Announcements: Grad assignments all have 3 day extension compared to original timing.
Grad assignment B due next Monday.

C1. Counting.

Notes originally prepared by Judy-anne Osborn.
Editing, expansion and additions by Malcolm Brooks.

Text Reference (Epp)	3ed:	Sections	6.1-7, 7.3
	4ed:	Sections	9.1-7
	5ed:	Sections	9.1-7

Recap: Matrices

{box with elements of a set in them $\mathbb{Q}$} $\overset{\text{bijection}}{\longleftrightarrow}$ {linear functions $\mathbb{Q}^n \to \mathbb{Q}^n$}

"2d" data

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This formal definition is suitable for generalisation to infinite sets (not all infinite sets have the same cardinality, as we shall see) and also points to some practical counting techniques (see next slide).

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- We have made a bijection to the set $\{1, 2, 3, ..., 11\}$, so $|S| = 11$.

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- Therefore $|S| = b - a + 1$.

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Since the composition of bijections is a bijection, the answer is 26.

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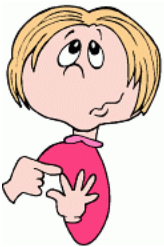
Examples:

Finite Sets	Infinite Sets
$\{1, 2, 3\}$	$\mathbb{N}$ ... natural numbers
{red, orange, yellow, green, blue, purple}	$\mathbb{Z}$ ... integers
$\{b$: b is a book in the Hancock library$\}$	$\mathbb{Q}$ ... rational numbers
$\{s$: s is a star in the Milky Way Galaxy$\}$	$\mathbb{R}$ ... real numbers
$\{\} = \emptyset$	$\mathcal{P}(\mathbb{R})$ ... power set of $\mathbb{R}$

Countability



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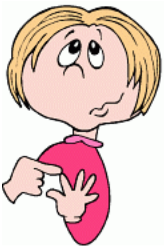

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$S \to \{1, \ldots, |S|\} \subset \mathbb{N}$

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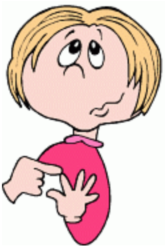

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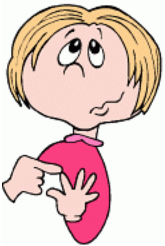

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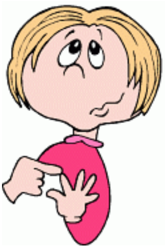

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- The sets $\mathbb{N}$ and $\mathbb{P}$ are each both countable and infinite. Such sets are called **countably infinite**.

Comparing cardinalities

Generalising from the case of finite sets, we say that two sets A and B have **the same cardinality**, written $|A| = |B|$, provided that there exists a bijection (one-to-one correspondence) from A to B.

$|S| = n$

$|T| = n$

$$S \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \{1, \ldots, n\} \underset{g^{-1}}{\overset{g}{\leftrightarrows}} T$$

$g^{-1} \circ f : S \to T$

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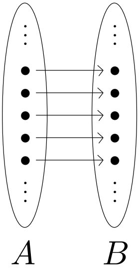

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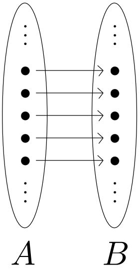

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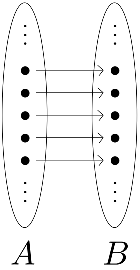

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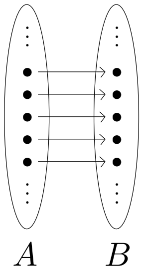

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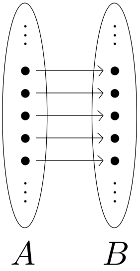

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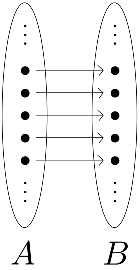

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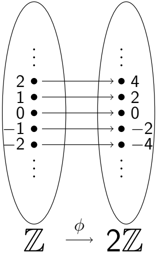

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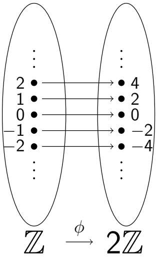

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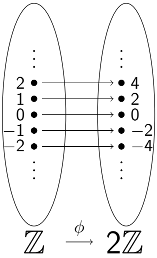

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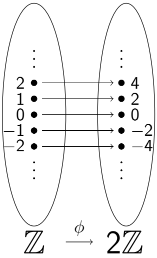

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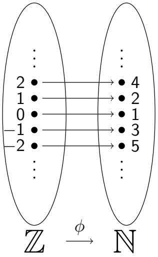
It not difficult to see that this is a bijection.



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Note: It follows from the result above (and its proof) that proving that an arbitrary infinite set S (not necessarily a subset of $\mathbb{N}$) is countably infinite amounts to showing that it can be '**well-ordered**'.

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Recall that we call a set S countably infinite if (and only if) it is infinite and there is a bijection from S to a subset of $\mathbb{N}$.
In other words, S has the same cardinality as some infinite subset of $\mathbb{N}$. However

Any infinite subset S of $\mathbb{N}$ has the same cardinality as $\mathbb{N}$ itself.

Proof: Since every bijection has an inverse, it is enough to establish a bijection $\phi : \mathbb{N} \to S$. This can be done recursively as follows:

$$\begin{aligned} \phi(1) &= \text{least member of } S \\ \phi(2) &= \text{least member of } S\backslash\{\phi(1)\} \quad (2^{\text{nd}} \text{ least member}) \\ \forall n \in \mathbb{N} \ \ \phi(n+1) &= \text{least member of } S\backslash\{\phi(1), \ldots, \phi(n)\}. \end{aligned}$$

Note: It follows from the result above (and its proof) that proving that an arbitrary infinite set S (not necessarily a subset of $\mathbb{N}$) is countably infinite amounts to showing that it can be '**well-ordered**'. This means that it is possible to order the elements of S in some (perhaps ingenious) way so that S and every subset of S has a 'least' member.

Georg Cantor: 'Small' and 'Big' infinities

(Non-assessable for MATH1005)

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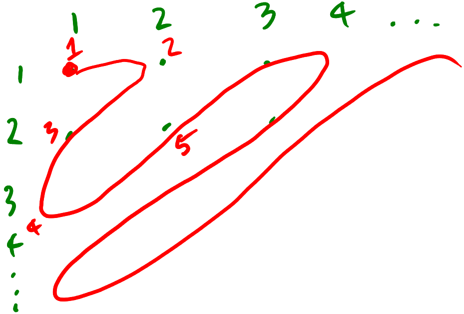
Positive rational numbers: $\frac{p}{q}$, $p, q \in \mathbb{N}$



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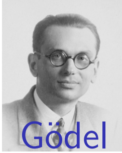
In fact it turns out that (in ZFC set theory) there is an infinite heirarchy of cardinalities, (an infinity of infinities)

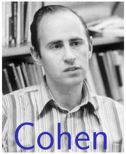
$$\aleph_0 < \aleph_1 < \aleph_2 < \ldots\ldots$$

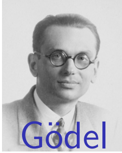
starting from $\aleph_0$ ('**aleph naught**'), the cardinality of $\mathbb{N}$.



The Continuum Hypothesis

(non-assessable)





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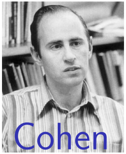

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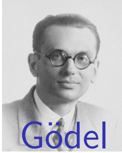

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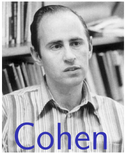

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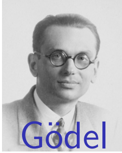

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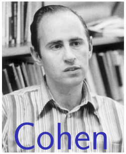

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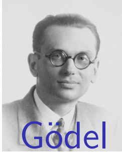

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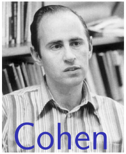

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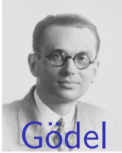

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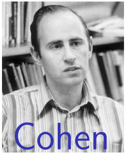

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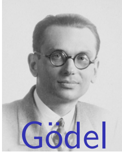

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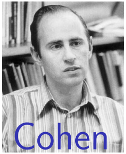

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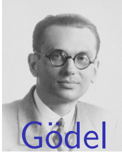

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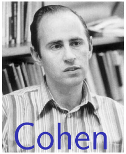

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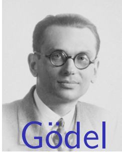

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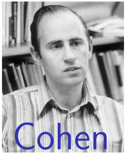

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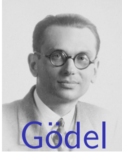

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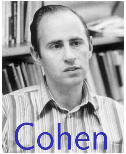

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Ref: "Cardinality": https://web.archive.org/web/20120822170123/http://personal.maths.surrey.ac.uk/st/H.Bruin/MMath/Cardinality.html

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(non-assessable for MATH1005)

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Theorem 2: $|S| \neq |\mathcal{P}(S)|$ for any set S.

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Examples:

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2. If a molecule can exist in 2 different configurations, and you have 10^9 such molecules, at least two of them must be in the same configuration.

Generalized Pigeonhole Principle

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More generally:

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Example:
In any set of a thousand words, there must be at least 39 words that start with the same letter, because $\left\lceil \frac{1000}{26} \right\rceil = \lceil 38.46 \rceil = 39$.

Harder pigeon hole example I

(Epp(4ed) Q9.4.33)

Let A be a set of six [distinct] positive integers each of which is less than 15. Show that there must be two distinct subsets of A whose elements when added up give the same sum.

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Harder pigeon hole example I

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Let A be a set of six [distinct] positive integers each of which is less than 15. Show that there must be two distinct subsets of A whose elements when added up give the same sum.

The phrase '*there must be two distinct subsets*' in the second sentence of the question suggests that the subsets should play the rôle of the pigeons.

Then the phrase '*give the same sum*' suggests that the possible sums should play the rôle of the pigeon holes. So

$$A = \{a, b, c, d, e, f \in \mathbb{N} \ : \ a < b < c < d < e < f < 15\}$$

Pigeons: subsets of A

Pigeon holes: possible element sums of subsets of A

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Now we have to count the pigeons and pigeon holes.

Harder pigeon hole example I (cont.)

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So there are more pigeons than pigeon holes.

Hence at least two pigeons share the same pigeon hole; *i.e* at least two subsets have the same element sum.

Harder pigeon hole example II

(Epp(4ed) Q9.4.35)

Given a set of 52 distinct integers, show that there must be two whose sum or difference is divisible by 100.

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But note that, assuming the integers are written in ordinary base 10 form, it is only their last two digits that matter.

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If a and b are two of the integers and if

$$a' = a \bmod 100, \qquad b' = b \bmod 100,$$

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(1) $a - b$ is divisible by 100 if and only if $a' = b'$;

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$$\underbrace{a'}_{\in\{0,\dots,99\}} = a \bmod 100, \qquad \underbrace{b'}_{\in\{0,\dots,99\}} = b \bmod 100,$$

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Requirement 3 eventually leads to the idea of numbering each pigeon hole with **two** two-digit numbers whose sum is 100:

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Thus at least two of the 52 pigeons share a hole and so have sum 100.

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In how many different orders could this queue be arranged?

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$\boxed{\text{There are } n! \text{ ways to arrange } n \text{ distinct objects in a list.}}$

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These ordered selections (lists) are called ***r*-permutations**.

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The 10 possibilities are:

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Combinations and binomial coefficients

Generalising the previous example we get

\# of ways to reorder

\# of ways to pick r in order

There are $\boldsymbol{C(n,r)} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$ ways to choose a set of r objects from a set of n candidates.
I.e. A set of cardinality n has $\frac{n!}{r!(n-r)!}$ subsets of cardinality r.

The subsets are called r-**combinations**.

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We say 'n choose r' for $C(n,r)$ and often write it $\underline{\binom{n}{r}}$.

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> *I.e.* A set of cardinality n has $\frac{n!}{r!(n-r)!}$ subsets of cardinality r.

The subsets are called **r-combinations**.

We say 'n choose r' for $C(n,r)$ and often write it $\dbinom{n}{r}$.

These numbers $\binom{n}{r}$ arise as coefficients in the algebraic expansion of the n-th power of the 'binomial' $(x+y)^n$ and are consequently also known as **binomial coefficients**.

Combinations and binomial coefficients

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These numbers $\binom{n}{r}$ arise as coefficients in the algebraic expansion of the n-th power of the 'binomial' $(x+y)$ and are consequently also known as **binomial coefficients**. The expansion is

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r .$$

$(x+y)^2$ $\binom{2}{0}$ $\binom{2}{2}$

$= x^2 + 2xy + y^2$

$\binom{2}{1}$

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Can you see how?

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(Epp(4ed) Q9.6.15)

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$||||\star\star\star\star\star\star\star\star\star\star$

$0, 0, 0, 0, 10$

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So the number of integers is $\binom{14}{4}-5=\dfrac{14\times 13\times 12\times 11}{4\times 3\times 2\times 1}-5=996.$

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Formally, a **size-r multiset** is a set S together with a 'multiplicity function' $m : S \to \mathbb{N}$, where,

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There are $\binom{r+n-1}{r}$ size-r multisets with members from a set of size n.

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(Epp(4ed) Q9.6.6)

If n is a positive integer, how many 5-tuples of integers from 1 through n can be formed in which the elements of the 5-tuple are written in non-increasing order?

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(2 2 3 1 1) ✗

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So by stars-and-bars, there are $\binom{5+n-1}{5} = \binom{n+4}{5}$ of these 5-tuples.

For example for $n = 3$ there are $\binom{7}{5} = \binom{7}{2} = 21$ such 5-tuples:
33333 33332 33331 33322 33321 33311 33222 33221 33211 33111
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New counts from old

- The Sum Rule
- The Product Rule
- Inclusion-Exclusion

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More generally, if $\{A_1, A_2, \ldots, A_m\}$, $m \in \mathbb{N}$, is a *partition* of the finite set A then

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Example:

For $U = \{-10, \ldots, 10\} \subseteq \mathbb{Z}$ and $S = \{n \in U \ : \ |20 - n^2| > 10\}$, find $|S|$.

The Sum Rule

If sets A and B are finite and *disjoint* then the cardinality of their union $A \cup B$ is the sum of the their individual cardinalities, i.e.

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$20-n^2 > 10$
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Observe that $|20 - n^2| > 10 \iff n^2 < 10 \ \vee \ n^2 > 30.$

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$n^2 < 10$ $\quad$ $n^2 > 30,\ n > 0$ $\quad$ $n^2 > 30,\ n < 0$

So $S = \{-3, -2, \ldots, 2, 3\} \cup \{6, 7, \ldots, 10\} \cup \{-10, -9, \ldots, -6\}$.

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Hence $|S| = 7 + 5 + 5 = 17$.

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$\underbrace{a_1 \in A_1}, \ldots, a_m \in A_m$

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The number of possible outcomes from throwing the three dice together is $|C \times O \times D| = |C| \times |O| \times |D| = 6 \times 8 \times 12 = 576.$

Inclusion-Exclusion

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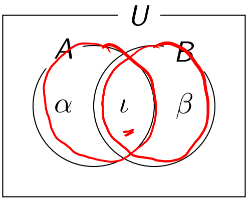
If A and B are finite sets which *may not be disjoint* the sum rule has to be modified to the **inclusion-exclusion rule:**

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$



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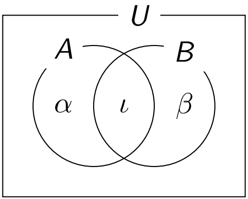

This is because the plain sum rule $|A \cup B| = |A| + |B|$ includes the intersection $A \cap B$ twice, so it has to be excluded once:

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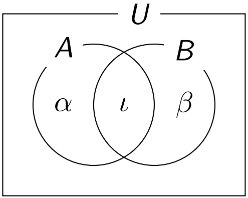

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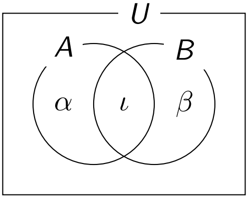

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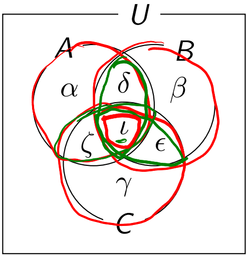
The inclusion-exclusion rule can be generalised to deal with more than two sets, but it quickly gets very messy.

Can you figure out how to extend the rule to deal with just three sets A, B and C?



Bit-String Example of Inclusion-Exclusion

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Task 1: Construct a string of length 8 that starts with '1'.

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Task 1: Construct a string of length 8 that starts with '1'.

- There is one way to choose the first bit (1)
- There are two ways to choose the second bit (0 or 1)
- There are two ways to choose the third bit (0 or 1)

$\vdots$

- There are two ways to choose the eighth bit (0 or 1)

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Product Rule: Task 1 can be done in $1 \times 2^7 = 128$ ways.

Bit-String Example of Inclusion-Exclusion

Task 2

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- There are two ways to choose the first bit (0 or 1)
- There are two ways to choose the second bit (0 or 1)

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- There are two ways to choose the sixth bit (0 or 1)
- There is one way to choose the seventh bit (0)
- There is one way to choose the eighth bit (0)

Bit-String Example of Inclusion-Exclusion

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- There are two ways to choose the sixth bit (0 or 1)
- There is one way to choose the seventh bit (0)
- There is one way to choose the eighth bit (0)

Product Rule: Task 2 can be done in $2^6 \times 1^2 = 64$ ways.

Bit-String Example of Inclusion-Exclusion

Task 3

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Is the answer $128+64 = 196$?

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We have to subtract off the cases we counted twice.

Task 3: Construct a string of length 8 that both starts with '1' and ends with '00'.

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- There are two ways to choose the sixth bit (0 or 1)
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Product Rule: Task 3 can be done in $1 \times 2^5 \times 1^2 = 32$ ways.

Bit-String Example of Inclusion-Exclusion

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- the number of ways to do task 2,

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Finally, the number of ways to construct a bit string of length 8 that starts with '1' or ends with '00' is equal to:

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Finally, the number of ways to construct a bit string of length 8 that starts with '1' or ends with '00' is equal to:

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$$128 + 64 - 32 = 160.$$

Note: An alternative, and quite different, way to solve this problem is to use **complementary counting**;

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Conclusion

Finally, the number of ways to construct a bit string of length 8 that starts with '1' or ends with '00' is equal to:

$S = A \cup B$

$S^c = A^c \cap B^c$

- the number of ways to do task 1, **plus**
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Can you see how to get the $1 \times 2^5 \times 3$?

END OF SECTION C1