These questions are for practice, in preparation for Workshop 2.

1. Let $E = \{a,b,c,d,e,f\}$, $A = \{a,b,c\}$, $B = \{d,e\}$, $C = \{\{a,b,c\},\{d,e\}\}$. Are the following statements true or false? Explain your answers.

(a) $A \subseteq E$. **True. Every member of A is a member of E.**

(b) $B \subset E$. **True. Every member of B is a member of E and $B \neq E$.**

(c) $C \subseteq E$. **False. The members of C are not *members* of E; they are *subsets* of E.**

(d) $A \subseteq C$. **False. A is a *member* of C, not a *subset* of C.**

2. Let $E = \{a,b,c,d,e,f\}$ be a universe of discourse. Let $A = \{a\}$, $B = \{b,c,d\}$, $C = \{f,a,d\}$. Compute the following.

(a) $A \cup B$. $\boldsymbol{\{a,b,c,d\}}$.

(b) $B \cap C$ $\boldsymbol{\{d\}}$.

(c) B^c. $\boldsymbol{\{a,e,f\}}$.

(d) $A \Delta C$. $\boldsymbol{\{f,d\}}$.

(e) $C \backslash A$. $\boldsymbol{\{f,d\}}$.

3. Let A, B, C be sets.
Prove that $(A \cap B)^c = A^c \cup B^c$.

LHS: $x \in (A \cap B)^c \iff x \notin A \cap B \iff \neg(x \in A \cap B) \iff \neg(x \in A \wedge x \in B)$
$\iff \neg(x \in A) \vee \neg(x \in B) \iff \boxed{(x \notin A) \vee (x \notin B)}$.

RHS: $x \in A^c \cup B^c \iff (x \in A^c) \vee (x \in B^c) \iff \boxed{(x \notin A) \vee (x \notin B)}$.

So LHS = RHS. (A full proof would give a reason for each if-and-only-if; e.g. "def of complement" for the 1st; "negation" for 2nd; etc. c.f. answer to Q9.)

4. Is $0 \in \varnothing$? Is $\{\varnothing\} \in \varnothing$? **No to both.**
Explain why. **By definition, $\varnothing$ has *no* members.**

5. Let $A = \{a,b,c,d\}$, $B = \{c,d,e\}$. Compute the following:

(a) $P(A)$. $\boldsymbol{\{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\},}$
$\boldsymbol{\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, A\}}$

(b) $P(A \cap B)$. $P(\{c,d\} = \boldsymbol{\{\varnothing, \{c\}, \{d\}, \{c,d\}\}}$

6. Let $A = \{a,b\}$, $B = \{1,2\}$. Compute the following:

(a) $A \cap B$. $\boldsymbol{\varnothing}$

(b) $P(A \cap B)$. $\{\boldsymbol{\varnothing}\}$

7. Let $A = \{0,1\}$. Compute $A \times A \times A \times A$. $= \{(s,t,u,v) : s,t,u,v \in A\}$
$= \boldsymbol{\{(0,0,0,0), (0,0,0,1), (0,0,1,0), (0,0,1,1), (0,1,0,0), (0,1,0,1), (0,1,1,0), (0,1,1,1),}$
$\boldsymbol{(1,0,0,0), (1,0,0,1), (1,0,1,0), (1,0,1,1), (1,1,0,0), (1,1,0,1), (1,1,1,0), (1,1,1,1)\}}$

8. Let $A = \{0,1\}$ and $B = \{a,b,c\}$. Are the following partitions of $A \times B$? Explain why or why not.

(a) $\{A_1, A_2\}$ where $A_1 = \{(0,a),(0,b),(0,c)\}$. $A_2 = \{(1,a),(1,b),(1,c)\}$.
Yes. $\boldsymbol{A_1 \cap A_2 = \varnothing}$ **and** $\boldsymbol{A_1 \cup A_2 = A \times B}$.

(b) $\{A_1, A_2\}$ where $A_1 = \{(0,a),(0,b),(0,c),(0,0)\}$. $A_2 = \{(1,a),(1,b),(1,c),(1,1)\}$.
No. $\boldsymbol{A_1 \nsubseteq A \times B}$ **since** $\boldsymbol{(0,0) \notin A \times B}$.

(c) $\{A_1, A_2, A_3\}$ where $A_1 = \{(0,a),(1,a)\}$. $A_2 = \{(0,b),(1,b)\}$. $A_3 = \{(0,c),(1,c)\}$.
Yes. $\boldsymbol{A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \varnothing}$ **and** $\boldsymbol{A_1 \cup A_2 \cup A_3 = A \times B}$.

(d) $\{A_1, A_2, A_3, A_4\}$ where $A_1 = \{(0,a),(0,b),(0,c)\}$. $A_2 = \{(0,a),(1,a)\}$. $A_3 = \{(0,b),(1,b)\}$. $A_4 = \{(0,c),(1,c)\}$. **No.** $\boldsymbol{A_1 \cap A_2 = \{(0,a)\} \neq \varnothing}$ **[*i.e.*** $\boldsymbol{A_1}$ **and** $\boldsymbol{A_2}$ **are not disjoint.]**

9. Prove that $A \cup (B \backslash A) = A \cup B$.

$$\begin{aligned}
x \in A \cup (B\backslash A) &\iff (x \in A) \vee (x \in B\backslash A) \quad \text{(definition of union)}\\
&\iff (x \in A) \vee ((x \in B) \wedge (x \notin A)) \quad \text{(definition of set difference)}\\
&\iff ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \notin A)) \quad \text{(distributive law)}\\
&\iff ((x \in A) \vee (x \in B)) \wedge T \quad \text{(since } p \vee \neg p \text{ is a tautology)}\\
&\iff (x \in A) \vee (x \in B) \quad \text{(since } p \wedge T \equiv p\text{)}\\
&\iff x \in A \cup B \quad \text{(definition of union)}
\end{aligned}$$

10. Find counterexamples to the following statements.

(a) $A \subseteq B \implies A^c \subseteq B^c$.
We need to find sets $\boldsymbol{A}$ **and** $\boldsymbol{B}$ **such that** $\boldsymbol{A \subseteq B}$ **but** $\boldsymbol{A^c \nsubseteq B^c}$.
The latter requires us to find an element $\boldsymbol{x \in A^c}$ **with** $\boldsymbol{x \notin B^c}$,
i.e. **an element** $\boldsymbol{x}$ **in** $\boldsymbol{B}$ **but not in** $\boldsymbol{A}$.
An easy way to do this, and at the same time ensure that
$\boldsymbol{A \subseteq B}$, **is just to take** $\boldsymbol{A}$ **empty and** $\boldsymbol{B}$ **non-empty.**

So the simplest counterexample is: $\boxed{\boldsymbol{A = \varnothing, \quad B = \{b\}}}$.

(b) $(A \nsubseteq B) \wedge (B \nsubseteq C) \implies A \nsubseteq C$.
We need to find sets $\boldsymbol{A, B, C}$ **such that** $\boldsymbol{A \nsubseteq B}$ **and** $\boldsymbol{B \nsubseteq C}$ **but** $\boldsymbol{A \subseteq C}$.
To satisfy the first two of these conditions $\boldsymbol{A}$ **needs and element** $\boldsymbol{a \notin B}$ **and** $\boldsymbol{B}$ **needs and element** $\boldsymbol{b \notin C}$.
We can satisfy the third condition by choosing $\boldsymbol{A = C}$ **provided this does not contradict the first two conditiions.**

The simplest counterexample is $\boxed{\boldsymbol{A = \{a\} = C, \quad B = \{b\}, \quad \text{with } a \neq b}}$.

(c) $(A \subseteq B) \wedge (B \nsubseteq C) \implies A \nsubseteq C$
We need to find sets $\boldsymbol{A, B, C}$ **such that** $\boldsymbol{A \subseteq B}$ **and** $\boldsymbol{A \subseteq C}$ **but** $\boldsymbol{B \nsubseteq C}$.
By combining ideas from (a) and (b) we can soon see that we can get what we need by taking $\boldsymbol{A}$ **and** $\boldsymbol{C}$ **empty but** $\boldsymbol{B}$ **non-empty.**

The simplest counterexample is $\boxed{\boldsymbol{A = C = \varnothing, \quad B = \{b\}}}$.