Question 1 (*Matrix algebra*) Let A, B, C be the matrices shown:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Compute those of the following that are defined:

(a) $A + B$ $\quad\boxed{\textit{Not defined}}$ *(shapes don't match).*

(b) $B + C$ $\quad\boxed{\textit{Not defined}}$ *(shapes don't match).*

(c) AB

$$\begin{bmatrix} 1\times1+2\times4 & 1\times2+2\times5 & 1\times3+2\times6 \\ 3\times1+4\times4 & 3\times2+4\times5 & 3\times3+4\times6 \end{bmatrix} = \boxed{\begin{bmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \end{bmatrix}}$$

(d) BC

$$\begin{bmatrix} 1\times1+2\times3+3\times5 & 1\times2+2\times4+3\times6 \\ 4\times1+5\times3+6\times5 & 4\times2+5\times4+6\times6 \end{bmatrix} = \boxed{\begin{bmatrix} 22 & 28 \\ 49 & 64 \end{bmatrix}}$$

(e) $A + BC$

$$\begin{bmatrix} 1+22 & 2+28 \\ 3+49 & 4+64 \end{bmatrix} = \boxed{\begin{bmatrix} 23 & 30 \\ 52 & 68 \end{bmatrix}}$$

(f) $\det(A)$ $\quad 1\times4 \; - \; 2\times3 \; = \; \boxed{-2}.$

(g) $\det(B)$ $\quad\boxed{\textit{Not defined}}$ *(B is not square).*

(h) $\det(C)$ $\quad\boxed{\textit{Not defined}}$ *(C is not square).*

(i) $\det(BC)$ $\quad 22\times64 \; - \; 28\times49 \; = \; 1408 - 1372 \; = \; \boxed{36}.$

(j) A^{-1}

$$\frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}}$$

Question 2 (*Systems of Linear equations*)

(a) Express the linear system
$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 5 \\ -x & + & 2y & + & z & = & 4 \\ x & - & y & + & z & = & 3 \end{array}$$

in matrix form $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is a 3×3 matrix and $\mathbf{x}$, $\mathbf{b}$ are 3×1 matrices.

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}.$$

(b) Verify that $\mathbf{A}^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -2 & -1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{bmatrix}$.

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{A} &= \frac{1}{4}\begin{bmatrix} 3 & -2 & -1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 3+2-1 & 3-4+1 & 3-2-1 \\ 2+0-2 & 2+0+2 & 2+0-2 \\ -1-2+3 & -1+4-3 & -1+2+3 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}. \end{aligned}$$

(c) Use $\mathbf{A}^{-1}$ to solve the linear system.

$$\begin{aligned} \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} &= \frac{1}{4}\begin{bmatrix} 3 & -2 & -1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 15-8-3 \\ 10+0-6 \\ -5+8+9 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}. \end{aligned}$$

So the solution is $x = 1$, $y = 1$, $z = 3$.

Question 3 (*Cardinality*) Let $\mathbb{Q}^+$ denote the set of positive rational numbers.
A relation '$\prec$' is defined on $\mathbb{Q}^+$ as follows: For any $q_1, q_2 \in \mathbb{Q}^+$ to determine whether $q_1 \prec q_2$ first write q_1, q_2 in 'lowest terms' $q_1 = \frac{a_1}{b_1}$, $q_2 = \frac{a_2}{b_2}$ where $a_1, b_1, a_2, b_2 \in \mathbb{N}$ and $\gcd(a_1, b_1) = \gcd(a_2, b_2) = 1$. Then:

$$q_1 \prec q_2 \Leftrightarrow \begin{cases} a_1 + b_1 < a_2 + b_2 & \text{or} \\ a_1 + b_1 = a_2 + b_2 & \text{and } a_1 < a_2 \end{cases}$$

(a) Verify that $\frac{5}{1} \prec \frac{2}{5} \prec \frac{3}{4}$.

Comparing $\frac{5}{1}$ *and* $\frac{2}{5}$: $5+1 = 6, \quad 2+5 = 7; \quad 6 < 7 \quad \therefore \quad \frac{5}{1} \prec \frac{2}{5}$

Comparing $\frac{2}{5}$ *and* $\frac{3}{4}$: $2+5 = 7 = 3+4; \quad 2 < 3 \quad \therefore \quad \frac{2}{5} \prec \frac{3}{4}$

(b) The relation $\prec$ 'well orders' $\mathbb{Q}^+$ in the sense that $\mathbb{Q}^+$ can be arranged 'in order' starting with $\frac{1}{1}$:

$$q_1 = \tfrac{1}{1} \prec q_2 = \tfrac{1}{2} \prec q_3 = \tfrac{2}{1} \prec q_4 \prec q_5 \prec \ldots\ldots$$

Write out the next nine fractions $q_4, \ldots q_{12}$ in this sequence.

i	1	2	3	4	5	6	7	8	9	10	11	12
a_i+b_i	2	3		4		5				6		7
a_i	1	1	2	1	3	1	2	3	4	1	5	1
b_i	1	2	1	3	1	4	3	2	1	5	1	6
q_i	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{1}$	$\mathbf{\frac{1}{3}}$	$\mathbf{\frac{3}{1}}$	$\mathbf{\frac{1}{4}}$	$\mathbf{\frac{2}{3}}$	$\mathbf{\frac{3}{2}}$	$\mathbf{\frac{4}{1}}$	$\mathbf{\frac{1}{5}}$	$\mathbf{\frac{5}{1}}$	$\mathbf{\frac{1}{6}}$

(c) Explain why this 'well ordering' $\prec$ shows that, surprisingly, $\mathbb{Q}^+$ is countable (*i.e.* $\mathbb{Q}^+$ has the same cardinality as $\mathbb{N}$).

Because the function $f : \mathbb{N} \to \mathbb{Q}^+$; $f(n) = q_n$ *is a bijection.*

It is injective because of the "lowest terms" requirement.

It is surjective because the sequence includes all fractions $\frac{a}{b}$, $a, b \in \mathbb{N}$.

(d) Is $\mathbb{Q}$ countable? Justify your answer.

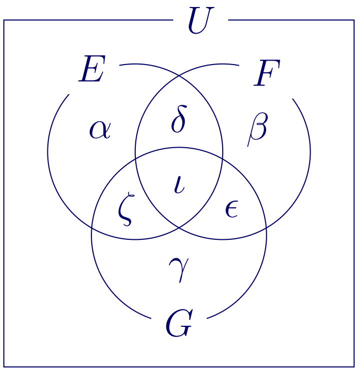
$\boxed{\textit{Yes}}$. *Define* $g : \mathbb{N} \to \mathbb{Q}$ *by*

$$\begin{aligned} g(1) &= 0 \\ g(2k) &= f(k) \ \forall k \in \mathbb{N} \\ g(2k+1) &= -f(k) \ \forall k \in \mathbb{N} \end{aligned}$$

Given that $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$ *it is fairly obvious that* g *is a bijection.*

Question 4 (*Inclusion-exclusion principle and the product rule*)

(a) The inclusion-exclusion principle for two sets A, B is $|A \cup B| = |A| + |B| - |A \cap B|$. Use a Venn diagram to find a similar formula for three sets A, B, C.



In the diagram at left (taken from the relevant section of the lecture notes) the Greek letters represent the number of members of $E \cup F \cup G$ in each of its subsets created by intersections. So:

$$
\begin{aligned}
&|E \cup F \cup G| = \alpha + \delta + \beta + \zeta + \iota + \epsilon + \gamma, \\
&|E| = \alpha + \delta + \zeta + \iota, \quad |E \cap F| = \delta + \iota, \\
&|F| = \delta + \beta + \iota + \epsilon, \quad |F \cap G| = \iota + \epsilon, \\
&|G| = \zeta + \iota + \epsilon + \gamma, \quad |G \cap E| = \zeta + \iota, \\
&\qquad\qquad \text{and } |E \cap F \cap G| = \iota.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|E| + |F| + |G| &= \alpha + 2\delta + \beta + 2\zeta + 3\iota + 2\epsilon + \gamma \\
&= |E \cup F \cup G| + \delta + \zeta + \epsilon + 2\iota
\end{aligned}
$$

Hence $\quad = |E \cup F \cup G| + |E \cap F| + |F \cap G| + |G \cap E| - \iota.$

$$\boxed{|E \cup F \cup G| = |E| + |F| + |G| - |E \cap F| - |F \cap G| - |G \cap E| + |E \cap F \cap G|}.$$

(b) A PIN is a number with four decimal digits, *e.g.* 2357, 0944 etc. A 'double digit' in a PIN is any pair of consecutive equal digits, such as the 44 in 0944.
Use inclusion-exclusion to count how many PINs have at least one double digit.
[There is another, slightly quicker, way to count these PINs. Can you see it?]

Let L be the set of all PINs with a double digit at the left end.
Let R be the set of all PINs with a double digit at the right end.
Let M be the set of all PINs with a double digit in the middle.

We need $|L \cup M \cup R|$:

$$
\begin{aligned}
|L| &= |\{sstu : s, t, u \in \{0, \ldots, 9\}\}| = 10 \times 10 \times 10 = 10^3 \\
|M| &= |\{sttu : s, t, u \in \{0, \ldots, 9\}\}| = 10 \times 10 \times 10 = 10^3 \\
|R| &= |\{stuu : s, t, u \in \{0, \ldots, 9\}\}| = 10 \times 10 \times 10 = 10^3 \\
|L \cap M| &= |\{ssst : s, t \in \{0, \ldots, 9\}\}| = 10 \times 10 = 10^2 \\
|M \cap R| &= |\{sttt : s, t \in \{0, \ldots, 9\}\}| = 10 \times 10 = 10^2 \\
|L \cap R| &= |\{sstt : s, t \in \{0, \ldots, 9\}\}| = 10 \times 10 = 10^2 \\
|L \cap M \cap R| &= |\{ssss : s \in \{0, \ldots, 9\}\}| = 10.
\end{aligned}
$$

So $|L \cup M \cup R| = 3(10^3) - 3(10^2) + 10 = \boxed{2710}.$

An alternative to the above is to use complementary counting. Without restictions there are 10^4 PINs. With the restiction of no *double digits there are $10 \times 9 \times 9 \times 9 = 7290$ PINs. So the number* with *double digits is* $10000 - 7290 = 2710$.

Question 5 (*Combinations and 'stars and bars'*) A TAW is a three letter 'word' whose letters are in alphabetical order. Letters are drawn from the standard lower case English 26-letter alphabet, and are allowed to repeat. A 'word' does not have to appear in any dictionary but must contain at least one vowel and at least one consonant. Letter y can count as a vowel or a consonant. Examples of TAWs are *abc*, *ccy*, *aoy* and *yyy*.

How many different TAWs are there?

Hint 1: Once three letters are chosen (possibly involving repeats) there is only one way to put them in alphabetical order.
Hint 2: As a first step ignore the requirement about vowels and consonants.

From the hints, we first need to count the multisets of size 3 *with members from the alphabet. (Recall that a "multiset" is an un-ordered list with repeats allowed.) We use 'stars and bars' for this.*

Take 26 buckets, one for each letter of the alphabet, and drop three stars into this collection. We need 25 bars to separate the buckets.

For example, $|||\star|||||\star\star\underbrace{||\cdots||}_{17\ bars}$ *represents* dii.

So the number of these multisets is $\binom{25+3}{3} = \binom{28}{3}$.

Now, using complementary counting, we must subtract the number of all-vowel 'words' and the number of all-consonant words.
We again use 'stars and bars', but with less buckets.

There are $5\,*$ *vowels, so* $\binom{4+3}{3} = \binom{7}{3}$ *all-vowel words.*

There are $20\,*$ *consonants, so* $\binom{19+3}{3} = \binom{22}{3}$ *all-consonant words.*

Thus the number of TAWs is

$$\begin{aligned}\binom{28}{3} - \binom{7}{3} - \binom{22}{3} &= \frac{28\cdot 27\cdot 26}{3.2.1} - \frac{7\cdot 6\cdot 5}{3.2.1} - \frac{22\cdot 21\cdot 20}{3.2.1} \\ &= 3276 - 35 - 1540 = \boxed{1701}.\end{aligned}$$

* *Since letter* y *is ambivalent, we do not count it amongst the vowels or the consanents when counting all-vowel and all-consonant words. For example we do not want to eliminate* eye *nor* cry.

Question 6 (*The pigeon hole principle*)

(a) Each year the ANU enrols new students from all eight States and Territories in Australia. How many new Australian students must the ANU enrol next year in order to ensure that there are at least 500 students from the same State or Territory?

The students are the pigeons; the States and Territories are the pigeon holes.

We need the smallest n for which $\left\lceil \dfrac{n}{8} \right\rceil = 500$.

So take $n = 8 \times 499 + 1 = \boxed{3993}$.

(b) Seven numbers are picked from the set $\{1, 2, \ldots, 12\}$ of the first twelve natural numbers. Prove that amongst the seven numbers picked it is guaranteed that two of them, say a and b satisfy $a - b = 3$.

Remark: This is easy once you've figured out what the pigeon holes should be, but figuring that out is perhaps not so easy!

*Make six pigeon holes, each labelled with **two** numbers, like this:*

1,4	*2,5*	*3,6*	*7,10*	*8,11*	*9,12*

Then every number in $\{1, 2, \ldots, 12\}$ *is part of exactly one label.*

The given seven numbers are the pigeons and each roosts in the hole with its number on it.

With seven pigeons and only six holes, at least one hole contains two pigeons. The corresponding numbers differ by 3.

[This question is from the preparatory question set for section C1.]