Announcements:

Next Monday public holiday.

3 Monday 6005 workshops

3 replacements + make-up workshop Friday afternoon

Attend any of the four options.

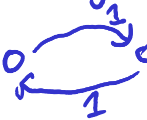
steady state:

$T S = S$

Recap:

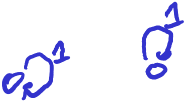
Markov processes.

steady states vs. long-run probabilities

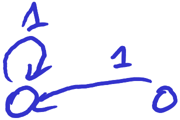


: has unique steady state.

Doesn't stabilise



: Every state is stationary



unique steady state

Any starting state tends to steady state

D1. Graph Theory

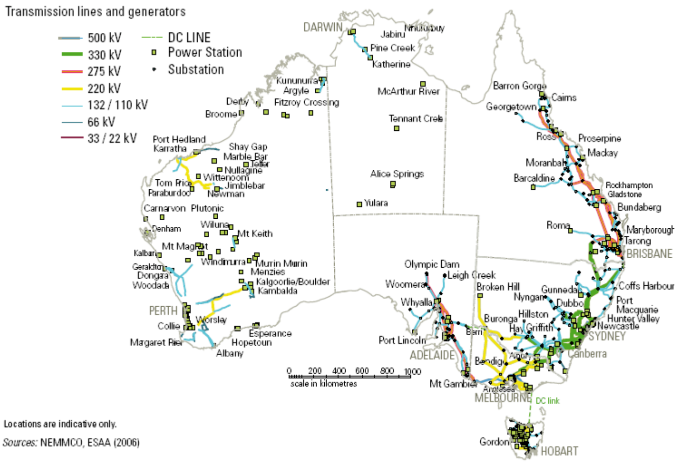
Notes originally prepared by Judy-anne Osborn.
Editing, expansion and additions by Malcolm Brooks.

Text Reference (Epp)	3ed:	Chapter 11
	4ed:	Chapter 10
	5ed:	Chapter 10

These references may not *completely* cover everything in this section, but they do have most it. They also contain a few items we do not cover.

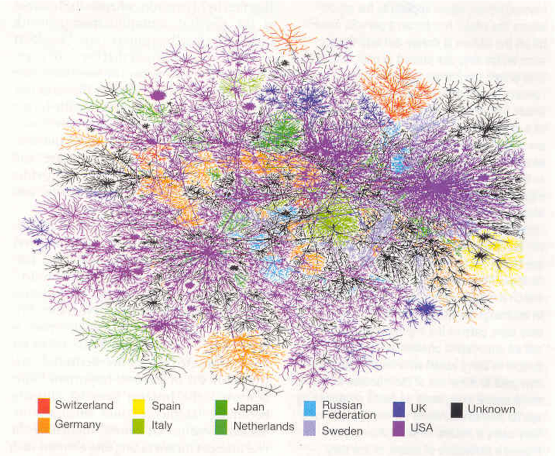
Real-world phenomena often modeled with graphs:

Australian Power Transmission Network

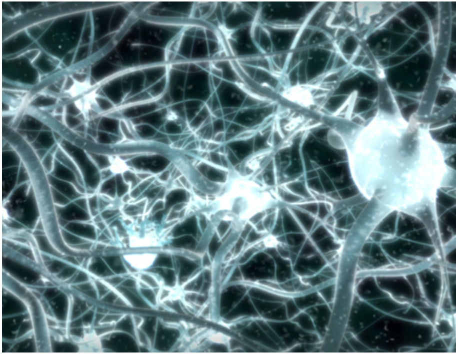


Complex Network Example: Internet

(William R. Cheswick)



Brain Network



from documentary, ‘Inside the living body’
http://abcnews.go.com/2020/popup?id=3560899

Graphs

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- Each edge is specified by a pair of vertices.
  (The two vertices could be the same.)
- In this course we only consider **finite** graphs;
  i.e. $V(G)$ and $E(G)$ are both finite sets.

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Diagrams of Graphs

To draw a graph we use:

- dots/circles for vertices
- lines for edges

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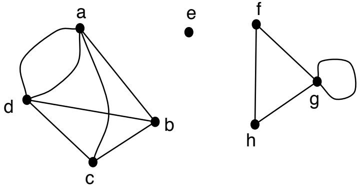

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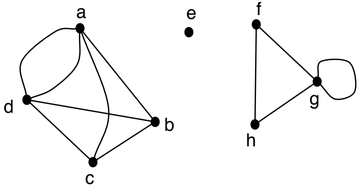

has vertex set $V(G) = \{a, b, c, d, e, f, g, h\}$
and edge multiset $E(G) =$
$\{\{a,b\}, \{a,c\}, \{a,d\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{f,g\}, \{f,h\}, \{g,g\}, \{g,h\}\}.$

A table of edges

- The same graph as a table of labelled edges:

Edge	Endpoints
e_1	$\{a, b\}$
e_2	$\{a, c\}$
e_3	$\{a, d\}$
e_4	$\{a, d\}$
e_5	$\{b, c\}$
e_6	$\{b, d\}$

Edge	Endpoints
e_7	$\{c, d\}$
e_8	$\{f, g\}$
e_9	$\{f, h\}$
e_{10}	$\{g, g\}$
e_{11}	$\{g, h\}$

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e_9	$\{f, h\}$
e_{10}	$\{g, g\}$
e_{11}	$\{g, h\}$

- Notice that e_3 is distinct from e_4 even though both edges have the same endpoints.

A vertex adjacency listing

- The same graph as a vertex adjacency listing:

Vertex	Adjacent to:
a	b, c, d, d
b	a, c, d
c	a, b, d
d	a, a, b, c
e	
f	g, h
g	f, g, h
h	f, g

An adjacency matrix

- An adjacency matrix for the same graph

$$
\begin{array}{c@{\;}c}
 & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array} &
\begin{bmatrix}
0 & 1 & 1 & \underline{2} & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\underline{2} & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

no change, because edges are undirected.

An adjacency matrix

- An adjacency matrix for the same graph

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

- The $(i, j)^{\text{th}}$ entry is the number of edges between vertices i and j.

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\begin{bmatrix}
0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

- The $(i, j)^{\text{th}}$ entry is the number of edges between vertices i and j.
- Thus $a_{i,j}$ is number of ways that i is adjacent to j.

Some Graph Terminology

- An edge connects its **endpoints**.

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- An edge with both endpoints the same is called a **loop**.
- Two edges may connect the same pair of endpoints, in which case they are said to be **parallel**.
- Two vertices are **adjacent** if they are connected by an edge; two edges are **adjacent** if they share an endpoint.

Some Graph Terminology

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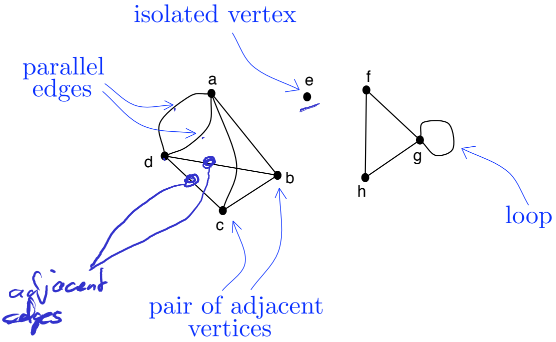
- An edge is **incident on** its endpoints.
- A vertex with no incident edges is **isolated**.
- A graph with no vertices (hence no edges) is **empty**.
- The **order** of a graph, G, is the number of vertices in it, i.e. $|V(G)|$.
  (A graph of order ‘0’ is empty.)

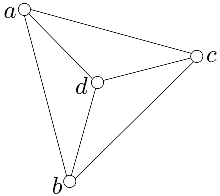
Some graph concepts illustrated



Another example

- Tetrahedron Graph

$G:$



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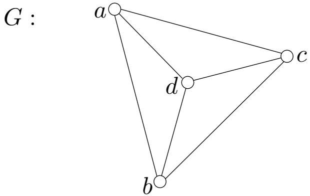
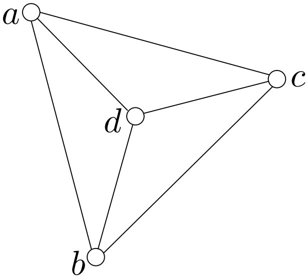


  - $V(G) = \{a, b, c, d\}$
  - $E(G) = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$



- Tetrahedron Graph:

Adjacency listing:

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b	a, c, d
c	a, b, d
d	a, b, c

Adjacency matrix:

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

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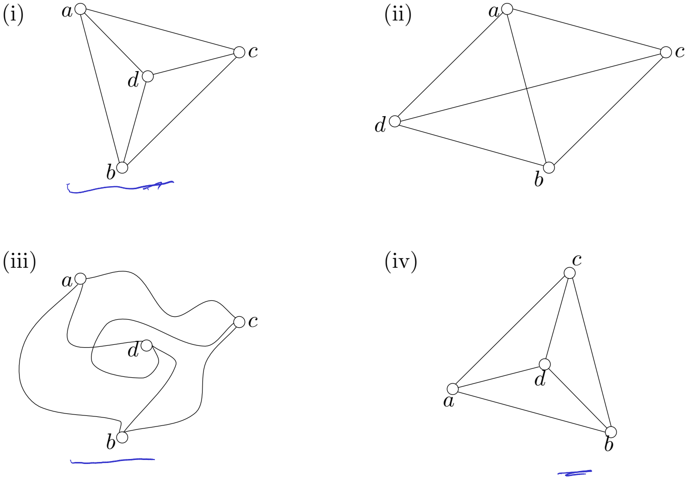
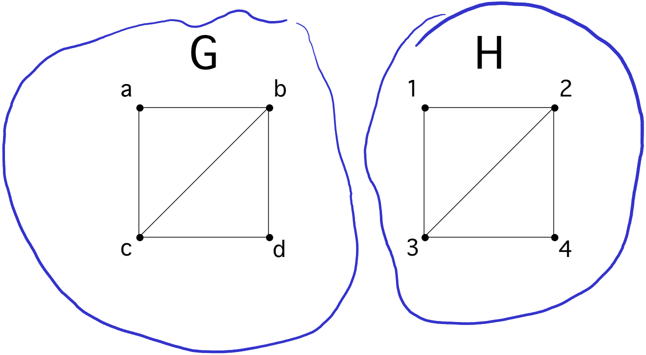
- For instance, the following diagrams all represent the same graph.

Four diagrams of the same graph



Isomorphic Graphs

Consider graphs G and H as pictured:



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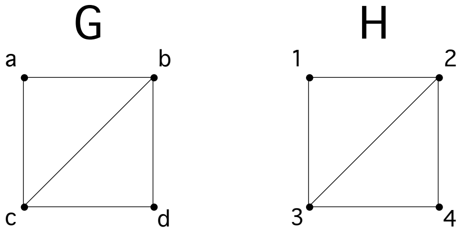

- They are different graphs because their vertex labels are different.

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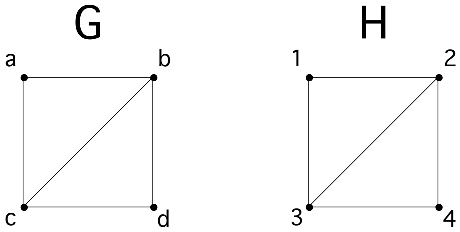

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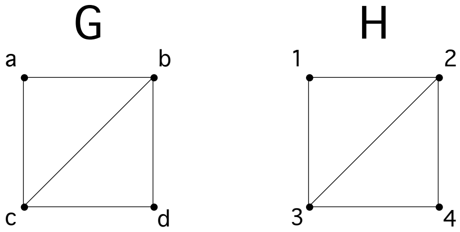

- They are different graphs because their vertex labels are different.
- But they are the same in some sense.
- Formally these graphs are called ‘isomorphic’.

Isomorphisms

An **isomorphism** between two graphs G_1 and G_2 is a bijection

$$f : V(G_1) \to V(G_2)$$

such that:

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such that:

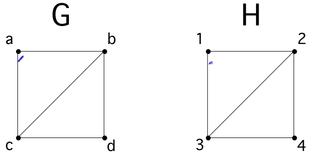
$\{u, v\}$ is an edge in $E(G_1)$

if and only if

with multiplicity

$\{f(u), f(v)\}$ is an edge in $E(G_2)$

Isomorphisms



An example of an isomorphism between G and H is the mapping

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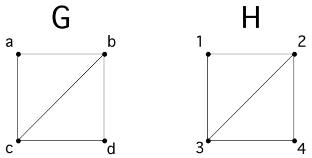
Isomorphisms



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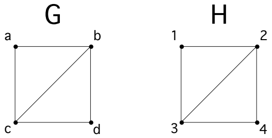
Isomorphisms



The mapping ‘preserves’ edges:

$$\begin{array}{cccc}
\{a,b\} & \mapsto & \{1,2\} & \checkmark \\
\{a,c\} & \mapsto & \{1,3\} & \checkmark \\
\{b,c\} & \mapsto & \{2,3\} & \checkmark \\
\{b,d\} & \mapsto & \{2,4\} & \checkmark \\
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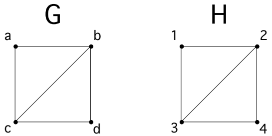
\end{array}$$

Isomorphisms



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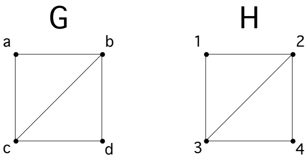
....and non-edges:

$$\{a,d\} \mapsto \{1,4\} \quad \checkmark$$

Isomorphisms



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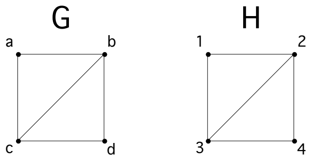
$$g : V(G) \quad \rightarrow \quad V(H)$$

Isomorphisms



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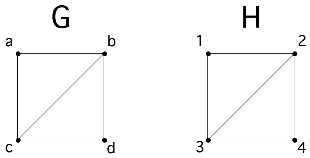
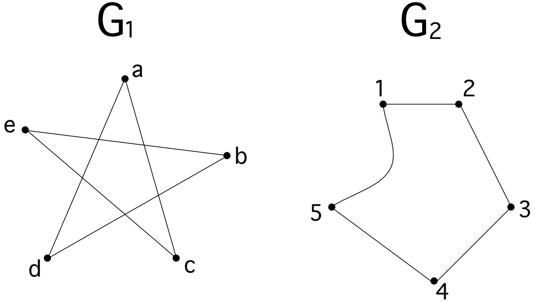
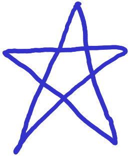
Example: Graphs G and H are isomorphic.



Isomorphic Graphs

- The following graphs pictured are isomorphic.





Isomorphic Graphs

$a \mapsto 1$
$c \mapsto 2$
$e \mapsto 3$
$b \mapsto 4$
$d \mapsto 5$

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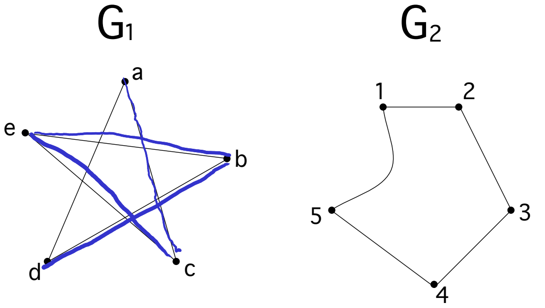

- Can you specify an explicit isomorphism between them?

Directed Graphs

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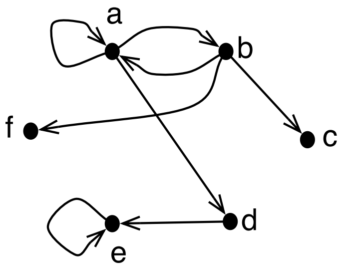
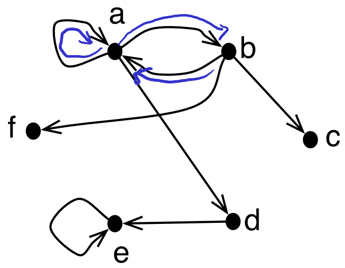
- Each edge has an **initial vertex** and a **final vertex**.
- Loops are still allowed.
- The edges of a digraph are sometimes called **arcs**.
- In a diagram of a digraph, the direction of an arc is given by an arrow.

Example



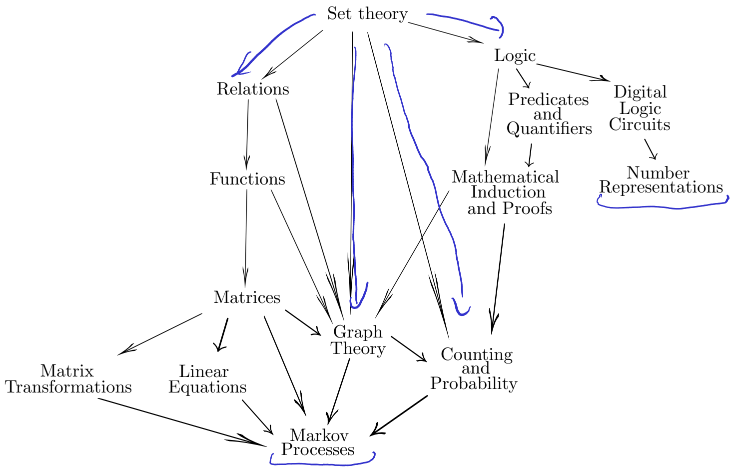
Example



The vertex set and edge set for this graph are:

$$
\begin{aligned}
V(G) &= \{a, b, c, d, e, f\} \\
E(G) &= \{(a,a), (a,b), (a,d), (b,a), (b,c), (b,f), (d,e), (e,e)\}
\end{aligned}
$$

An application: Recording Information Dependencies



An arrow from A to B means that B depends upon A in some way.

Foodwebs

- An application of digraphs in ecology is in describing a *foodweb*.

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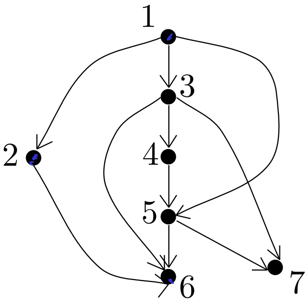
- The next slide shows a foodweb developed by Parsons and LeBrasseur, as adapted by Cohen, pertaining to the following species in the Strait of Georgia, British Columbia.

KEY	SPECIES
1.	Juvenile pink salmon
2.	P. Minutus
3.	Calanusand Euphausiid Burcillia
4.	Euphausiid Eggs
5.	Euphausiids
6.	Chaetoceros Socialis and Debilis
7.	Mu-Flagellates

Example

An arrow from i to j means *'i eats j'*:



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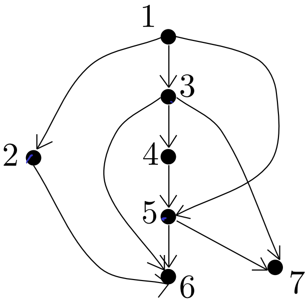

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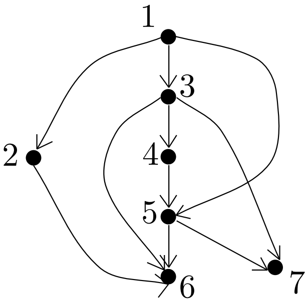

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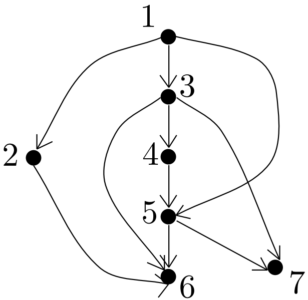

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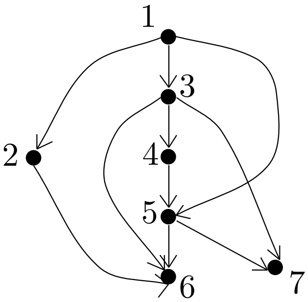

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We shall use the first convention unless stated otherwise.

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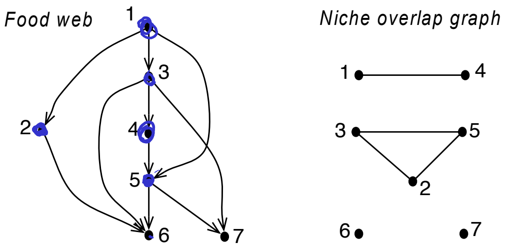
- The following niche overlap graph is constructed from the Food Web data of the previous example.

Food Webs and Niche Overlap Graphs



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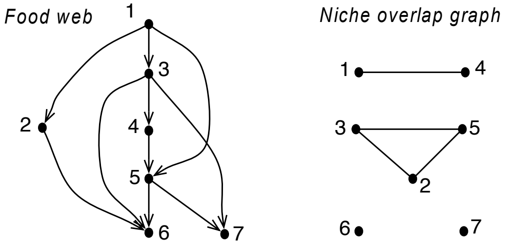

- For example:
  species 1 and 4 compete for food (species 5), so are connected by an edge in the niche overlap graph.

Types of Graphs and Digraphs

Sometimes it is useful to restrict our attention to two types of graphs and digraphs, called:

- **Simple Graphs** and
- **Simple Digraphs**

Simple Graphs

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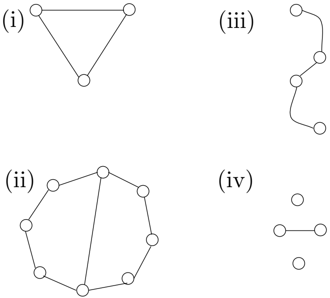
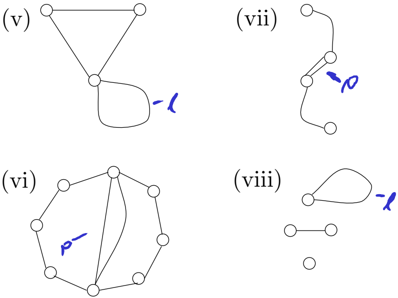


Some simple Graphs



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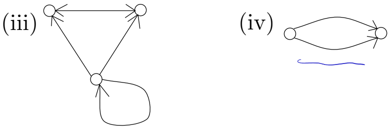

Some non-simple Digraphs

Don't allow duplicate ordered pairs in our edge set

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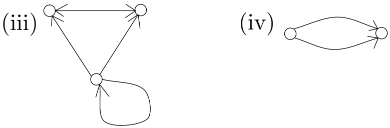

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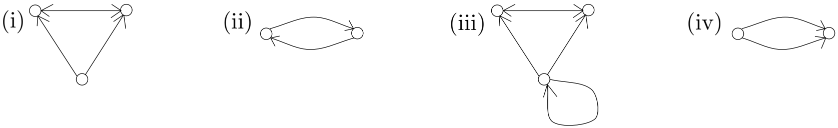

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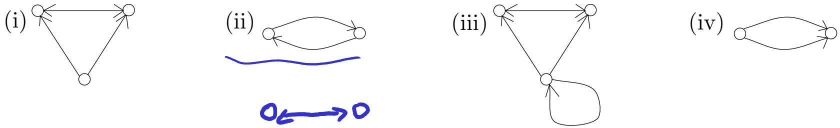

Some simple Digraphs *Some non-simple Digraphs*

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- It is okay to have both (a, b) and (b, a) - these are not parallel;
- It is not okay to have (a, b) twice - those would be parallel.
- We sometimes draw a single edge with an arrow at each end to indicate a pair of edges (a, b) and (b, a), instead of drawing two distinct lines.

Special simple graphs I: Complete Graphs

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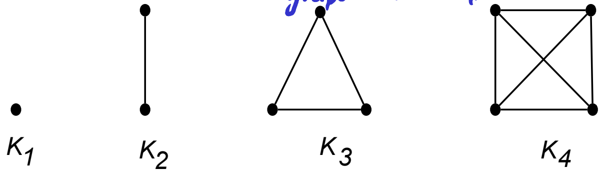
If G, H have n vertices and are complete, then any bijection $f : G \to H$ is going to preserve edges so is a graph isomorphism

Examples:



How many edges in K_n?

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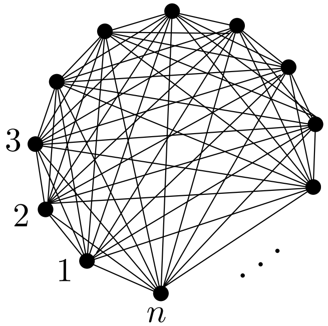

$\binom{n}{2}$

since we need an edge for every pair of vertices.

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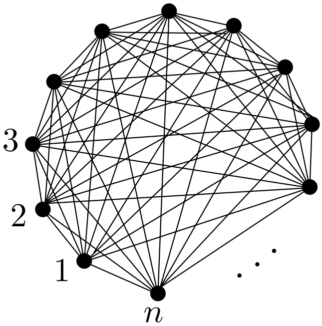

The answer is $\boxed{\binom{n}{2} = \frac{n(n-1)}{2}}$. ~~Why?~~

Special simple graphs II: Bipartite Graphs

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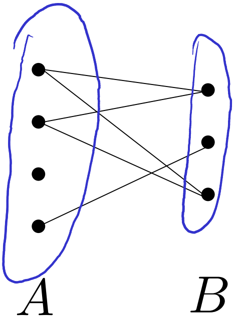

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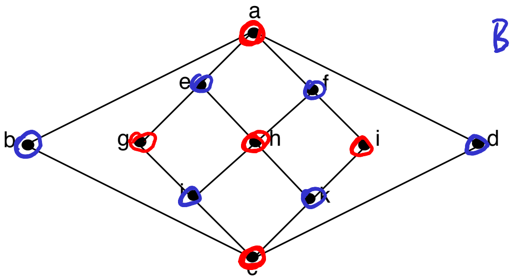

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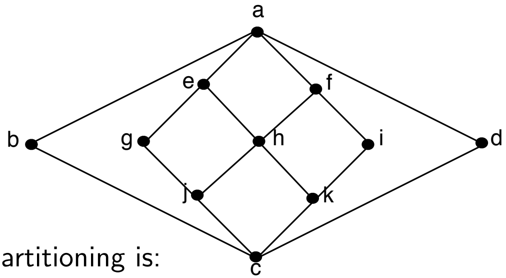


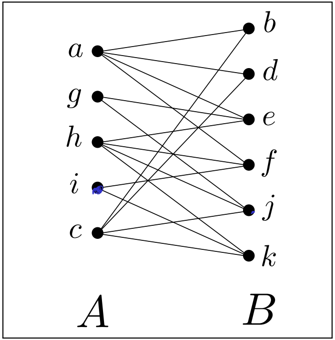
The vertex partitioning is:

$$\boxed{\begin{aligned} A &= \{a, g, h, i, c\} \\ B &= \{b, d, e, f, j, k\} \end{aligned}}$$

Larger example continued

planar graphs
are all 4-partite
4 colourable

This is the same graph, redrawn.



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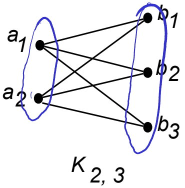

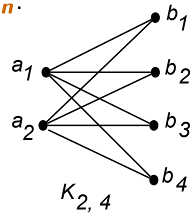

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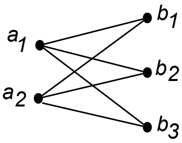
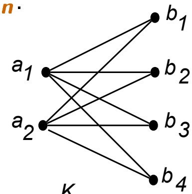


$K_{2,3}$

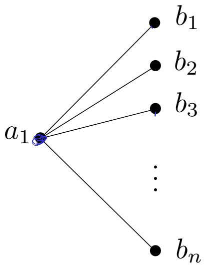


$K_{2,4}$

Caution: The name "Complete Bipartite Graph" is misleading. Except for $K_{1,1}$, such graphs are **not complete graphs**.
The adjective 'complete' is qualifies 'bipartite', not 'graph'.

How many edges in $K_{1,n}$?

- *How many edges in the complete bipartite graph $K_{1,n}$?* n



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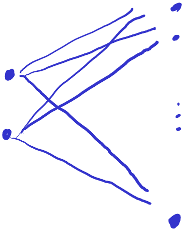

- *How many edges in* $K_{1,n}$*?*
- *How many edges in* $K_{2,n}$*?* $2n$

$\vdots$

- *How many edges in* $K_{m,n}$*?* mn

Subgraphs

- A **subgraph**, S, of a graph G, is a graph whose vertices are a subset of $V(G)$ and whose edges are a subset of $E(G)$, i.e.

$$V(S) \subseteq V(G)$$
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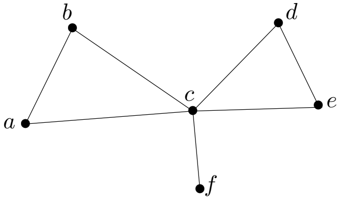
G is a subgraph
of G

- Since S is a graph, if the edge

$$\{a, b\} \in E(S),$$

we require its endpoints to be in $V(S)$, i.e. $a, b \in V(S)$.

A subgraph example:

Let G be the graph: 

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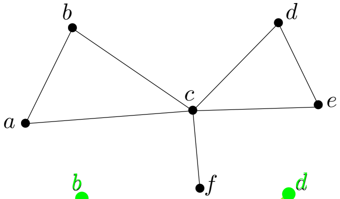
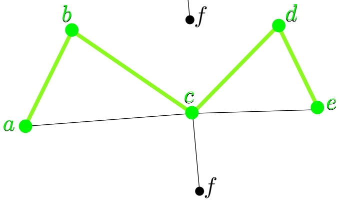


Select some edges and vertices:



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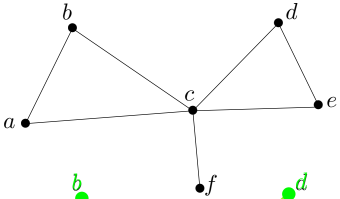

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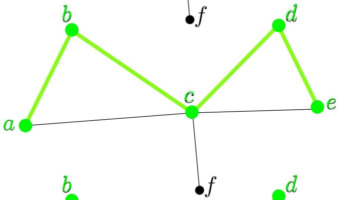
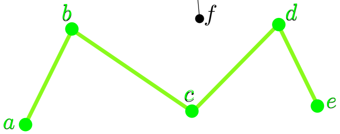


Now S is a subgraph of G:



Another subgraphs example:

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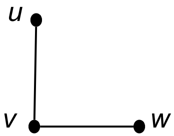

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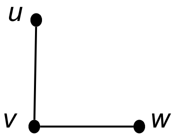

The following five graphs (numbered 1-5) are each subgraphs of G:

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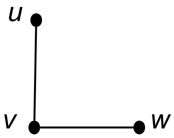

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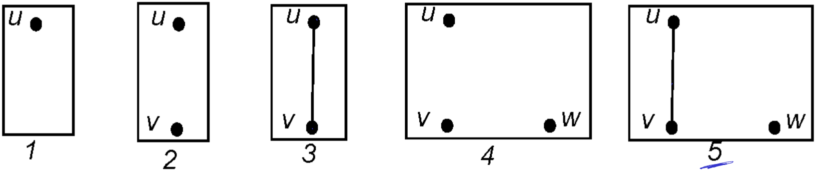

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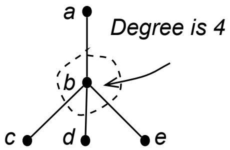

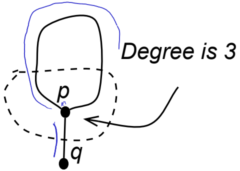

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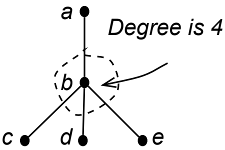

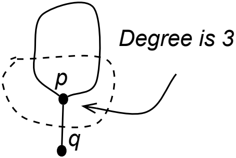

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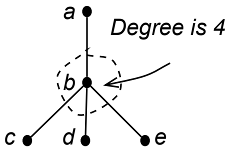

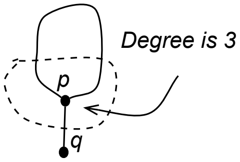

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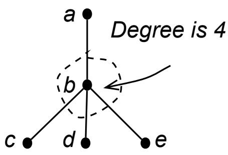

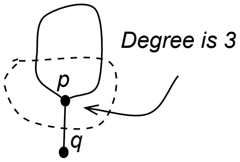

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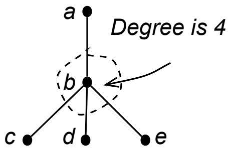

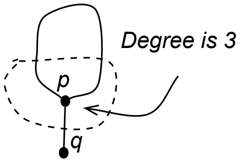

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We shall always use the 'adding two' version.

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The **total degree** of a graph G is the sum of the degrees of all its vertices, $\sum_{v \in V(G)} \deg(v)$.

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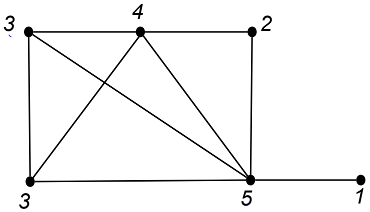

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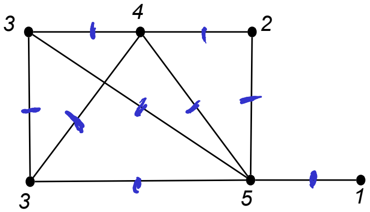

edges = 9

The total degree of the graph is $3 + 4 + 2 + 3 + 5 + 1 = 18$.

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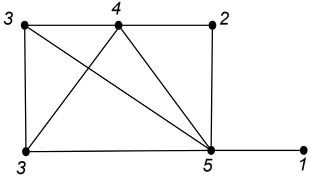

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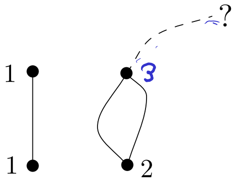
- However we try, we always end up with an edge that doesn't have a vertex to connect to:



A useful abbreviation

We often abbreviate

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Note that

- for graphs, ab means *the same* as ba, since $\{a, b\} = \{b, a\}$; but
- for digraphs, ab is *different* from ba since $(a, b) \neq (b, a)$.

Walks on Graphs

The bridges of Königsberg

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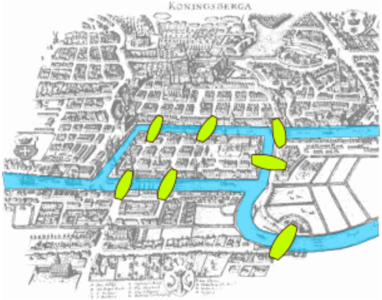

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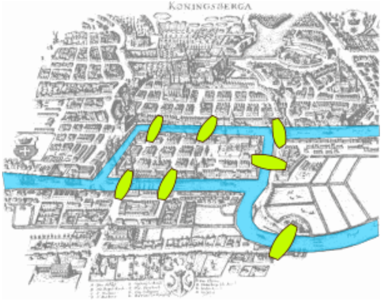

Adapted from:

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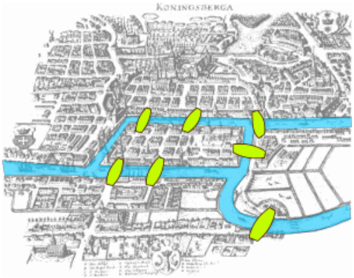
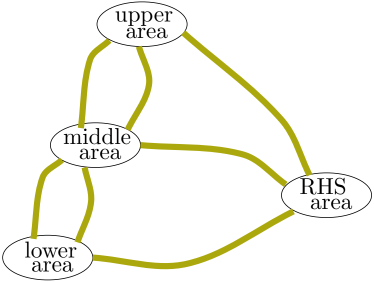
The bridges of Königsberg

Leonard Euler realized that the task can be modeled as a problem in graph theory, which he invented for the purpose.





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Walks

- A **walk** in a graph is a sequence of vertices alternating with edges:

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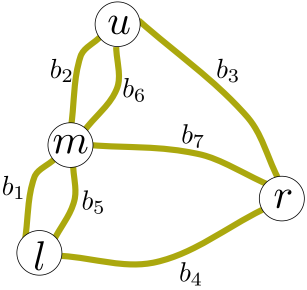
- A **trivial walk**, say v_0, contains no edges; hence has length 0.

Walks on the Königsberg Graph

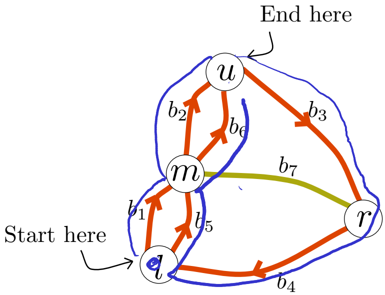
The question becomes, 'Is there a walk on the Königsberg Graph which traverses each edge exactly once?'



Walks on the Königsberg Graph

Try starting in the lower part of town, going via bridge b_1 to the middle island, then via bridge b_2 to the upper part, then via bridge b_3 to the right-most part; continuing as in the listed walk:

$$l b_1 m b_2 u b_3 r b_4 l b_5 m b_6 u$$



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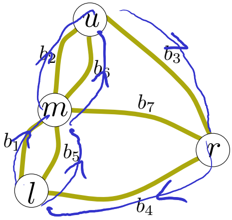

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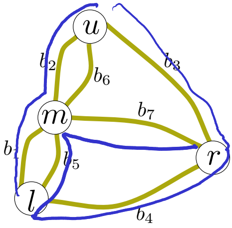

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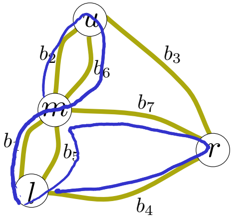

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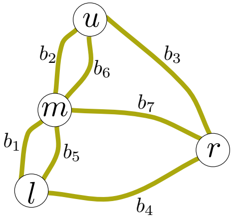

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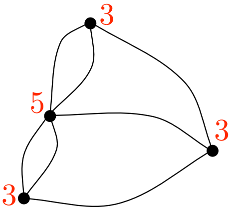
- *How? Think about the degrees of the vertices:*



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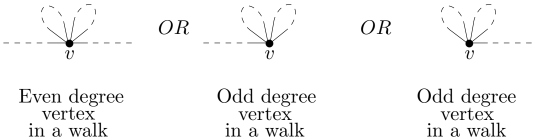
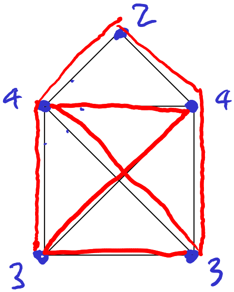


A related puzzle

You may have come across the following puzzle:

Can you draw the following 'house' diagram without taking your pen off the paper or overwriting edges?

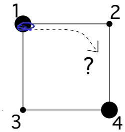


Can you?

Yes, start and end at an odd degree vertex.

Walks, paths and circuits

Counting Walks with Adjacency Matrices

The graph 

has adjacency matrix $M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}.$

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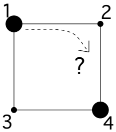
The graph  has adjacency matrix $M = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \multicolumn{4}{c}{\multirow{4}{*}{$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$}} \\ 2 \\ 3 \\ 4 \end{array}$.

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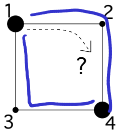
The graph 

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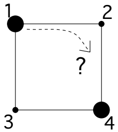
The graph 

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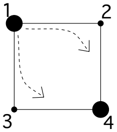

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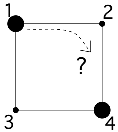
The graph  has adjacency matrix
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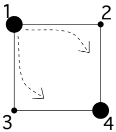

$$M^2 = \begin{array}{c} \begin{array}{cccc} 1\text{-}1 & 1\text{-}2 & 1\text{-}3 & 1\text{-}4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \begin{array}{cccc} & & & 1\text{-}4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ 2\text{-}4 \\ 3\text{-}4 \\ 4\text{-}4 \end{array}$$

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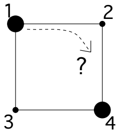
The graph 

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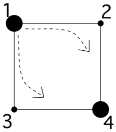

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For any graph, the number of ways to walk from vertex i to vertex j in t steps is given in terms of its adjacency matrix M by the $(i, j)^{\text{th}}$ entry of M^t.

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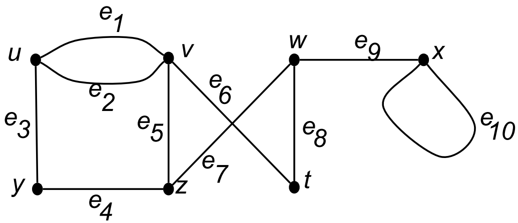
We will now look at each of these potential properties in turn, with examples using the graph G below.



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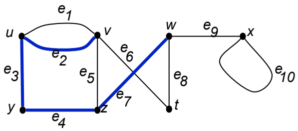

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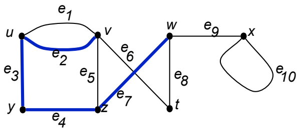

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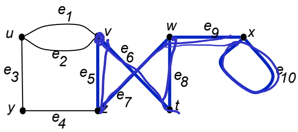

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$$v\ \mathbf{e_6}\ t\ \mathbf{e_8}\ w\ \mathbf{e_9}\ x\ \mathbf{e_{10}}\ x\ \mathbf{e_9}\ w\ \mathbf{e_7}\ z\ \mathbf{e_5}\ v$$

has length 7 and **is** closed because $v = v_0 = v_n = v_7 = v$.

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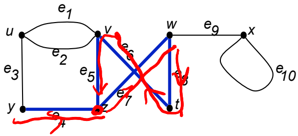

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has length 5 and is a **path** because the five edges are all different, but is **not simple** because $z = v_1 = v_5$

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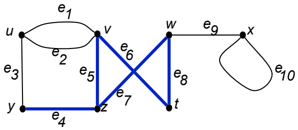

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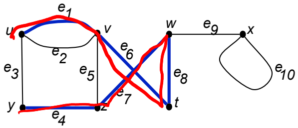

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has length 5 and is a **simple path** because the six vertices are all different.

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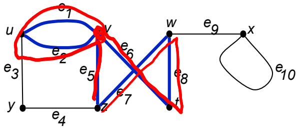

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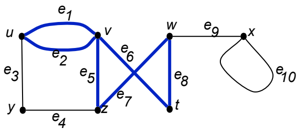

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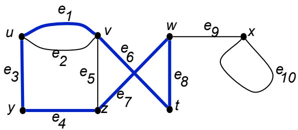

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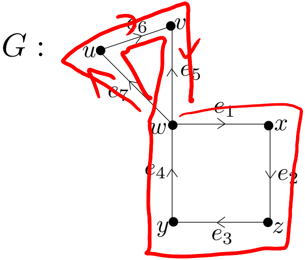

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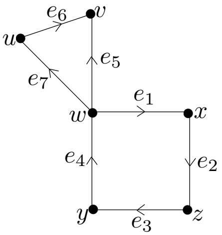

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- Note that for a simple graph or digraph (with no loops and no parallel edges), **any** walk is uniquely determined by its sequence of vertices.

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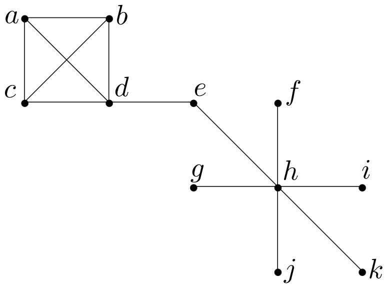
Here is an example of a connected graph:



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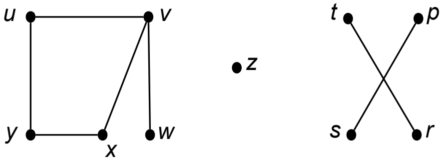

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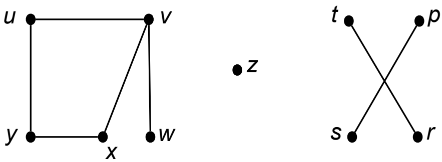

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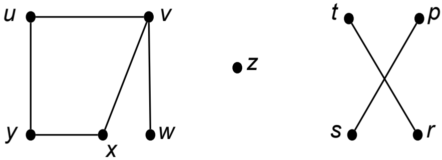

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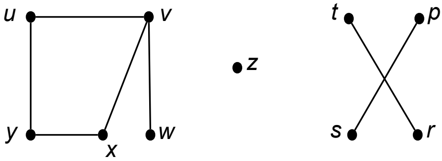

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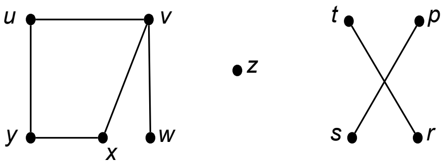

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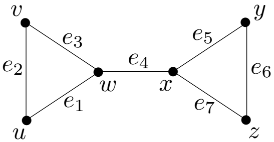

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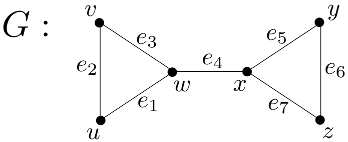

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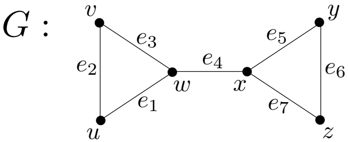

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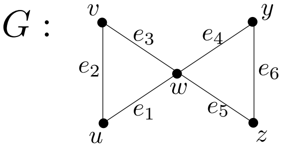

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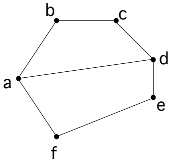

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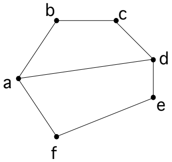

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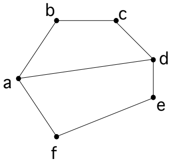

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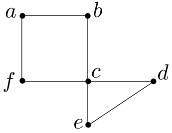

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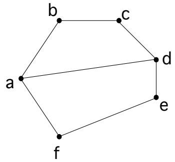

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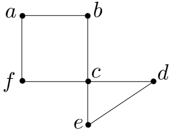

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Note: By convention, an Euler path must be open, *i.e* not a circuit.

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- Always leave an edge to return to the start vertex as the last step.

Fleury's Algorithm for finding Euler Circuits

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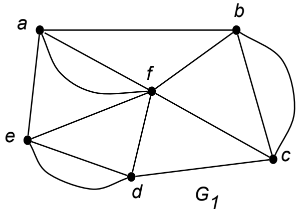
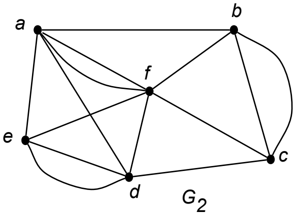
5. Repeat steps 2 - 4 until all edges have been traversed, and you are back to the starting vertex.

Candidate Graphs for Fleury's Algorithm

The graph G_1 below satisfies the criterion that all vertices have even degree, so it contains an Euler circuit and Fleury's algorithm can be used to find that circuit.





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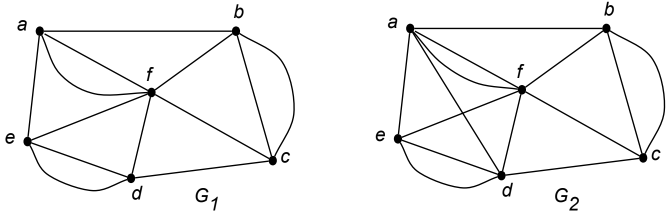
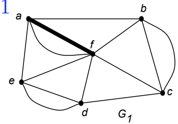


The graph G_2 has two vertices of odd degree. Fleury's algorithm can be modified to find an Euler path in this graph. The only modification needed is that the first vertex must be one of the vertices of odd degree.

Fleury's Algorithm example in pictures

Start at f (just because we feel like it!)

1



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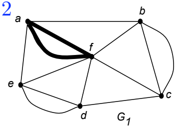

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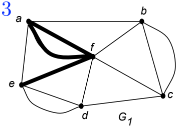

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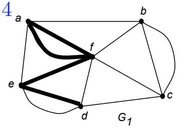

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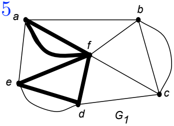

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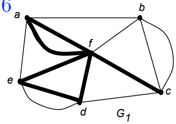

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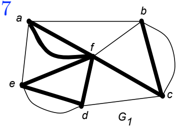
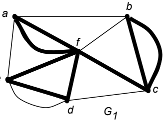


Notice that from here
we MAY NOT step to f
but either a or c is allowed

Fleury's Algorithm example in pictures

At step 9 we're forced to go to d; and then all steps are forced.

8


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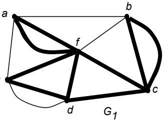

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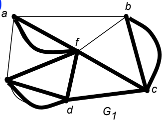

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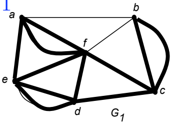

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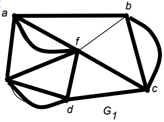

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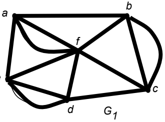

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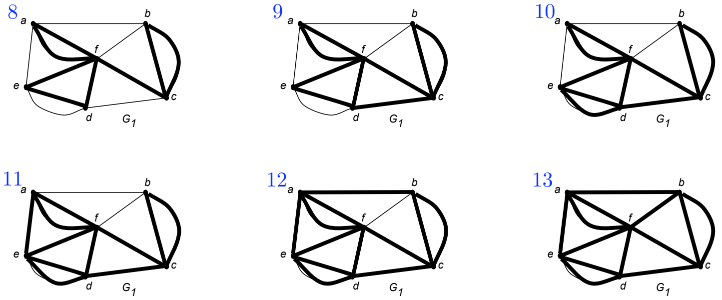

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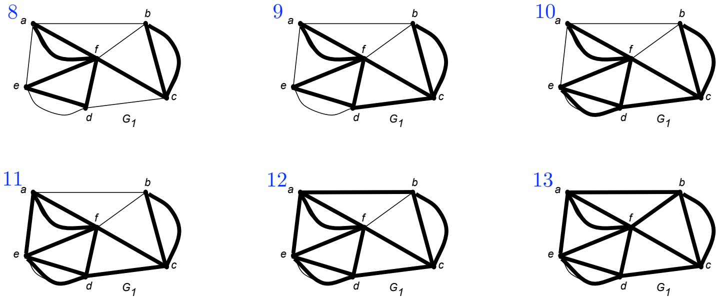

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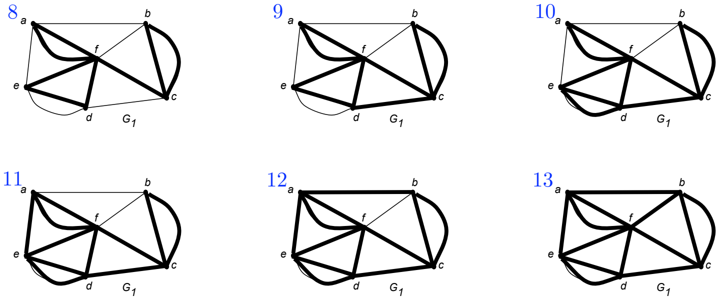

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(As mentioned earlier, we do not call this path an Euler *path*, because, by convention, Euler paths ar not closed.)

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- In any implementation, we never have to back-track, so the algorithm is quite fast; as are some other algorithms to solve this problem.
- By contrast, the following problem – that of finding a *Hamilton* path or circuit – has no known 'fast' algorithm.

Hamilton Paths and Circuits

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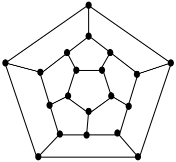
Note: By convention, a Hamilton path must be open, *i.e* not a circuit.

Graphs and Hamilton Paths / Circuits

- This graph has a Hamilton Circuit. *Can you find one?*



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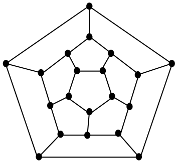
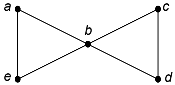


- This graph has no Hamilton circuit. *Why not?*



Types of Walks on Graphs – Summary

For a walk $v_0, e_1, v_1, e_2, \ldots, e_n, v_n$ on a graph G:

Properties					
Name:	**closed**	—	**Euler**	**simple**	**Hamilton**
Description:	—	no repeated edges	uses all edges	no repeated vertices	uses all vertices
Requirement:	$v_0 = v_n$	$i \neq j \implies e_i \neq e_j$	$\forall e \in E(G)$ $\exists i\ e_i = e$	$i \neq j \implies v_i \neq v_j$ ($v_0 = v_n$ OK)	$\forall v \in V(G)$ $\exists i\ v_i = v$
path		✓			
simple path		✓		✓	
Euler path		✓	✓		
Hamilton path		✓		✓	✓
closed walk	✓				
circuit	✓	✓			
simple circuit	✓	✓		✓	
Euler circuit	✓	✓	✓		
Hamilton circuit	✓	✓		✓	✓

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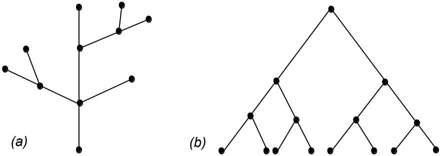

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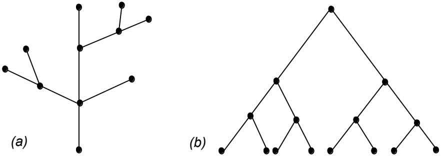

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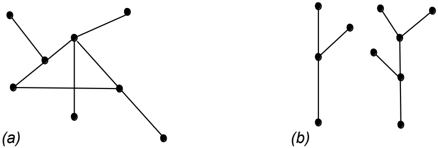

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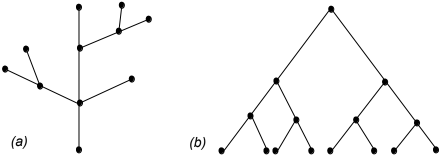

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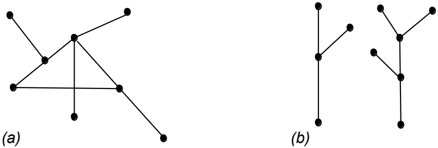

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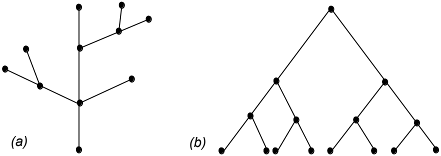

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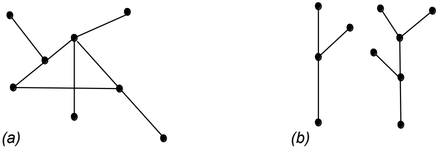

  - Example (a) contains a circuit, so is not a tree.
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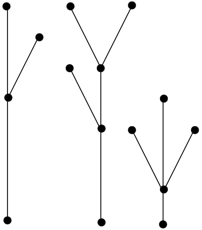

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- In the representation of such molecules we use C to represent a carbon atom and H to represent a hydrogen atom. These will be used instead of dots for the vertices.

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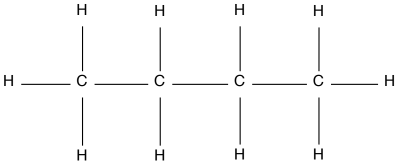

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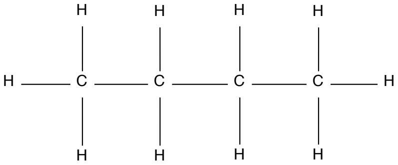

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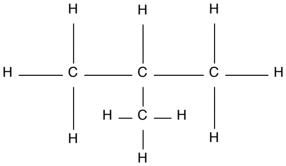

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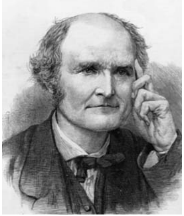
Some History of Trees



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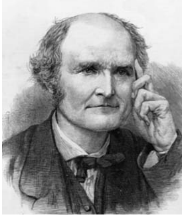

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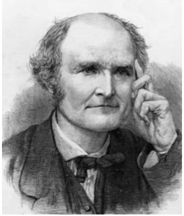

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- You will explore a proof of this formula in Workshops next week.

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Theorem: Let T be a graph with n vertices. The following statements are logically equivalent:

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(vi) T contains no non-trivial circuits, but the addition of any new edge (connecting an existing pair of vertices) creates a simple circuit.

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- To prove the Theorem, we should show that any statement in the list is derivable from any other statement. One way to do this is to show the chain of implications:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i).$$

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- *Can you prove that* $(i) \Rightarrow (j)$ *for any of the statements in the Tree-Characterising Theorem?*

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  Lemma A: Any tree that has more than one vertex has at least one vertex of degree 1.

  Lemma B: If G is any connected graph, C is a non-trivial circuit in G, and one of the edges of C is removed, then the subgraph that remains is connected.

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- Problems of efficiently planning routes for mail delivery, garbage pickup, diagnostics in computer networks, and so on, can be solved using models that involve graphs.
- A subgraph of a connected graph that provides a unique path between any two vertices is a *spanning tree*. These trees have applications in many fields, including engineering.

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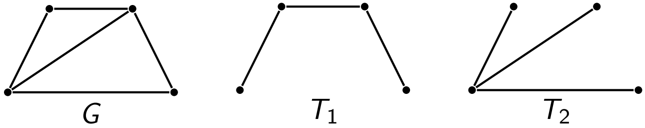

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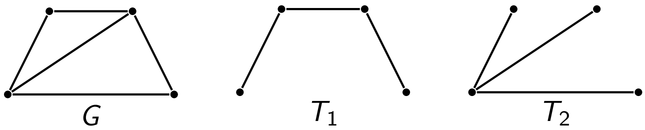

How many more spanning trees can you find for G?

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   Hint: See (ii) or (iii) of the tree characterisation theorem.

A method to make a spanning tree

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2. Pick an edge.

A method to make a spanning tree

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The above description of the algorithm is intended for ‘pen-and-paper’ work.

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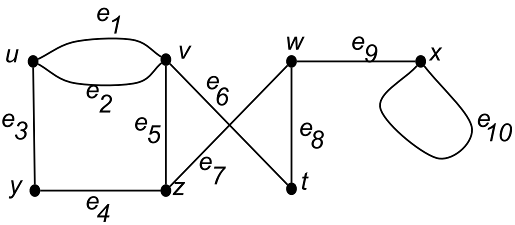
The above description of the algorithm is intended for 'pen-and-paper' work.

For computer implementation it is necessary to **also**:
— at step 1 initialize a 'pool of potential edges' P to $E(G)$,
— at step 2 ensure the picked edge e comes from P and
— after step 3 remove e from P (whether it contributes to T or not).

Building a spanning tree: example

To demonstrate the spanning tree algorithm we will use a graph we have seen before:



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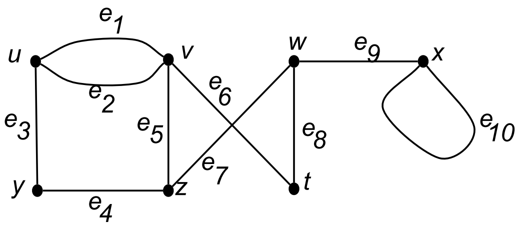
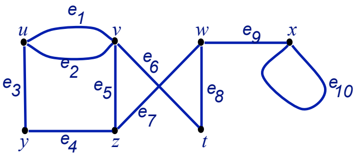


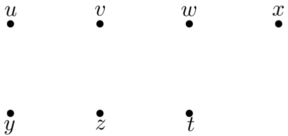
Initialize T to

- vertex set $V(T) = \{u, v, w, x, y, z, t\}$,
- edge set $E(T) = \{\}$:

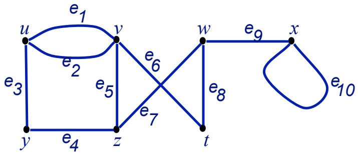
Building a spanning tree for



$E(T) = \{\}$: $\quad T$:



Building a spanning tree for



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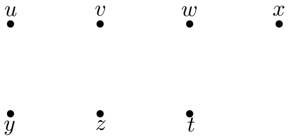
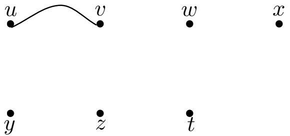
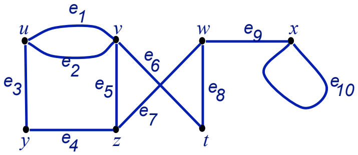


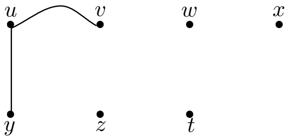
$E(T) = \{e_1\}$ $T:$



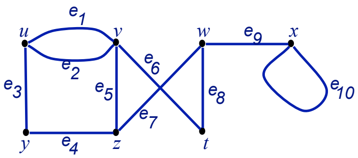
Building a spanning tree for



$E(T) = \{e_1, e_3\}$: T :



Building a spanning tree for



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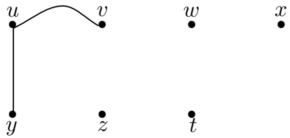
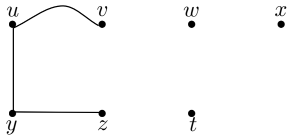
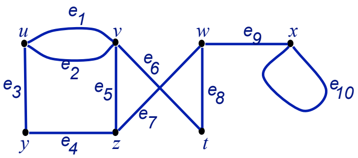


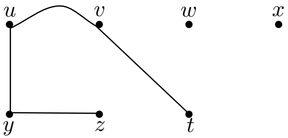
$E(T) = \{e_1, e_3, e_4\}$ $T:$



Building a spanning tree for



$E(T) = \{e_1, e_3, e_4, e_6\}$: $\quad T:$



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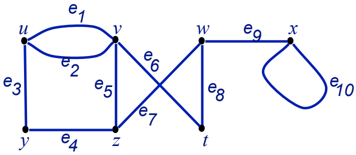

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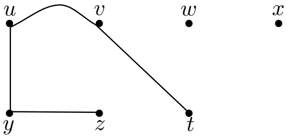
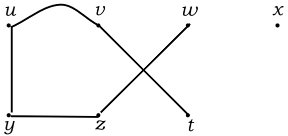
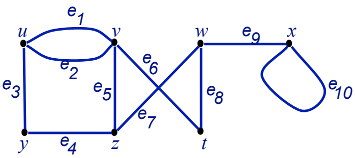


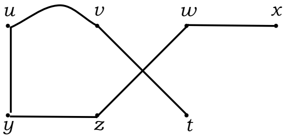
$E(T) = \{e_1, e_3, e_4, e_6, e_7\}$ T :



Completed spanning tree for



$E(T) = \{e_1, e_3, e_4, e_6, e_7, e_9\}$: T :



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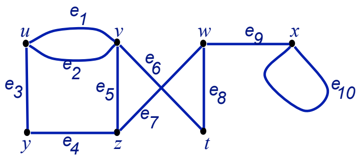

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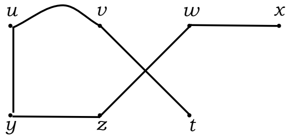
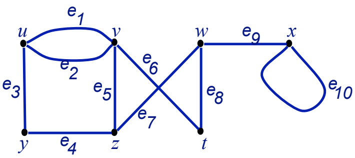


Our tree now has $7 - 1 = 6$ edges, so we are done (as you can see).

Completed spanning tree for



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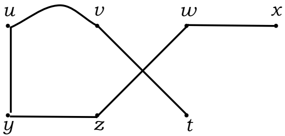

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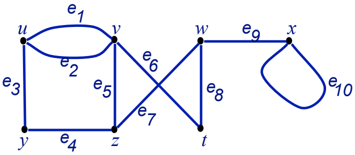

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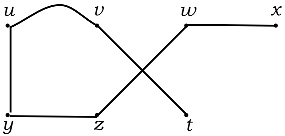

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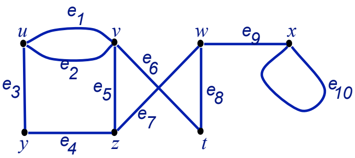
Can you make a spanning tree in which the longest path has length 4?

Completed spanning tree for



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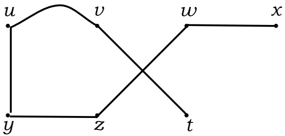

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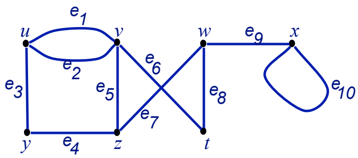
Can you make a spanning tree in which the longest path has length 4? Length 3?

Completed spanning tree for



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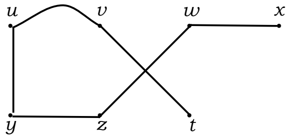

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END OF SECTION D1