1 Admin: - Mock mid-sem exam in workshops (1hr)
- Collaboratively assessing solutions (1hr)

Recap: Counting, cardinality, different ∞s, permutations, combinations, stars + bars, pigeon-hole principle

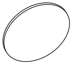
C2. Probability

Notes originally prepared by Judy-anne Osborn and Pierre Portal.
Editing, expansion and additions by Malcolm Brooks.

Text Reference (Epp)	3ed:	Sections	6.7-9
	4ed:	Sections	9.7-9
	5ed:	Sections	9.7-9

(Only the last part of §9 on 'independence' is relevant for this course)

A thought experiment:



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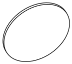

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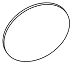

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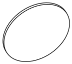

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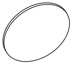

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- What is the probability of ‘Heads’?
- We say it is

$$\mathbb{P}(\text{Heads}) = \frac{1}{2}.$$ *Why?*

Methods of assigning probabilities

Method 1: Use relative frequencies

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same answer

LoLN

- Eg. assume equally likely outcomes

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Law of Large Numbers

- There is much more to be said on ‘relative frequencies’, but for this course we will focus on making ‘models’.

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  for

  Heads or Tails. $p + p = 1$

A model for coin tossing: equal likelihood

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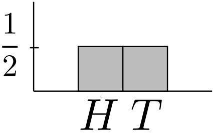
We can represent this situation graphically as



prob. density

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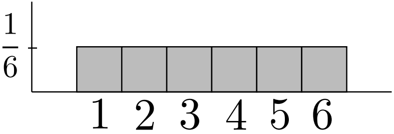
An equal likelihood model for die-tossing



What's an event?

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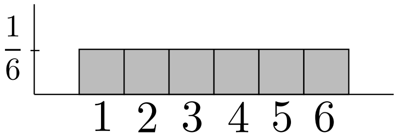

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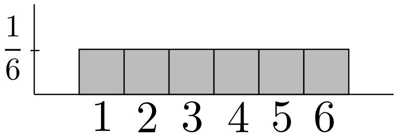

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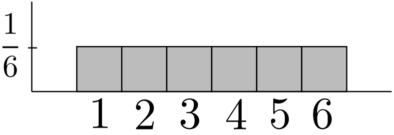

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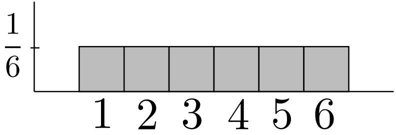

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$$\boxed{\mathbb{P}(\{3, 6\}) = \frac{\overset{2}{|\{3, 6\}|}}{\underset{6}{|\{1, 2, 3, 4, 5, 6\}|}} = \frac{2}{6} = \frac{1}{3}}$$

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uniform distribution on S

where $|E|$ is the number of outcomes in E, and
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singleton events

i.e. $\displaystyle\sum_{s \in S} \mathbb{P}(\{s\}) = 1$

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MATH3029 "probability theory"

*For infinite sets, this isn't necessarily true. 'Measure theory' explains why.

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$$\mathbb{P}(\{1\}) + \ldots + \mathbb{P}(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

The Sum and Product Rules for Probability

The Sum Rule

Sum Rule: If events $E_1, \ldots, E_n$ are mutually disjoint, i.e. $E_i \cap E_j = \emptyset$ for all $i \neq j$, then

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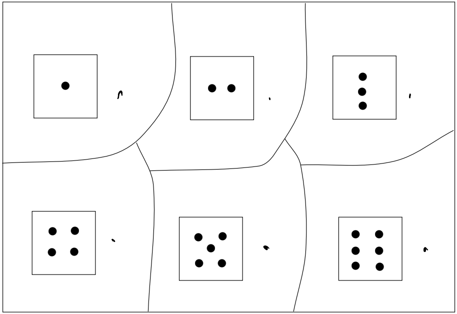
Disjoint events exclude one another in the sense that they cannot happen at the same time.

Sum Rule for probability: another die-tossing example

What is the probability that the outcome from a single toss of a die is an odd number?

The six possible outcomes are all disjoint (cannot occur simultaneously).



Thus the sum rule applies.

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{⚀, ⚂, ⚄}

- The probability that the die lands with an odd number up is

$$
\begin{aligned}
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= \quad & \frac{1}{2}
\end{aligned}
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by the sum rule.

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$$\mathbb{P}(\text{the number is negative}) + \mathbb{P}(\text{the number is positive})$$
$$= \frac{n}{2n+1} + \frac{n}{2n+1} = \frac{2n}{2n+1}.$$

The Product Rule

- Product Rule: If events $E_1, ..., E_n$ are 'independent' of each other; then the probability of composite event 'E_1 and E_2 and ... and E_n' is

$$\mathbb{P}(E_1 \cap E_2 \cap \ldots \cap E_n) = \mathbb{P}(E_1) \times \mathbb{P}(E_2) \times \ldots \mathbb{P}(E_n).$$

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stochastic

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almost

$$\mathbb{P}(E_1 \wedge E_2 \wedge \ldots \wedge E_n) = \mathbb{P}(E_1) \times \mathbb{P}(E_2) \times \ldots \mathbb{P}(E_n).$$ (defn of independent)

causal

- To see what we mean by 'independent', consider a procedure that can be broken down into successive tasks, each of which could be done in a number of ways. If the choice of the way to do any one task had no influence on the choice of ways to do any other of the tasks, then the tasks would be independent.

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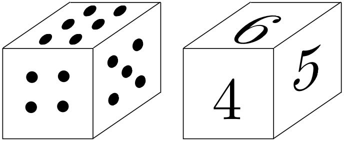
- A formal definition of independence will be given later.

Product Rule probability example: Tossing two dice



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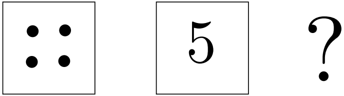

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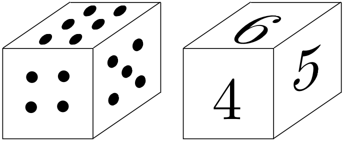

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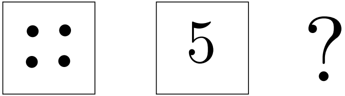

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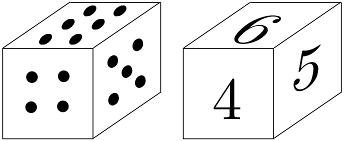

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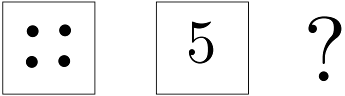

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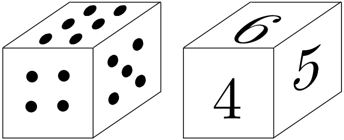

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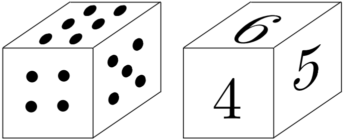

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An example of the Sum and Product Rules used together

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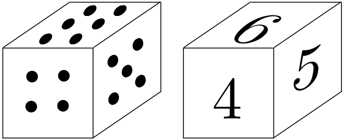

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  - the first die must give even and the second die odd.
- These two possibilities are **disjoint**, so the sum rule applies:

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- But now consider $\mathbb{P}(\text{1st odd, 2nd even})$. The events

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- Putting it all together,

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Density and Distribution

Frequency Histograms

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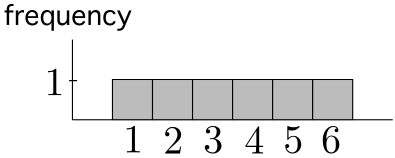
- We have already seen some simple examples, like tossing a die with equally likely possible outcomes: $1, 2, 3, 4, 5, 6$:



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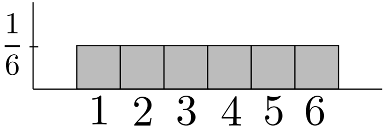

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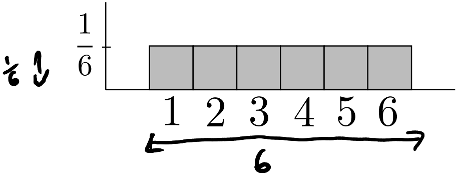

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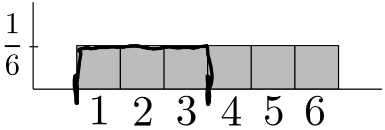

What is the area under the curve? Why?

Because area = probability

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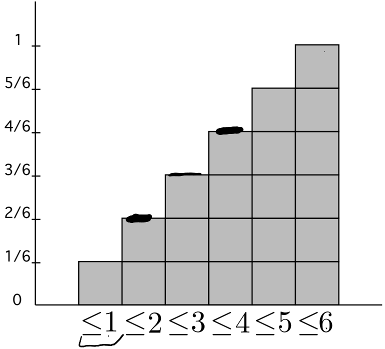

P(between 3 and 4)

$P(\leq 4) - P(\leq 2)$

- We will only use of cumulative distributions when looking up probability values in tables or online.

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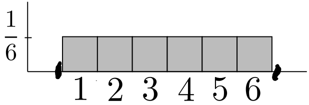

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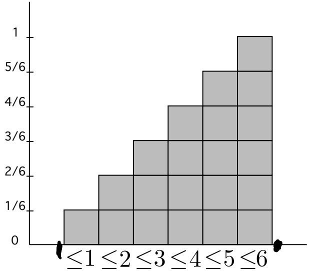

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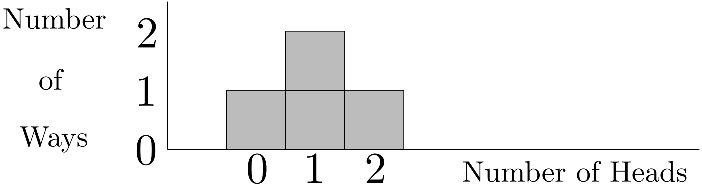

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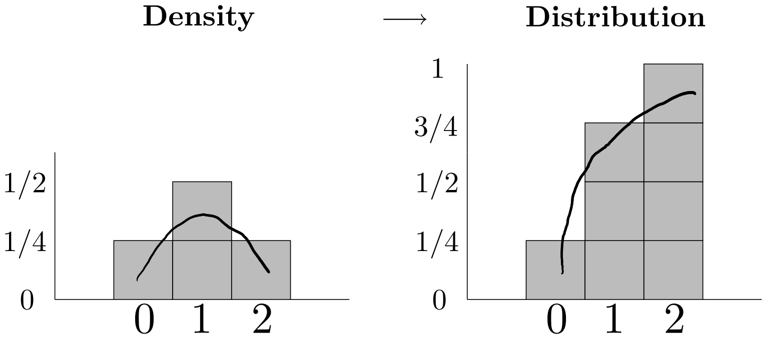

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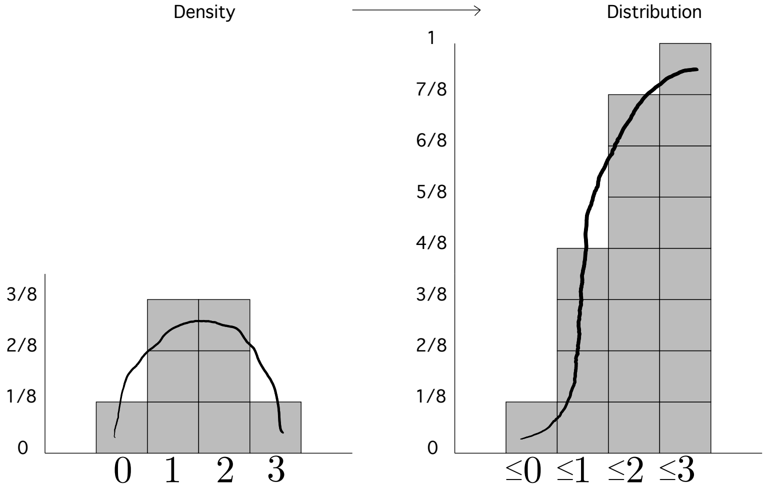
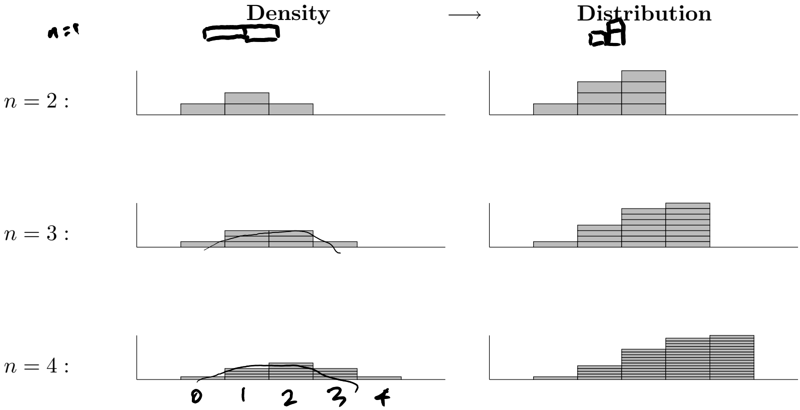
Three Fair Coins: Density and Distribution Functions



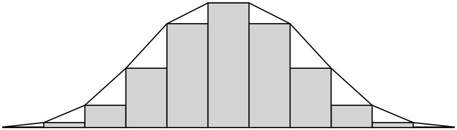
Binomial Probability Distributions

The family of functions that come from coin-tossing are all examples of binomial densities/distributions:



Bell-like curves for large n

As n gets larger and larger these **binomial probability density functions** get closer and closer to the famous Bell Curve:



which is the so-called **'Normal' Probability Density Function**.

Central limit theorem

Pascal's Triangle and Coin Tossing

Pascal's Triangle

r ↓

$0 \rightarrow$ $\quad 1 = \binom{n}{r}$

$1 \rightarrow$ $\quad 1 \quad 1$

n

$2 \rightarrow$ $\quad \overset{0H}{1} \quad \overset{1H}{2} \quad \overset{2H}{1}$

$3 \rightarrow$ $\quad \overset{0H}{1} \quad \overset{1H}{3} \quad \overset{2H}{3} \quad \overset{3H}{1}$

$4 \rightarrow$ $\quad 1 \quad 4 \quad 6 \quad 4 \quad 1$

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

Pascal's Triangle

- Frequencies in Coin-Tossing are numbers in Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

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\end{array}
$$

- Each row is generated by expanding a binomial, eg:

$$(y + x)^4 = y^4 + 4y^3x + 6y^2x^2 + 4yx^3 + x^4.$$

$(H + T)^2 = HH + HT + TH + TT$

$HH + 2\,HT + TT$

Pascal's Triangle

- We've seen these numbers before in 'combinations': $\binom{n}{k}$:

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & + & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & + & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

The Binomial Theorem

- The Binomial Theorem states that

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and gives the rows of Pascal's Triangle in its coefficients.

Idea of Proof of Binomial Theorem:

$$(y+x)(y+x)(y+x)$$

$$= yyy + yyx + yxy + yxx + xyy + xyx + xxy + xxx$$

$$= \underbrace{yyy}_{\binom{3}{0}\ x\text{'s}} + \underbrace{yyx + yxy + xyy}_{\binom{3}{1}\ x\text{'s}} + \underbrace{yxx + xyx + xxy}_{\binom{3}{2}\ x\text{'s}} + \underbrace{xxx}_{\binom{3}{3}\ x\text{'s}}$$

$$\square\square\square \quad \square\square\boxtimes \quad \square\boxtimes\square \quad \boxtimes\square\square \quad \square\boxtimes\boxtimes \quad \boxtimes\square\boxtimes \quad \boxtimes\boxtimes\square \quad \boxtimes\boxtimes\boxtimes$$

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$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots \binom{n}{n}?$$

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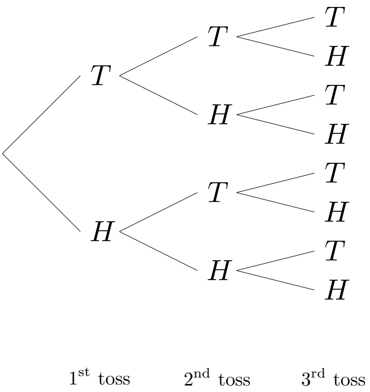
- Probabilities like these can be looked up in tables rather than calculated. Examples will be found in worksheet and assignment questions.

Slide42

Tree Diagrams, Fair and Unfair Coins, and the General Binomial Distribution

A tree representation of Coin-tossing

- Another way to list all the outcomes of an event is to draw a Tree Diagram of the Possibilities

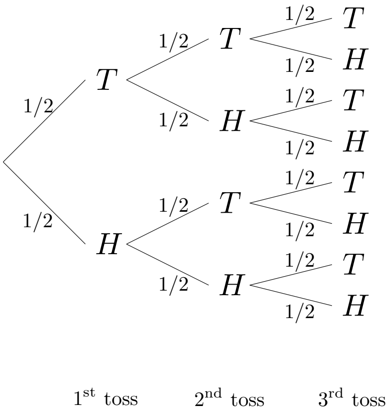


Outcome:

TTT
TTH
THT
THH
HTT
HTH
HHT
HHH

Three tosses of a fair coin

- This allows us to deal with fair coins, as before:



Outcome:

$\Pr(TTT) = 1/8$

$\Pr(TTH) = 1/8$

$\Pr(THT) = 1/8$

$\Pr(THH) = 1/8$

$\Pr(HTT) = 1/8$

$\Pr(HTH) = 1/8$

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Three tosses of a fair coin

Collecting possibilities from the tree and using the sum rule gives

$$\boxed{\mathbb{P}(0\text{heads}) = \tfrac{1}{8},\ \ \mathbb{P}(1\text{head}) = \tfrac{3}{8},\ \ \mathbb{P}(2\text{heads}) = \tfrac{3}{8},\ \ \mathbb{P}(3\text{heads}) = \tfrac{1}{8}}$$

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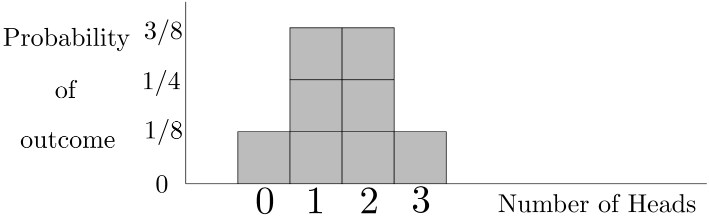

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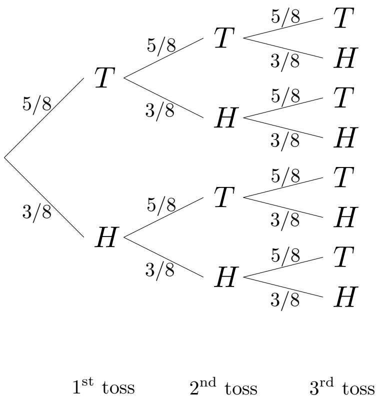

Outcome:

$\Pr(TTT) = 125/512$
$\Pr(TTH) = 75/512$
$\Pr(THT) = 75/512$
$\Pr(THH) = 45/512$
$\Pr(HTT) = 75/512$
$\Pr(HTH) = 45/512$
$\Pr(HHT) = 45/512$
$\Pr(HHH) = 27/512$

Three tosses of an unfair coin

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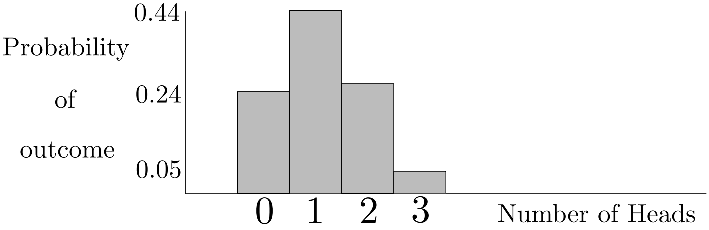

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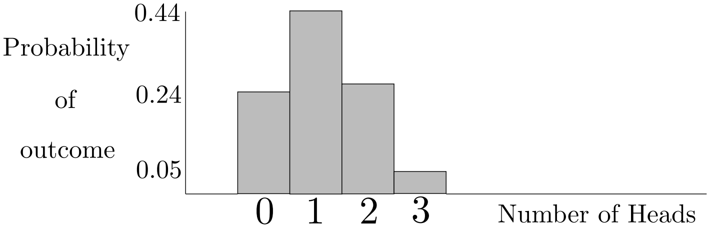

The general binomial density function for n trials (*e.g.* tosses) with probability p of a success (*e.g.* head) on each trial is given by

$$\boxed{\mathbb{P}(k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}.}$$

Review of Probability Density Functions with More Challenging Examples

Probability for equally likely outcomes (Review)

For a finite non-empty set S and $E \subseteq S$. the **probability of E for equally likely outcomes** is the number

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Then $$\mathbb{P}(\{\omega_1, \ldots, \omega_m\}) = \sum_{j=1}^{m} \mathbb{P}(\omega_j) = \sum_{j=1}^{m} \frac{1}{n} = \frac{m}{n} = \frac{|\{\omega_1, \ldots, \omega_n\}|}{|S|}.$$

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So $\mathbb{P}$ is the probability of equally likely outcomes.

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In a group of 10 students, 5 are studying computer science, 2 are studying art history, and 3 are studying mathematics. We pick a student from this group and ask what her/his major is.

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Theorem: Let S be a finite set and $\mathbb{P} : S \to \mathbb{Q}_+$ a probability density function. Then:

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(iii) For *any* $E, F \in \mathcal{P}(S), \quad \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$.

(iv) $\mathbb{P}(E^c) = 1 - \mathbb{P}(E) \quad \forall E \in \mathcal{P}(S)$.

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Thus

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c) \sim 0.97.$$

There is a 97% chance that two people will have the same birthday.

Random Variables, Expected Values and Independence

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$S = \{\text{H,T}\}^3 =$ set of outcomes of tossing three coins.

$X((a, b, c)) =$ number of H's amongst a, b, c.

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Relative to a probability density function $\mathbb{P} : S \to \mathbb{Q}_+$ the **expected value** $\mathbb{E}(X)$ of a random variable X is defined by

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$$\mathbb{E}(X) = (\tfrac{1}{8})0 + (\tfrac{3}{8})1 + (\tfrac{3}{8})2 + (\tfrac{1}{8})3 = \frac{12}{8} = 1.5.$$

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Thus the expected value of X is just the mean (average) number of heads obtained when three coins are tossed.

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If you play this game 30 times, you should expect to *loose* $30(\frac{1}{3}) = 10$ dollars.

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  G, K are **not** independent (again as we would expect) since
  $$\mathbb{P}(G \cap K) = \mathbb{P}(\{\text{HT,TH}\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq \frac{9}{16} = \frac{3}{4} \times \frac{3}{4} = \mathbb{P}(G) \times \mathbb{P}(K).$$

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Then for *any* $a, b \in \{0, 1\}$:

$\mathbb{P}(\{X=a\}) = \mathbb{P}(\{Y=b\}) = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$ and $\mathbb{P}(\{X=a\} \cap \{Y=b\}) = \frac{1}{4}$,

and hence the events $\{X=a\}, \{Y=b\}$ are independent because

$$\mathbb{P}(\{X=a\}\cap\{Y=b\}) = \frac{1}{4} = \frac{1}{2}\times\frac{1}{2} = \mathbb{P}(\{X=a\})\times\mathbb{P}(\{Y=b\}).$$

Thus, by the above definition, X, Y are independent.

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s	1	2	3	4	5	6
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Table 2: Probabilities

a	0	1	2
$\mathbb{P}(\{X = a\})$	$\frac{1}{2}$	$\frac{1}{2}$	0

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Table 1: Definition of X and Y

s	1	2	3	4	5	6
$s \bmod 2 = X(s)$	1	0	1	0	1	0
$s \bmod 3 = Y(s)$	1	2	0	1	2	0

Table 2: Probabilities

a	0	1	2
$\mathbb{P}(\{X=a\})$	$\frac{1}{2}$	$\frac{1}{2}$	0
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$b =$	0	1	2
$a = 0$	6	4	2
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Thus, by definition, the random variables X, Y are independent.

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Let $Y, Z : S \to \mathbb{Q}$ be random variables as follows:

Table 1: Definition of Y and Z

s	1	2	3	4	5	6
$s \bmod 3 = Y(s)$	1	2	0	1	2	0

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Table 2: Probabilities

a	0	1	2	3
$\mathbb{P}(\{Y = a\})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0

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Notice that, in the Table 1, many potential columns are not present.

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For example there is no value of s for which $Y(s)=0$ and $Z(s)=0$.

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Notice that, in the Table 1, many potential columns are not present.
For example there is no value of s for which $Y(s) = 0$ and $Z(s) = 0$.
Using this, and also Table 2, we now have

$$\mathbb{P}(\{Y = 0\} \cap \{Z = 0\}) = 0 \neq \frac{1}{3} \times \frac{1}{6} = \mathbb{P}(\{Y = 0\}) \times \mathbb{P}(\{Z = 0\}),$$

and so the events $\{Y = 0\}$, $\{Z = 0\}$ are not independent.

It follows that the random variables Y, Z are not independent.

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It follows that the random variables Y, Z are not independent.

Challenge: Are the random variables X, Z independent?

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It follows that the random variables Y, Z are not independent.

Challenge: Are the random variables X, Z independent?

END OF SECTION C2