Question 1

(a) Explain why every finite tree of order 2 or more has at least one vertex of degree 1.

(b) Prove by mathematical induction on n that every tree of order $n \geq 1$ has exactly $(n-1)$ edges. (You will need part (a).)

Question 2 Recall that a hydrocarbon molecule consists of joined-up carbon and hydrogen atoms. If the molecule contains c carbon atoms and h hydrogen atoms we write its formula as C_cH_h. The way the atoms are joined up can be modelled by a connected graph with the atoms as vertices and the joins as edges. In such a graph every carbon vertex has degree 4 and every hydrogen vertex has degree 1. The graph may have parallel edges but cannot have loops. A hydrocarbon molecule is called *saturated* if it contains the maximum possible number of hydrogen atoms for its number of carbon atoms.

(a) Draw the graphs of saturated hydrocarbon molecules containing 1 (methane CH_4), 2 (ethane C_2H_6) and 3 (propane C_3H_8) carbon atoms.

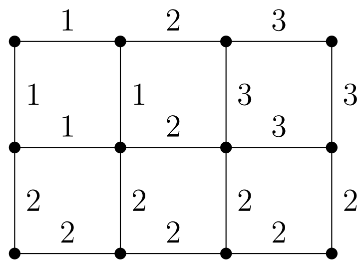
(b) Explain why the graph of a saturated hydrocarbon molecule must be a tree. [Hint: What is the defining property of a tree?

(c) Recall that a tree with n vertices has exactly $n-1$ edges. (See Question 1). Use this to explain why the total degree of the graph of a saturated hydrocarbon molecule is $2c + 2h - 2$.

(d) Give another formula for the total degree of the graph of *any* hydrocarbon molecule.

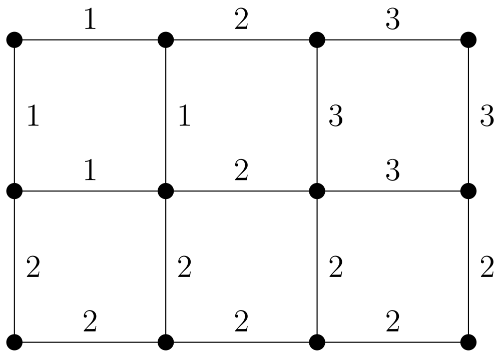
(e) Use (c) and (d) to prove Cayley's result that every saturated hydrocarbon molecule has formula C_cH_{2c+2}.

Question 3 A weighted graph G is shown at right.



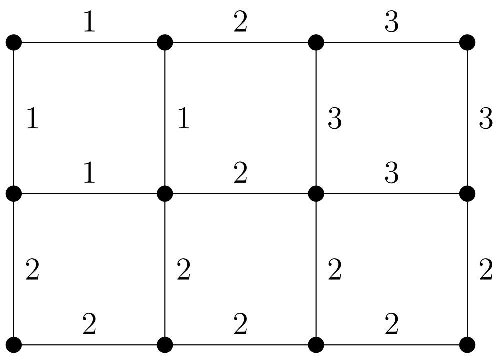
(a) Find a minimal spanning tree for G and calculate its total weight.

Trace out your answer below:



(b) Prove or disprove that G has a minimal spanning tree containing two vertices of degree 4.

(c) Find a maximal spanning tree for G and calculate its total weight.
Trace out your answer below:



Question 4 An irrigation system is being laid down to connect 7 fields. The cost of the pipelines between field i and field j is given by the matrix at right.

Find a set of pipelines that connects all fields at a minimal cost. Branching is allowed.

$$\begin{pmatrix} 0 & 1 & 4 & 5 & 3 & 6 & 1 \\ 1 & 0 & 1 & 8 & 2 & 3 & 7 \\ 4 & 1 & 0 & 6 & 1 & 3 & 5 \\ 5 & 8 & 6 & 0 & 6 & 1 & 4 \\ 3 & 2 & 1 & 6 & 0 & 8 & 9 \\ 6 & 3 & 3 & 1 & 8 & 0 & 4 \\ 1 & 7 & 5 & 4 & 9 & 4 & 0 \end{pmatrix}$$

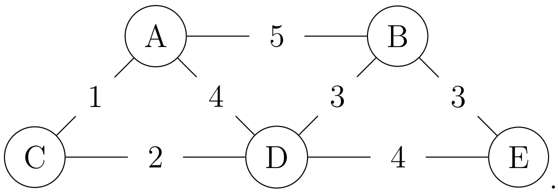
Question 5 To see how bad the (greedy) Nearest Neighbour algorithm can be consider this deliberately pathological example: The complete graph K_{10} has vertices labelled 1,. . . ,10 and each edge $\{a, b\}$ is given weight ab. Do not attempt to draw this graph.

(a) Starting at the vertex determined by the last digit of your ANU ID (use vertex 10 if that digit is 0) , apply the Nearest Neighbour algorithm to find a Hamilton circuit of supposedly low total weight. Give your answer as a list of vertices, and calculate the total weight.

(b) By avoiding large products such as 9×10, find (without the help of any algorithm) a Hamilton circuit of much lower total weight. Again give your answer as a list of vertices, and calculate the total weight.

(c) How many different Hamilton circuits would you need to consider in order to prove your answer to (b) has the lowest possible total weight? Circuits are different if and only if there is at least one edge on one that is not on the other.

Question 6 Let G be the weighted graph



(a) Show that Dijkstra's algorithm correctly finds that the shortest path from A to B is the direct path using just the edge AB. You can do all the work on a single diagram, but, to show that you have used the algorithm correctly, if an annotation needs updating do not erase it — just put a line through it and write the new annotation above that.

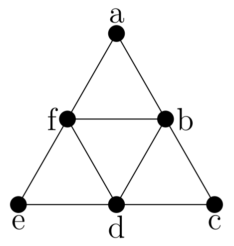
(b) In what order are the vertices added to the tree?

(c) Notice that the algorithm does not, in this instance, generate a spanning tree. Which vertex or vertices are missing?

(d) Extend the use of the algorithm until the shortest distance from A to each other vertex is established, and a spanning tree is thereby generated. Draw this tree.

(e) Is the spanning tree you have generated using Dijkstra's algorithm a minimal spanning tree? Justify your answer.

Question 7 For the graph H at right:



(a) Is the graph simple? Justify your answer.

(b) Is the walk abcdb

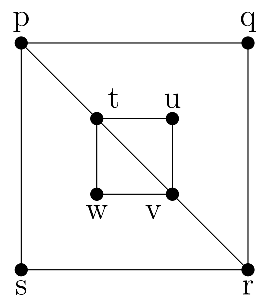
(i) a path?
(ii) closed?
(iii) simple?

(c) Find simple circuits of lengths 3, 4 and 5.

(d) Find an Euler circuit.

(e) Find a Hamilton circuit.

Question 8 For the graph J at right:



(a) Prove that J has no Euler circuit.

(b) Prove that J has no Hamilton circuit.

(c) Suppose that Fleury's algorithm is used to find an Euler path from p to r. What feature of the algorithm prevents the path starting ptvr?