Question 1

	Location		Approximate air distance (000km) to:			
City	degrees North of Equator	degrees East of Gr'nwich	Auckland	Chennai	Marrakesh	Winnipeg
Auckland	-39	175	0	11	19	13
Chennai	13	80	11	0	9	13
Marrakesh	32	-8	19	9	0	7
Winnipeg	50	-97 ($\equiv 263$)	13	13	7	0

Relations R_1, R_2 and R_3 on the set of four cities tabulated above are defined below. Draw a directed graph for each relation (use arrows between 'A', 'C', 'M' and 'W') and say whether or not the relation is a function. If not a function, say why not.

(a) $xR_1y \Leftrightarrow x$ is more than 10 000km from y.

(b) $xR_2y \Leftrightarrow x$ is South of y.

(c) $xR_3y \Leftrightarrow x$ is between 80° and 100° East of y.

(a) R_1*:*

$$\begin{tikzcd} & & W \arrow[dd, leftrightarrow] \\ M \arrow[r, leftrightarrow] & A \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & \\ & & C \end{tikzcd}$$

$\boxed{\textit{Not a function.}}$

E.g. There are multiple arrows starting at A.

(b) R_2*:*

$$\begin{tikzcd} C \arrow[r] \arrow[dr] & M \arrow[d] \\ A \arrow[u] \arrow[r] \arrow[ur] & W \end{tikzcd}$$

$\boxed{\textit{Not a function.}}$

E.g. There are multiple arrows starting at A.
(Also, none starting at W.)

(c) R_3*:*

$$\begin{tikzcd} C \arrow[r] & M \arrow[d] \\ A \arrow[u] & W \arrow[l] \end{tikzcd}$$

$\boxed{\textit{This is a function.}}$

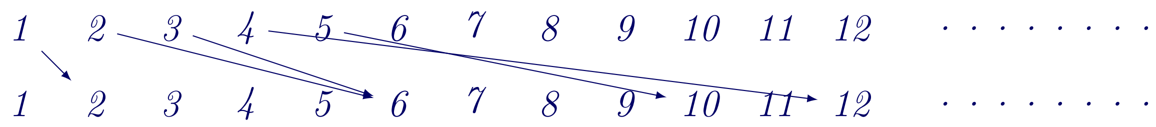
There is exactly one arrow starting at each city.

Question 2 Functions $a, b, c, d \; : \mathbb{N} \to \mathbb{N}$ are defined by the rules below. In each case decide whether the function is injective (one-to-one), surjective (onto), neither or both (bijective). Justify your answers.

$$a(n) = \begin{cases} 2n & \text{if } n \text{ is odd} \\ 3n & \text{if } n \text{ is even} \end{cases} \qquad b(n) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases} \qquad c(n) = n^2 \qquad d(n) = \lfloor \sqrt{n} \rfloor$$

For d, $\lfloor x \rfloor$ denotes the 'floor' of x, the greatest integer not greater than x. E.g. $\lfloor \sqrt{7} \rfloor = 2$.

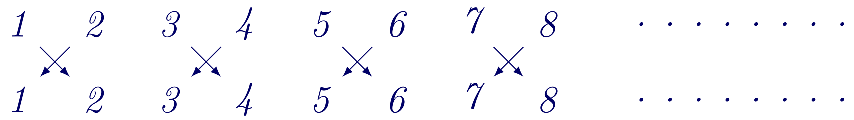
a:



$\boxed{\textit{Not injective}}$ *since* $a(2) = a(3)$.

$\boxed{\textit{Not surjective}}$ *since* $\nexists n \in \mathbb{N} \; a(n) = 1$.

b:

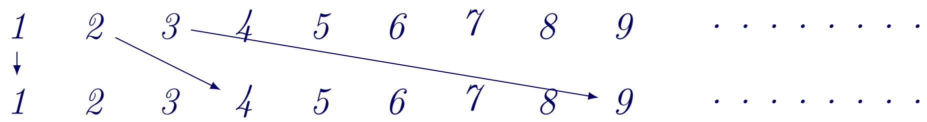


$\boxed{\textit{Injective}}$ *since* $\forall m, n \in \mathbb{N} \; m \neq n \implies b(m) \neq b(n)$.

$\boxed{\textit{Surjective}}$ *since* $\forall n \in \mathbb{N} \; \exists m \in \mathbb{N} \; b(m) = n$.

$\boxed{\textit{...hence Bijective}}$ [$m = n+1$ *if* n *is odd,* $m = n-1$ *if* n *is even.*]

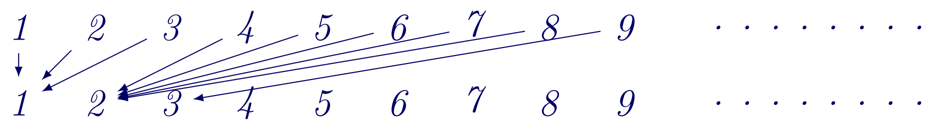
c:



$\boxed{\textit{Injective}}$ *since* $\forall m, n \in \mathbb{N} \; m \neq n \implies c(m) \neq c(n)$.

$\boxed{\textit{Not surjective}}$ *since* $\nexists n \in \mathbb{N} \; c(n) = 2$.

d:



$\boxed{\textit{Not injective}}$ *since* $d(1) = d(2)$.

$\boxed{\textit{Surjective}}$ *since* $\forall n \in \mathbb{N} \; \exists m \in \mathbb{N} \; d(m) = n$ [*namely* $m = n^2$.]

Question 3 Let $A = \{x, y\}$ and let $B = \{4, 7, 9\}$.

1. Use set roster notation to list all possible functions from A to B. Give each of your functions a different name.
2. Use set roster notation to list all possible functions from B to A. Give each of your functions a different name.
3. Which, if any, of your functions from part 1 are injective?
4. Which, if any, of your functions from part 1 are surjective?
5. Which, if any, of your functions from part 2 are injective?
6. Which, if any, of your functions from part 2 are surjective?

1. *The nine functions from A to B are:*

$$f_1 = \{(x,4),(y,4)\} \qquad f_6 = \{(x,7),(y,9)\}$$
$$f_2 = \{(x,4),(y,7)\} \qquad f_7 = \{(x,9),(y,4)\}$$
$$f_3 = \{(x,4),(y,9)\} \qquad f_8 = \{(x,9),(y,7)\}$$
$$f_4 = \{(x,7),(y,4)\} \qquad f_9 = \{(x,9),(y,9)\}$$
$$f_5 = \{(x,7),(y,7)\}$$

2. *The eight functions from B to A are:*

$$g_1 = \{(4,x),(7,x),(9,x)\} \qquad g_5 = \{(4,y),(7,x),(9,x)\}$$
$$g_2 = \{(4,x),(7,x),(9,y)\} \qquad g_6 = \{(4,y),(7,x),(9,y)\}$$
$$g_3 = \{(4,x),(7,y),(9,x)\} \qquad g_7 = \{(4,y),(7,y),(9,x)\}$$
$$g_4 = \{(4,x),(7,y),(9,y)\} \qquad g_8 = \{(4,y),(7,y),(9,y)\}$$

3. *The functions $f_2, f_3, f_4, f_6, f_7, f_8$ are injective.*
4. *None of the functions are surjective.*
5. *None of the functions are injective*
6. *The functions $g_2, g_3, g_4, g_5, g_6, g_7$ are surjective.*

Question 4 Prove or disprove each of the following statements:

1. There exists a bijection from $\mathbb{N}$ to $\mathbb{Z}$.
2. Every function from $\mathbb{Z}$ to $\mathbb{N}$ is surjective.
3. For any nonempty set A, any function from A to A is a bijection.
4. For any nonempty set A, there exists a bijection from A to A.

1. The statement is true. For example, consider the function $f : \mathbb{N} \to \mathbb{Z}$ defined as follows

n	*1*	*2*	*3*	*4*	*5*	...
f(n)	*0*	*1*	*-1*	*2*	*-2*	...

This function is surjective and injective, and hence a bijection.

2. The statement is false. Consider the function $f : \mathbb{Z} \to \mathbb{N}$ defined by $f(z) = 12$. Note that Range $f = \{12\}$. Since Range $f \neq \mathbb{N}$, f is not surjective.

3. This is false. Consider $A = \{1, 2\}$ and the function $f : A \to A$ defined by $f(1) = f(2) = 1$. Then f is a function on A, but f is not surjective.

4. This is true. Let A be a nonempty set. The identity function $Id_A : A \to A$ defined by $Id_A(a) = a$ is a bijection on A.

Question 5 Let $x = 11010_2,\ y = 10111_2,\ s = x + y,\ d = x - y,\ p = xy.$

(a) Calculate s, d and p directly in binary. Keep the answers in binary.

(b) Convert x, y to hexadecimal and recalculate s, d and p directly in hexadecimal.

(c) Finally convert x, y to decimal, recalculate s, d and p (in decimal) and convert the answers to hexadecimal and thence to binary.
Use the results to check your answers to (a) and (b).

(a)

$$\begin{array}{rrrrrr} & 1 & 1 & 0 & 1 & 0 \\ + & 1 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 & 1 \\ \hline {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \end{array} \qquad \begin{array}{rrrrrr} & 1 & 1 & {}^{1}0 & {}^{1}1 & {}^{1}0 \\ - & 1 & 0_1 & 1_1 & 1_1 & 1 \\ \hline & 0 & 0 & 0 & 1 & 1 \\ \hline \end{array}$$

$$\begin{array}{rrrrrrrrrr} & & & & & 1 & 1 & 0 & 1 & 0 \\ & & & & \times & 1 & 0 & 1 & 1 & 1 \\ \hline & & & & & 1 & 1 & 0 & 1 & 0 \\ & & & & 1 & 1 & 0 & 1 & 0 & 0 \\ & & & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ \hline & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 10} & {\scriptstyle 1} & {\scriptstyle 1} & & & & \end{array}$$

(b)

$$x = 11010_2 = \overbrace{0001}^{1}\overbrace{1010_2}^{A} = 1A_{16}$$

$$y = 10111_2 = \overbrace{0001}^{1}\overbrace{0111_2}^{7} = 17_{16}$$

$$\begin{array}{rrr} & 1 & A \\ + & 1 & 7 \\ \hline & 3 & 1 \\ \hline & {\scriptstyle 1} & \end{array} \qquad \begin{array}{rrr} & 1 & A \\ - & 1 & 7 \\ \hline & & 3 \\ \hline \end{array} \qquad \begin{array}{rrr} & 1 & A \\ \times & 1 & 7 \\ \hline & B & 6 \\ & {\scriptstyle (4)} & \\ 1 & A & 0 \\ \hline 2 & 5 & 6 \\ \hline {\scriptstyle 1} & & \end{array}$$

(c)

$$x = 1A_{16} = (16 + 10)_{10} = 26_{10}$$
$$y = 17_{16} = (16 + 7)_{10} = 23_{10}$$

$$\begin{array}{rrr} & 2 & 6 \\ + & 2 & 3 \\ \hline & 4 & 9 \\ \hline \end{array} \qquad \begin{array}{rrr} & 2 & 6 \\ - & 2 & 3 \\ \hline & & 3 \\ \hline \end{array} \qquad \begin{array}{rrr} & 2 & 6 \\ \times & 2 & 3 \\ \hline & 7 & 8 \\ & {\scriptstyle (1)} & \\ 5 & 2 & 0 \\ {\scriptstyle (1)} & & \\ \hline 5 & 9 & 8 \\ \hline \end{array}$$

$$49_{10} = (3 \times 16 + 1)_{10} = 31_{16}\ \checkmark = \underbrace{3}_{0011}\ \underbrace{1}_{0001} = 110001_2.\ \checkmark$$

$$3_{10} = 3_{16}\ \checkmark = 11_2.\ \checkmark$$

$$\begin{aligned} 598_{10} = (512 + 86)_{10} &= (2 \times 256 + 80 + 6)_{10} \\ &= (2 \times 16^2 + 5 \times 16 + 6)_{10} \\ &= 256_{16}\ \checkmark = \underbrace{2}_{0010}\ \underbrace{5}_{0101}\ \underbrace{6}_{0110} = 1001010110_2.\checkmark \end{aligned}$$

Question 6 This question is about the toggle-plus-one method as it relates to the storage of integers in computer words. It also shows how this method avoids the need for separate subtraction circuits.

As demonstrated in lectures, for a binary word W, toggle-plus-one means:

toggle:	replace every 1 by 0, every 0 by 1, then
add one:	treating W as a binary number, add 1.
	Ignore any carry beyond the length of the word.

For $l \in \mathbb{N}$ let $S_l = \{n \in \mathbb{Z} : -2^{l-1} \leq n < 2^{l-1}\}$. Then S_l is the set of all integers that can be stored in words of length l bits, using the standard computer representation. The rules governing the storage of an integer n in a word W may be summarised as:

Rule 1:	n is negative if and only the left-most bit of W is 1
Rule 2:	$-n$ is stored as the word obtained from W by toggle-plus-one.
	This is true even when n is negative.
Rule 3:	If the left-most bit of W is 0 than $n = W_2$.
	i.e. n is retrieved by treating W as a binary number.

In computer arithmetic, subtraction uses negation and addition: $x - y = x + (-y)$.
In the addition, any carry beyond the length of the word is ignored.
As an example, take $x = 11010_2$, $y = 10111_2$ and use 8-bit computer arithmetic on:

(a) $x - y$ (b) $y - x$ (c) $-x - y$

Check your results by expressing x, y and your answers in decimal.

$x = 11010_2 \longrightarrow$ 00011010 $\qquad x = 16 + 8 + 2 = 26_{10}$.

$-x \longrightarrow$ 11100101
$\qquad + 1$
$\boxed{11100110}$

$y = 10111_2 \longrightarrow$ 00010111 $\qquad y = 16 + 4 + 2 + 1 = 23_{10}$.

$-y \longrightarrow$ 11101000
$\qquad + 1$
$\boxed{11101001}$

(a)

$x :$ 00011010
$+(-y) :$ 11101001
(1)$\boxed{00000011}$
1 1 1 1

$\longrightarrow 3_{10} = 26 - 23\checkmark$

(b)

$y :$ 00010111
$+(-x) :$ 11100110
$\boxed{11111101}$

00000010
+1
00000011

$\rightarrow -3_{10} = 23 - 26\checkmark$

(c)

$-x :$ 11100110
$+(-y) :$ 11101001
(1)$\boxed{11001111}$

00110000
+1
00110001

$-49_{10} = -26 - 23\checkmark$

Question 7 Let A, B, C be nonempty sets and let $f : A \to B$ and $g : B \to C$ be functions. Prove or disprove each of the following statements:

1. If f and g are injective, then $g \circ f$ is injective.
2. If f is injective and g is surjective, then $g \circ f$ is surjective.

1. *The statement is true. Suppose that f and g are injective. Let $a_1, a_2 \in A$. Suppose that $(g \circ f)(a_1) = (g \circ f)(a_2)$. By the definition of composition, we have $g(f(a_1)) = g(f(a_2))$. Since g is injective and $g(f(a_1)) = g(f(a_2))$, we have $f(a_1) = f(a_2)$. Since f is injective and $f(a_1) = f(a_2)$, we have $a_1 = a_2$.*
2. *The statement is false. Consider $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and $C = \{1, 2, 3\}$. Let $f : A \to B$ be defined by $f(a) = a$, and let $g : B \to C$ be defined by $g(b) = b$. Then f is injective, g is surjective, but $g \circ f$ is not surjective (the range of $g \circ f$ is $\{1, 2\}$, which is not equal to C).*

Question 8 Let $Q = \mathbb{Z} \times (\mathbb{Z} \setminus 0)$. Let $\sim$ be the relation on Q defined by

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc.$$

1. Use set roster notation to describe the set

$$F_{(2,3)} = \{(x, y) \in Q \mid (x, y) \sim (2, 3)\}$$

2. Use set roster notation to describe the set

$$F_{(-3,6)} = \{(x, y) \in Q \mid (x, y) \sim (-3, 6)\}$$

3. You have encountered the relation $\sim$ before, although you probably did not use the notation of relations. In what context have you seen this relation?

1. $F_{(2,3)} = \{(2, 3), (-2, -3), (4, 6), (-4, -6), (6, 9), (-6, -9), \ldots\}$

2. $F_{(-3,6)} = \{(1, -2), (-1, 2), (2, -4), (-2, 4), (3, -6), (-3, 6), \ldots\}$

3. If we think of a fraction $\frac{a}{b}$ as corresponding to an ordered pair (a, b), then every fraction corresponds to an element of Q. The relation $\sim$ tells us when two fractions determine the same rational number.