# D2. Weighted Graphs

Notes by Malcolm Brooks,
partly inspired by notes of Pierre Portal.

| | | |
|---|---|---|
| Text Reference (Epp) | 3ed: | Chapter 11 |
| | 4ed: | Chapter 10 |
| | 5ed: | Chapter 10 |

Some of the work in this section is not covered in our text by Epp.
I have based some examples on ones from:
Kolman, Busby & Ross *Discrete Mathematical Structures*
Johnsonbaugh *Discrete Mathematics*

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  - **Fibre telecommunications:** Vertices are primary routing stations; there are edges between every pair of vertices (a complete graph); weights are the costs of laying fibre between stations.
  - **The internet:** Vertices are internet nodes; edges are all direct connections between nodes; weights are times (in milliseconds) for a packet to travel across a connection.

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We will also look at a different kind of problem on a weighted **directed** graph: Maximal Flow. Details later.

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4. Repeat steps 2 and 3 until $T$ has $n-1$ edges.

# Example: Applying Kruskal's algorithm

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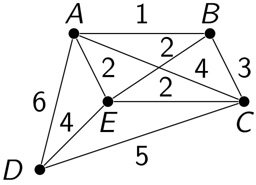

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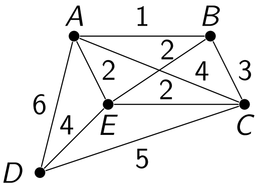


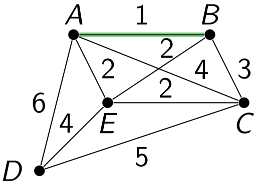


$W = 1$

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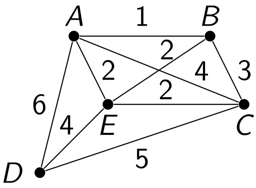
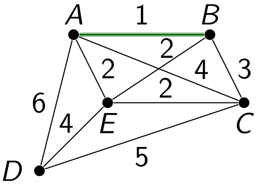
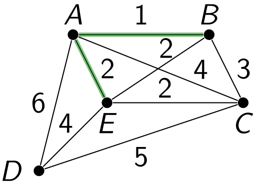



$W = 1$

$\Longrightarrow$

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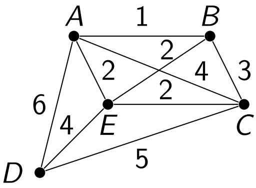
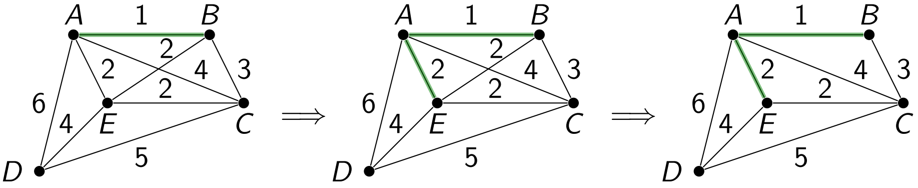
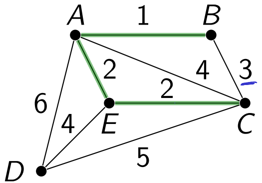



$W = 1$ $\qquad$ $W = 3$ $\qquad$ $W = 3$

# Example: Applying Kruskal's algorithm (cont.)

$\Longrightarrow$




$W = 5$

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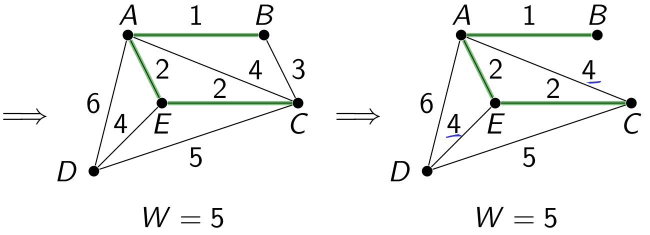

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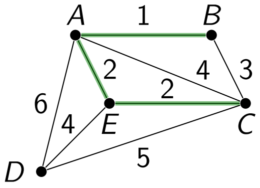
$\Longrightarrow$ 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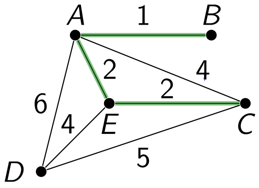
$\Longrightarrow$ 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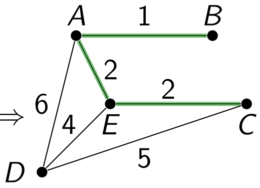
$\Longrightarrow$ 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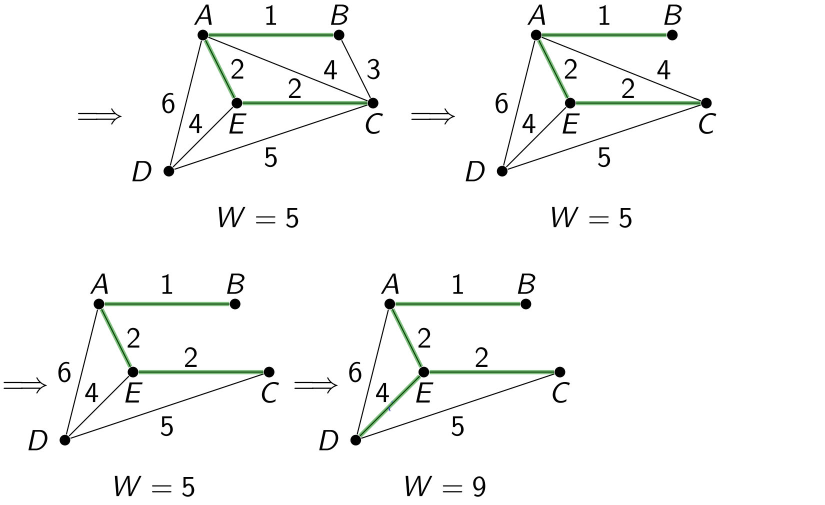

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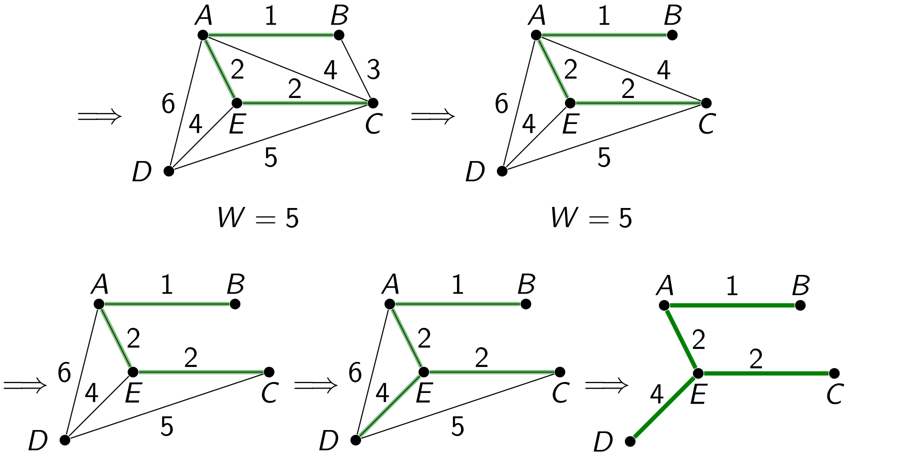


$W = 5$ $W = 5$

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(The generated spanning tree)

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- Kruskal's algorithm always succeeds! (Non-obvious theorem omitted)

  That is, it always finds a minimal spanning tree, given any weighted connected (finite) graph.

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- This problem originates from a time when all salespeople were men !!
- The salesman needs to visit $n$ towns on a shortest possible ‘circular tour’.
- Given: a table of distances between every pair of towns.
- **Model:** Graph $K_n$ with towns as vertices and edges weighted by the the inter-town distances.

  Find a Hamilton circuit of minimum possible total weight.

# The 'Nearest Neighbour' algorithm (for the travelling salesman problem)

**Input:** Weighted complete graph $G$ with $n$ vertices.

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6. Add weight$(L(n), L(1))$ to $W$. Append $L(1)$ to $L$ as $L(n + 1)$.

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- Greed doesn't always pay !!

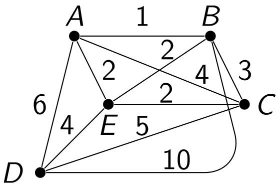
$P = NP$ ?

- In fact, no efficient successful algorithm for the travelling salesman problem is known at this time. Finding one, or proving that none exists, is a major outstanding problem in mathematics.

# Example: Applying the Nearest Neighbour algorithm

Find a minimal Hamilton circuit (tour) for this weighted graph:

**Note:** This graph is as for the minimal spanning tree example but with the addition of an edge $BD$ to make it complete.



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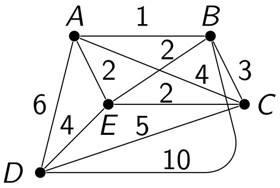


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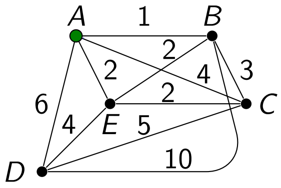


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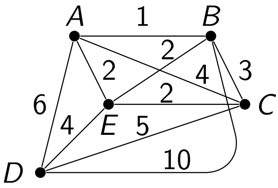


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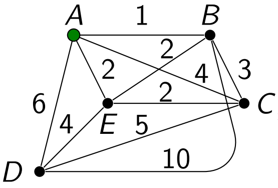


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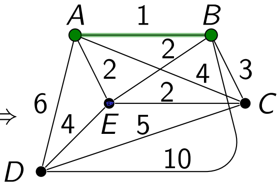


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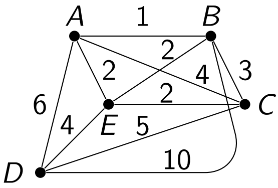


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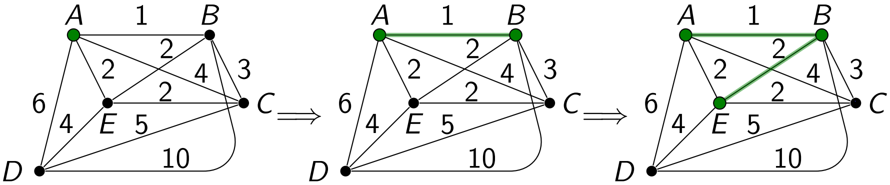
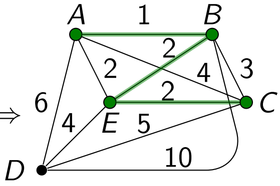



$L(1) = A,\ W = 0$ $\quad$ $L(2) = B,\ W = 1$ $\quad$ $L(3) = E,\ W = 3$

# Example: Applying the Nearest Neighbour algorithm (cont.)

$\Longrightarrow$ 


$L(4) = C, \; W = 5$

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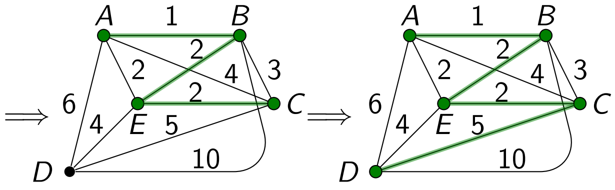


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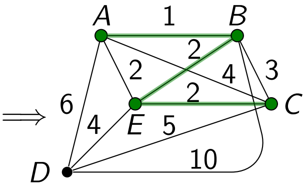


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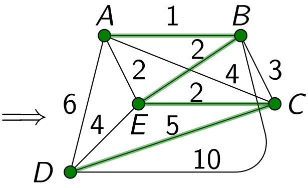
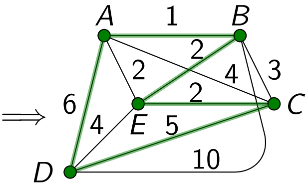



$L(5) = D, \ W = 10$




$L(6) = A, \ W = 16$

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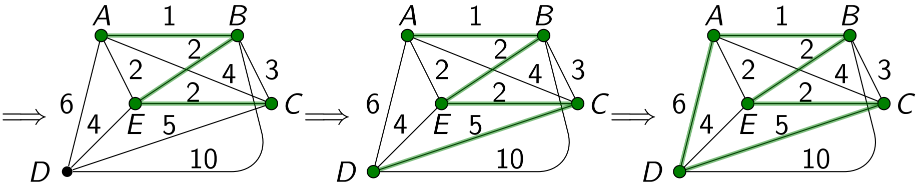


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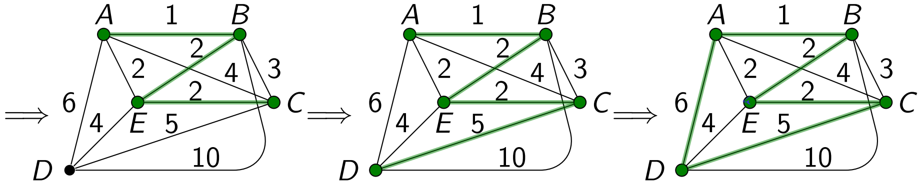


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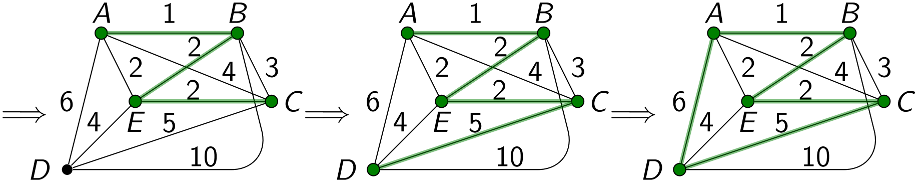


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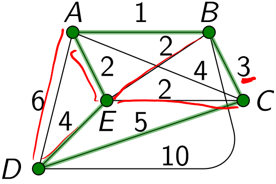


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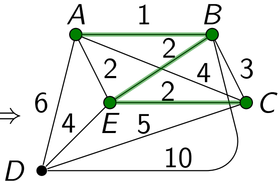


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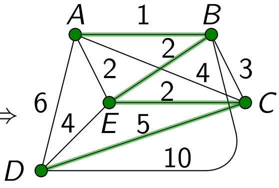


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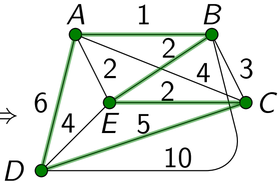


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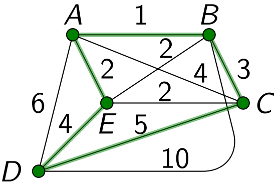


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Note that Nearest Neighbour may generate this tour if we start at $D$ instead of $A$. Then $L(2) = E$ and it just depends on the choice for $L(3)$.

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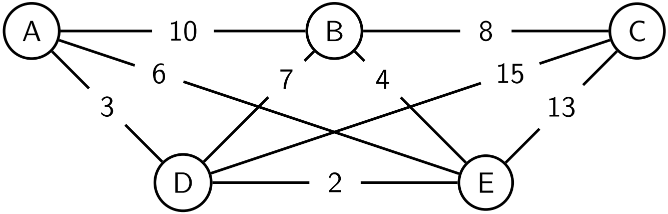
Consider this weighted graph:



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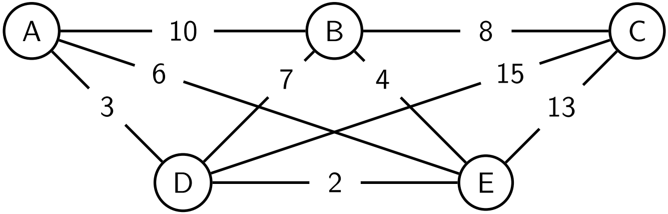


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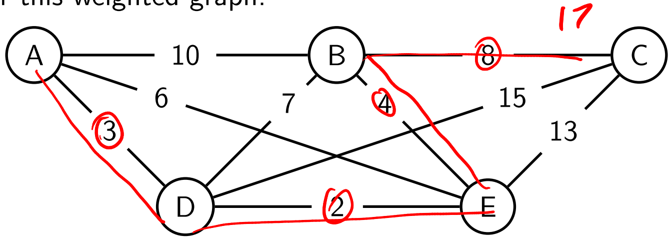


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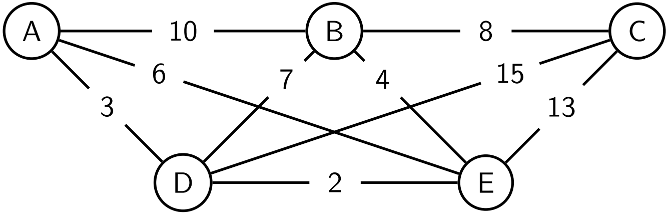


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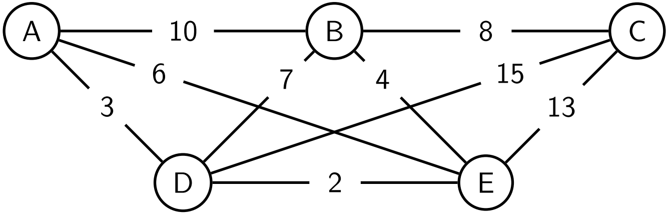


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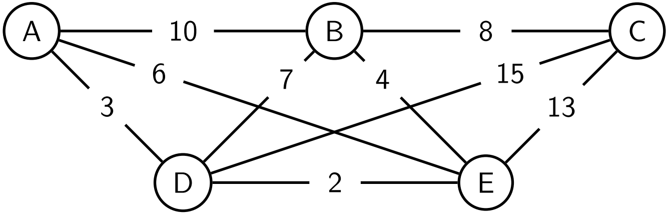


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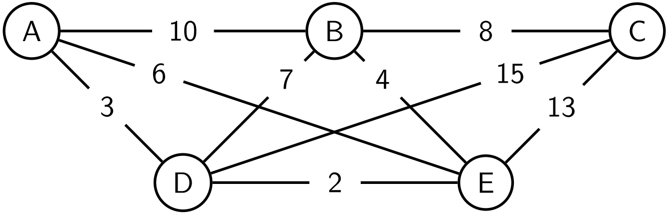


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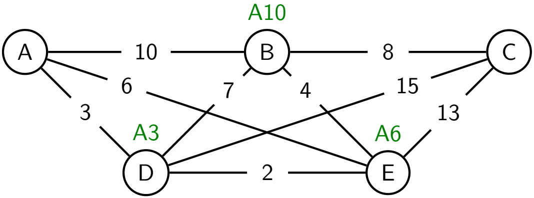
For large graphs this approach is not practical. We need an *algorithm*.

# Dijkstra's Algorithm




Edsger Dijkstra 1930 - 2002

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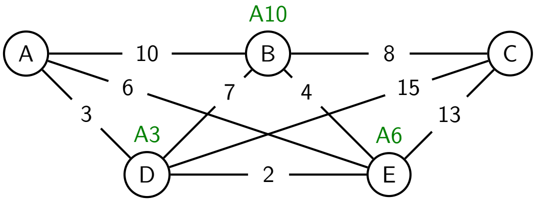


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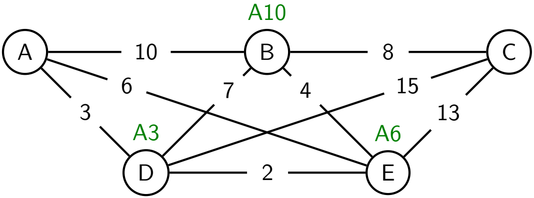


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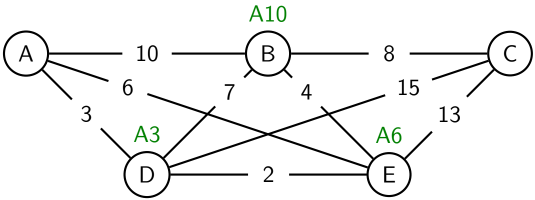


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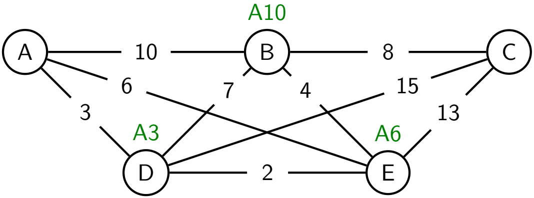


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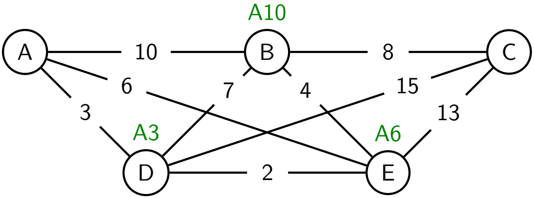


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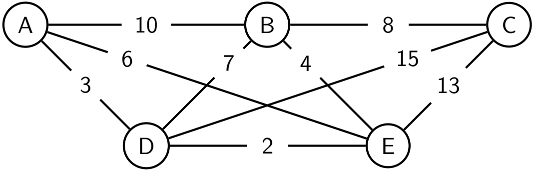
Dijkstra's algorithm has some similarities to Kruskal's algorithm for finding a minimum spanning tree but is a little more complicated. So I will delay setting out the algorithm formally and launch straight in to demonstrating its use on the above example.

# Example 1 — Slide 1

We seek a minimal weight path from A to C in the graph below.



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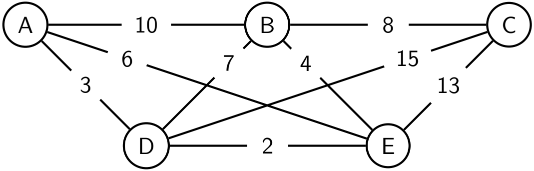


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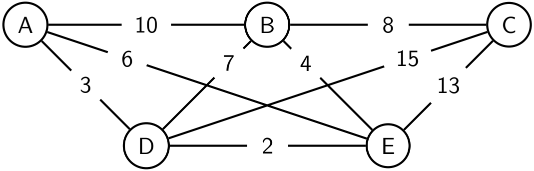


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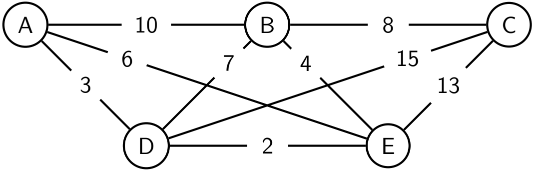


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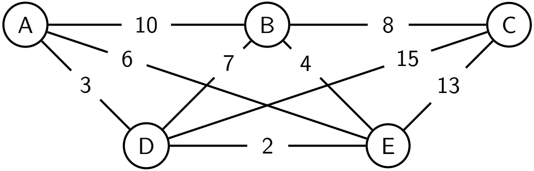


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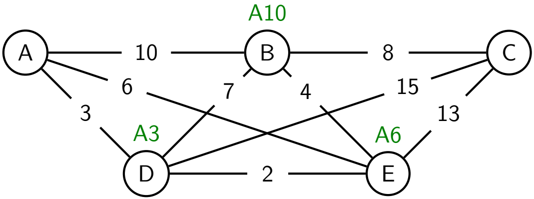

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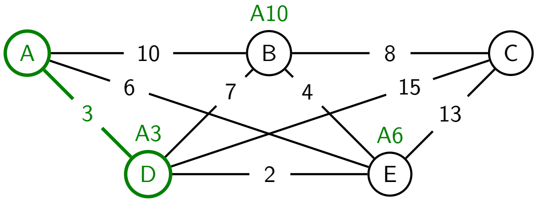

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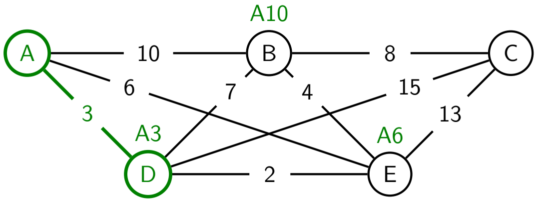


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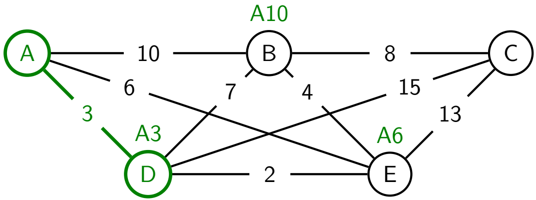


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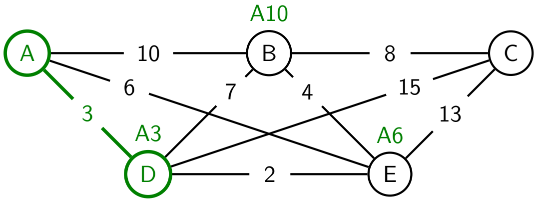


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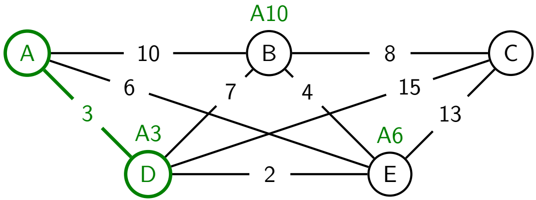


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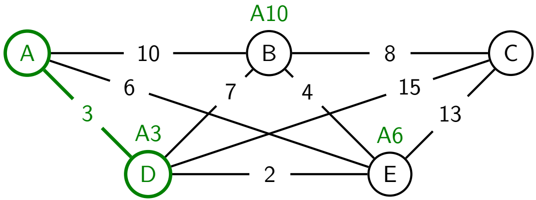


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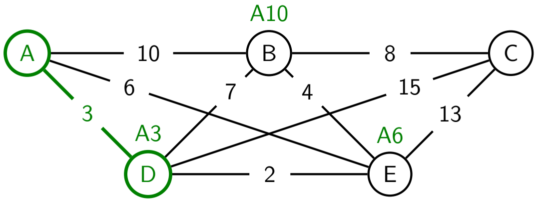


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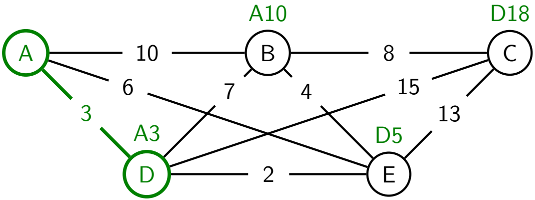
So we leave the A10 above B as it is.

The annotated graph now looks like this:



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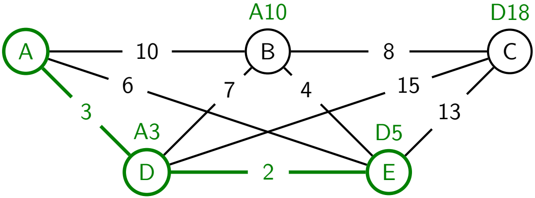

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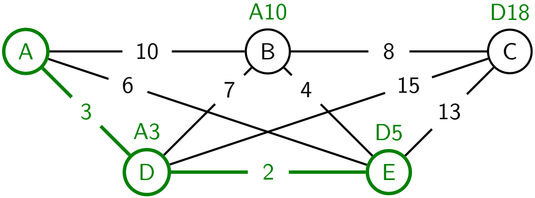


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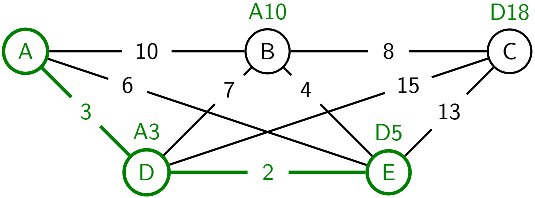


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We now repeat the process applied to the previous current vertex D.

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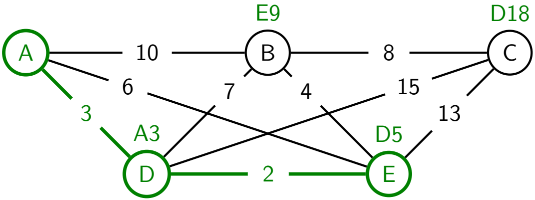

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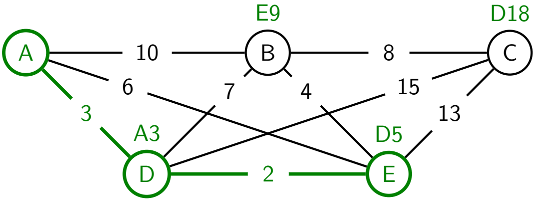
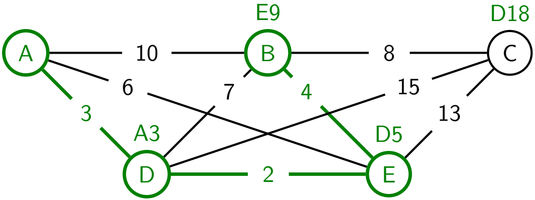



The lowest fringe value is now 9 on B, so we lock in B and its lead-in edge EB (next slide).

# Example 1 — Slide 6



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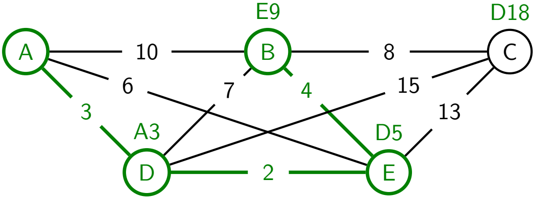


The new current vertex is the just locked-in B.

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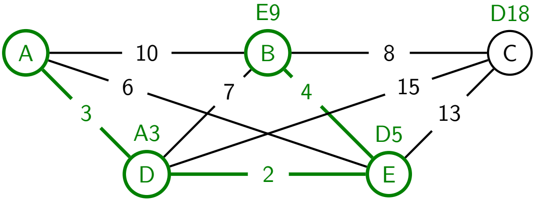


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There is only one vertex adjacent to B that has not already been locked in, namely C.

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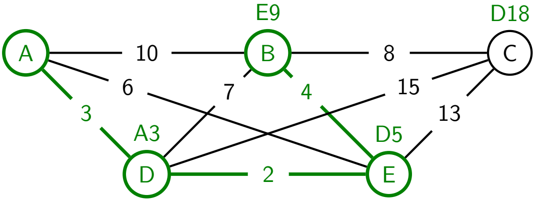


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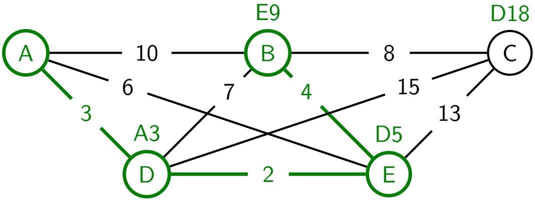


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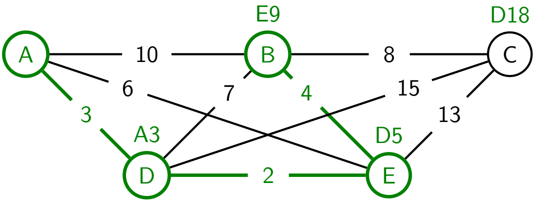


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Since vertex C is the only vertex in the fringe it has the lowest label by default. So C and its lead-in edge BC are locked in.

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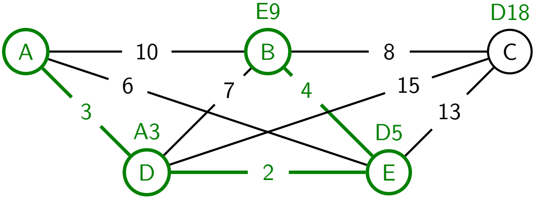


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We have now locked in the minimal distance 17 into our 'target' vertex C, so we can stop.

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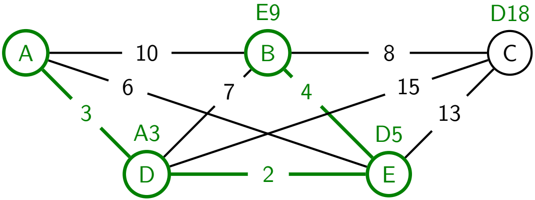


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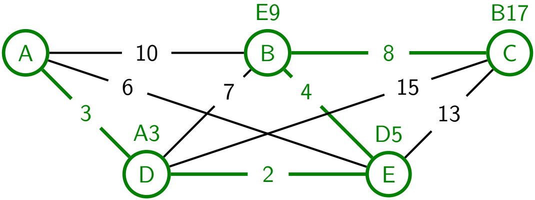
# Example 1 — Slide 7; Results and Comments



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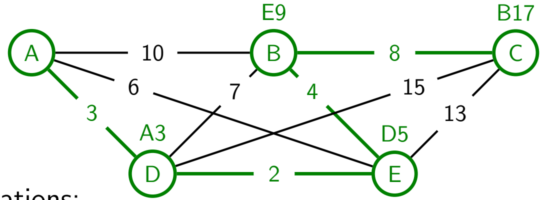


Some Observations:

- Besides the shortest path from A to C, the solution provides the shortest path to all the vertices along that path. For this example that happens to be the entire vertex set.

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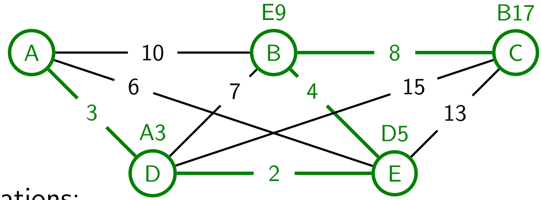


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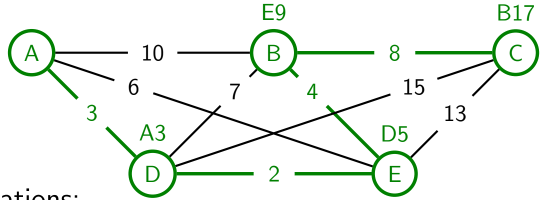


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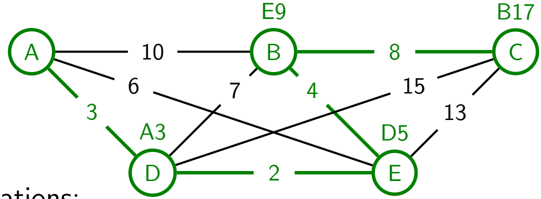


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- With all vertices locked, the solution provides a spanning tree for the graph.

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While $c \neq \mathrm{Z}$:

4. For each vertex $v$ adjacent to $c$ but not in $T$:
   If $v$ is unmarked (*i.e.* $M(v) = \text{blank}$)
   or if $L(v) > L(c) + \mathrm{dist}(\{c, v\})$
   set $M(v) = c$, $L(v) = L(c) + \mathrm{dist}(\{c, v\})$.

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5. From all marked $v \in G \setminus T$ (*i.e.* $M(v) \neq$ blank and $v \notin T$)
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This completes the formal description of Dijkstra's shortest path algorithm.

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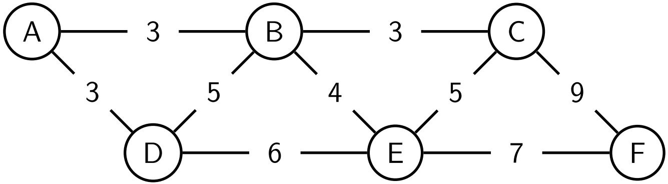
Next a second example, but this time with less commentary.

# Example 2 – Slide 1

Find the shortest path for A to F:



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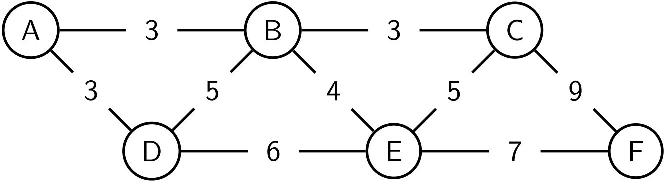


First annotate the vertices adjacent to the start vertex A:

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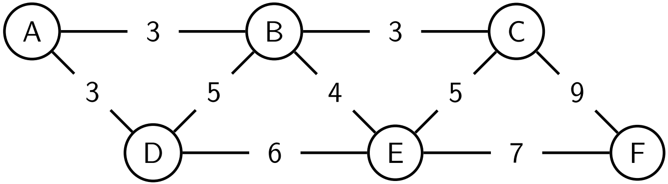


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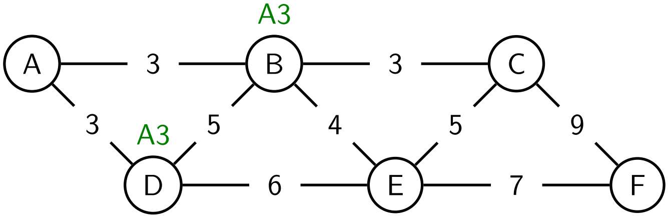

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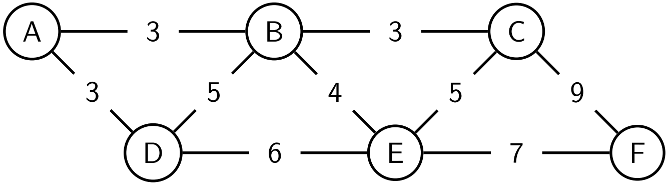


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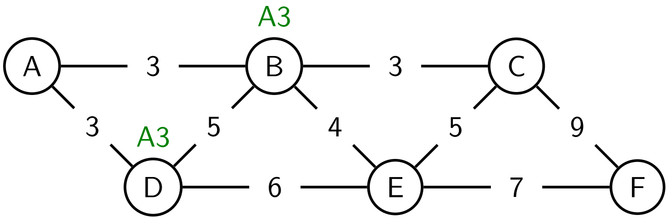
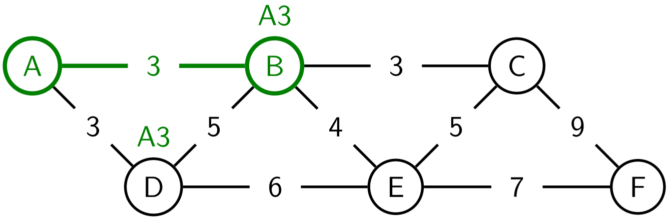



Vertices B and D have equal lowest label; let's lock in B:

# Example 2 – Slide 2



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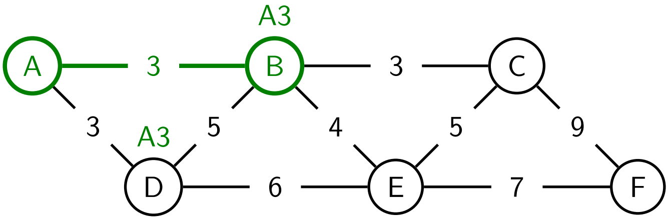


Current vertex is now B. Fringe vertices will be C,D,E.

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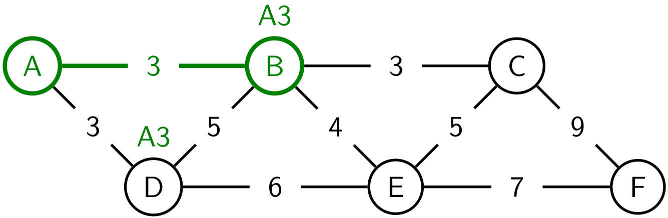


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Annotations required for C and E but D's does not need updating.

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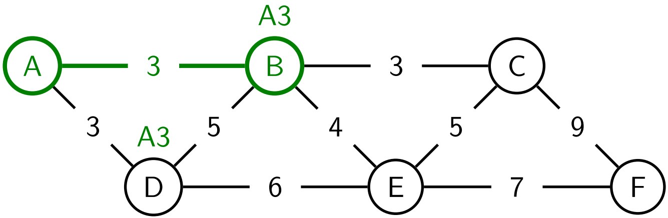


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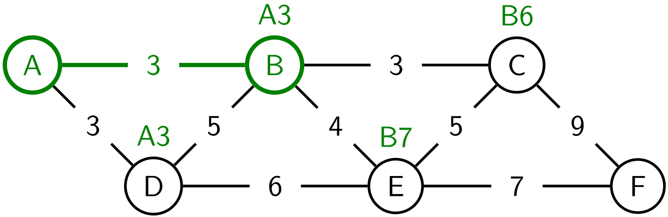

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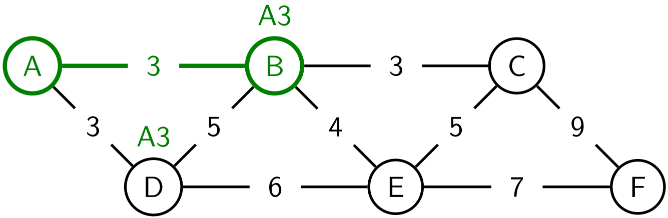


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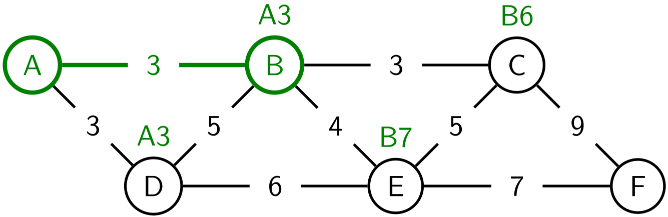
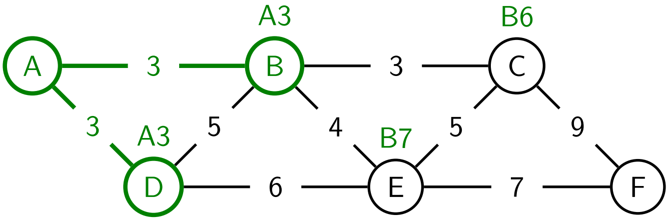



D now has lowest label so needs locking in next:

# Example 2 – Slide 3



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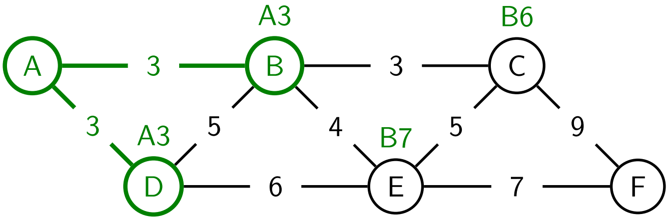
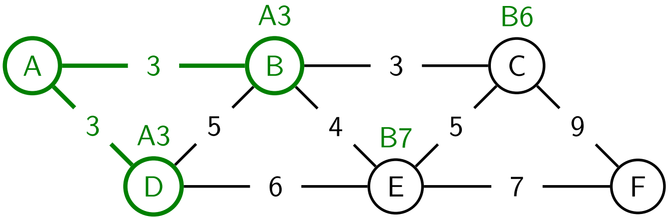



Current vertex is now D. Its only un-locked neighbour is E but no updating is required.

# Example 2 – Slide 3

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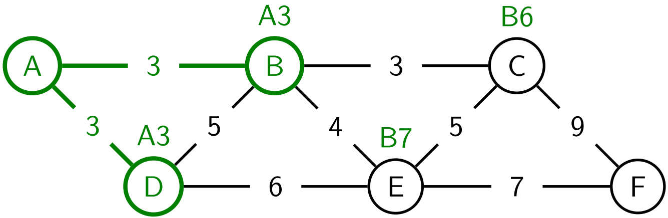


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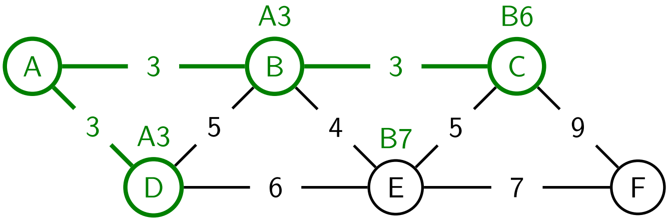

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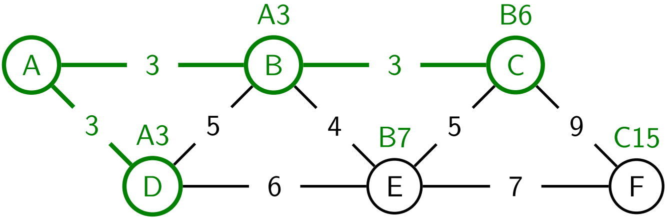

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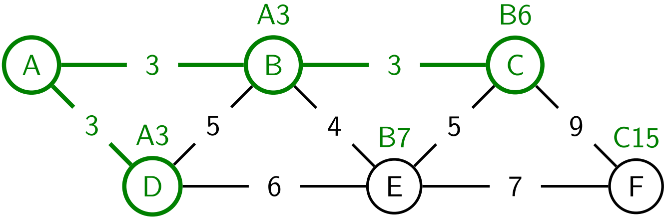
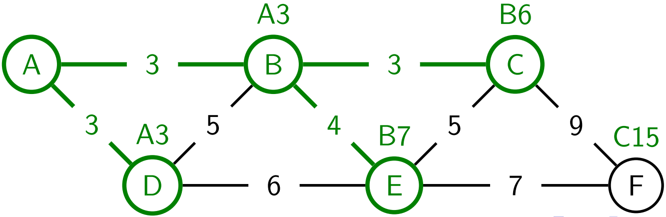



Of the two fringe vertices, E has the lower label value so is locked in. Its lead-in vertex is marked as B, so edge BE is also locked in.



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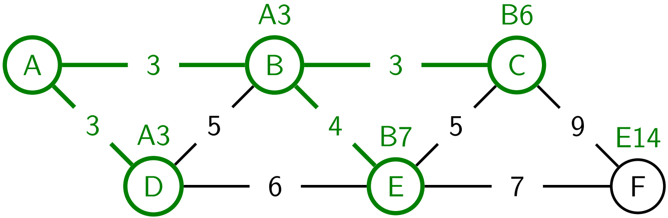

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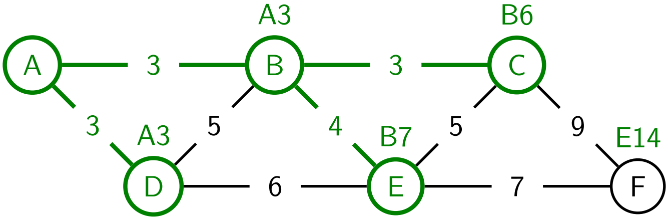
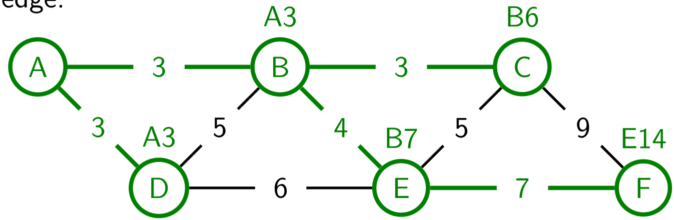



Now F is the only fringe vertex, so F is locked in, together with its lead-in edge.



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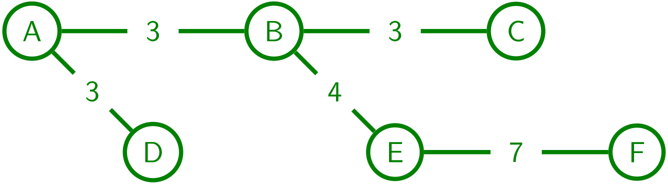

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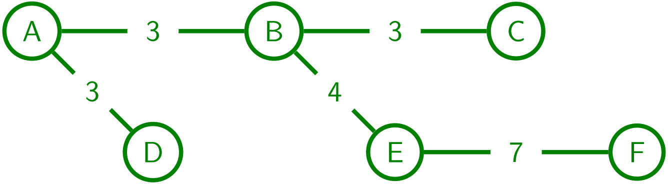


As it happens, this is a minimal spanning tree. However, in general a spanning tree produced by Dijkstra's algorithm will not be minimial.

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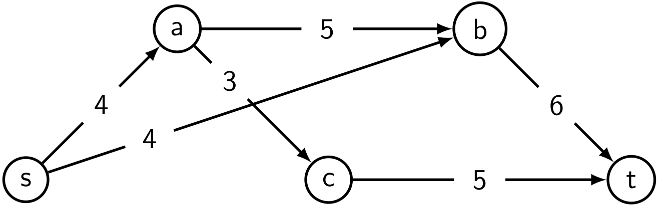

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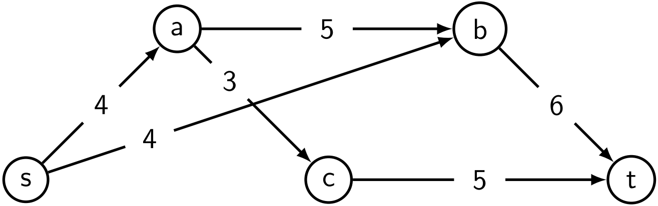


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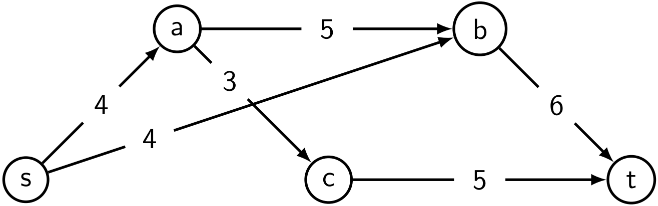


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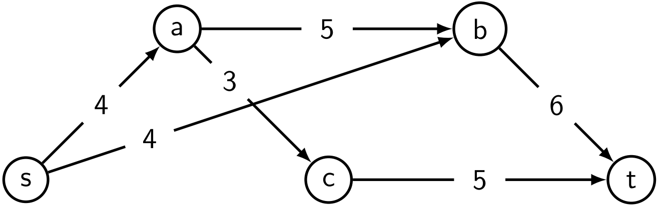


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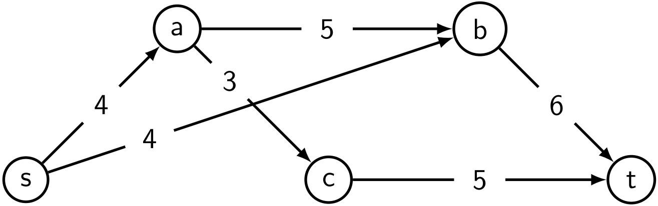


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(3) Total flow into a node equals total flow out, except for nodes $s, t$. $[\forall v \in V(D) \backslash \{s, t\} \;\; \sum_{e \in v_{\text{in}}} F(e) = \sum_{e \in v_{\text{out}}} F(e)$, where $v_{\text{in}}, v_{\text{out}}$ are the sets of edges coming **in** to, and **out** of, $v$, respectively.]

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For any flow $F$ the constrains on the network imply that the total flow out of the source must equal the total flow into the target:

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At stage $i$, flow $F_i$ is constructed as $F_i = F_{i-1} + f_i$, where the incremental flow $f_i$ is based on a constant $k_i \in \mathbb{Q}^+$ and a simple path $p_i$ from $s$ to $t$:

$$f_i(e) = \begin{cases} k_i & \text{for every edge } e \text{ on the path } p_i \\ 0 & \text{for evey other edge } e. \end{cases}$$

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Given a transport network $D$ with capacity and flow functions $C, F$, we know that $F(e) \leq C(e)$ for every edge $e \in E(D)$. I will call the non-negative difference $S(e) = C(e) - F(e)$ the **spare capacity** of the edge $e$. (Some authors use the term "excess capacity".)

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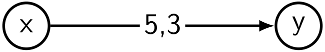
To depict a flow I will follow the capacity value $C(e)$ on each (directed) edge $e$ with the flow value $F(e)$ for that edge. For example




represents a flow of 3 in the edge from x to y, with spare capacity 2.

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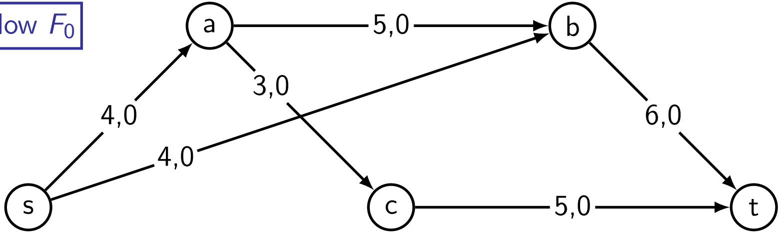
Flow $F_0$



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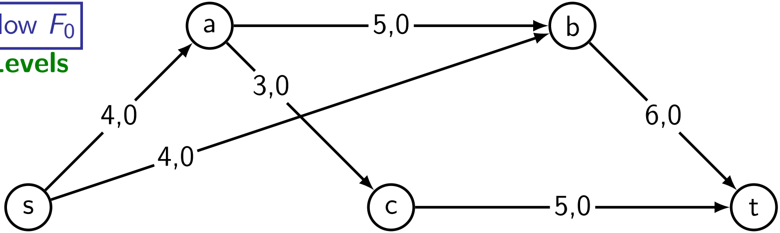

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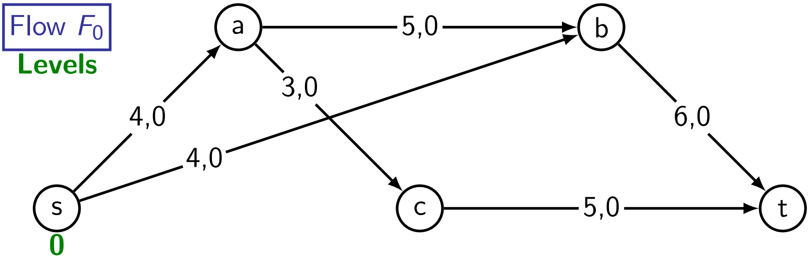

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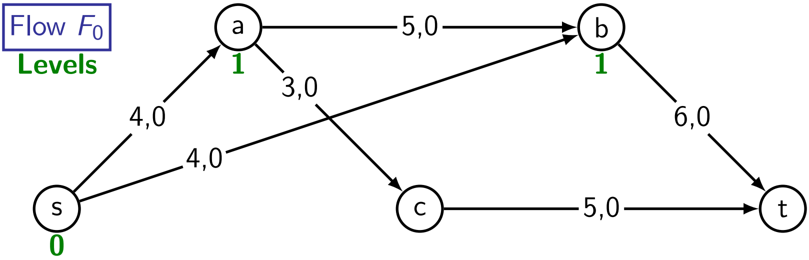

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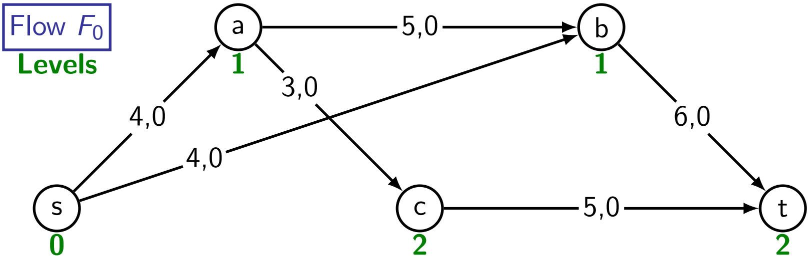

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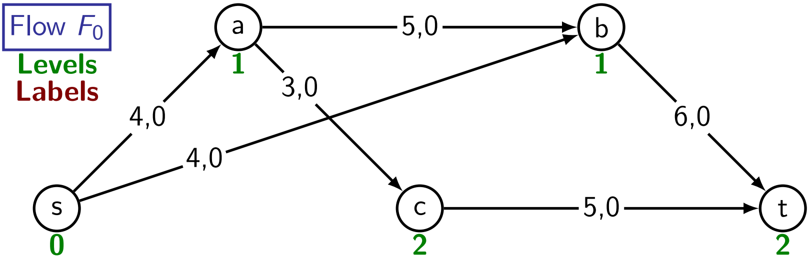

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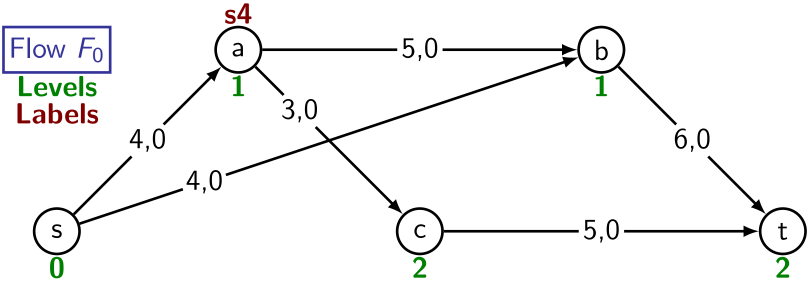

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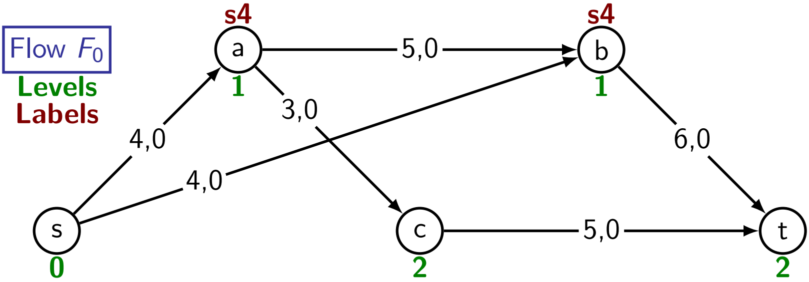

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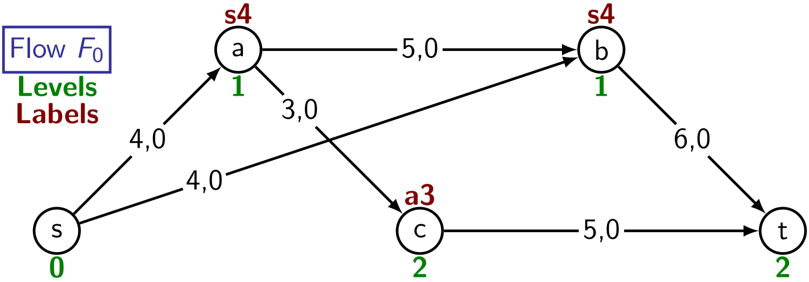

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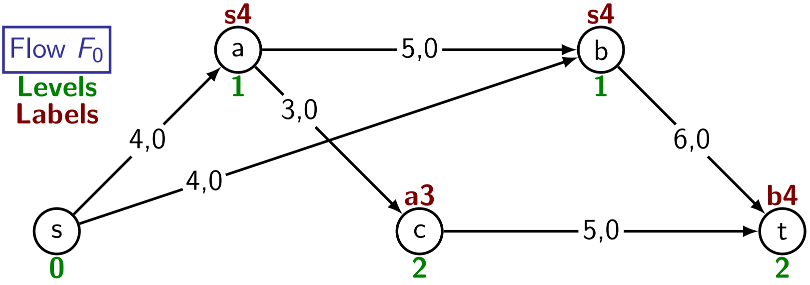

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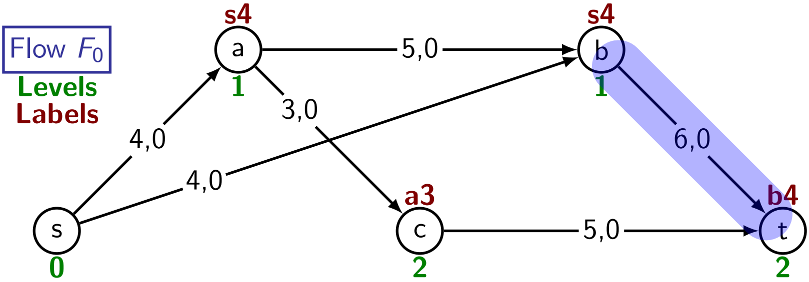

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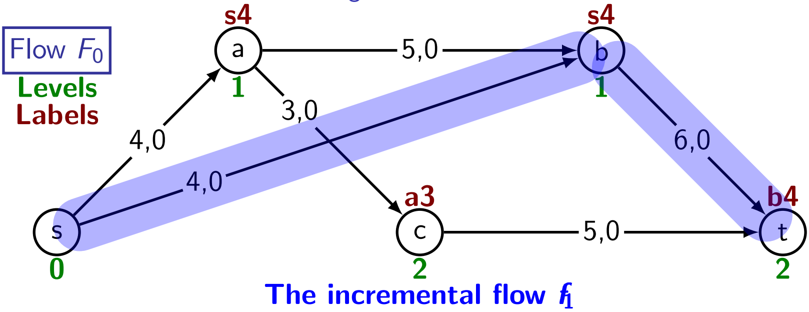

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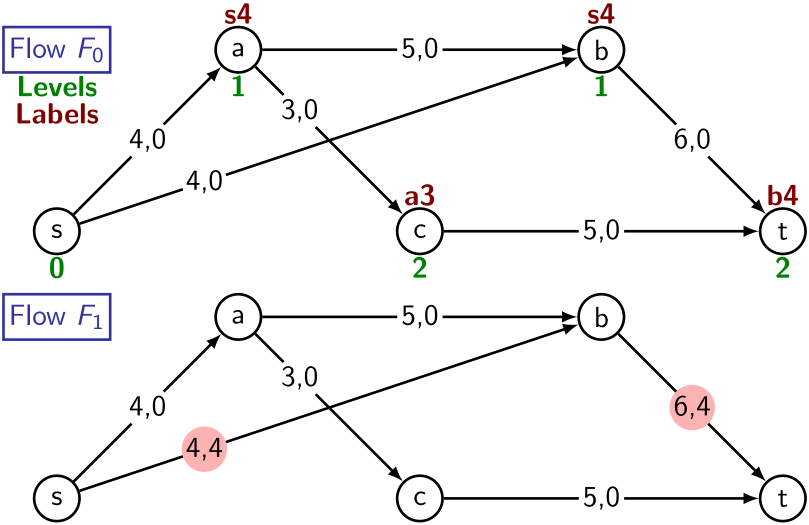

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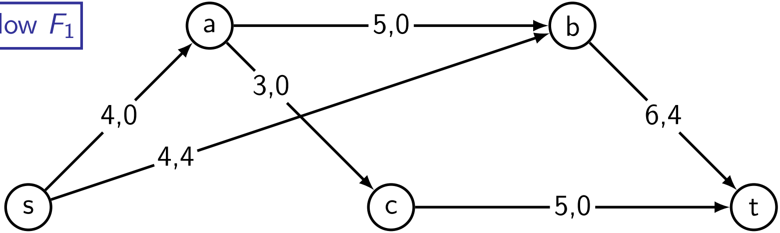

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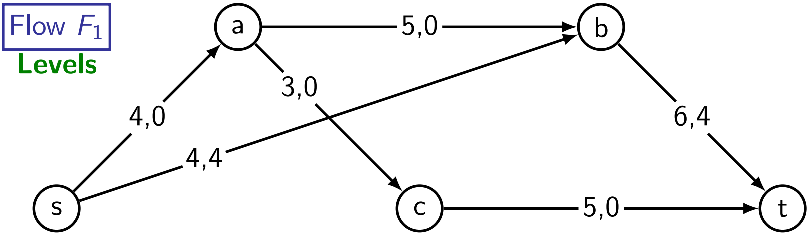

# Vertex labelling algorithm, Example 1

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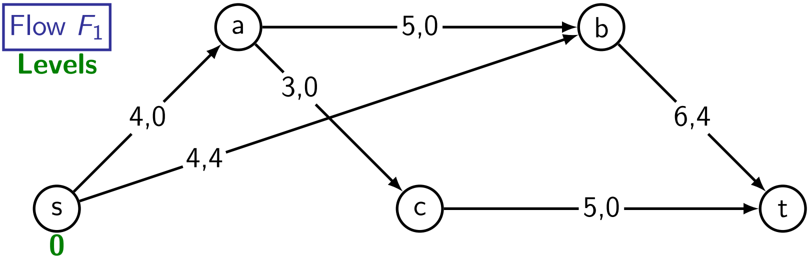

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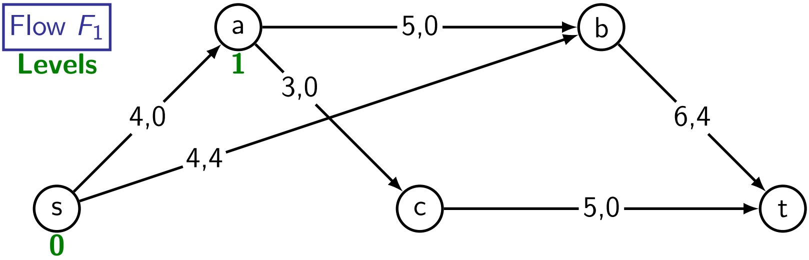

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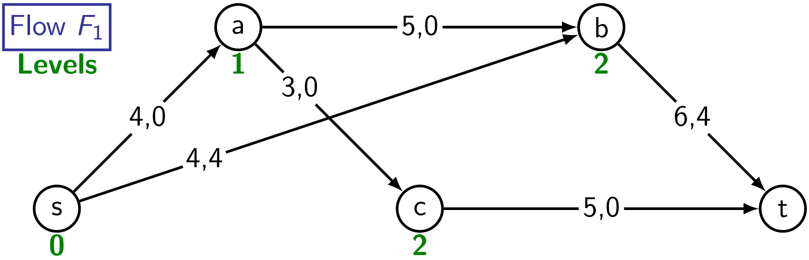

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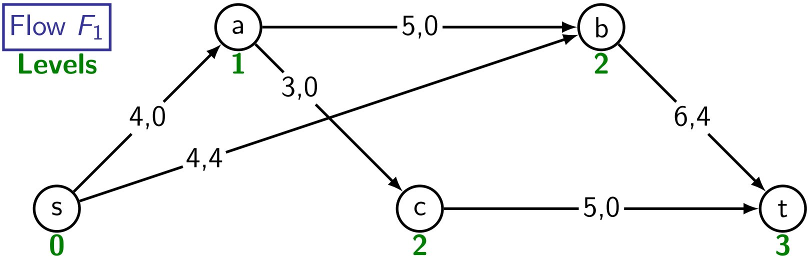

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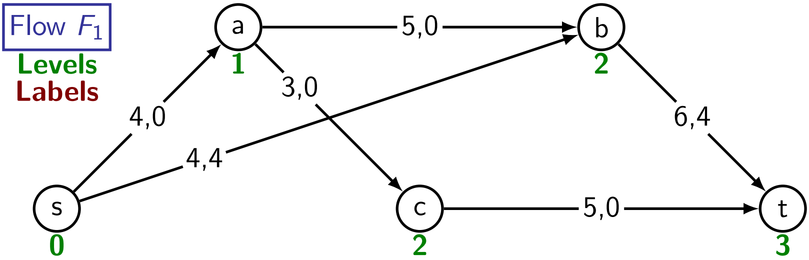

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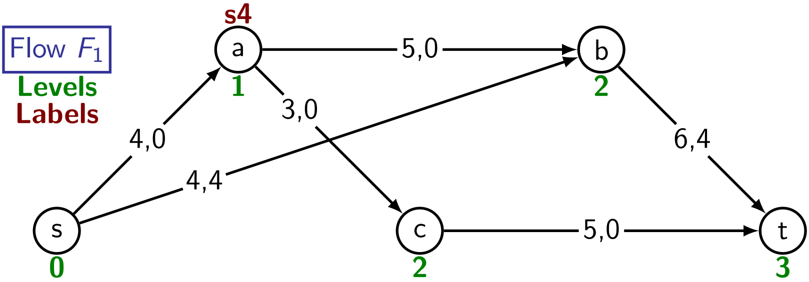

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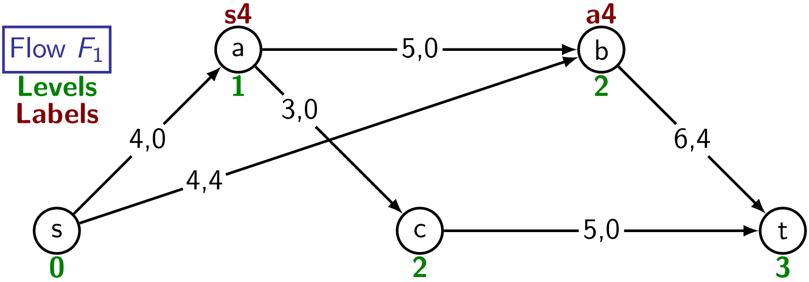

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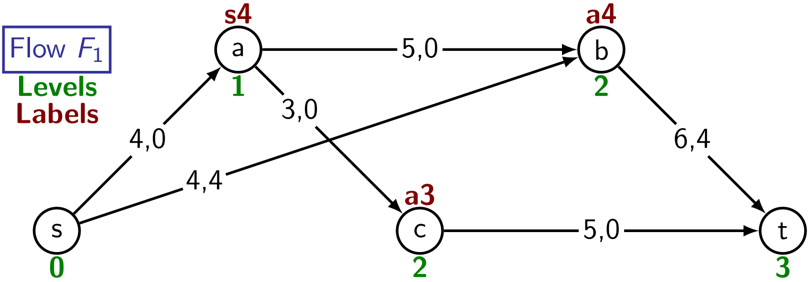

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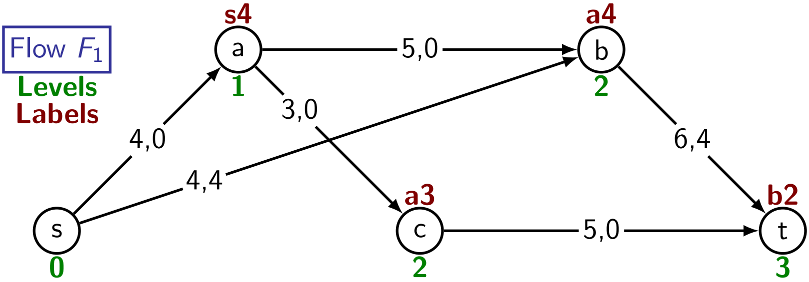

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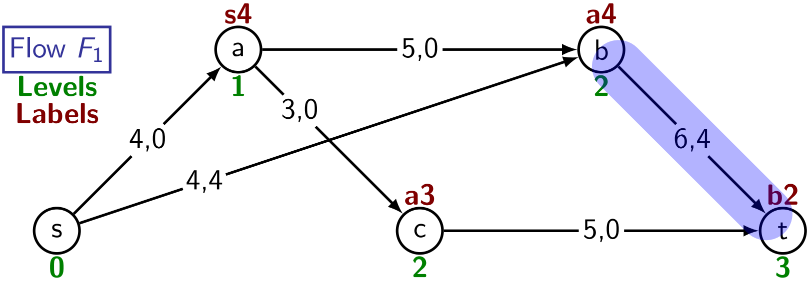

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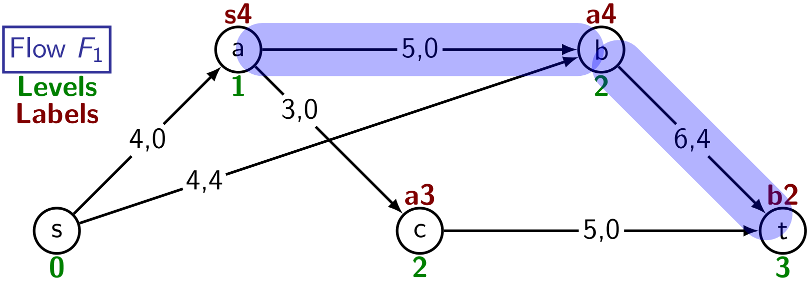

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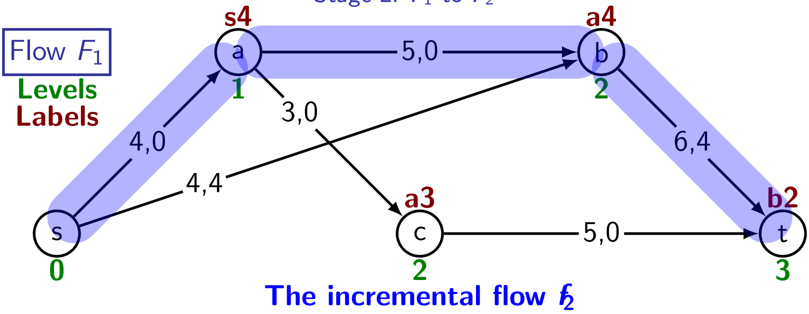

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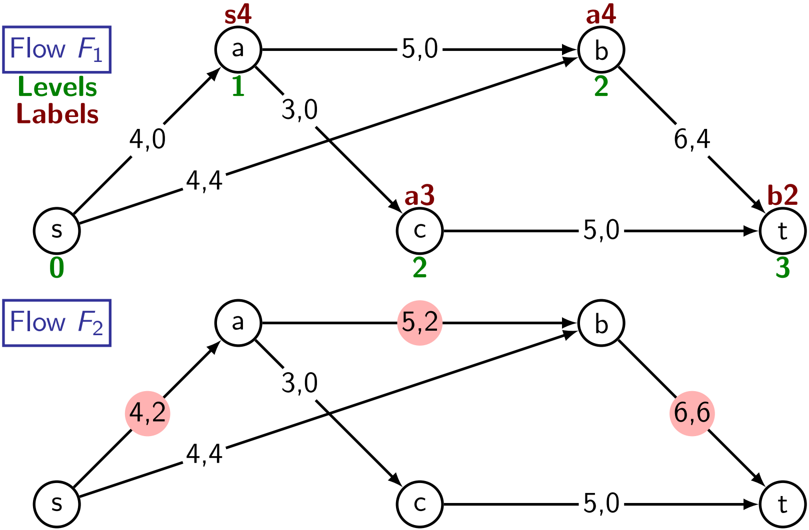

# Vertex labelling algorithm, Example 1

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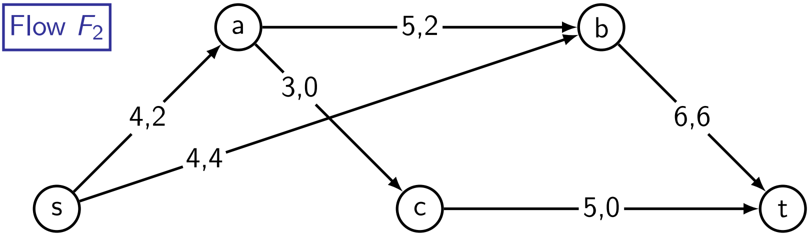

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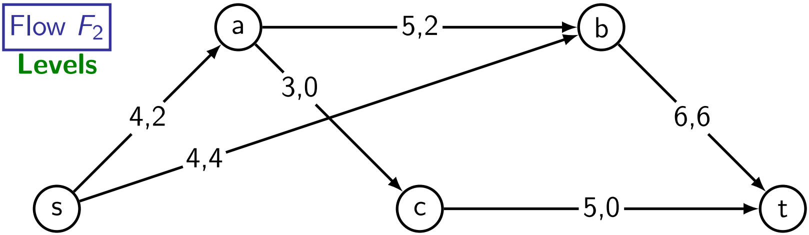

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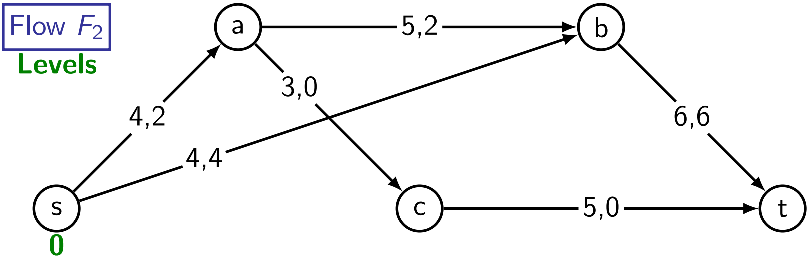

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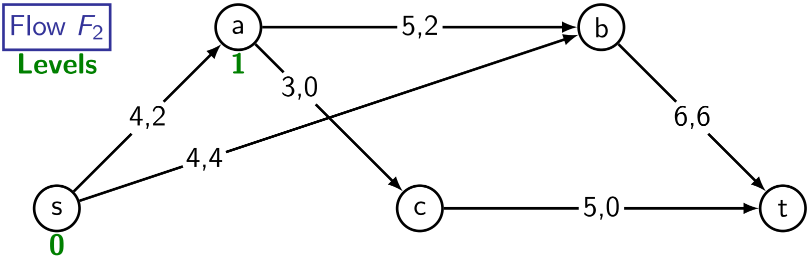

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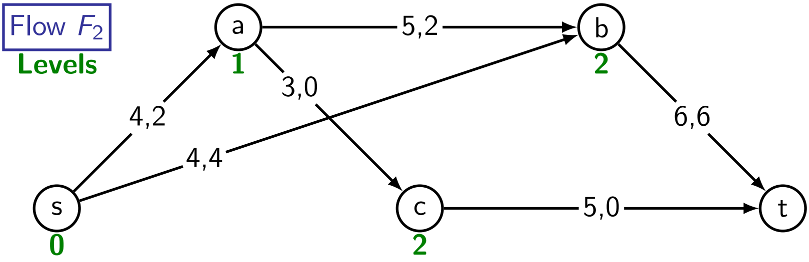

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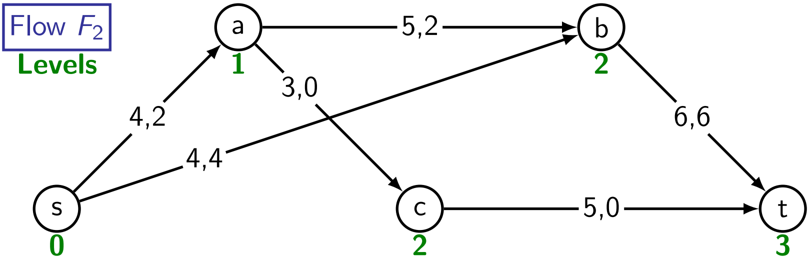

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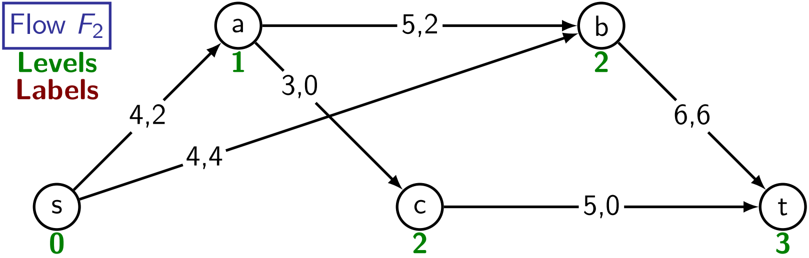

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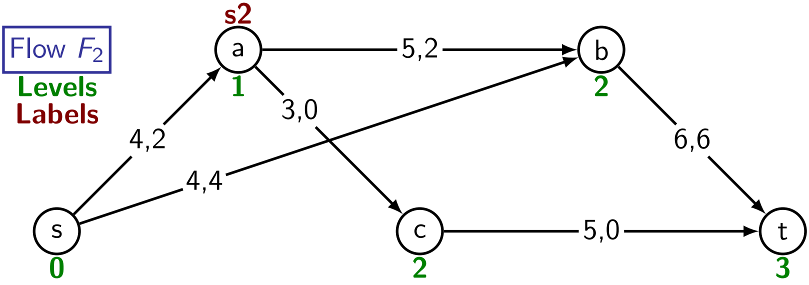

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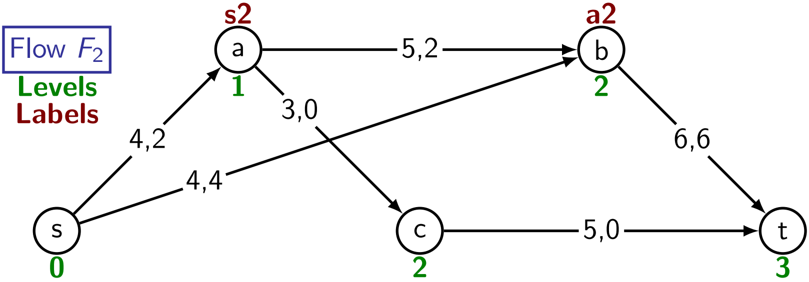

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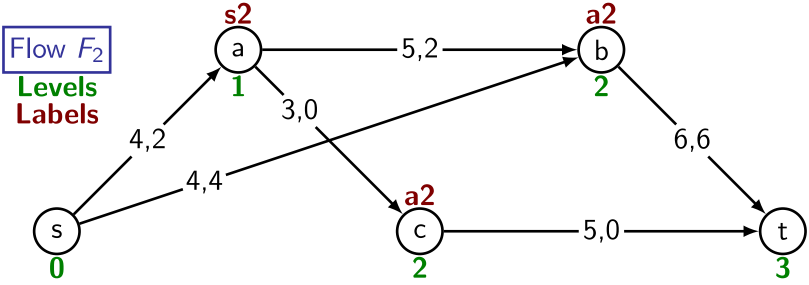

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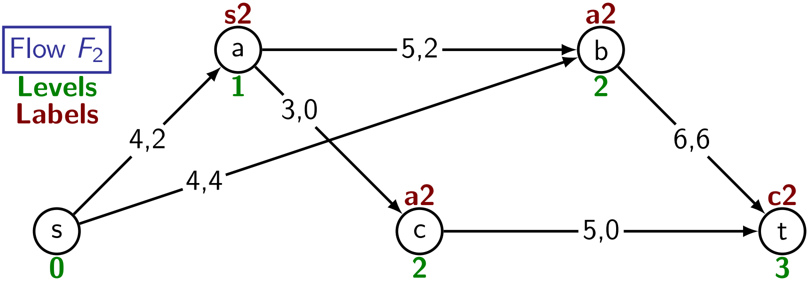

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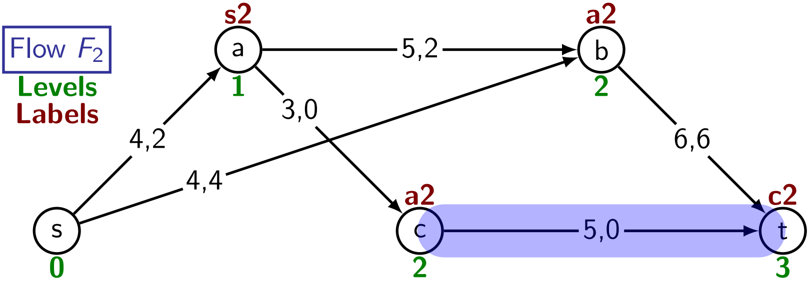

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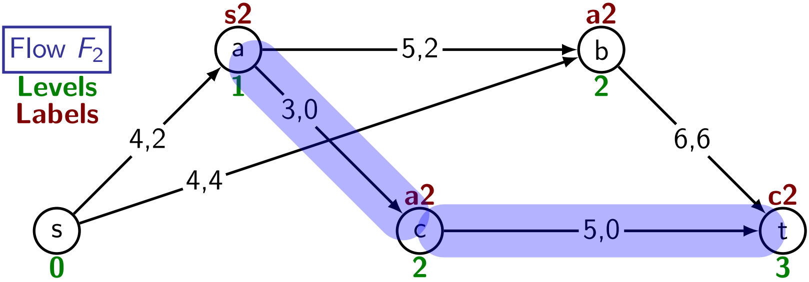

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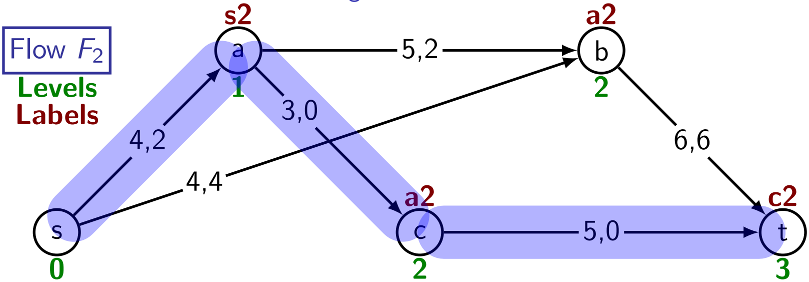


**The incremental flow $f_3$**

# Vertex labelling algorithm, Example 1

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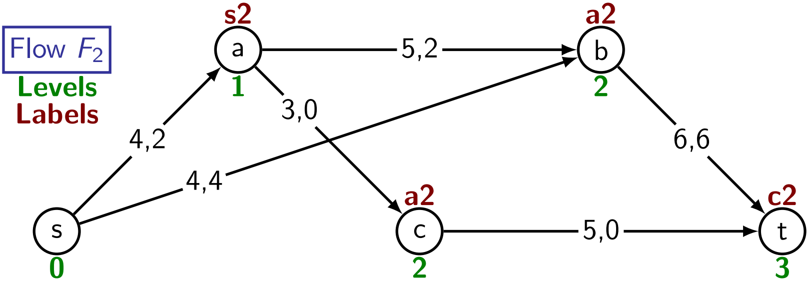


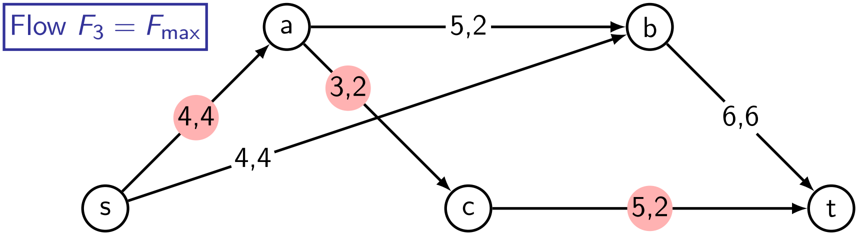

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3. Next slide....

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5. Let $p_i$ be the path $\text{u}_0\text{u}_1 \ldots \text{u}_n$ where $\text{u}_n = \text{t}$ and for $0 < j \leq n$ $\text{u}_j$ has label $\text{u}_{j-1}k_j$.

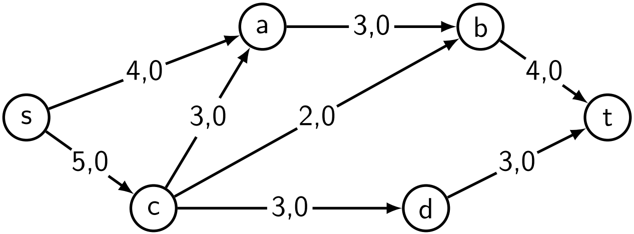
   Define $f_i$ to be the incremental flow on $p_i$ with flow value $k_n$.

End of Method

# Vertex labelling algorithm, Example 2

Stage 1: $F_0$ to $F_1$

$\boxed{\text{Flow } F_0}$

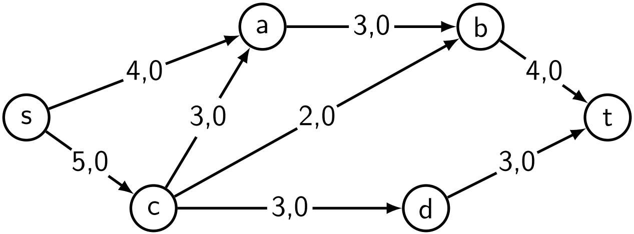


# Vertex labelling algorithm, Example 2

## Stage 1: $F_0$ to $F_1$

$\boxed{\text{Flow } F_0}$

**Levels**



# Vertex labelling algorithm, Example 2

Stage 1: $F_0$ to $F_1$

Flow $F_0$

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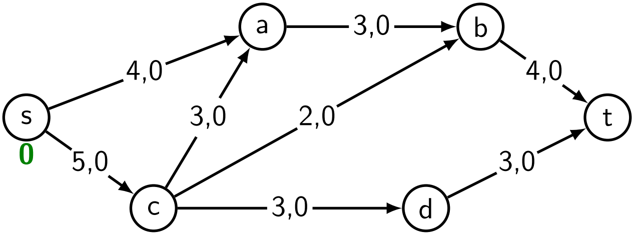

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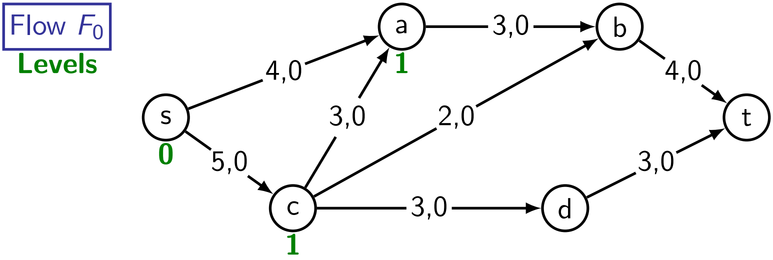

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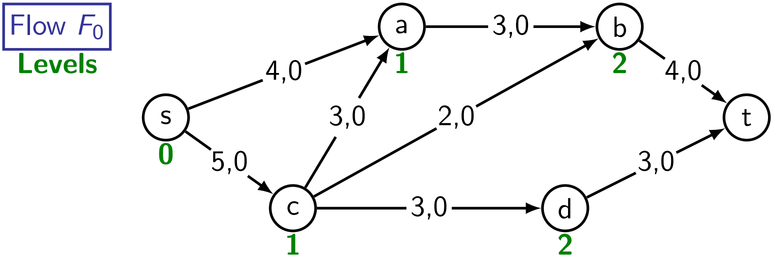

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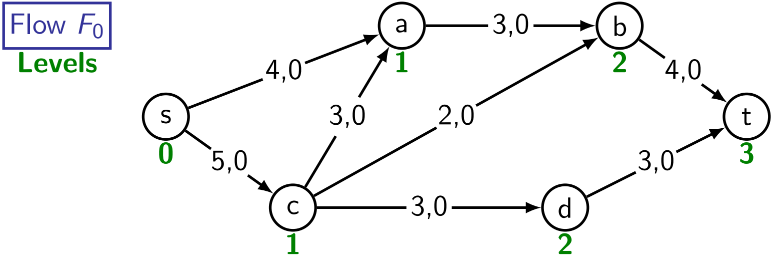


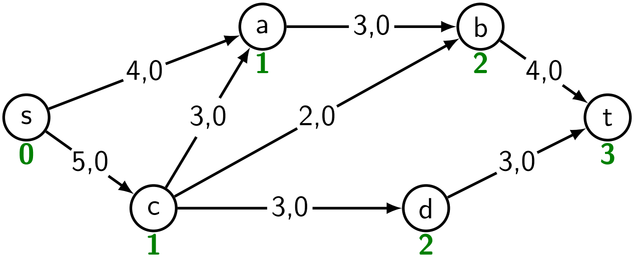
# Vertex labelling algorithm, Example 2

## Stage 1: $F_0$ to $F_1$

Flow $F_0$

**Levels**

**Labels**



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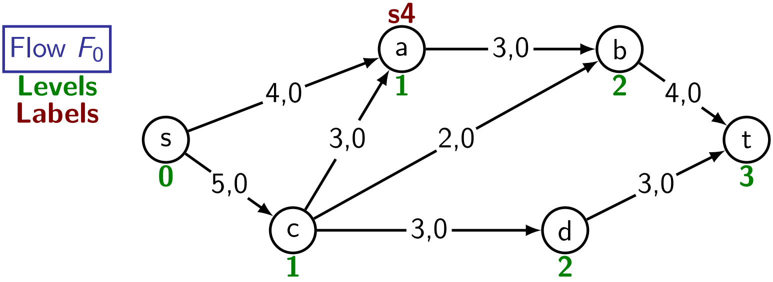

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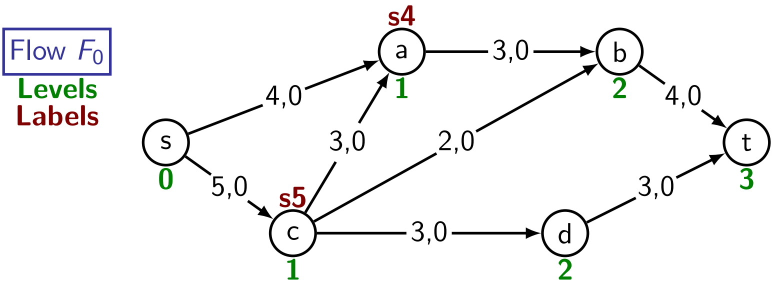

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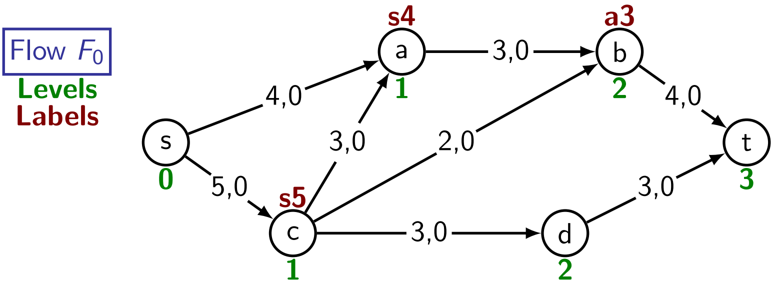

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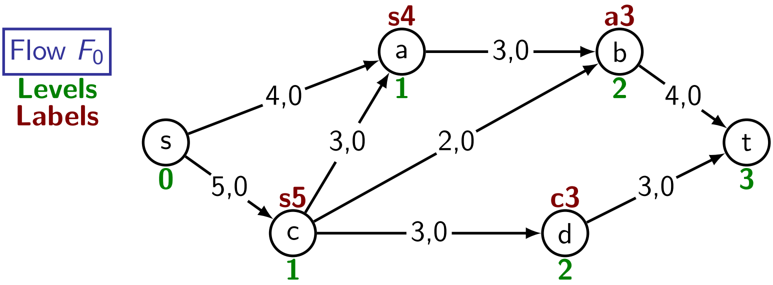

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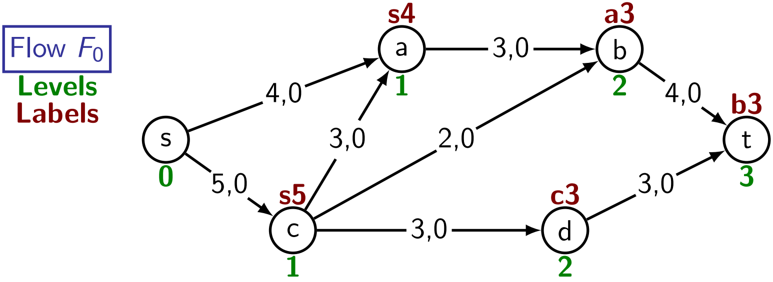

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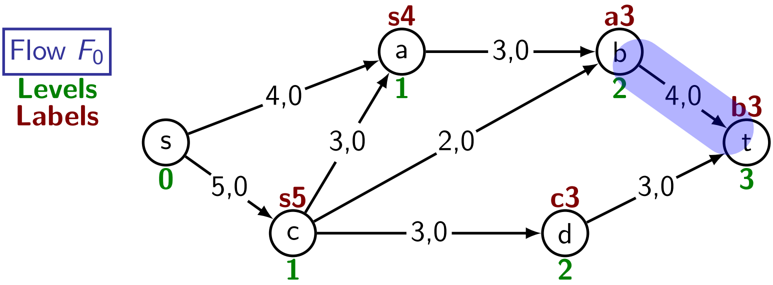

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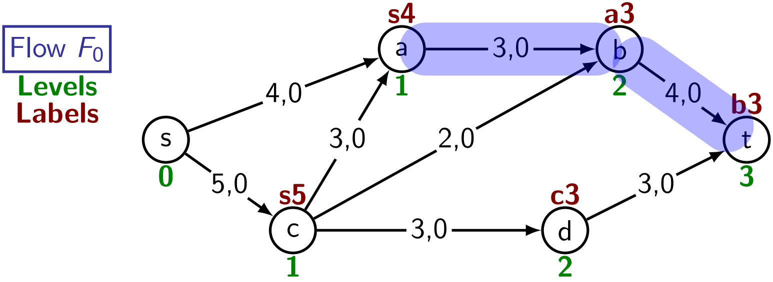
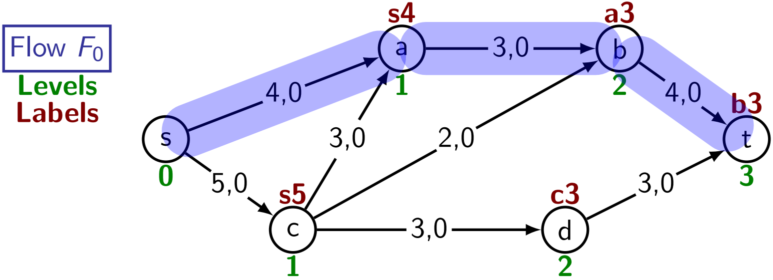


# Vertex labelling algorithm, Example 2

Stage 1: $F_0$ to $F_1$




**The incremental flow $f_1$**

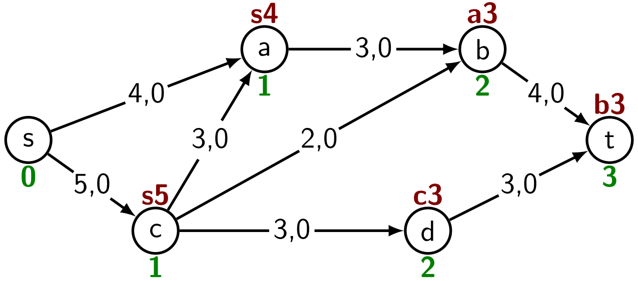
# Vertex labelling algorithm, Example 2

## Stage 1: $F_0$ to $F_1$

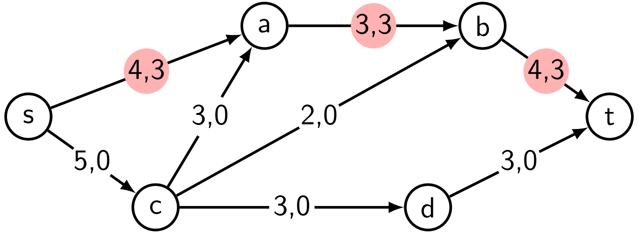
Flow $F_0$

**Levels**

**Labels**



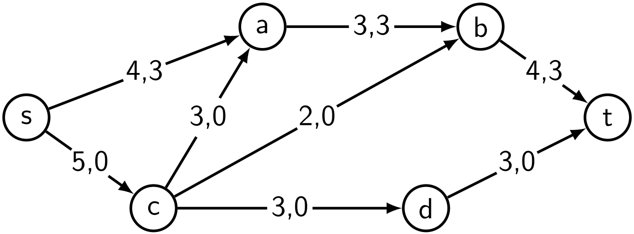

Flow $F_1$



# Vertex labelling algorithm, Example 2

## Stage 2: $F_1$ to $F_2$

$\boxed{\text{Flow } F_1}$

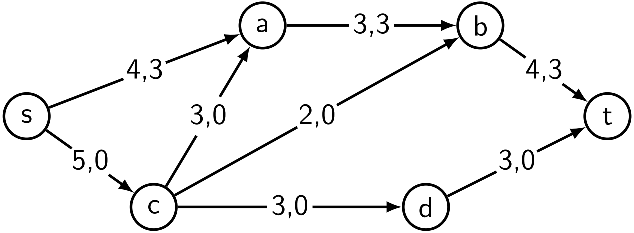


# Vertex labelling algorithm, Example 2

## Stage 2: $F_1$ to $F_2$

Flow $F_1$

**Levels**



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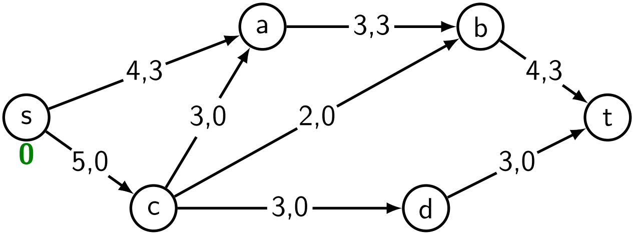

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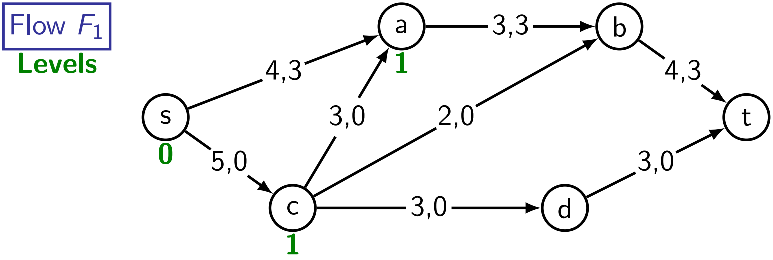
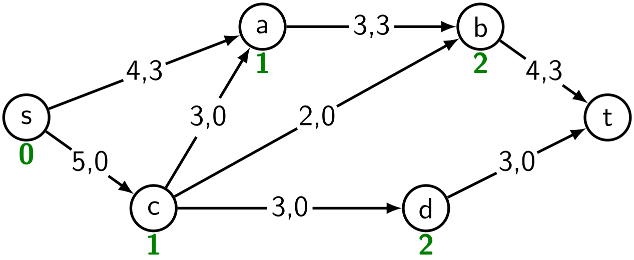


# Vertex labelling algorithm, Example 2

Stage 2: $F_1$ to $F_2$

Flow $F_1$

**Levels**



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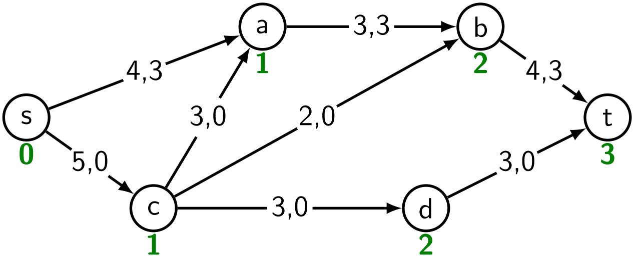
Stage 2: $F_1$ to $F_2$

Flow $F_1$

**Levels**



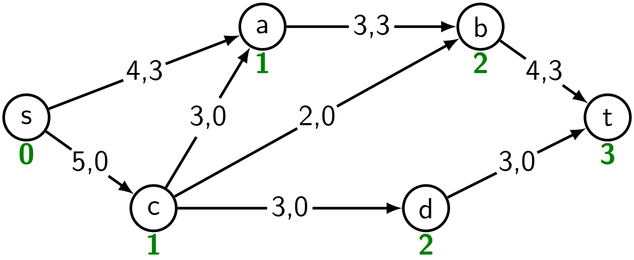
# Vertex labelling algorithm, Example 2

Stage 2: $F_1$ to $F_2$

Flow $F_1$

**Levels**

**Labels**



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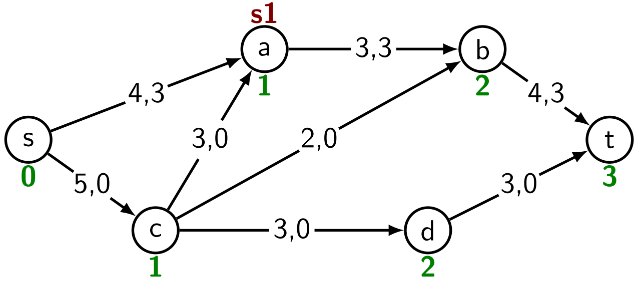
Stage 2: $F_1$ to $F_2$

Flow $F_1$

**Levels**

**Labels**



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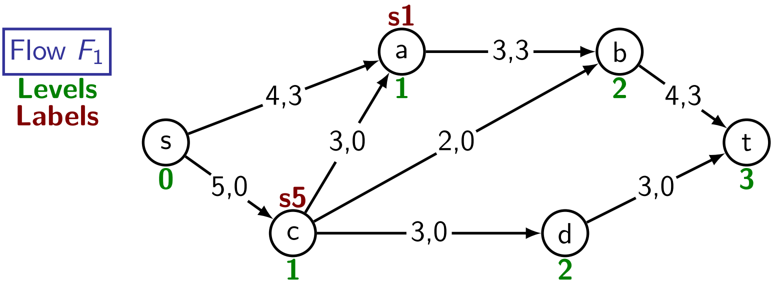
Stage 2: $F_1$ to $F_2$



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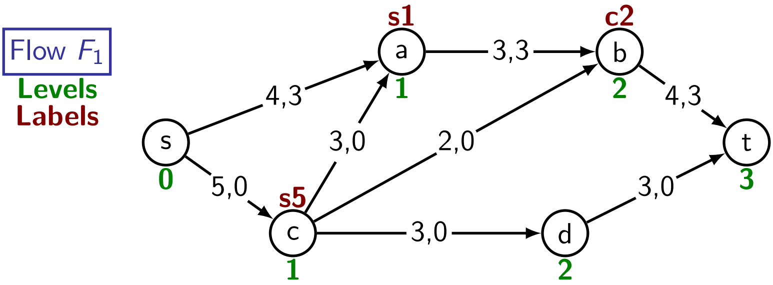
Stage 2: $F_1$ to $F_2$



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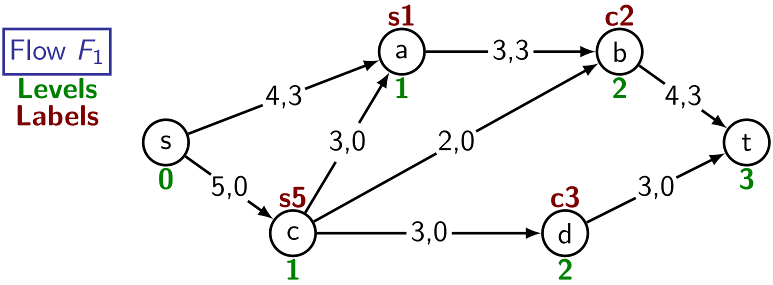
Stage 2: $F_1$ to $F_2$



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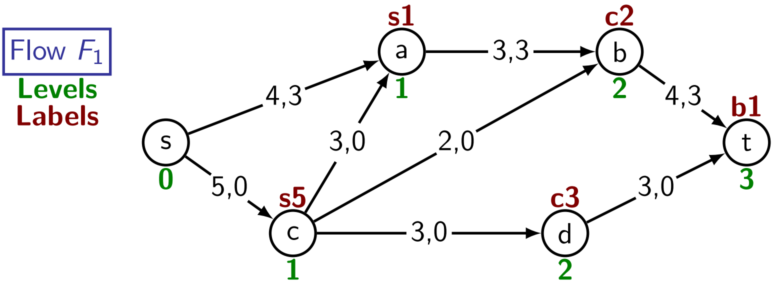

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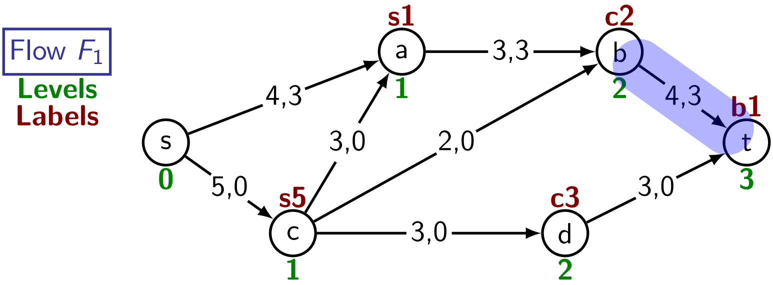

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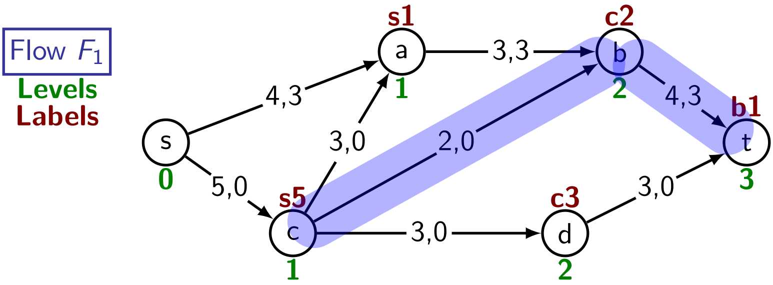

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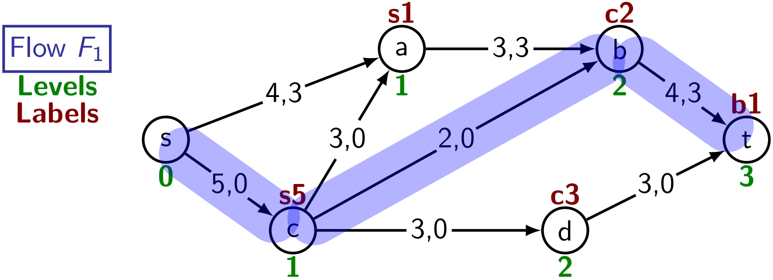
Stage 2: $F_1$ to $F_2$



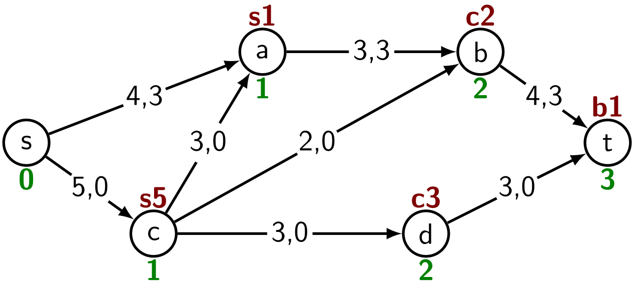

**The incremental flow $f_2$**

# Vertex labelling algorithm, Example 2

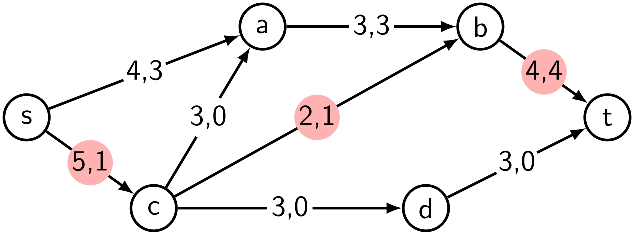
Stage 2: $F_1$ to $F_2$

Flow $F_1$

**Levels**
**Labels**



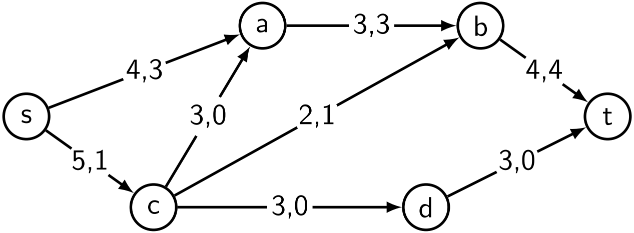

Flow $F_2$



# Vertex labelling algorithm, Example 2

## Stage 3: $F_2$ to $F_3$

Flow $F_2$

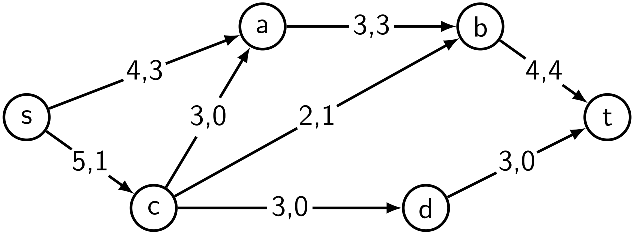


# Vertex labelling algorithm, Example 2

Stage 3: $F_2$ to $F_3$

$\boxed{\text{Flow } F_2}$

**Levels**



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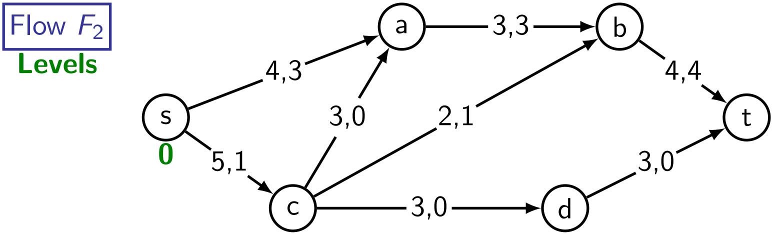

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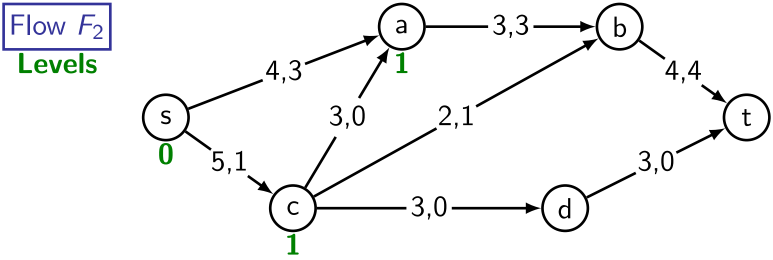

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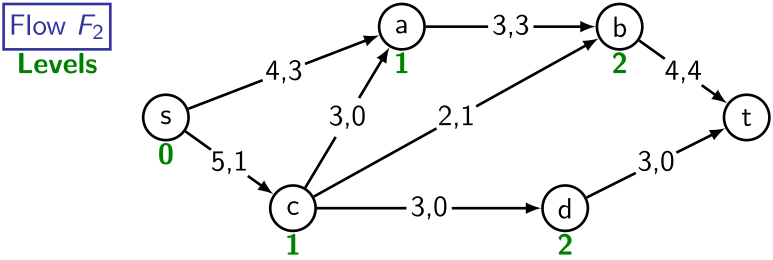

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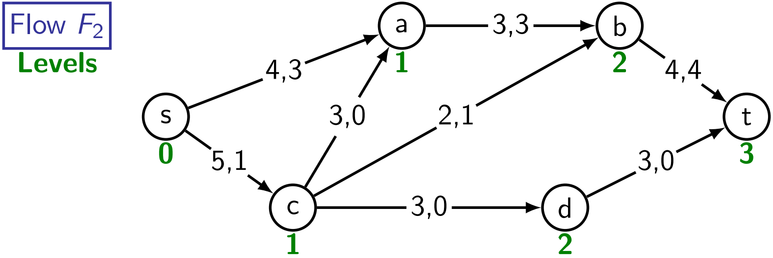

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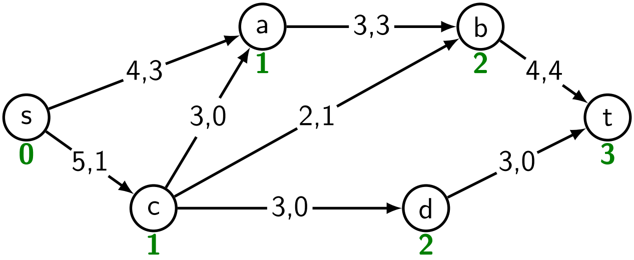
Flow $F_2$

**Levels**

**Labels**



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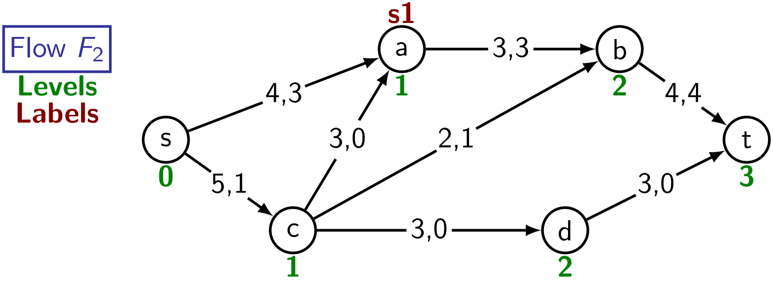

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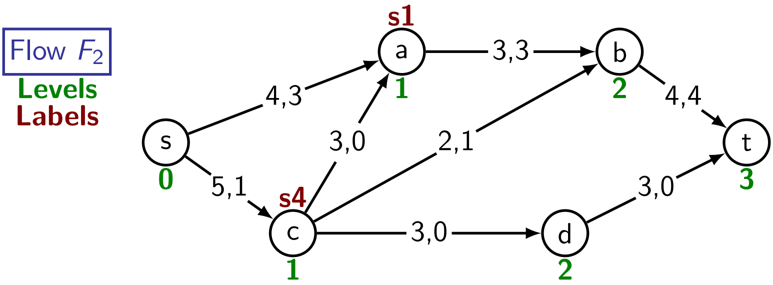

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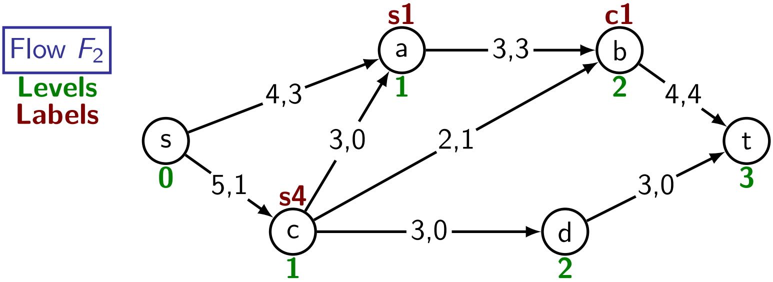

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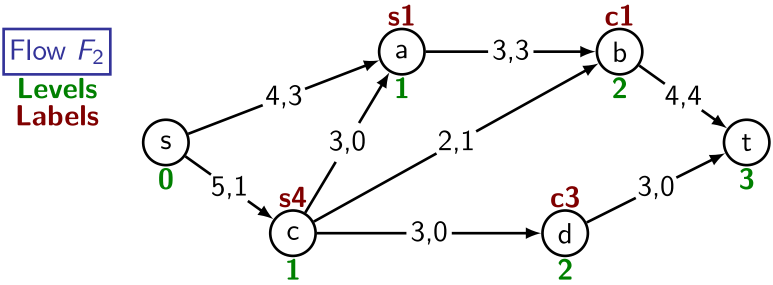

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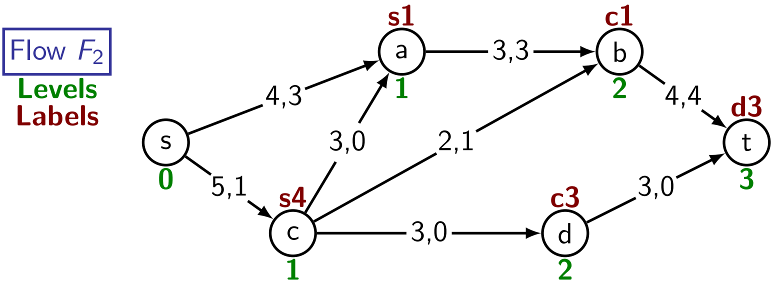

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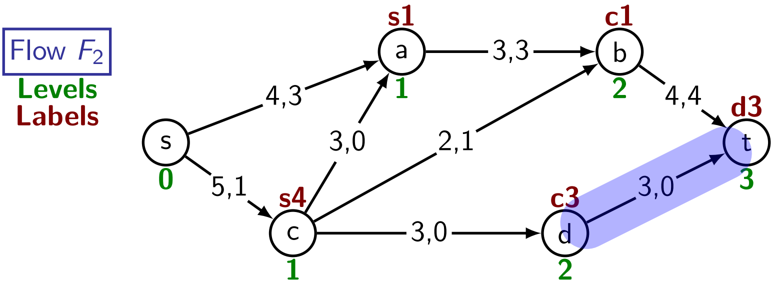

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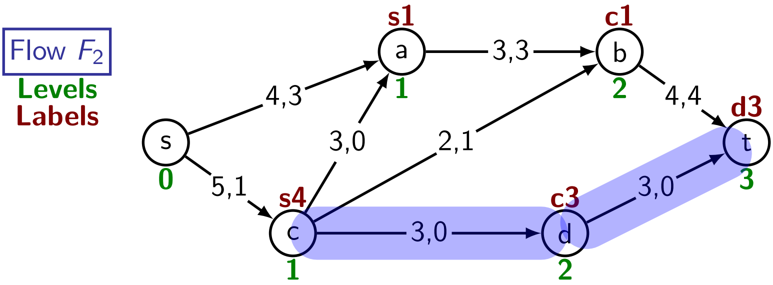

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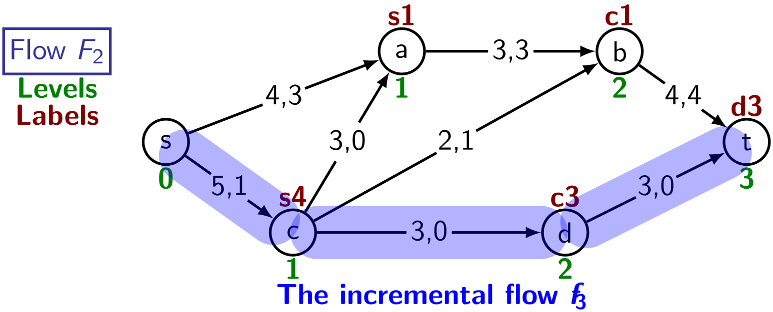

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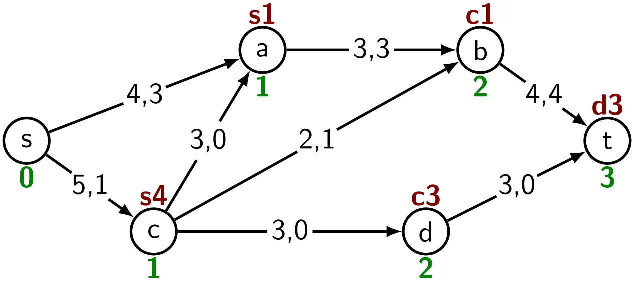
Stage 3: $F_2$ to $F_3$

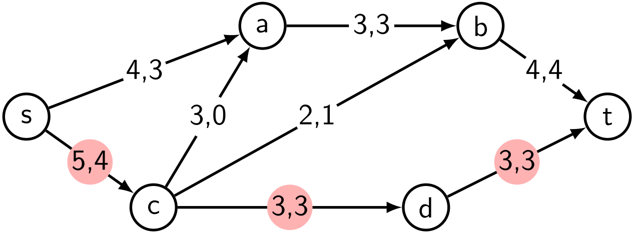
Flow $F_2$

**Levels**

**Labels**



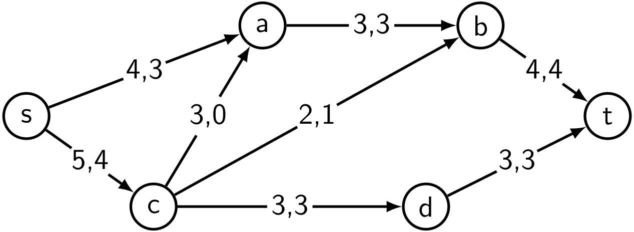

Flow $F_3$



# Vertex labelling algorithm, Example 2

Stage 4: $F_3$ is $F_{max}$

Flow $F_3$



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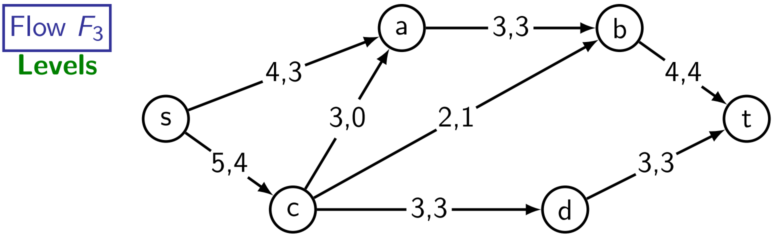

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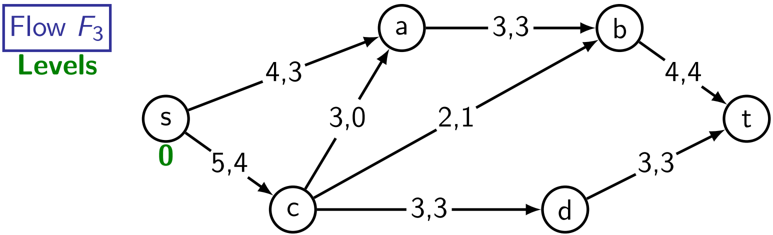

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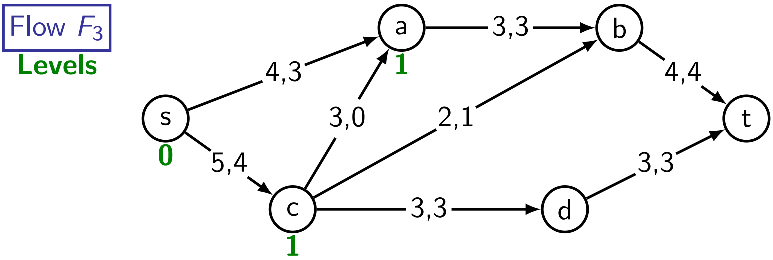

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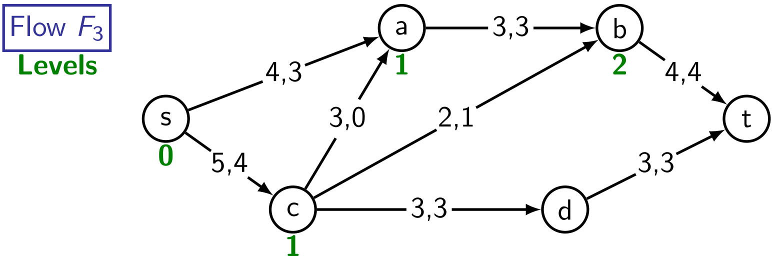

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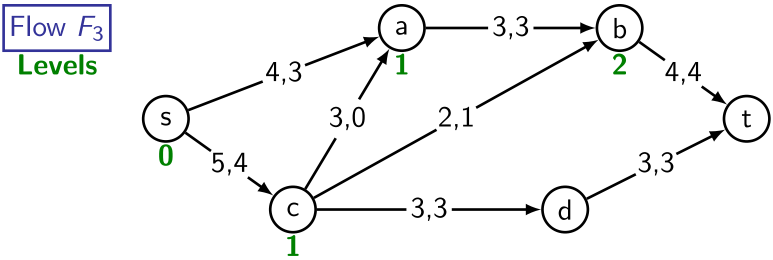


**No level can be assigned to t !**

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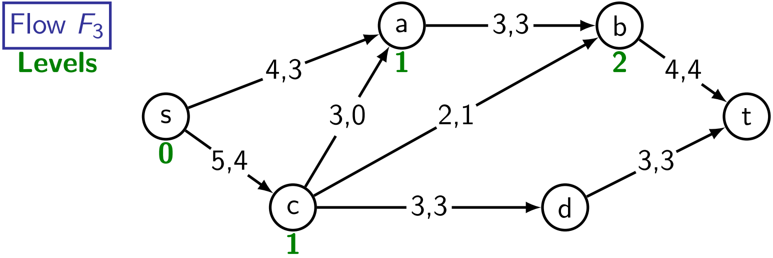


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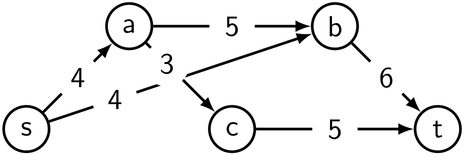
So the algorithm terminates with $F_{\max} = F_3$.

# Cuts

Consider again Example 1 at right.



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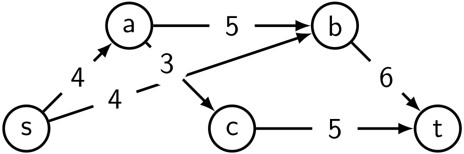


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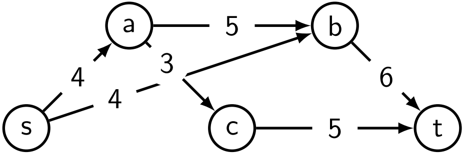


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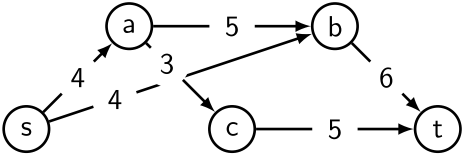


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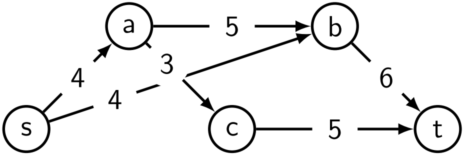


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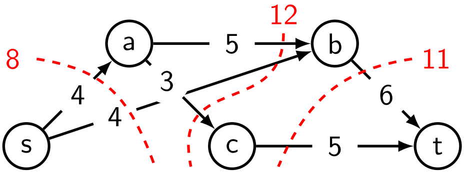


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Some examples, with values, are shown on the digraph above.

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**Max flow min cut theorem:** For any transport network:

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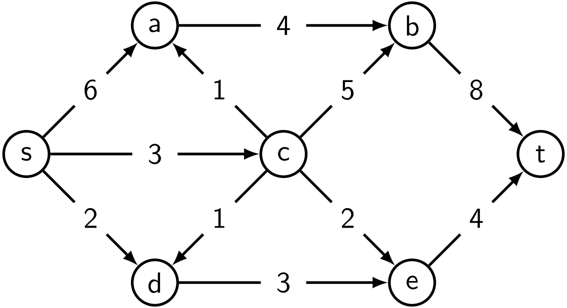
Though highly plausible, this theorem is little tricky to prove, and the proof will be omitted, as will the proof that the vertex labelling algorithm always finds a maximum flow.

# Max flow min cut: Class example 1

What is the maximum flow value for this transport network?



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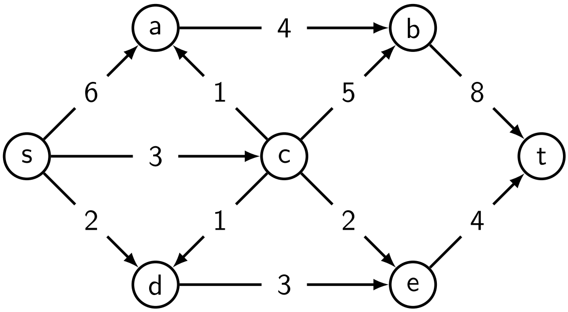


**Answer:**

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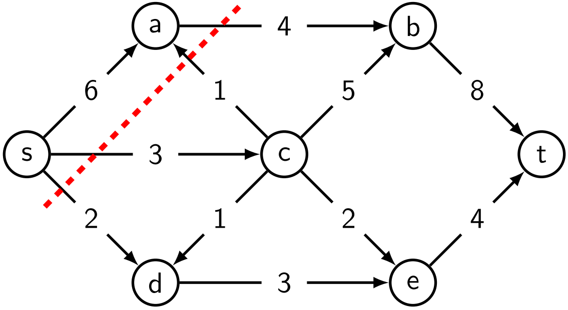


**Answer:** After some searching we find the minimum cut shown, for which $S = \{s, a\}$ and $T = \{b, c, d, e, t\}$.

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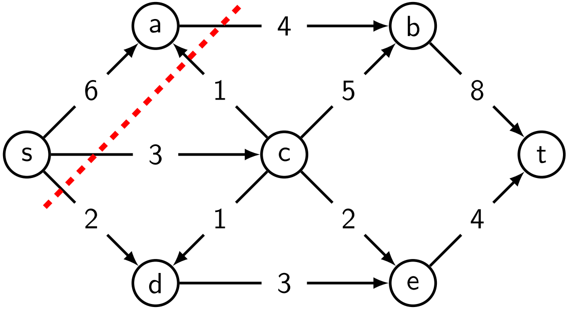


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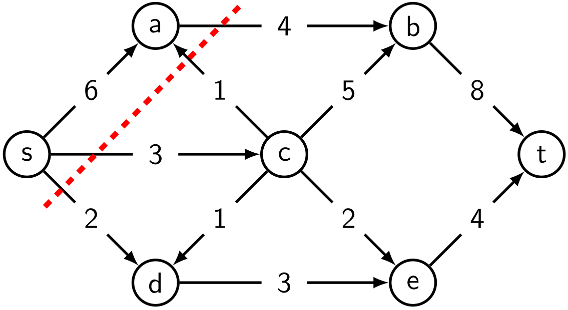


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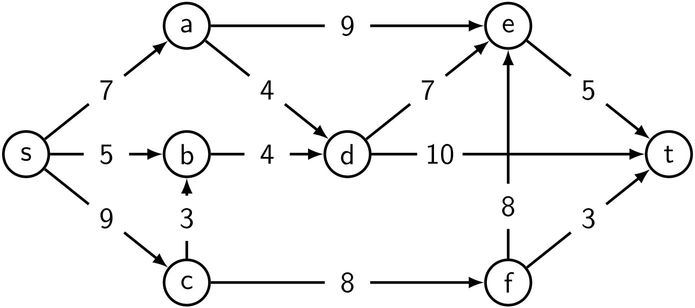
The capacity of this cut is $4+3+2=9$, so this is the maximum flow.
**Note:** Edge (c,a) is not in the cut since it's in the wrong direction.

# Max flow min cut: Class example 2

from Kolman, Busby & Ross

What is the maximum flow value for this transport network?



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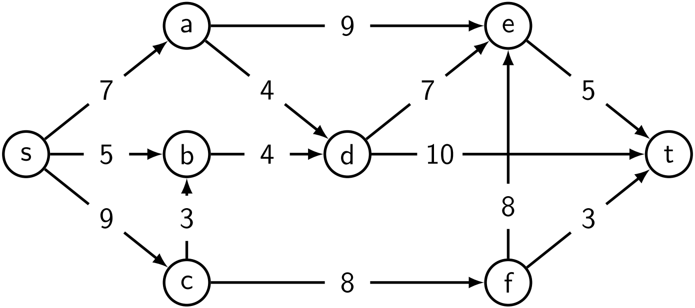


**Hint:**

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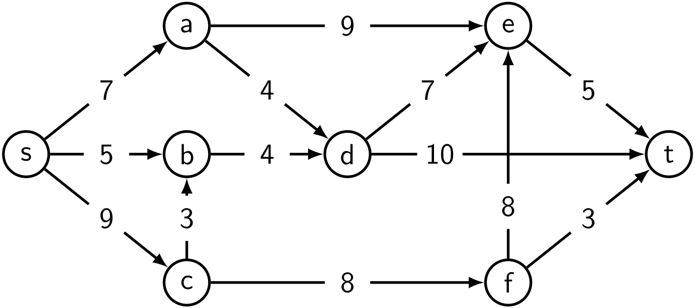


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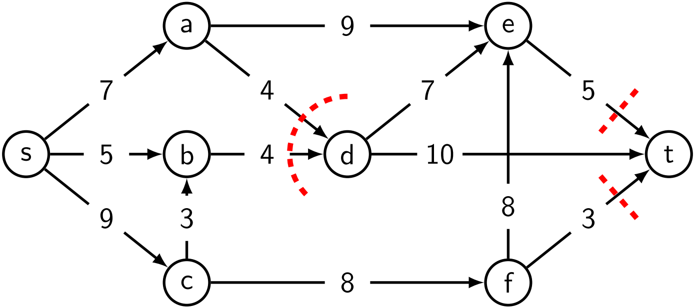


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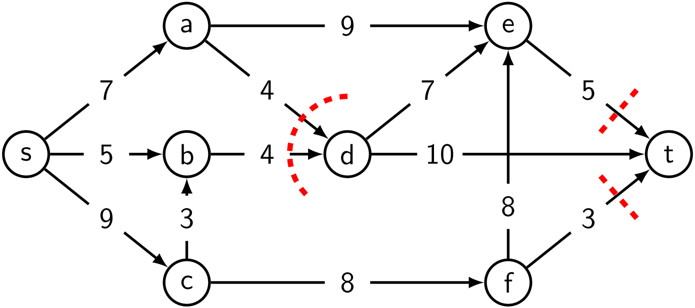


**Hint:** This time the minimum cut cannot be drawn as a single line!
**Answer:** Use all the edges from $S = \{s, a, b, c, e, f\}$ to $T = \{d, t\}$.
The capacity of this cut is $4+4+5+3=16$, so this is the max flow.

# Introduction to virtual flows

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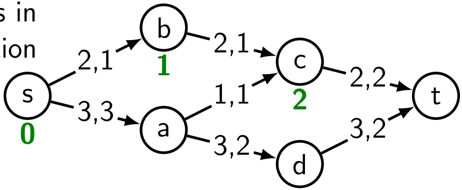

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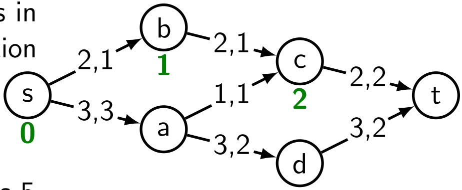

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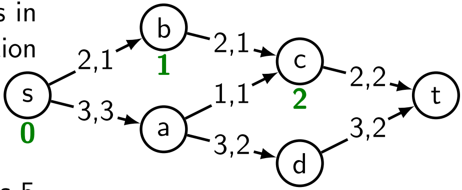


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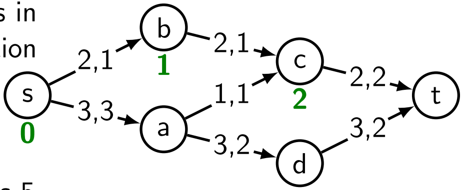


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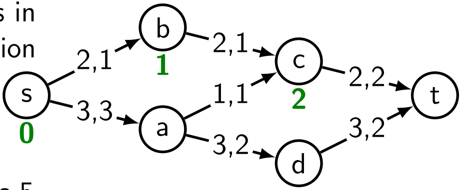


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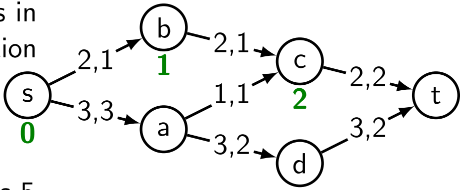


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First, the definition and an explanation of how the algorithm is modified.

# The complete vertex labelling algorithm

Let (u,v) be a (directed) edge in a transport network $D$, and suppose there is currently a flow of $f > 0$ along this edge. The vertex labelling algorithm can reduce this flow to $g < f$ by introducing a **virtual flow** of $f - g$ in the opposite direction, *i.e.* from v to u.

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$$\boxed{S((\mathrm{u,v})) = \begin{cases} C((\mathrm{u,v})) - F((\mathrm{u,v})) & \text{if } (\mathrm{u,v}) \in E(D) \\ F((\mathrm{v,u})) & \text{if } (\mathrm{v,u}) \in E(D) \\ 0 & \text{otherwise} \end{cases}}$$

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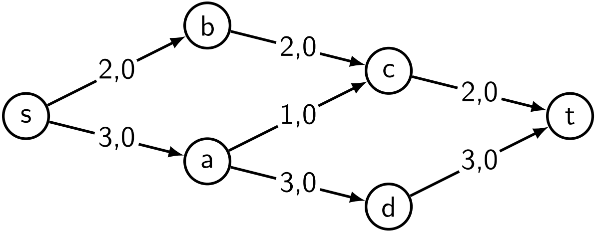
When $(\mathrm{v,u}) \in E(D)$, $S((\mathrm{u,v}))$ is called a **virtual capacity**.

# Vertex labelling algorithm, Example 3

## Stage 1: $F_0$ to $F_1$

$\boxed{\text{Flow } F_0}$



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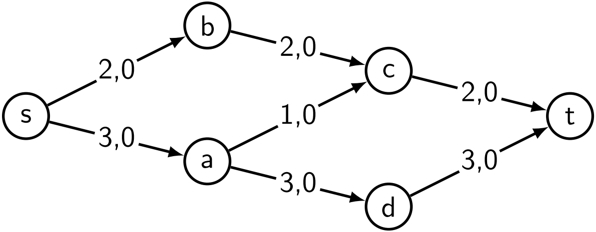
**Levels**



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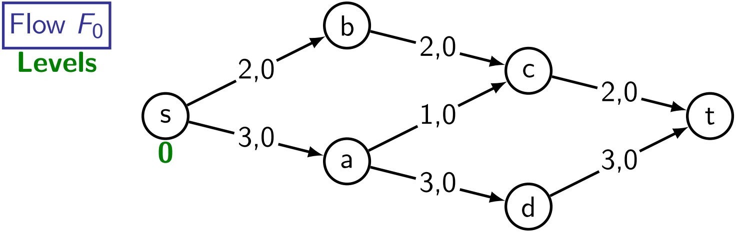

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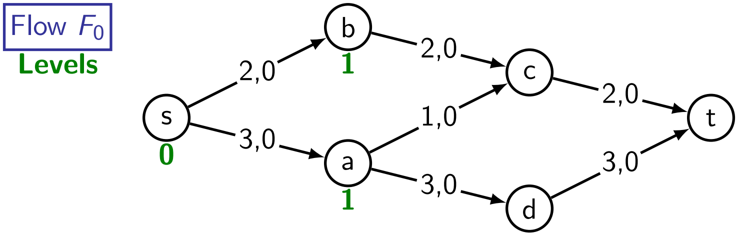

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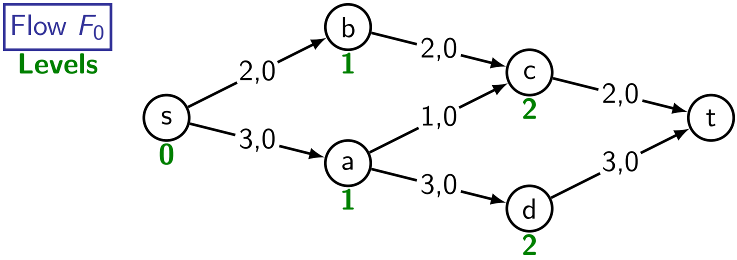

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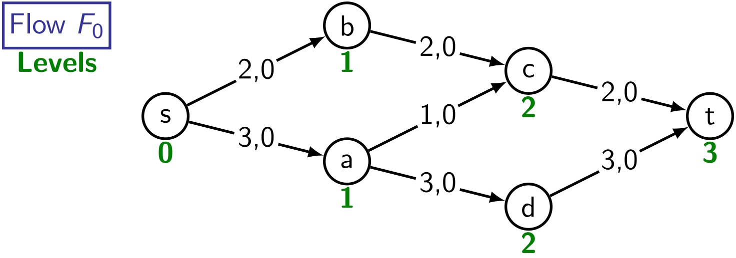

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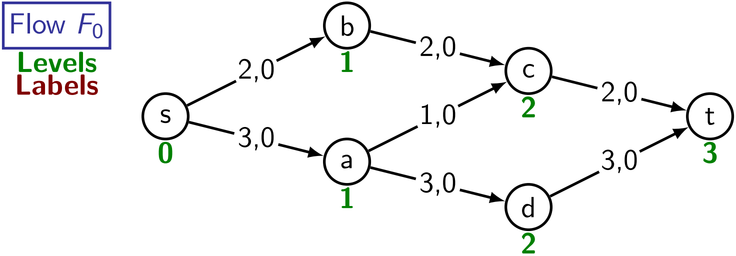

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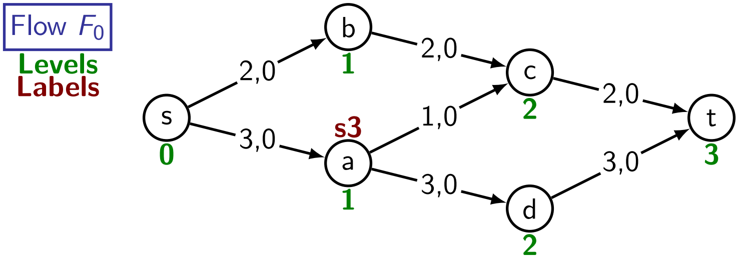

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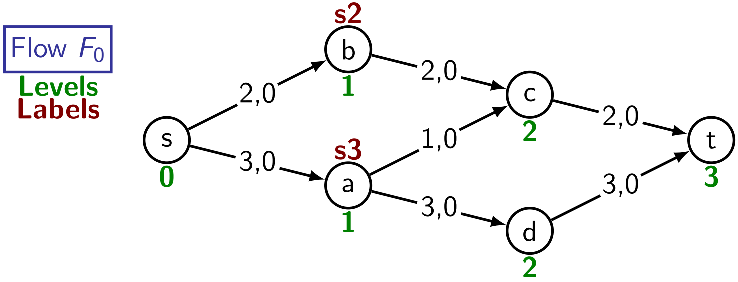

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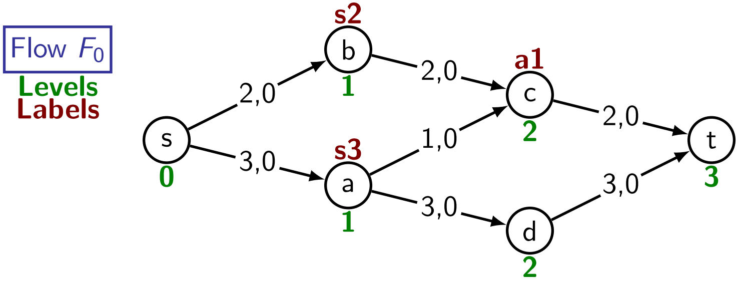

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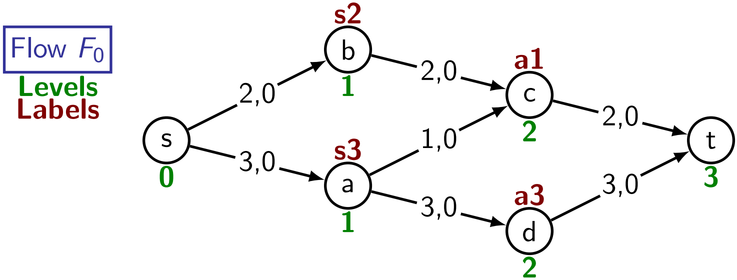

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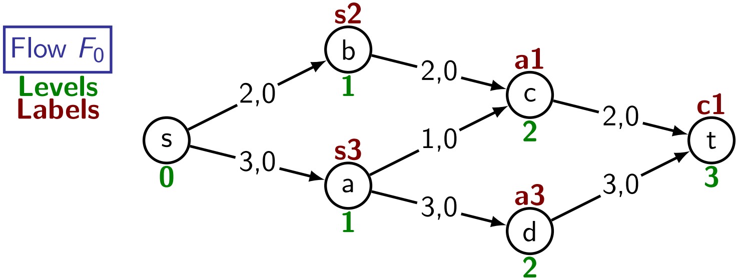

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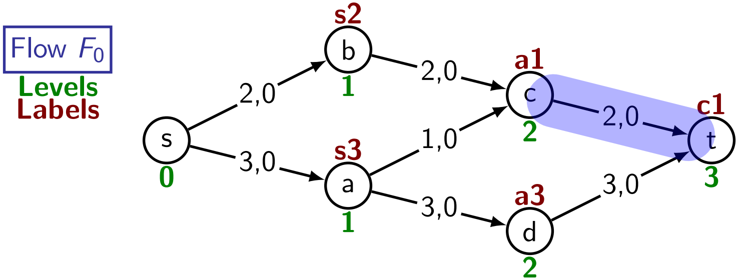

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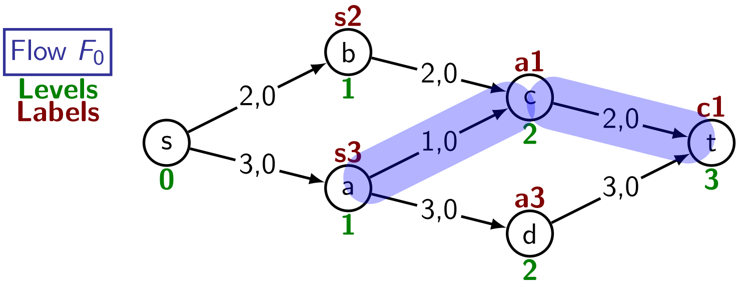

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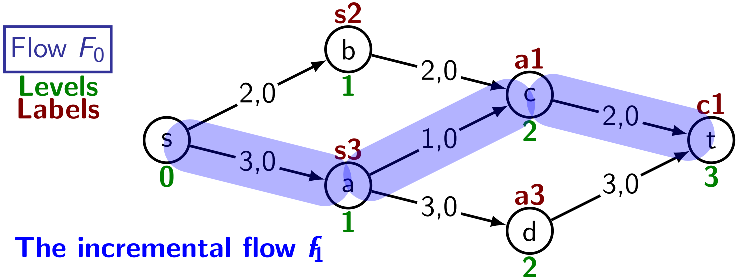


**The incremental flow $f_1$**

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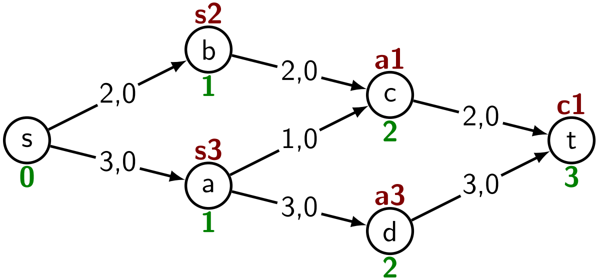
Stage 1: $F_0$ to $F_1$

Flow $F_0$

Levels
Labels

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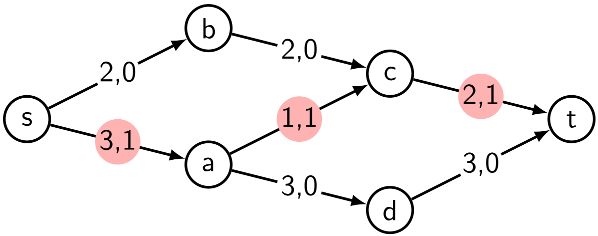
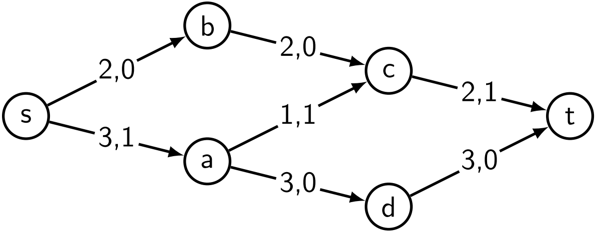


# Vertex labelling algorithm, Example 3

## Stage 2: $F_1$ to $F_2$

$\boxed{\text{Flow } F_1}$



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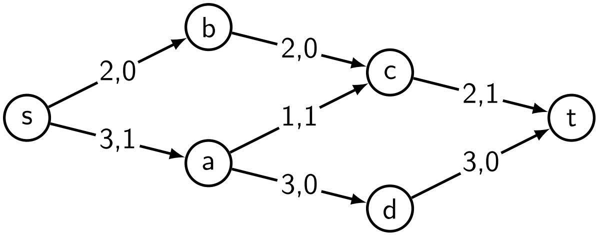

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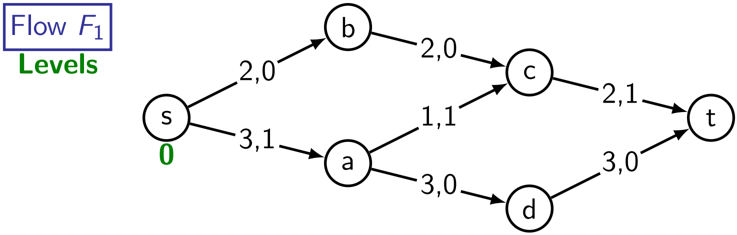

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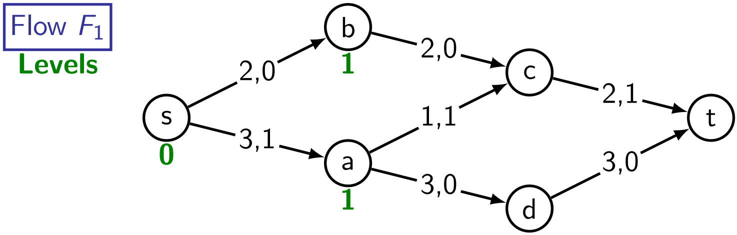

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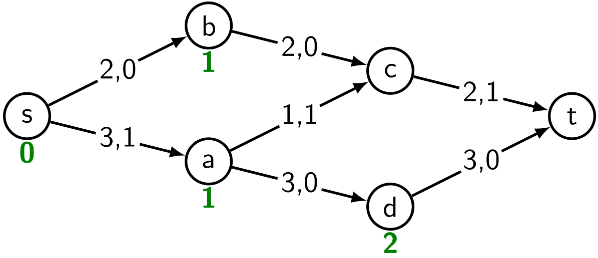

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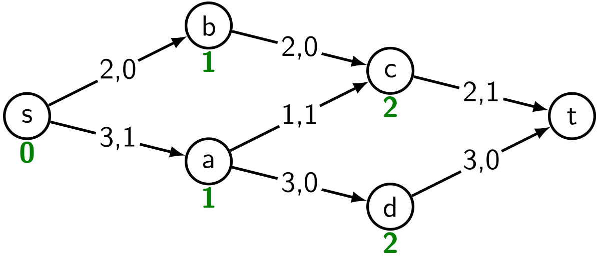

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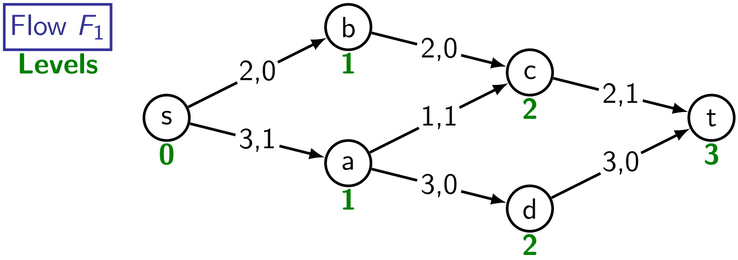

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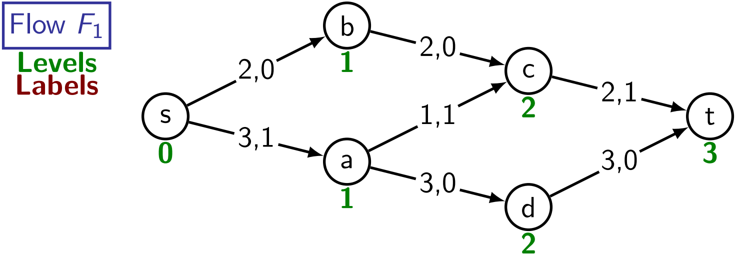

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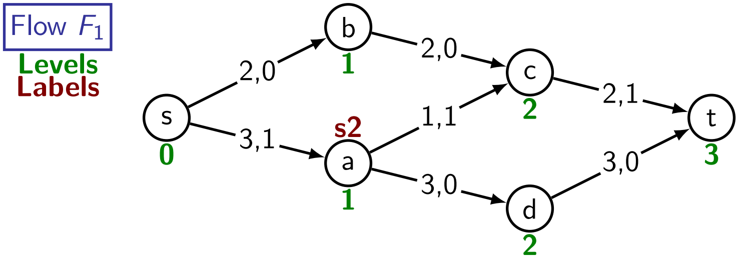

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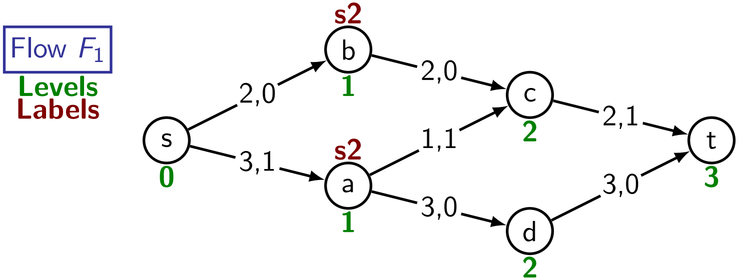

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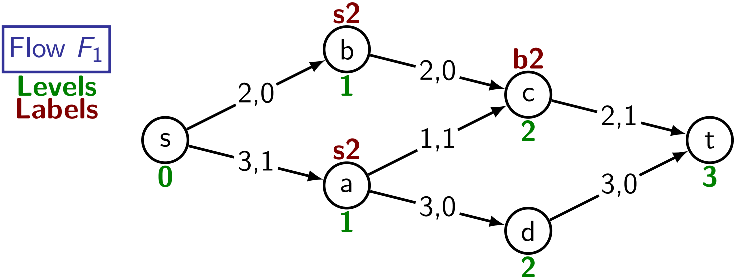

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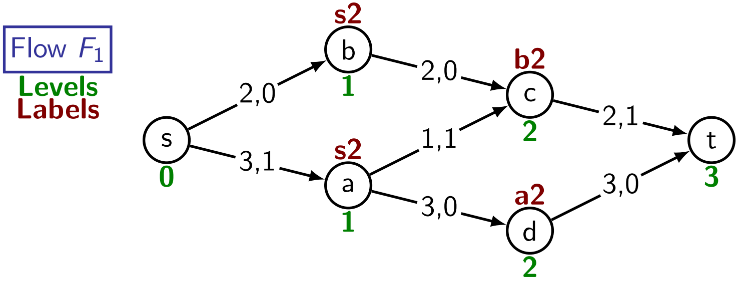

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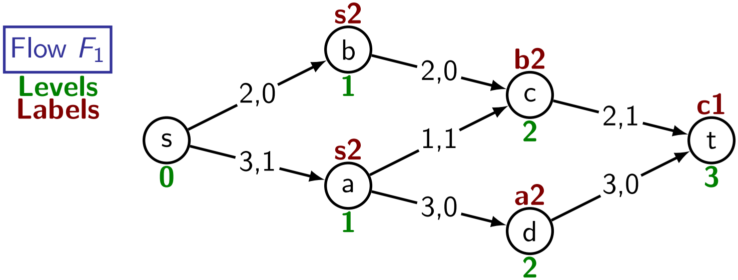
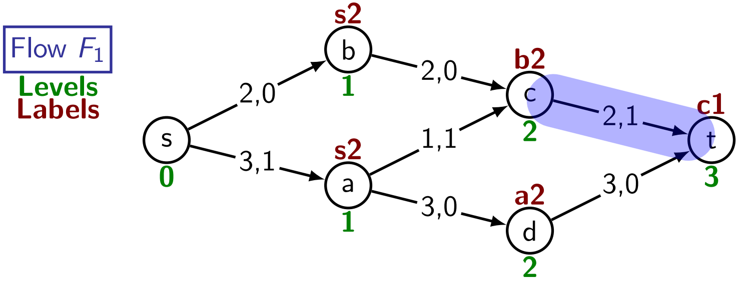


# Vertex labelling algorithm, Example 3

Stage 2: $F_1$ to $F_2$



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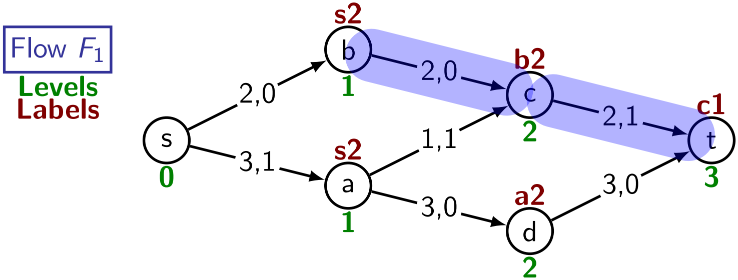

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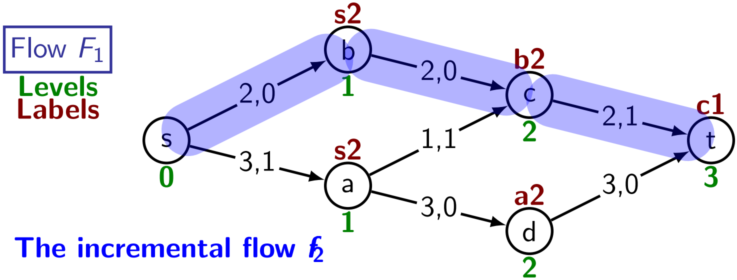
Stage 2: $F_1$ to $F_2$



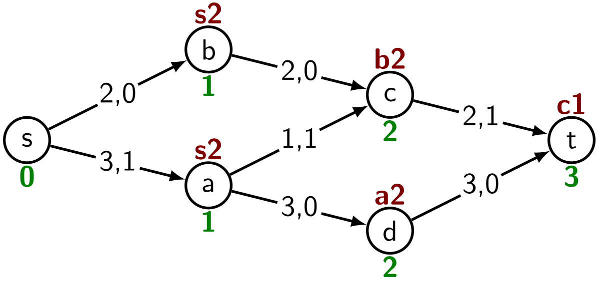

**The incremental flow $f_2$**

# Vertex labelling algorithm, Example 3

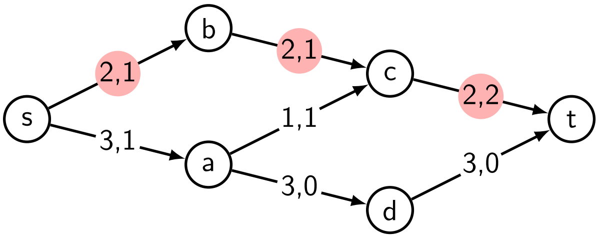
Stage 2: $F_1$ to $F_2$

Flow $F_1$

**Levels**
**Labels**

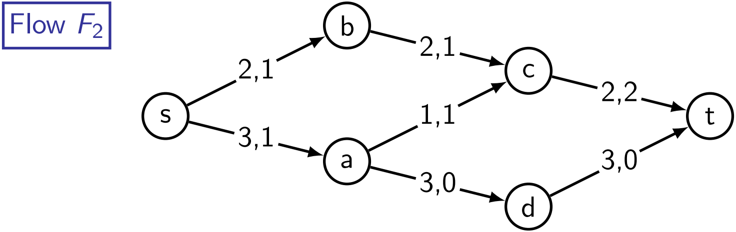



Flow $F_2$



# Vertex labelling algorithm, Example 3

## Stage 3: $F_2$ to $F_3$

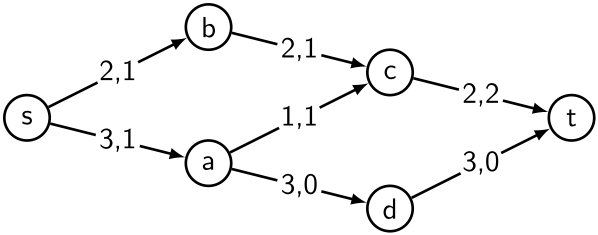


# Vertex labelling algorithm, Example 3

Stage 3: $F_2$ to $F_3$

Flow $F_2$

**Levels**

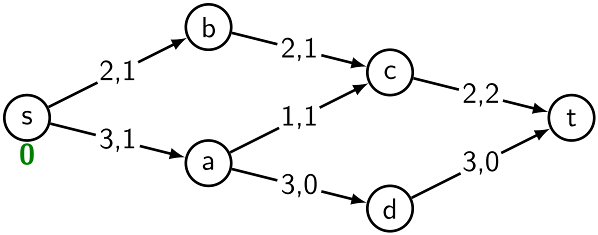


# Vertex labelling algorithm, Example 3

Stage 3: $F_2$ to $F_3$

$\boxed{\text{Flow } F_2}$

**Levels**



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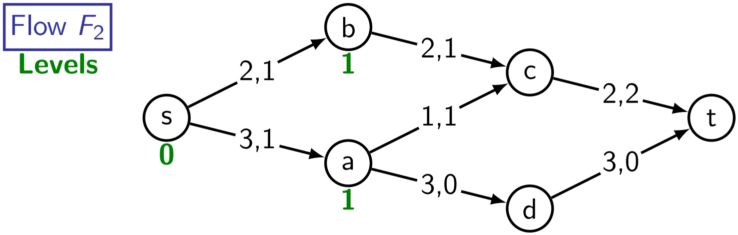

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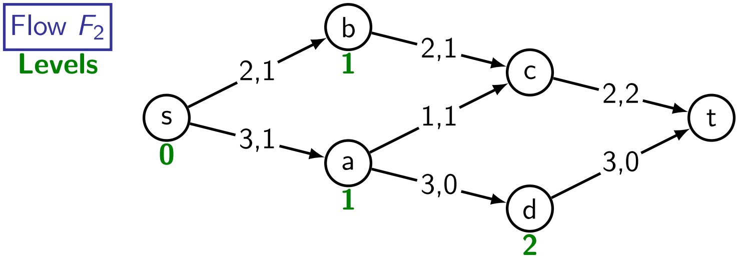

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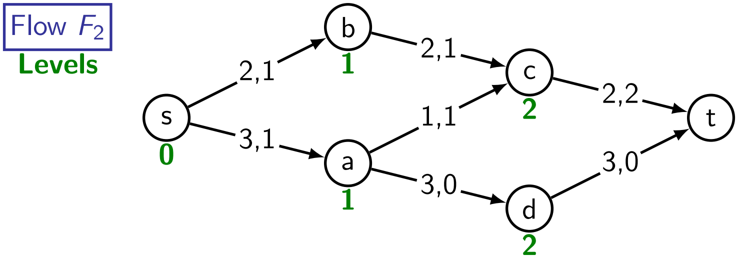

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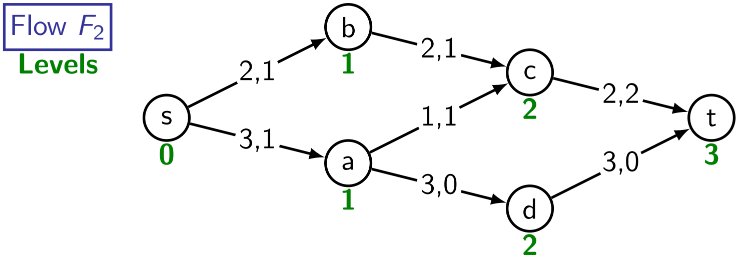

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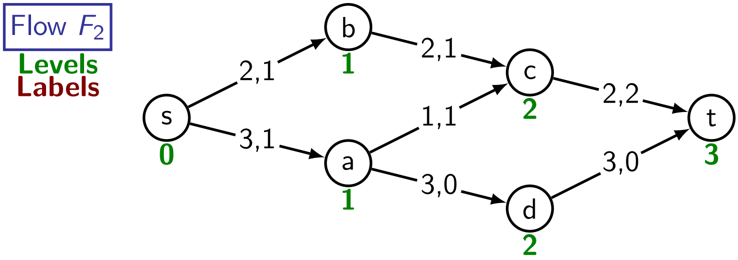

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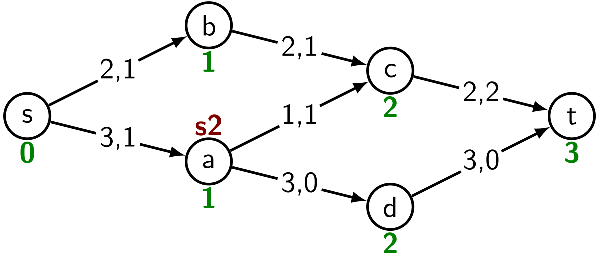
Stage 3: $F_2$ to $F_3$

Flow $F_2$

**Levels**
**Labels**



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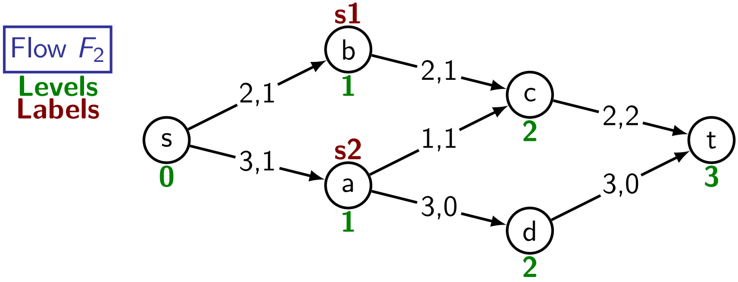

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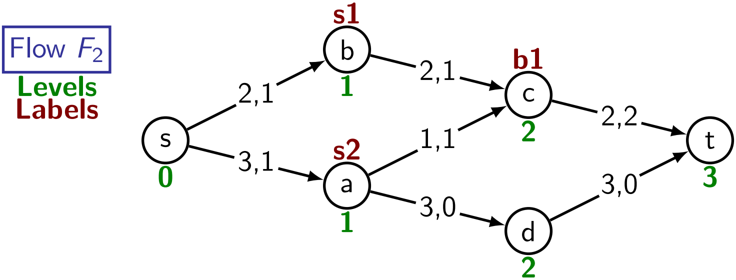

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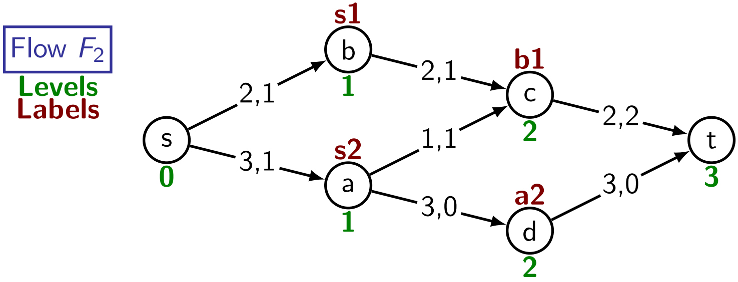

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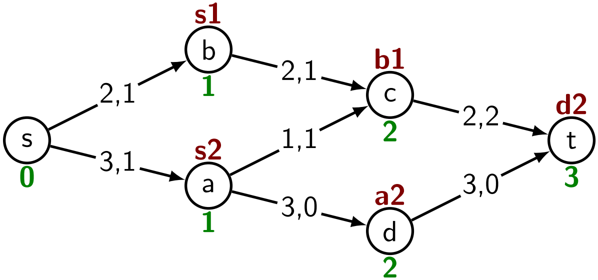
Stage 3: $F_2$ to $F_3$

$\boxed{\text{Flow } F_2}$

**Levels**
**Labels**



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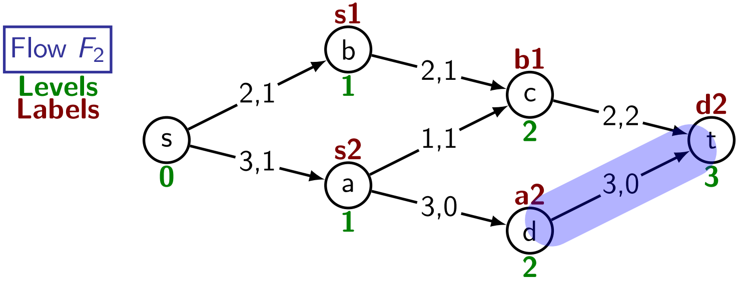

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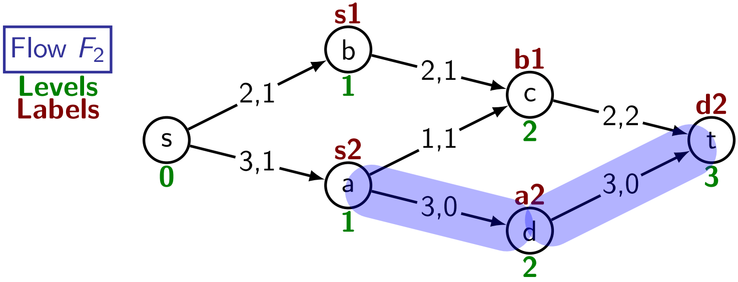
Stage 3: $F_2$ to $F_3$



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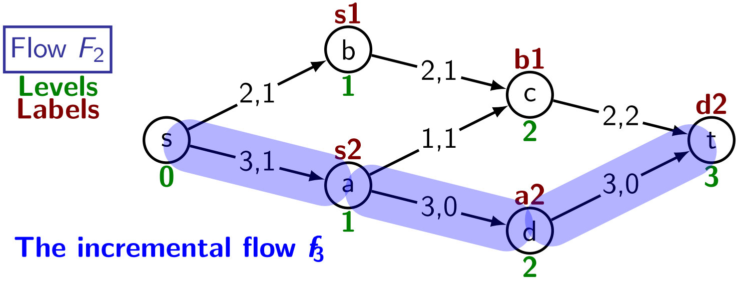
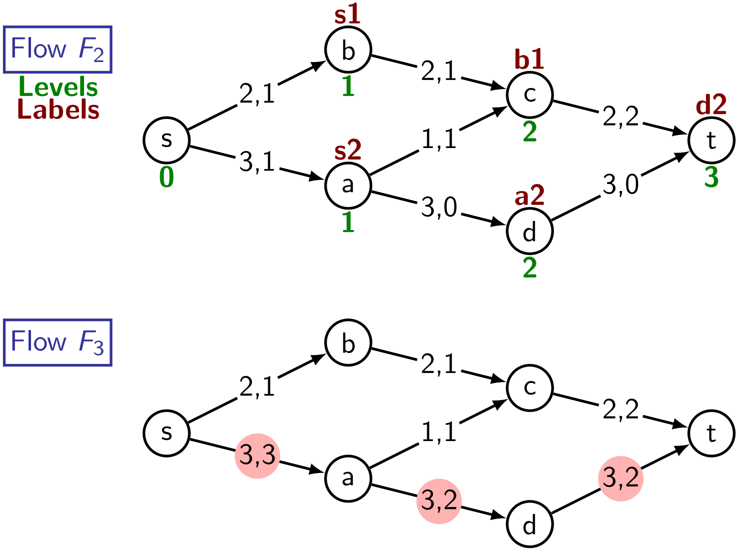
Stage 3: $F_2$ to $F_3$




**The incremental flow $f_3$**

# Vertex labelling algorithm, Example 3

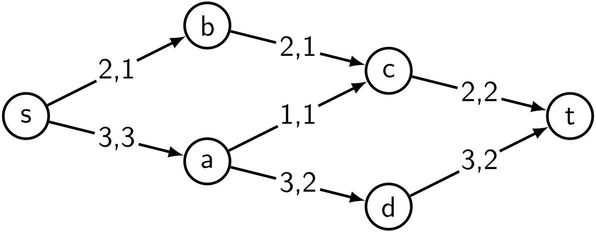
Stage 3: $F_2$ to $F_3$



# Vertex labelling algorithm, Example 3

## Stage 4: $F_3$ to $F_4$

Flow $F_3$

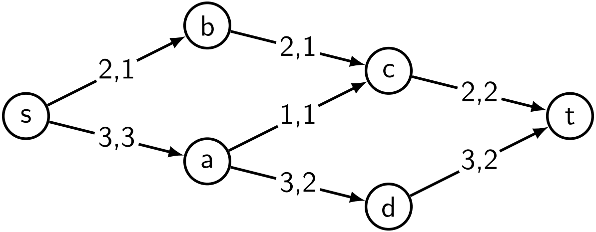


# Vertex labelling algorithm, Example 3

Stage 4: $F_3$ to $F_4$

Flow $F_3$

**Levels**



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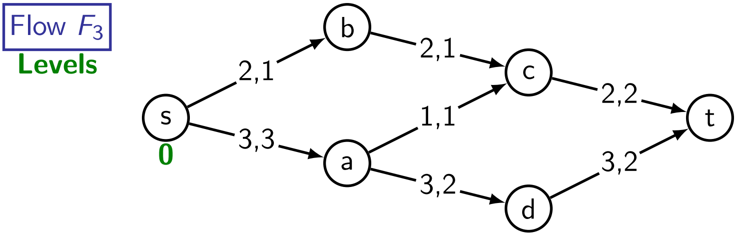
Stage 4: $F_3$ to $F_4$



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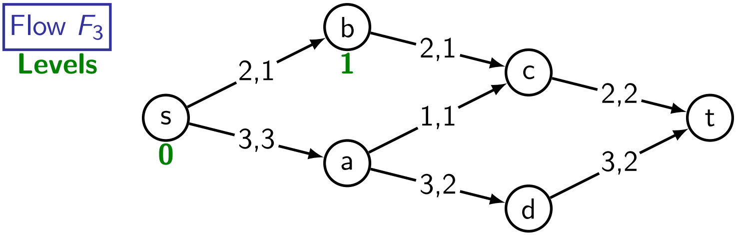

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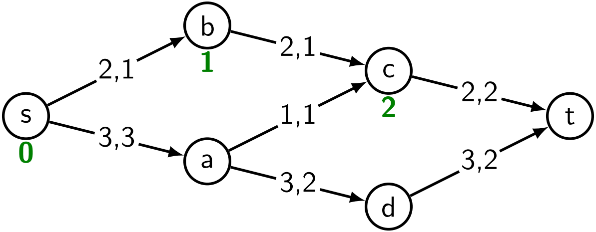
$\boxed{\text{Flow } F_3}$

**Levels**



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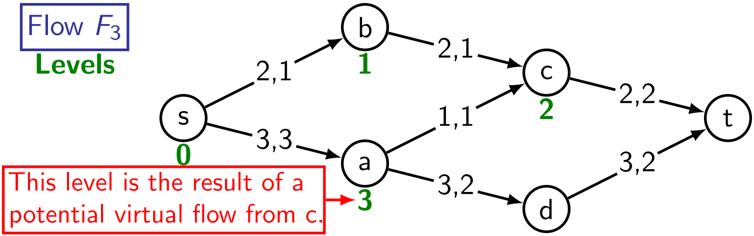

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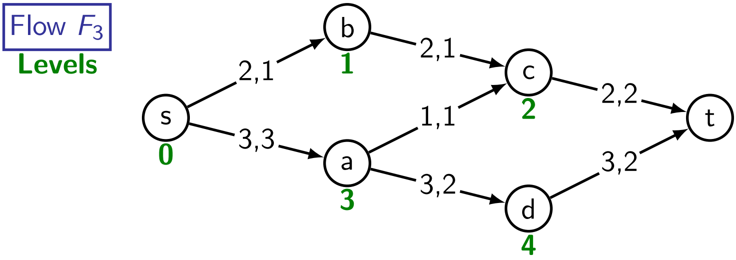

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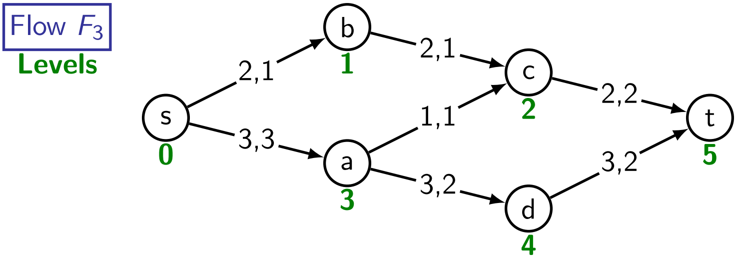

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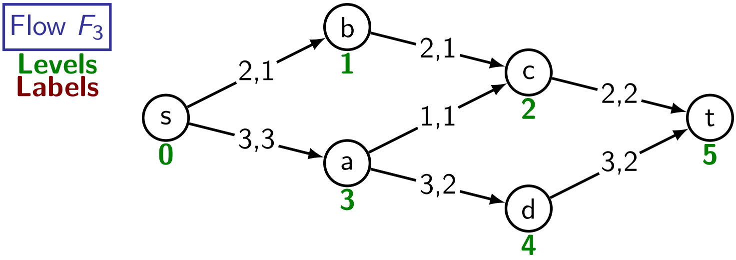

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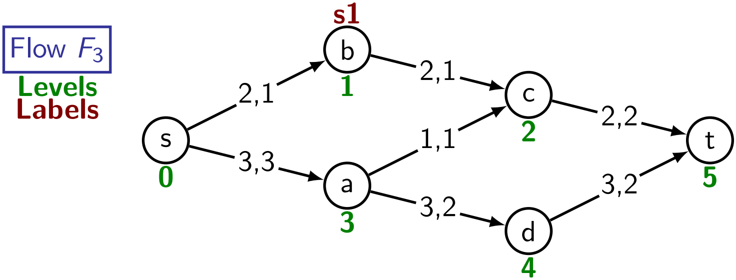

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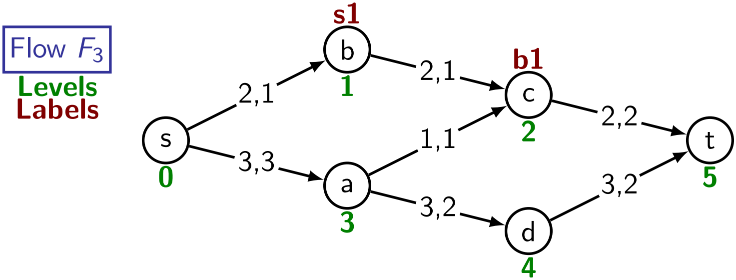

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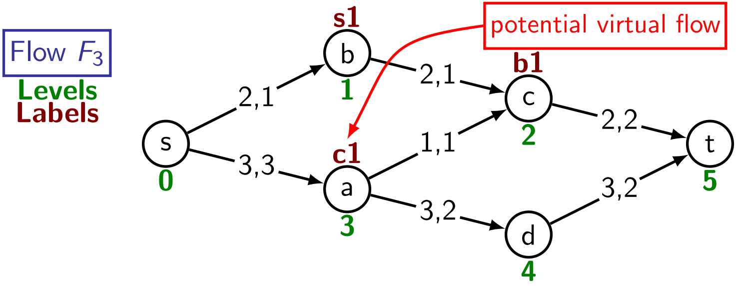

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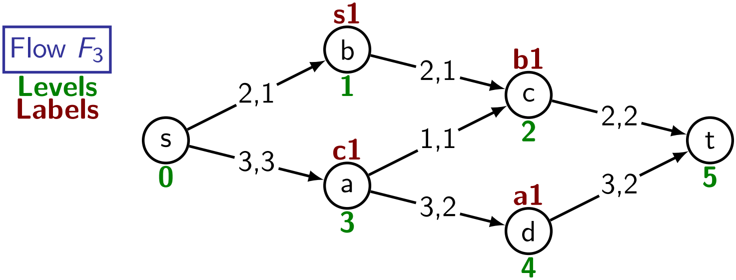

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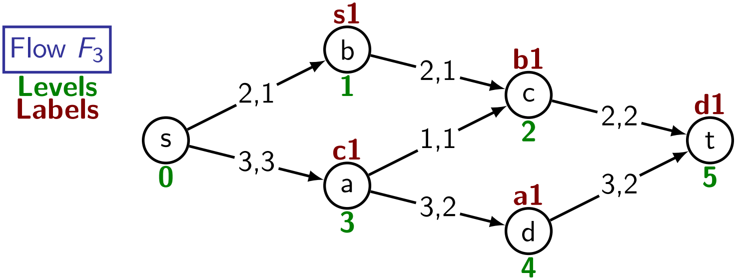

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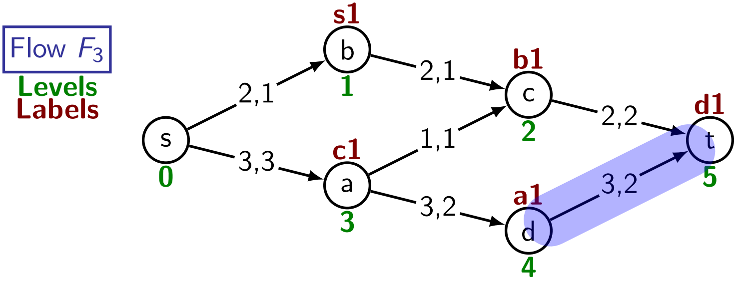

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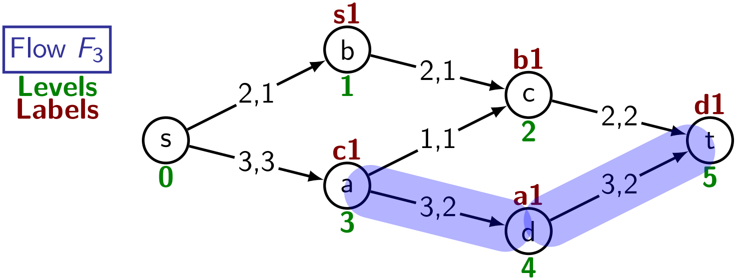

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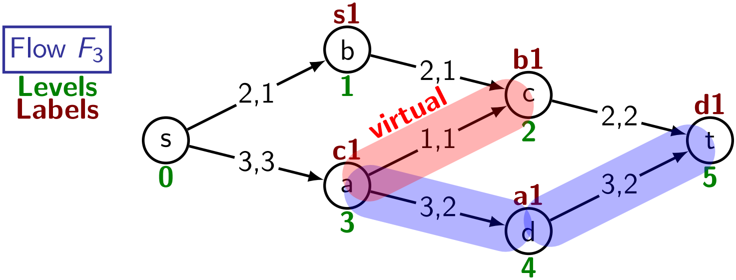

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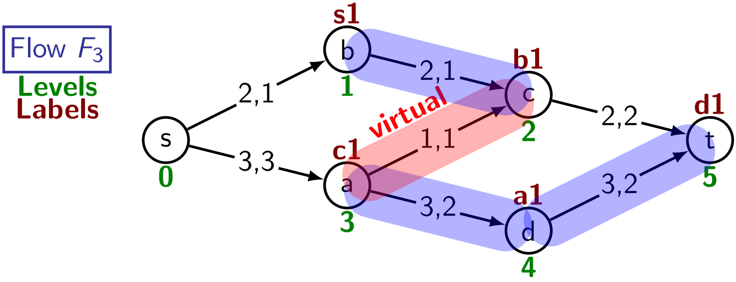

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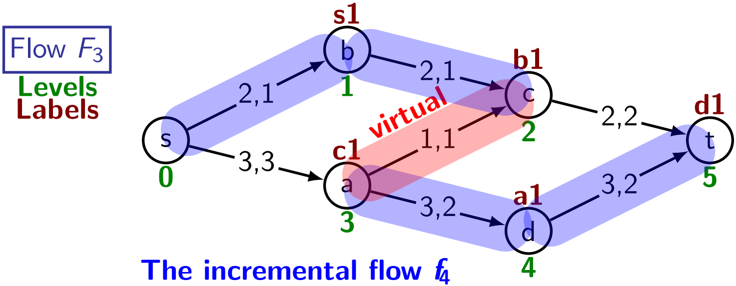


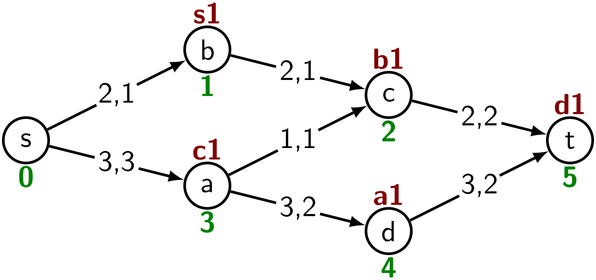

**The incremental flow $f_4$**

# Vertex labelling algorithm, Example 3

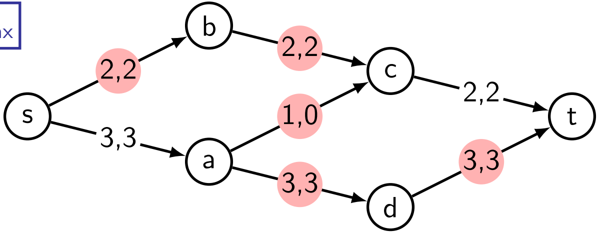
Stage 4: $F_3$ to $F_4$

Flow $F_3$

**Levels**
**Labels**



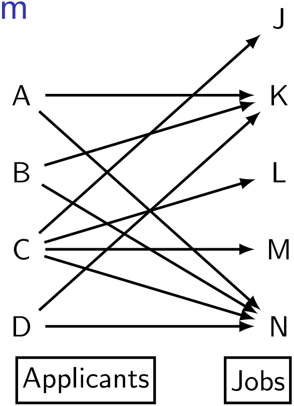

Flow $F_4 = F_{max}$



# A matching problem

from Johnsonbaugh

Four people A,B,C,D are each interested in one or more of five jobs J,K,L,M,N on offer. The diagram indicates who is interested in what job. Is it possible to satisfy everyone?



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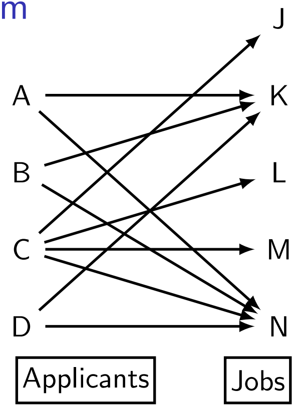

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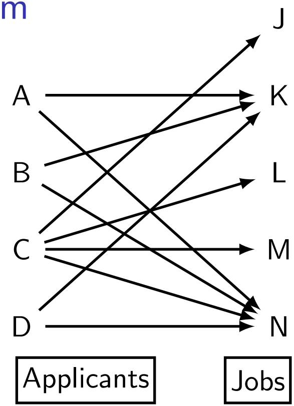

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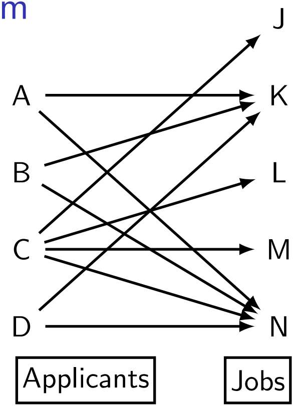

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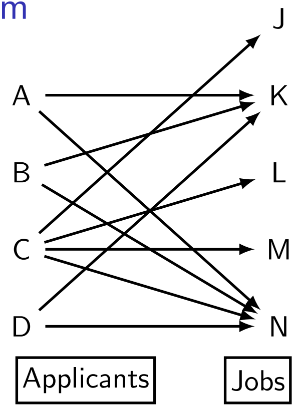


One answer:

| $x$ | A | B | C | D |
|---|---|---|---|---|
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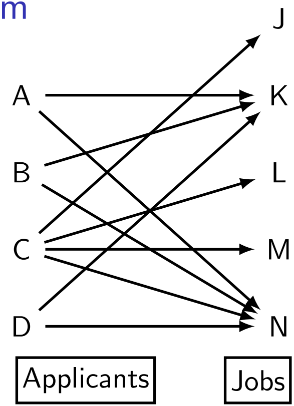


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More generally, given a relation $R \subseteq S \times T$ a **matching problem** seeks a **maximal matching function** (or just a 'matching') $m \subseteq R$.

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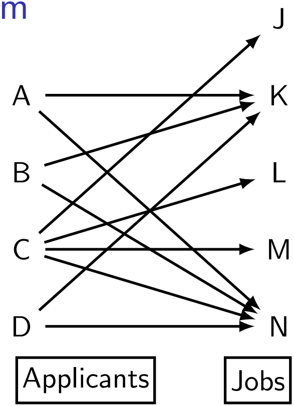


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More generally, given a relation $R \subseteq S \times T$ a **matching problem** seeks a **maximal matching function** (or just a 'matching') $m \subseteq R$. This is a **injective** (one-to-one) function $f : S' \to T$ with domain $S' \subseteq S$ as large as possible subject to $m$ being an injective subset of $R$.

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   - a **supertarget** t and edges ($y$,t) for each $y \in T$.

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The matching problem for $R \subseteq S \times T$ can be converted to the max flow problem for a transport network as follows:

1. Create a digraph $D$ by adding to the arrow diagram for $R$
   - a **supersource** s and edges (s,$x$) for each $x \in S$ and
   - a **supertarget** t and edges ($y$,t) for each $y \in T$.
2. Assign capacity 1 to every edge of $D$. (*i.e.* $\forall e \in E(D)\ C(e) = 1$.)

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A solution to the max flow problem provides the matching:

$$\boxed{m = \{(x, y) \in S \times T \ : \ F_{\max}((x, y)) = 1\}.}$$

# Vertex labelling for matching; Example 1

Here is how vertex labelling would be used on the Johnsonbaugh problem.

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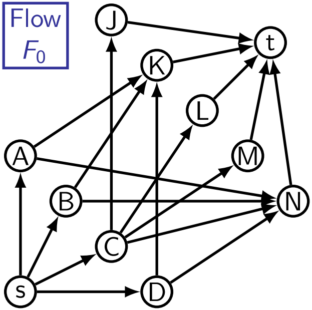

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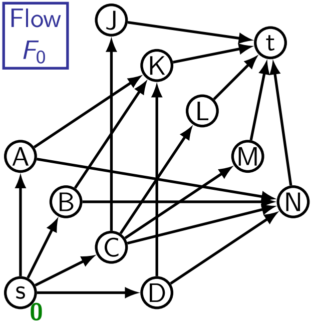
$F_0$



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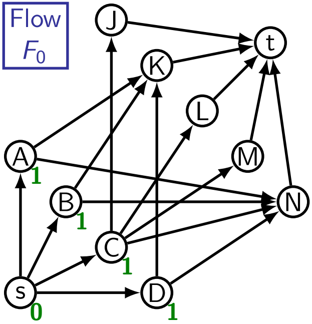

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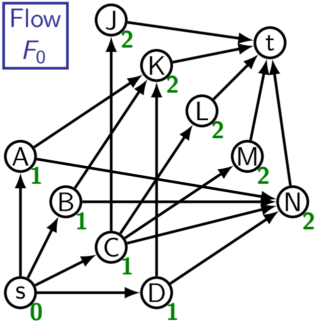

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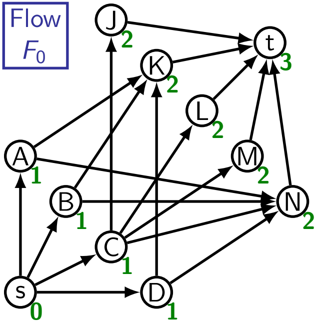

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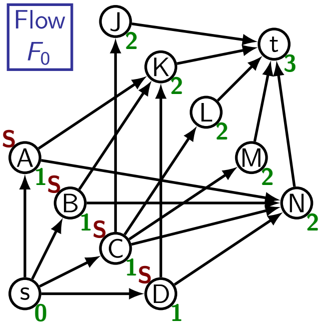

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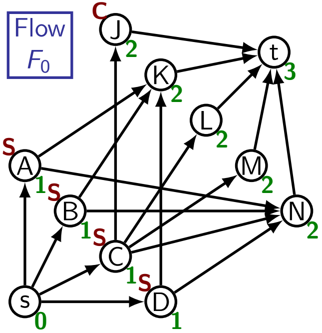

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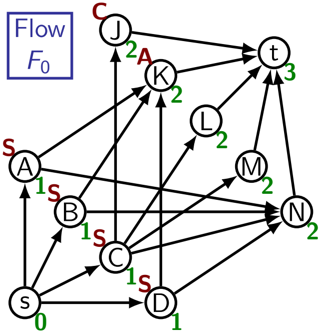

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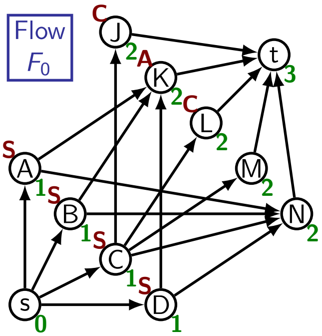

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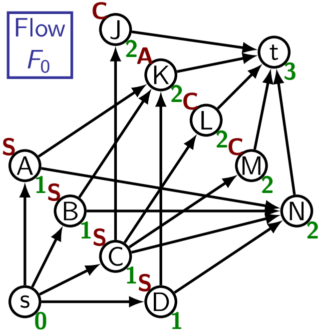

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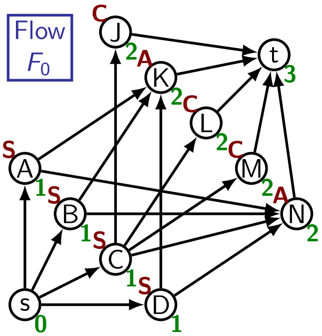

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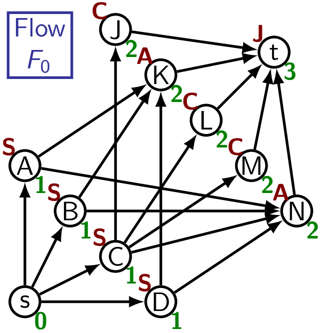

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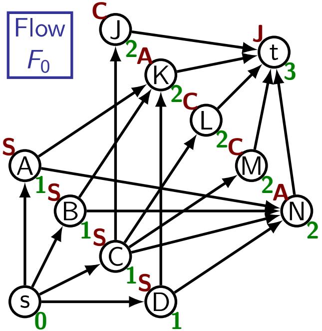


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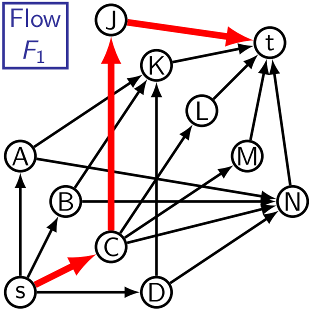

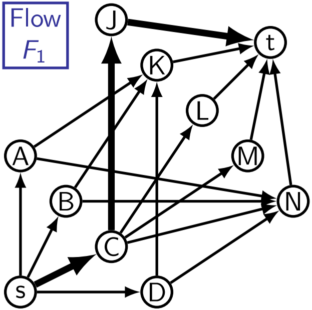

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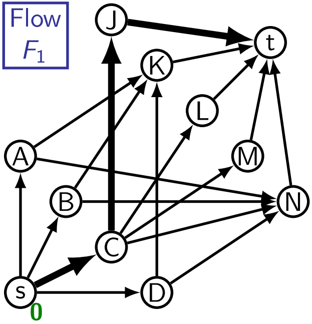
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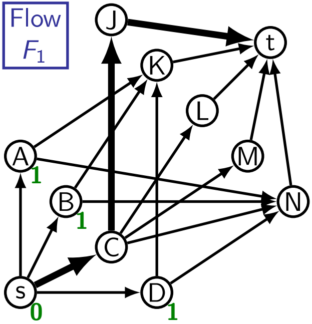
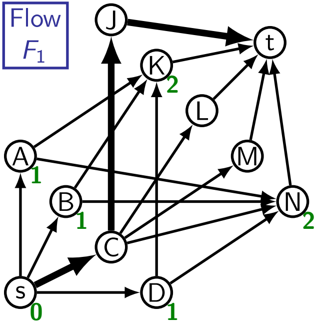
Flow $F_1$





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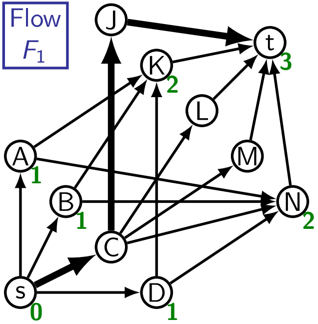
Flow $F_1$



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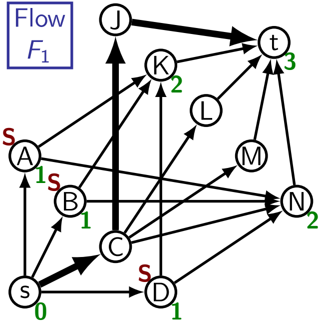

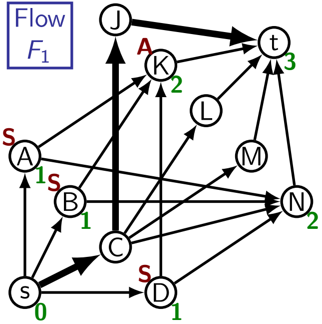
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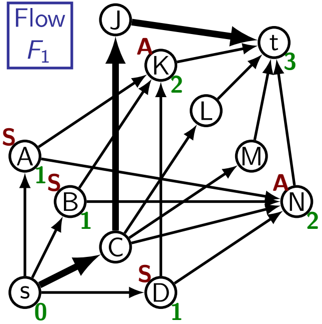

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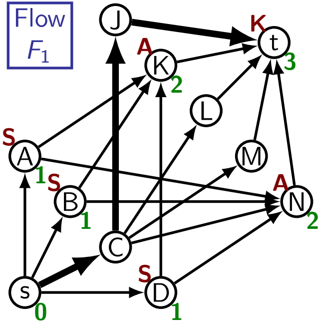

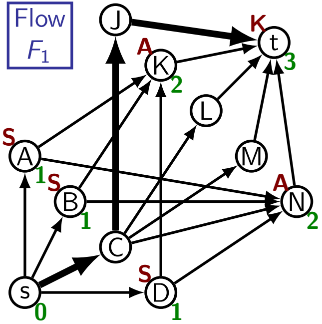


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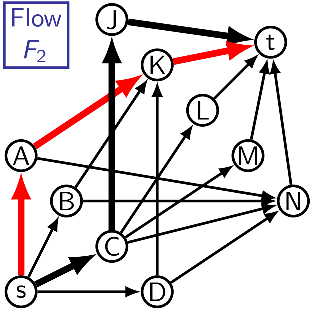

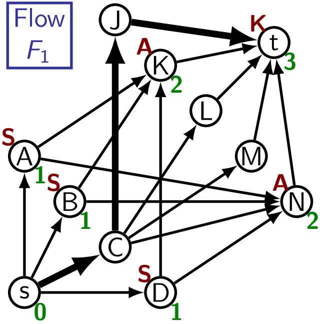
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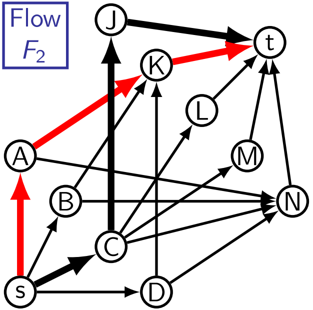
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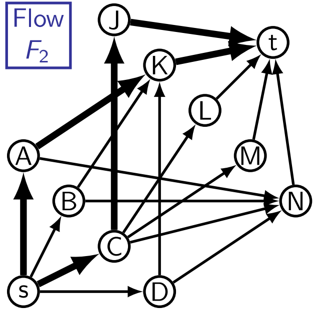
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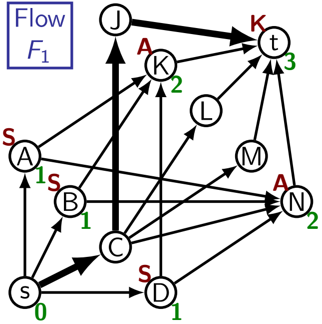
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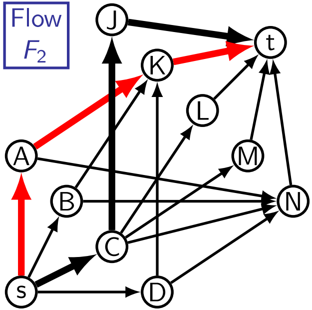
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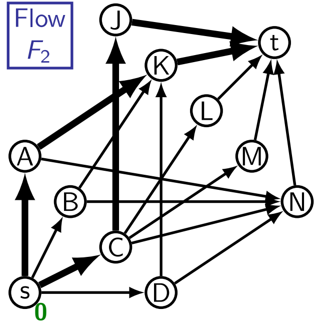
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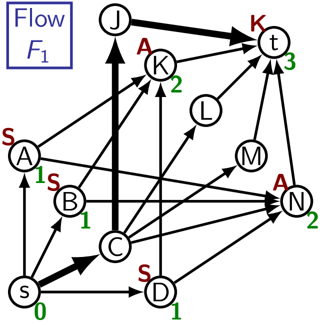
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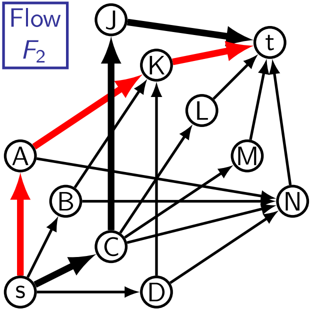
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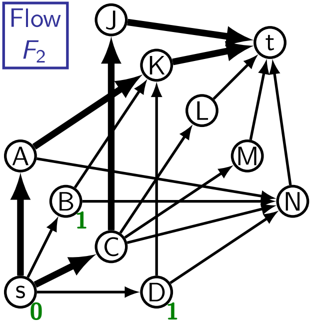
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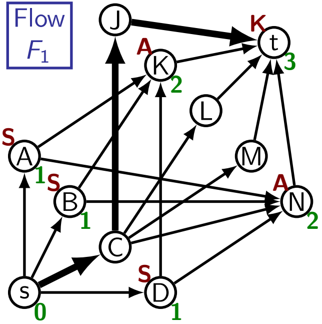
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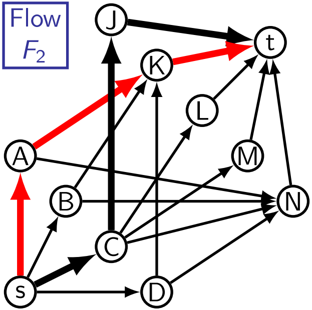
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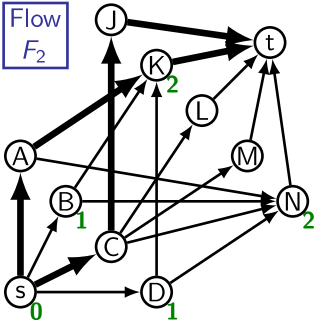
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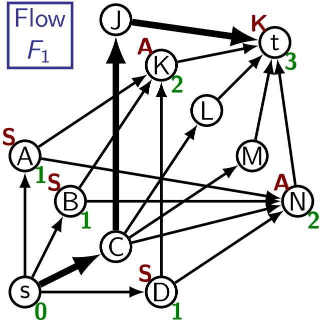
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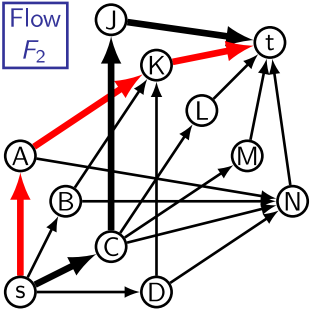
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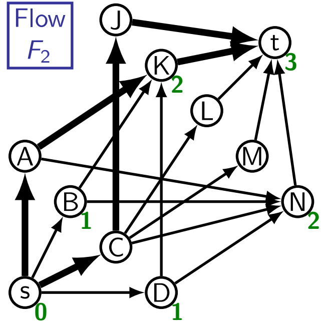
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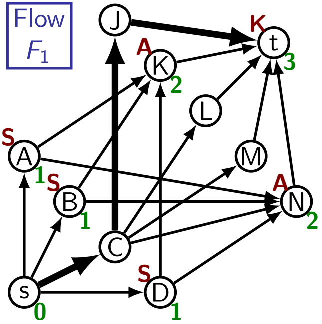
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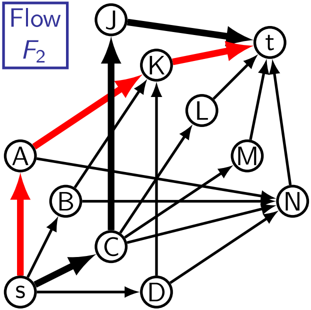
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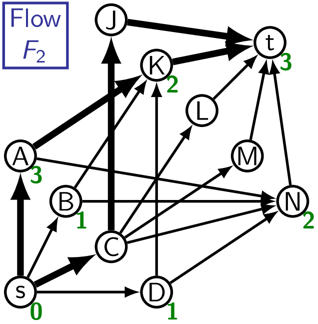
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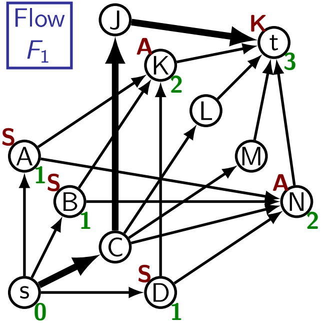
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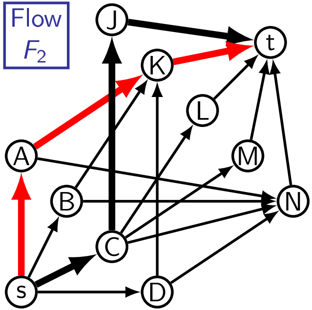
1

⇒


Flow $F_2$
J
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B
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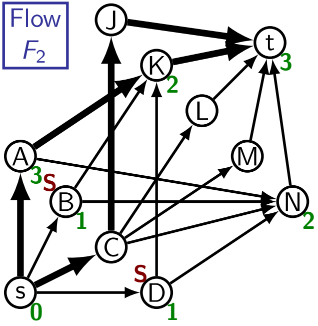
Flow $F_2$
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K
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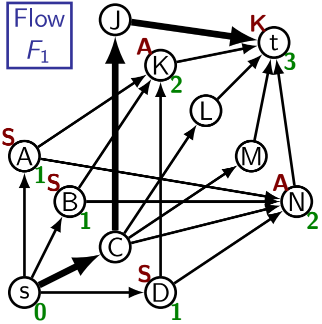
Flow
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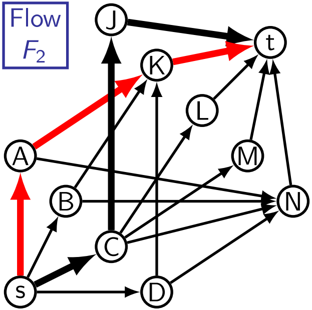
Flow
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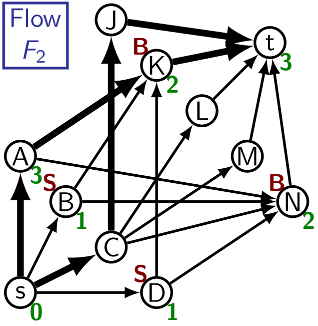
Flow
F2
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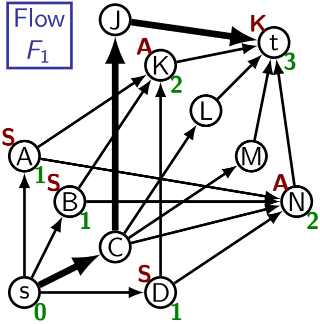
Flow
$F_1$
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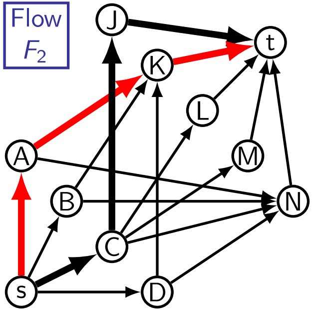
$\Rightarrow$


Flow
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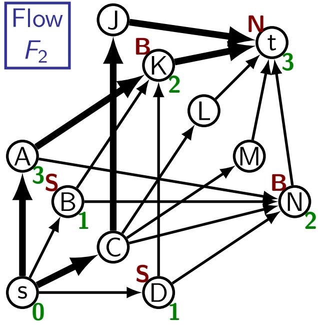
Flow
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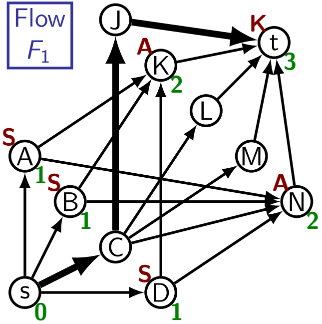
Flow $F_1$
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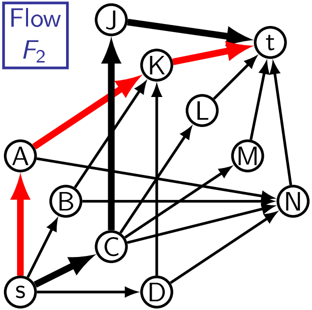
⇒


Flow $F_2$
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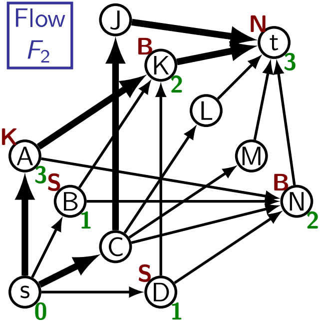
Flow $F_2$
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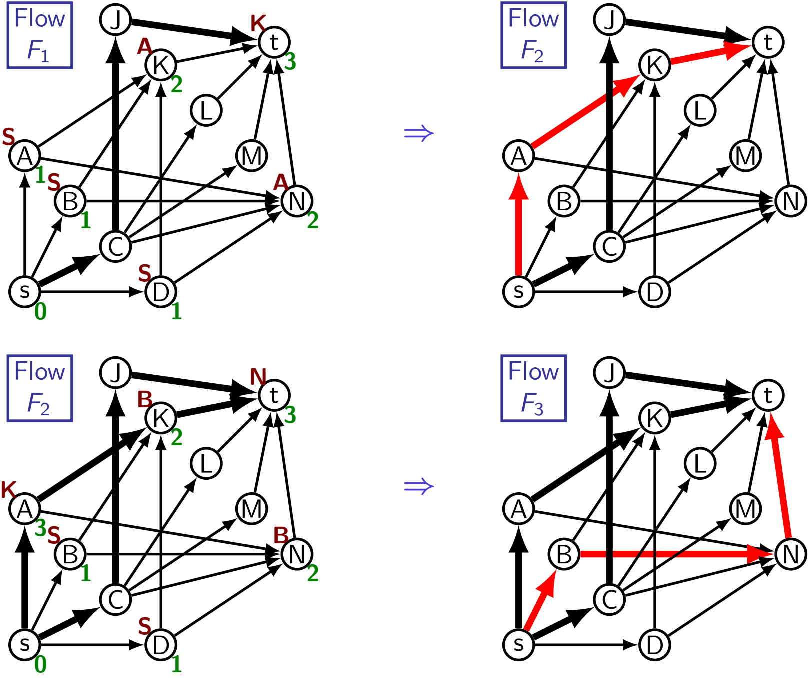
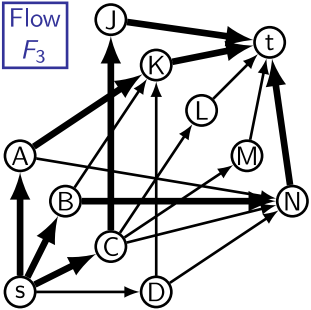


Flow $F_1$
Flow $F_2$
⇒
Flow $F_2$
Flow $F_3$
⇒


Flow
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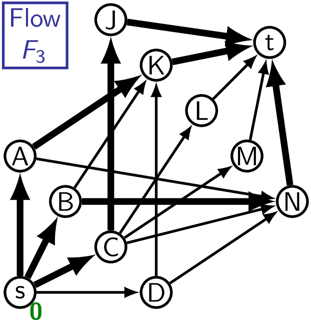
Flow
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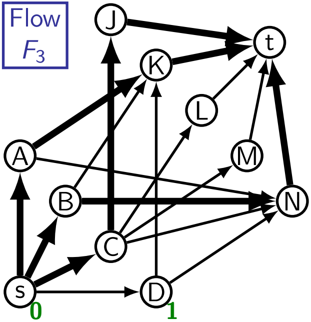
Flow
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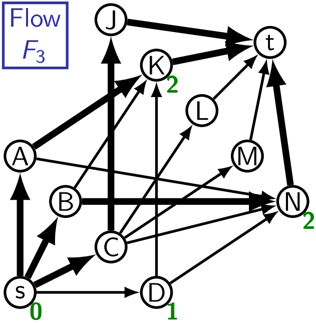
Flow
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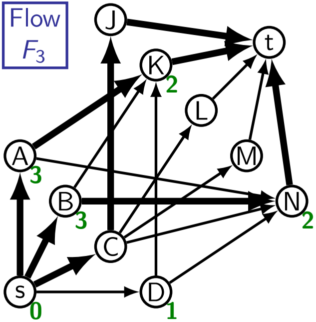
Flow
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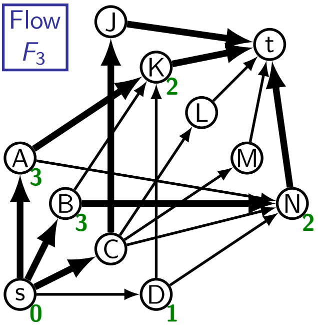
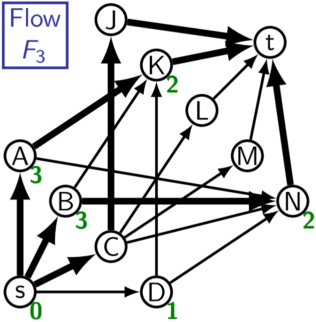



$\Rightarrow$

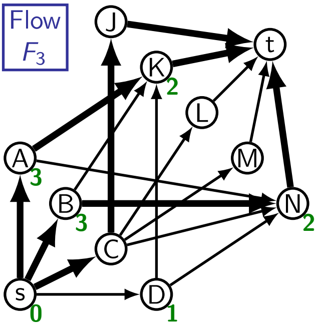
Even using virtual capacities to reach vertices of level 3, it isn't possible to assign a level to t, so the algorithm terminates.




$\Rightarrow$

Even using virtual capacities to reach vertices of level 3, it isn't possible to assign a level to t, so the algorithm terminates.

The matchings are indicated by the edges between $\{A, B, C, D\}$ and $\{J, K, L, M, N\}$ that have flow 1 (the thick edges).



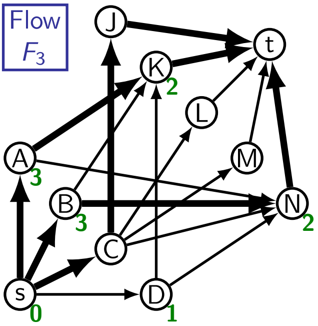

$\Rightarrow$

Even using virtual capacities to reach vertices of level 3, it isn't possible to assign a level to t, so the algorithm terminates.

The matchings are indicated by the edges between $\{A, B, C, D\}$ and $\{J, K, L, M, N\}$ that have flow 1 (the thick edges).

So the matching is

$$m = \{(A, K), (B, N), (C, J)\}.$$

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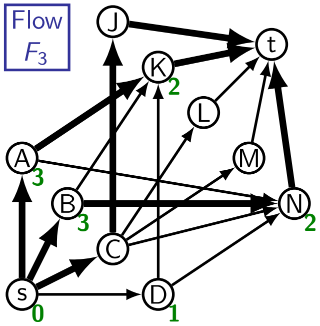
So the matching is

$$m = \{(A, K), (B, N), (C, J)\}.$$

As remarked earlier, the above solution is very easy to get without the use of the vertex labelling algorithm. We have just matched the first member of $T$ with the first 'available' member of $S$, then the next member of $T$ with the next 'available' member of $S$ and so on.




$\Rightarrow$

Even using virtual capacities to reach vertices of level 3, it isn't possible to assign a level to t, so the algorithm terminates.

The matchings are indicated by the edges between $\{A, B, C, D\}$ and $\{J, K, L, M, N\}$ that have flow 1 (the thick edges).

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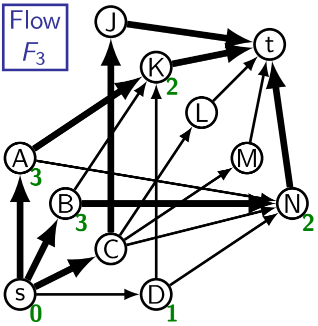
The benefit of using the algorithm lies in its ability to 'undo' initial pairings and replace them by new ones when this becomes necessary to enable later pairings.




$\Rightarrow$

Even using virtual capacities to reach vertices of level 3, it isn't possible to assign a level to t, so the algorithm terminates.

The matchings are indicated by the edges between $\{A, B, C, D\}$ and $\{J, K, L, M, N\}$ that have flow 1 (the thick edges).

So the matching is

$$m = \{(A, K), (B, N), (C, J)\}.$$

As remarked earlier, the above solution is very easy to get without the use of the vertex labelling algorithm. We have just matched the first member of $T$ with the first 'available' member of $S$, then the next member of $T$ with the next 'available' member of $S$ and so on.

The benefit of using the algorithm lies in its ability to 'undo' initial pairings and replace them by new ones when this becomes necessary to enable later pairings.

The final example shows how it does this in the simplest possible case.

# Vertex labelling for matching; Example 2

Find a (maximal) matching function $m$
for the relation $R = \{(a, p), (a, q), (b, p)\}$.

# Vertex labelling for matching; Example 2

Find a (maximal) matching function $m$
for the relation $R = \{(a, p), (a, q), (b, p)\}$.

Obviously the only answer is $m = \{(a, q), (b, p)\}$,

# Vertex labelling for matching; Example 2

Find a (maximal) matching function $m$
for the relation $R = \{(a, p), (a, q), (b, p)\}$.

Obviously the only answer is $m = \{(a, q), (b, p)\}$, but the vertex labelling algorithm will first match $a$ with $p$.

# Vertex labelling for matching; Example 2

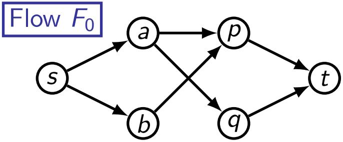
## Find a (maximal) matching function $m$ for the relation $R = \{(a,p),(a,q),(b,p)\}$.

Obviously the only answer is $m = \{(a,q),(b,p)\}$, but the vertex labelling algorithm will first match $a$ with $p$. However it will then use a virtual flow to undo this and match $a$ with $q$, allowing $b$ to be matched with $p$ instead.

# Vertex labelling for matching; Example 2

## Find a (maximal) matching function $m$ for the relation $R = \{(a, p), (a, q), (b, p)\}$.

Obviously the only answer is $m = \{(a, q), (b, p)\}$, but the vertex labelling algorithm will first match $a$ with $p$. However it will then use a virtual flow to undo this and match $a$ with $q$, allowing $b$ to be matched with $p$ instead. Here are the diagrams:



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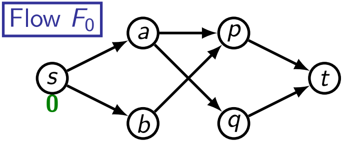
## Find a (maximal) matching function $m$ for the relation $R = \{(a, p), (a, q), (b, p)\}$.

Obviously the only answer is $m = \{(a, q), (b, p)\}$, but the vertex labelling algorithm will first match $a$ with $p$. However it will then use a virtual flow to undo this and match $a$ with $q$, allowing $b$ to be matched with $p$ instead. Here are the diagrams:



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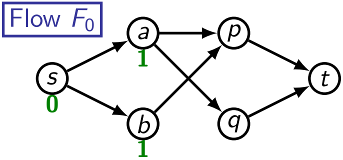

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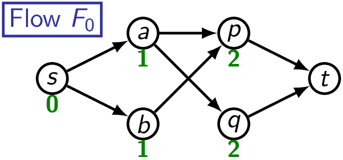

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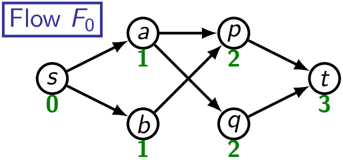

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## Find a (maximal) matching function $m$ for the relation $R = \{(a,p),(a,q),(b,p)\}$.

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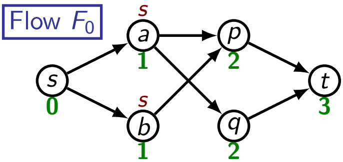

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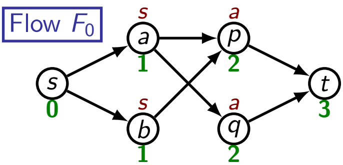

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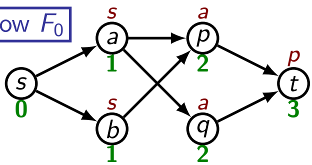
Flow $F_0$



# Vertex labelling for matching; Example 2

Find a (maximal) matching function $m$
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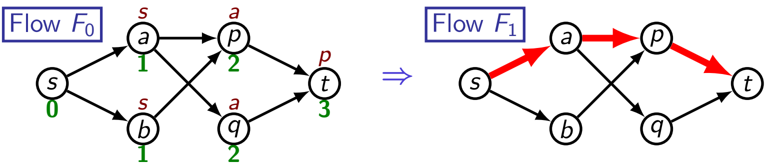
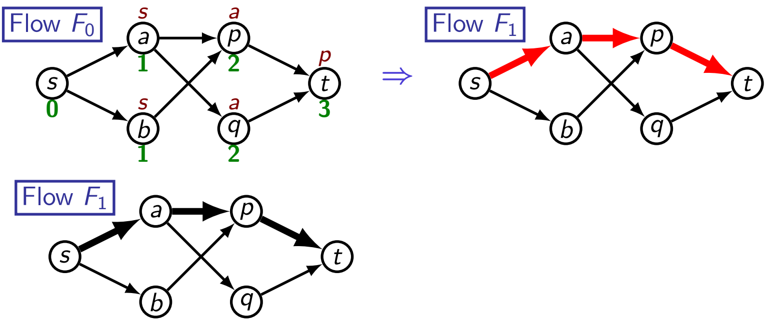
Obviously the only answer is $m = \{(a,q),(b,p)\}$, but the vertex labelling algorithm will first match $a$ with $p$. However it will then use a virtual flow to undo this and match $a$ with $q$, allowing $b$ to be matched with $p$ instead. Here are the diagrams:



# Vertex labelling for matching; Example 2

## Find a (maximal) matching function $m$ for the relation $R = \{(a, p), (a, q), (b, p)\}$.

Obviously the only answer is $m = \{(a, q), (b, p)\}$, but the vertex labelling algorithm will first match $a$ with $p$. However it will then use a virtual flow to undo this and match $a$ with $q$, allowing $b$ to be matched with $p$ instead. Here are the diagrams:



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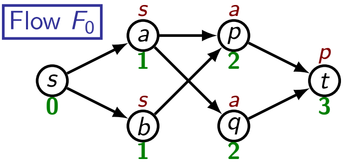
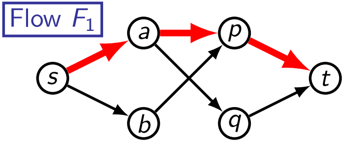
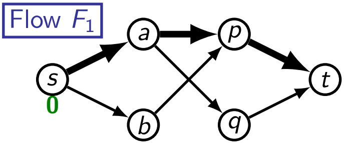



$\Rightarrow$



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## Find a (maximal) matching function $m$ for the relation $R = \{(a, p), (a, q), (b, p)\}$.

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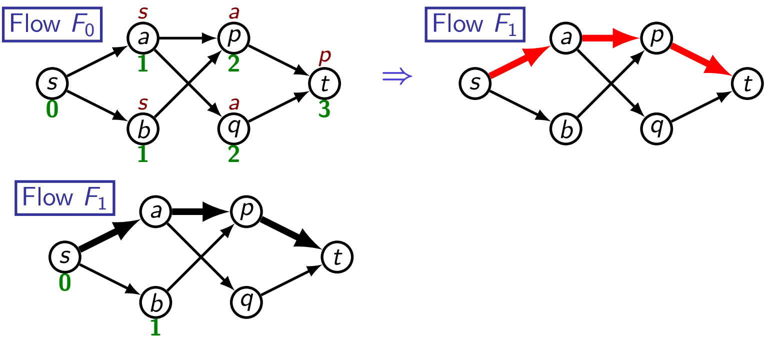

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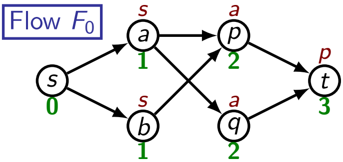
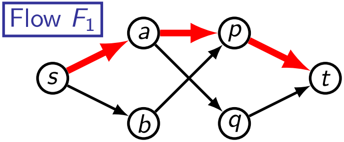
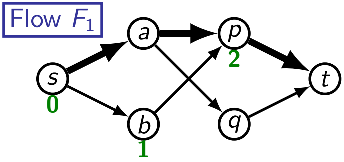



$\Rightarrow$



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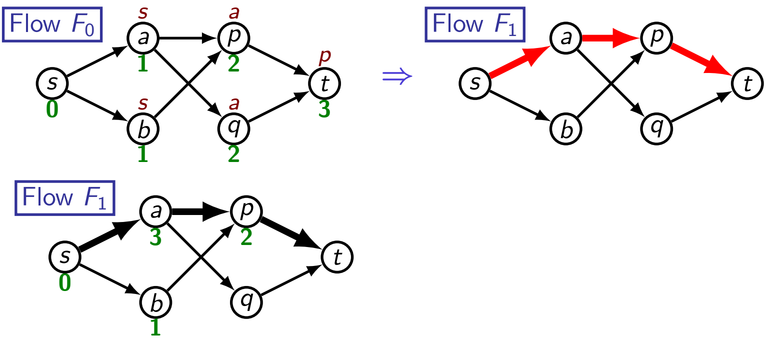

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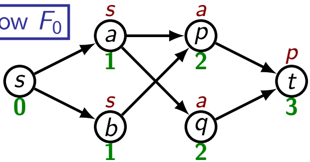
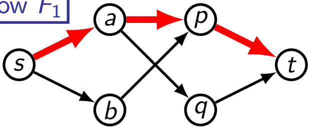
Obviously the only answer is $m = \{(a,q),(b,p)\}$, but the vertex labelling algorithm will first match $a$ with $p$. However it will then use a virtual flow to undo this and match $a$ with $q$, allowing $b$ to be matched with $p$ instead. Here are the diagrams:

Flow $F_0$

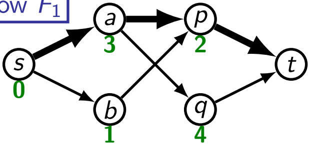



$\Rightarrow$

Flow $F_1$



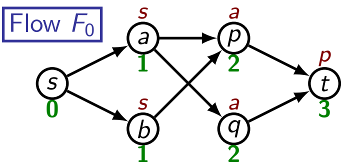
Flow $F_1$



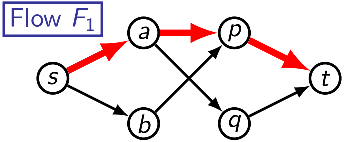
# Vertex labelling for matching; Example 2

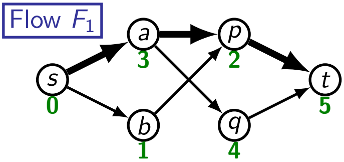
## Find a (maximal) matching function $m$ for the relation $R = \{(a,p),(a,q),(b,p)\}$.

Obviously the only answer is $m = \{(a,q),(b,p)\}$, but the vertex labelling algorithm will first match $a$ with $p$. However it will then use a virtual flow to undo this and match $a$ with $q$, allowing $b$ to be matched with $p$ instead. Here are the diagrams:




$\Rightarrow$



# Vertex labelling for matching; Example 2

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for the relation $R = \{(a, p), (a, q), (b, p)\}$.

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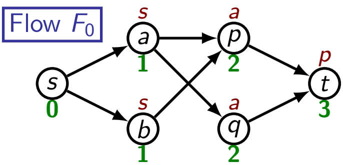
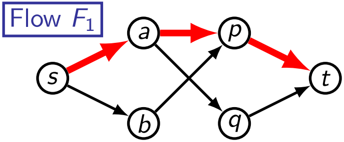
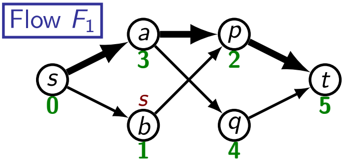



$\Rightarrow$



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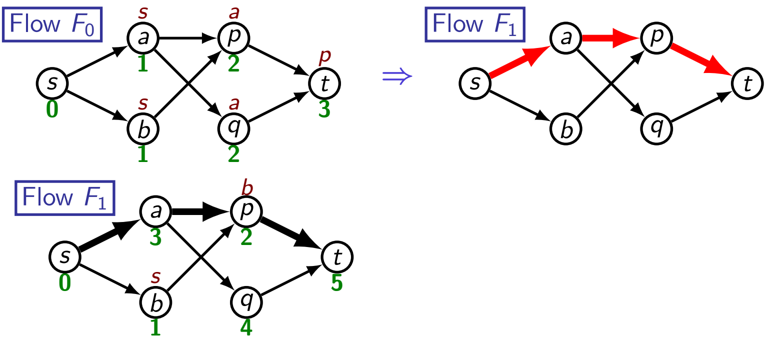

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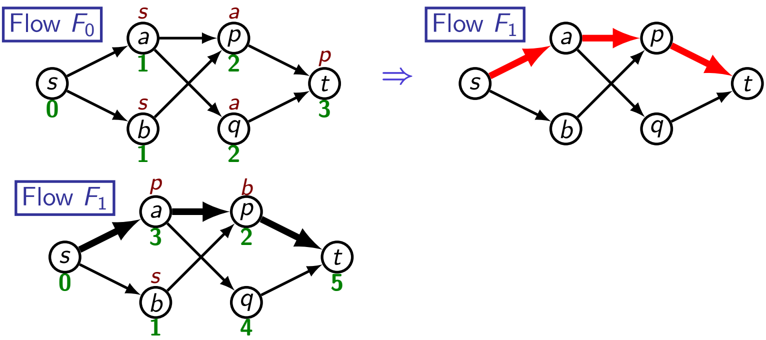

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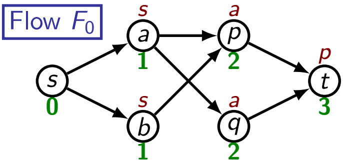
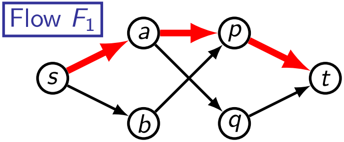
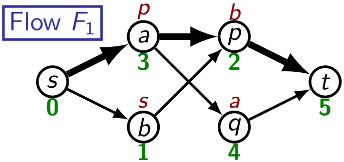



$\Rightarrow$



# Vertex labelling for matching; Example 2

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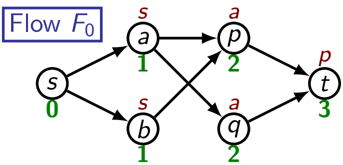
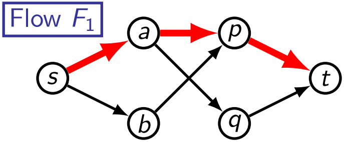
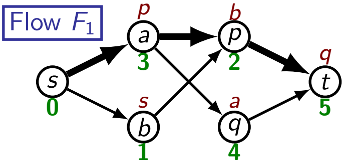



$\Rightarrow$



# Vertex labelling for matching; Example 2

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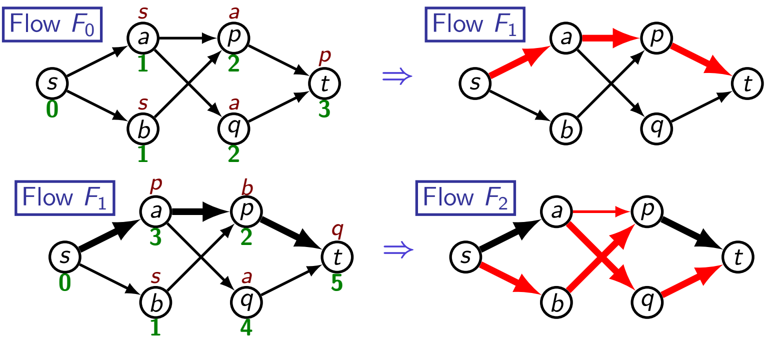

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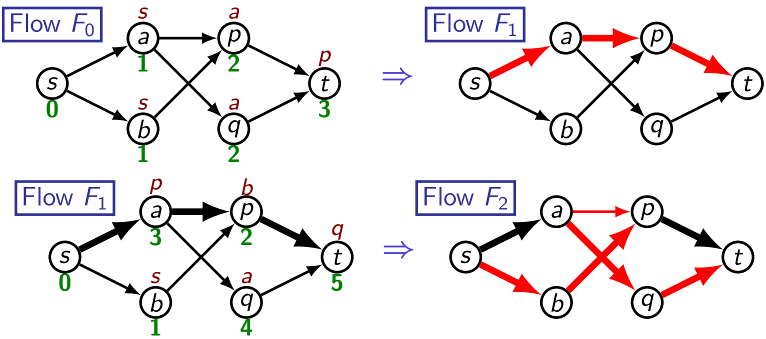


**END OF SECTION D2**