

1 Recap: Weighted graphs: Traveling salesman problem  
Shortest path ~ Dijkstra's algorithm  
Minimal spanning trees  
Transport networks (max. flow  
min. "cut")

### D3. Random walks on graphs.

Notes by Malcolm Brooks,  
expanded from notes of Pierre Portal  
with influences from Judy-anne Osborn .

Unfortunately, Random walks are not covered in our text by Epp, nor in the books by Johnsonbaugh or Kolman et. al.

Announcements: Poll to appear on Wattle to select  
review lecture topics. (for next week)

# Announcements

- Revision topics poll Watthe

- In Announcement Forum: post regarding exam revision

## Random walks: idea

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At time  $k$  you are at a vertex  $t$  and there is a walk of length  $k$  from  $x$  to  $t$ .

Before each step, you choose where to go next probabilistically :

If you are at a vertex  $i$  you go to vertex  $j$  with probability  $p_{ij}$ .

[If  $(i, j) \notin E$ , then, of course,  $p_{ij} = 0$ .]



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Associated with an  $n$ -vertex directed graph  $G$ , let  $T = (p_{ij})_{1 \leq i, j \leq n}$  be an  $n \times n$  stochastic matrix satisfying  $p_{ij} = 0 \ \forall (i, j) \notin E(G)$ .

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For any given  $n$  let  $B_n$  denote the set of **basis vectors**  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  where  $\mathbf{e}_i$  is the  $n \times 1$  vector with 1 as the  $i$ -th entry (*i.e.* in row  $i$ ) and all other entries zero. *E.g.*, for  $n = 3$ :  $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

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Then  $X_k = (T')^k \mathbf{e}_i = (q_j)_{1 \leq j \leq n}$  say gives, for  $1 \leq j \leq n$ , the probability  $q_j$  of being at the vertex  $j$  after  $k$  steps, starting from vertex  $i$ .

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approaches  $S$  as  $N$  gets large,

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↑  
average  
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↑  
state at time  $k$ .

i.e.  $T^k X \rightarrow S$   
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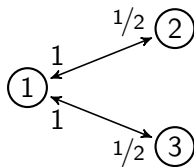
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**Idea for the Google algorithm:** If  $G$  is the Web graph, then, for some suitable transition matrix  $T$ ,  $q_j$  is the relative importance of the page  $j$ . We will explore this idea later, but first some examples of random walks.

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Consider a graph  $G$  with adjacency matrix  $A$  and a random walk on  $G$  with transition matrix  $T$ , where

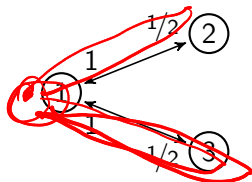
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad T = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



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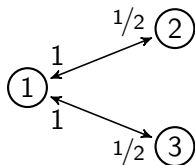


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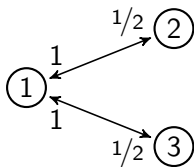
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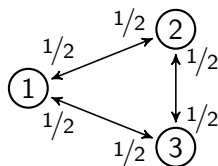
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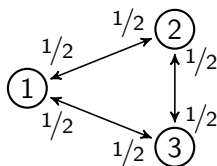


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We see that no one vertex is favoured over any other.

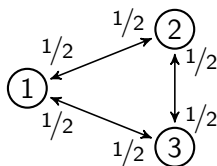
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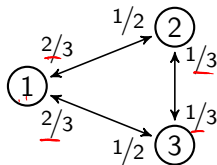
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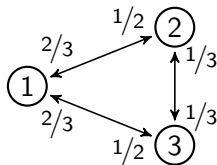
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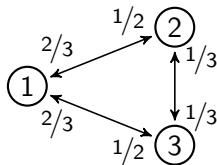


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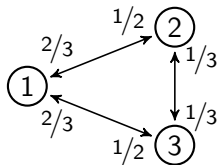
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But what should the probabilities  $p$  and  $q$  be?

Can you guess?

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The steady state vector is  $S = \underbrace{\begin{bmatrix} 4/10 \\ 3/10 \\ 3/10 \end{bmatrix}}$ . As you can check,  $T'S = S$ .

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**PageRank** is a link analysis algorithm, named after Larry Page, co-founder of Google, used by the Google Internet search engine.





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**PageRank** is a link analysis algorithm, named after Larry Page, co-founder of Google, used by the Google Internet search engine. It assigns a numerical weighting to each element of a hyperlinked set of documents with the purpose of “measuring” its relative importance within the set. The numerical weight that it assigns to any given element E is referred to as the PageRank of E.



## The Webgraph and PageRank

The **webgraph** describes the directed links between pages of the World Wide Web. The webgraph is a directed graph, whose vertices correspond to the pages of the WWW, and a directed edge connects page X to page Y if there exists a hyperlink on page X, referring to page Y.

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The name “PageRank” is a trademark of Google, and the PageRank process has been patented (U.S. Patent 6,285,999).

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## Rationale of PageRank

The idea is that the importance of a page is indicated by the number of links to it, modified by the idea that links from important pages should carry more weight than links from less important pages.

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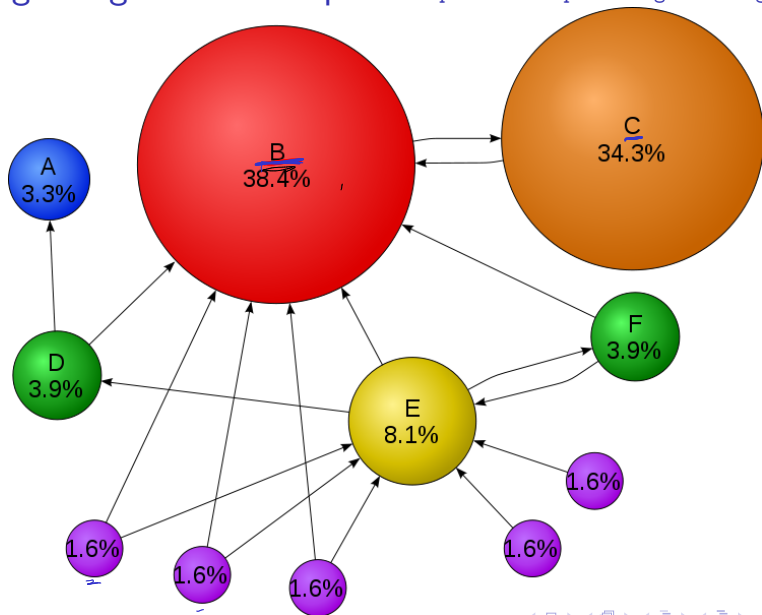
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In the following diagram the sizes of the vertices indicate their importance, as calculated by the PageRank algorithm. The only input to the algorithm was the digraph itself plus a 'damping' factor of 85%, to be discussed later.

# Google PageRank Example

<http://en.wikipedia.org/wiki/PageRank>





## Basic transition probabilities

The PageRank algorithm assumes that our random walker (random *surfer* !) is equally likely to follow any link on a page, and, if there are no links, is equally likely to 'teleport' to any other page on the web.

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Formally, for any webgraph  $G$  let  $n$  be the number of vertices (pages) and for each vertex  $i$  let  $n_i$  be the number vertices to which  $i$  is linked:

$$n = |V(G)|$$

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Then the basic probability  $p_{ij}$  of a transition from vertex  $i$  to  $j$  is given by

$$p_{ij} = \begin{cases} 1/n_i & \text{if } n_i \neq 0 \text{ and } (i, j) \in E(G) \quad \leftarrow \\ 1/(n-1) & \text{if } n_i = 0 \text{ and } i \neq j \quad (\text{but see footnote})^1 \quad \leftarrow \\ 0 & \text{otherwise} \end{cases}$$

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<sup>1</sup>When  $n$  is large (as in the WWW) this line can be simplified to:  $1/n$  if  $n_i = 0$ . 

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The basic transition matrix is  $T = (p_{ij})_{1 \leq i,j \leq n}$ .

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## Example 4A

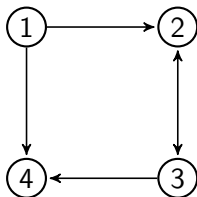
Here is a tiny example of basic transition probabilities - with just  $n = 4$  vertices :

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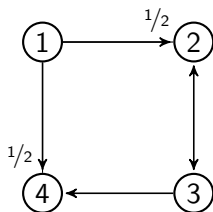
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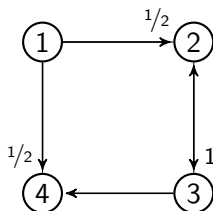
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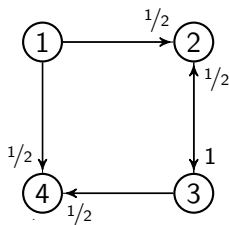




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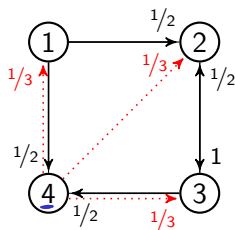
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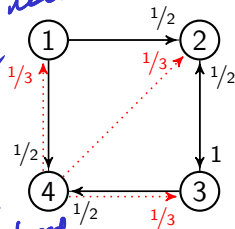


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needs to be replaced.

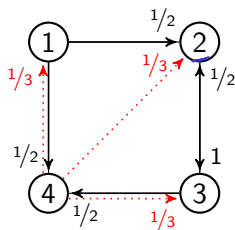


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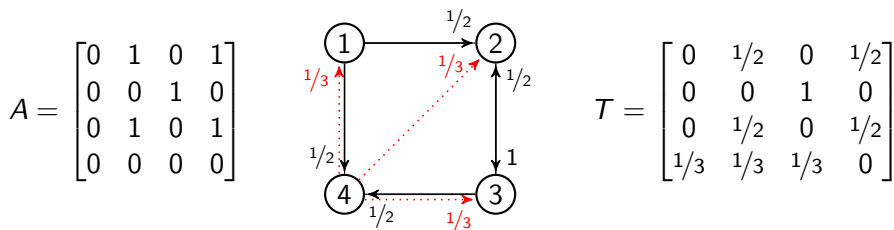


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Solving, by computer,  $(T' - I)S = 0$  with the usual replacement last equation, gives steady state solution  $S = \frac{1}{13} \begin{bmatrix} 1 \\ 4 \\ 5 \\ 3 \end{bmatrix} \approx \begin{bmatrix} .08 \\ .31 \\ .38 \\ .23 \end{bmatrix}$ .

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So on this basis, vertex 3 is most important and vertex 1 least.

## The damping factor $(1 - \alpha)$

The PageRank algorithm assumes that, at any time  $k$ , there is a small probability  $\alpha$  that, irrespective of what links are available at the current page, the surfer chooses to teleport randomly to any page on the web; *i.e.* the surfer acts as if there were *no* links from the current page.

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Thus the modified probability for transition from vertex  $i$  to  $j$  is

$$m_{ij} = \underbrace{\alpha/n}_{\text{random jump}} + (1 - \alpha) \underbrace{p_{ij}}_{\text{follow hyperlinks.}}$$

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$$m_{ij} = \alpha/n + (1 - \alpha)p_{ij}.$$

In practice, Google uses a damping factor of 85%, *i.e.*  $\alpha = 0.15$ .

## The modified transition matrix $M$ and PageRank vector $\mathcal{PR}$

The modified transition probabilities lead to a modified transition matrix

$$M = (m_{ij})_{1 \leq i, j \leq n} = \left( \underbrace{\alpha/n} + \underbrace{(1 - \alpha)p_{ij}} \right)_{1 \leq i, j \leq n}$$

## The modified transition matrix $M$ and PageRank vector $\mathcal{PR}$

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$$\begin{aligned} M = (m_{ij})_{1 \leq i, j \leq n} &= (\alpha/n \mathbf{1} + (1 - \alpha)p_{ij})_{1 \leq i, j \leq n} \\ &= (\alpha/n) \mathbf{U} + (1 - \alpha) \mathbf{T}, \end{aligned}$$

where  $\mathbf{U}$  is the  $n \times n$  all-1's matrix and  $\mathbf{T}$  is the basic transition matrix.

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As indicated earlier, the PageRank algorithm defines the rank of page  $i$  of the webgraph to be the  $i$ -th entry in the **PageRank vector**  $\mathcal{PR}$ , which in turn is defined as the steady state vector for the random walk on the webgraph with transition matrix  $M$ .

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Thus  $PR$  is defined as the probability vector solution to the equation

$$M'PR = PR.$$

and  $PR$  sums to 1, is non-negative.

## Calculating $PR$

Expanding the defining equation  $M'PR = PR$  gives

$$\begin{aligned} PR &= ((\alpha/n)U + (1 - \alpha)T')PR && \text{since } U' = U \\ &= (\alpha/n)\underline{UPR} + (1 - \alpha)T'PR && (**) \end{aligned}$$

$$U'PR = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} p_1 + \dots + p_n \\ \vdots \\ p_1 + \dots + p_n \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

## Calculating $\mathcal{R}$

Expanding the defining equation  $M'\mathcal{R} = \mathcal{R}$  gives

$$\begin{aligned}\mathcal{R} &= ((\alpha/n)U + (1 - \alpha)T')\mathcal{R} && \text{since } U' = U \\ &= (\alpha/n)U\mathcal{R} + (1 - \alpha)T'\mathcal{R} && (\star\star)\end{aligned}$$

Now each entry of the product  $U\mathcal{R}$  is the sum of all the entries in  $\mathcal{R}$ , and since  $\mathcal{R}$  is a probability vector that sum is 1. Hence  $U\mathcal{R} = \mathbf{1}$  where  $\mathbf{1}$  is the  $n \times 1$  vector of all 1's.



## Calculating $\underline{PR}$

Expanding the defining equation  $M'\underline{PR} = \underline{PR}$  gives

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$$\boxed{(I - (1-\alpha)T')\underline{PR} = (\alpha/n)\mathbf{1}}$$

(subtract  $(1-\alpha)T'\underline{PR}$  from both sides)

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For small to moderate  $n$  this equation can be solved directly (by computer).

## Calculating $\overline{PR}$

Expanding the defining equation  $M'\overline{PR} = \overline{PR}$  gives

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Now each entry of the product  $U\overline{PR}$  is the sum of all the entries in  $\overline{PR}$ , and since  $\overline{PR}$  is a probability vector that sum is 1. Hence  $U\overline{PR} = \mathbf{1}$  where  $\mathbf{1}$  is the  $n \times 1$  vector of all 1's. So equation  $(\star\star)$  can be rearranged as

$$\boxed{(\underline{I - (1 - \alpha)T'})\overline{PR} = (\alpha/n)\mathbf{1}}$$

*invariant if  $\alpha \neq 0$*

For small to moderate  $n$  this equation can be solved directly (by computer).

- Notes:
- When  $\alpha = 0$  this equation reverts to the basic steady state equation.
  - When  $\alpha \neq 0$  the fact that we have used  $U\overline{PR} = \mathbf{1}$  means that it is no longer necessary, nor appropriate, to replace the last row of this matrix equation by all 1's, as in the case of  $\alpha = 0$ .

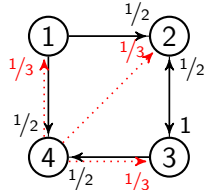
### Example 4B

For example 4A we had:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{c} \text{Diagram: 4 nodes (1,2,3,4) in a square. Solid black edges: 1→2 (1/2), 2→3 (1/2), 3→4 (1/2), 4→1 (1/2). Dotted red edges: 1→3 (1/3), 2→4 (1/3).} \end{array} \quad T = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix} \quad S = \begin{bmatrix} .08 \\ .31 \\ .38 \\ .23 \end{bmatrix}$$

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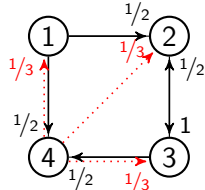
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Let's see what happens if we apply a damping factor of 90%; *i.e.*  $\alpha = 0.1$ .

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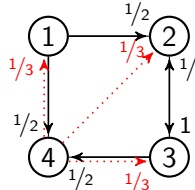
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We need to solve the equation  $(I - (1 - \alpha)T')PR = (\alpha/n)\mathbf{1}$  where:

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We need to solve the equation  $(I - (1 - \alpha)T')$  $PR = (\alpha/n)\mathbf{1}$  where:

$$(I - (1 - \alpha)T') = I - (0.9)T'$$

$$= \begin{bmatrix} 1 & 0 & 0 & -0.3 \\ -0.45 & 1 & -0.45 & -0.3 \\ 0 & -0.9 & 1 & -0.3 \\ -0.45 & 0 & -0.45 & 1 \end{bmatrix}$$

### Example 4B

For example 4A we had:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{c} \text{Diagram: A square with nodes 1 (top-left), 2 (top-right), 3 (bottom-right), 4 (bottom-left).} \\ \text{Solid black edges: } 1 \xrightarrow{1/2} 2, 2 \xrightarrow{1/2} 3, 3 \xrightarrow{1} 4, 4 \xrightarrow{1/2} 1. \\ \text{Dotted red edges: } 1 \xrightarrow{1/3} 4, 4 \xrightarrow{1/3} 3, 3 \xrightarrow{1/3} 2, 2 \xrightarrow{1/3} 1. \end{array} \quad T = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix} \quad S = \begin{bmatrix} .08 \\ .31 \\ .38 \\ .23 \end{bmatrix}$$

Let's see what happens if we apply a damping factor of 90%; *i.e.*  $\alpha = 0.1$ .

We need to solve the equation  $(I - (1 - \alpha)T')PR = (\alpha/n)\mathbf{1}$  where:

$$(I - (1 - \alpha)T') = I - (0.9)T' \quad \parallel \parallel \quad (\alpha/n)\mathbf{1} = \frac{0.1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.025 \\ 0.025 \\ 0.025 \\ 0.025 \end{bmatrix}$$



## Example 4B (cont.)

Solving the equation

$$\begin{bmatrix} 1 & 0 & 0 & -0.3 \\ -0.45 & 1 & -0.45 & -0.3 \\ 0 & -0.9 & 1 & -0.3 \\ -0.45 & 0 & -0.45 & 1 \end{bmatrix} \mathcal{R} = \begin{bmatrix} 0.025 \\ 0.025 \\ 0.025 \\ 0.025 \end{bmatrix}$$

on the computer gives,

## Example 4B (cont.)

Solving the equation

$$\begin{bmatrix} 1 & 0 & 0 & -0.3 \\ -0.45 & 1 & -0.45 & -0.3 \\ 0 & -0.9 & 1 & -0.3 \\ -0.45 & 0 & -0.45 & 1 \end{bmatrix} \mathcal{R} = \begin{bmatrix} 0.025 \\ 0.025 \\ 0.025 \\ 0.025 \end{bmatrix}$$

on the computer gives, to two decimal places,

$$\mathcal{R} = \begin{bmatrix} .10 \\ .30 \\ .37 \\ .23 \end{bmatrix}$$

## Example 4B (cont.)

Solving the equation

$$\begin{bmatrix} 1 & 0 & 0 & -0.3 \\ -0.45 & 1 & -0.45 & -0.3 \\ 0 & -0.9 & 1 & -0.3 \\ -0.45 & 0 & -0.45 & 1 \end{bmatrix} \mathcal{R} = \begin{bmatrix} 0.025 \\ 0.025 \\ 0.025 \\ 0.025 \end{bmatrix}$$

on the computer gives, to two decimal places,

$$\mathcal{R} = \begin{bmatrix} .10 \\ .30 \\ .37 \\ .23 \end{bmatrix} \quad \text{compared to} \quad S = \begin{bmatrix} .08 \\ .31 \\ .38 \\ .23 \end{bmatrix} \quad \text{without damping.}$$

## Example 4B (cont.)

Solving the equation

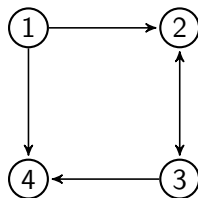
$$\begin{bmatrix} 1 & 0 & 0 & -0.3 \\ -0.45 & 1 & -0.45 & -0.3 \\ 0 & -0.9 & 1 & -0.3 \\ -0.45 & 0 & -0.45 & 1 \end{bmatrix} \mathbf{PR} = \begin{bmatrix} 0.025 \\ 0.025 \\ 0.025 \\ 0.025 \end{bmatrix}$$

on the computer gives, to two decimal places,

$$\mathbf{PR} = \begin{bmatrix} .10 \\ .30 \\ .37 \\ .23 \end{bmatrix} \quad \text{compared to} \quad \mathbf{S} = \begin{bmatrix} .08 \\ .31 \\ .38 \\ .23 \end{bmatrix} \quad \text{without damping.}$$

The PageRank of vertex 1 increases because it now has teleporting 'inputs' from all vertices, not just vertex 4.

*PR is a "smoothed out" version of S.*



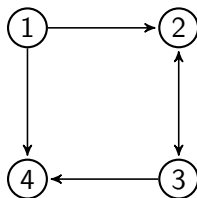
## Example 4B (cont.)

Solving the equation

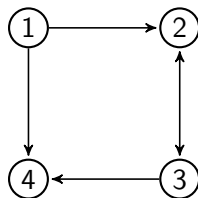
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The PageRank of vertex 1 increases because it now has teleporting ‘inputs’ from all vertices, not just vertex 4. This increase is at the expense of the stronger vertices 2 and 3.

$$\begin{bmatrix} 1 & 0 & 0 & -0.3 \\ -0.45 & 1 & -0.45 & -0.3 \\ 0 & -0.9 & 1 & -0.3 \\ -0.45 & 0 & -0.45 & 1 \end{bmatrix} \mathcal{R} = \begin{bmatrix} 0.025 \\ 0.025 \\ 0.025 \\ 0.025 \end{bmatrix}$$

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The PageRank of vertex 1 increases because it now has teleporting ‘inputs’ from all vertices, not just vertex 4. This increase is at the expense of the stronger vertices 2 and 3. Vertex 4 gains about as much as it loses.

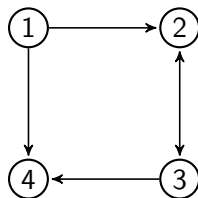
## Example 4B (cont.)

Solving the equation

$$\begin{bmatrix} 1 & 0 & 0 & -0.3 \\ -0.45 & 1 & -0.45 & -0.3 \\ 0 & -0.9 & 1 & -0.3 \\ -0.45 & 0 & -0.45 & 1 \end{bmatrix} \mathbf{PR} = \begin{bmatrix} 0.025 \\ 0.025 \\ 0.025 \\ 0.025 \end{bmatrix}$$

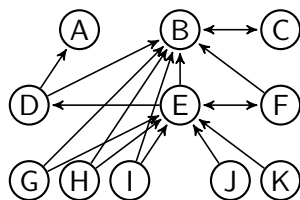
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The PageRank of vertex 1 increases because it now has teleporting ‘inputs’ from all vertices, not just vertex 4. This increase is at the expense of the stronger vertices 2 and 3. Vertex 4 gains about as much as it loses. This is what you expect with damping.

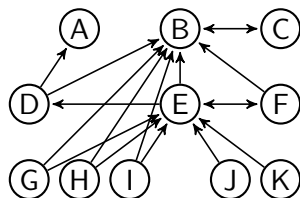
## Example 5



At left is the Wikipedia example we saw earlier of a miniweb of 11 pages and 17 hyperlinks. Colours, variable sizes and PageRanks have been removed and the bottom five vertices have been labelled G to K, following the given labelling of the top six vertices. The layout is similar to that in the original.



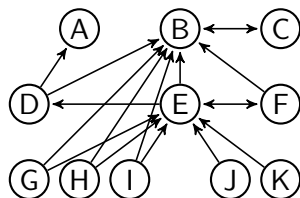
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Using the steady state method we have been discussing, we will derive the PageRanks given on the Wikipedia diagram.

## Example 5

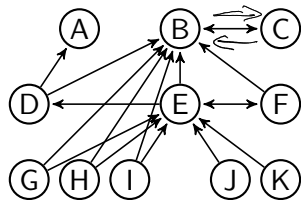


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**Step 1:** Compile the adjacency matrix  $A$ .

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Using the steady state method we have been discussing, we will derive the PageRanks given on the Wikipedia diagram.

**Step 1:** Compile the adjacency matrix  $A$ .

$$A = \begin{matrix} & \begin{matrix} A & B & C & D & E & F & G & H & I & J & K \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \\ J \\ K \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \underline{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

## Example 5 (cont.)

**Step 2:** Compile the basic transition matrix  $T$ .

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In each row  $i$  of  $A$ , count the number  $n_i$  of 1's in the row then:

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- If  $n_i = 0$  then
  - if  $n$  (the total number of pages) is small (less than 10 say) replace all but the  $i$ -th (diagonal) entry by  $1/(n-1)$  (we did this in Example 4B)
  - but for  $n \geq 10$  (as here and for WWW) replace every entry by  $1/n$ .

Jump  
to different  
page

Jump to  
any page  
(including  
current)

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For our Wikipedia example we get

$$T = \begin{bmatrix} 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 1/11 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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For each row  $i$  of  $T$

1. multiply each entry by  $(1 - \alpha)$  and put the *negative* of this in the corresponding position in the  $i$ -th *column* of  $(I - (1 - \alpha)T')$ ;
2. add 1 to each diagonal entry of the resulting matrix.

## Example 5 (cont.)

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Google uses an 85% damping factor, so we set  $(1 - \alpha) = 0.85$ .

### Example 5 (cont.)

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**Step 4:** Compile vector  $(\alpha/n)\mathbf{1}$ .

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Thus  $\alpha/n = \frac{.15}{11} = .01\overline{36}$

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$$(\alpha/n)\mathbf{1} = \begin{bmatrix} .01\overline{36} \\ .01\overline{36} \\ .01\overline{36} \\ .01\overline{36} \\ .01\overline{36} \\ .01\overline{36} \\ .01\overline{36} \\ .01\overline{36} \\ .01\overline{36} \\ .01\overline{36} \\ .01\overline{36} \end{bmatrix}$$

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**Step 5:** Use a computer to solve  $(I - (1 - \alpha)T')PR = (\alpha/n)\mathbf{1}$



## Example 5 (cont.)

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For example 'Gauss-Jordan Elimination' in the *Matrix Reshish* online matrix calculator, gives

$$PR = \begin{bmatrix} 0.032919 \\ 0.384644 \\ 0.343127 \\ 0.039112 \\ 0.080937 \\ 0.039112 \\ 0.016180 \\ 0.016180 \\ 0.016180 \\ 0.016180 \\ 0.016180 \end{bmatrix} \approx \begin{bmatrix} 3.3\% \\ 38.4\% \\ 34.3\% \\ 3.9\% \\ 8.1\% \\ 3.9\% \\ 1.6\% \\ 1.6\% \\ 1.6\% \\ 1.6\% \\ 1.6\% \end{bmatrix} \begin{matrix} -17 \\ -6 \\ \\ \\ \\ \\ \end{matrix}$$

## Iterative Approximation method

When  $n$  is huge, as it is with the WWW, solving the the  $n \times n$  linear system  $(I - (1 - \alpha)T')\underline{R} = (\alpha/n)\mathbf{1}$  becomes computationally infeasible.

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A computationally simpler method starts from the defining equation:

$$M'\mathcal{R} = \mathcal{R}.$$

Recall that  $M = (\alpha/n)U + (1 - \alpha)T$  is the modified transition matrix.

## Iterative Approximation method

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As with Markov chains in general we can attempt to find  $\overline{PR}$  by iteratively calculating the chain of probability vectors  $\underbrace{P_0, P_1, \dots}$  where  $P_0$  is arbitrary and  $P_k = M'P_{k-1}$  for  $k \geq 1$  (so  $P_k = (M')^k P_0$ ).

A steady state is reached when  $P_k \approx P_{k-1}$ . Then declare that  $\overline{PR} \approx P_k$ .

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$$\begin{aligned} \text{Now } P_k &= M'P_{k-1} = [(\alpha/n)U + (1 - \alpha)T']P_{k-1} \\ &= (\alpha/n)\mathbf{1} + (1 - \alpha)T'P_{k-1} \end{aligned}$$

and it is natural to start with all ranks equal.

## Iterative Approximation method

inserting is difficult

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and it is natural to start with all ranks equal. So the iterative scheme is

$$P_0 = \underline{(1/n)\mathbf{1}}; \quad P_k = \underline{\alpha P_0 + (1 - \alpha)T'P_{k-1}}, \quad k \geq 1.$$

## Iterative Approximation method

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and it is natural to start with all ranks equal. So the iterative scheme is

$$\boxed{P_0 = (1/n)\mathbf{1}; \quad P_k = \alpha P_0 + \underbrace{(1 - \alpha)T'P_{k-1}}_{\text{hyperlinking}}, \quad k \geq 1.}$$

Each iteration takes a weighted average of teleporting and hyperlinking.



## Example 4C

In Example 4B we used the equation-solving method to find  $\mathcal{PR}$ .

For  $T = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$  and  $\alpha = 0.1$  we found  $\mathcal{PR} = \begin{bmatrix} .10 \\ .30 \\ .37 \\ .23 \end{bmatrix}$  to 2d.p.

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Let's try the same problem using iterative approximation:

$$P_0 = \begin{bmatrix} .25 \\ .25 \\ .25 \\ .25 \end{bmatrix} \quad P_k = \begin{bmatrix} .025 \\ .025 \\ .025 \\ .025 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & .3 \\ .45 & 0 & .45 & .3 \\ 0 & .9 & 0 & .3 \\ .45 & 0 & .45 & 0 \end{bmatrix} P_{k-1}$$

## Example 4C

In Example 4B we used the equation-solving method to find  $\mathcal{R}$ .

For  $T = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$  and  $\alpha = 0.1$  we found  $\mathcal{R} = \begin{bmatrix} .10 \\ .30 \\ .37 \\ .23 \end{bmatrix}$  to 2d.p.

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Results of the first ten iterations, rounded to 2d.p. Calcs used 15d.p.

$k$	1	2	3	4	5	6	7	8	9	10
$P_k$	.10	.10	.09	.10	.09	.10	.09	.10	.09	.10
	.33	.29	.31	.30	.31	.30	.31	.30	.30	.30
	.33	.39	.35	.38	.36	.37	.36	.37	.37	.37
	.25	.22	.25	.22	.24	.23	.24	.23	.24	.23

## Example 4C

In Example 4B we used the equation-solving method to find  $\overline{PR}$ .

For  $T = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$  and  $\alpha = 0.1$  we found  $\overline{PR} = \begin{bmatrix} .10 \\ .30 \\ .37 \\ .23 \end{bmatrix}$  to 2d.p.

Let's try the same problem using iterative approximation:

$$P_0 = \begin{bmatrix} .25 \\ .25 \\ .25 \\ .25 \end{bmatrix} \quad P_k = \begin{bmatrix} .025 \\ .025 \\ .025 \\ .025 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & .3 \\ .45 & 0 & .45 & .3 \\ 0 & .9 & 0 & .3 \\ .45 & 0 & .45 & 0 \end{bmatrix} P_{k-1}$$

Results of the first ten iterations, rounded to 2d.p. Calcs used 15d.p.

$k$	1	2	3	4	5	6	7	8	9	10
$P_k$	.10	.10	.09	.10	.09	.10	.09	.10	.09	.10
	.33	.29	.31	.30	.31	.30	.31	.30	.30	.30
	.33	.39	.35	.38	.36	.37	.36	.37	.37	.37
	.25	.22	.25	.22	.24	.23	.24	.23	.24	.23

Pretty good after just 2 iterations! Within 1%-point after 4 iterations.

### Example 4C

In Example 4B we used the equation-solving method to find  $\mathcal{PR}$ .

For  $T = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$  and  $\alpha = 0.1$  we found  $PR = \begin{bmatrix} .10 \\ .30 \\ .37 \\ .23 \end{bmatrix}$  to 2d.p.

Let's try the same problem using iterative approximation:

$$P_0 = \begin{bmatrix} .25 \\ .25 \\ .25 \\ .25 \end{bmatrix} \quad P_k = \begin{bmatrix} .025 \\ .025 \\ .025 \\ .025 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & .3 \\ .45 & 0 & .45 & .3 \\ 0 & .9 & 0 & .3 \\ .45 & 0 & .45 & 0 \end{bmatrix} P_{k-1}$$

Results of the first ten iterations, rounded to 2d.p. Calcs used 15d.p.

$k$	1	2	3	4	5	6	7	8	9	10
$P_k$	.10	.10	.09	.10	.09	.10	.09	.10	.09	.10
	.33	.29	.31	.30	.31	.30	.31	.30	.30	.30
	.33	.39	.35	.38	.36	.37	.36	.37	.37	.37
	.25	.22	.25	.22	.24	.23	.24	.23	.24	.23

Pretty good after just 2 iterations! Within 1%-point after 4 iterations.

End of Course Notes.