B3. Matrices.

Notes originally prepared by Pierre Portal.
Editing and expansion by Malcolm Brooks.

Text Reference (Epp)	3ed: Section	11.3
	4ed: Section	10.3
	5ed: Section	10.2

Last time:

→ Selection sort $O(n^2)$
intuitionistic sorting algorithm.
Repeatedly move the smallest element to the start. Slow.

→ Merge sort $O(n \log_2(n))$
Divide and conquer
Sort short lists and merges them preserving "sortedness"
Much quicker

Announcements

Assignment marking within the next week (hopefully)

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Unfortunately these sections are part of chapters on Graph Theory, that we have not yet covered, so the examples may seem unfamiliar.

Also they do not go quite as far as we do, in that matrix inverses are not discussed.

What is a matrix (plural: matrices) ?

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$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \pi/2 \\ -\pi/2 \end{bmatrix} \qquad \mathbf{C} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

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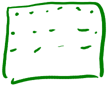

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Examples: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in M_2(\mathbb{N})$, $\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \in M_3(\{a, b, c\})$.

Indexing

A generic member of $M_{m \times n}(S)$ is written

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Example: For the matrix $\mathbf{A} = \begin{bmatrix} 2 & 7 \\ 0 & -3 \end{bmatrix}$ we have

$$a_{1,1} = 2, \qquad a_{1,2} = 7, \qquad a_{2,1} = 0, \qquad a_{2,2} = -3.$$

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Elements of $M_n(S)$ correspond to functions

$$\begin{aligned} a : \underline{\{1, ..., n\} \times \{1, ..., n\}} &\to S \\ (i, j) &\mapsto \underline{a_{i,j}}. \end{aligned}$$

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This is 2-dimensional information: information which depends on 2 numbers, i and j.

Examples

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  Let C be the set of colours.
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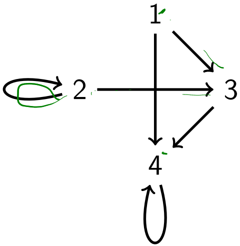

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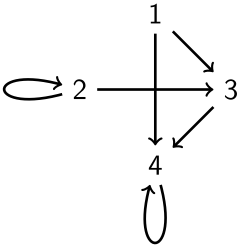

is represented by

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Another example

- A matrix $(a_{i,j}) \in M_n(\mathbb{Q})$ can define a weighted relation. Let us consider 4 companies, called 1,2,3,4, and let $a_{i,j}$ be the money (\$) received by i from j in a year.

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$$\begin{tikzcd}
1 \arrow[d, shift left, "10^5"] \arrow[dr, bend left, "10^4"] & 2 \arrow[l, "10^4"'] \\
4 \arrow[u, shift left, "10^5"] \arrow[r, "10^4"'] \arrow[ur, bend right, "10^5"'] & 3
\end{tikzcd}$$

Vectors and vector arithmetic

$\mathbb{R}, \mathbb{C}$

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dot

cross

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There are a number of ways to define the product of two vectors (*e.g* the 'inner' and the 'outer' products) but we will not use them in this course. However we do need to define the product of a number λ $\in \mathbb{Q}$ and a vector. In this context the number λ is referred to as a **scalar**, to distinguish it from a vector, and the product $\lambda\mathbf{x}$ is called a **scalar product**.

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There are a number of ways to define the product of two vectors (*e.g* the 'inner' and the 'outer' products) but we will not use them in this course. However we do need to define the product of a number λ and a vector. In this context the number λ is referred to as a **scalar**, to distinguish it from a vector, and the product $\lambda\mathbf{x}$ is called a **scalar product**. It is also defined element-wise:

scale

$$\forall \lambda \in \mathbb{Q} \quad \lambda\mathbf{x} = \lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n).$$

Examples of vectors and vector arithmetic

- Let $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{Q}^3$ represent the state of an ecosystem with p_1, p_2, p_3 being the sizes of the populations of three different species.

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  $$3\mathbf{a} = 3(a_1, ..., a_n),$$
  represents to the same sound, but three times stronger.

Addition and scalar multiplication of matrices

The same can be done with matrices.

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For matrices $\mathbf{A} = (a_{i,j})$ and $\mathbf{B} = (b_{i,j})$ in $M_{m \times n}(\mathbb{Q})$, and $\lambda \in \mathbb{Q}$, we define the **sum** $\mathbf{A} + \mathbf{B}$ and **scalar product** $\lambda\mathbf{A}$ by

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Examples:

$$\begin{pmatrix} 1 & 2 \\ 3 & \underline{4} \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & \underline{8} \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & \underline{12} \end{pmatrix}$$

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Linear functions

Definition: A function $F : \mathbb{Q}^n \to \mathbb{Q}^n$ is called **linear** if and only if it satisfies the following two conditions:

- $F(x+y) = F(x) + F(y) \quad \forall x, y \in \mathbb{Q}^n$.
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respects +

respecting scaling

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Then for $m \in \mathbb{N}$ with $m \leq n$ the function F specified by

$$\begin{array}{rcl} F : \mathbb{Q}^n & \to & \mathbb{Q}^n \\ (a_1, ..., a_n) & \mapsto & (a_1, ..., a_m, 0, 0, ...0) \end{array}.$$

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Filters are linear functions. (Check!)

Linear functions: another example

Let $(p_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}^2$ represent the state of an ecosystem with two species at time n; say $p_n = (x_n, y_n)$, where x_n is the size of the population of species 1, and y_n the size of the population of species 2.

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Assume that the ecosystem evolves as follows, due to a predator-prey relationship between the two species:

$$\forall n \in \mathbb{N} \quad \begin{cases} x_{n+1} = \underbrace{4x_n} - \underbrace{y_n}, \\ y_{n+1} = \underbrace{y_n} + \underbrace{2x_n}. \end{cases}$$

← get eaten by species2

↖ eat species 1

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Then $p_{n+1} = F(p_n) \quad \forall n \in \mathbb{N}$, where $F(x, y) = (4x - y, 2x + y)$.
The function F is linear. (Check!)

Check F is linear

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implicit

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The function F is linear. (Check!)

We will return to this example several times in this section on matrices.

Multiplying a vector by a matrix: motivation

We now explore the possibility of expressing the function

$$\begin{array}{rcl} F : \mathbb{Q}^2 & \to & \mathbb{Q}^2 \\ (x, y) & \mapsto & (4x - 1y, 2x + 1y) \end{array}$$

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Writing (x, y) as the column vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, we want

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That's exactly what we do next.

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For a matrix $\underbrace{\mathbf{A} = (a_{i,j})}$ $\in M_n(\mathbb{Q})$ and a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Q}^n$ we define the **matrix-vector product Ax** as the vector given by

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By convention, in this context, we normally write the vectors as columns. Thus

$$\begin{array}{r} \\ \\ \\ \\ \text{row} \end{array}\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{array}{c} \text{column} \\ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \end{array} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n \end{bmatrix}$$

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$M_{m\times n}$ $M_{n\times k}$

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Example:
$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}.$$

Linear functions expressed using matrices

came up with matrix

started with linear function

Example: $$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x - y \\ 2x + y \end{pmatrix} = F(x, y)$$

where, as we have seen, the function $F : \mathbb{Q} \to \mathbb{Q}$ so defined is linear.

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$\{\text{linear functions } \mathbb{Q}^n \to \mathbb{Q}^n\} \xrightarrow[\text{bijection}]{f} M_n(\mathbb{Q})$ linear how to specify

Theorem (proof omitted): To each linear function $F : \underbrace{\mathbb{Q}^n}_{} \to \mathbb{Q}^n$ there is a matrix $\mathbf{M} \in M_n(\mathbb{Q})$ such that

column vectors.

$$F(\mathbf{x}) = \underline{\mathbf{Mx}} \quad \forall \mathbf{x} \in \mathbb{Q}^n.$$

Conversely, every function $F : \mathbb{Q}^n \to \mathbb{Q}^n$ defined using a matrix in this way is linear.

x row vector

xM another row vector

Matrix multiplication: motivation

Question: Given $\mathbf{M} \in M_n(\mathbb{Q})$, how, if at all, should $\mathbf{M}^2$ be defined?

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Example: For $\mathbf{M} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$:

$$\begin{aligned}
\mathbf{M}(\mathbf{Mx}) &= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \left[\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right] \\
&= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4x - y \\ 2x + y \end{pmatrix}
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&= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4x - y \\ 2x + y \end{pmatrix} \\
&= \begin{pmatrix} 14x - 5y \\ 10x - y \end{pmatrix} = \underbrace{\begin{pmatrix} 14 & -5 \\ 10 & -1 \end{pmatrix}}_{M^2} \underset{\begin{bmatrix} x \\ y \end{bmatrix}}{\begin{pmatrix} x \\ y \end{pmatrix}}
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So we want $\mathbf{M}^2 = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 14 & -5 \\ 10 & -1 \end{pmatrix}$.

Matrix multiplication: definition

For matrices $\mathbf{A} = (a_{i,j})$ and $\mathbf{B} = (b_{i,j})$ in $M_n(\mathbb{Q})$ the **product** $\mathbf{AB} = \mathbf{C} = (c_{i,j}) \in M_n(\mathbb{Q})$ is defined by

$$\boxed{c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} \quad \forall i, j \in \{1, .., n\}.}$$

compose the linear functions

$x \mapsto Bx \mapsto A(Bx)$
$= Cx$

$A\cdot : \mathbb{Q}^n \to \mathbb{Q}^n$
$B\cdot : \mathbb{Q}^n \to \mathbb{Q}^n$

$(A\cdot) \circ (B\cdot) : \mathbb{Q}^n \to \mathbb{Q}^n$ linear

so it is represented by some matrix.

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Two Examples:

(a) First, let's check that this formula produces what we were looking for with $\mathbf{M}^2$ on the previous slide:

$$\mathbf{M}^2 = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4\times 4 + (-1)\times 2 & 4\times(-1) + (-1)\times 1 \\ 2\times 4 + 1\times 2 & 2\times(-1) + 1\times 1 \end{pmatrix} = \begin{pmatrix} 14 & -5 \\ 10 & -1 \end{pmatrix}.$$

Matrix multiplication: definition

For matrices $\mathbf{A} = (a_{i,j})$ and $\mathbf{B} = (b_{i,j})$ in $M_n(\mathbb{Q})$ the **product** $\mathbf{AB} = \mathbf{C} = (c_{i,j}) \in M_n(\mathbb{Q})$ is defined by

$$\boxed{c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} \quad \forall i, j \in \{1, .., n\}.}$$

Two Examples:

(a) First, let's check that this formula produces what we were looking for with $\mathbf{M}^2$ on the previous slide:

$$\mathbf{M}^2 = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \times 4 + (-1) \times 2 & 4 \times (-1) + (-1) \times 1 \\ 2 \times 4 + 1 \times 2 & 2 \times (-1) + 1 \times 1 \end{pmatrix} = \begin{pmatrix} 14 & -5 \\ 10 & -1 \end{pmatrix}.$$

(b) This example demonstrates the product formula more clearly:

$$\begin{array}{ll} \text{row} \rightarrow & \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \end{array} \begin{array}{l} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \quad \uparrow \text{column} \end{array} = \begin{pmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{pmatrix}.$$

Identity matrices

Observe that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ acts as an 'identity' in the sense that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

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More generally, for $n \in \mathbb{N}$, we define the $n \times n$ **identity matrix** I_n by

$$\mathbf{I}_n = (\delta_{i,j}) \in M_n(\mathbb{Q}) \text{ with } \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Kronecker delta

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So $I_1 = \begin{bmatrix}1\end{bmatrix}$, $I_2 = \begin{bmatrix}1&0\\0&1\end{bmatrix}$, $I_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, etc.

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By applying the matrix product formula we can immediately establish that, for any $n \in \mathbb{N}$, the identity matrix I_n does indeed have the identity property:

$$\forall n \in \mathbb{N},\ \forall \mathbf{M} \in M_n(\mathbb{Q}) \quad \mathbf{I}_n\mathbf{M} = \mathbf{M} = \mathbf{M}\mathbf{I}_n.$$

Remark: When the value of n is clear from the context, we abbreviate $\mathbf{I}_n$ to just $\mathbf{I}$.

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$$E.g. \text{ take } \mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \mathbf{N} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

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Let x be the quantity of R1, y quantity of R2.

$$\begin{cases} \text{from (a): } x = 2y \\ \text{from (b): } \frac{x}{2} + \frac{y}{3} = 5 \end{cases} \iff \begin{cases} x - 2y = 0 \\ 3x + 2y = 30 \end{cases}$$

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We can solve these equations by elimination, but consider the equivalent matrix equation

$$\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 30 \end{pmatrix}.$$

Q: Can we solve this matrix equation, just using matrices?

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Our chemical reaction example is a case in point:

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This matrix $\mathbf{A}^{-1}$ is an 'inverse' of $\mathbf{A}$ in the following sense:

An **inverse**, if one exists, of a matrix $\mathbf{A} \in M_n(\mathbb{Q})$ is a matrix $\mathbf{A}^{-1} \in M_n(\mathbb{Q})$ with the property that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$.

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Note the if an $\mathbf{A}^{-1}$ exists and $\mathbf{Ax} = \mathbf{b}$ then

$$\mathbf{x} = \mathbf{Ix} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{x} = \mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}\mathbf{b}.$$

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In this course we only consider the 2×2 case:

The **determinant** of a matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q})$ is the number

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Lemma: For any $\mathbf{A}, \mathbf{B} \in M_2(\mathbb{Q})$, $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$.
Proof: Multiply out both sides.

Calculating Inverses

A matrix $\mathbf{A}$ can have at most one inverse, because if $\mathbf{B}$ and $\mathbf{C}$ are both inverses then $\mathbf{BA} = \mathbf{I}$ and $\mathbf{AC} = \mathbf{I}$ and so

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Theorem: A matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q})$ has an inverse if and only if $\det(\mathbf{A}) \neq 0$ and in this case

$$\boxed{\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}} \quad \textit{e.g.} \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}^{-1} = \frac{1}{8} \begin{pmatrix} 2 & 2 \\ -3 & 1 \end{pmatrix}$$

Proof: If $\mathbf{A}$ has an inverse then

$$1 = \det(\mathbf{I}_2) = \det(\mathbf{AA}^{-1}) = \det(\mathbf{A}) \det(\mathbf{A}^{-1}),$$

so $\det(\mathbf{A})$ cannot be zero.

Calculating Inverses

A matrix $\mathbf{A}$ can have at most one inverse, because if $\mathbf{B}$ and $\mathbf{C}$ are both inverses then $\mathbf{BA} = \mathbf{I}$ and $\mathbf{AC} = \mathbf{I}$ and so

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What about $n > 2$?

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What about $n > 2$? See Math1013 or Math1115.

Back to population dynamics

As a final example involving matrix multiplication and matrix inverses, we return to the simple ecosystem model

$$\forall n \in \mathbb{N} \quad \begin{cases} x_{n+1} = 4x_n - y_n, \\ y_{n+1} = y_n + 2x_n, \end{cases}$$

where x_n, y_n are the populations of two species after n time steps.

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This is an implicit definition of a sequence of vectors. We will use mathematical induction to establish an explicit formula, stated and proved on the next slide.

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R1: $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ [prove by multiplying out the RHS]

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R2: $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \dfrac{1}{2-1} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ [formula for inverse of 2×2 matrix]

Claim: $\forall n \in \mathbb{N}^{\star}$ $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

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Proof:

Basis step: When $n = 0$ the RHS becomes (using R2)

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END OF SECTION B3