AEF_Assignment_2

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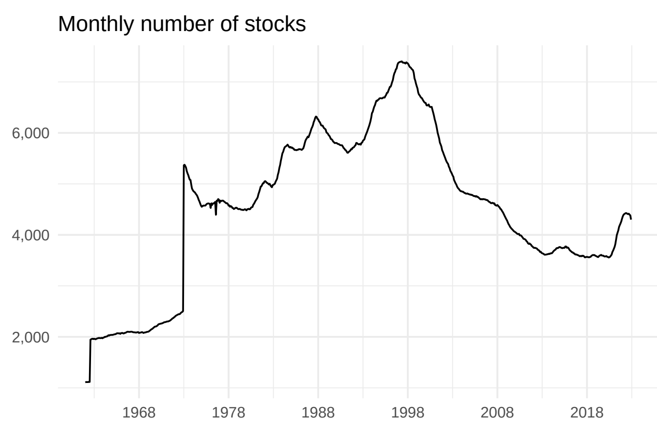
1 Load Data

For the empirical analysis, use the monthly CRSP dataset. Select all stocks from the CRSP universe with uninterrupted history to obtain a balanced panel of monthly (excess) returns for the entire time series from 1962 until 2020. Briefly summarize the sample: How many stocks does your investment universe consist of?



10 Industry portfolios are used as data as we do not have time to do it with CRSP. We use 10 industry portfolios from the Fama/French database for now as data, but we do want to be using the CRSP universe of stocks. We did gather the data from CRSP earlier, but now we will fall back on the industry data.

2 Transaction-cost adjusted portfolio

We derive a closed-form solution for the mean-variance efficient portfolio.

$$\begin{aligned} & arg\ \max \quad U && s.t.\ \ \omega'\iota = 1 \\ = & arg\ \max \quad \omega'\mu - \frac{\gamma}{2}\omega'\Sigma\omega && s.t.\ \ \omega'\iota = 1 \end{aligned} \tag{1}$$

With transaction costs (Equation 1) becomes

$$arg\ \max \quad \omega'\mu - \nu_t - \frac{\gamma}{2}\omega'\Sigma\omega \quad s.t.\ \ \omega'\iota = 1$$

where transaction costs ν_t are given by

$$\nu_t = \lambda(\omega - \omega_{t^+})'\Sigma(\omega - \omega_{t^+}) \tag{2}$$

We expand the equation for transaction cost

$$\nu_t = \lambda\omega'\Sigma\omega - 2\lambda\omega'\Sigma\omega_{t^+} + \lambda\omega'_{t^+}\Sigma\omega_{t^+} \tag{3}$$

We insert the equation for transaction cost (Equation 2 and then Equation 3) in the standard equation for utility (Equation 1)

$$\begin{aligned} & arg\ \max \quad \omega'\mu - \frac{\gamma}{2}\omega'\Sigma\omega - \lambda\omega'\Sigma\omega && s.t.\ \ \omega'\iota = 1 \\ = & arg\ \max \quad \omega'\mu - \frac{\gamma}{2}\omega'\Sigma\omega - (\lambda\omega'\Sigma\omega - 2\lambda\omega'\Sigma\omega_{t^+} + \lambda\omega'_{t^+}\Sigma\omega_{t^+}) && s.t.\ \ \omega'\iota = 1 \end{aligned} \tag{4}$$

We create two new variables μ^* and Σ^*

$$\begin{aligned} \mu^* &= \mu + 2\lambda\Sigma\omega_{t^+} \\ \Sigma^* &= \Sigma \cdot \left(1 + 2\tfrac{\lambda}{\gamma}\right) \end{aligned} \tag{5}$$

Since the only variable subject to optimizing in Equation 4 is ω, as ω_{t^+} is a constant, namely the portfolio weights before reallocating assets, ω_{t+1} does not depend on ω_{t^+} and thus not on $-\lambda\omega'_{t^+}\Sigma\omega_{t^+}$. By substituting (Equation 5) in (Equation 4), we get

$$\begin{aligned} & arg\ \max \quad \omega'\mu^* - \frac{\gamma}{2}\omega'\Sigma^*\omega - \left(\lambda\omega'_{t^+}\Sigma^*\omega_{t^+}\right) && s.t.\ \ \omega'\iota = 1 \\ = & arg\ \max \quad \omega'\mu^* - \frac{\gamma}{2}\omega'\Sigma^*\omega && s.t.\ \ \omega'\iota = 1 \end{aligned} \tag{6}$$

The solution to the simple optimal portfolio problem as stated in Equation 1 is

$$\omega^*_{t+1} = \frac{1}{\gamma}\left(\Sigma^{-1} - \frac{\Sigma^{-1}\iota\iota'\Sigma^{-1}}{\iota'\Sigma^{-1}\iota}\right)\mu + \frac{\Sigma^{-1}\iota}{\iota'\Sigma^{-1}\iota} \tag{7}$$

As the problem with transaction costs descriped in Equation 7 is identical to Equation 1, the solution is also the equivalent. The only difference is that Equation 7 assumes transaction costs proportional to risk and Equation 1 assumes no market frictions such as transaction costs. The added transaction costs are included by changeing return parameters μ and Σ to μ^* and Σ^*. As a result, adjusting the optimal portfolio problem for transaction costs implies a standard mean-variance optimal portfolio choice with the adjusted return parameters μ^* and Σ^*.

Also, since the true expected excessive return μ and their covariance matrix Σ are unknown, making μ^* and Σ^* equally unknown, we have to estimate them. We estimate μ and Σ from historic prices and shrink them, such that the estimation variance decreases vastly with the unfortunate side effect of introducing an increasing bias. Our applied shrinkage of μ and Σ will be explained further in section 3.

The equation for the optimal portfolio with transaction costs given by (Equation 2) as the solution to (Equation 6) is thus

$$\omega_{t+1}^* = \frac{1}{\gamma}\left(\hat{\Sigma}^{*-1} - \frac{\hat{\Sigma}^{*-1}\iota\iota'\hat{\Sigma}^{*-1}}{\iota'\hat{\Sigma}^{*-1}\iota}\right)\hat{\mu}^* + \frac{\hat{\Sigma}^{*-1}\iota}{\iota'\hat{\Sigma}^{*-1}\iota} \tag{8}$$

The expected excessive return μ^* in Equation 7 of an asset is now increased proportional to the current allocation to the asset, decreasing the asset allocation change from instant to instant when combined with Σ^*.

The assumption of transaction costs proportional to volatility (Σ) has some nice propaties. First, transaction costs vastly stems from moving the market slightly in the direction that the portfolio is changed in. This is because there are a limited number of market participants offering each asset at various prices. When increasing asset allocation in a direction, the cheapest offers are bought first and increasing the order size, increases the prices of the latest assets bought, as they are bought from market participants who are willing to sell to a (discrete) increasing slightly higher price. This is why transaction costs have a positive second derivative, in this case they are assumed to be proportional to the squared asset allocation change in each asset. However, as one stock increases in price, given no arbitrage and not assumed to be a liquidity issue, other stocks will increase with their covariance to the stock changing price. Thus having the transaction costs increase with volatility makes sence when looking at covariances between different stocks.

Transaction costs increases when liquidity is low, as less shares are offered close to the current price and bid-ask spread is thin. This is why transaction costs are most often assumed to be decreasing in trading volume, as a more traded asset can easier offset the trading pressure of an investor.

If we assume volatility and liquidity is decreasing in each other, this would also support increasing trading costs in volatility. If prices move fast (volatility is high), maybe because market participants are disagreeable on the right price, buying or selling pressure will generally move the market more easily, increasing illiquidity and thus transaction costs.

We write a function that returns the optimal portfolio weights after calculating the closed-form solution to Equation 8.

We also write a function that returns the same optimal mean-variance portfolio with transaction costs by finding a numerical solution to the maximum of the utility function stated in Equation 4, thus not a closed form solution, but a numerical one

We test with 5 industry portfolios if the two functions actually finds identical optimal weights

	closed___form	numerical___opt
nodur	1.2001583	1.2001583
durbl	0.0013712	0.0013712
manuf	-0.7880743	-0.7880743
enrgy	0.3544185	0.3544185
hitec	0.2321263	0.2321263

The two methods calculated identical weights, however the closed-form solution is much faster and elegant. Since the two functions returns the same weights, we will solely use the closed-form function, witch was also asked for explicitly since the closed-form solution do exist for this problem (Equation 4).

3 Implementation of a full-fledged portfolio backtesting strategy

We apply Ledoit-Wolf linear shrinkage to Σ towards the equicorrelation matrix and linear shrinkage of μ towards the mean return $\bar{\mu}$.

We create functions used to create the following variables needed for computeing the Ledoit-Wolf shrinked covariance matrix:

- $\hat{\pi}$, an estimator for π that denotes the sum of asymptotic variances of the entries of the sample covariance matrix scaled by $\sqrt{T}$
- $\hat{\gamma}$, an estimator for the parameter that measures the misspecification of the (population) shrinkage target
- $\hat{\rho}$, an estimator for the parameter sum of asymptotic covariances of the entries of the shrinkage target with the entries of the sample covariance matrix scaled by $\sqrt{T}$
- $\hat{\kappa}$, an estimator for the constant used to create the shrinkage intensity by deviding by T
- $\hat{\vartheta}$, a consistent estimator for $AsyCov[\sqrt{T}s_{ii}, \sqrt{T}s_{ij}]$

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We define five strategies:

a) naive equal weighted portfolio with daily rebalancing

b) theoretical optimal portfolio with transaction costs

c) mean-variance efficient portfolio with no short-selling

d) theoretical optimal portfolio without transaction costs

We create a backtesting function that takes the first ten years and calculates the optimal portfolio. Then the function uses the new portfolio and calculates the next portfolio with the new month but excluding the very first month, and so on. This way the data period length stays constant.

We get the performance table:

```
# A tibble: 5 x 7
  Strategy      leverage Mean_brutto Mean_net    SD `Sharpe ratio` Turnover
  <chr>            <dbl>       <dbl>    <dbl> <dbl>          <dbl>    <dbl>
1 1) Buy_&_hold     1           11.3     11.3  14.5          0.780     0
2 2) Naive          1           11.8     11.8  15.2          0.776     2.32
3 3) MV (TC)        2.40        11.0     11.0  23.2          0.474     2.80
4 4) MV (NS)        1           11.5      8.28 17.5          0.474    16.6
5 5) MV             7.88         7.79   -89.9  93.4         NA       153.
```

We see that the transaction cost strategy adjusted mean-variance portfolio strategy performed much better than the industry portfolios given the assumed transaction costs, net of transaction costs and actually also before costs.

We have assumed transaction costs follow Equation 2, but that is at best a decent approximation, as order book information would be needed for each stock for the approximation to be great. Information about the liquidity is lacking and even if we had perfect market information, it would still be an approximation, as trading in the market changes the market. Including a reasonable transaction cost assumption, makes our results more realistic.

Assuming we did the backtesting with CRSP data, it would still not be 100% true out of sample, as much of the theory relies on the CRSP data in the first place, as such it is slightly biased. Also, we did not look at alphas, solely on sharpe ratios and returns after considering approximate transaction costs. We did cheat slightly, as returns in the previous period is not entirely known at time of rebalancing to get the next returns. For that, investors would have to sell at the closing price and buy at the same closing price, where closing prices for the day are already known. The closing price are not known when buying and selling in the day, as closing prices are defined as the very last trade or sometimes (at least in the CRSP data, see their WRDS-website) as the middle of bid and ask prices.

Thus some biases was introduced, but this portfolio backtest was not bad at all. Using monthly data also decreases these buy and sell closing price issues a lot compared to daily data, as only once a month the bias is introduced.

The Sharpe Ratios are including transaction costs. Also lambda was set to 10000 * (200/10000).

We see that the buy and hold strategy was good, but the naive portfolio was better. The buy and hold portfolio has decreased diversification as time pases. The naive had a higher shape ratio than the transaction cost adjusted mean variance portfolio. No short portfolio did about the same as the transaction adjusted mean variance portfolio, but had a higher turnover, and as such, accured higher transaction costs. They where however offset by lower volatility even though it had lower returns net of transaction costs. The mean variance without transaction cost optimizing and shrink did very poorly with a way higher turnover and thus much higher transacton costs than the other. It was also way more leveraged than the other portfolios.