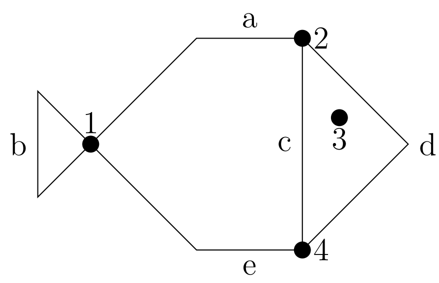
Question 1 For the 'fish' graph G at right find:



(a) $V(G)$.

$\boxed{\{1, 2, 3, 4\}}$.

(The set of vertices.)

(b) $|E(G)|$.

$\boxed{5}$.

(The number of edges.)

(c) All edges incident on vertex 2.

$\boxed{\{a, c, d\}}$.

(d) All vertices adjacent to vertex 2.

$\boxed{\{1, 4\}}$.

(e) All edges adjacent to edge c.

$\boxed{\{a, d, e\}}$.

(f) A loop.

$\boxed{b}$.

(g) An isolated vertex.

$\boxed{3}$.

(h) A pair of parallel edges.

$\boxed{\{c, d\}}$.

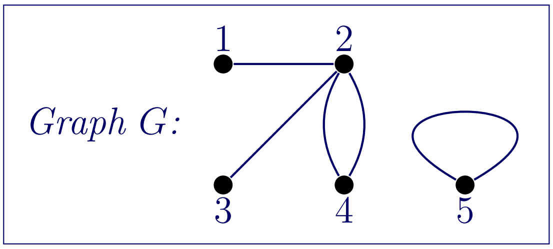
(i) The adjacency matrix of G.

$$\boxed{\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}}.$$

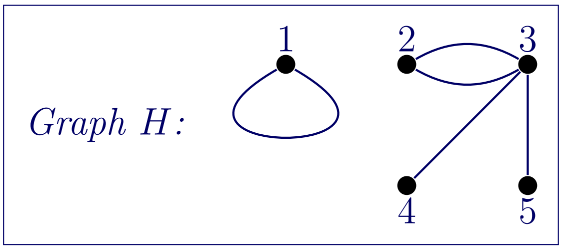
Question 2

(a) Draw diagrams for the graphs with adjacency matrices below:

(i) $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$



(ii) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$



(b) Are the graphs of (a) isomorphic? Justify your answer.

$\boxed{\textit{Yes.}}$ *An isomorphism $F : G \to H$ is given by*

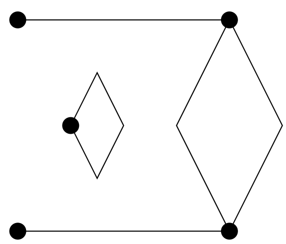
v	1	2	3	4	5
$f(v)$	4	3	5	2	1

Observe that this is a bijection $V(G) \to V(H)$ and that $\forall x, y \in V(G)$

$$\{x, y\} \in E(G) \Leftrightarrow \{f(x), f(y)\} \in E(H),$$

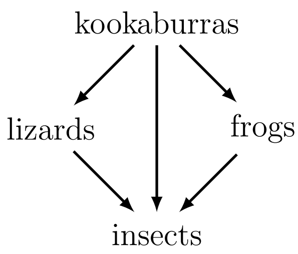
as required for a graph isomorphism.

(c) Is the graph (a)(i) isomorphic to the graph with diagram at right? Justify your answer.



$\boxed{\textit{No.}}$ *For example G has a vertex of degree 4 (vertex 2) but this graph does not. For an isomorphism f the degree of any vertex v is equal to the degree of $f(v)$.*

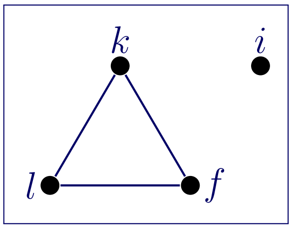
Question 3A The digraph W at right represents a foodweb.
An edge from a to b indicates that a eats b.
Let k, l, f, i denote kookaburras, lizards, frogs and insects.



(a) Which of these are edges of W: $(k, l), (l, f), (f, i), (i, k)$?

$\boxed{(k, l) \text{ and } (f, i)}$ *only.*

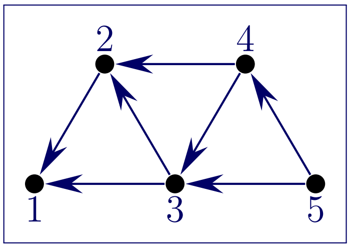
(b) Draw the niche overlap graph for W.



(k, l, f have a common food source.)

Question 3B A relation $R \subseteq \{1, 2, 3, 4, 5\}^2$ is defined by $xRy \Leftrightarrow 0 < x - y < 3$.

(a) Draw a digraph representing R.

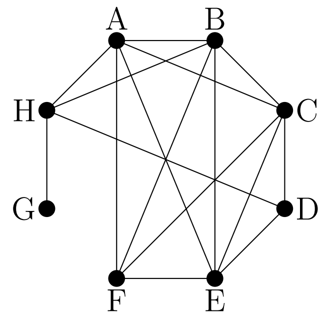


(Any orientation of the graph diagram is OK.)

(b) Write out the adjacency matrix for R.

$$\boxed{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}$$

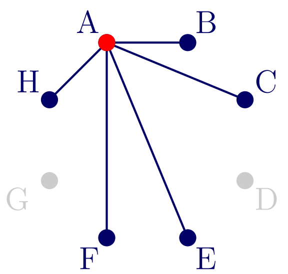
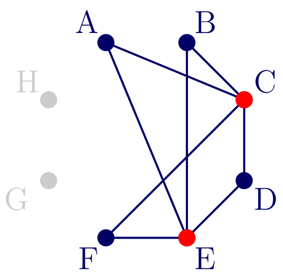
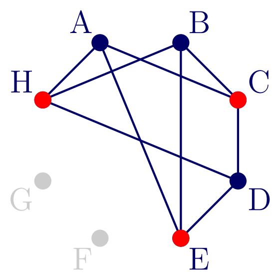
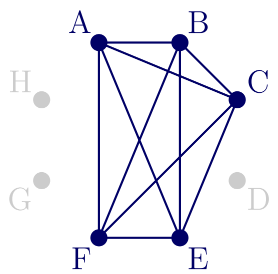
Question 4A For the graph G at right:



(a) State the degree of each vertex.

Vertex:	A	B	C	D	E	F	G	H
Degree:	5	5	5	3	5	4	1	4

(b) Exhibit subgraphs of G isomorphic to $K_{1,5}$, $K_{2,4}$, $K_{3,3}$ and K_5.

$K_{1,5}$	$K_{2,4}$	$K_{3,3}$	K_5
			

(These are examples. For some cases other examples exist.)

(c) Prove that G does not have a subgraph isomorphic to $K_{3\,4}$.
Hint: Concentrate on vertices G and H.

Suppose there were such a subgraph.

Since $3+4=7$*, exactly one vertex must be omitted. This must be vertex G, since its degree is less than* 3*.*

Omitting G leaves H with degree 3 *so in the* $K_{3,4}$*-subgraph all its adjacent vertices must have degree* 4*. However D is adjacent to H but its degree is only 3.*

This contradiction shows that such a subgraph does not exist.

Question 4B Draw a graph with five vertices, each of degree 3, or say why you believe this to be impossible.

Impossible.

By the corollary to the handshake theorem, no graph can have an odd number of vertices of odd degree.

Question 5A A pair of fair 6-sided dice, one red and one blue, are rolled. What is the probability that the sum of the numbers showing face up is 6, given that both of the numbers are odd?
An excellent response will identify: the experiment; how an outcome is recorded; the sample space; how probabilities are assigned to events; and any particular events of interest.

The probability experiment is to roll a pair of fair 6-sided dice, one red and one blue.

An outcome will be recorded as an element of $\{1,2,3,4,5,6\}\times\{1,2,3,4,5,6\}$*, with the first element recording the result of the red die and the second digit the result of rolling the blue die. For example, the outcome* $(2,4)$ *records that we rolled a 2 on the red die and a 4 on the blue die.*

The sample space is the set $S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$*. By the product rule for counting,* $|S| = 6^2 = 36$.

Since the dice are 'fair', it is reasonable to suppose that outcomes are equally likely. We then have that $\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{|E|}{36}$ *for each event* $E \subseteq S$.

Let B *be the event that the sum of the numbers showing face up is 6; that is,* $B = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$*. Let* A *be the event that both of the numbers rolled are odd; that is,*

$$A = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5)\}.$$

We compute:

$$\begin{aligned}
&\textit{The probability that the sum of the numbers showing face up is 6}\\
&\textit{given that both numbers are odd}\\
=&\mathbb{P}(B|A) \qquad \textit{(translating into notation)}\\
=&\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)} \qquad \textit{(defn of conditional prob.)}\\
=&\frac{\mathbb{P}(\{(1,5),(3,3),(5,1)\})}{\mathbb{P}(\{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5)\})}\\
=&\frac{3/36}{9/36}\\
=&\frac{1}{3}.
\end{aligned}$$

$\square$

Question 6A [This problem is Problem 14 in Excercise set 9.9 of Epp. (2019). Discrete Mathematics with Applications, metric Edition. Cengage. (our optional text)] A drug-screening test is used in a large population of people of whom 4% actually use drugs. Suppose that the false positive rate is 3% and the false negative rate is 2%. Thus a person who uses drugs tests positive for them 98% of the time, and a person who does not use drugs tests negative for them 97% of the time

(i) What is the probability that a randomly chosen person who tests positive for drugs actually uses drugs?

(ii) What is the probability that a randomly chosen person who tests negative for drugs does not use drugs?

(i) We consider an experiment in which a member of the population is selected at random and then tested for drugs. The outcome of the experiment is the person chosen, which we represent by their name, and the sample space S is the set of all names of people in the population. Since the person is selected at random, each outcome is equally likely, and for any event $E \subset S$, we may compute

$$\mathbb{P}(E) = \frac{|E|}{|S|}.$$

Let P be the event that the person tests positive for drugs, let N be the event that the person tests negative for drugs, let U be the event that the person uses drugs and let D be the event that the person does not use drugs. We note that $\{D, U\}$ is a partition of S. The data we are given may now be expressed in terms of the events we have introduced and conditional probabilities. We have:

$$\begin{aligned}
\mathbb{P}(P|U) &= 0.98 &&\textit{(this is a true positive)}\\
\mathbb{P}(P|D) &= 0.03 &&\textit{(this is a false positive)}\\
\mathbb{P}(N|U) &= 0.02 &&\textit{(this is a false negative)}\\
\mathbb{P}(N|D) &= 0.97 &&\textit{(this is a true positive)}
\end{aligned}$$

Since 4% of the population uses drugs, we have

$$\mathbb{P}(U) = 0.04 \textit{ and } \mathbb{P}(D) = 0.96$$

We compute

$$\begin{aligned}
&\textit{The probability that a randomly chosen person who tests positive}\\
&\qquad\qquad \textit{for drugs actually uses drugs}\\
=&\mathbb{P}(U|P) \quad \textit{(Converting to notation)}\\
=&\frac{\mathbb{P}(P|U)P(U)}{\mathbb{P}(P|U)P(U)+\mathbb{P}(P|D)P(D)} \quad \textit{(By Bayes' Theorem)}\\
=&\frac{0.98\times 0.04}{0.98\times 0.04+0.03\times 0.96} \quad \textit{(Using the data above)}\\
=&0.576\\
=&57.6\%
\end{aligned}$$

(ii) Using the setup and notation from our solution to (i), we compute

$$\begin{aligned}
&\textit{The probability that a randomly chosen person who tests negative}\\
&\qquad\qquad \textit{for drugs does not use drugs}\\
=&\mathbb{P}(D|N) \quad \textit{(Converting to notation)}\\
=&\frac{\mathbb{P}(N|D)P(D)}{\mathbb{P}(N|U)P(U)+\mathbb{P}(N|D)P(D)} \quad \textit{(By Bayes' Theorem)}\\
=&\frac{0.97\times 0.96}{0.97\times 0.96+0.02\times 0.04} \quad \textit{(Using the data above)}\\
=&0.999\\
=&99.9\%
\end{aligned}$$

Question 7A Recall the following from our lecture notes.

Lemma:
For any probability experiment with sample space S, and for any events $A, B \subseteq S$, if $\mathbb{P}(A) \neq 0$ then

$$\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A).$$

Theorem:[Bayes' Theorem]
For any probability experiment with sample space S, for any $n \in \mathbb{N}$, for any partition $\{B_1, B_2, \ldots, B_n\}$ of S and for any event $A \subseteq S$, if $\mathbb{P}(A) \neq 0$ and for all $i \in \{1, 2, \ldots, n\}$ we have $\mathbb{P}(B_i) \neq 0$, then for all $k \in \{1, 2, \ldots, n\}$ we have

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i)}$$

(i) Write out the structural part of a proof of the lemma that proceeds directly.

(ii) Complete the proof of the lemma you started in part (i).
HINT: Use the definition of conditional probability.

(iii) Write out the structural part of a proof of Bayes' Theorem that proceeds directly.

(iv) In our lecture proof of Bayes' Theorem, we wrote a sequence of 8 equalities to establish that, under the hypotheses made, the conclusion of Bayes' Theorem holds. Each equality was accompanied by a justification. Below is a table of the 8 justifications in the order in which they appeared in our lecture proof of Bayes' Theorem. Use these to finish the proof of Bayes' Theorem you started in part(ii).

#	Justification for algebraic manipulation
1	(By defn of $\mathbb{P}(B_k\|A)$)
2	(Applying the lemma, which is OK because $\mathbb{P}(B_k) \neq 0$)
3	(Because $A \cap S = A$)
4	(Because $\{B_1, \ldots, B_n\}$ is a partition of S, we have $S = B_1 \cup B_2 \cup \cdots \cup B_n$)
5	($\cap$ distributes over $\cup$)
6	(Applying the sum rule, which is OK because $B_1, \ldots, B_n$ are mutually disjoint)
7	(Applying the lemma n times, which is OK because $\mathbb{P}(B_i) \neq 0$ for $i \in \{1, 2, \ldots, n\}$)
8	(Using Σ notation)

(i) **Proof:** *Consider a probability experiment with sample space S. Let $A, B \subseteq S$. Suppose that $\mathbb{P}(A) \neq 0$.*

$\vdots$

Hence $\mathbb{P}(B \mid A)\mathbb{P}(A) = \mathbb{P}(A \cap B)$. $\square$

(ii) **Proof:** *Consider a probability experiment with sample space S. Let $A, B \subseteq S$. Suppose that $\mathbb{P}(A) \neq 0$. Since $\mathbb{P}(A) \neq 0$, the conditional probability $\mathbb{P}(B|A)$ is defined. The definition gives*

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

Multiplying both sides by $\mathbb{P}(A)$ gives $\mathbb{P}(B|A)\mathbb{P}(A) = \mathbb{P}(A \cap B)$. $\square$

(iii) **Proof:** *Consider a probability experiment with sample space S. Let $n \in \mathbb{N}$, let $\{B_1, B_2, \ldots, B_n\}$ be a partition of S and let $A \subseteq S$. Suppose that $\mathbb{P}(A) \neq 0$ and for all $i \in \{1, 2, \ldots, n\}$ we have $\mathbb{P}(B_i) \neq 0$. Let $k \in \{1, 2, \ldots, n\}$.*

$\vdots$

Hence

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i)} \quad \square$$

(iv) **Proof:** *Consider a probability experiment with sample space S. Let $n \in \mathbb{N}$, let $\{B_1, B_2, \ldots, B_n\}$ be a partition of S and let $A \subseteq S$. Suppose that $\mathbb{P}(A) \neq 0$ and for all $i \in \{1, 2, \ldots, n\}$ we have $\mathbb{P}(B_i) \neq 0$. Let $k \in \{1, 2, \ldots, n\}$. Now*

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(B_k \cap A)}{\mathbb{P}(A)} \qquad (\#1)$$

$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A)} \qquad (\#2)$$

$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A \cap S)} \qquad (\#3)$$

$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A \cap (B_1 \cup B_2 \cup \cdots \cup B_n))} \qquad (\#4)$$

$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}((A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n))} \qquad (\#5)$$

$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A \cap B_1) + \mathbb{P}(A \cap B_1) + \cdots + \mathbb{P}(A \cap B_n)} \qquad (\#6)$$

$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) \cdots + \mathbb{P}(A|B_n)\mathbb{P}(B_n)} \qquad (\#7)$$

$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i)} \qquad (\#8) \qquad \square$$