1 Admin: - Mock mid-sem exam in workshops (1hr)
- Collaboratively assessing solutions (1hr)

Recap: Counting, cardinality, different ∞s, permutations, combinations, stars + bars, pigeon-hole principle

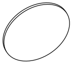
C2. Probability

Notes originally prepared by Judy-anne Osborn and Pierre Portal.
Editing, expansion and additions by Malcolm Brooks.

Text Reference (Epp)	3ed:	Sections	6.7-9
	4ed:	Sections	9.7-9
	5ed:	Sections	9.7-9

(Only the last part of §9 on ‘independence’ is relevant for this course)

A thought experiment:



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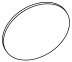

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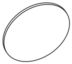

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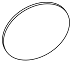

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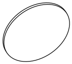

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- What is the probability of ‘Heads’?
- We say it is

$$\mathbb{P}(\text{Heads}) = \frac{1}{2}. \qquad \textit{Why?}$$

Methods of assigning probabilities

Method 1: Use relative frequencies

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same answer

LoLN

- Eg. assume equally likely outcomes

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Law of Large Numbers

- There is much more to be said on 'relative frequencies', but for this course we will focus on making 'models'.

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  for

  Heads or Tails. $p + p = 1$

A model for coin tossing: equal likelihood

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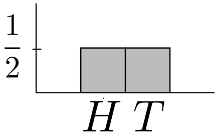
We can represent this situation graphically as



prob. density

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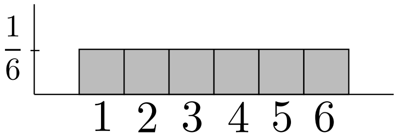
An equal likelihood model for die-tossing



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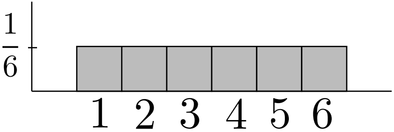

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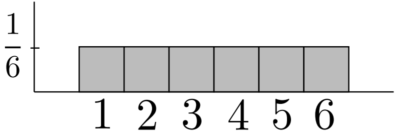

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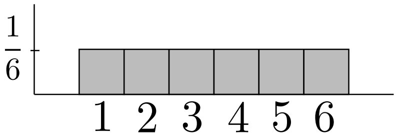

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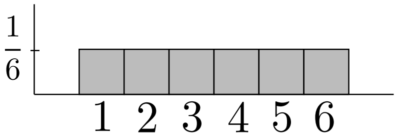

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$$\boxed{\mathbb{P}(\{3, 6\}) = \frac{|\{3, 6\}|}{|\{1, 2, 3, 4, 5, 6\}|} = \frac{2}{6} = \frac{1}{3}}$$

6

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uniform distribution on S

where $|E|$ is the number of outcomes in E, and
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singleton events

i.e. $\sum_{s \in S} \mathbb{P}(\{s\}) = 1$

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MATH3029 "probability theory"

*For infinite sets, this isn't necessarily true. 'Measure theory' explains why.

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$$\mathbb{P}(\{1\}) + \ldots + \mathbb{P}(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

The Sum and Product Rules for Probability

The Sum Rule

Sum Rule: If events $E_1, ..., E_n$ are mutually disjoint, i.e. $E_i \cap E_j = \emptyset$ for all $i \neq j$, then

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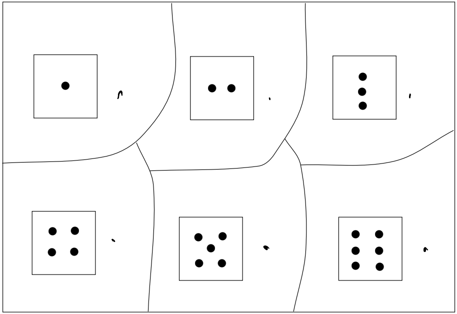
Disjoint events exclude one another in the sense that they cannot happen at the same time.

Sum Rule for probability: another die-tossing example

What is the probability that the outcome from a single toss of a die is an odd number?

The six possible outcomes are all disjoint (cannot occur simultaneously).



Thus the sum rule applies.

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{⚀, ⚂, ⚄}

- The probability that the die lands with an odd number up is

$$
\begin{aligned}
& \Pr(\text{⚀}) + \Pr(\text{⚂}) + \Pr(\text{⚄}) \\
= \; & \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\
= \; & \frac{1}{2}
\end{aligned}
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by the sum rule.

Sum Rule for probability. Example: Non-zero numbers

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$$\mathbb{P}(\text{the number is negative}) + \mathbb{P}(\text{the number is positive})$$

$$= \frac{n}{2n+1} + \frac{n}{2n+1} = \frac{2n}{2n+1}.$$

The Product Rule

- Product Rule: If events $E_1, ..., E_n$ are 'independent' of each other; then the probability of composite event 'E_1 and E_2 and ... and E_n' is

$$\mathbb{P}(E_1 \cap E_2 \cap \ldots \cap E_n) = \mathbb{P}(E_1) \times \mathbb{P}(E_2) \times \ldots \mathbb{P}(E_n).$$

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stochastic

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almost

$$\mathbb{P}(E_1 \wedge E_2 \wedge \ldots \wedge E_n) = \mathbb{P}(E_1) \times \mathbb{P}(E_2) \times \ldots \mathbb{P}(E_n).$$ (defn of independent)

causal

- To see what we mean by 'independent', consider a procedure that can be broken down into successive tasks, each of which could be done in a number of ways. If the choice of the way to do any one task had no influence on the choice of ways to do any other of the tasks, then the tasks would be independent.

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- A formal definition of independence will be given later.

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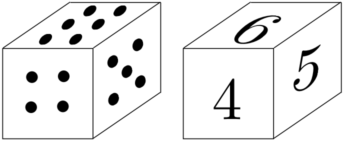

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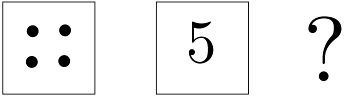

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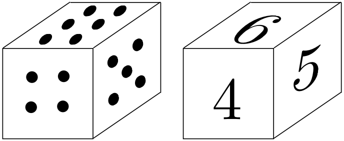

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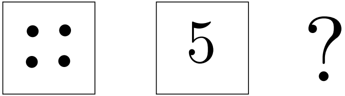

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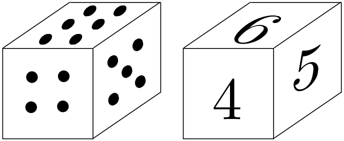

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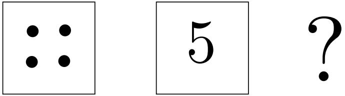

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- Hence the product rule applies.

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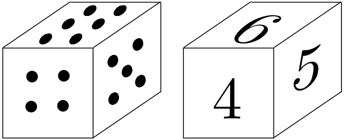

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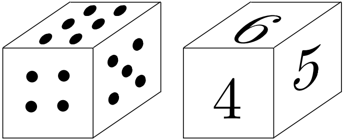

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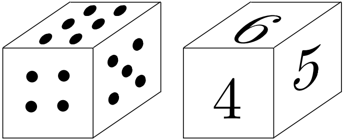

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- These two possibilities are **disjoint**, so the sum rule applies:

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Density and Distribution

Frequency Histograms

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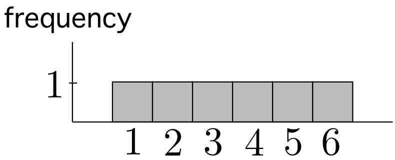
- We have already seen some simple examples, like tossing a die with equally likely possible outcomes: $1, 2, 3, 4, 5, 6$:



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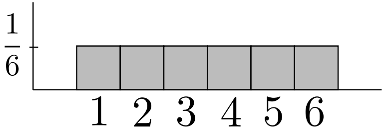

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pdf

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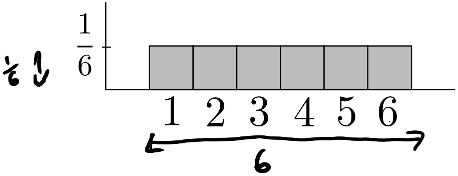

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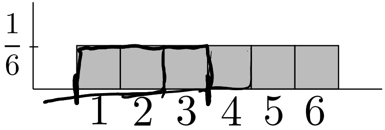

What is the area under the curve? Why?

Because area = probability

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CDF

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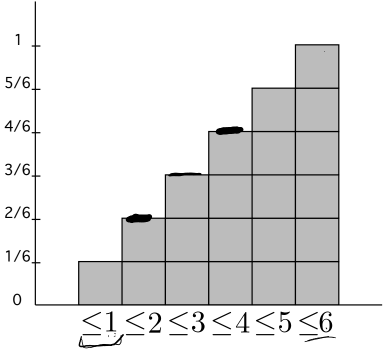

P(between 3 and 4)

$P(\leq 4) - P(\leq 2)$

- We will only use of cumulative distributions when looking up probability values in tables or online.

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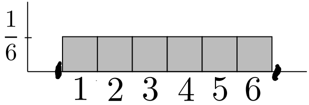

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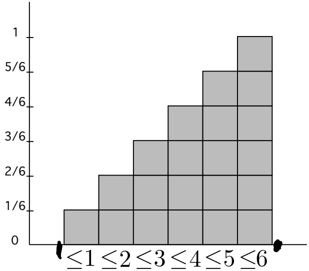

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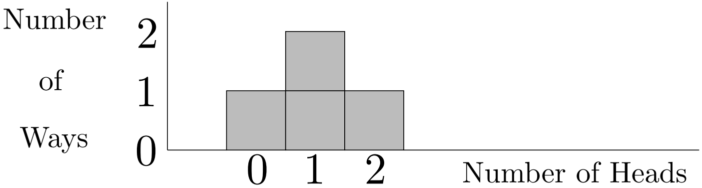

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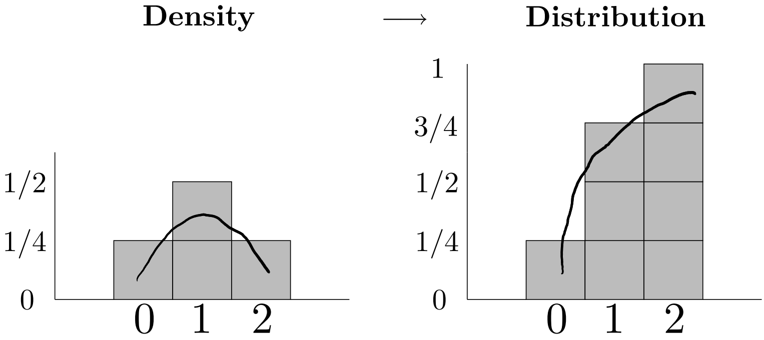

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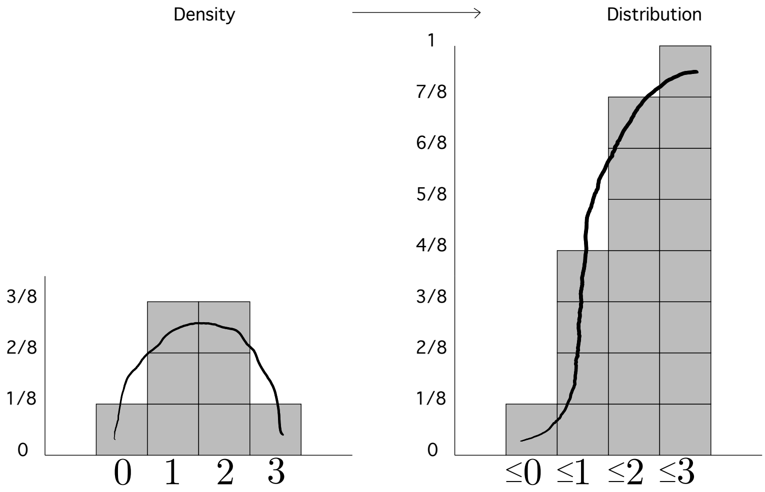
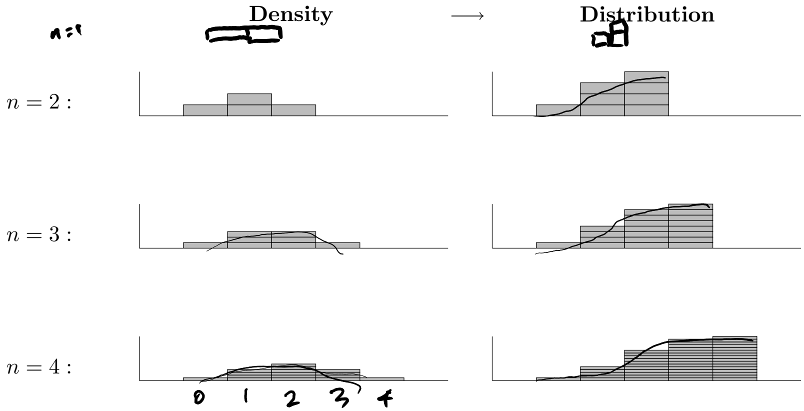
Three Fair Coins: Density and Distribution Functions



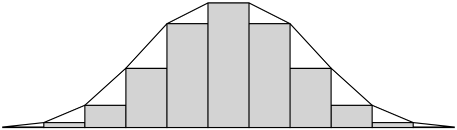
Binomial Probability Distributions

The family of functions that come from coin-tossing are all examples of binomial densities/distributions:



Bell-like curves for large n

As n gets larger and larger these **binomial probability density functions** get closer and closer to the famous Bell Curve:



which is the so-called **'Normal' Probability Density Function**.

Central limit theorem

Pascal's Triangle
and
Coin Tossing

Pascal's Triangle

r

0 → $1 = \binom{n}{r}$

1 → $1 \quad 1$

n

2 → 0H: 1 1H: 2 2H: 1

3 → 0H: 1 1H: 3 2H: 3 3H: 1

4 → $1 \quad 4 \quad 6 \quad 4 \quad 1$

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

Pascal's Triangle

- Frequencies in Coin-Tossing are numbers in Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & \binom{0}{0} & & \\
 & & \binom{1}{0} & 1 & & 1 & \binom{1}{1} & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

$+$

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$\binom{4}{2}$

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\end{array}
\quad \leftarrow 4 = n
$$

- Each row is generated by expanding a binomial, eg:

$x\,y\,y\,y$

$y\,x\,y\,y$

$y\,y\,x\,y$

$y\,y\,y\,x$

$$(y+x)^4 = y^4 + 4y^3x + 6y^2x^2 + 4yx^3 + x^4.$$

$(H+T)^2 = HH + HT + TH + TT$

$= HH + 2\,HT + TT$

Pascal's Triangle

- We've seen these numbers before in 'combinations': $\binom{n}{k}$:

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

The Binomial Theorem

- The Binomial Theorem states that

$$(y+x)^n = \binom{n}{0} y^n x^0 + \binom{n}{1} y^{n-1} x^1 + \cdots + \binom{n}{n} y^0 x^n$$

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and gives the rows of Pascal's Triangle in its coefficients.

Idea of Proof of Binomial Theorem:

$(y+x)^3$

$$(y+x)(y+x)(y+x)$$

$$= yyy + yyx + yxy + yxx + xyy + xyx + xxy + xxx$$

$$= \underbrace{yyy}_{\binom{3}{0}\ x\text{'s}} + \underbrace{yyx + yxy + xyy}_{\binom{3}{1}\ x\text{'s}} + \underbrace{yxx + xyx + xxy}_{\binom{3}{2}\ x\text{'s}} + \underbrace{xxx}_{\binom{3}{3}\ x\text{'s}}$$

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multinomial

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    $\vdots$
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- I.e. what is

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2} + \ldots \binom{n}{n}x^ny^0 ?$$

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so that $$\boxed{\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.}$$

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\# of ways to get 3 heads $\to$
$$\frac{\binom{6}{3}}{2^6} = \frac{\frac{6\cdot5\cdot4}{3\cdot2\cdot1}}{64} = \frac{20}{64} = {}^5\!/_{16}.$$
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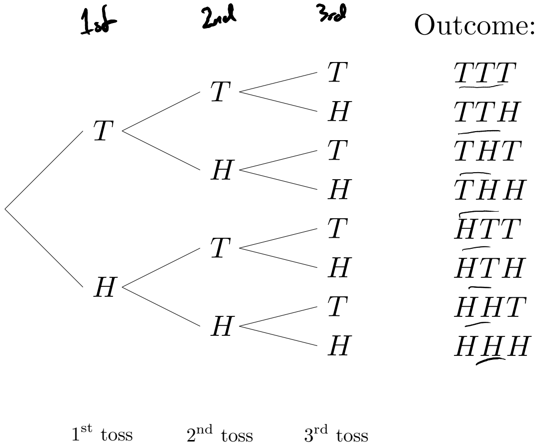
- Probabilities like these can be looked up in tables rather than calculated. Examples will be found in worksheet and assignment questions.

Slide42

Tree Diagrams, Fair and Unfair Coins, and the General Binomial Distribution

A tree representation of Coin-tossing

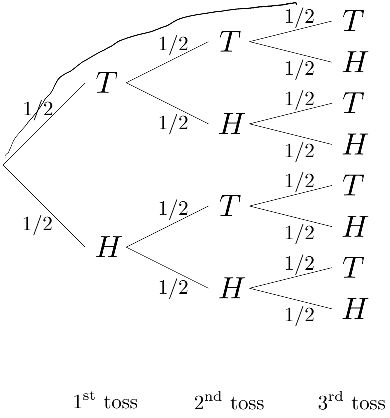
- Another way to list all the outcomes of an event is to draw a Tree Diagram of the Possibilities



Three tosses of a fair coin

Assuming independence

- This allows us to deal with fair coins, as before:



Outcome:

$\Pr(TTT) = 1/8$

$\Pr(TTH) = 1/8$

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$\Pr(THH) = 1/8$

$\Pr(HTT) = 1/8$

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Three tosses of a fair coin

Collecting possibilities from the tree and using the sum rule gives

$$\boxed{\mathbb{P}(0\text{heads}) = \frac{1}{8},\ \ \mathbb{P}(1\text{head}) = \frac{3}{8},\ \ \mathbb{P}(2\text{heads}) = \frac{3}{8},\ \ \mathbb{P}(3\text{heads}) = \frac{1}{8}}$$

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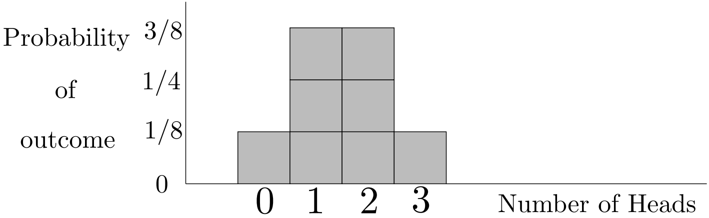

Three tosses of an unfair coin

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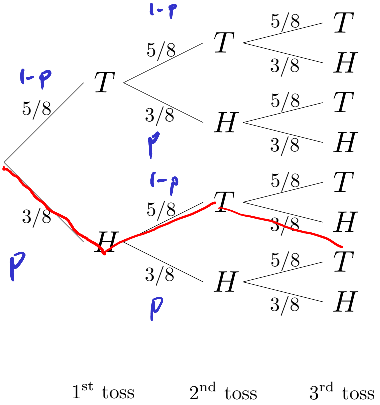

Outcome:

$\Pr(TTT) = \underline{125/512}$

$\Pr(\underline{TTH}) = \underline{75/512}$ –

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$\Pr(THH) = 45/512$

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$\Pr(HHH) = 27/512$

sum = 1

Three tosses of an unfair coin

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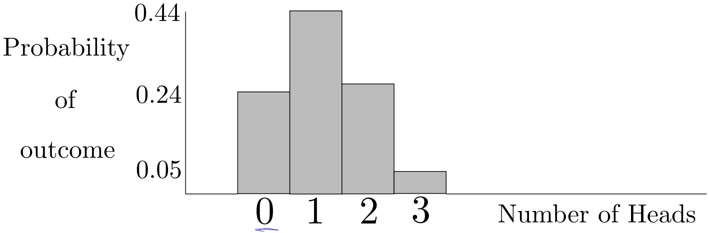
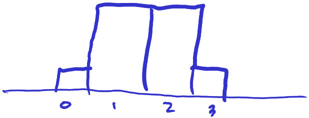


fair coin



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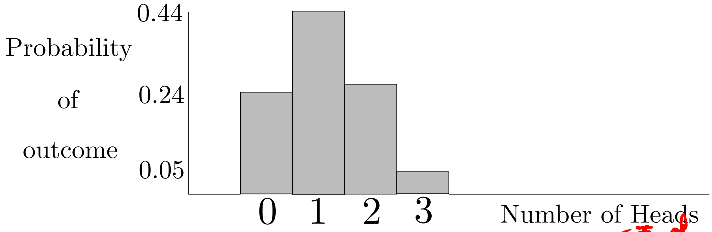

# of ways to get k successes

The general binomial density function for n trials (*e.g.* tosses) with probability p of a success (*e.g.* head) on each trial is given by

$$\boxed{\mathbb{P}(k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}.}$$

Review of Probability Density Functions with More Challenging Examples

Probability for equally likely outcomes (Review)

For a finite non-empty set S and $E \subseteq S$. the **probability of E for equally likely outcomes** is the number

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General probabilities on finite sets

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So $\mathbb{P}$ is the probability of equally likely outcomes.

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In a group of 10 students, 5 are studying computer science, 2 are studying art history, and 3 are studying mathematics. We pick a student from this group and ask what her/his major is.

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The associated event probabilities are

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The Monty Hall problem car, goats

A game: Three doors, with a prize behind one of them.
The contestant chooses one door.
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Theorem: Let S be a finite set and $\mathbb{P} : S \to \mathbb{Q}_+$ a probability density function. Then:

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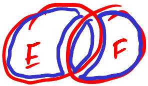

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(ii) For $E, F \in \mathcal{P}(S)$ with $E \cap F = \emptyset, \quad \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F).$

(iii) For *any* $E, F \in \mathcal{P}(S), \quad \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F).$

Compare to cardinality of sets.

Properties of probabilities

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(iv) $\mathbb{P}(E^c) = 1 - \mathbb{P}(E) \quad \forall E \in \mathcal{P}(S)$.

The birthday problem

paradox

In a group of 50 people, what is the probability that two people have the same birthday (assuming all birthdays are equally likely)?

The birthday problem

ignore leap years.

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$$\mathbb{P}(E^c) = \frac{|E^c|}{|S|} = \frac{\cancel{P(365, 50)}\ {}^{365}P_{50}}{365^{50}} = \frac{365!}{365^{50} \times 315!} \sim 0.03.$$

The birthday problem

prob > $\frac{1}{2}$ with 31 people

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Thus

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c) \sim 0.97.$$

There is a 97% chance that two people will have the same birthday.

Random Variables, Expected Values and Independence

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$S = \{\text{H,T}\}^3$ = set of outcomes of tossing three coins.

$X((a, b, c))$ = number of H's amongst a, b, c.

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average outcome if we measure X many times

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Example(cont.):

$$\mathbb{E}(X) = (\tfrac{1}{8})0 + (\tfrac{3}{8})1 + (\tfrac{3}{8})2 + (\tfrac{1}{8})3 = \frac{12}{8} = 1.5.$$

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Thus the expected value of X is just the mean (average) number of heads obtained when three coins are tossed.

Die roll example of expected value

Game: \$ 2 to play. Roll a die. Win \$10 if you get a 6.

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alternatively:

$= 8\, \mathbb{P}(\{X = 8\}) - 2\, \mathbb{P}(\{X = -2\})$

$= 8 \times \frac{1}{6} - 2 \times \frac{5}{6}$

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If you play this game 30 times, you should expect to *lo~~o~~se* $30(\frac{1}{3}) = 10$ dollars.

Independent Events

For a sample space S with probability density function $\mathbb{P} : S \to \mathbb{Q}_+$, $E, F \in \mathcal{P}(S)$ are called **independent events** when

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- $G = \{\text{HT,TH,HH}\}$ (at least one Head), $\mathbb{P}(G) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$,
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Stochastic independence

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  G, K are **not** independent (again as we would expect) since
  $$\mathbb{P}(G \cap K) = \mathbb{P}(\{\text{HT,TH}\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq \frac{9}{16} = \frac{3}{4} \times \frac{3}{4} = \mathbb{P}(G) \times \mathbb{P}(K).$$

Independent random variables

For a sample space S with probability density function $\mathbb{P} : S \to \mathbb{Q}_+$, $X, Y : S \to \mathbb{Q}$ are called **independent random variables** when

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Then for *any* $a, b \in \{0, 1\}$:

$\mathbb{P}(\{X=a\}) = \mathbb{P}(\{Y=b\}) = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$ and $\mathbb{P}(\{X=a\} \cap \{Y=b\}) = \frac{1}{4}$,

and hence the events $\{X=a\},\{Y=b\}$ are independent because

$$\mathbb{P}(\{X=a\}\cap\{Y=b\}) = \frac{1}{4} = \frac{1}{2}\times\frac{1}{2} = \mathbb{P}(\{X=a\})\times\mathbb{P}(\{Y=b\}).$$

Thus, by the above definition, X, Y are independent.

Independent random variables — Example

Toss a regular fair die. $S = \{1, \ldots, 6\}$, $\mathbb{P}(i) = \frac{1}{6}$, $i = 1, \ldots, 6$.

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Let $X, Y : S \to \mathbb{Q}$ be random variables as follows:

Table 1: Definition of X and Y

s	1	2	3	4	5	6
$s \bmod 2 = X(s)$	1	0	1	0	1	0

Independent random variables — Example

Toss a regular fair die. $S = \{1, \dots, 6\}$, $\mathbb{P}(i) = \frac{1}{6}$, $i = 1, \dots, 6$.
Let $X, Y : S \to \mathbb{Q}$ be random variables as follows:

Table 1: Definition of X and Y

s	1	2	3	4	5	6
$s \bmod 2 = X(s)$	1	0	1	0	1	0
$s \bmod 3 = Y(s)$	1	2	0	1	2	0

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Table 2: Probabilities

a	0	1	2
$\mathbb{P}(\{X = a\})$	$\frac{1}{2}$	$\frac{1}{2}$	0

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Table 3: s

$b =$	0	1	2
$a = 0$	6	4	2
$a = 1$	3	1	5

Independent random variables — Example

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Table 2: Probabilities

a	0	1	2
$\mathbb{P}(\{X=a\})$	$\frac{1}{2}$	$\frac{1}{2}$	0
$\mathbb{P}(\{Y=a\})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

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Using Table 2 it now follows that, for any $a \in \{0,1\}$ $b \in \{0,1,2\}$ the events $\{X=a\}, \{Y=b\}$ are independent because

Independent random variables — Example

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$$\mathbb{P}(\{X=a\}\cap\{Y=b\})=\frac{1}{6}=\frac{1}{2}\times\frac{1}{3}=\underbrace{\mathbb{P}(\{X=a\})}_{\frac{1}{2}}\times\mathbb{P}(\underbrace{\{Y=b\}}_{\frac{1}{3}}).$$

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$$\mathbb{P}(\{X=a\} \cap \{Y=b\}) = \frac{1}{6} = \frac{1}{2} \times \frac{1}{3} = \mathbb{P}(\{X=a\}) \times \mathbb{P}(\{Y=b\}).$$

Thus, by definition, the random variables X, Y are independent.

Non-independent random variables — Example

Let's modify the previous example just a little:

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Toss a regular fair die. $S = \{1, \ldots, 6\}$, $\mathbb{P}(i) = \frac{1}{6}$, $i = 1, \ldots, 6$.
Let $Y, Z : S \to \mathbb{Q}$ be random variables as follows:

Table 1: Definition of Y and Z

s	1	2	3	4	5	6
$s \bmod 3 = Y(s)$	1	2	0	1	2	0

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Toss a regular fair die. $S = \{1, \ldots, 6\}$, $\mathbb{P}(i) = \frac{1}{6}$, $i = 1, \ldots, 6$.
Let $Y, Z : S \to \mathbb{Q}$ be random variables as follows:

Table 1: Definition of Y and Z

s	1	2	3	4	5	6
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$s \bmod 4 = Z(s)$	1	2	3	0	1	2

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s	1	2	3	4	5	6
$s \bmod 3 = Y(s)$	1	2	0	1	2	0
$s \bmod 4 = Z(s)$	1	2	3	0	1	2

Table 2: Probabilities

a	0	1	2	3
$\mathbb{P}(\{Y = a\})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0

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Table 2: Probabilities

a	0	1	2	3
$\mathbb{P}(\{Y = a\})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
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$\mathbb{P}(\{Z = a\})$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Notice that, in the Table 1, many potential columns are not present.

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Notice that, in the Table 1, many potential columns are not present.
For example there is no value of s for which $Y(s) = 0$ and $Z(s) = 0$.

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Table 2: Probabilities

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Notice that, in the Table 1, many potential columns are not present.
For example there is no value of s for which $Y(s) = 0$ and $Z(s) = 0$.
Using this, and also Table 2, we now have

$$\mathbb{P}(\{Y = 0\} \cap \{Z = 0\}) = 0 \neq \frac{1}{3} \times \frac{1}{6} = \mathbb{P}(\{Y = 0\}) \times \mathbb{P}(\{Z = 0\}),$$

and so the events $\{Y = 0\}$, $\{Z = 0\}$ are not independent.

It follows that the random variables Y, Z are not independent.

Non-independent random variables — Example

Let's modify the previous example just a little:

Toss a regular fair die. $S = \{1, \ldots, 6\}$, $\mathbb{P}(i) = \frac{1}{6}$, $i = 1, \ldots, 6$.
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Table 2: Probabilities

a	0	1	2	3
$\mathbb{P}(\{Y=a\})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
$\mathbb{P}(\{Z=a\})$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Notice that, in the Table 1, many potential columns are not present.
For example there is no value of s for which $Y(s)=0$ and $Z(s)=0$.
Using this, and also Table 2, we now have

$$\mathbb{P}(\{Y=0\} \cap \{Z=0\}) = 0 \neq \frac{1}{3} \times \frac{1}{6} = \mathbb{P}(\{Y = 0\}) \times \mathbb{P}(\{Z = 0\}),$$

and so the events $\{Y = 0\}$, $\{Z = 0\}$ are not independent.

It follows that the random variables Y, Z are not independent.

Challenge: Are the random variables $\underset{\text{mod } 2}{X}, \underset{\text{mod } 4}{Z}$ independent?

Non-independent random variables — Example

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Table 1: Definition of Y and Z

s	1	2	3	4	5	6
$s \bmod 3 = Y(s)$	1	2	0	1	2	0
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Table 2: Probabilities

a	0	1	2	3
$\mathbb{P}(\{Y=a\})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
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Notice that, in the Table 1, many potential columns are not present.
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$$\mathbb{P}(\{Y=0\} \cap \{Z=0\}) = 0 \neq \frac{1}{3} \times \frac{1}{6} = \mathbb{P}(\{Y = 0\}) \times \mathbb{P}(\{Z = 0\}),$$

and so the events $\{Y = 0\}$, $\{Z = 0\}$ are not independent.

It follows that the random variables Y, Z are not independent.

Challenge: Are the random variables X, Z independent?

END OF SECTION C2

2 die, $S = \{1, \dots, 6\}^2$, $\mathbb{P}(j) = \frac{1}{36}$ $\forall j \in S$

X = first die roll, Z = second die roll

Y = sum of rolls

$\forall j \in \{1, \dots, 6\}$,

$\mathbb{P}(\{X = j\}) = \frac{1}{6}$, $\mathbb{P}(\{Y = 7\}) = \frac{1}{6}$,

$$\mathbb{P}(\{X=j\} \cap \{Y=7\})$$
$$= \mathbb{P}(\{X=j\} \cap \{Z = 7-j\}) \quad X, Z \text{ indep.}$$
$$= \mathbb{P}(\{X=j\})\, \mathbb{P}(\{Z = 7-j\})$$
$$= \underline{\frac{1}{6} \times \frac{1}{6}}$$

, might be indep.

$\mathbb{P}(\{X = 1\}) = \frac{1}{6}$, $\mathbb{P}(\{Y = 8\}) = \frac{5}{36}$,

$$\mathbb{P}(\{X=1\} \cap \{Y=8\}) = 0$$

not independent.

Non-independent random variables — Example

Let's modify the previous example just a little:

Toss a regular fair die. $S = \{1, \dots, 6\}$, $\mathbb{P}(i) = \frac{1}{6}$, $i = 1, \dots, 6$.
Let $Y, Z : S \to \mathbb{Q}$ be random variables as follows:

Table 1: Definition of Y and Z

s	1	2	3	4	5	6
$s \bmod 3 = Y(s)$	1	2	0	1	2	0
$s \bmod 4 = Z(s)$	1	2	3	0	1	2

Table 2: Probabilities

a	0	1	2	3
$\mathbb{P}(\{Y = a\})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
$\mathbb{P}(\{Z = a\})$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Notice that, in the Table 1, many potential columns are not present.
For example there is no value of s for which $Y(s) = 0$ and $Z(s) = 0$.
Using this, and also Table 2, we now have

$$\mathbb{P}(\{Y = 0\} \cap \{Z = 0\}) = 0 \neq \frac{1}{3} \times \frac{1}{6} = \mathbb{P}(\{Y = 0\}) \times \mathbb{P}(\{Z = 0\}),$$

and so the events $\{Y = 0\}$, $\{Z = 0\}$ are not independent.

It follows that the random variables Y, Z are not independent.

Challenge: Are the random variables X, Z independent?

END OF SECTION C2