

B2. Sequences.

Notes originally prepared by Pierre Portal
Editing and expansion by Malcolm Brooks.

Text Reference (Epp)

- 3ed: Sections 4.1-4, 8.1-3 (Sequences and induction),
9.3,5 (Sorting)
- 4ed: Sections 5.1-4,6-8, (Sequences and induction),
11.3,5 (Sorting)
- 5ed: Sections 5.1-4,6-7, (Sequences and induction),
11.3,5 (Sorting)

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4. U set of users. $(u_n)_{n \in \{1,2,3,4,5\}} \subseteq U$: a list of 5 users.

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Continuing to argue in this manner gives $P(n)$ for all $n \in \mathbb{N}$.

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Inductive step: Assume the formula is correct for up to and including some fixed n .

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Claim: $\forall n \in \mathbb{N} \ a_n = 3^n$.

Proof by mathematical induction that the claim is correct:

Basis step: For $n=1$, formula gives $a_1=3^1=3$, agreeing with the implicit definition.

Inductive step: Assume the formula is correct for up to and including some fixed n . Then

$$a_{n+1} = 3a_n \quad (\text{from the implicit definition})$$

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and so the formula is also correct for $n+1$.

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$$(1) \sum_{n=1}^{10} n = 1 + 2 + 3 + 4 + \dots + 9 + 10 = 55.$$

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(4) $\prod_{n=1}^8 n^2 = 4 \times 9 \times 16 \times \dots \times 64 = 1\,625\,702\,400.$

Geometric Sequences and Series

Example 1 (Slide 6) is a special case of

Geometric Sequence	
Implicit Definition	Explicit Definition
$a_k = a$ (a is the first term) $a_{n+1} = ra_n, \forall n \geq k$ $(r$ is the common ratio)	$\forall n \geq k$ $a_n = ar^{n-k}$

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When terms of a sequence are summed, we get a **series**. In particular

Geometric Series	
Series of N terms	Sum of N terms
$b_n := \sum_{n=k}^{k+(N-1)} ar^{n-k} = \underline{a+ar+\dots+ar^{N-1}}$ $[\text{Usually } k=0 \text{ or } k=1.]$	$\begin{cases} \frac{a(1-r^N)}{(1-r)} & \text{if } r \neq 1 \\ Na & \text{if } r = 1 \end{cases}$

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From implicit to explicit definitions; Example 2

A sequence is defined implicitly by
$$\begin{cases} b_{n+1} = b_n + 5 & \forall n \in \mathbb{N}, \\ b_1 = 0. \end{cases}$$

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Arithmetic Sequences and Series

Example 2 (Slide 9) is a special case of

Arithmetic Sequence	
Implicit Definition	Explicit Definition
$a_k = \underline{a}$ (a is the first term) $a_{n+1} = a_n + \underline{d}$, $\forall n \geq k$ (d is the common difference)	$\forall n \geq k$ $\underline{a_n = a + (n - k)d}$

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When the terms of an arithmetic sequence are summed, we get:

Arithmetic Series	
Series of N terms	Sum of N terms
$\sum_{n=k}^{k+(N-1)} [a + (n-k)d] = a + (a+d) + \cdots + (a+(N-1)d)$ [Usually $k=0$ or $k=1$.]	$\frac{N}{2} [2a + (N-1)d]$ $= N \left[\frac{a + (a+(N-1)d)}{2} \right]$

Exercise: prove this by induction.

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sum = number of terms times average of first and last

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Example: $1 + 3 + 5 + 7 + \cdots + 99$

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Example: $1 + 3 + 5 + 7 + \cdots + 99 = 50 \left(\frac{1 + 99}{2} \right)$

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Example: $1 + 3 + 5 + 7 + \cdots + 99 = 50 \left(\frac{1 + 99}{2} \right) = 2500.$

Simplest Malthusian model in population dynamics

(Exponential growth)



T. Robert Malthus 1766 - 1834

Let p_n be number of individuals in a population after n years.

Simplest Malthusian model in population dynamics

(Exponential growth)

bacteria



T. Robert Malthus 1766 - 1834

Let p_n be number of ~~individuals~~ in a population after n years.

Assume the population doubles every year,

Simplest Malthusian model in population dynamics

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Let p_n be number of individuals in a population after n years.

Assume the population doubles every year, i.e. $\forall n \in \mathbb{N}^* \quad p_{n+1} = 2p_n$.

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Let p_n be number of individuals in a population after n years.

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Assume the current population is five million.

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Questions: 1. What will the population size be in 10 years time?

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Assume the current population is five million. i.e. $p_0 = 5 \times 10^6$.

- Questions:
1. What will the population size be in 10 years time?
 2. When will the population size reach 10^9 ?

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Assume the population doubles every year, i.e. $\forall n \in \mathbb{N}^* \quad p_{n+1} = 2p_n$.

Assume the current population is five million. i.e. $p_0 = 5 \times 10^6$.

- Questions:
1. What will the population size be in 10 years time?
 2. When will the population size reach 10^9 ?

We have a geometric sequence with $k = 0$, $a = 5 \times 10^6$, $r = 2$.

Simplest Malthusian model in population dynamics

(Exponential growth)



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Answers: 1. $a_{10} = 5 \times 10^6 \times 2^{10} = 5.12 \times 10^9$

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From 1. $a_8 = \frac{1}{4}(5.12 \times 10^9) = 1.28 \times 10^9$

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From 1. $a_8 = \frac{1}{4}(5.12 \times 10^9) = 1.28 \times 10^9$

and $a_7 = \frac{1}{8}(5.12 \times 10^9) = 0.64 \times 10^9$.

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So $\mathbf{N = 8}$ i.e. eight years.

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$$c_{10} = 2 \times 10^4 \times 1.03^{10} = \text{\$26 878:33}.$$

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Assume the capital yields $\overbrace{3\%}$ interest every year, i.e.

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Now suppose the capital yields $\frac{3}{12}\%$ interest per *month*.

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$$d_{120} = 2 \times 10^4 \times 1.0025^{120} = \underline{\underline{\$26\,987.71}}.$$

10 years

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$$d_{120} = 2 \times 10^4 \times 1.0025^{120} = \text{\$26 987:71}. \quad \text{Slightly better!}$$

Better models

Compound Interest: If the bank is charging a fee of \$10 per year, the compound interest model becomes

$$\begin{cases} c_{n+1} = \underline{1.03c_n} - 10, & \forall n \in \mathbb{N}^*, \\ c_0 = 2 \times 10^4. \end{cases}$$

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We need to start again from scratch.

From implicit to explicit definitions; Example 3

We seek an explicit formula for the population given by the implicit formula at right, where d and p are shorthand for 10^3 and 5×10^6 .

$$\begin{cases} p_{n+1} = \underline{2}p_n + \underline{d} \quad \forall n \in \mathbb{N}^*, \\ p_0 = p. \end{cases}$$

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We start by generating the first few terms of the sequence:

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We start by generating the first few terms of the sequence:

$$\begin{aligned} p_0 &= p, & p_1 &= 2p + d, & p_2 &= 2(2p + d) + d = 4p + 3d, \\ p_3 &= 8p + 7d, \end{aligned}$$

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$$\text{Claim: } \forall n \in \mathbb{N}^* \quad p_n = \underbrace{2^n p} + \underbrace{(2^n - 1)d}.$$

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Proof by mathematical induction that the claim is correct:

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Proof by mathematical induction that the claim is correct:

Basis step: For $n = 0$, formula gives $\underline{p_0} = \underline{2^0 p + (2^0 - 1)d} = \underline{p}$, agreeing with the implicit definition.

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$$p_{n+1} = 2p_n + d \quad (\text{from the implicit definition})$$

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$$\begin{aligned} p_{n+1} &= 2p_n + d \quad (\text{from the implicit definition}) \\ &= 2(\underbrace{2^n p + (2^n - 1)d}_{\text{by the inductive assumption}}) + d \end{aligned}$$

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$$\begin{aligned} p_{n+1} &= 2p_n + d && \text{(from the implicit definition)} \\ &= 2(2^n p + (2^n - 1)d) + d && \text{(by the inductive assumption)} \\ &= \underbrace{2^{n+1}} p + (2^{n+1} - 1)d && \text{(by algebraic simplification)} \end{aligned}$$

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and so the formula is also correct for $n+1$.

So by mathematical induction, $\forall n, P(n)$

From implicit to explicit definitions; Example 4

We seek an explicit formula for the investment capital given by the implicit formula at right, where

$$\begin{cases} c_{n+1} = \underline{r}c_n + d \quad \forall n \in \mathbb{N}^*, \\ c_0 = c. \end{cases}$$

$r=1.03$, $d=-10$ and $c=2 \times 10^4$.

From implicit to explicit definitions; Example 4

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$$\begin{cases} c_{n+1} = rc_n + d \quad \forall n \in \mathbb{N}^*, \\ c_0 = c. \end{cases}$$

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We start by generating the first few terms of the sequence:

$$c_0 = c,$$

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$$\underline{c_{10}} = (1.03)^{10} (2 \times 10^4) - \left(\frac{1 - (1.03)^{10}}{1 - 1.03} \right) 10 = \underline{26\,878.33} - \underline{114.64}.$$

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So the \$10 annual fee over 10 years costs the investment \$114.64.

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It takes very little extra analysis to generalise the previous example:

Mixed Geometric-Arithmetic Sequence	
Implicit Definition	Explicit Definition
$a_k = a$ (a is the first term) $a_{n+1} = \underline{ra_n} + \underline{d}$, $\forall n \geq k$ ($r \neq 1$ is the multiplier and d is the offset)	$\forall n \geq k$ $a_n = ar^{n-k} + \left(\frac{1-r^{n-k}}{1-r} \right) d$

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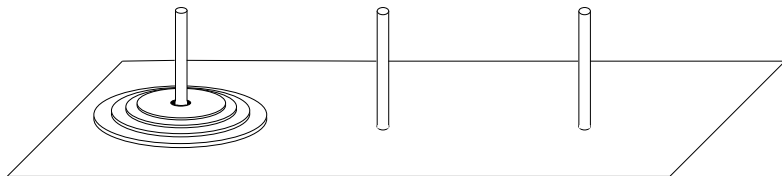
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Remark: For this sequence, as n increases a_n approaches 4 ever more closely. In fact the value 4 is called the **steady state** of the sequence, because if $a_n = 4$ then from the implicit definition $a_{n+1} = \left(\frac{1}{2}\right)4 + 2 = 4$, so the sequence values remain at 4 for ever.

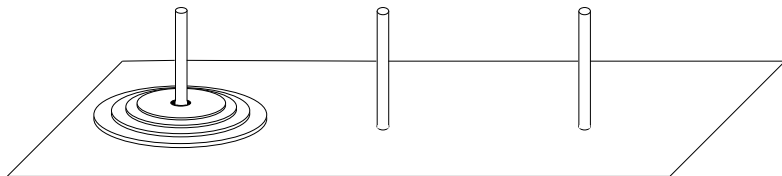
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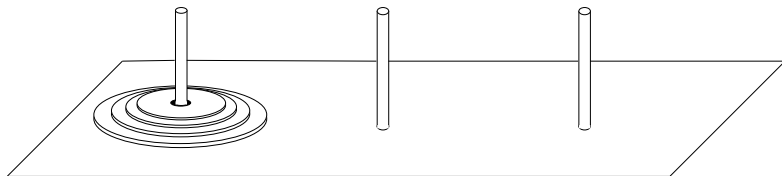
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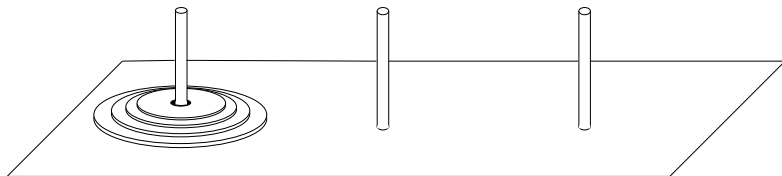


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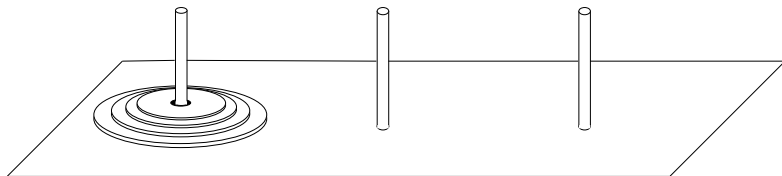


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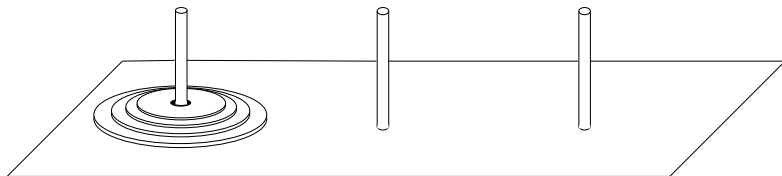


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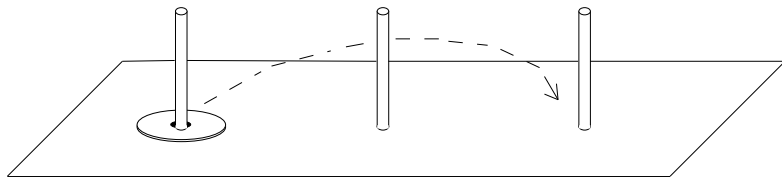
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At one move per second, how fast can you solve a puzzle with 64 discs?

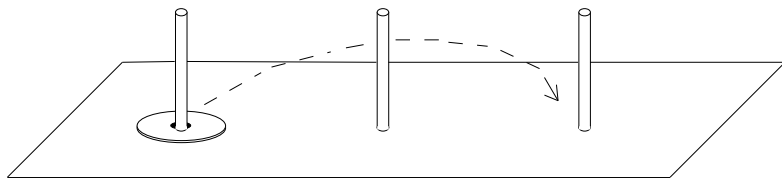
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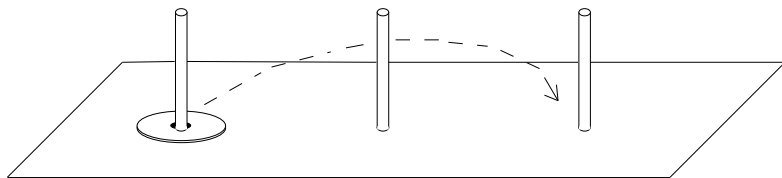


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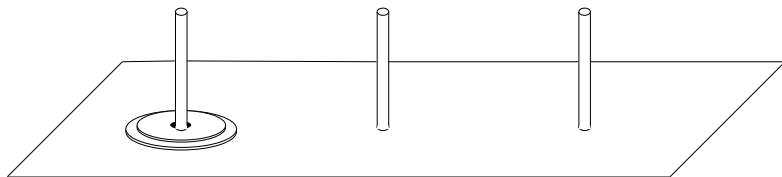


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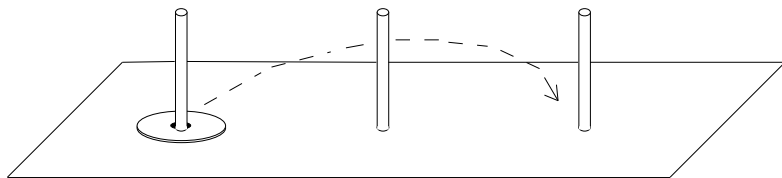
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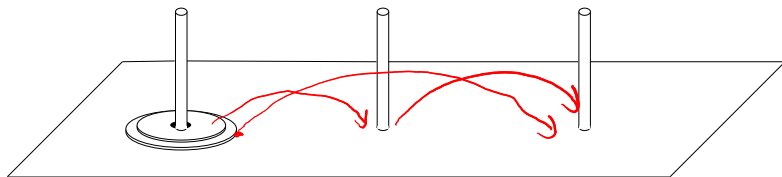
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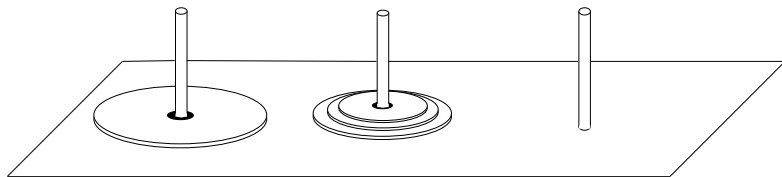
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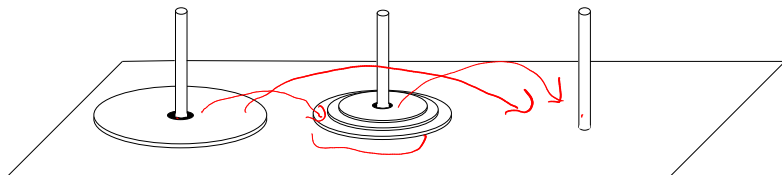
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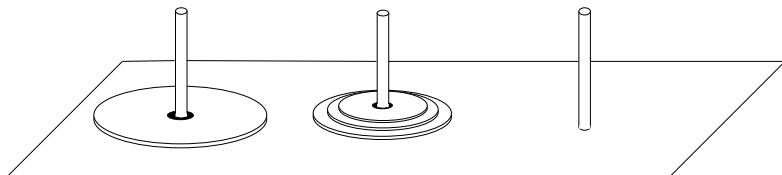
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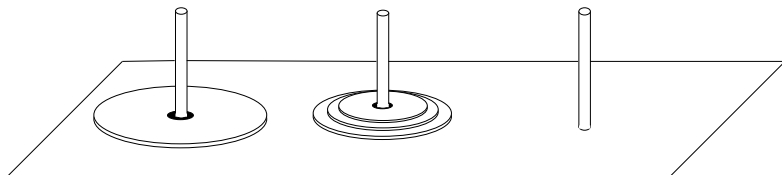


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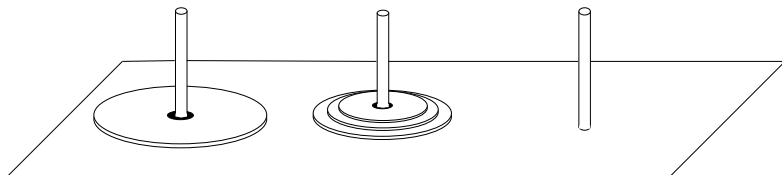
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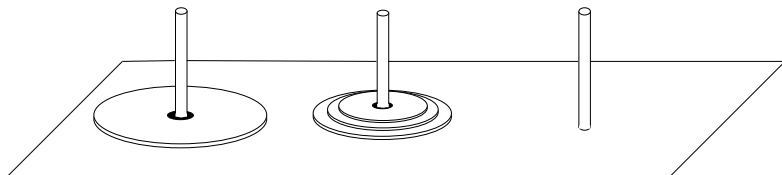
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In particular $x_{64} = \underline{(2^{64} - 1)}$ seconds $\sim 5.8 \times 10^{11}$ years.

Sorting algorithms

Let $N \in \mathbb{N}$, S be a set, and $(x_n)_{n \in \{1, \dots, N\}} \subseteq S$.

$(x_n) \in S \times \dots \times S$
 N copies

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Example:

$(x_n)_{n \in \{1, \dots, 5\}} = \text{Jane, Fred, Jo, Jane, Ann}$

$(y_n)_{n \in \{1, \dots, 5\}} = \text{Ann, Fred, Jane, Jane, Jo}$ (in alphabetical order)

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where $s, f \in \mathbb{N}^*$, $s \leq f$, are the **start index** and the **finish index**.

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Example: Suppose $\forall n \in \mathbb{N} \ a_n = 2n + 1$. Then $(a_n)_{3..6} = 7, 9, 11, 13$.

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where $s, f \in \mathbb{N}^*$, $s \leq f$, are the **start index** and the **finish index**.

Example: $I = \{3, 4, 5, 6\}$ ($s = 3$, $f = 6$)

For $I = \{s, \dots, f\}$ we may denote the sequence $(a_n)_{n \in I}$ by $(a_n)_{s..f}$.

Example: Suppose $\forall n \in \mathbb{N} \ a_n = 2n + 1$. Then $(a_n)_{3..6} = 7, 9, 11, 13$.

An **index permutation** on an index set I is a bijection (one-to-one correspondence) $\pi : I \rightarrow I$.

Sorting preliminaries

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Example:

$$\pi = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 5 \end{pmatrix} \text{ means } I = \{3, 4, 5, 6\} \\ \pi(3)=6, \pi(4)=4, \pi(5)=3, \pi(6)=5.$$

More sorting preliminaries

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Example: (names example recast using an index permutation)

If $(x_n)_{n \in \{1, \dots, 5\}} = \text{Jane}, \text{Fred}, \text{Jo}, \text{Jane}, \text{Ann}$

then $(x_{\pi(n)})_{n \in \{1, \dots, 5\}} = \text{Ann}, \text{Fred}, \text{Jane}, \text{Jane}, \text{Jo}$

where $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{pmatrix}$ sorts the sequence into alphabetical order.

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$i \leftarrow s + 1$. [Initialisation]

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Loop: If $i = f + 1$ stop.

If $x_{\pi(i)} < x_{\pi(m)}$ then $m \leftarrow i$.

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Repeat loop

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Swap the values of $\pi(s)$ and $\pi(m)$.

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		i	1	2	3	4	5	6
BEFORE		$\pi(i)$	1	2	3	4	5	6
		$x_{\pi(i)}$	F	D	C	E	B	C

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		i	1	2	3	4	5	6
BEFORE	$\pi(i)$		1	2	3	4	5	6
	$x_{\pi(i)}$		F	D	C	E	B	C

Trace:		i	2
		m	1
		$x_{\pi(i)}$	D
		$x_{\pi(m)}$	F

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Trace:

i	2	3
m	1	2
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$x_{\pi(m)}$	F	D

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i	2	3	4
m	1	2	3
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$x_{\pi(m)}$	F	D	C

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$x_{\pi(i)}$	D	C	E	B	C
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Trace:		i	2	3	4	5	6	7
		m	1	2	3	3	5	5
		$x_{\pi(i)}$	D	C	E	B	C	-
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BEFORE	$\pi(i)$		1	2	3	4	5	6
	$x_{\pi(i)}$		F	D	C	E	B	C

Trace:	i	2	3	4	5	6	7
	m	1	2	3	3	5	5
	$x_{\pi(i)}$	D	C	E	B	C	-
	$x_{\pi(m)}$	F	D	C	C	B	B

		i	1	2	3	4	5	6
AFTER	$\pi(i)$		5	2	3	4	1	6
	$x_{\pi(i)}$		B	D	C	E	F	C

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Loop: If $s = n$ stop.

Run least element

algorithm on $(x_{\pi(i)})_{s..n}$

$s \leftarrow s + 1$

Repeat loop

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Repeat loop

Example:

	i:	1	2	3	4	5	6
Input:	$\pi(i)$	1	2	3	4	5	6
$(n = 6)$	$x_{\pi(i)}$	F	D	C	E	B	C

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Repeat loop

Example:

	i:	1	2	3	4	5	6
Input: ($n = 6$)	$\pi(i)$	1	2	3	4	5	6
	$x_{\pi(i)}$	F	D	C	E	B	C

$s = 1$

After 1st iteration	$\pi(i)$	5	2	3	4	1	6
	$x_{\pi(i)}$	B	D	C	E	F	C

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Example:

	i:	1	2	3	4	5	6
Input: ($n = 6$)	$\pi(i)$	1	2	3	4	5	6
	$x_{\pi(i)}$	F	D	C	E	B	C
$s = 1$							
After 1st iteration	$\pi(i)$	5	2	3	4	1	6
	$x_{\pi(i)}$	B	D	C	E	F	C
$s = 2$							
After 2nd iteration	$\pi(i)$	5	3	2	4	1	6
	$x_{\pi(i)}$	B	C	D	E	F	C

Selection sort algorithm

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 S and an index function
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Output: Modification to π ,
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non-decreasing order
 $x_{\pi(1)} \leq x_{\pi(2)} \leq \dots \leq x_{\pi(n)}.$

Method:

$$s \leftarrow 1 \text{ [Initialisation]}$$

Loop: If $s = n$ stop.

Run least element

algorithm on $(x_{\pi(i)})_{s..n}$

$$s \leftarrow s + 1$$

Repeat loop

Example:

i: 1 2 3 4 5 6

Input: ($n = 6$)	$\pi(i)$	1	2	3	4	5	6
	$x_{\pi(i)}$	F	D	C	E	B	C

$$s = 1$$

After 1st iteration	$\pi(i)$	5	2	3	4	1	6
	$x_{\pi(i)}$	B	D	C	E	F	C

$$s = 2$$

After 2nd iteration	$\pi(i)$	5	3	2	4	1	6
	$x_{\pi(i)}$	B	C	D	E	F	C

$$s = 3$$

After 3rd iteration	$\pi(i)$	5	3	6	4	1	2
	$x_{\pi(i)}$	B	C	C	E	F	D

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	$x_{\pi(i)}$	F	D	C	E	B	C
$s = 1$							
After 1st iteration	$\pi(i)$	5	2	3	4	1	6
	$x_{\pi(i)}$	B	D	C	E	F	C
$s = 2$							
After 2nd iteration	$\pi(i)$	5	3	2	4	1	6
	$x_{\pi(i)}$	B	C	D	E	F	C
$s = 3$							
After 3rd iteration	$\pi(i)$	5	3	6	4	1	2
	$x_{\pi(i)}$	B	C	C	E	F	D
$s = 4$							
After 4th iteration	$\pi(i)$	5	3	6	2	1	4
	$x_{\pi(i)}$	B	C	C	D	F	E

Selection sort algorithm

Input: Sequence $(x_i)_{1..n} \subseteq S$,
 an ordering rule “ \leq ” for
 S and an index function
 π on $\{1, \dots, n\}$.

Output: Modification to π ,
 so that $(x_{\pi(i)})_{1..n}$ is in
 non-decreasing order
 $x_{\pi(1)} \leq x_{\pi(2)} \leq \dots \leq x_{\pi(n)}$.

Method:

$s \leftarrow 1$ [Initialisation]

Loop: If $s = n$ stop.

Run least element

algorithm on $(x_{\pi(i)})_{s..n}$

$s \leftarrow s + 1$

Repeat loop

Example:

		i:	1	2	3	4	5	6
Input: ($n = 6$)	$\pi(i)$	1	2	3	4	5	6	
	$x_{\pi(i)}$	F	D	C	E	B	C	
$s = 1$								
After 1st iteration	$\pi(i)$	5	2	3	4	1	6	
	$x_{\pi(i)}$	B	D	C	E	F	C	
$s = 2$								
After 2nd iteration	$\pi(i)$	5	3	2	4	1	6	
	$x_{\pi(i)}$	B	C	D	E	F	C	
$s = 3$								
After 3rd iteration	$\pi(i)$	5	3	6	4	1	2	
	$x_{\pi(i)}$	B	C	C	E	F	D	
$s = 4$								
After 4th iteration	$\pi(i)$	5	3	6	2	1	4	
	$x_{\pi(i)}$	B	C	C	D	F	E	
$s = 5$								
After final iteration	$\pi(i)$	5	3	6	2	4	1	
	$x_{\pi(i)}$	B	C	C	D	E	F	

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So:	1st	iteration uses	$n-1$	comparisons
	2nd	iteration uses	$n-2$	comparisons

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Hence the total number of comparisons, T_n say, is given by

$$1 + 2 + \dots + (n-1) = (n-1) \left(\frac{1+(n-1)}{2} \right) \text{ (sum of an arithmetic series).}$$

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That is: $\forall n \in \mathbb{N} \quad T_n = \frac{n(n-1)}{2}.$

Other sorting algorithms

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In order to keep the description simple, I will not use an indexing function π in specifying the algorithm, though it is possible, and often preferable, to do so.

As with Selection Sort, Merge Sort makes use of a sub-algorithm, which we treat first.

Merge algorithm

Input: Two lists (sequences) $(a_i)_{i \in \{1, \dots, n\}} \subseteq S$ and $(b_j)_{j \in \{1, \dots, p\}} \subseteq S$ pre-sorted according to an ordering rule " \leq " on S .

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Output: In-order list (sorted sequence) $(z_k)_{k \in \{1, \dots, n+p\}}$ that merges the two input lists.

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Method:

$i, j, k \leftarrow 1$. [Initialize the indices for the a -, b - and z - lists]

Loop: If $k = n + p + 1$ stop. [there will be $n + p$ items in the z -list]

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Else $[z_k \leftarrow b_j, j \leftarrow j + 1]$ [else take item from b -list]

$k \leftarrow k + 1$ [prepare to add next item to z -list]

Repeat loop.

Merge algorithm: example of execution

Example: Merge $(1, 3, 7)$ and $(2, 3, 6, 8, 9)$.

After iteration	i	j	k	a_i [green]	b_j [green]	(z_1, \dots, z_{k-1})
0	1	1	1	$(\mathbf{1}, 3, 7)$	$(\mathbf{2}, 3, 6, 8, 9)$	$()$

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After iteration	i	j	k	a_i [green]	b_j [green]	(z_1, \dots, z_{k-1})
0	1	1	1	(1 , 3, 7)	(2 , 3, 6, 8, 9)	()
1	2	1	2	(1, 3 , 7)	(2 , 3, 6, 8, 9)	(1)

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1	2	1	2	(1, 3 , 7)	(2 , 3, 6, 8, 9)	(1)
2	2	2	3	(1, 3 , 7)	(2, 3 , 6, 8, 9)	(1, 2)

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3	2	3	4	$(1, \mathbf{3}, 7)$	$(2, 3, \mathbf{6}, 8, 9)$	$(1, 2, 3)$

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3	2	3	4	$(1, \mathbf{3}, 7)$	$(2, 3, \mathbf{6}, 8, 9)$	$(1, 2, 3)$
4	3	3	5	$(1, 3, \mathbf{7})$	$(2, 3, \mathbf{6}, 8, 9)$	$(1, 2, 3, 3)$

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5	3	4	6	$(1, 3, \mathbf{7})$	$(2, 3, 6, \mathbf{8}, 9)$	$(1, 2, 3, 3, 6)$

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6	4	4	7	(1, 3, 7)	(2, 3, 6, 8 , 9,)	(1, 2, 3, 3, 6, 7)

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8	4	6	9	(1, 3, 7)	(2, 3, 6, 8, 9)	(1, 2, 3, 3, 6, 7, 8, 9)

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(For lists whose length is not a power of 2, see workshop question.)

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Method: There are r steps.

If $r < 3$ adjust the description below accordingly.

- Step 1: Apply the Merge algorithm 2^{r-1} times with inputs $\{(x_1), (x_2)\}, \{(x_3), (x_4)\}, \dots, \{(x_{2^{r-1}-1}), (x_{2^{r-1}})\}$.
This gives 2^{r-1} in-order lists of length 2.

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This gives 2^{r-2} in-order lists of length $2 \times 2 = 2^2$.
- Steps 3 to r : Continue in this vein until you have just one ($= 2^{r-r}$) in-order list with 2^r elements.

Merge sort: example

Example: merge sort (1, 2, 6, 1, 7, 9, 4, 5).

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Step 2:

Merging $\{(1, 2), (1, 6)\}$ gives (1, 1, 2, 6).

Merging $\{(7, 9), (4, 5)\}$ gives (4, 5, 7, 9).

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Merging $\{(1, 2), (1, 6)\}$ gives (1, 1, 2, 6).

Merging $\{(7, 9), (4, 5)\}$ gives (4, 5, 7, 9).

Step 3:

Merging $\{(1, 1, 2, 6), (4, 5, 7, 9)\}$ gives (1, 1, 2, 4, 5, 6, 7, 9).

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Claim: $\forall r \in \mathbb{N} \quad T_r = r2^r$. Verify by induction!

(The sequence $(T_r)_{r \in \mathbb{N}}$ is neither geometric, arithmetic nor mixed.)

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For short lists on high speed computers, slower speed may not matter.

END OF SECTION B2