 Broken recordings:

B2. Sequences.

Notes originally prepared by Pierre Portal
Editing and expansion by Malcolm Brooks.

Text Reference (Epp)

3ed:	Sections	4.1-4, 8.1-3 (Sequences and induction), 9.3,5 (Sorting)
4ed:	Sections	5.1-4,6-8, (Sequences and induction), 11.3,5 (Sorting)
5ed:	Sections	5.1-4,6-7, (Sequences and induction), 11.3,5 (Sorting)

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3. $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}^\star$ sequence of amplitudes. a_n: amplitude of the harmonic of frequency $n \times f$ (f fundamental frequency).
4. U set of users. $(u_n)_{n\in\{1,2,3,4,5\}} \subseteq U$: a list of 5 users.

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Continuing to argue in this manner gives $P(n)$ for all $n \in \mathbb{N}$.

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Proof by mathematical induction that the claim is correct:

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Inductive step: Assume the formula is correct for up to and including some fixed n.

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Basis step: For $n = 1$, formula gives $a_1 = 3^1 = 3$, agreeing with the implicit definition.

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Geometric Sequences and Series

Example 1 (Slide 6) is a special case of

Geometric Sequence	
Implicit Definition	Explicit Definition
$\boldsymbol{a_k = a}$ (a is the **first term**) $\boldsymbol{a_{n+1} = ra_n}$, $\forall n \geq k$ (r is the **common ratio**)	$\forall n \geq k$ $\boldsymbol{a_n = ar^{n-k}}$

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When terms or a sequence are summed, we get a **series**. In particular

Geometric Series	
Series of N terms	Sum of N terms
$b_N := \displaystyle\sum_{n=k}^{k+(N-1)} ar^{n-k} = a + ar + \cdots + ar^{N-1}$ [Usually $k=0$ or $k=1$.]	$\begin{cases} \dfrac{a(1-r^N)}{(1-r)} & \text{if } r \neq 1 \\ Na & \text{if } r = 1 \end{cases}$

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$$6+3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8} = \sum_{n=0}^{4} 6\left(\frac{1}{2}\right)^n$$

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From implicit to explicit definitions; Example 2

A sequence is defined implicitly by $\begin{cases} \boldsymbol{b_{n+1} = b_n + 5} & \forall n \in \mathbb{N}, \\ \boldsymbol{b_1 = 0.} \end{cases}$

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Arithmetic Sequences and Series

Example 2 (Slide 9) is a special case of

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$\boldsymbol{a_k = a}$ (a is the **first term**) $\boldsymbol{a_{n+1} = a_n + d}$, $\forall n \geq k$ (d is the **common difference**)	$\forall n \geq k$ $\boldsymbol{a_n = a + (n-k)d}$

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When the terms or an arithmetic sequence are summed, we get:

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Series of $\boldsymbol{N}$ terms	Sum of $\boldsymbol{N}$ terms
$\displaystyle\sum_{n=k}^{k+(N-1)} [\boldsymbol{a+(n-k)d}] = a+(a+d)+\cdots+(a+(N-1)d)$ [Usually $k=0$ or $k=1$.]	$\frac{N}{2}[2a+(N-1)d]$ $= N\left[\frac{a \;+\; (a+(N-1)d)}{2}\right]$

Exercise: prove this by induction.

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sum = number of terms times average of first and last

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Example: $1+3+5+7+\cdots+99$

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Example: $1+3+5+7+\cdots+99 = 50\left(\frac{1+99}{2}\right)$

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Example: $1+3+5+7+\cdots+99 = 50\left(\dfrac{1+99}{2}\right) = 2500.$

Simplest Malthusian model in population dynamics

(Exponential growth)


T. Robert Malthus 1766 - 1834

Let p_n be number of individuals in a population after n years.

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Assume the population doubles every year,

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Let p_n be number of individuals in a population after n years.

Assume the population doubles every year, *i.e.* $\forall n \in \mathbb{N}^\star\ p_{n+1} = 2p_n$.

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Let p_n be number of individuals in a population after n years.

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Assume the current population is five million.

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Assume the current population is five million. *i.e.* $p_0 = 5 \times 10^6$.

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So $\boldsymbol{N = 8}$ *i.e.* eight years.

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Now suppose the capital yields $\frac{3}{12} = 0.25\%$ interest per *month*.

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$$\boldsymbol{d_0 = 2 \times 10^4,\quad \forall n \in \mathbb{N}\ d_{n+1} = d_n + 0.0025d_n = 1.0025d_n}.$$

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$$\underbrace{d_{120}}_{\text{10 years}} = 2 \times 10^4 \times 1.0025^{120} = \underline{\mathbf{\$26\,987.71}}.$$

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$$d_{120} = 2\times 10^4 \times 1.0025^{120} = \mathbf{\$26\,987:71}. \quad \textbf{Slightly better!}$$

Better models

Compound Interest: If the bank is charging a fee of \$10 per year, the compound interest model becomes

$$\begin{cases} c_{n+1} = 1.03c_n - 10 \quad \forall n \in \mathbb{N}^\star, \\ c_0 = 2 \times 10^4. \end{cases}$$

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The sequences defined by these models are neither geometric nor arithmetic, but are a generalisation of both.
We need to start again from scratch.

From implicit to explicit definitions; Example 3

We seek an explicit formula for the population given by the implicit formula at right, where d and p are shorthand for 10^3 and 5×10^6.

$$\begin{cases} p_{n+1} = 2p_n + d \ \forall n \in \mathbb{N}^\star, \\ p_0 = p. \end{cases}$$

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Claim: $\forall n \in \mathbb{N}^\star \quad p_n = \underbrace{2^n p} + \underbrace{(2^n - 1)d}$.

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Proof by mathematical induction that the claim is correct:

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Basis step: For $n = 0$, formula gives $p_0 = 2^0 p + (2^0 - 1)d = p$, agreeing with the implicit definition.

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$$\begin{aligned} p_{n+1} &= 2p_n + d \quad \text{(from the implicit definition)} \\ &= 2(\underline{2^n p + (2^n - 1)d}) + d \quad \text{(by the inductive assumption)} \end{aligned}$$

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$$\begin{aligned} p_{n+1} &= 2p_n + d \quad \text{(from the implicit definition)} \\ &= 2(2^n p + (2^n - 1)d) + d \quad \text{(by the inductive assumption)} \\ &= 2^{n+1} p + (2^{n+1} - 1)d \quad \text{(by algebraic simplification)} \end{aligned}$$

and so the formula is also correct for $n+1$.

So by mathematical induction, $\forall n$, $P(n)$

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It takes very little extra analysis to generalise the previous example:

Mixed Geometric-Arithmetic Sequence	
Implicit Definition	Explicit Definition
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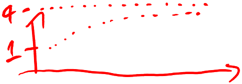

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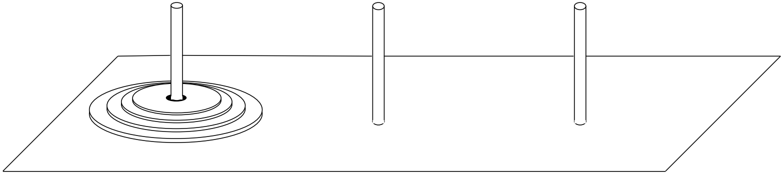
Counting moves: Towers of Hanoi

"The Towers of Hanoi" is a puzzle with 3 pegs and a number of punctured discs of decreasing sizes initially on the leftmost peg.



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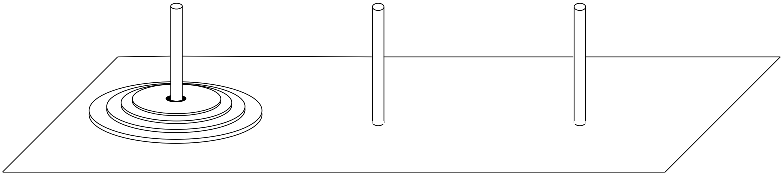

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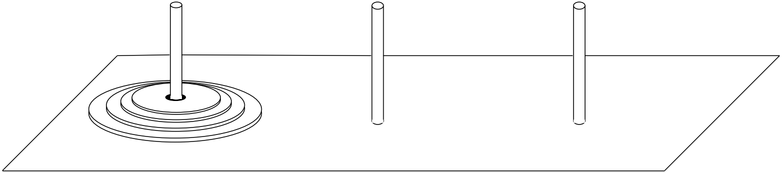

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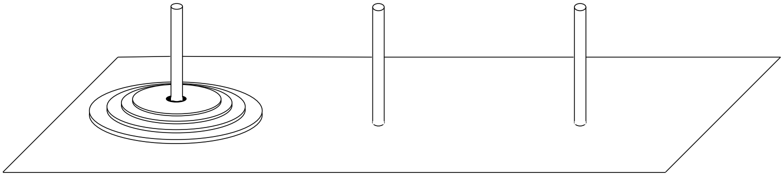

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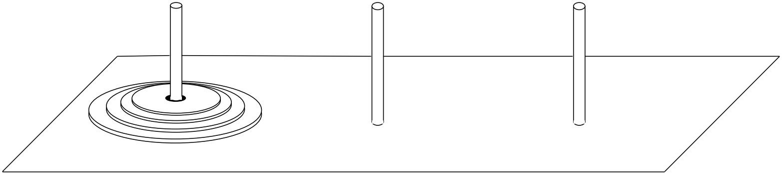

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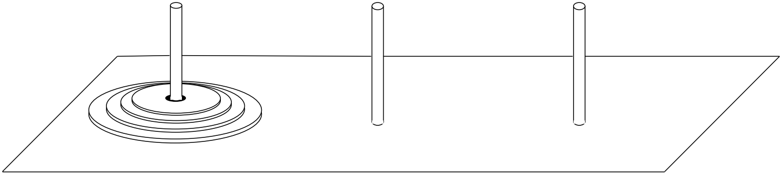

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At one move per second, how fast can you solve a puzzle with 64 discs?

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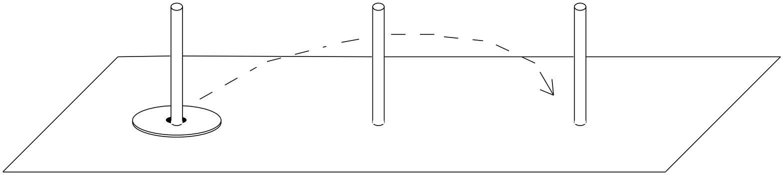

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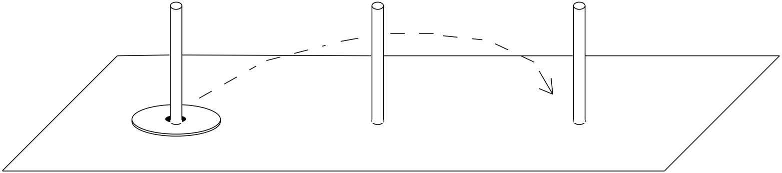

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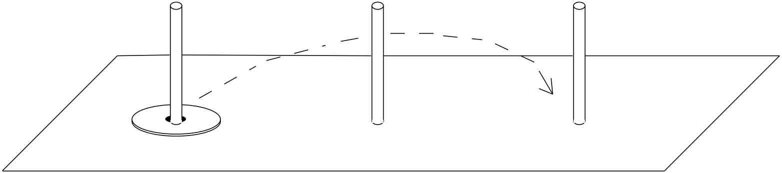

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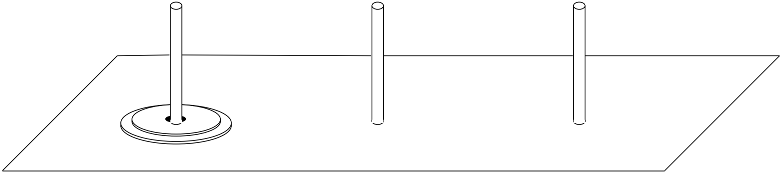

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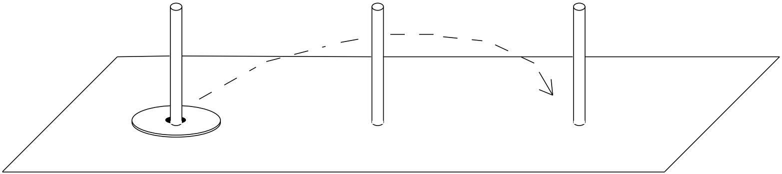

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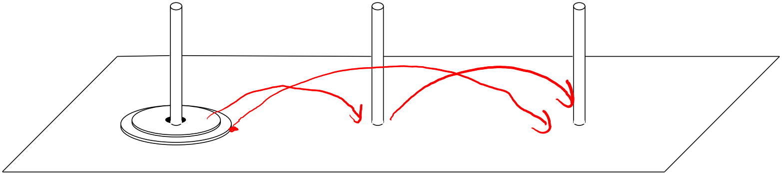

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Towers of Hanoi: Solution

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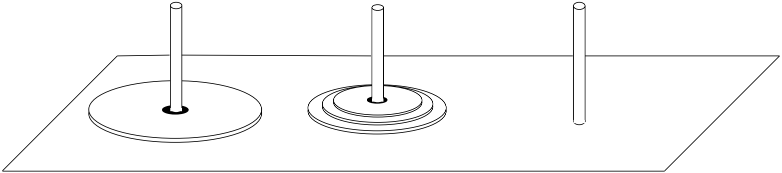

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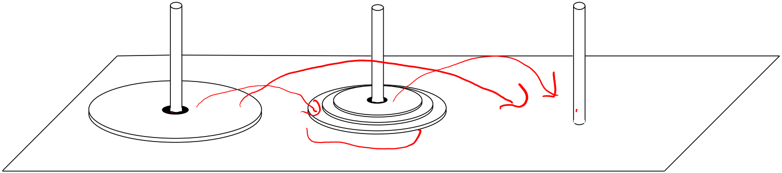

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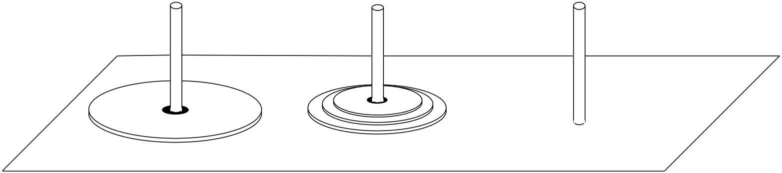

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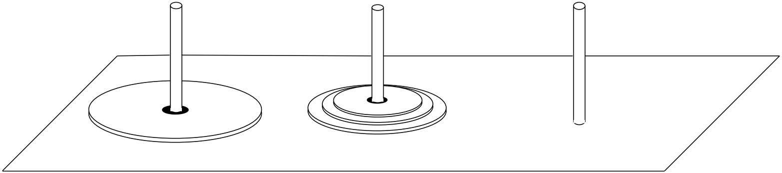

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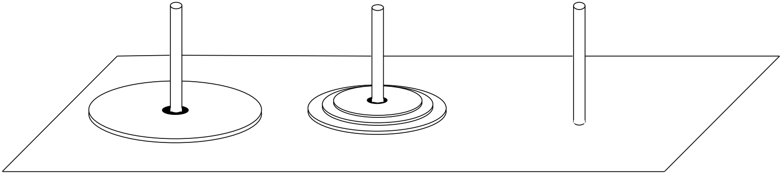

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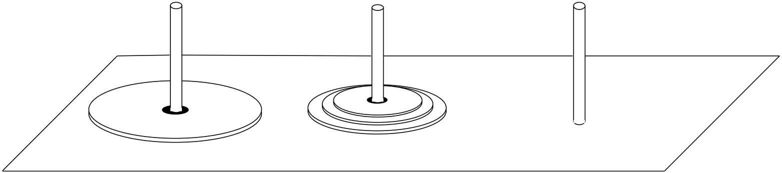

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In particular $x_{64} = \underline{(2^{64} - 1) \text{ seconds}} \sim 5.8 \times 10^{11}$ years.

Sorting algorithms

$(x_n) \in \underbrace{S \times \cdots \times S}_{N \text{ copies}}$

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A **sorting algorithm** is a procedure for sorting a sequence into increasing order according to some specified ordering rule (*e.g.* numerical, alphabetical, etc.)

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A **sorting algorithm** is a procedure for sorting a sequence into increasing order according to some specified ordering rule (*e.g.* numerical, alphabetical, etc.) *i.e.* it replaces $(x_n)_{n\in\{1,\ldots,N\}}$ by a rearrangement $(y_n)_{n\in\{1,\ldots,N\}}$ with

$$y_1 \leq y_2 \leq y_3 \cdots y_{N-1} \leq y_N$$

where "$\leq$" denotes the ordering rule.

Sorting algorithms

Let $N \in \mathbb{N}$, S be a set, and $(x_n)_{n\in\{1,\ldots,N\}} \subseteq S$.

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Example:
$(x_n)_{n\in\{1,\ldots,5\}}$ = Jane, Fred, Jo, Jane, Ann
$(y_n)_{n\in\{1,\ldots,5\}}$ = Ann, Fred, Jane, Jane, Jo (in alphabetical order)

Sorting preliminaries

An **index set** I is a set of the form

$$I = \{i \in \overset{\mathbb{Z}_{\geq 0}}{\cancel{\mathbb{N}}}^\star \ : \ s \leq i \leq f\} = \{s, \ldots, f\}$$

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Example:

$\pi = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 5 \end{pmatrix}$ means $\begin{array}{l} I = \{3, 4, 5, 6\} \\ \pi(3) = 6,\ \pi(4) = 4,\ \pi(5) = 3,\ \pi(6) = 5 \end{array}$.

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Example: (names example recast using an index permutation)
If $(x_n)_{n\in\{1,\ldots,5\}}$ = Jane, Fred, Jo, Jane, Ann
then $(x_{\pi(n)})_{n\in\{1,\ldots,5\}}$ = Ann, Fred, Jane, Jane, Jo
where $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{pmatrix}$ sorts the sequence into alphabetical order.

Least element algorithm

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Repeat loop

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Example: $(s = 1,\ f = 6)$

	i	1	2	3	4	5	6
BEFORE	$\pi(i)$	1	2	3	4	5	6
	$x_{\pi(i)}$	F	D	C	E	B	C

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Trace:

i	2
m	1
$x_{\pi(i)}$	D
$x_{\pi(m)}$	F

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Trace:

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i	2	3	4	5	6	7
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Trace:

i	2	3	4	5	6	7
m	1	2	3	3	5	5
$x_{\pi(i)}$	D	C	E	B	C	-
$x_{\pi(m)}$	F	D	C	C	B	B

	i	1	2	3	4	5	6
AFTER	$\pi(i)$	**5**	2	3	4	**1**	6
	$x_{\pi(i)}$	**B**	D	C	E	**F**	C

Selection sort algorithm

Input: Sequence $(x_i)_{1..n} \subseteq S$, an ordering rule "$\leq$" for S and an index function π on $\{1, \ldots, n\}$.

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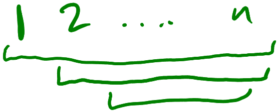
Loop: If $s = n$ stop.
Run least element algorithm on $(x_{\pi(i)})_{s..n}$
$s \leftarrow s + 1$

Repeat loop



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Repeat loop

Example:

	i:	1	2	3	4	5	6
Input:	$\pi(i)$	1	2	3	4	5	6
$(n = 6)$	$x_{\pi(i)}$	F	D	C	E	B	C

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Repeat loop

Example:

	i:	1	2	3	4	5	6
Input: ($n = 6$)	$\pi(i)$	1	2	3	4	5	6
	$x_{\pi(i)}$	F	D	C	E	B	C
$s = 1$							
After 1st iteration	$\pi(i)$	**5**	2	3	4	**1**	6
	$x_{\pi(i)}$	**B**	D	C	E	**F**	C

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Method:

$s \leftarrow 1$ [Initialisation]

Loop: If $s = n$ stop.

Run least element algorithm on $(x_{\pi(i)})_{s..n}$

$s \leftarrow s + 1$

Repeat loop

Example:

	i:	1	2	3	4	5	6
Input: ($n = 6$)	$\pi(i)$	1	2	3	4	5	6
	$x_{\pi(i)}$	F	D	C	E	B	C
$s = 1$							
After 1st iteration	$\pi(i)$	**5**	2	3	4	**1**	6
	$x_{\pi(i)}$	**B**	D	C	E	**F**	C
$s = 2$							
After 2nd iteration	$\pi(i)$	5	**3**	**2**	4	1	6
	$x_{\pi(i)}$	B	**C**	**D**	E	F	C

Selection sort algorithm

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$s = 1$							
After 1st iteration	$\pi(i)$	**5**	2	3	4	**1**	6
	$x_{\pi(i)}$	**B**	D	C	E	**F**	C
$s = 2$							
After 2nd iteration	$\pi(i)$	5	**3**	**2**	4	1	6
	$x_{\pi(i)}$	B	**C**	**D**	E	F	C
$s = 3$							
After 3rd iteration	$\pi(i)$	5	3	**6**	4	1	**2**
	$x_{\pi(i)}$	B	C	**C**	E	F	**D**

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	$x_{\pi(i)}$	**B**	D	C	E	**F**	C
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	$x_{\pi(i)}$	B	**C**	**D**	E	F	C
$s = 3$							
After 3rd iteration	$\pi(i)$	5	3	**6**	4	1	**2**
	$x_{\pi(i)}$	B	C	**C**	E	F	**D**
$s = 4$							
After 4th iteration	$\pi(i)$	5	3	6	**2**	1	**4**
	$x_{\pi(i)}$	B	C	C	**D**	F	**E**

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After 1st iteration	$\pi(i)$	**5**	2	3	4	**1**	6
	$x_{\pi(i)}$	**B**	D	C	E	**F**	C
$s = 2$							
After 2nd iteration	$\pi(i)$	5	**3**	**2**	4	1	6
	$x_{\pi(i)}$	B	**C**	**D**	E	F	C
$s = 3$							
After 3rd iteration	$\pi(i)$	5	3	**6**	4	1	**2**
	$x_{\pi(i)}$	B	C	**C**	E	F	**D**
$s = 4$							
After 4th iteration	$\pi(i)$	5	3	6	**2**	1	**4**
	$x_{\pi(i)}$	B	C	C	**D**	F	**E**
$s = 5$							
After final iteration	$\pi(i)$	5	3	6	2	**4**	**1**
	$x_{\pi(i)}$	B	C	C	D	**E**	**F**

Selection sort: number of operations

How many operations are required by Selection Sort?

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So:	1st	iteration uses	$n-1$	comparisons
	2nd	iteration uses	$n-2$	comparisons

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So:	1st	iteration uses	$n-1$	comparisons
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	$\vdots$	$\vdots$	$\vdots$	$\vdots$
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$$\text{So:} \quad \begin{array}{rcccl} \text{1st} & \text{iteration uses} & n-1 & \text{comparisons} \\ \text{2nd} & \text{iteration uses} & n-2 & \text{comparisons} \\ \vdots & \vdots & \vdots & \vdots \\ \text{last} & \text{iteration uses} & 1 & \text{comparison} \end{array}$$

Hence the total number of comparisons, T_n say, is given by

$$1 + 2 + \cdots + (n-1) = (n-1)\left(\frac{1+(n-1)}{2}\right) \text{ (sum of an arithmetic series).}$$

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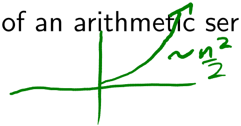

That is: $\quad \forall n \in \mathbb{N} \quad T_n = \dfrac{n(n-1)}{2}.$

Other sorting algorithms

There are many different sorting algorithms, with various pros and cons. A full study of the topic belongs in course on algorithms and data structures.

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As with Selection Sort, Merge Sort makes use of a sub-algorithm, which we treat first.

Merge algorithm

Input: Two lists (sequences) $(a_i)_{i\in\{1,\ldots,n\}} \subseteq S$ and $(b_j)_{j\in\{1,\ldots,p\}} \subseteq S$ pre-sorted according to an ordering rule "$\leq$" on S.

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Input: Two lists (sequences) $(a_i)_{i\in\{1,\ldots,n\}} \subseteq S$ and $(b_j)_{j\in\{1,\ldots,p\}} \subseteq S$ pre-sorted according to an ordering rule "$\leq$" on S.

Output: In-order list (sorted sequence) $(z_k)_{k\in\{1,\ldots,n+p\}}$ that merges the two input lists.

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If $i = n+1$ then $[\, z_k \leftarrow b_j,\ j \leftarrow j+1 \,]$ [a-list empty; take from b-list]

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Else [$z_k \leftarrow b_j$, $j \leftarrow j+1$] [else take item from b-list]

$k \leftarrow k+1$ [prepare to add next item to z-list]

Repeat loop.

Merge algorithm: example of execution

Example: Merge $(1, 3, 7)$ and $(2, 3, 6, 8, 9)$.

After iteration	i	j	k	a_i [green]	b_j [green]	$(z_1, \ldots, z_{k-1})$
0	1	1	1	$(\mathbf{1}, 3, 7)$	$(\mathbf{2}, 3, 6, 8, 9)$	$()$

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3	2	3	4	$(1, \mathbf{3}, 7)$	$(2, 3, \mathbf{6}, 8, 9)$	$(1, 2, 3)$
4	3	3	5	$(1, 3, \mathbf{7})$	$(2, 3, \mathbf{6}, 8, 9)$	$(1, 2, 3, 3)$

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7	4	5	8	$(1, 3, 7)$	$(2, 3, 6, 8, \mathbf{9})$	$(1, 2, 3, 3, 6, 7, 8)$

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8	4	6	9	$(1, 3, 7)$	$(2, 3, 6, 8, 9)$	$(1, 2, 3, 3, 6, 7, 8, 9)$

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- Steps 3 to r: Continue in this vein until you have just one $(= 2^{r-r})$ in-order list with 2^r elements.

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Example: merge sort $(1, 2, 6, 1, 7, 9, 4, 5)$.

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$(1, 1, 2, 4, 5, 6, 7, 9)$	6 comps
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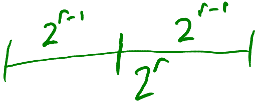

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Claim: $\forall r \in \mathbb{N} \quad T_r = r2^r$. Verify by induction!

(The sequence $(T_r)_{r\in\mathbb{N}}$ is neither geometric, arithmetic nor mixed.)

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For short lists on high speed computers, slower speed may not matter.

END OF SECTION B2