Question 1

(a) Evaluate: (i) $\sum_{k=1}^{5}(k-1)^2$ (ii) $\prod_{k=1}^{5}\left(\frac{k+1}{k}\right)$

(b) Calculate a_6 given that

$$a_0 = 2, \quad a_1 = 3, \quad \text{and} \quad \forall n \in \mathbb{N} \quad a_{n+1} = a_n(a_{n-1} - 1).$$

(a) *(i)* $0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 0 + 1 + 4 + 9 + 16 = \boxed{30}.$

(ii) $\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right)\left(\frac{6}{5}\right) = \frac{\cancel{2}}{1}\frac{\cancel{3}}{\cancel{2}}\frac{\cancel{4}}{\cancel{3}}\frac{\cancel{5}}{\cancel{4}}\frac{6}{\cancel{5}} = \frac{6}{1} = \boxed{6}.$

(b)
$$\begin{aligned}
a_2 &= a_1(a_0-1) = 3(2-1) = 3(1) = 3.\\
a_3 &= a_2(a_1-1) = 3(3-1) = 3(2) = 6.\\
a_4 &= a_3(a_2-1) = 6(3-1) = 6(2) = 12.\\
a_5 &= a_4(a_3-1) = 12(6-1) = 12(5) = 60.\\
a_6 &= a_5(a_4-1) = 60(12-1) = 60(11) = \boxed{660}.
\end{aligned}$$

Question 2 A sequence $(a_n)_{n\in\{0,\dots,N\}}$ defined implicitly by

$$a_0 = a, \qquad \forall n \in \mathbb{N} \quad n \le N \;\Rightarrow\; a_n = a_{n-1} + d$$

is called *arithmetic*.

(a) Write out the structural part of a proof by mathematical induction that every a_n is given by the formula $a_n = a + nd$.

Leave plenty of space on the board (or your page) to show where the body of the proof will go. The following should be made explicit and clear in your answer: the predicate that will be proved to hold for all values from a set; the base step; the inductive step; the moment that you make the inductive hypothesis; the part where you "invoke" the principle of mathematical induction.

(b) Using a different colour marker or pen, complete the body of the proof.

(Answer to part (a) is in bold.)

Let

$$\mathbf{P(n) : a_n = a + nd}$$

We will use mathematical induction to prove $\forall \mathbf{n} \in \mathbb{Z}_{\ge 0} \;\; \mathbf{P(n)}$.

Basis step: We will show that $\mathbf{P(0)}$ holds.

$$\begin{aligned} \textit{LHS of } P(0) &= a_0 \\ &= a \qquad \textit{(by the inductive defn of the sequence).} \\ &= a + 0d \\ &= \textit{RHS of } P(0). \end{aligned}$$

Hence $P(0)$ holds.

Inductive step: Let $\mathbf{n} \in \mathbb{N}$. Suppose that $\mathbf{P(0), P(1), \dots, P(n)}$ all hold. We will show that $\mathbf{P(n+1)}$ holds.

$$\begin{aligned} \textit{LHS of } P(n+1) &= a_{n+1} \\ &= a_n + d \qquad \textit{(by the inductive defn of the sequence).} \\ &= (a + nd) + d \qquad \textit{(Using } P(n)\textit{)} \\ &= a + nd + d \\ &= a + (n+1)d \\ &= \textit{RHS of } P(n+1). \end{aligned}$$

Hence $\mathbf{P(n+1)}$ holds.

By the principle of mathematical induction, $\mathbf{P(n)}$ holds for all $\mathbf{n} \in \mathbb{Z}_{\ge 0}$. □

Question 3

(a) Prove by induction that $\forall n \in \mathbb{N} \quad \sum_{k=0}^{n-1} k = \frac{n(n-1)}{2}$.

(b) Using (a), or otherwise, prove that the sum of any arithmetic sequence is equal to its number of terms times the average of its first and last terms. *e.g.*

$$2+5+8+11+14+17+20+23+26 \;=\; 9\left(\frac{2+26}{2}\right) = 9 \times 14 = 126.$$

(a) Let

$$P(n): \quad \sum_{k=0}^{n-1} k = \frac{n(n-1)}{2}.$$

We will use mathematical induction to show $\forall n \in \mathbb{N} \;\; P(n)$.

Basis Step: *LHS of P(1)* $= \sum_{k=0}^{1-1} k = \sum_{k=0}^{0} k = 0$;

RHS of P(1) $= \frac{1(1-1)}{2} = \frac{0}{2} = 0$.

Hence P(1) holds.

Inductive Step: *Let* $n \in \mathbb{N}$. *Suppose that* $P(1), P(2), \ldots, P(n)$ *all hold. We will show that* $P(n+1)$ *also holds:*

LHS of P(n+1)

$$= \sum_{k=0}^{(n+1)-1} k = \sum_{k=0}^{n} k = \sum_{k=0}^{n-1} k + n \qquad (\textit{summation propety})$$

$$= \frac{n(n-1)}{2} + n \qquad (\textit{using } P(n))$$

$$= \frac{n(n-1)+2n}{2} = \frac{n^2-n+2n}{2} = \frac{n^2+n}{2}$$

$$= \frac{(n+1)n}{2} = \frac{(n+1)((n+1)-1)}{2}$$

$= \textit{RHS of } P(n+1)$.

Hence $P(n+1)$ *holds*

newpage By the principle of mathematical induction, $P(n)$ *holds for all* $n \in \mathbb{N}$.

(b) *Let $a_0 = a$ be the first term of the arithmetic sequence, d be the common difference, and n the number of terms. Then*

$$
\begin{aligned}
\textit{sum of sequence} &= \sum_{k=0}^{n-1}(a+kd) \\
&= \sum_{k=0}^{n-1} a + \left(\sum_{k=0}^{n-1} k\right) d \quad (\textit{properties of addition}) \\
&= na + \left(\frac{n(n-1)}{2}\right) d \quad \begin{array}{l}\textit{(since } a \textit{ is constant,} \\ \textit{and using part (a))}\end{array} \\
&= \overbrace{n}^{\textit{number of terms}} \underbrace{\left(\frac{\overbrace{a}^{\textit{first term}} + \overbrace{(a+(n-1)d)}^{\textit{last term}}}{2}\right)}_{\textit{average of first and last terms}}
\end{aligned}
$$

Question 4 I borrow \$100 000 at an interest rate of 6% per annum and agree to pay back \$1000 per month. Assuming interest compounding monthly, my debt, in dollars, after n months is given implicitly by

$$a_0 = 100\,000 = a \qquad a_n = (1.005)a_{n-1} - 1000 = ra_{n-1} - f$$

say, where $a = 100\,000$, $r = 1.005$ and $f = 1000$.

(a) Prove by mathematical induction that $\forall n \in \mathbb{N} \quad a_n = ar^n - f\sum_{k=0}^{n-1} r^k$.

(b) Given that for $r \neq 1$ and $n \in \mathbb{N}$, $\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r-1}$, how much will I owe in 10 years time?

(a) Let

$$P(n): \quad a_n = ar^n - f\sum_{k=0}^{n-1} r^k.$$

We will use mathematical induction to prove that $\forall n \in \mathbb{N}\ \ P(n)$ ***Basis Step:***

$$LHS \text{ of } P(1) = a_1 = ra_0 - f \quad \text{(using the inductive defn).}$$

$$RHS \text{ of } (P(1) = ra^1 - f\sum_{k=0}^{1-1} r^k = ra - f\sum_{k=0}^{0} r^k = ra - fr^0 = ra - f.$$

Hence $P(1)$ holds.

Inductive step: *Let $n \in \mathbb{N}$. Suppose that $P(1), P(2), \ldots, P(n)$ all hold. We will show that $P(n+1)$ holds.*

$$LHS \text{ of } P(n+1) = a_{n+1}$$

$$ra_n - f \qquad \text{(using the implicit definition)}$$

$$= r\left(ar^n - f\sum_{k=0}^{n-1} r^k\right) - f \qquad \text{(using } P(n))$$

$$= \qquad ar^{n+1} - f\left(\sum_{k=0}^{n-1} r^{k+1} + 1\right)$$

$$= ar^{n+1} - f\sum_{k=0}^{n} r^k = ar^{n+1} - f\sum_{k=0}^{(n+1)-1} r^k$$

Hence $P(n+1)$ holds

By the principle of mathematical induction, $P(n)$ holds for all $n \in \mathbb{N}$.

(b)

$$\begin{aligned} a_{120} &= 100\,000(1.005)^{120} - 1000\left(\frac{(1.005)^{120}-1}{1.005-1}\right) \\ &= 100\,000(1.005)^{120} - 200\,000((1.005)^{120}-1) \\ &= 100\,000(2-(1.005)^{120}) = \boxed{\$18\,060.33}. \end{aligned}$$

Question 5 The letters of the word `TROUNCED` form the list $(X_i)_{1..8} = $ (T,R,O,U,N,C,E,D). This list is to be sorted into alphabetical order using Selection sort. The sorting is to be achieved by progressively modifying an index function π, rather than by shuffling members of the list itself. So initially

$$(X_i)_{1..8} = (X_{\pi(i)})_{1..8} \text{ where } \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

and when sorting is complete π is sufficiently changed so that $(X_{\pi(i)})_{1..8}$ is in order.

(a) First apply the Least Element algorithm to $(X_i)_{1..8}$. Demonstrate the application by completing the trace table at right.

i	2	3	4	5	6	7	8	9
m	1	2	3	3	5	6	6	6
$x_{\pi(i)}$	R	O	U	N	C	E	D	-
$x_{\pi(m)}$	T	R	O	O	N	C	C	C

(b) Write out the modified index function π resulting from (a). $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 3 & 4 & 5 & 1 & 7 & 8 \end{pmatrix}$

(c) Now apply the Least Element algorithm to $(X_{\pi(i)})_{2..8}$ using this modified π, again demonstrating the application by a trace table.

(d) Write out the newly modified index function π resulting from (c).

(e) Without making trace tables, write out the state of index function π after each of the remaining applications of the Least element algorithm needed to complete the Selection sort of (T,R,O,U,N,C,E,D).

(f) What is the total number of comparisons used during this sort?

(g) By contrast, how many comparisons, in total, would be used to sort (T,R,O,U,N,C,E,D) using the Merge sort algorithm? To find out, carry out the Merge sort algorithm on (T,R,O,U,N,C,E,D) and carefully count the comparisons, remembering that when the Merge algorithm reaches a stage where one of its input lists is empty, it does not need any more comparisons to complete its task.

(c)

i	3	4	5	6	7	8	9
m	2	3	3	5	5	7	8
$x_{\pi(i)}$	O	U	N	T	E	D	-
$x_{\pi(m)}$	R	O	O	N	N	E	D

(d) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 3 & 4 & 5 & 1 & 7 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & \mathbf{7} & 4 & 5 & 1 & \mathbf{3} & 2 \end{pmatrix}$ [C D E U N T O R] $\to$ $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 7 & \mathbf{5} & \mathbf{4} & 1 & 3 & 2 \end{pmatrix}$ [C D E N U T O R] $\to$ $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 7 & 5 & \mathbf{3} & 1 & \mathbf{4} & 2 \end{pmatrix}$ [C D E N O T U R]

$\to$ $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 7 & 5 & 3 & \mathbf{2} & 4 & \mathbf{1} \end{pmatrix}$ [C D E N O R U T] $\to$ $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 7 & 5 & 3 & 2 & \mathbf{1} & \mathbf{4} \end{pmatrix}$ [C D E N O R T U]

(f) $8(7)/2 = \boxed{28}$.

(g)
(T) (R) (O) (U) (N) (C) (E) (D)
(R,T) (O,U) (C,N) (D,E) 1+1+1+1 = 4
(O,R,T,U) (C,D,E,N) 3+3 = 6
(C,D,E,N,O,R,T,U) 4 $\boxed{\textit{Total } 14}$.

Question 6 In lectures we saw how use the Merge sort algorithm to sort a sequence of length $n = 2^r$ into ascending order. In fact the algorithm can be applied to sequences of any length $n \in \mathbb{N}$. At each iteration the current sorted sub-sequences are merged in pairs as for the 2^r case but if there are an odd number of sub-sequences then the 'left over' one just joins, unchanged, the newly formed sub-sequences at the next iteration. This will mean that the merge algorithm will sometimes need to merge sequences of unequal lengths, but this causes no problems.

For example, if Merge sort is used to sort the letters of the word `PROVISIONAL` into alphabetical order then the subsequences at each stage will be:

after 0th iteration	`(P),(R),(O),(V),(I),(S),(I),(O),(N),(A),(L);`
after 1st iteration	`(P,R),(O,V),(I,S),(I,O),(A,N),(L);`
after 2nd iteration	`(O,P,R,V),(I,I,O,S),(A,L,N);`
after 3rd iteration	`(I,I,O,O,P,R,S,V),(A,L,N);`
after 4th iteration	`(A,I,I,L,N,O,O,P,R,S,V).`

(a) Apply the Merge sort algorithm to sort the letters of the word `APPROPRIATION` into alphabetical order, showing the results of each iteration as in the example above.

(b) How many comparison operations are used to merge sort `APPROPRIATION`? As in Q5, remember that when the merge algorithm reaches the stage where one of its input lists is empty, it does not need any more comparisons to complete its task. For example, for `PROVISIONAL` there are only 5 comparisons during the first iteration, 8 in the 2nd, 7 in the 3rd and 5 in the last.

(c) How many comparison operations would be used if `APPROPRIATION` were sorted using the Selection sort algorithm?

(a) & (b):

`(A) (P) (P) (R) (O) (P) (R) (I) (A) (T) (I) (O) (N)`

`(A,P) (P,R) (O,P) (I,R) (A,T) (I,O) (N)` $6\times1=6$

`(A,P,P,R)`* `(I,O,P,R) (A,I,O,T) (N)` $3\times3^\star=9$

`(A,I,O,P,P,P,R,R) (A,I,N,O,T)` $7+3=10$

`(A,A,I,I,N,O,O,P,P,P,R,R,T)` 12

$\textit{Total} \;= 6 + 9 + 10 + 12 = \boxed{37}$.

* *When comparing equal items, we 'take from the right'.*
This (arbitrary) choice is dictated by the particular way the psuedo code for the MergeSort algorithm was formulated in the lecture notes. See B2 Slide27.

(c) $13(12)/2 = \boxed{78}$.

Question 7 So far, when using mathematical inducton we have made limited use of the inductive hypothesis. That is, we have only used $P(n)$ when deducing $P(n+1)$. Work this example to see that sometimes you need to use all of the inductive hypothesis $P(1) \wedge \cdots \wedge P(n)$ (or in this case $P(2) \wedge \cdots \wedge P(n)$).

Prove the following: Every integer that is at least two is prime or a product of two or more primes.

If you are not sure how to start, fall back on the safety steps we have reiterated in the course: write out the logical structure of the statement to be proved, with quantification; write out the structural part of a proof that is appropriate; then try to fill in the body of the proof.

Let

$$P(n) : (n \text{ is prime }) \vee (n \text{ is a product of two or more primes}).$$

We shall use mathematical induction to prove that $\forall n \in \{2, 3, 4, \dots\}\ \ P(n)$.

Basis step: *Since* 2 *is prime,* $P(2)$ *holds.*

Inductive step: *Let* $n \in \{2, 3, 4, \dots\}$. *Suppose that* $P(2), P(3), \dots, P(n)$ *all hold. We will show that* $P(n+1)$ *also holds. We consider cases.*
Case $n+1$ *is prime: It is clear that* $P(n+1)$ *holds in this case.*
Case $n+1$ *is not prime: By the definition of prime, there exist integers* $j, k \in \{2, 3, \dots, n\}$ *such that* $n + 1 = j \times k$.
Since $p(j)$ *holds, there exist* $x \in \mathbb{N}$ *and primes* $p_1, \dots, p_x$ *such that* $j = p_1 \times \cdots \times p_x$.
Since $p(k)$ *holds, there exist* $y \in \mathbb{N}$ *and primes* $q_1, \dots, q_y$ *such that* $k = q_1 \times \cdots \times q_y$.
Then

$$n + 1 = j \times k = p_1 \times \cdots \times p_x \times q_1 \times \cdots \times q_y.$$

Hence $n+1$ *is a product of two or more primes. Hence* $P(n+1)$ *holds.*
In all cases $P(n+1)$ *holds.*

By the principle of mathematical induction, $P(n)$ *holds for all* $n \in \{2, 3, 4, \dots\}$.
$\square$