Question 1 Define the logical variables a, d and f as follows:

$$\begin{aligned} a &= \text{“The ASX is at least 4000”} \\ d &= \text{“The Dow is at least 1200”} \\ f &= \text{“The FTSE is at least 5000”} \end{aligned}$$

Express each of the following in symbols, $\boxed{\text{as succinctly as you can}}$.

(a) The ASX is at least 4000, and either the Dow is at least 1200 or the FTSE is at least 5000, but not both.

(b) When the FTSE is below 5000 the Dow is below 1200.

(c) Either the ASX is at least 4000 and the Dow is at least 1200, or the ASX is below 4000 and the Dow is below 1200. [*Try to express this with just one logical connective.*]

(a) Directly: $a \wedge (d \vee f) \wedge \neg(d \wedge f)$.

Most succinctly: $\boxed{a \wedge (d \oplus f).}$

(b) Directly: $\neg f \Rightarrow \neg d$. *(or* $f \vee \neg d$*)*

Most succinctly, using contrapositive: $\boxed{d \Rightarrow f.}$

(c) Directly: $(a \wedge d) \vee (\neg a \wedge \neg d)$.

Most succinctly: $\boxed{a \Leftrightarrow d.}$

Question 2 Negate each of the statements below.
Use as natural sounding English as you can manage, and try to avoid using the word 'not'. Do not use symbols.

(a) She will win silver or gold.

(b) She will win silver if she fails to win gold.

(c) She will win gold in the 100m event and in the 200m event.

(a) Directly: She will not win silver and not win gold.

One of several more natural ways to say this:

$\boxed{\textit{She will win neither silver nor gold.}}$

(b) The given statement has the form $\neg g \implies s$*, which is equivalent to* $\neg\neg g \vee s$ *and hence to* $s \vee g$*, which is the statemnt given for (a). So the negation is*

$\boxed{\textit{Same answer as for (a).}}$

(c) Directly: She will not win gold in the 100m event or she will not win gold in the 200m event.

NB: *The statement "She will not win gold in the 100m event or in the 200m event" is at best ambiguous and in practice usually interpreted as a conjunction ("She won't win gold in either event"), which is* ***not*** *the negation of the given statement.*

One of several more succinct and precise negations is:

$\boxed{\textit{She will win at most one gold from the 100m and 200m events.}}$

Question 3 When solving the equation $2x - 6 = 0$ we might write out the solution method in 'shorthand' something like this:

$$2x - 6 = 0 \quad \Rightarrow \quad 2x = 6 \quad \Rightarrow \quad x = 3.$$

In our model of logic, this appears to have the form $p \Rightarrow q \Rightarrow r$. However, this statement form is ambiguous because the $\Rightarrow$ connective is not associative; *i.e.* $(p \Rightarrow q) \Rightarrow r$ and $p \Rightarrow (q \Rightarrow r)$ mean different things.

(a) Use truth table(s) to show that $[(p \Rightarrow q) \Rightarrow r] \not\equiv [p \Rightarrow (q \Rightarrow r)]$.

(b) Is the shorthand solution scheme notation $p \Rightarrow q \Rightarrow r$ correctly represented by either $(p \Rightarrow q) \Rightarrow r$ or $p \Rightarrow (q \Rightarrow r)$? If so, which one; if not, what should it be?

(a)

p	q	r	$(p$	$\Rightarrow$	$q)$	$\Rightarrow$	r	p	$\Rightarrow$	$(q$	$\Rightarrow$	$r)$
T	T	T	$\cancel{T}$	$\cancel{T}$	$\cancel{T}$	**T**	$\cancel{T}$	$\cancel{T}$	**T**	$\cancel{T}$	$\cancel{T}$	$\cancel{T}$
T	T	F	$\cancel{T}$	$\cancel{T}$	$\cancel{T}$	**F**	$\cancel{F}$	$\cancel{T}$	**F**	$\cancel{T}$	$\cancel{F}$	$\cancel{F}$
T	F	T	$\cancel{T}$	$\cancel{F}$	$\cancel{F}$	**T**	$\cancel{T}$	$\cancel{T}$	**T**	$\cancel{F}$	$\cancel{T}$	$\cancel{T}$
T	F	F	$\cancel{T}$	$\cancel{F}$	$\cancel{F}$	**T**	$\cancel{F}$	$\cancel{T}$	**T**	$\cancel{F}$	$\cancel{T}$	$\cancel{F}$
F	T	T	$\cancel{F}$	$\cancel{T}$	$\cancel{T}$	**T**	$\cancel{T}$	$\cancel{F}$	**T**	$\cancel{T}$	$\cancel{T}$	$\cancel{T}$
F	T	F	$\cancel{F}$	$\cancel{T}$	$\cancel{T}$	**F**	$\cancel{F}$	$\cancel{F}$	**T**	$\cancel{T}$	$\cancel{F}$	$\cancel{F}$
F	F	T	$\cancel{F}$	$\cancel{T}$	$\cancel{F}$	**T**	$\cancel{T}$	$\cancel{F}$	**T**	$\cancel{F}$	$\cancel{T}$	$\cancel{T}$
F	F	F	$\cancel{F}$	$\cancel{T}$	$\cancel{F}$	**F**	$\cancel{F}$	$\cancel{F}$	**T**	$\cancel{F}$	$\cancel{T}$	$\cancel{F}$

The two resulting columns (shown bold) are not identical (they differ in rows 6 and 8), so non-equivalence is established.

(b) The solution is saying that q follows from p and r follows from q, so the correct representation is

$$\boxed{(p \Rightarrow q) \wedge (q \Rightarrow r)}$$

This is equivalent to neither $(p \Rightarrow q) \Rightarrow r$ nor $p \Rightarrow (q \Rightarrow r)$. (For example, consider the case $p, q, r = T, F, T$.)

*Note that we automatically infer $p \Rightarrow r$. This is because the argument at right is valid.

$$\begin{array}{c} p \Rightarrow q \\ q \Rightarrow r \\ \hline p \Rightarrow r \end{array}$$

$\star$: Not required.

Question 4 Let p : "If the new drug succeeds, diabetes rates will fall".

(a) Write out the converse of p. Is this equivalent to p?

(b) Write out the contrapositive of p. Is this equivalent to p?

(c) Express p using the phrase "necessary condition".

(a) *(i) Directly: If diabetes rates will fall, (then) the new drug succeeds. This sounds unnatural English. Better is*

$\boxed{\textit{If dabetes rates fall, the new drug will have succeeded.}}$

(ii) This is $\boxed{\textit{not}}$ *equivalent to the original because* $(a \Rightarrow b) \not\equiv (b \Rightarrow a)$.

(b) *(i) Negating each component of the answer to (a) above gives*

$\boxed{\textit{If diabetes rates don't fall, the new drug will have failed.}}$

(ii) This $\boxed{\textit{is}}$ *equivalent to the original because* $(a \Rightarrow b) \equiv (\neg b \Rightarrow \neg a)$.

(c) For any conditional statement, say $s \Rightarrow n$*, we say that* s *is a sufficient condition for* n *and that* n *is a necessary condition for* s.

In particular, another way to express p *is:*

$\boxed{\textit{Falling diabetes rates is a necessary condition for the drug's success.}}$

[In this context 'requirement' might be a more appropriate word than 'condition'.]

Question 5 For each of the following sentences, say whether the sentence is a true statement, a false statement or a predicate. Also give the negation of each sentence.

(a) If $x^2 > 0$ then $x > 0$.

(b) $\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ \ x = y^2$.

(c) $\exists! x \in \mathbb{N}\ \ 3x - x^2 = 2$. [*This one is tricky!*]

(a) *(i) Strictly speaking, this is a* $\boxed{\textit{predicate}}$ *because variable x is unquantified (and so requires a value to be specified before truth can be determined.)*
However, in practice, the context of a sentence like this often implies a tacit *universal quantifier whose domain is to be 'understood' from the context. In this case the resulting statement would be true for domain $\mathbb{N}^\star = \{0, 1, 2, 3, \ldots\}$ but false for domain $\mathbb{R}$.*

(ii) Negation: Strictly speaking, $\boxed{x^2 > 0 \wedge x \leq 0}$.
But if a tacit quantifier $\forall x$ applies, then the negation needs the quantifier $\exists x$.

(b) *(i) This is a* $\boxed{\textit{false statement}}$. *A counterexample is $x = 2$ because 2 is not the square of any natural number.*

(ii) Negation: $\boxed{\exists x \in \mathbb{N}\ \forall y \in \mathbb{N}\ x \neq y^2}$.
Note that this just asserts the existence of a counterexample.

(c) *(i) This statement asserts that the equation $3x - x^2 = 2$ has a unique solution in $\mathbb{N}$. This is a* $\boxed{\textit{false statement}}$ *because in fact the equation has two solutions in $\mathbb{N}$,* viz *1 and 2. (Solve the quadratic equation $x^2 - 3x + 2 = 0$.)*

(ii) The negation of unique existence is non-existence or multiple existence. Thus:
$\boxed{(\forall x \in \mathbb{N}\ 3x - x^2 \neq 2) \vee (\exists x, y \in \mathbb{N}\ (3x - x^2 = 3y - y^2 = 2) \wedge (x \neq y))}$.

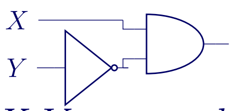
Question 6

inputs		output
X	Y	
1	1	0
1	0	1
0	1	1
0	0	0

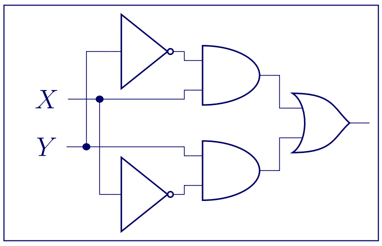
(a) Construct a circuit diagram corresponding to the input-output (truth) table at right. Do this by employing the standard method of first writing out a logical expression, in disjunctive normal form (*i.e.* the disjunction of several conjunctions), that has the given truth table, and then converting this to a circuit using only AND, NOT and OR gates.

(b) (*Challenge*) Construct a circuit diagram corresponding to the same input-output table but this time using only NAND gates. Try to use as few gates as you can.

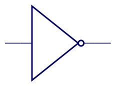
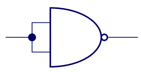
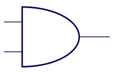
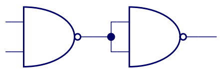
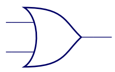
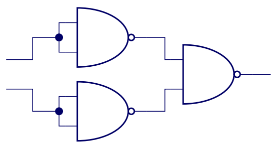
(a) *(i) There are two 'true' outputs;* $(X, Y) = (1, 0)$ *and* $(X, Y) = (0, 1)$. *So the corresponding logical expression is* $\boxed{(X \wedge \neg Y) \vee (\neg X \wedge Y)}$. *[Although this is equivalent to the simpler expression* $X \oplus Y$*, that expression is not in disjuctive normal form.]*

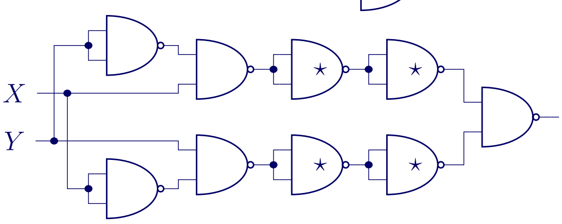
(ii) A circuit for $X \wedge \neg Y$ *is* 

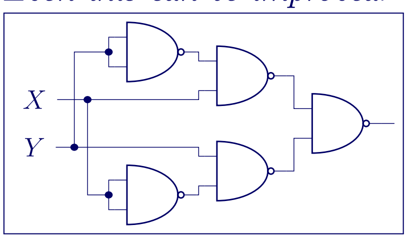
A similar circuit, with X,Y *reversed, gives* $\neg X \wedge Y$. *Combining these two circuits using an OR gate gives:*

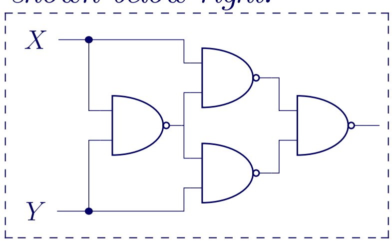


(b)

- *Recall that* NAND, *meaning 'not and', is represented by* $X \,|\, Y \equiv \neg(X \wedge Y)$.
- *It follows that* $X \,|\, X \equiv \neg X$, *so*  *can be replaced by* .
- *Since* $X \wedge Y \equiv \neg(X \,|\, Y)$,  *can be replaced by* .
- *By DeMorgan* $X \vee Y \equiv \neg(\neg X \wedge \neg Y) \equiv \neg X \,|\, \neg Y$ *so*  *can be replaced by* .
- *Applying these substitutions to the circuit found for (a) gives the* 9-NAND *circuit at right. But the four starred* NAND *gates form two sets of double negations, so they can be eliminated, leading to the* 5-NAND *circuit below left. Even this can be improved. The best solution is shown below right.*







Question 7 Determine whether or not the following argument is valid.

> If Adam's kids stay up past their bedtime, then they are grumpy in the morning. If the kids are grumpy in the morning, then Adam is late for work. Adam was late for work, therefore the kids stayed up past their bedtime.

Let

$$\begin{aligned} b &: \textit{Adam's kids stay up past their bedtime} \\ g &: \textit{Adam's kids are grumpy in the morning} \\ \ell &: \textit{Adam is late for work.} \end{aligned}$$

The argument has the form

$$[b \to g, g \to \ell, \ell \;\therefore\; b].$$

By definition, the argument is valid if and only if

$$\big((b \to g) \wedge (g \to \ell) \wedge \ell\big) \to b$$

is a tautology. Adding some extra parentheses to decide an order in which to evaluate the ANDs, we see our compound statement is logically equivalent to:

$$\Big(\big((b \to g) \wedge (g \to \ell)\big) \wedge \ell\Big) \to b$$

So we make a truth table, and we only fill in enough entries to evaluate the compound statement in each row. We compute:

b	g	ℓ	$\Big(\big((b \to g)$	$\wedge$	$(g \to \ell)\big)$	$\wedge$	$\ell\Big)$	$\to$	b
T	T	T						**T**	T
T	T	F						**T**	T
T	F	T						**T**	T
T	F	F						**T**	T
F	T	T	T	T	T	T	T	**F**	F
F	T	F				F	F	**T**	F
F	F	T		F	F	F		**T**	F
F	F	F				F	F	**T**	F

Since the compound statement is false in the 5th row, it is not a tautology. The argument is not valid.