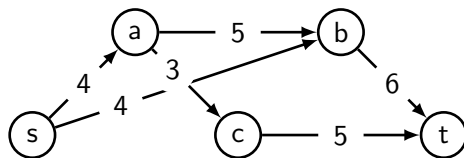


No level can be assigned to t !

So the algorithm terminates with $F_{\max} = F_3$.

Cuts

Consider again Example 1
at right.

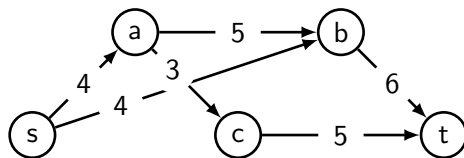


Cuts

Consider again Example 1 at right.

Since the total capacity of edges leading from the source is $4 + 4 = 8$ it is

clear that the maximum flow value cannot be more than 8.



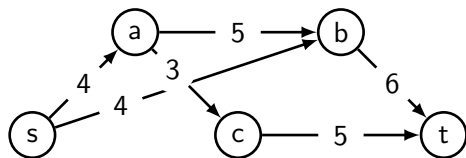
Cuts

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In fact the maximum flow value *is* 8 as a flow with that value is easily found. (We found it using the vertex labelling algorithm, but that isn't really needed on such a simple example.)



Cuts

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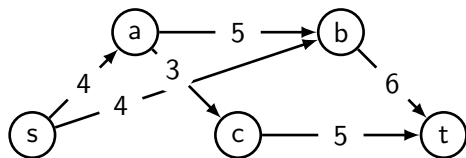
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The set of edges leading out of the source is an example of a network *cut*, so called because removing the edges would 'cut' the network in two, separating the source from the target.

The $4 + 4$ is the 'capacity' of the cut.



Cuts

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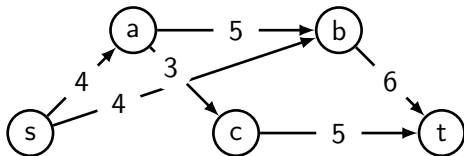
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A cut may be indicated by drawing a (red) line through its edges.



Cuts

Consider again Example 1 at right.

Since the total capacity of edges leading from the source is $4 + 4 = 8$ it is

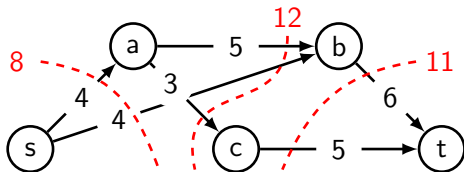
clear that the maximum flow value cannot be more than 8.

In fact the maximum flow value *is* 8 as a flow with that value is easily found. (We found it using the vertex labelling algorithm, but that isn't really needed on such a simple example.)

The set of edges leading out of the source is an example of a network *cut*, so called because removing the edges would 'cut' the network in two, separating the source from the target.

The $4 + 4$ is the 'capacity' of the cut.

A cut may be indicated by drawing a (red) line through its edges. Some examples, with values, are shown on the digraph above.



Max Flow Min Cut

Observe that for Example 1 the minimum cut and maximum flow have the same value (8).

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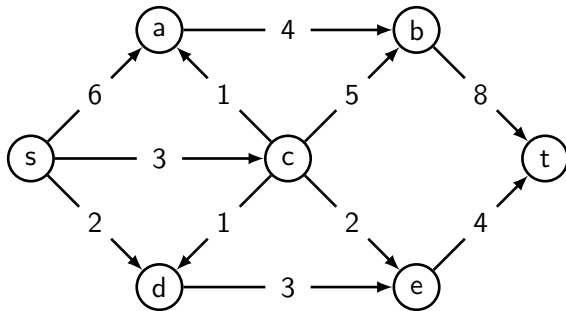
Max flow min cut theorem: For any transport network:

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Though highly plausible, this theorem is little tricky to prove, and the proof will be omitted, as will the proof that the vertex labelling algorithm always finds a maximum flow.

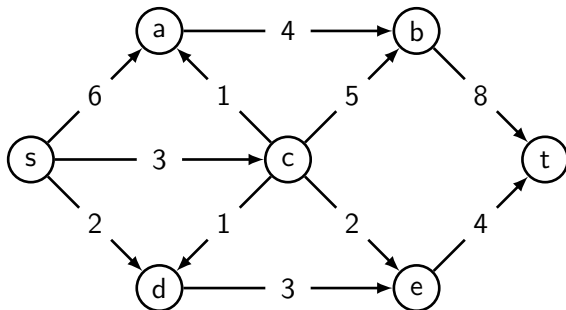
Max flow min cut: Class example 1

What is the maximum flow value for this transport network?



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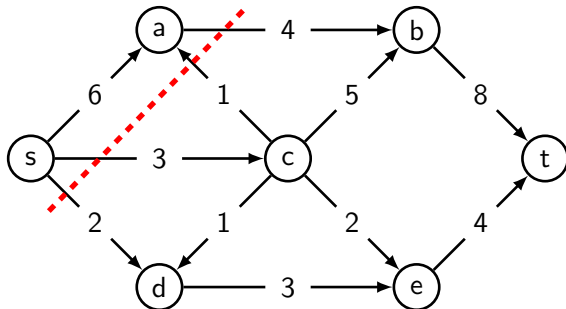
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Answer:

Max flow min cut: Class example 1

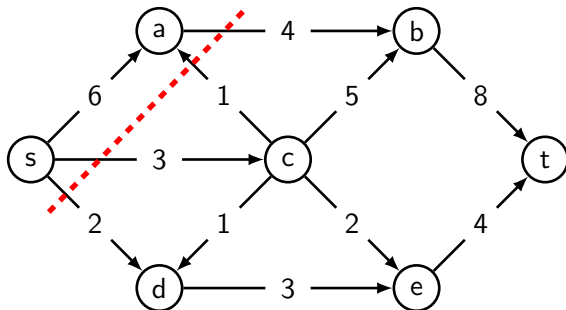
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Answer: After some searching we find the minimum cut shown, for which $S = \{s, a\}$ and $T = \{b, c, d, e, t\}$.

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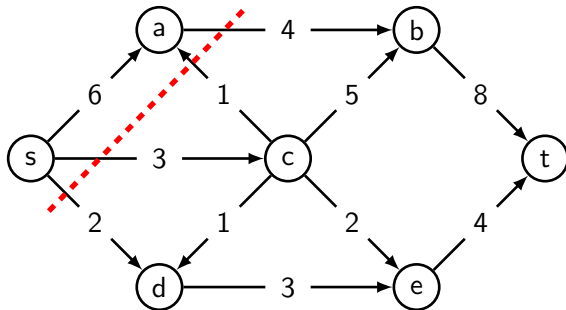


Answer: After some searching we find the minimum cut shown, for which $S = \{s, a\}$ and $T = \{b, c, d, e, t\}$.

The capacity of this cut is $4+3+2=9$, so this is the maximum flow.

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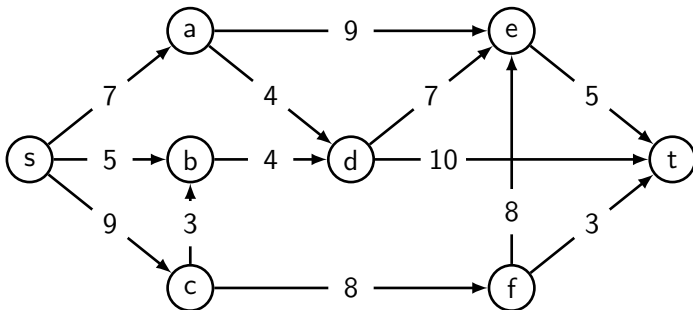
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Note: Edge (c,a) is not in the cut since it's in the wrong direction.

Max flow min cut: Class example 2

from Kolman, Busby & Ross

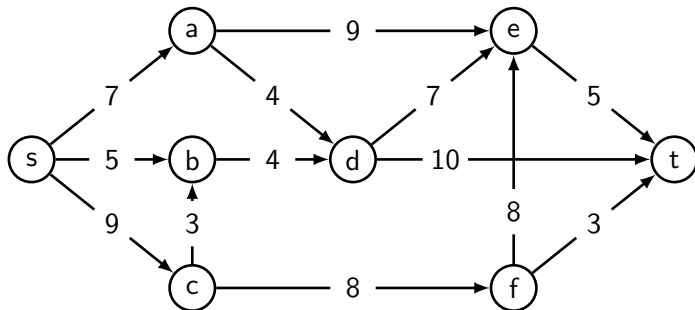
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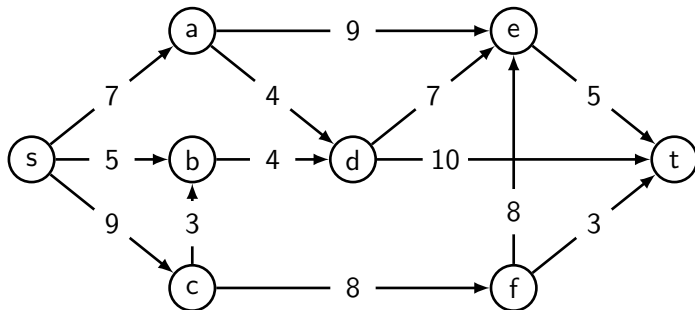


Hint:

Max flow min cut: Class example 2

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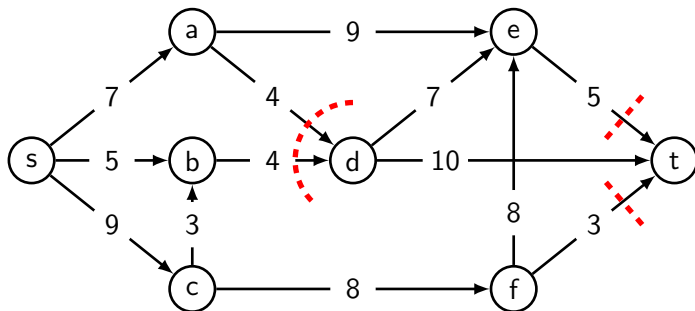


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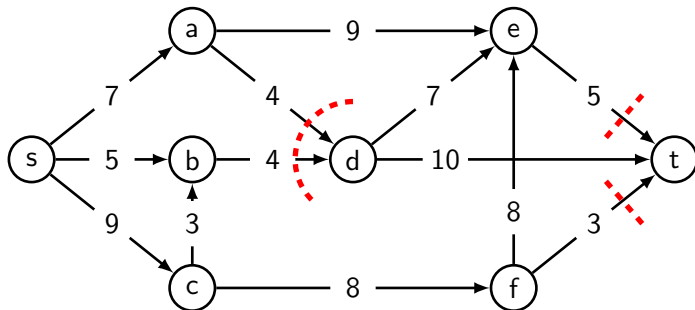
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What is the maximum flow value for this transport network?



Hint: This time the minimum cut cannot be drawn as a single line!

Answer: Use all the edges from $S = \{s, a, b, c, e, f\}$ to $T = \{d, t\}$. The capacity of this cut is $4 + 4 + 5 + 3 = 16$, so this is the max flow.

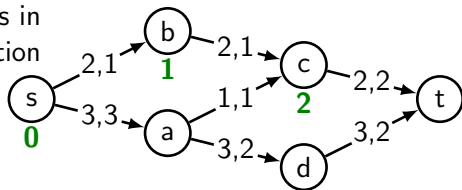
Introduction to virtual flows

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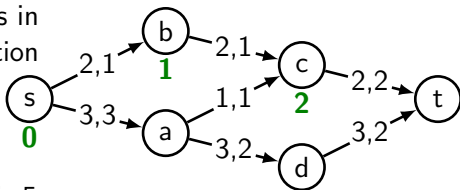
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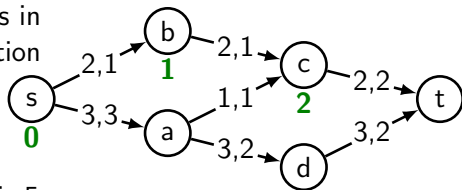
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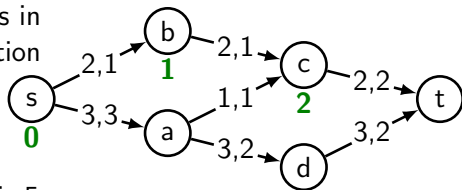


To escape from this we need to 'divert' the flow $a \rightarrow c \rightarrow t$ to $a \rightarrow d \rightarrow t$, thereby allowing another 1 unit of flow $s \rightarrow b \rightarrow c \rightarrow t$.

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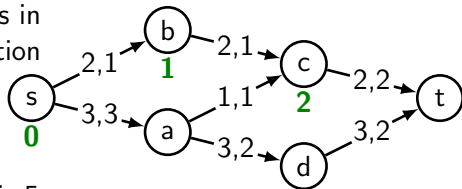
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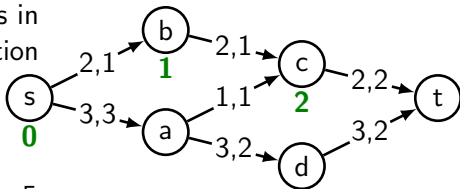
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First, the definition and an explanation of how the algorithm is modified.

The complete vertex labelling algorithm

Let (u,v) be a (directed) edge in a transport network D , and suppose there is currently a flow of $f > 0$ along this edge. The vertex labelling algorithm can reduce this flow to $g < f$ by introducing a **virtual flow** of $f - g$ in the opposite direction, *i.e.* from v to u .

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For vertices u,v of D , where D has capacity and flow functions C, F :

$$S((u,v)) = \begin{cases} C((u,v)) - F((u,v)) & \text{if } (u,v) \in E(D) \\ F((v,u)) & \text{if } (v,u) \in E(D) \\ 0 & \text{otherwise} \end{cases}$$

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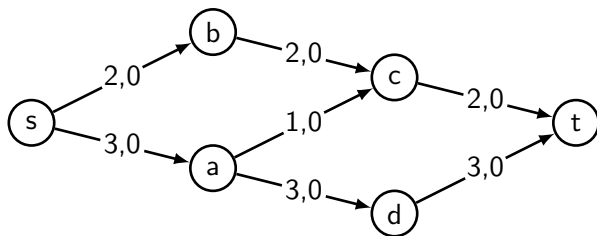
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When $(v,u) \in E(D)$, $S((u,v))$ is called a **virtual capacity**.

Vertex labelling algorithm, Example 3

Stage 1: F_0 to F_1

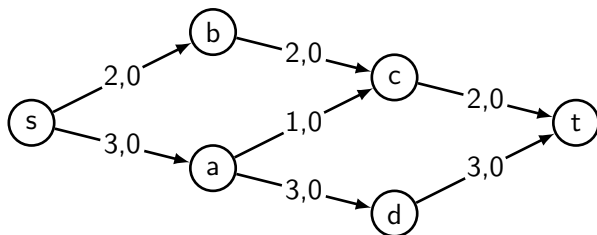
Flow F_0



Vertex labelling algorithm, Example 3

Stage 1: F_0 to F_1

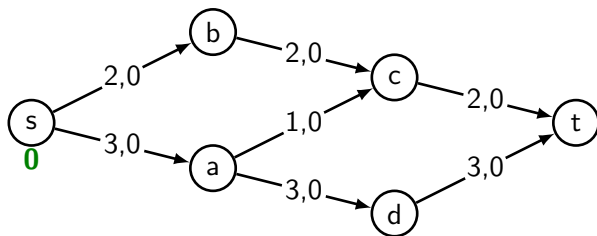
Flow F_0
Levels



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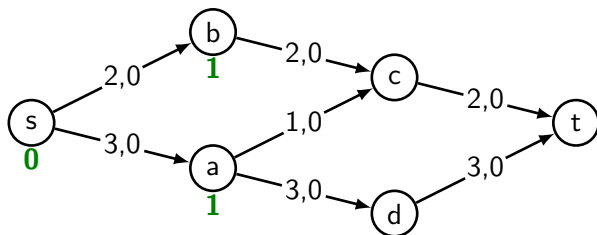
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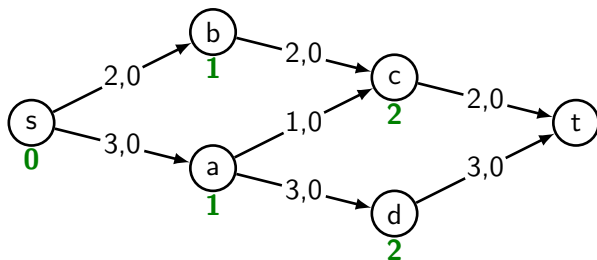
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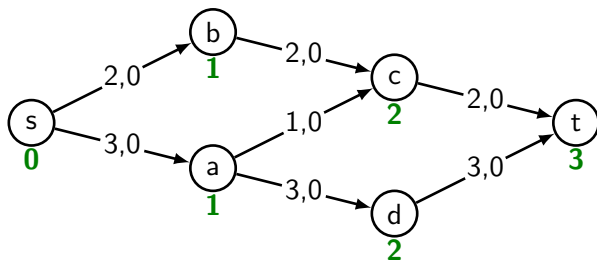
Flow F_0
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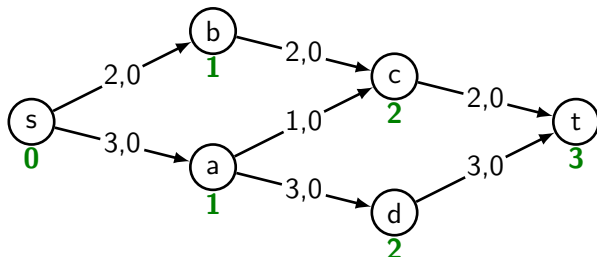


Vertex labelling algorithm, Example 3

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Levels
Labels

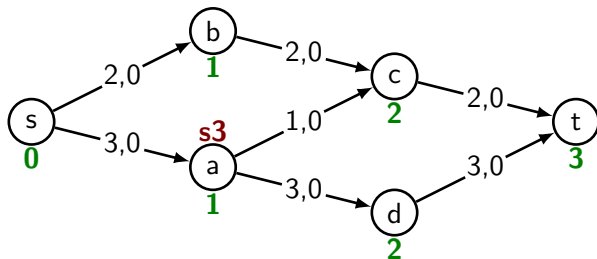


Vertex labelling algorithm, Example 3

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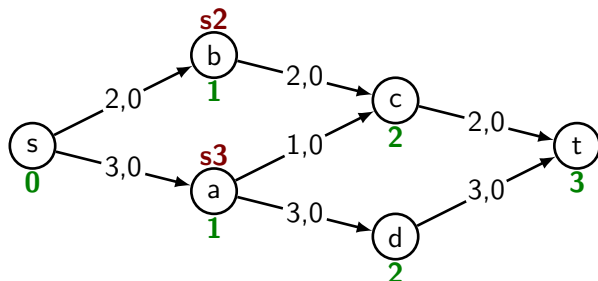


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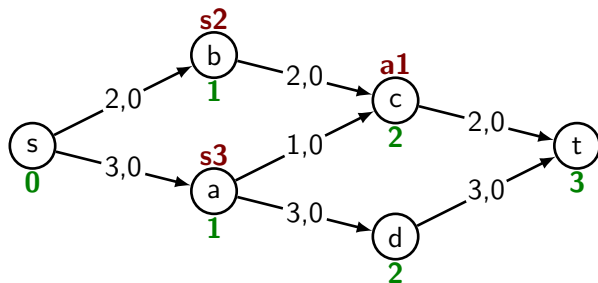


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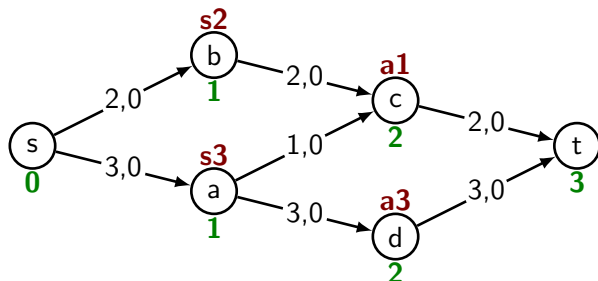


Vertex labelling algorithm, Example 3

Stage 1: F_0 to F_1

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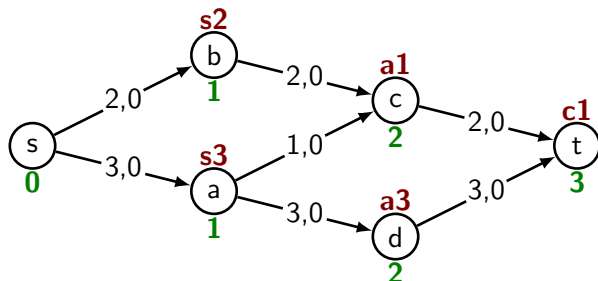


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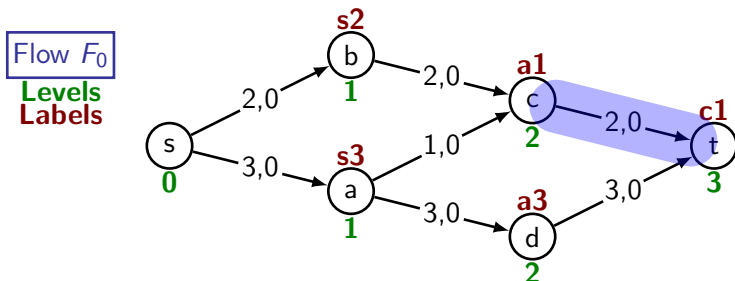
Flow F_0

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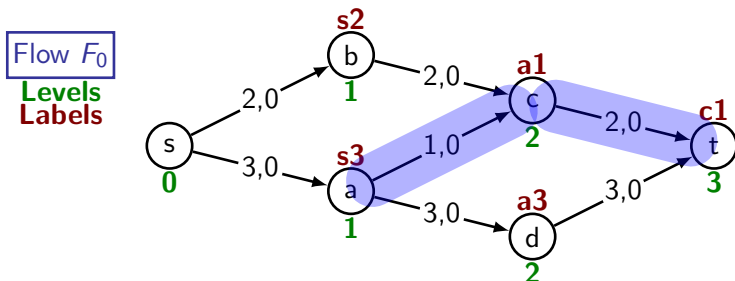
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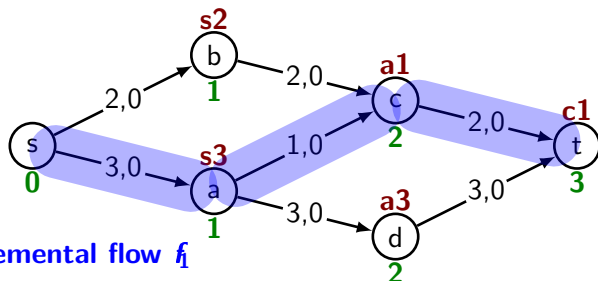


Vertex labelling algorithm, Example 3

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Labels



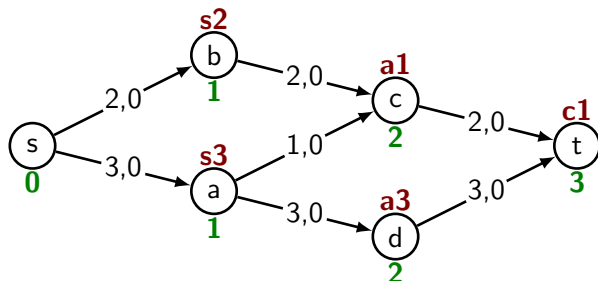
The incremental flow f_1

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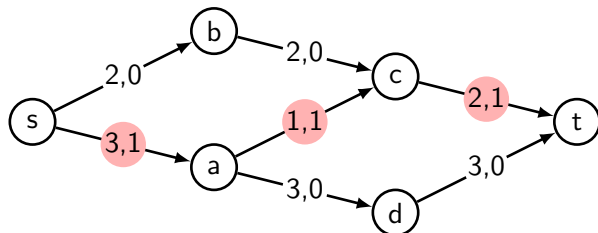
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Flow F_0

Levels
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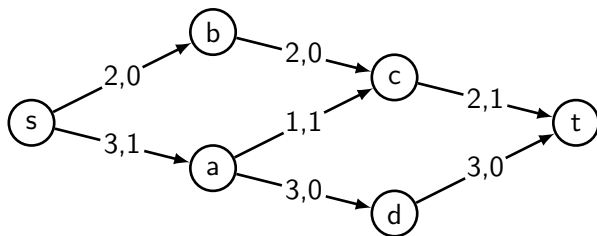
Flow F_1



Vertex labelling algorithm, Example 3

Stage 2: F_1 to F_2

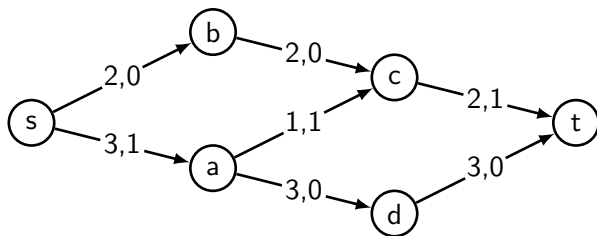
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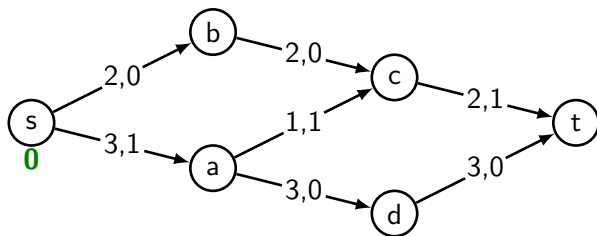
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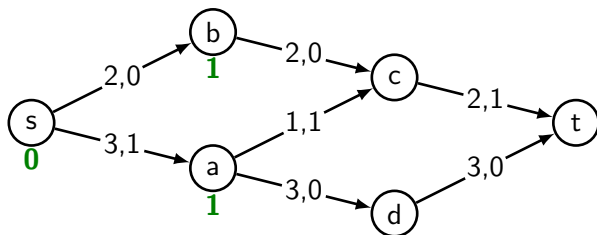
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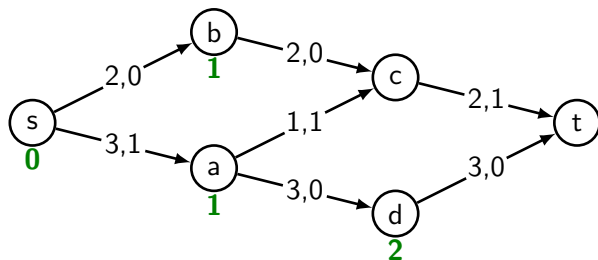
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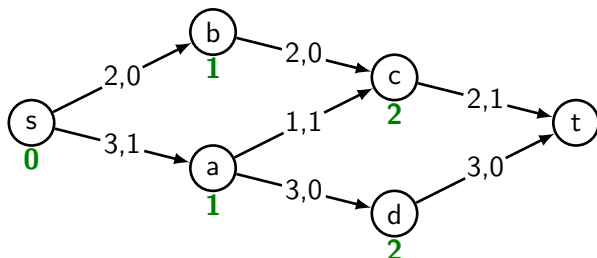
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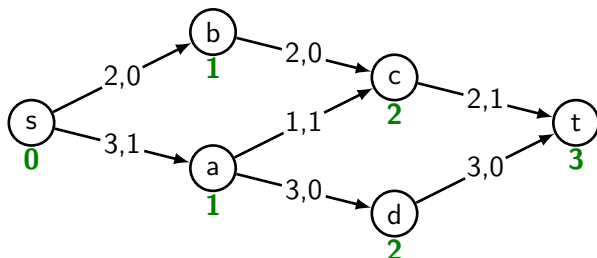
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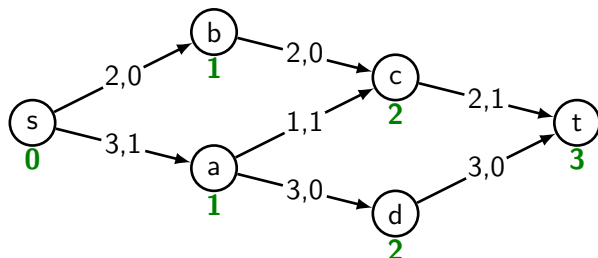


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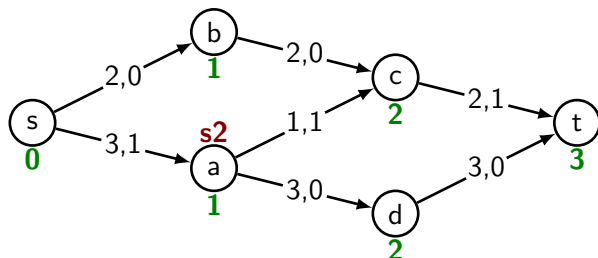


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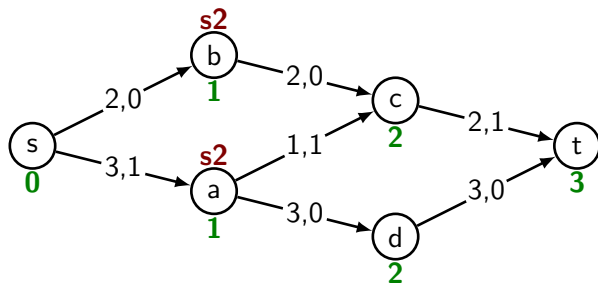


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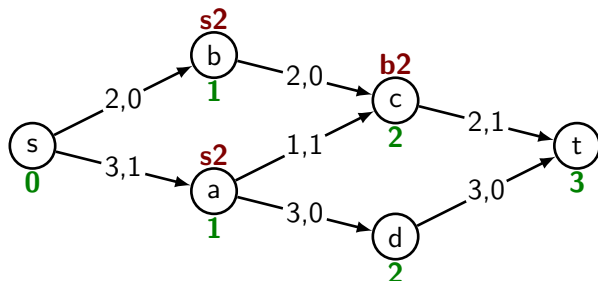


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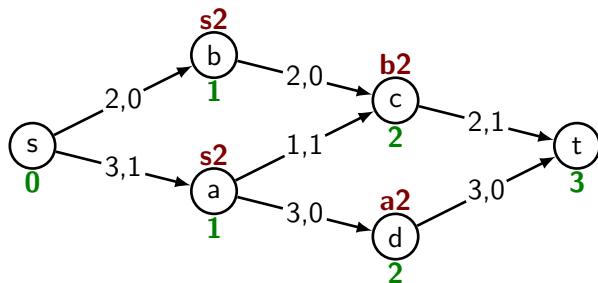


Vertex labelling algorithm, Example 3

Stage 2: F_1 to F_2

Flow F_1

Levels
Labels

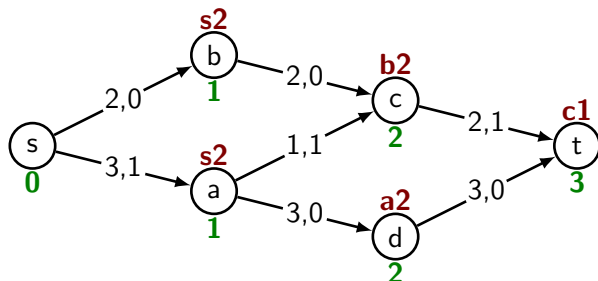


Vertex labelling algorithm, Example 3

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Flow F_1

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Labels

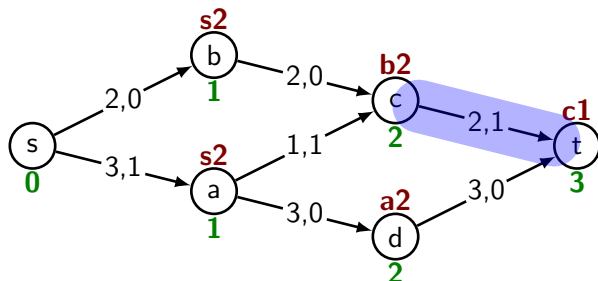


Vertex labelling algorithm, Example 3

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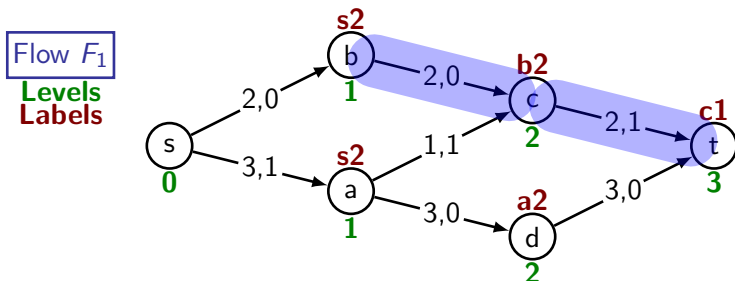
Flow F_1

Levels
Labels



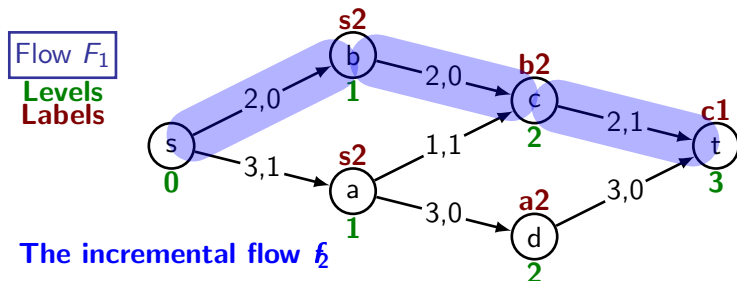
Vertex labelling algorithm, Example 3

Stage 2: F_1 to F_2



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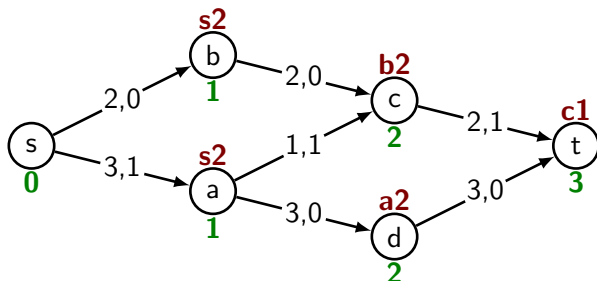


Vertex labelling algorithm, Example 3

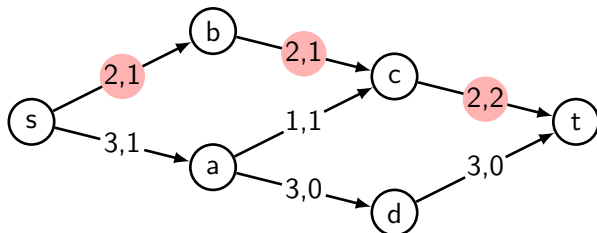
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Levels
Labels



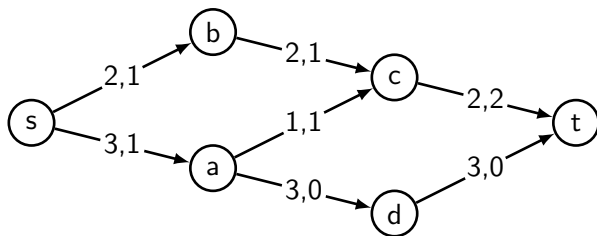
Flow F_2



Vertex labelling algorithm, Example 3

Stage 3: F_2 to F_3

Flow F_2

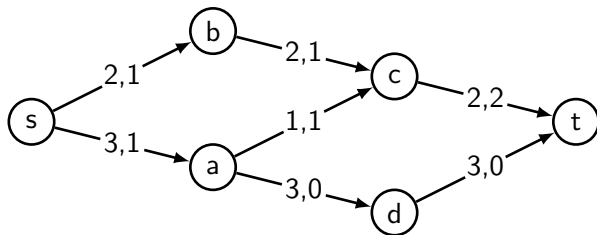


Vertex labelling algorithm, Example 3

Stage 3: F_2 to F_3

Flow F_2

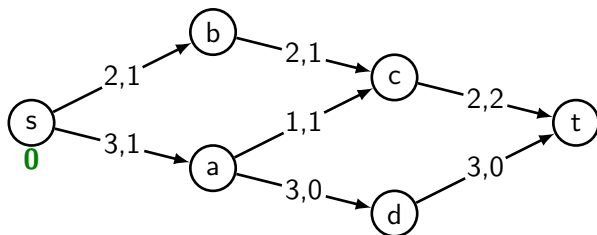
Levels



Vertex labelling algorithm, Example 3

Stage 3: F_2 to F_3

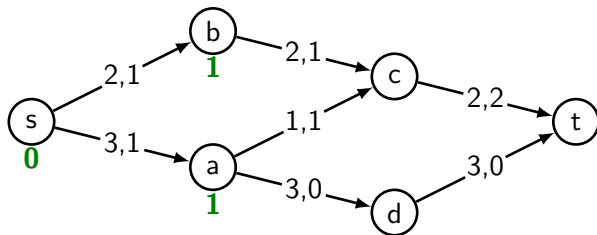
Flow F_2
Levels



Vertex labelling algorithm, Example 3

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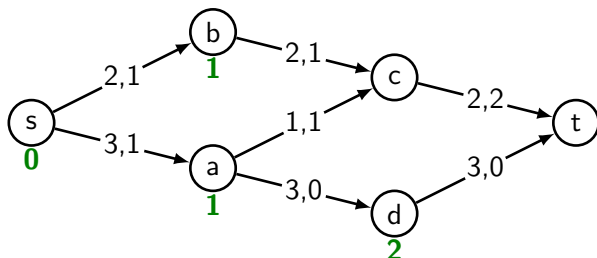
Flow F_2
Levels



Vertex labelling algorithm, Example 3

Stage 3: F_2 to F_3

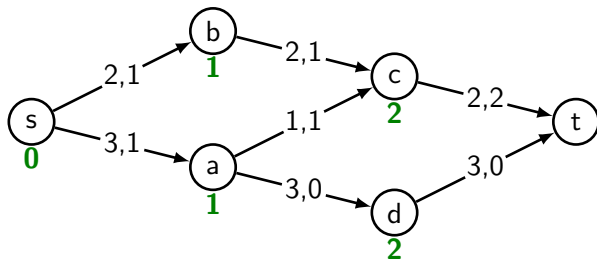
Flow F_2
Levels



Vertex labelling algorithm, Example 3

Stage 3: F_2 to F_3

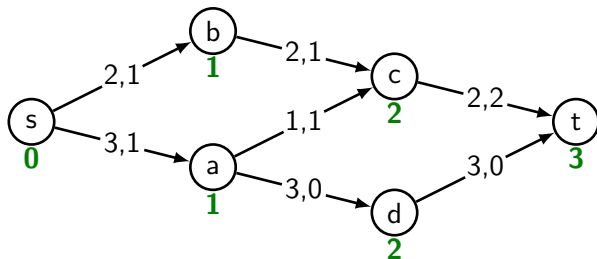
Flow F_2
Levels



Vertex labelling algorithm, Example 3

Stage 3: F_2 to F_3

Flow F_2
Levels

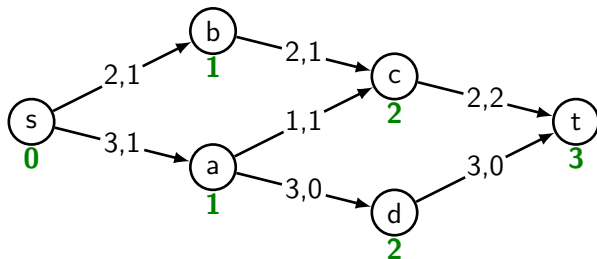


Vertex labelling algorithm, Example 3

Stage 3: F_2 to F_3

Flow F_2

Levels
Labels

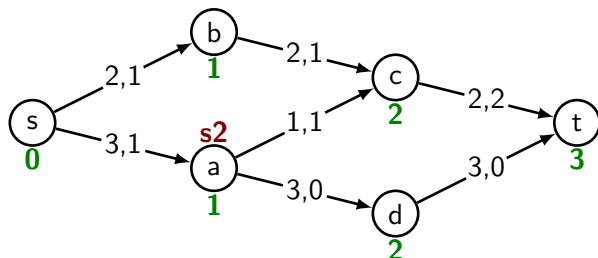


Vertex labelling algorithm, Example 3

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Flow F_2

Levels
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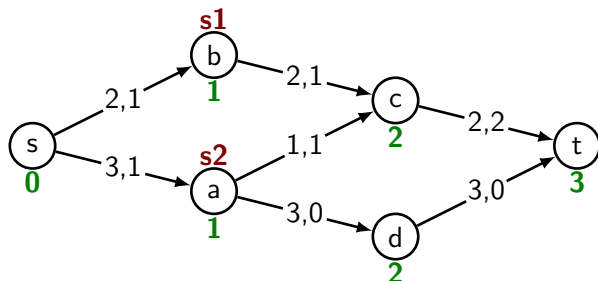


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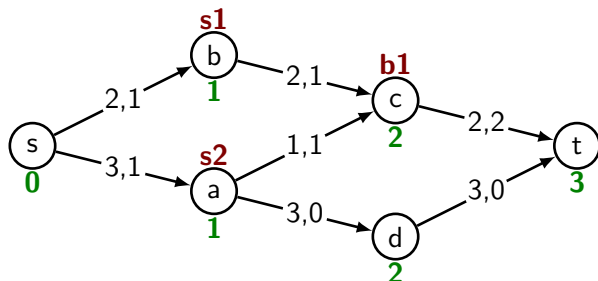


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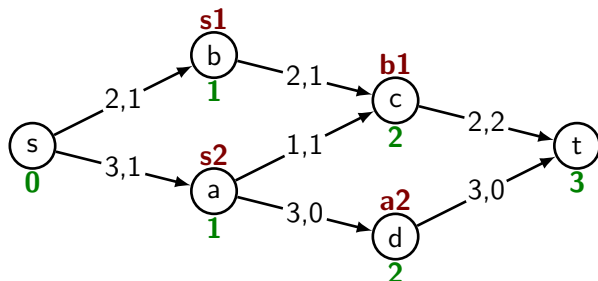


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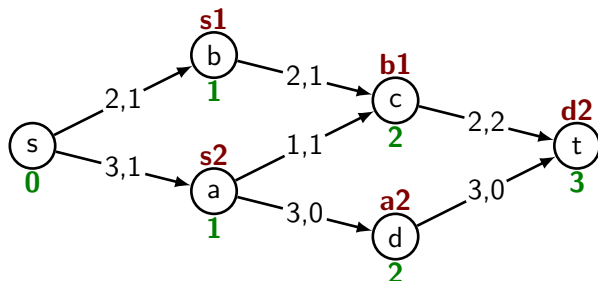


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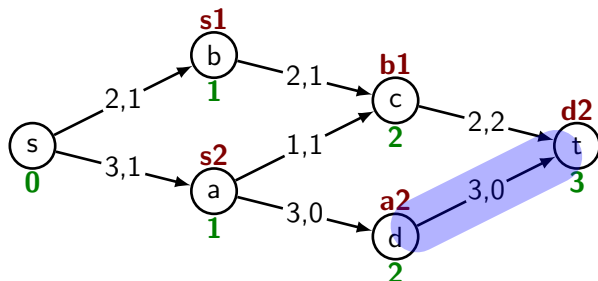


Vertex labelling algorithm, Example 3

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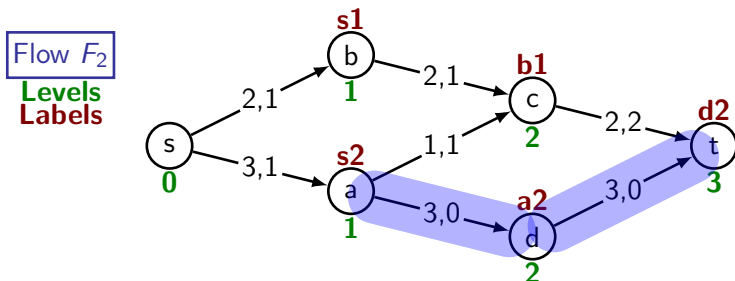
Flow F_2

Levels
Labels



Vertex labelling algorithm, Example 3

Stage 3: F_2 to F_3

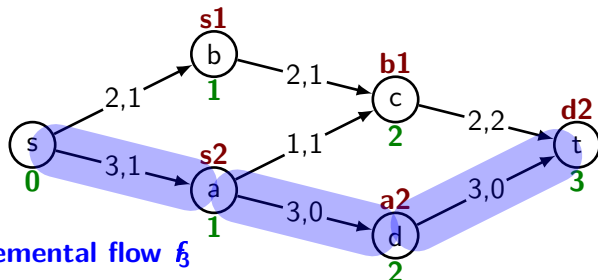


Vertex labelling algorithm, Example 3

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Flow F_2

Levels
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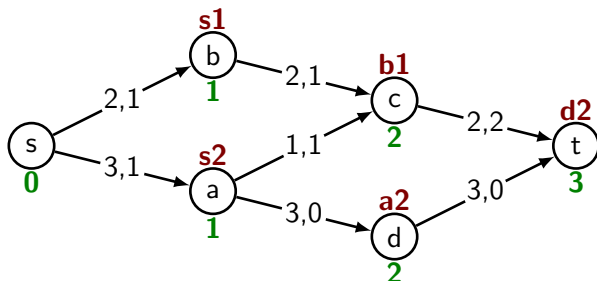
The incremental flow f_3

Vertex labelling algorithm, Example 3

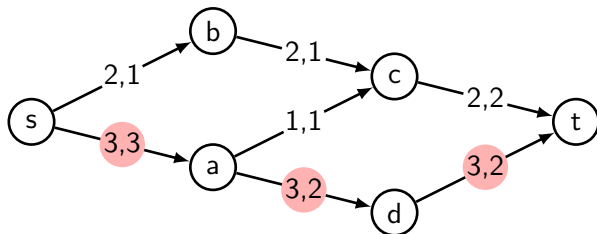
Stage 3: F_2 to F_3

Flow F_2

Levels
Labels



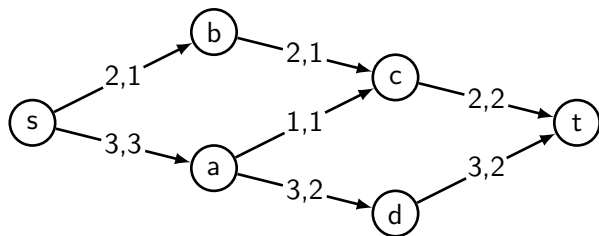
Flow F_3



Vertex labelling algorithm, Example 3

Stage 4: F_3 to F_4

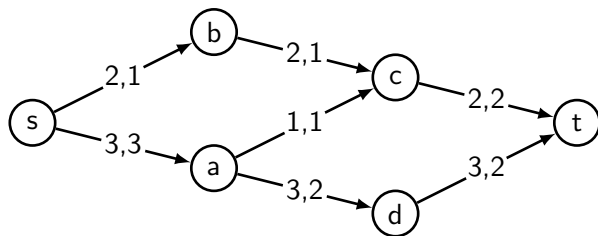
Flow F_3



Vertex labelling algorithm, Example 3

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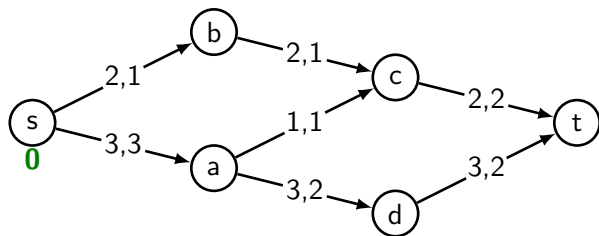
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Levels



Vertex labelling algorithm, Example 3

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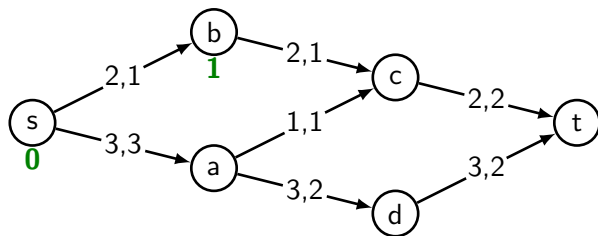
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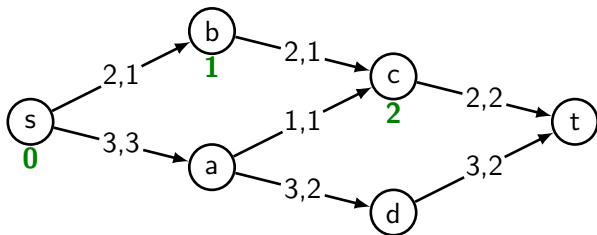
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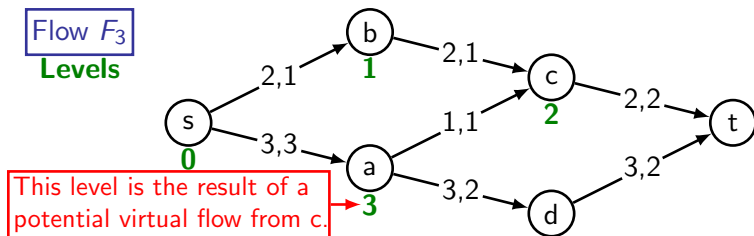
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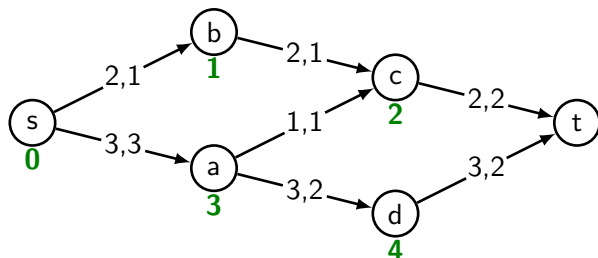
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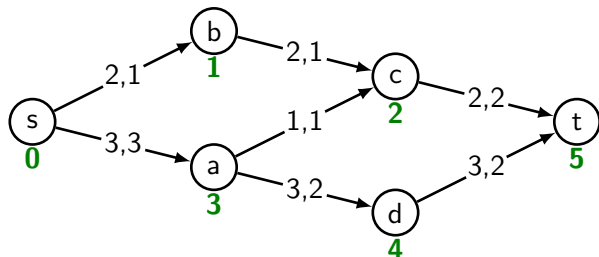
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Vertex labelling algorithm, Example 3

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Flow F_3
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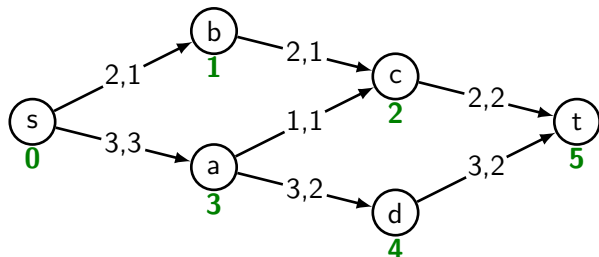


Vertex labelling algorithm, Example 3

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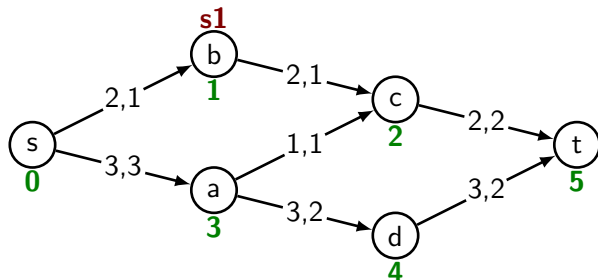


Vertex labelling algorithm, Example 3

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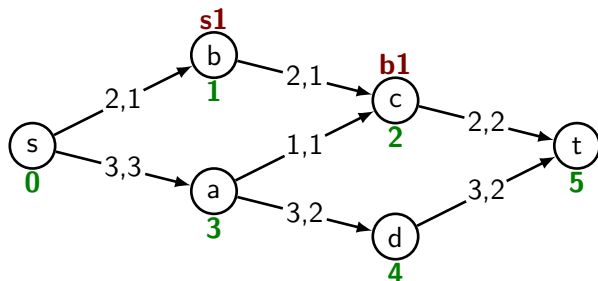


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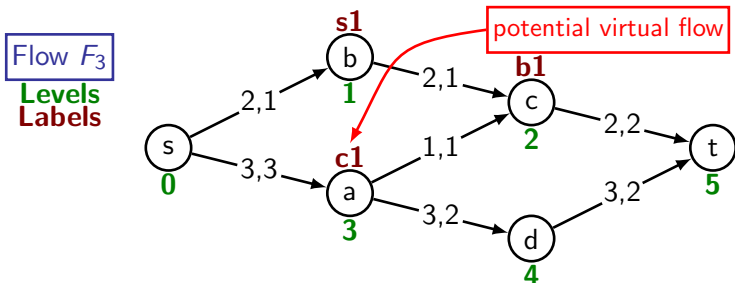
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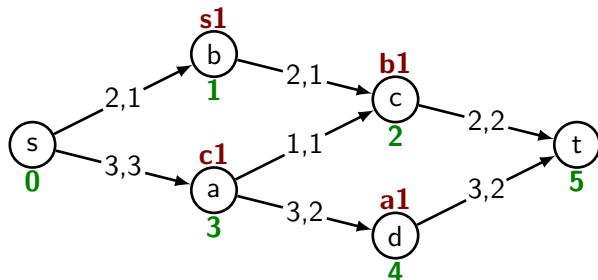


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Flow F_3

Levels
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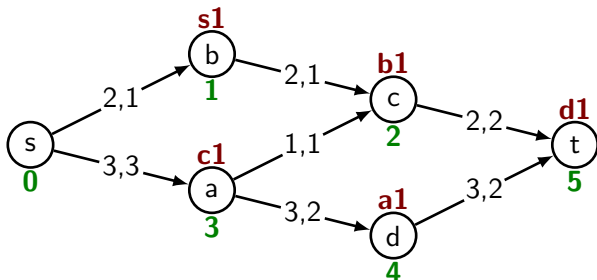


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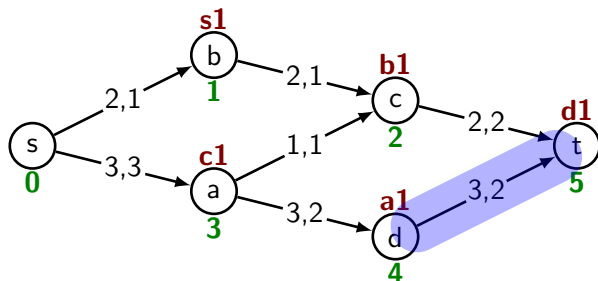


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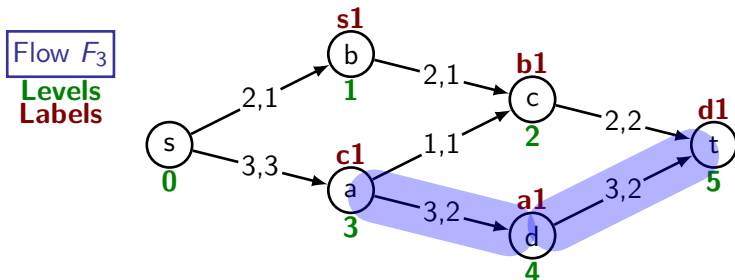
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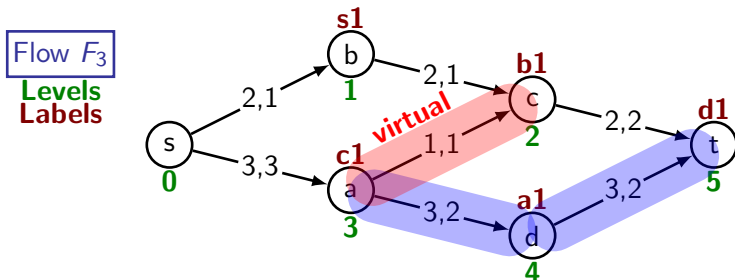
Vertex labelling algorithm, Example 3

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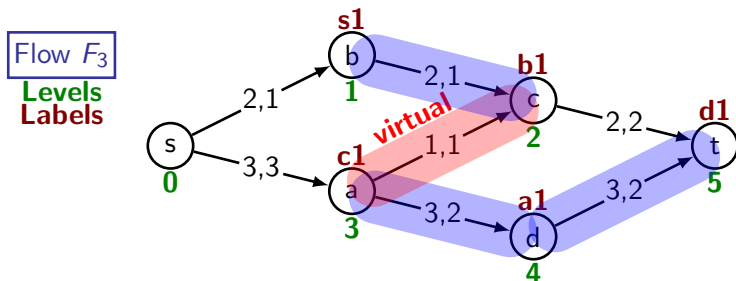
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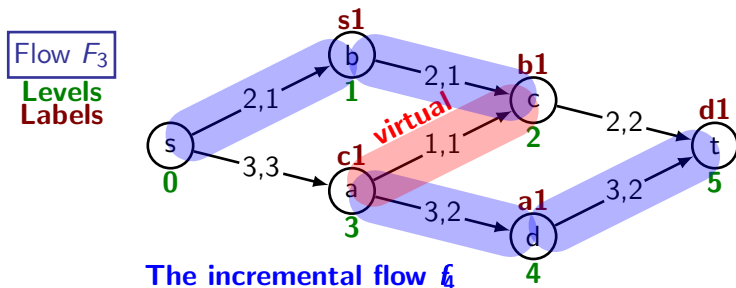
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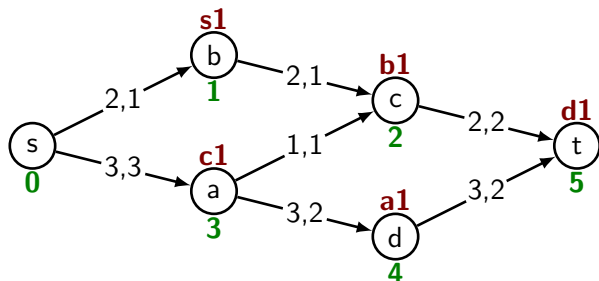


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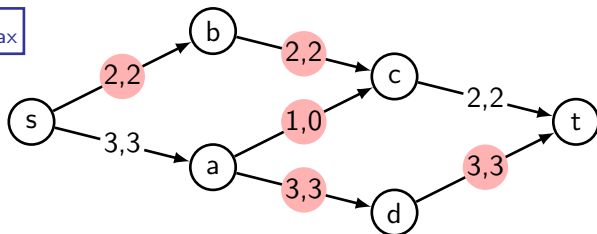
Stage 4: F_3 to F_4

Flow F_3

Levels
Labels



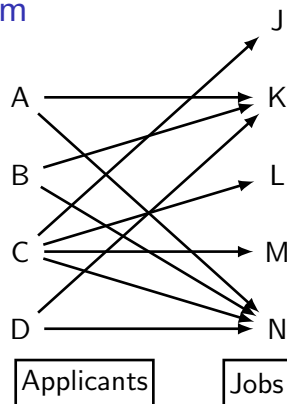
Flow $F_4 = F_{\max}$



A matching problem

from Johnsonbaugh

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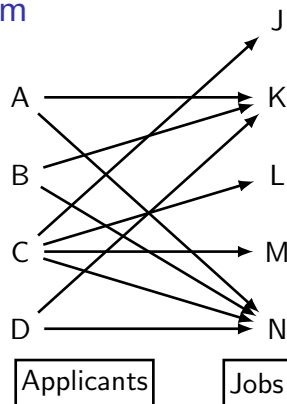


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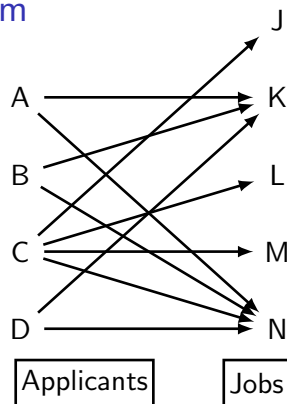


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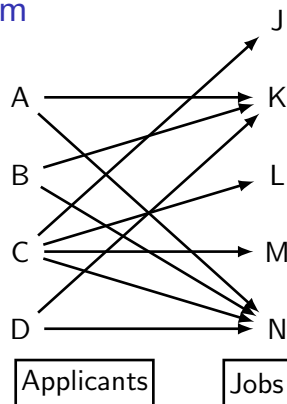
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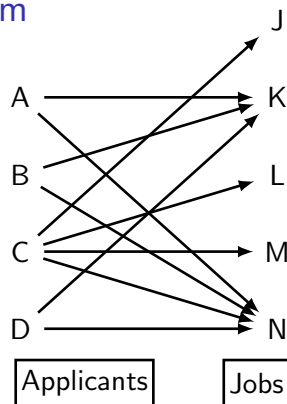
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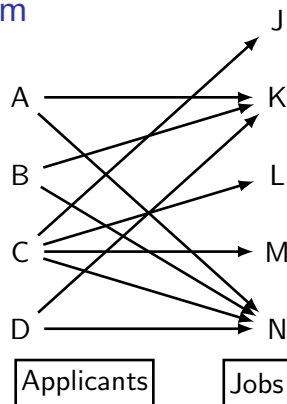
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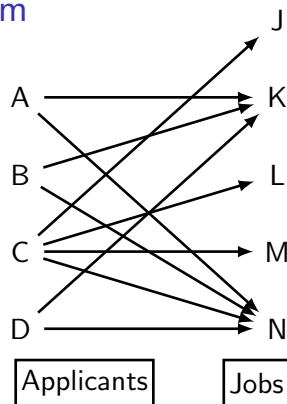
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This is a **injective** (one-to-one) function $f : S' \rightarrow T$ with domain $S' \subseteq S$ as large as possible subject to m being an injective subset of R .

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A solution to the max flow problem provides the matching:

$$m = \{(x, y) \in S \times T : F_{\max}((x, y)) = 1\}.$$

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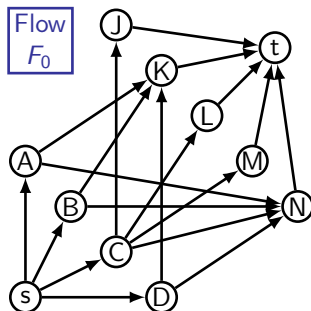
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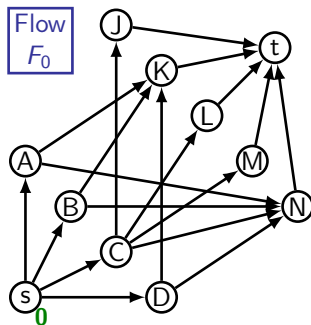


Vertex labelling for matching; Example 1

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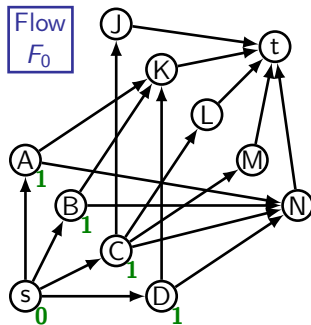


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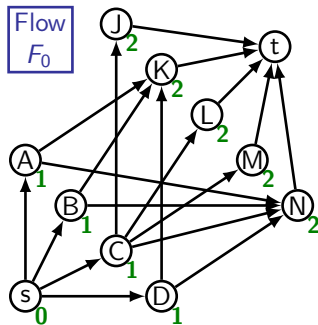


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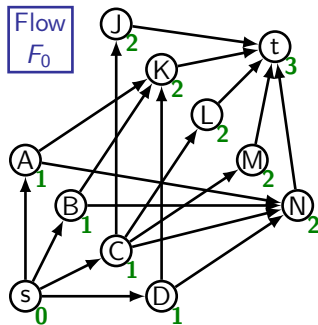


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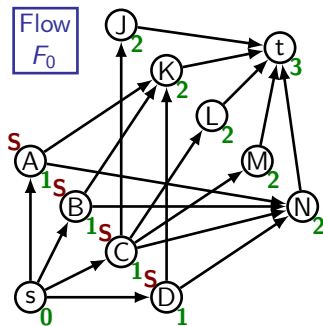


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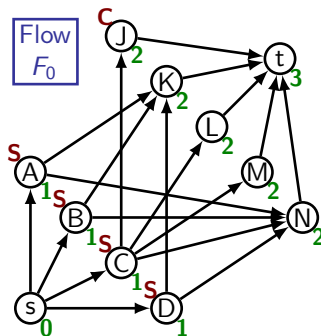


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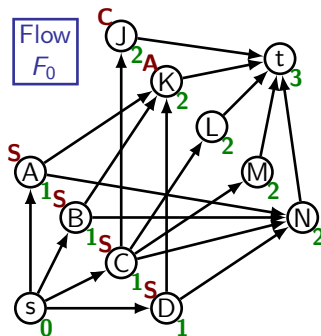


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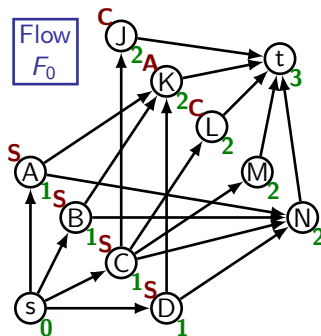


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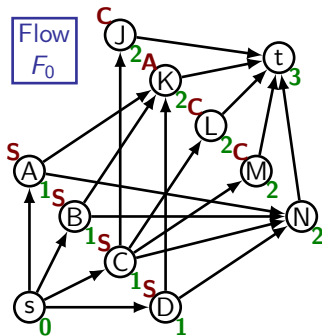


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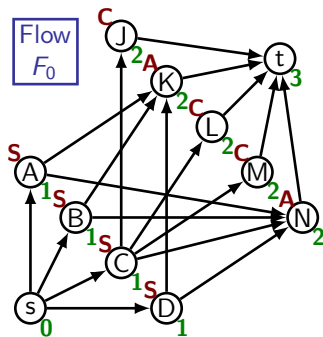


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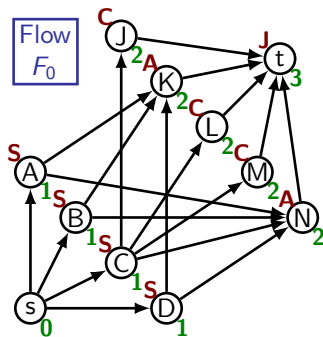


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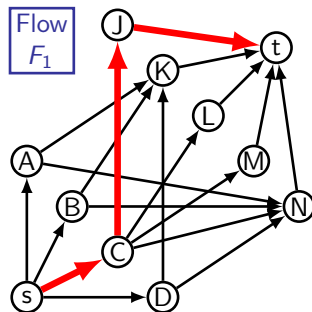
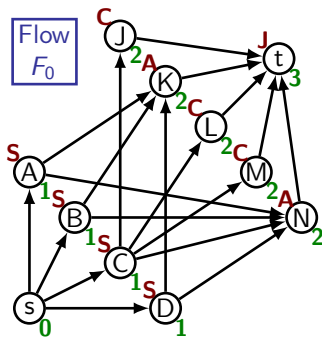


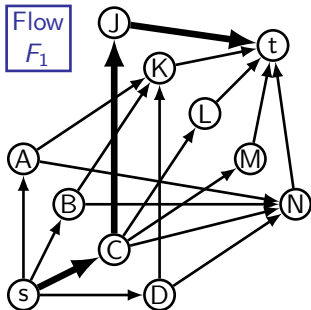
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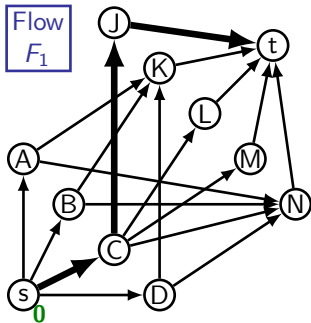
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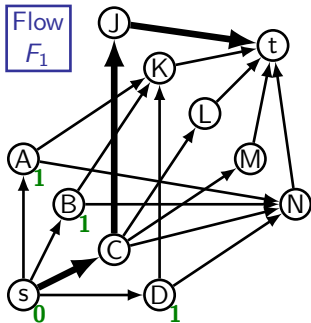
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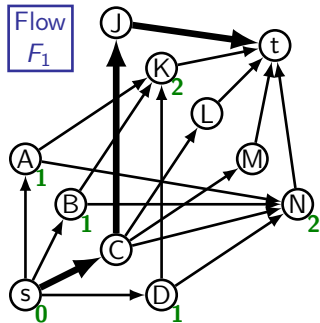
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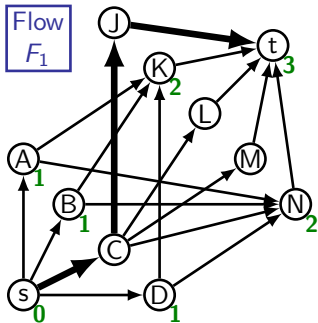


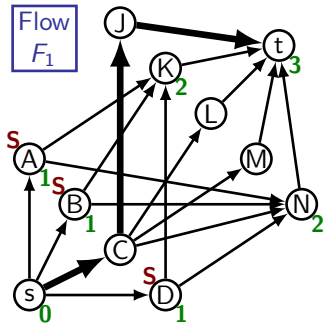


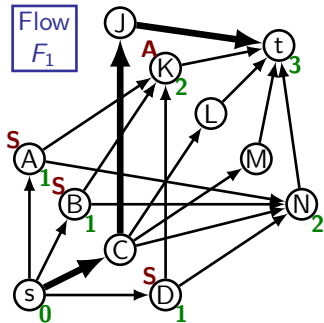


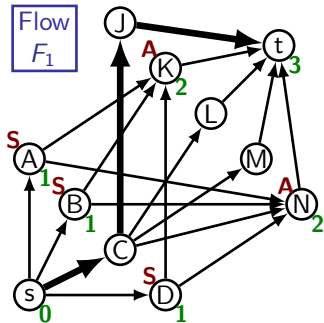


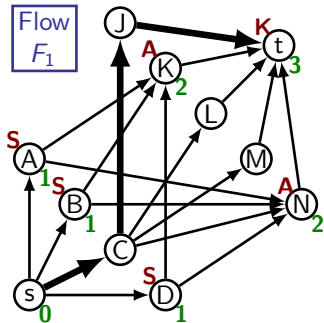


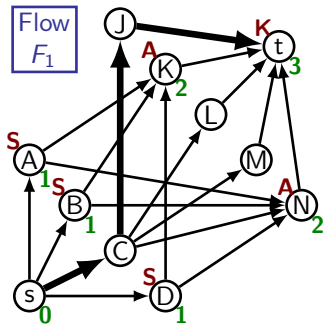
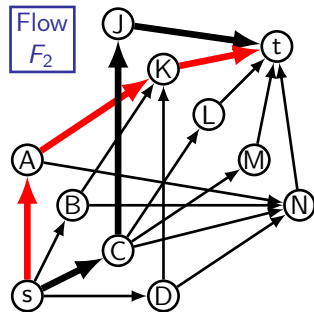


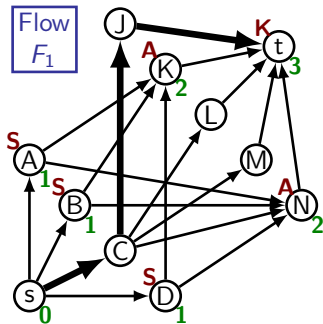
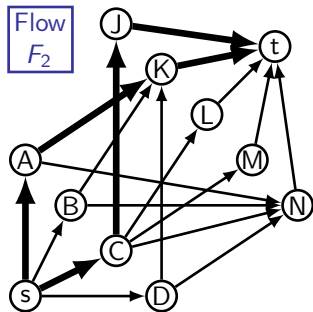
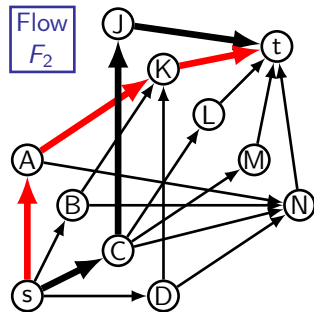


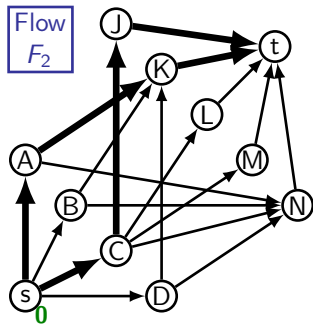
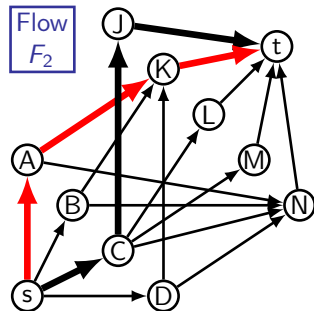
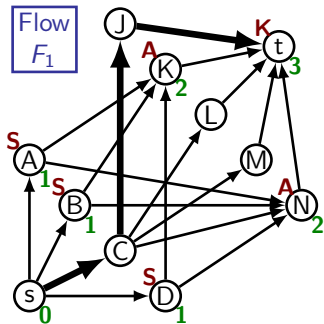


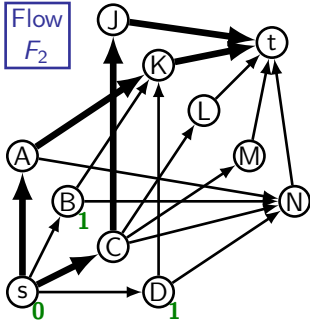
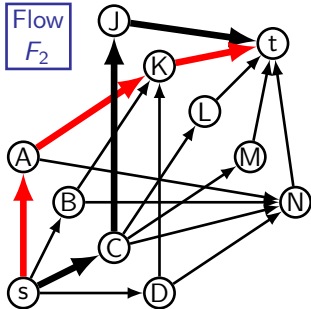
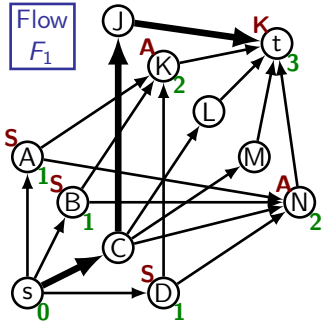


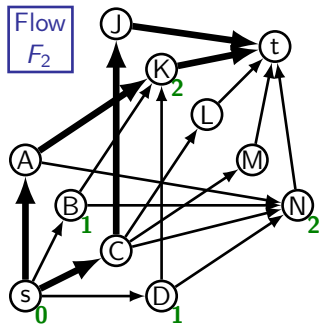
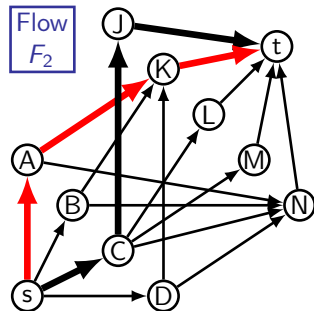
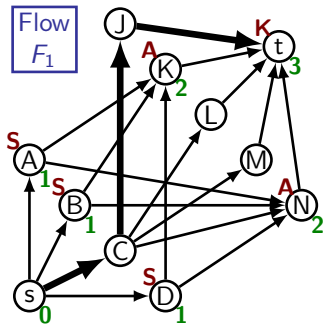


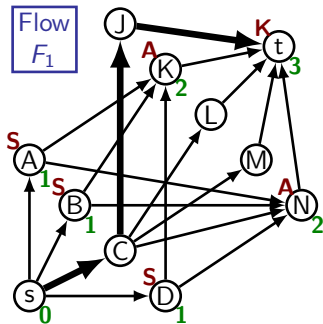
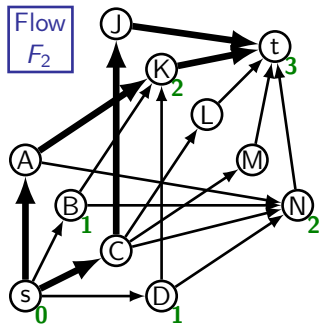
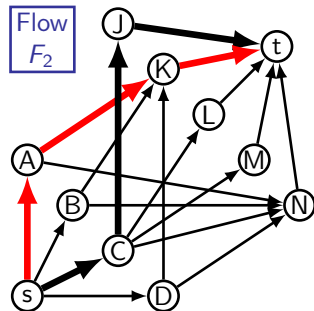

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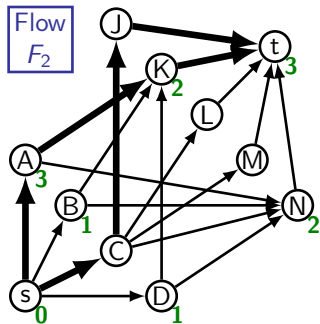
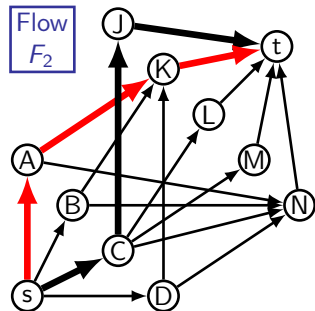
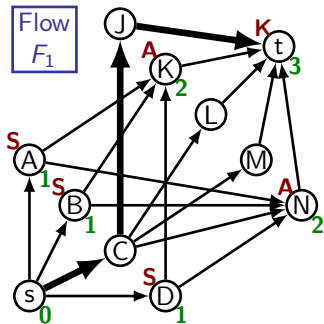

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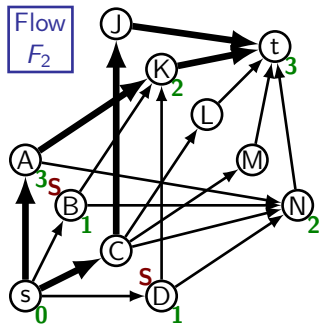
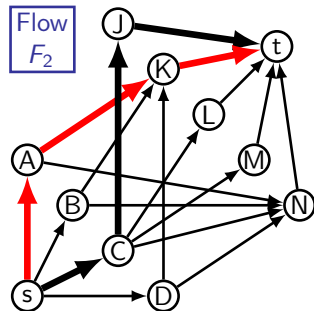
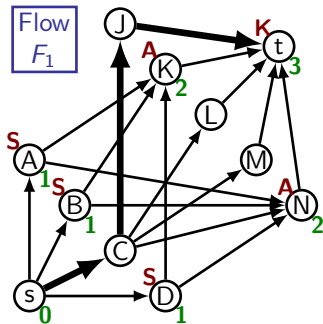


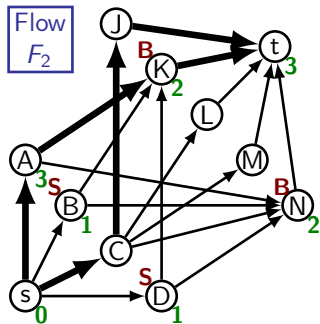
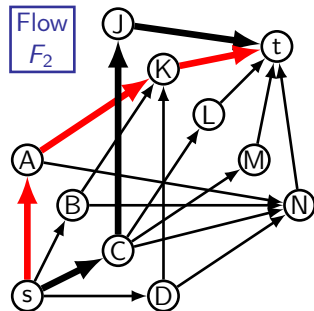
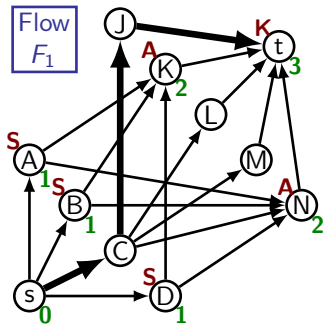


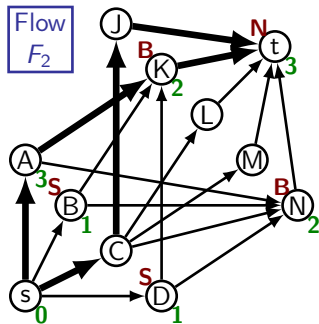
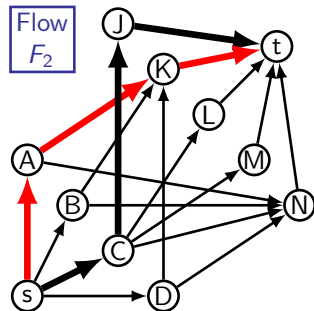
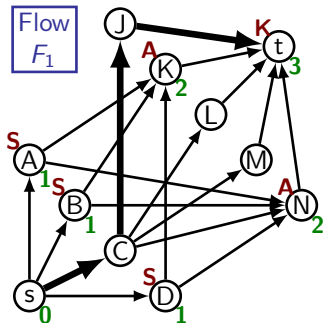


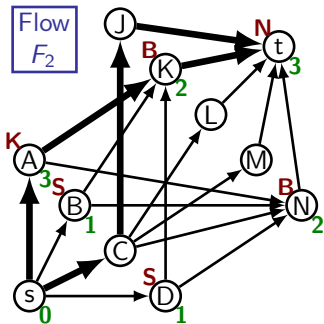
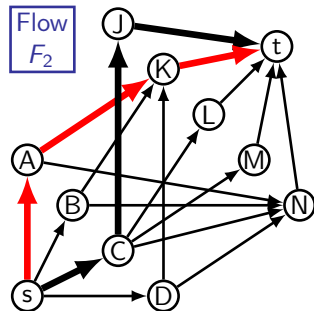
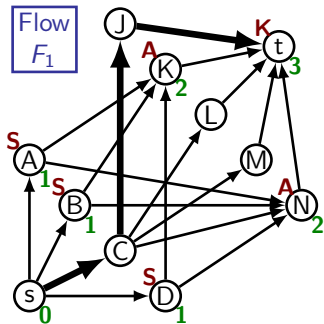

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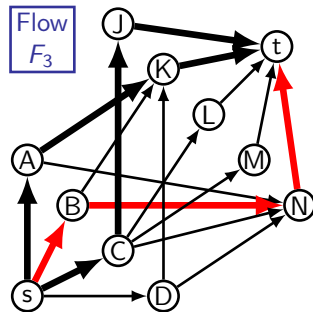
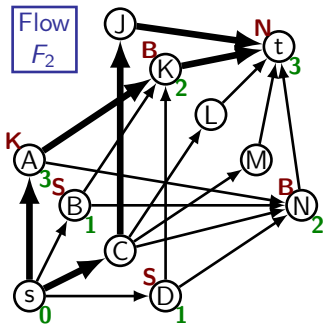
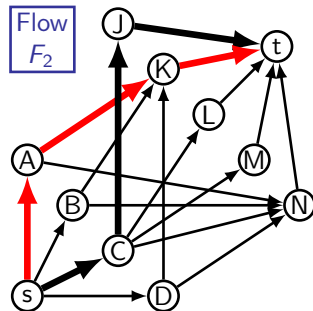
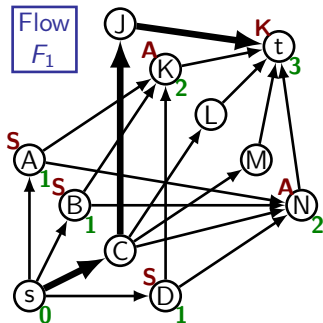


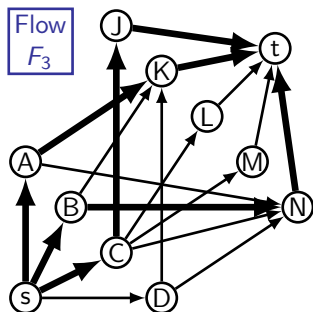


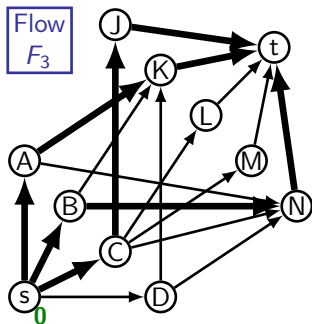


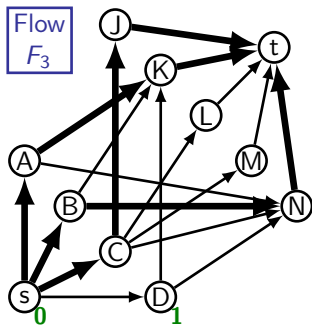


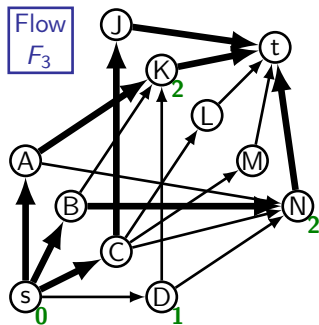


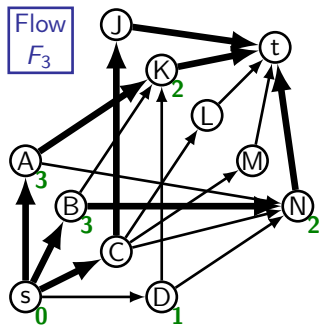


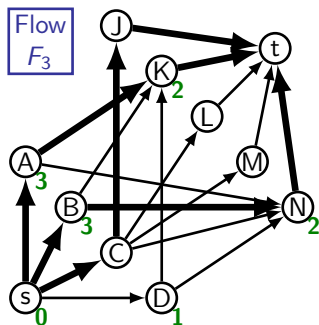




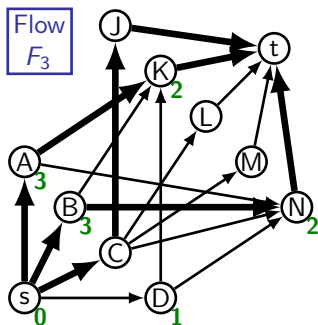






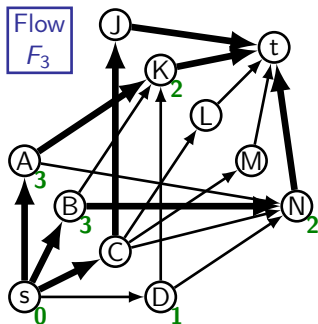


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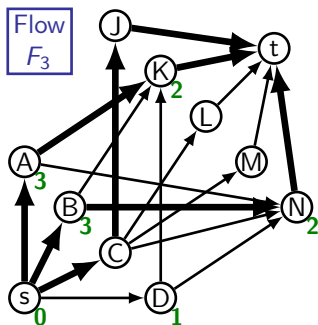


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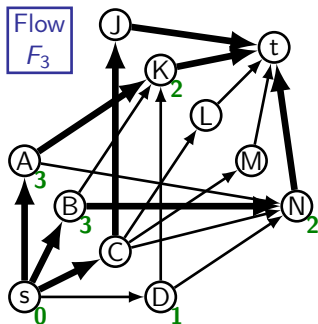
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As remarked earlier, the above solution is very easy to get without the use of the vertex labelling algorithm. We have just matched the first member of T with the first 'available' member of S , then the next member of T with the next 'available' member of S and so on.



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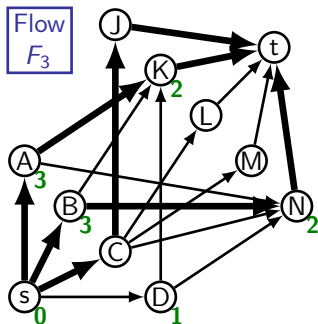
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The final example shows how it does this in the simplest possible case.

Vertex labelling for matching; Example 2

Find a (maximal) matching function m
for the relation $R = \{(a, p), (a, q), (b, p)\}$.

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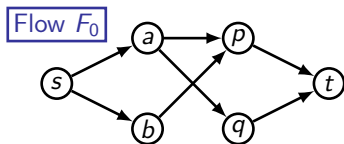
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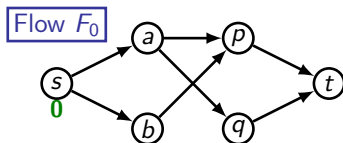
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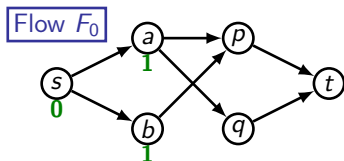
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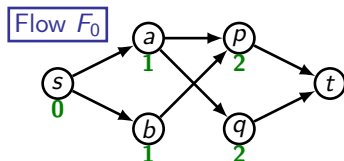
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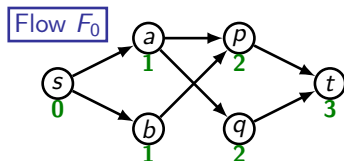
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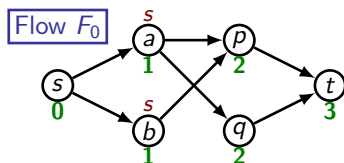
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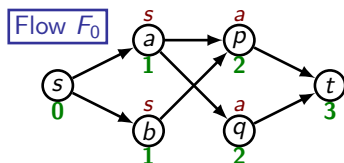
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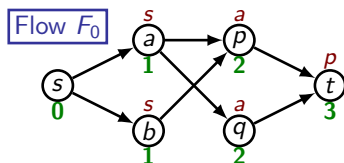
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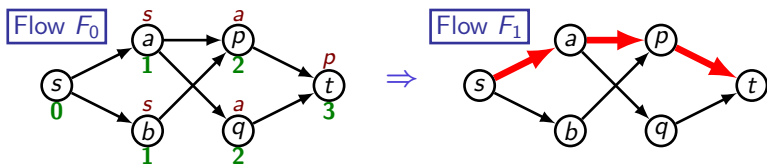
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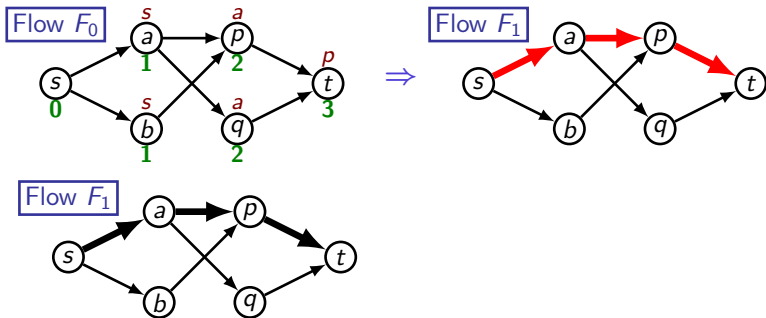
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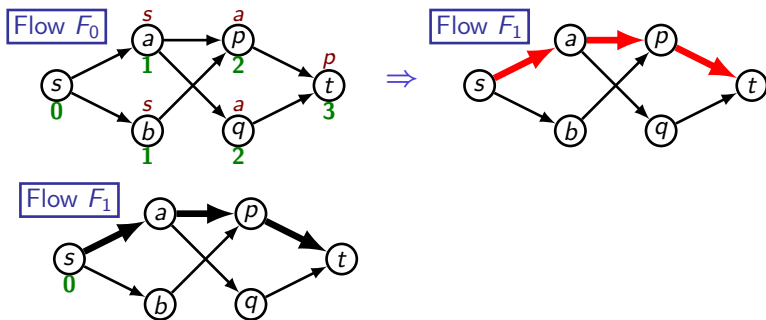
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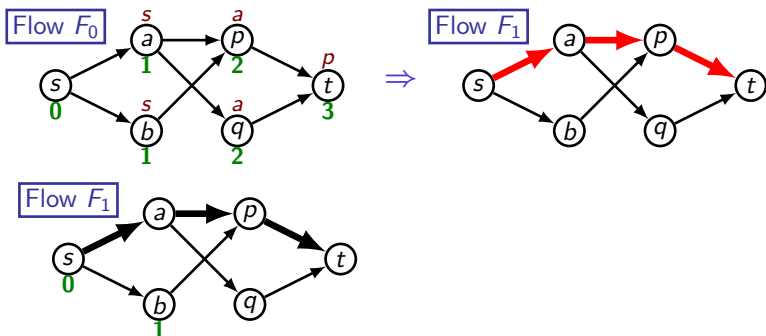
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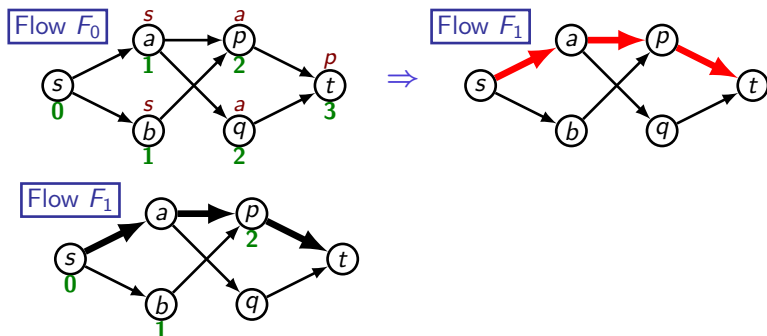
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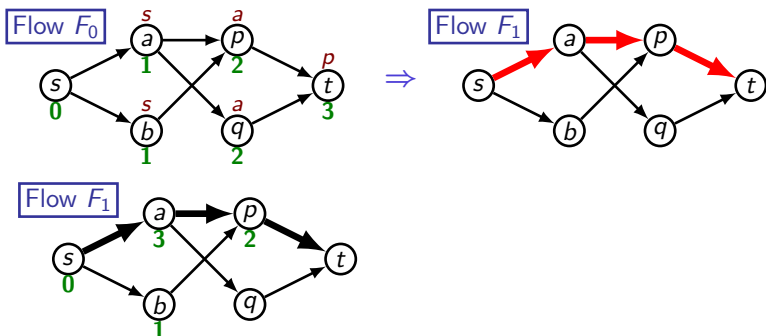
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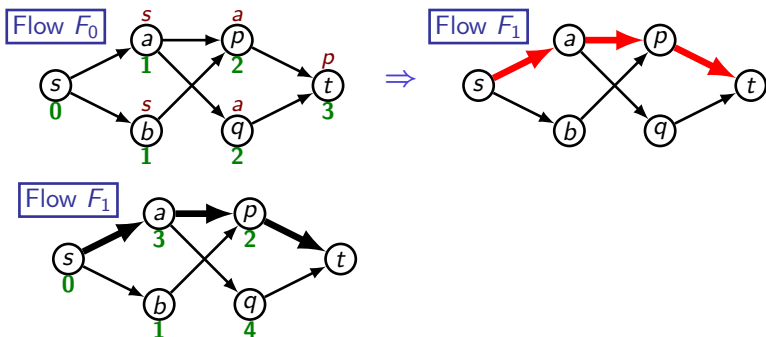
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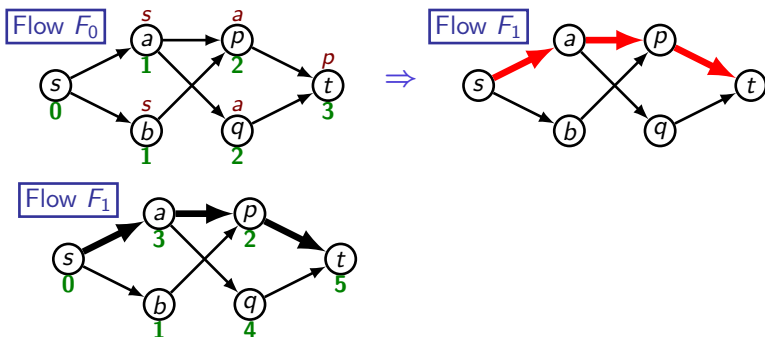
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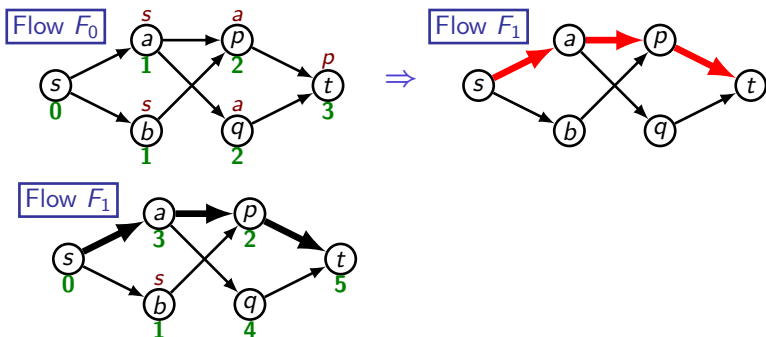
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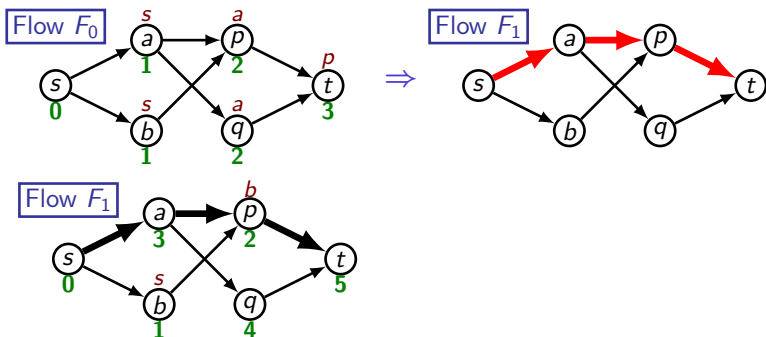
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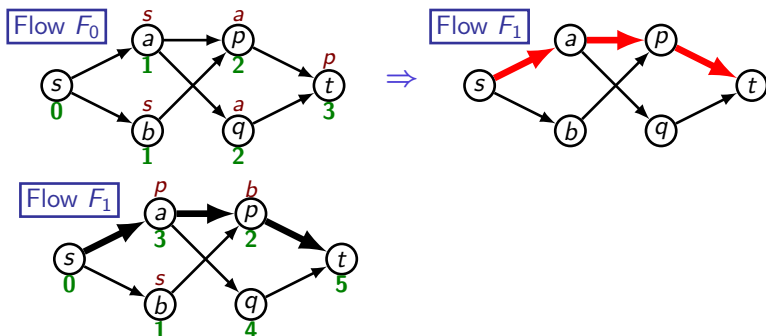
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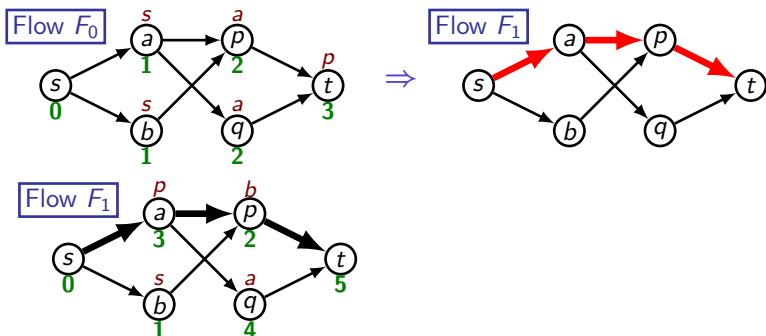
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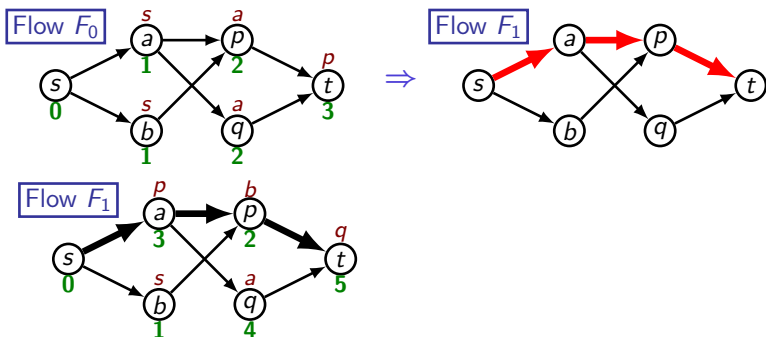
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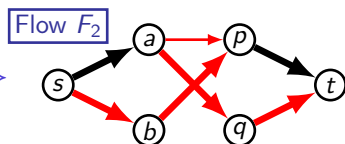
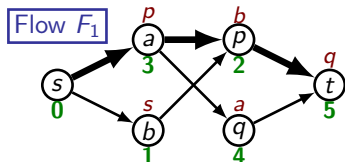
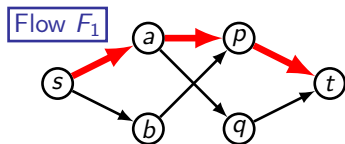
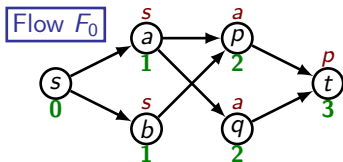
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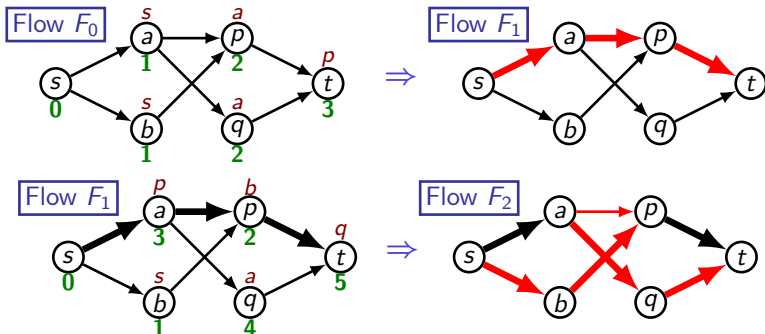
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END OF SECTION D2