Throughout this exam, and as we did throughout the course, we write N for the set of positive integers.

Problem 1 (10 marks) (a) Use a truth table to prove or disprove the following logical equivalence:

$$((p \wedge q) \to r) \equiv (p \to (q \to r))$$

p	q	r	$(p \wedge q)$	$\to r$	p	$\to$	$(q \to r)$
T	T	T		T	T	T	T
T	T	F		F	T	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	F	T	F	T	
F	T	F	F	T	F	T	
F	F	T	F	T	F	T	
F	F	F	F	T	F	T	

The logical equivalence holds since the truth tables agree

(b) Let $\mathcal{X}$ denote the set of all finite graphs. Consider the following statement:

For all finite graphs G, if G is planar and has no loops, then the chromatic number of G is at most 4.

Write down the structural part of a proof that proceeds via the contrapositive.

Please note, some of the terms in this statement were not introduced in our course. You do not need to know what they mean in order to complete the question.

Let $G \in \mathcal{X}$. Suppose that the chromatic number of G is greater than 4

⋮

Hence G is not planar or G has loops $\square$

(c) Let $U = \{x, y, z\}$ and $S = \{(a, W) \in U \times \mathcal{P}(U) \mid a \notin W\}$. Use set-roster notation to describe S.

$$S = \{(x,\emptyset), (x,\{y\}), (x,\{z\}), (x,\{y,z\}), (y,\emptyset), (y,\{x\}), (y,\{z\}), (y,\{x,z\}), (z,\emptyset), (z,\{x\}), (z,\{y\}), (z,\{x,y\})\}$$

(d) Consider the following statement:

For any universal set U and for any $A, B, C \in \mathcal{P}(U)$, we have $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

Either: provide a counterexample to disprove the statement; or use an element proof, the definition of set operations and relations, and any of the logical equivalences below you may need to prove the statement.

(Use two Venn diagrams to decide that the statement is true, and then ...)

The statement is true.

Proof. Let U be a universal set.

Let $A, B, C \in \mathcal{P}(U)$. Let $x \in U$. Then

$$
\begin{aligned}
& x \in (A \setminus B) \setminus C & \\
\Leftrightarrow\ & x \in A \setminus B \wedge \neg(x \in C) & \text{(defn of } \setminus) \\
\Leftrightarrow\ & (x \in A \wedge \neg(x \in B)) \wedge \neg(x \in C) & \text{(defn of } \setminus) \\
\Leftrightarrow\ & x \in A \wedge (\neg(x \in B) \wedge \neg(x \in C)) & \text{(Associative Law)} \\
\Leftrightarrow\ & x \in A \wedge \neg(x \in B \vee x \in C) & \text{(De Morgan's Law)} \\
\Leftrightarrow\ & x \in A \wedge \neg(x \in B \cup C) & \text{(defn of } \cup) \\
\Leftrightarrow\ & x \in A \setminus (B \cup C) & \text{(defn of } \setminus) \quad \square
\end{aligned}
$$

Given any statement variables p, q, and r, a tautology t and a contradiction c, the following logical equivalences hold.

1. Commutative laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Identity laws:	$p \wedge t \equiv p$	$p \vee c \equiv p$
5. Negation laws:	$p \vee \neg p \equiv t$	$p \wedge \neg p \equiv c$
6. Double negative law:	$\neg(\neg p) \equiv p$	
7. Idempotent laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. Universal bound laws:	$p \vee t \equiv t$	$p \wedge c \equiv c$
9. De Morgan's laws:	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
10. Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. Negations of t and c:	$\neg t \equiv c$	$\neg c \equiv t$

(e) Consider the following statement:

For any universal set U and for any $A, B \in \mathcal{P}(U)$, we have $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

Either: provide a counterexample to disprove the statement; or use an element proof, the definition of set operations and relations, and any of the logical equivalences below you may need to prove the statement.

The statement is false, as shown by the following.

Let $U = \{a, b\}$ and $A = \{a\}$ and $B = \{b\}$. Then

$\mathcal{P}(A \cup B) = \mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

and

$\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(\{a\}) \cup \mathcal{P}(\{b\}) = \{\emptyset, \{a\}\} \cup \{\emptyset, \{b\}\}$
$= \{\emptyset, \{a\}, \{b\}\}$

Since $\{a, b\} \in \mathcal{P}(A \cup B)$ and $\{a, b\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$

$\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$

Given any statement variables p, q, and r, a tautology t and a contradiction c, the following logical equivalences hold.

1. Commutative laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Identity laws:	$p \wedge t \equiv p$	$p \vee c \equiv p$
5. Negation laws:	$p \vee \neg p \equiv t$	$p \wedge \neg p \equiv c$
6. Double negative law:	$\neg(\neg p) \equiv p$	
7. Idempotent laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. Universal bound laws:	$p \vee t \equiv t$	$p \wedge c \equiv c$
9. De Morgan's laws:	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
10. Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. Negations of t and c:	$\neg t \equiv c$	$\neg c \equiv t$

Problem 2 (10 marks) (a) Use induction to prove the following:

For all $n \in \mathbb{N}$, n is a non-negative integer power of 2 or n can be written as a sum of distinct non-negative integer powers of 2.

Let $P(n)$: n is a non-negative integer power of 2 or n can be written as a sum of distinct non-negative integer powers of 2.

We shall use induction to prove $P(n)$ for all $n \in \mathbb{N}$.

Base Step: Since $1 = 2^0$, $P(1)$ holds.

Induction Step: Let $n \in \mathbb{N}$. Suppose that $P(1), P(2), \ldots, P(n)$ all hold. We consider $P(n+1)$.

We consider cases

Case $n+1$ is a non-negative integer power of 2: In this case $P(n+1)$ holds immediately.

Case $n+1$ is not a non-negative integer power of 2: Let i be the largest integer such that $2^i < n+1$. It follows that $2^i < n+1 < 2 \times 2^i$ (otherwise i is not the largest). It follows in turn that $0 < (n+1) - 2^i < 2^i$. Let $m = (n+1) - 2^i$. By $P(m)$, $m = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_k}$ for some distinct $i_1, i_2, \ldots, i_k \in \{0, 1, \ldots, i-1\}$. Note $n+1 = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_k} + 2^i$.

Hence $P(n+1)$ holds. By the Principle of mathematical induction, $P(n)$ holds for all $n \in \mathbb{N}$.

Note: By completing the problem above, you have proved that every positive integer can be represented by a bit string using binary positional notation.

(b) A certain bank has 31,700 customers. Each customer at the bank has chosen a 4-digit PIN to identify themselves at the ATM. Examples of PINs are 0678, 2189, 9834. Prove that there is at least one PIN shared by four or more customers.

We apply the generalised pigeon-hole principle. Customers are like pigeons and PINs are like pigeonholes. There are 10000 four digits numbers, so 10000 PINs. By the generalised PHP, at least one PIN (pigeonhole) is occupied by $\lceil \frac{31700}{10000} \rceil = \lceil 3.17 \rceil = 4$ customers (pigeons)

(c) A FAW is a four letter 'word' whose letters are in alphabetical order. Letters are drawn from the standard lower case English 26-letter alphabet, and are allowed to repeat. A 'word' does not have to appear in any dictionary. Examples of FAWs are abcd, ccyz, aooy and yyyy. A FAW is said to contain a repeated letter if there is at least one letter which appears more than once in the word.

If a FAW is chosen at random, what is the probability that it contains a repeated letter? You may give you answer in the form of a mathematical expression (you do not have to compute a final number)

The experiment is to choose a FAW at random. An outcome is represented by the FAW chosen. The sample space S is therefore the set of all the FAWs. Let E denote the event that the FAW chosen has at least one repeated letter. Since the FAW is chosen at random, outcomes are equally likely and $\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{|S| - |E^c|}{|S|}$

Each FAW may be represented by an arrangement of 25 bars and four stars - the 25 bars separate the line into 26 regions, one for each letter, and the stars indicate how many letters from each region are chosen. There are $\binom{25+4}{4} = \binom{29}{4}$ ways to arrange 25 bars and 4 stars, so $|S| = \binom{29}{4}$. A FAW in E^c has 4 different letters so choosing a FAW from E^c is like choosing a set of size 4 from the alphabet. Hence $|E^c| = \binom{26}{4}$. Hence $\mathbb{P}(E) = \frac{\binom{29}{4} - \binom{26}{4}}{\binom{29}{4}}$

(d) A four-sided die is a tetrahedron with faces marked 1, 2, 3 and 4. When a four-sided die is thrown, one face is not visible—the number on the face that is not visible is said to be the 'number thrown.' Consider a probability experiment in which we toss a red four-sided die and a blue four-sided die simultaneously.

(i) Describe how we may represent outcomes so that outcomes are equally likely, and then use set-roster notation to describe the sample space of the experiment.

An outcome may be represented as an ordered pair of integers; the first component records the number thrown on the red die and the second component records the number thrown on the blue die. So

$$S = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$$

(ii) Let T be the random variable on S which assigns to each outcome the absolute value of the difference between the numbers thrown on the two dice. Let $E_1 = \{s \in S \mid T(s) > 1\}$ and let E_2 be the event that a 2 is thrown on the red die. Are the events E_1 and E_2 independent?

Since outcomes are equally likely,

$$\mathbb{P}(E) = \frac{|E|}{|S|} \text{ for any event } E.$$

We compute

$$\mathbb{P}(E_1) = \frac{|\{(1,3),(1,4),(2,4),(3,1),(4,1),(4,2)\}|}{|S|} = \frac{6}{16} = \frac{3}{8}$$

and

$$\mathbb{P}(E_2) = \frac{|\{(2,1),(2,2),(2,3),(2,4)\}|}{|S|} = \frac{4}{16} = \frac{1}{4}$$

and

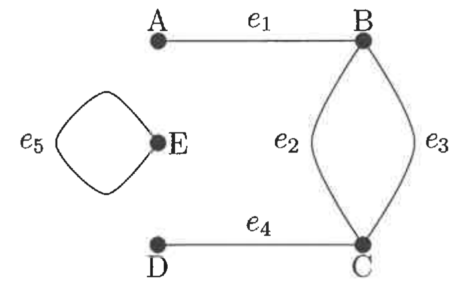
$$\mathbb{P}(E_1 \cap E_2) = \frac{|\{(2,4)\}|}{|S|} = \frac{1}{16}$$

Since $\mathbb{P}(E_1) \times \mathbb{P}(E_2) = \frac{3}{8} \times \frac{1}{4} = \frac{3}{32} \neq \frac{1}{16} = \mathbb{P}(E_1 \cap E_2)$,

the events are <u>not</u> independent

Problem 3 (10 marks) Enumerative Combinatorics and Graph Theory

(a) Let G be the graph shown below.



(i) How many different walks in G start at A and end at D? If possible, list them all.

There are infinitely many such walks, so we cannot list them all

(ii) How many different walks in G are paths which start at A and end at D? If possible, list them all. There are two.

- $A e_1 B e_2 C e_4 D$
- $A e_1 B e_3 C e_4 D$

(iii) How many different walks in G are circuits (including the trivial circuits)? If possible, list them all. There are ten

- A
- B
- C
- D
- E
- $B e_2 C e_3 B$
- $B e_3 C e_2 B$
- $C e_2 B e_3 C$
- $C e_3 B e_2 C$
- $E e_5 E$

(b) Hypercubes were introduced in the course as a family of graphs which may represent a good solution to which problem?

How may we connect CPUs into a network with good properties for parallel computing

(c) (i) If G and H are graphs, what is an isomorphism between G and H?

An isomorphism between G and H is a bijection $f: V(G) \to V(H)$ such that for all $u_1, u_2 \in V(G)$, $\{u_1, u_2\}$ appears in $E(G)$ exactly as many times as $\{f(u_1), f(u_2)\}$ appears in $E(H)$

(ii) Let $m, n \in \mathbb{N}$ be such that $m \neq n$, and let G be a graph that is isomorphic to the complete bipartite graph $K_{m,n}$. How many different isomorphisms are there between G and $K_{m,n}$? Justify your answer.

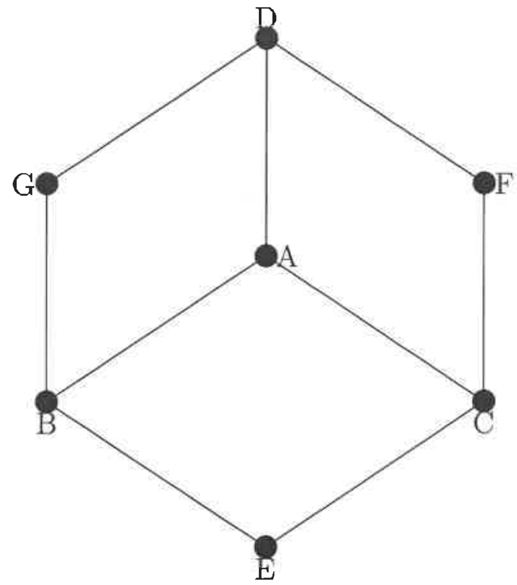
The vertex set of $K_{m,n}$ may be partitioned $\{A, B\}$ such that $|A| = m$ and $|B| = n$, and every vertex in A is adjacent to every vertex in B

Since G is isomorphic to $K_{m,n}$, the vertex set of G may be partitioned $\{X, Y\}$ such that $|X| = m$ and $|Y| = n$ and every vertex in X is adjacent to every vertex in Y.

Any isomorphism between G and $K_{m,n}$ must map vertices in X to vertices in A, and vertices in Y to vertices in B. We may construct an isomorphism between G and $K_{m,n}$ by choosing a bijection from X to A (there are $m!$ ways to do this) and then choosing a bijection from Y to B (there are $n!$ ways to do this)

Hence there are $m! \cdot n!$ different isomorphisms between G and $K_{m,n}$.

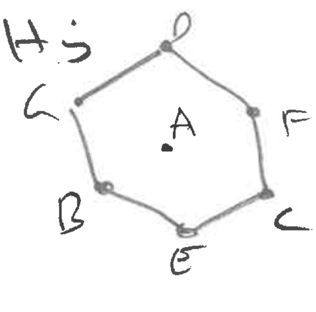
(d) Let W be the graph shown below.



Prove or disprove the following: W has a Hamilton circuit.

The statement is false. Suppose W has a Hamilton circuit. Let H be the subgraph of W which contains every vertex in W and exactly those edges traversed by the Hamilton circuit. We note that H is connected and every vertex in H has degree two (in H).

Since the circuit must visit F, H must contain the edges $\{D,F\}$ and $\{F,C\}$. Similarly since the circuit must visit E and G, H must contain $\{B,E\}$, $\{E,C\}$, $\{B,G\}$ and $\{G,D\}$. Since B, C, D have degree two in H, H may not contain $\{A,B\}$, $\{A,C\}$, $\{A,D\}$. So H is as shown, contradicting the fact that H is connected.



Problem 4 (10 marks) (a) Describe the input and output to the Nearest Neighbour algorithm.

INPUT: Weighted Complete graph G with n vertices

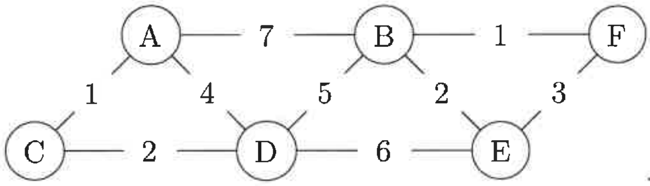
OUTPUT: Hamilton Circuit for G as a list L of vertices

Total weight W of this circuit

(b) Let G be a connected graph. State a condition on G that is both necessary and sufficient for Fleury's algorithm to be successful at identifying an Euler path in G.

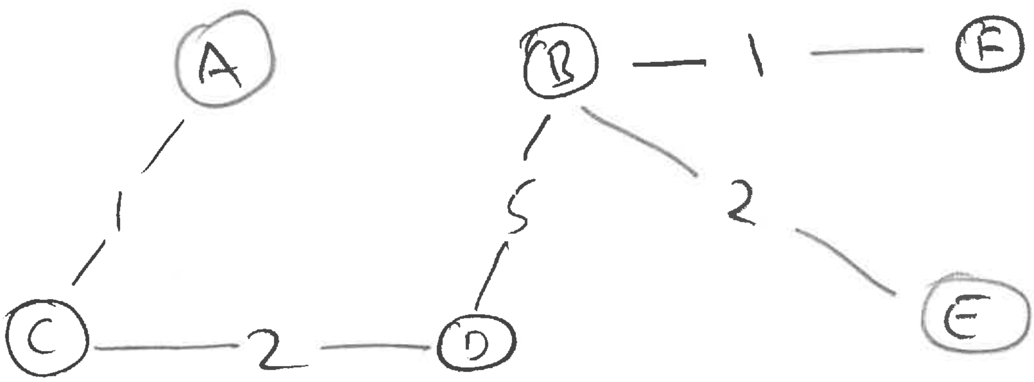
Every vertex of G has even degree.

(c) Let G be the weighted graph

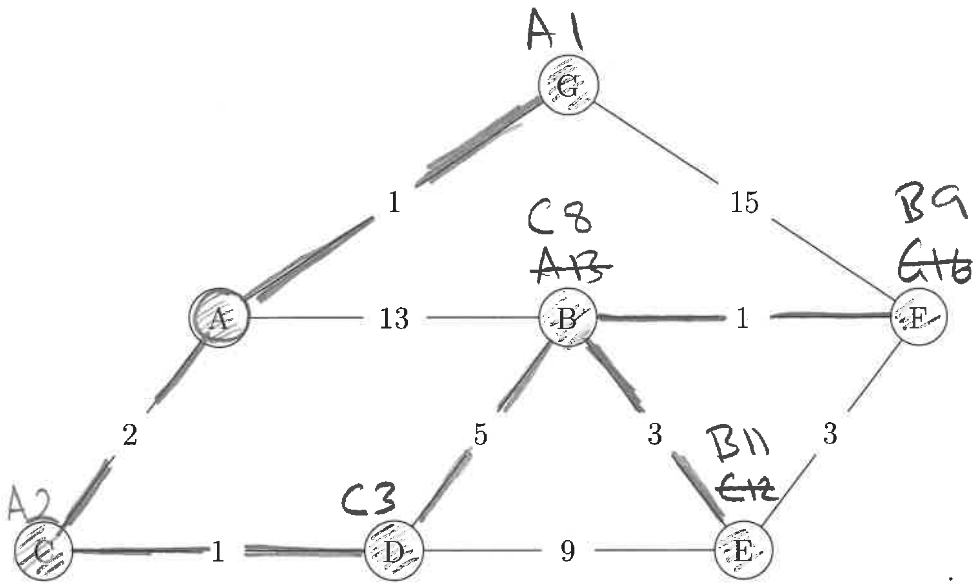


Use one of the algorithms discussed in the course to find a minimal spanning tree for G. Write the name of the algorithm, and draw the minimal spanning tree it produces, in the space below.

Kruskal's algorithm yields the following



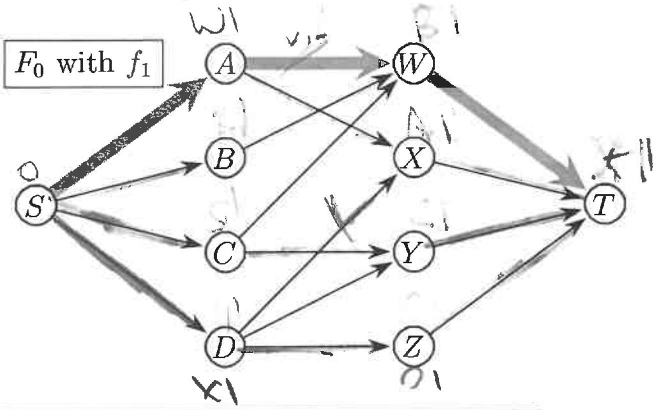
(d) Let H be the weighted graph



Use Dijkstra's algorithm to find the shortest path in H from A to E (pseudocode for the Dijkstra's algorithm is given at the end of the exam paper). To show that you applied the algorithm correctly, you should complete the table below

The vertices are 'locked in' in the following order	A, G, C, D, B, F, E
The shortest path found is	A, C, D, B, E

(e) A matching problem has been turned into a maximum flow problem for a transport network using the method described in the course. The resulting directed graph is shown below. Use the vertex labelling algorithm described in the course to solve the matching problem (pseudocode for the vertex-labeling algorithm is given at the end of the exam paper). The first incremental flow is shown on the diagram below and recorded in the first table below. Record each subsequent incremental flow in the first table below (use only as many rows as you need), and record the final matching in the second table.



working not shown

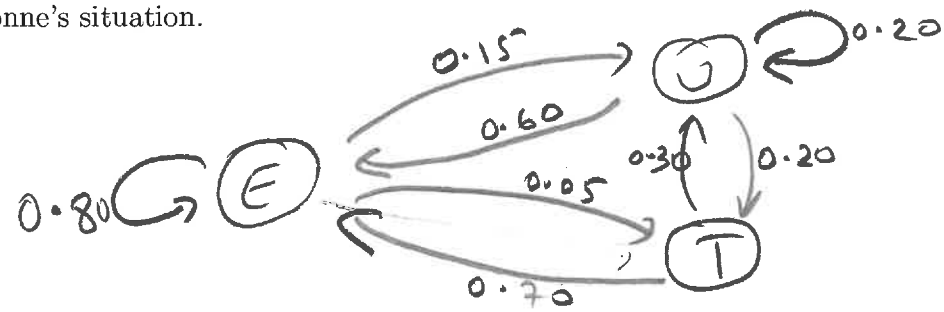
incremental flow label	path of incremental flow
f_1	SAWT
f_2	SDXT
f_3	SCYT
f_4	SBWAXDZT

vertex	The vertex, if any, with which it is matched
A	X
B	W
C	Y
D	Z

Problem 5 (10 marks) (a) A freelance computer network consultant, let's call her Yvonne, is employed in weekly contracts. Each week she is either: employed (E), unemployed (U) or training in new technology (T). Yvonne's records support the following assumptions:

- If she's employed this week, then next week she'll be employed with probability 0.80, and training in new technology with probability 0.05.
- If she's unemployed this week, then next week she'll be employed with probability 0.60 and training in new technology with probability 0.20.
- If she is training in new technology this week, then next week she'll be employed with probability 0.70.
- She never trains in new technology for two consecutive weeks.

We can model Yvonne's situation by a Markov process. Make a transition diagram to model Yvonne's situation.



(arrows leaving a state have labels that sum to 1)

(b) A corporation, BIG CORP INDUSTRIES, seeks to understand which employees perform tasks that are the most important to the corporation's operation. Every employee has submitted a response to the following survey question: "List the names of colleagues whose work is important to yours."

A 'PageRank-like approach' will be used to rank the importance of employees from this data.

(i) Describe a directed graph (What is the set of vertices? When is there an edge from one vertex to another?) related to the survey data that may play the role of a 'webgraph' in a PageRank-like approach to ranking the importance of employees.

- One vertex for each employee.
- A directed edge from employee X to employee Y if employee X listed employee Y on the survey

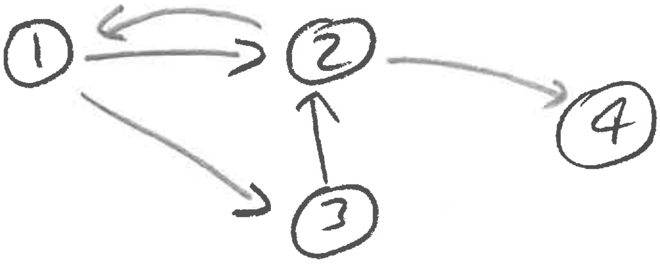
(ii) State at least two hypotheses concerning the data and the importance of employees that, if assumed true, would justify the claim that a PageRank-like approach will be an effective way to rank the importance of employees.

- Each time employee X is listed on a survey, is saying that X is a bit important
- Employee X being listed on an important employee's survey is saying more than employee X being listed on a less important employee's survey

(c) Let G be the webgraph with the adjacency matrix A shown below, and suppose that we are using the page rank algorithm with a damping factor of 80% (0.80) to rank the pages in G.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) Draw a picture of G.



(ii) Write down the basic transition matrix T associated to G.

$$T = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

(iii) The modified transition matrix M associated to G is $M = \begin{bmatrix} \frac{3}{60} & \frac{27}{60} & \frac{27}{60} & \frac{3}{60} \\ \frac{27}{60} & \frac{3}{60} & \frac{3}{60} & \frac{27}{60} \\ \frac{3}{60} & \frac{51}{60} & \frac{3}{60} & \frac{3}{60} \\ \frac{19}{60} & \frac{19}{60} & \frac{19}{60} & \frac{3}{60} \end{bmatrix}$.

Your friend says the following: "The page rank vector associated to G is $\mathbf{v} = \begin{bmatrix} \frac{3}{10} \\ \frac{5}{10} \\ \frac{1}{10} \\ \frac{1}{10} \end{bmatrix}$."

In no more than three sentences, explain how you could determine whether or not your friend is correct.

$\bar{v}'$ is the page rank vector if and only if $M'\bar{v}' = \bar{v}'$. I could simply compute $M'\bar{v}'$ and compare it to $\bar{v}'$.