This is Assignment 6 for MATH1005 students in a Friday workshop. It is due at 6 pm on the Thursday after Workshop 6 (6 days after was released).

There are four problems. The numbering of the problems is strange because the numbering is taken from a much larger document that has many problems from which I can select. As long as you can see four different problems, then you have the complete assignment.

You should write your best solutions to the problems here, and then upload your solutions before the due time. Here are three ways you may complete the assignment:

1. Print the assignment sheet. Write your solutions in pen or pencil on the print out. Scan your completed assignment, turn the file into a single .pdf file, then upload your solution file to Wattle.

2. Write your solutions in pen or pencil on blank paper. You should clearly label your solutions and you should write them in the order in which the problems appear in your assignment. Scan your completed assignment, turn the file into a single .pdf file, then upload your solution file to Wattle.

3. Download the assignment sheet to a tablet. Annotate the file using your favourite annotation software. **Flatten the file**–this makes your annotations a permanent part of the file, and if you do not do this then we see only a blank assignment in our grading software. Upload your flattened solution file to Wattle.

In all cases, the file you upload must be a .pdf file.

Please remember to plan your time carefully so you are not trying to submit your assignment at the last minute. No late work is accepted.

Please enjoy,

AP

Question $1^\star$ (*Matrix algebra*) Let A, B, C be the matrices shown:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Compute those of the following that are defined:

(a) $A + C$

(b) $(A + B) + D$

(c) AB

(d) BC

(e) CD

(f) $\det(A)$

(g) $\det(B)$

(h) $\det(C)$

(i) A^{-1}

(j) B^{-1}

Question 3^+ (*Cardinality*) Let $\mathcal{P}^\star(\mathbb{N})$ denote the set of all *finite* subsets of $\mathbb{N}$ and define a 'sum' function $\sigma : \mathcal{P}^\star(\mathbb{N}) \to \mathbb{N}^\star$ by the rule that for any set $S \in \mathcal{P}^\star(\mathbb{N})$, $\sigma(S) = \sum_{s \in S} s$. For example $\sigma(\{1,3,4,8\}) = 1+3+4+8=16$.
Now define a relation $\prec$ on $\mathcal{P}^\star(\mathbb{N})$ by:

$$S \prec T \Leftrightarrow \begin{cases} \sigma(S) < \sigma(T) \quad \text{or} \\ \sigma(S) = \sigma(T) \text{ and } |S| < |T| \quad \text{or} \\ \sigma(S) = \sigma(T) \text{ and } |S| = |T| \text{ and } \min(S \backslash T) < \min(T \backslash S) \end{cases}.$$

[Here $\min(S \backslash T)$ is the least member of S that is not in T. Similarly for $\min(T \backslash S)$.]

For example $\{1,3,4,8\} \prec \{1,4,5,7\}$ because for $S = \{1,3,4,8\}$ and $T = \{1,4,5,7\}$:
$\sigma(S) = \sigma(T) = 16$; $|S| = |T| = 4$; $\min(S \backslash T) = \min(\{3,8\}) = 3 < 5 = \min(\{5,7\}) = \min(T \backslash S)$.

(a) Let $A = \{1,2,3,4,5\}$, $B = \{3,4,9\}$, $C = \{4,5,7\}$ and $D = \{1,3,5,7\}$. Complete:

$A \prec B$ because

$B \prec C$ because

$C \prec D$ because

(b) The relation $\prec$ 'well orders' $\mathcal{P}^\star(\mathbb{N})$ in the sense that $\mathcal{P}^\star(\mathbb{N})$ can be arranged 'in order':

$$S_1 = \emptyset \prec S_2 = \{1\} \prec S_3 = \{2\} \prec S_4 = \{3\} \prec S_5 = \{1,2\} \prec S_6 = \ldots\ldots$$

Write out the next fifteen sets $S_6, \ldots S_{20}$ in this sequence.

$S_6 =$	$S_7 =$	$S_8 =$	$S_9 =$	$S_{10} =$
$S_{11} =$	$S_{12} =$	$S_{13} =$	$S_{14} =$	$S_{15} =$
$S_{16} =$	$S_{17} =$	$S_{18} =$	$S_{19} =$	$S_{20} =$

(c) Explain why this 'well ordering' $\prec$ shows that $\mathcal{P}^\star(\mathbb{N})$ is countable.
[*i.e.* $\mathcal{P}^\star(\mathbb{N})$ has the same cardinality as $\mathbb{N}$].

(d) Is $\mathcal{P}(\mathbb{N})$ countable? Justify your answer.

Question 5$^{\#}$ (*Combinations, including 'stars and bars'*)

Problem: *How many PINs have digit sum 20?* (A PIN is string $abcd$ of 4 decimal digits *e.g.* 6806.) As a first attempt at answering this:

(a) How many different solutions in non-negative integers has the equation $a+b+c+d=20$?
Hint: 20 stars and 3 bars.

The count in (a) is much too big for our problem because it includes many solutions that contain non-decimal digits; *i.e.* values of a, b, c or d that are greater than 9. We need to eliminate these. For example:

(b) How many of the solutions to $a+b+c+d=20$ have one of the 'digits' equal to 15?
Hint: In how many places can the 15 occur, and what's left when you remove it?

(c) How many of the solutions to $a+b+c+d=20$ have one or two of the 'digits' equal to 10?

(d) Given that $\binom{2}{2}+\binom{3}{2}+\ldots\binom{n}{2}=\binom{n+1}{3}$ for any $n>1$ (this can be proved by maathematical induction), use all the previous answers to calculate the answer to the original problem. As a check, you should find that the answer has digit sum 12.

Question 6* (*The pigeon hole principle*)

(a) Fifty points lie in a 10mm cube. Prove that seven of these points lie in a 5mm cube.

(b) The diagram at right comprises 25 small triangles. Suppose that 35 of the edges of these triangles are coloured red. Prove that there is a triangle with all its edges red.