Mandatory Assignment 1

Asbjørn Fyhn & Emil Beckett Kolko

2024-03-08

In the following, we complete the exercises for Mandatory Assignment 1.

Exercise 1 and 2
We retrieve the daily adjusted prices of all the constituents of the Dow Jones index from January 1, 2000, to December 31, 2023 from Yahoo!Finance and remove all tickers that do not have continuous trading history throughout the period. Effectively, we are left with 27 tickers.

We calculate the monthly returns of each ticker based on the price of the last trading day: $r_i = p_{i,t}/p_{i,t-1} - 1$. Next, we use these return series to calculate the sample mean, μ, and the sample variance-covariance matrix, Σ, as:

$$\mu = \frac{1}{T}\Sigma_{t=1}^{T} r_t \text{ and } \Sigma = \frac{1}{T-1}\Sigma_{t=1}^{T}(r_t - \mu)(r_t - \mu)$$

For each ticker i in the dataset we calculate the annualized sharpe ratio, S_i, as $S_i = \frac{r_i}{\sigma_i}$, where r_i is the annualized return and σ_i is the annualized standard deviation for ticker i. We find that the stock with the highest Sharpe ratio is AAPL with a ratio of 0.804.

Exercise 3
First, we create a function 'compute_efficient_frontier' which takes two inputs: a $N \times N$ variance-covariance matrix and a $N \times 1$ vector of expected return. The function then returns the weights, w_{mvp}, of N assets in the *minimum variance portfolio* (MVP) and the weights for the assets in the *efficient portfolio* (EFF) which delivers twice the expected return of the MVP. Lastly, it returns the efficient frontier.

The weights of the MVP minimize the following problem:

$$\omega_{mvp} = \arg\min \omega'\Sigma\omega \;\; s.t. \; \Sigma_{i=1}^{N}\omega_i = 1$$

which has the solution $\omega_{mvp} = \frac{\Sigma^{-1}\iota}{\iota'\Sigma^{-1}\iota}$.

The weights of the EFF solve the following problem:

$$\omega_{eff} = \arg\min \omega'\Sigma\omega \;\; s.t. \; \omega'\iota = 1 \text{ and } \omega'\mu \geq 2\omega_{mvp}'\mu$$

which has the solution $\omega_{eff} = \omega_{mvp} + \frac{\tilde{\lambda}}{2}\left(\Sigma^{-1}\mu - \frac{D}{C}\Sigma^{-1}\iota\right)$, where $C \equiv \iota'\Sigma^{-1}\iota,\; D \equiv \iota'\Sigma^{-1}\mu,\; E \equiv \mu'\Sigma^{-1}\mu,$ and $\tilde{\lambda} = 2\frac{2\omega_{mvp}'\mu - D/C}{E - D^2/C}$

The efficient frontier is then computed as a linear combination of MVP and EFF. For any portfolio c on the efficient frontier we have:

$$\omega_c = \lambda \omega_{mvp} + (1 - \lambda)\omega_{eff}$$

where λ is the fraction of the MVP in the portfolio. We compute the efficient frontier for λ ranging from -0.2 to 1.2.

Based on our μ and Σ, we then compute the weights of the MVP, EFF and for 141 portfolios on the efficient frontier.

The minimum variance portfolio has an expected return of 8.78 and a volatility of 15.31. While the volatility of the efficient portfolio that delivers two times the expected return of the minimum variance is 16.85. We notice that the Sharpe ratio increases from 0.57 to 1.04.

Table 1: Portfolio Statistics

	Tangency Portfolio	Efficient Frontier Portfolio	Minimum Variance Portfolio
Return	0.51	0.18	0.09
Volatility	0.37	0.17	0.15
Sharpe Ratio	1.38	1.04	0.57

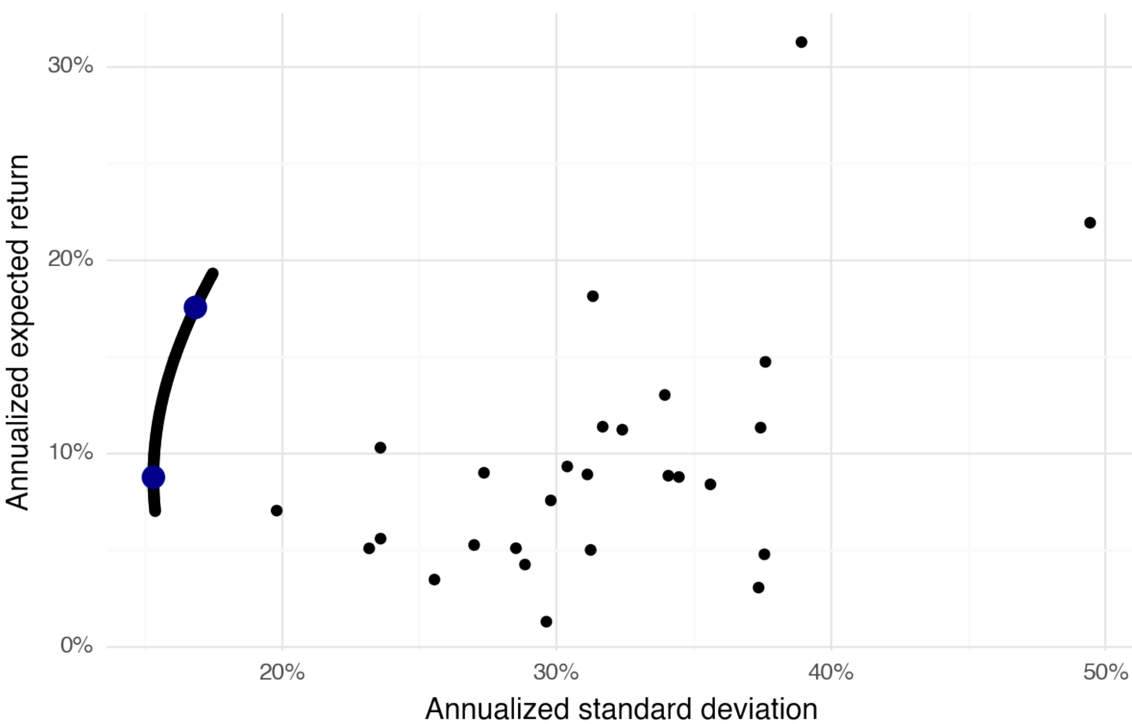

Figure 1: Efficient frontier for DOW index constituents

Exercise 4
We calculate the efficient tangency portfolio weights as $w_{tgc} = \frac{\Sigma^{-}\mu}{\iota'\Sigma^{-}\mu}$ under the assumption that

the risk free rate is 0. The weights are presented in Figure 2 below.

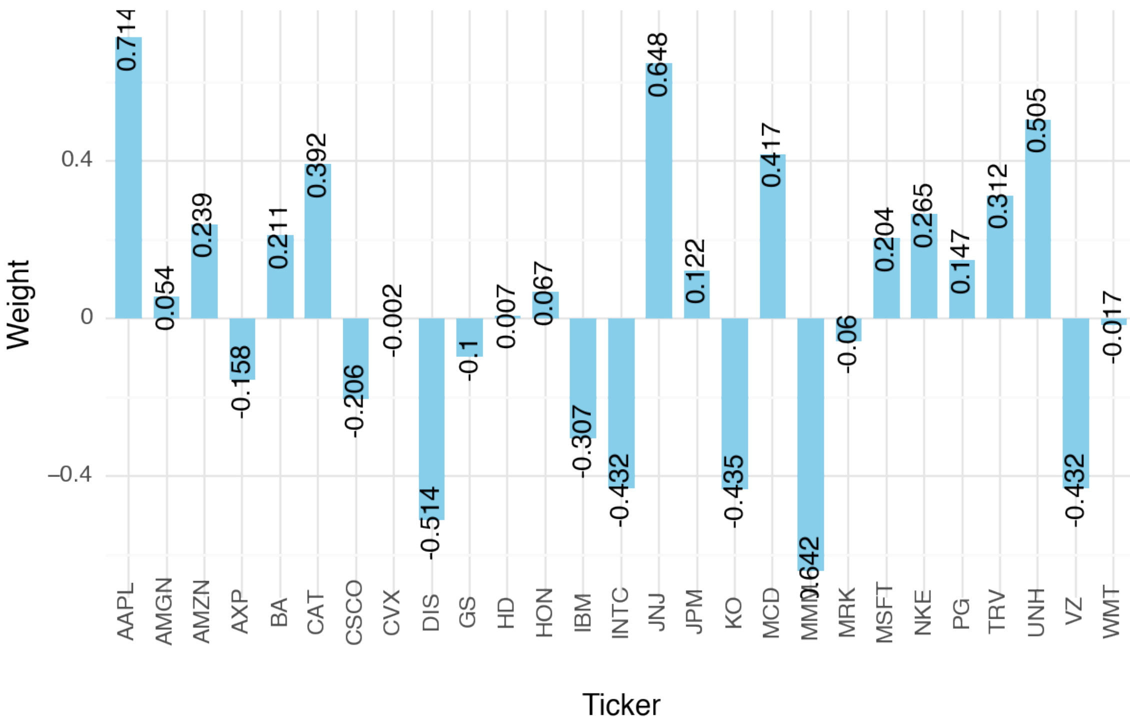

Figure 2: The weights of the Efficient Tangency Portfolio

The tangency portfolio weights seem well-balanced considering the limited diversification opportunities in the estimation. Poorly performing stocks are shorted to increase the weight of well performing stocks. However, the portfolio weights are not consistent under CAPM assumptions. Here, the efficient tangent portfolio is the market portfolio where all stocks are weighted according to their market values: $w_i = \frac{P_i}{\Sigma_{j=1}^{n} P_j}$, where w_i is the weight of stock i, P_i is the price of stock i, and $\Sigma_{j=1}^{n} P_j$ is the sum of the price of all stocks (the value of the entire market). Therefore, theoretically, there should be no shorting needed to obtain the efficient tangent portfolio. However, as Figure 2 shows, we find that it requires significant shorting of stocks to obtain the tangent portfolio. We did not expect a result in accordance with CAPM because our portfolio is composed of only 27 large-cap stocks which is a small subset of the entire market of which the CAPM theory builds upon.

When implementing this efficient tangent portfolio in reality, one should consider issues such as transactions costs and estimation issues. The portfolio weights must be updated constantly because the stock prices are fluctuating, which results in high transactions costs. Furthermore, both μ and Σ used for optimization are subject to estimation uncertainty, increasing risk and deviations from the expected outcomes.

However, the efficient tangent portfolio has the maximum attainable Sharpe ratio of 1.38 when comprising a portfolio of only the Dow Jones 30 index. The Sharpe Ratio is significantly higher than that of AAPL, which makes sense as AAPL is only one single stock. AAPL is included in the efficient tangent portfolio, so we could at least get that Sharpe Ratio. Instead, the efficient tangent portfolio optimizes the risk/return tradeoff across all 27 stocks.

Exercise 5

Provided with the function simulate_returns, we're able to simulate monthly returns of stocks for a given number of stocks. The function takes three parameters: periods, mu and sigma. Firstly, periods define the number of monthly return for each stock the function should return. Secondly, mu define the mean of each draw which comes from a normal distribution. Thirdly, sigma is the variance-covariance which is used to define the standard deviation for the distribution. But also the covariance comes into play as the function np.random.multivariate_normal takes the covariance into account. In our implementation of the simulation draw, we set the seed to 100.

Exercise 6

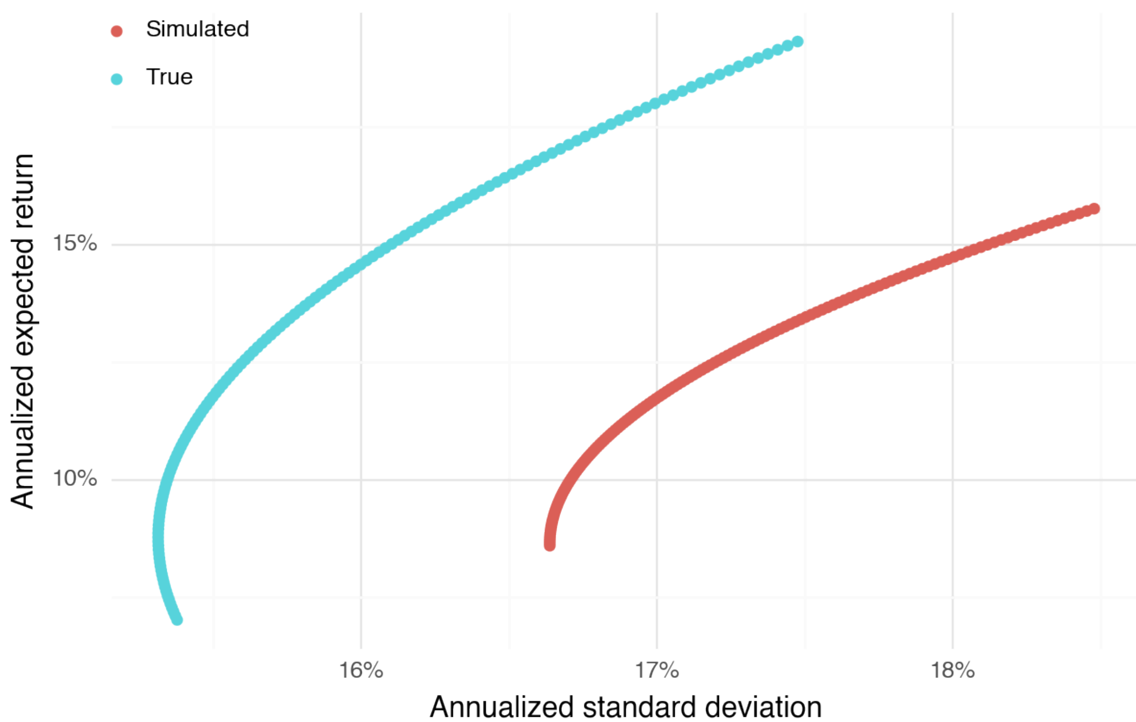

Figure 3: Single observed and true frontier

Exercise 7

This section explores the deviations between the theoretical efficient frontier and its estimates obtained through sample data. We achieve this by simulating multiple sample return series and constructing the corresponding efficient frontiers.

Simulation Process:

1. We employ a multivariate normal distribution to generate 100 hypothetical samples of asset returns, each with a size of 200 periods. The parameters for the distribution are set to the expected returns and covariance matrix of the actual assets.

2. For each simulated sample, we estimate the sample mean and sample covariance matrix.

3. Utilizing these estimated parameters, we compute the corresponding efficient frontier.

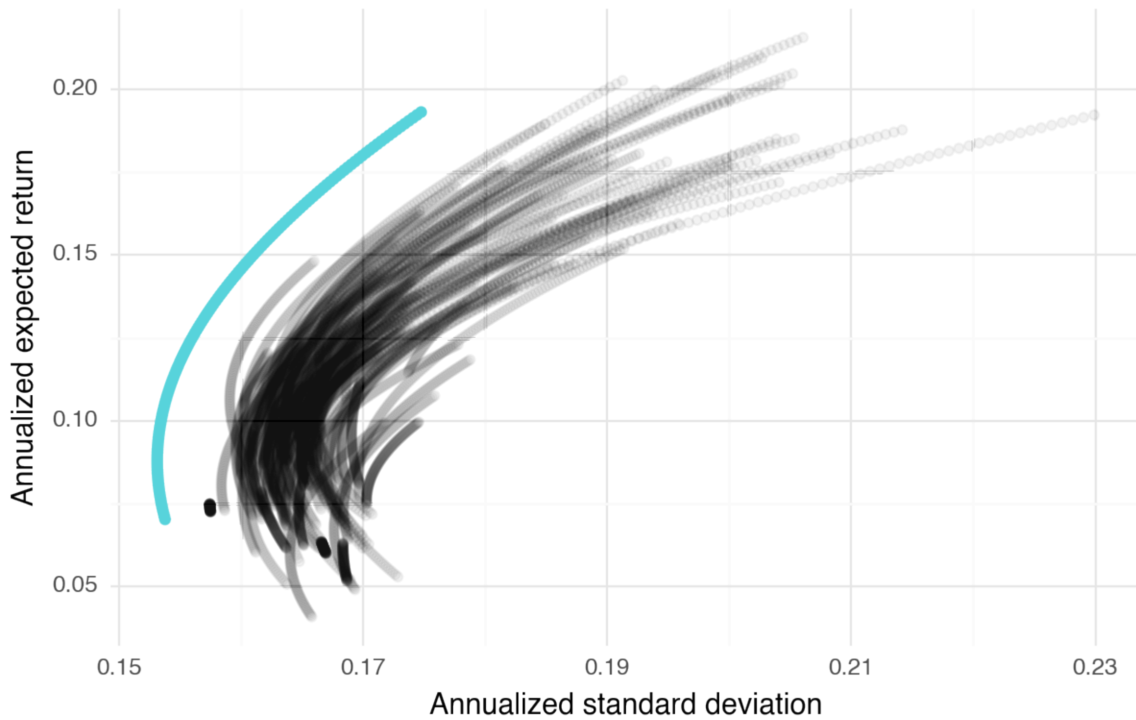

Figure 4: Observed frontiers and true frontier

4. Additionally, the tangency portfolio weights and Sharpe ratio are calculated for each simulated efficient frontier.

Analysis of the Results:

1. We visually compare the first simulated efficient frontier with the theoretically optimal frontier obtained from the population parameters. This initial comparison highlights the departure of the estimated frontier due to sampling error.
2. Subsequently, we plot all 100 simulated efficient frontiers alongside the true frontier. This visualization reveals the distribution and variability of the estimated frontiers around the theoretical optimum.

Observations and Inferences:

The simulated frontiers demonstrate a deviation from the true efficient frontier. This discrepancy arises due to the inherent uncertainty associated with using sample estimates of the population mean and covariance.

In conclusion, this simulation exercise underscores the importance of considering the limitations of sample-based estimates when constructing the efficient frontier. While the true frontier represents the optimal allocation for maximizing expected return for a given level of risk, practical implementation relies on estimates derived from available data. The presented results emphasize the uncertainty associated with these estimates and the potential deviations from the true efficient frontier.

Exercise 8 & 9

We compute the efficient tangent portfolio for each simulated return sample, assuming a zero risk-free rate and utilizing the estimated covariance matrix $\hat{\Sigma}$ and mean vector $\hat{\mu}$. The portfolio weights are derived as earlier described. With these weights, we calculate the annualized Sharpe ratio using true parameters μ and Σ, employing the formula $\mathrm{SR} = \sqrt{12}\frac{\omega^{tg}\mu}{\sqrt{\omega^{*}tg\Sigma\omega^{*}_{tg}}}$. The resultant Sharpe ratios are stored and visualized in a histogram, providing insight into portfolio performance variability across simulations.

The histogram shows Sharpe ratios for the 100 simulated tangency portfolios. The red dashed line marks the true Sharpe ratio derived from true parameters. We note that for most of the simulations, the Sharpe ratios are below the one derived with true parameters.

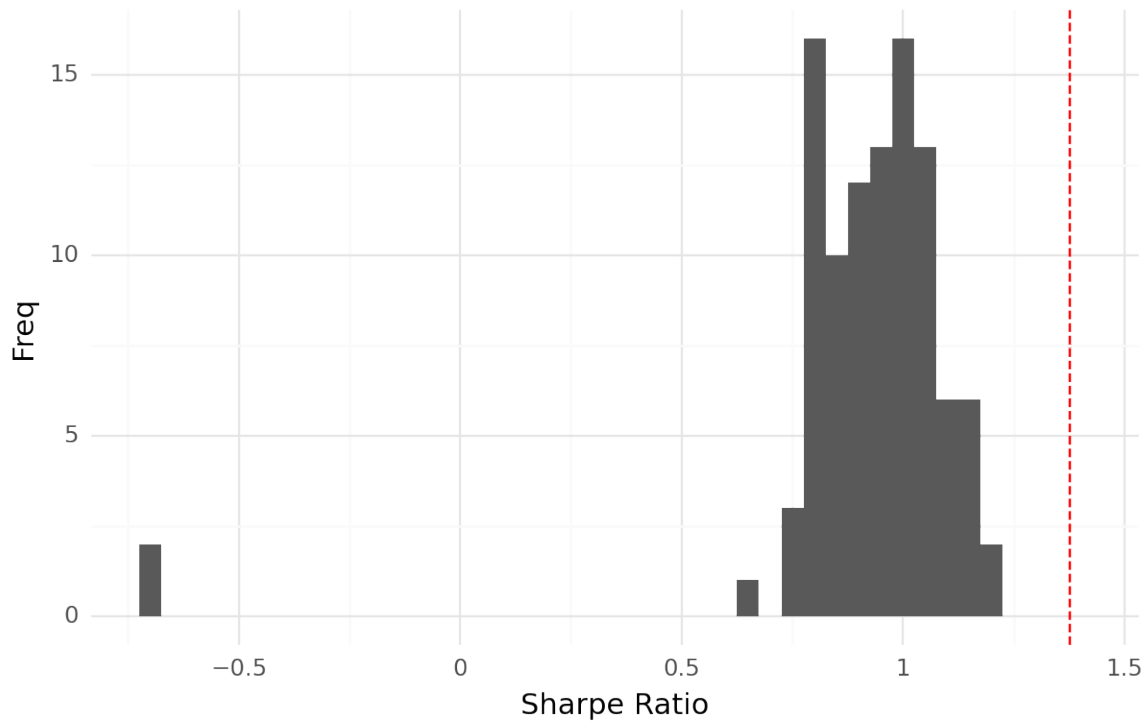

Figure 5: Distribution of sharpe-ratios

Exercise 10
When we increase the sample size periods, our results asymptotically move towards their true value.

The figure shows that the estimated frontiers are not on par with the true efficient frontier. This is because the efficient market portfolio is derived from past data which is not a precise indicator of future returns and volatility.

Unfortunately we were not able to find any alternative allocation strategies to improve the estimates' shortfall. Therefore, we were not able to complete the rest of exercise 10.

```
(0.09422823783351483, 0.22251395435812218)
```