Question 1 (*Matrix algebra*)

Let A, B, C be the matrices shown:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Compute those of the following that are defined:

(a) $A + B$

(b) $B + C$

(c) AB

(d) BC

(e) $A + BC$

(f) $\det(A)$

(g) $\det(B)$

(h) $\det(C)$

(i) $\det(BC)$

(j) A^{-1}

Question 2 (*Systems of Linear equations*)

(a) Express the linear system
$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 5 \\ -x & + & 2y & + & z & = & 4 \\ x & - & y & + & z & = & 3 \end{array}$$
in matrix form $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A}$ is a 3×3 matrix and $\mathbf{x}$, $\mathbf{b}$ are 3×1 matrices.

(b) Verify that $\mathbf{A}^{-1} = \dfrac{1}{4} \begin{bmatrix} 3 & -2 & -1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{bmatrix}$.

(c) Use $\mathbf{A}^{-1}$ to solve the linear system.

Question 3 (*Cardinality*) Let $\mathbb{Q}^+$ denote the set of positive rational numbers.
A relation '$\prec$' is defined on $\mathbb{Q}^+$ as follows: For any $q_1, q_2 \in \mathbb{Q}^+$ to determine whether $q_1 \prec q_2$ first write q_1, q_2 in 'lowest terms' $q_1 = \frac{a_1}{b_1}$, $q_2 = \frac{a_2}{b_2}$ where $a_1, b_1, a_2, b_2 \in \mathbb{N}$ and $\gcd(a_1, b_1) = \gcd(a_2, b_2) = 1$. Then:

$$q_1 \prec q_2 \Leftrightarrow \begin{cases} a_1 + b_1 < a_2 + b_2 & \text{or} \\ a_1 + b_1 = a_2 + b_2 & \text{and } a_1 < a_2 \end{cases}$$

(a) Verify that $\frac{5}{1} \prec \frac{2}{5} \prec \frac{3}{4}$.

(b) The relation $\prec$ 'well orders' $\mathbb{Q}^+$ in the sense that $\mathbb{Q}^+$ can be arranged 'in order' starting with $\frac{1}{1}$:

$$q_1 = \tfrac{1}{1} \prec q_2 = \tfrac{1}{2} \prec q_3 = \tfrac{2}{1} \prec q_4 \prec q_5 \prec \ldots\ldots$$

Write out the next nine fractions $q_4, \ldots q_{12}$ in this sequence.

(c) Explain why this 'well ordering' $\prec$ shows that, surprisingly, $\mathbb{Q}^+$ is countable (*i.e.* $\mathbb{Q}^+$ has the same cardinality as $\mathbb{N}$).

(d) Is $\mathbb{Q}$ countable? Justify your answer.

Question 4 (*Inclusion-exclusion principle and the product rule*)

(a) The inclusion-exclusion principle for two sets A, B is $|A \cup B| = |A| + |B| - |A \cap B|$. Use a Venn diagram to find a similar formula for three sets A, B, C.

(b) A PIN is a number with four decimal digits, *e.g.* 2357, 0944 etc. A 'double digit' in a PIN is any pair of consecutive equal digits, such as the 44 in 0944.
Use inclusion-exclusion to count how many PINs have at least one double digit.
[There is another, slightly quicker, way to count these PINs. Can you see it?]

Question 5 (*Combinations and 'stars and bars'*) A TAW is a three letter 'word' whose letters are in alphabetical order. Letters are drawn from the standard lower case English 26-letter alphabet, and are allowed to repeat. A 'word' does not have to appear in any dictionary but must contain at least one vowel and at least one consonant. Letter y can count as a vowel or a consonant. Examples of TAWs are *abc*, *ccy*, *aoy* and *yyy*.

How many different TAWs are there?

Hint 1: Once three letters are chosen (possibly involving repeats) there is only one way to put them in alphabetical order.
Hint 2: As a first step ignore the requirement about vowels and consonants.

Question 6 (*The pigeon hole principle*)

(a) Each year the ANU enrols new students from all eight States and Territories in Australia. How many new Australian students must the ANU enrol next year in order to ensure that there are at least 500 students from the same State or Territory?

The students are the pigeons; the States and Territories are the pigeon holes.

(b) Seven numbers are picked from the set $\{1, 2, \ldots, 12\}$ of the first twelve natural numbers. Prove that amongst the seven numbers picked it is guaranteed that two of them, say a and b satisfy $a - b = 3$.

Remark: This is easy once you've figured out what the pigeon holes should be, but figuring that out is perhaps not so easy!