Recap:

- 4-bit signed integers
- Adder circuits, addition algorithm,
- Multiplication algorithm, in base 2.

4-bit signed integer

Half-adder → no carry in

full-adder → carry in

$\underbrace{1}_{-\text{sign}}\underbrace{011}$ $= (3 - 2^{4-1})_{10} = (-5)_{10}$

$\underbrace{0}_{+\text{ve}}\underbrace{111}$ $= (7)_{10}$

Modular arithmetic

A Theorem

Theorem

$$\underbrace{\forall z \in \mathbb{Z}}_{\text{for all integers } z}\ \underbrace{\forall d \in \mathbb{N}}_{\text{for all positive integers } d}\ \underbrace{\exists! q \in \mathbb{Z}}_{\text{there exists a unique integer } q}\ \exists! r \in \mathbb{Z}\ \ (z = qd + r) \wedge (0 \leq r < d)$$

for all integers z (for all positive integers d) there exists a unique integer q there exists a unique integer r such that $z = qd + r$ and $r \in \{0, \ldots, d-1\}$

A Theorem

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$$\forall z \in \mathbb{Z}\, \forall d \in \mathbb{N}\, \exists! q \in \mathbb{Z}\, \exists! r \in \mathbb{Z}\ \ (z = qd + r) \wedge (0 \leq r < d)$$

Theorem: (The Quotient-Remainder Theorem). Given any integer z and any positive integer d, there is exactly one way to express z in the form $z = qd + r$, where q is an integer and $r \in \{0, 1, \ldots, d-1\}$.

In the expression $z = qd + r$, q is called the **quotient** (when z is divided by d) and r is called the **remainder** (when z is divided by d).

Picturing q and r

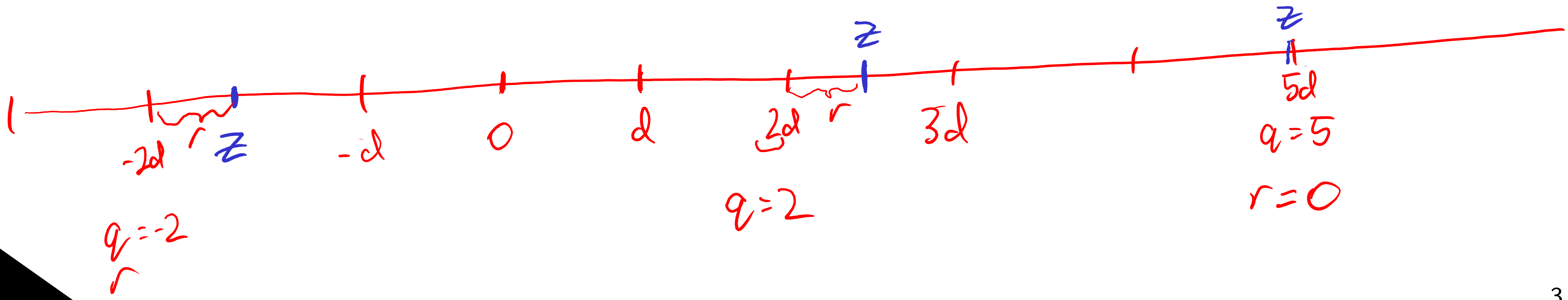
Fix a choice of $z \in \mathbb{Z}$ and $d \in \mathbb{N}$.

Now picture a number line with the integers marked.

Now: qd is the integer multiple of d that is closest to z but NOT to the right of z; and r is the distance between qd and z.

A picture will help.

$r \geq 0, \quad r < d$



‘mod’ and ‘div’

We define: $q = z\ \textbf{div}\ d$; $r = z\ \textbf{mod}\ d$.

You may like to say that:

- z div d gives the **quotient** when z is divided by d;
- z mod d gives the **remainder** (residue) when z is divided by d. Mod is short for “modulo”.

i.e. z modulo d

Examples

Q: Evaluate the following expressions:

87 mod 13

-100 div 13

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A:

$\in \{0, \ldots, 13-1\}$

Since $87 = 6(13) + 9$, 87 mod 13 $=$ 9.

Since $-100 = \underbrace{(-8)}(13) \underbrace{+\ 4}$, -100 div 13 $=$ -8.

$(-8)(12) - r'$

$r' \in \{0, \ldots, 13-1\}$

The division algorithm

The ‘primary school’ method of finding quotient and remainder is to use *repeated subtraction*. This only works for non-negative z.

divisor

Input: $z \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{N}$.
Output: $q = z$ div d and $r = z$ mod d.

Method:

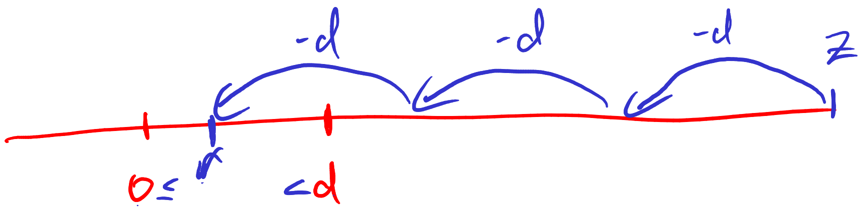
Set $r = z$, $q = 0$.

Loop: If $r < d$ stop.

Replace r by $r - d$.

Replace q by $q + 1$.

Repeat loop



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 Loop: If $r < d$ stop.
  Replace r by $r - d$.
  Replace q by $q + 1$.
 Repeat loop

Q: Some small modifications to the algorithm allow it cope also with negative z. Could you make them?

Congruence modulo d

mod

Let $d \in \mathbb{N}$. The **congruence modulo** d relation $R_d \subseteq \mathbb{Z} \times \mathbb{Z}$ is defined by

$$aR_db \Leftrightarrow (\exists k \in \mathbb{Z} \;\; a = b + kd).$$

We have unusual notation for this relation. We write

$$a \overset{\text{congruent}}{\equiv} b \pmod{d}$$

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Understanding the relation:

- Two integers are congruent modulo d IFF their difference is a multiple of d.
- Two integers are congruent modulo d IFF they leave the same remainder upon division by d.

Proof

Let $a, b \in \mathbb{Z}$ and let $d \in \mathbb{N}$.

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$[p \to q, \quad q \to p \quad \therefore \quad p \leftrightarrow q]$

Let $a, b \in \mathbb{Z}$ and let $d \in \mathbb{N}$. Suppose that $a \equiv b \pmod{d}$. Then there exists $k \in \mathbb{Z}$ such that $a = b + kd$. By the Quotient-Remainder Theorem, there exist unique integers q_1, r_1 such that $b = q_1 d + r_1$ and $0 \leq r_1 < d$. Then we have

$$a = b + kd = (q_1 d + r_1) + kd = \overbrace{(q_1 + k)}^{\in \mathbb{Z}} d + r_1 . \quad \in \{0, \ldots, d-1\}$$

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$$a = b + kd = (q_1d + r_1) + kd = (q_1 + k)d + r_1.$$

The uniqueness part of the Quotient-Remainder Theorem gives that r_1 is the remainder when a is divided by d. Thus a and b leave the same remainder upon division by d.
Now suppose that a and b leave the same remainder upon division by d. Then there exist $q_1, q_2, r \in \mathbb{Z}$ such that $a = q_1d + r$ and $b = q_2d + r$ and $0 \leq r < d$. Now $r = b - q_2d$, so

$$a = q_1d + r = q_1d + b - q_2d = b + \underbrace{\overset{\in\mathbb{Z}}{(q_1 - q_2)}}_{= k}d.$$

$a = b + kd$

Hence, by definition, $a \equiv b \pmod{d}$. $\square$

Example

Example: $-17 \equiv 15 \pmod 8$ since $(-17) - 15 = -32 = (-4)8$.

$\equiv$ partitions the integers

For any $d \in \mathbb{N}$ and any $a \in \mathbb{Z}$ the **congruence class** $[\mathbf{a}]_{\mathbf{d}}$ (or 'equivalence class') of a modulo d is defined by

pairwise disjoint,
none are empty,
union is all of $\mathbb{Z}$

$$[a]_d = \{m \in \mathbb{Z} \mid m \equiv a \pmod d\}. \subseteq \mathbb{Z}$$

Lemma: R_d induces the partition $\{[0]_d, [1]_d, \ldots, [d-1]_d\}$ on $\mathbb{Z}$

Example: $\mathcal{P} = \{[0]_5, [1]_5, [2]_5, [3]_5, [4]_5\}$ is a partition of $\mathbb{Z}$

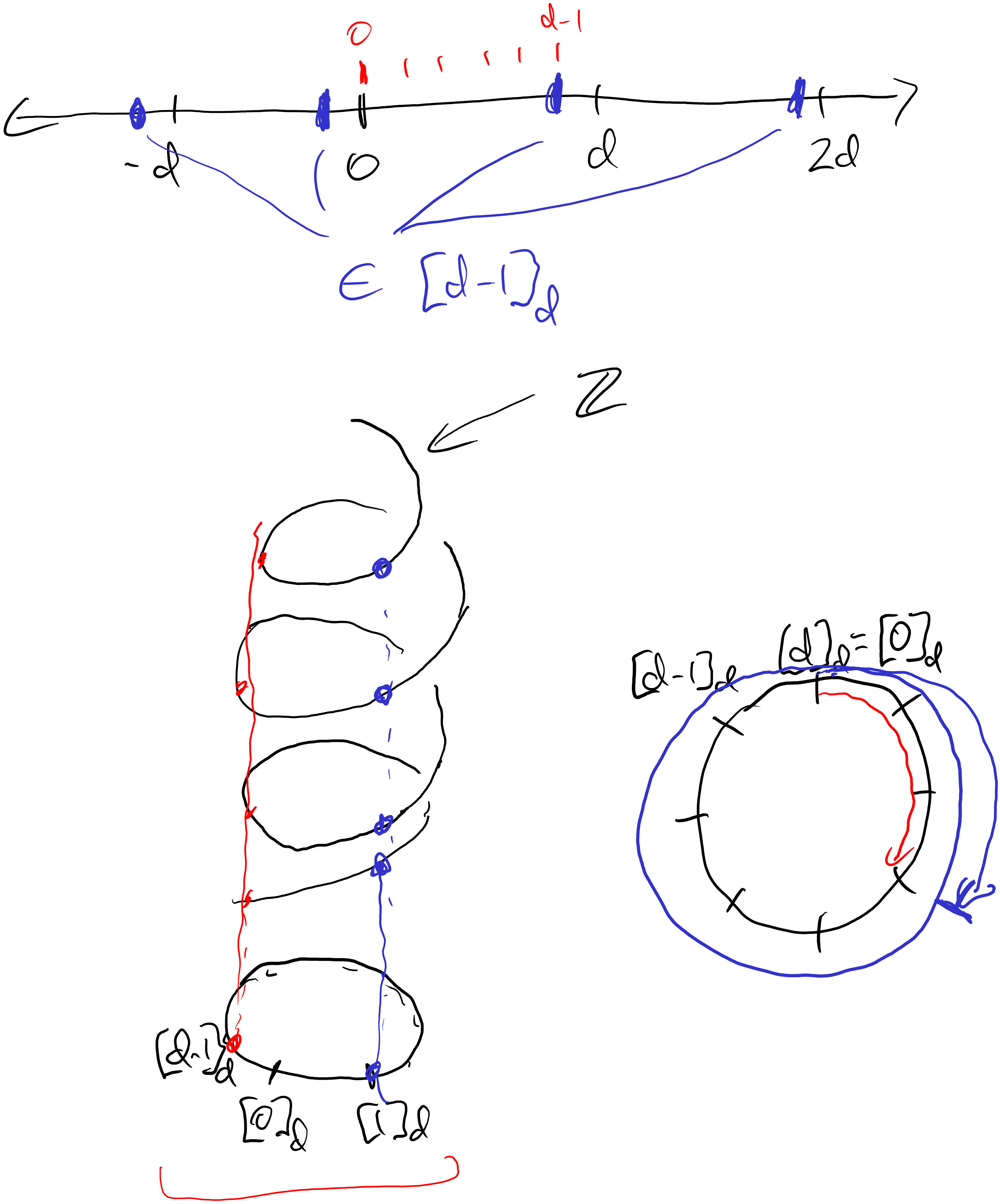
Q: Can you see how the Q-R theorem can be used to prove the lemma?

Q-R : $a = qd + r$

$a = r + qd$

$\Leftrightarrow a \equiv r \pmod d$

$\in \{0, \ldots, d-1\}$



0
d-1
-d
0
d
2d
∈ [d-1]_d
Z
[d-1]_d
[0]_d
[1]_d
[d-1]_d
[d]_d = [0]_d

Modular arithmetic

Theorem: For any $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ and $d \in \mathbb{N}$:

$$\boxed{\begin{array}{l} a_1 \equiv a_2 \pmod d \\ b_1 \equiv b_2 \pmod d \end{array}} \implies \underbrace{\boxed{\begin{array}{l} a_1 \pm b_1 \equiv a_2 \pm b_2 \pmod d \\ a_1 \times b_1 \equiv a_2 \times b_2 \pmod d \end{array}}}$$

Example:

Since

$$\underbrace{\overset{6 \pmod 7}{\overset{|||}{27}} \equiv -1 \pmod 7} \text{ and } \underbrace{36 \equiv 1 \pmod 7}$$

we have

$$\underbrace{\underbrace{27}+\underbrace{36}} \equiv \underbrace{-1}+\underbrace{1} \equiv 0 \pmod 7$$

and

$$\underbrace{\underbrace{27}\times 36} \equiv -1 \times 1 \equiv \underbrace{-1} \pmod 7$$

$\equiv 6 \pmod 7$

A key idea

When computing "modulo d", you may at any time replace a number by something to which it is equivalent. In this way, you may simplify computations so that you never have to work with large integers.

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Examples:

$$\underbrace{379}_{} - \underbrace{803}_{} \equiv \underbrace{1-5}_{} \equiv -4 \equiv 3 \pmod 7$$

and

$$379 \times 803 \equiv 1 \times 5 \equiv 5 \pmod 7$$

and

$25 = 3 \times 7 + 4$

$$803^5 \equiv 5^5 \equiv \underbrace{25 \times 25}_{} \times \underbrace{5}_{} \equiv \underbrace{4 \times 4 \times 5}_{} \equiv \underbrace{80}_{} \equiv 3 \pmod 7$$

$\equiv 4 \times 20 \equiv 4 \times 6 \equiv 24 \equiv 3 \pmod 7$

An important problem

The following problem is called the **discrete logarithm problem**: Given $d \in \mathbb{N}$ and $A, Q \in \{1, \ldots, d-1\}$, find $x \in \{1, \ldots, d-1\}$ such that $A^x \equiv Q \pmod{d}$.

A naive solution to the problem: Compute $A^1 \mod d, A^2 \mod d, \ldots$ until one of your computations produces Q.

A randomised naive solution to the problem: Repeatedly, select a number t from $\{1, 2, \ldots, d-1\}$ at random and compute $A^t \mod d$. Stop when one of your computations produces Q.

Your privacy on the internet often relies on the following fact: For certain choices of d, A and Q, the naive solution to the discrete logarithm problem will take a very long time and the randomised naive solution to the discrete logarithm problem will almost certainly take a very long time.