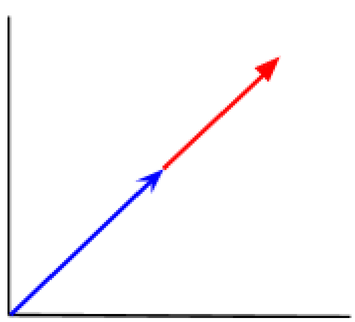
LINEAR ALGEBRA

Matrices can be represented as vectors in a given space.
The visual representation of matrices is important.

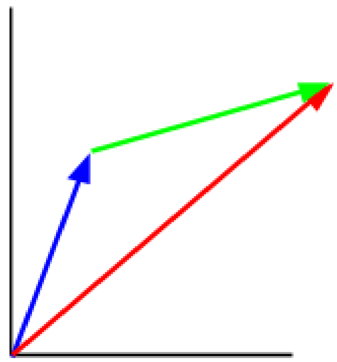
BASIS VECTORS
A Basis vectors are the ones which are used to represent any vector in the given space eg. i, j, k

SPAN
Span of 2 or more vectors is the space that the combination of all the vectors can trace

1. **Span in 2-D:**
    - **Parallel Vectors:** The span of such vectors is a straight line (Linearly Dependent)



    - **Non-Parallel Vectors:** Span of such vectors is the whole 2-D space (Linearly Independent).

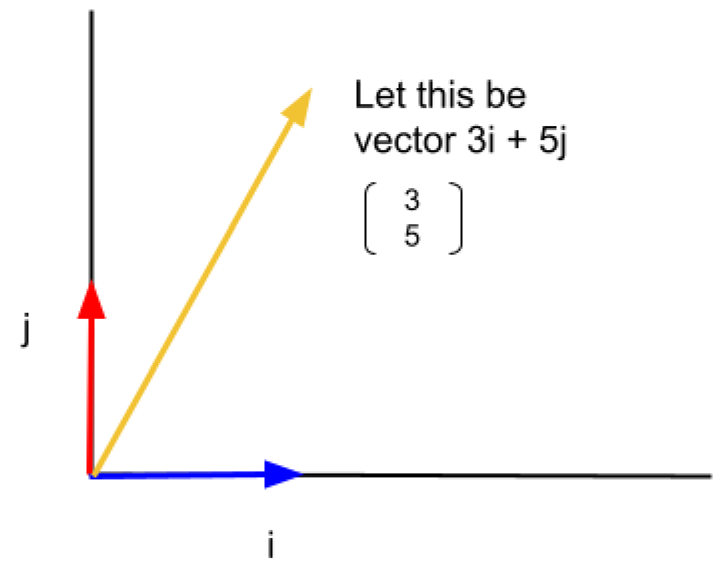


2. **Span in 3-D:**
    - **2-Vectors:** Trace out a Plane.
    - **3-Vectors (Third in the span of the first 2):** Trace out the same plane
    - **3-Vectors (Third in different span):** Trace the whole 3-D space

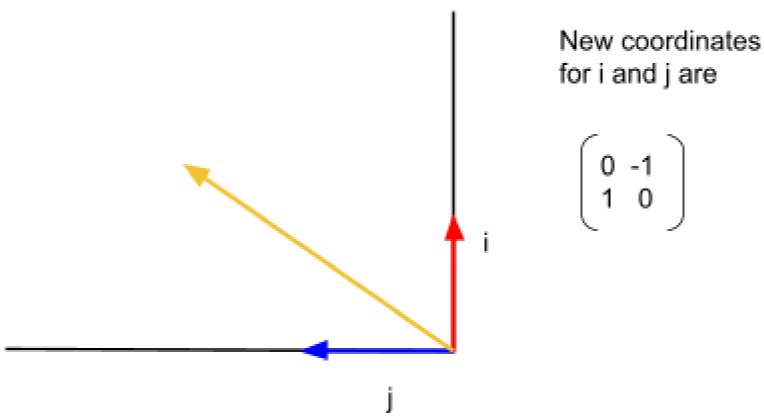
LINEAR TRANSFORMATION

Transform the given space of vectors such that every vector(matrix) that we have has a new coordinate in space according to the transformed space.
If the lines, vectors get curved on transformation then it is not a linear transformation. (Grid lines remain parallel and evenly spaced)
Therefore it is like a function for matrices.

We can determine the new coordinates of the transformed vector if we happen to know the whereabouts of the basis vectors.
For example:



If this space is now shifted by 90 degrees then.



Therefore now the new coordinates of the vector is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0\text{x}3 - 1\text{x}5 \\ 1\text{x}3 + 0\text{x}5 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

Therefore we can find the new coordinates (transformations) of any vector in space as long as we know the positions of the basis vectors.

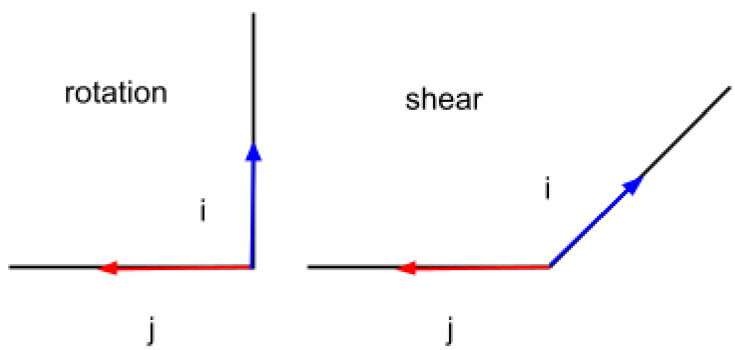
MULTIPLE LINEAR TRANSFORMATIONS
When a linear transformation is succeeded by another linear transformation then the total effect of those transformations is the matrix multiplication of them, with the first transformation starting from right
Like in functions we read functions from right to left
Eg. f(g(u(x)))

So if a transformation of rotation and then shear is applied then the resultant transformation is:

$$\underbrace{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}_{\text{rotation}} \Rightarrow \underbrace{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}}_{\text{shear}}$$

Therefore the resultant transformation looks like:

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \underbrace{\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}}_{\text{resultant}}$$



Matrix Multiplication is basically the resultant linear transformation.

DETERMINANT
Determinant is the factor by which the area between 2 basis vectors increases or decreases after transformation. (2D)
Negative determinant means that in the transformation the plane is been flipped (2D)

In 3D the determinant represents the factor by which volume changes.

In 3D positive determinant represents right hand rule for coordinates and negative determinant represents left hand rule for coordinates,

LINEAR EQUATIONS

Linear equations imply the need to find a vector x such that when it is taken through a transformation of A it lands on the vector V.
For example:

$$\left.\begin{array}{l}2x + 2y = -4\\ 1x + 3y = -1\end{array}\right\}$$

Given is a set of linear equation in 2 variables which can be written in matrix form as:

$$\underbrace{\begin{pmatrix}2 & 2\\ 1 & 3\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}x\\ y\end{pmatrix}}_{x} = \underbrace{\begin{pmatrix}-4\\ -1\end{pmatrix}}_{V}$$

A is the transformation
x is the vector to be transformed
V is the required vector after transformation

WHEN DETERMINANT ≠ 0

If there exists a matrix A then $A^{-1}$is the inverse.

I.e A^{-1} is the inverse transformation which when done after applying A reverts the effect of the transformation.

And then the matrix multiplication of A and A^{-1} (first transformation A and then transformation A^{-1}) Gives a matrix that does nothing, called the identity matrix.

I.e AA^{-1}= I
Where I is the identity matrix equal to $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ In 3D and $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ in 2D

WHEN DETERMINANT = 0

The transformation in this case squishes the plane into a single line of a point in 2D and squishes the space onto a plane in 3D. When det(A) = 0 then there exists no inverse for A.

RANK
Rank refers to the number of dimensions in the output of a transformation.
I.e if a transformation squishes into a line then the rank is 1.
If a transformation squishes into a plane then the rank is 2.

The maximum rank of a matrix is it's order.
For a 3x3 matrix the maximum rank is 3 and for 2x2 it is 2.
Here we are talking only about square matrices

COLUMN SPACE
It is the set of all possible outputs after a transformation.
In simple words column space is the span of the columns of our matrices.

<u>So coming back to **RANK** it is precisely the number of dimensions in the column space.</u>

NULL SPACE (KERNEL)
If a 2D transformation squishes a 2D plane on a line then there are a whole bunch of vectors in the opposite direction of the line that get squished onto the origin.

Similarly if 3D space is squished onto a plane then there is a whole line of vectors that land on the origin and when 3D space is squished onto a line then there is a whole plane of vectors that land on the origin.

This set of vectors that land on the origin is called the null space or the Kernel of a matrix.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So if in a system of linear equations in two variables v is

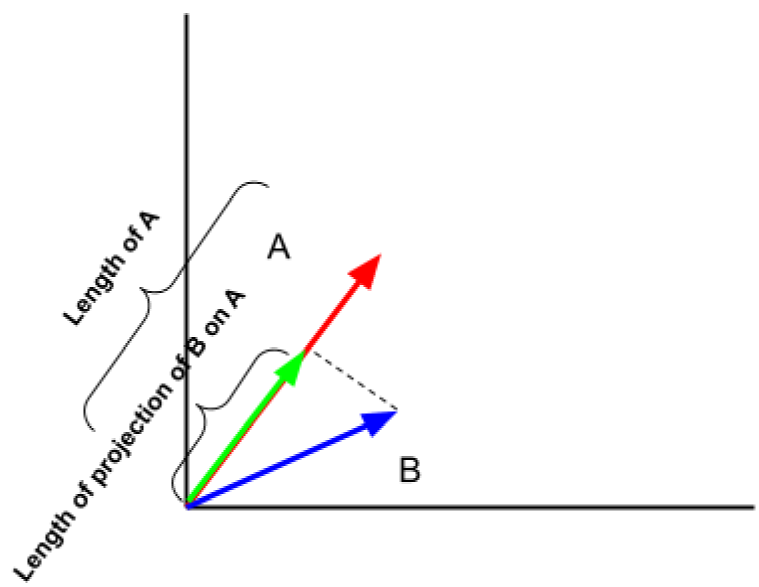
Then there exists a line full of vectors as solution i.e the null space (Infinitely many solutions).

DOT PRODUCT

Dot product is the length of projection of one vector on the second vector multiplied with the length of the second vector.

Eg let there be two vectors A and B.

A.B = (Length of projection of B on A / A on B).(Length of A/B)



Negative dot product represents that the vectors are pointing in opposite directions.

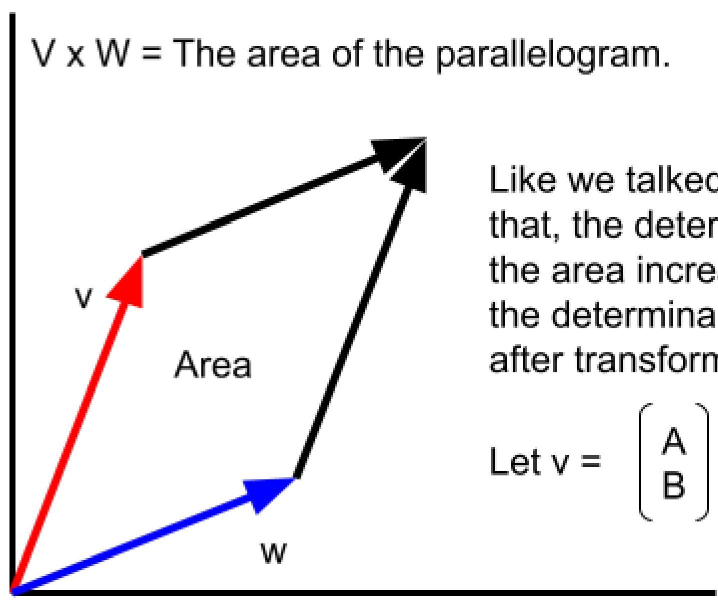
Dot product actually makes one of the vectors as transformation from 2D space to 1D space.
Eg.

Let there be 2 vectors X and A such that:

$$\begin{bmatrix} X \\ Y \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} \equiv X.A + Y.B \qquad \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \equiv X.A + Y.B$$

The dot product makes the vector X into a transformation from 2D space into a 1D space

CROSS PRODUCT
Magnitude of cross product is the area of the parallelogram between the vectors i.e the determinant of the matrix made of the 2 vectors.



Like we talked in the determinant section that, the determinant is the factor by which the area increases in a transformation, here the determinant is the area i.e cross product after transformation

Let $v = \begin{bmatrix} A \\ B \end{bmatrix}$ Let $w = \begin{bmatrix} X \\ Y \end{bmatrix}$

Then $|V \times W| = \det \left[\begin{pmatrix} A & X \\ B & Y \end{pmatrix} \right]$

The magnitude gives the length of the vector obtained from the cross product which is perpendicular to both V and W.

The actual cross product of of V and W which is a vector is obtained by:

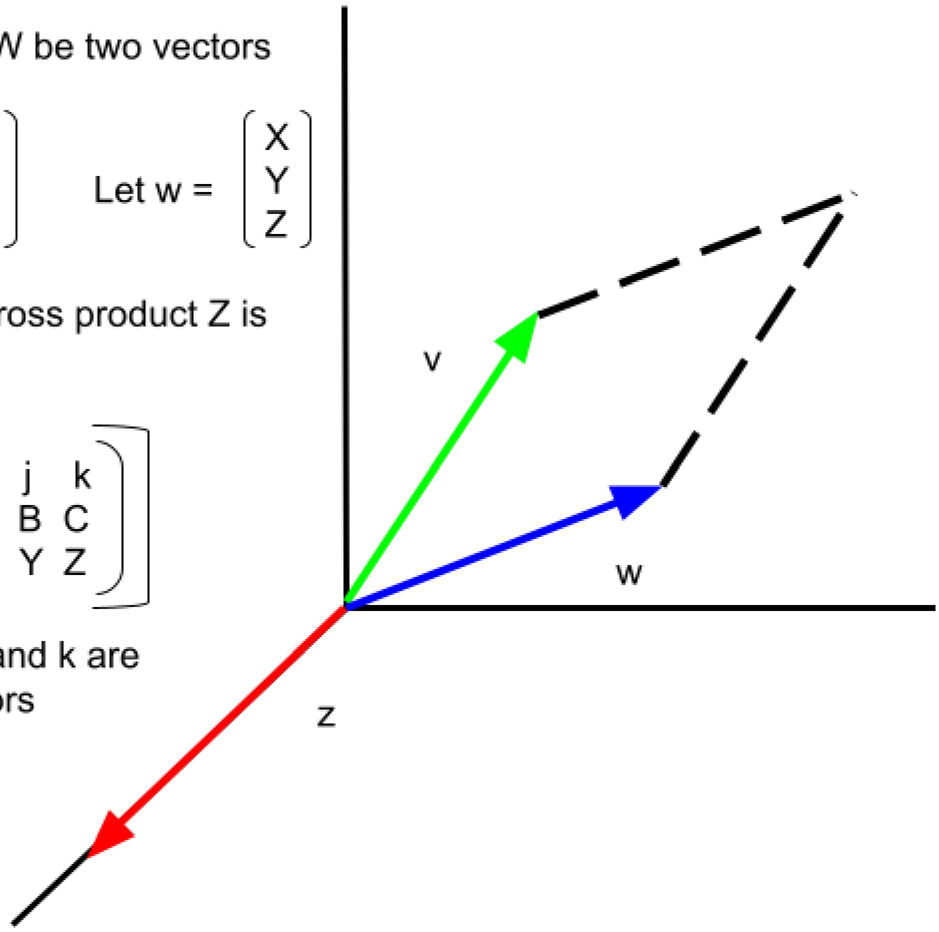
Let V and W be two vectors such that:

Let v = $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ Let w = $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

Then the cross product Z is given by:

$$\det \left[\begin{pmatrix} i & j & k \\ A & B & C \\ X & Y & Z \end{pmatrix} \right]$$

Where i, j and k are basis vectors



CHANGE OF BASIS

After a transformation the perspective of the transformer gives out a new set of basis.

Eg let there be a transformation $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

The basis vectors i and j are therefore taken to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Now if we want to flip a vector in this new coordinate system by 90 degrees

We have to follow the following process:

Let V be a vector in the new coordinate system and T be the transformation (In this case anticlockwise 90 degrees rotation)

$$\underset{A}{\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}} \underset{V}{\begin{pmatrix} X \\ Y \end{pmatrix}} \longrightarrow \text{This will give the vector in normal coordinates}$$

$$\underset{T}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \underset{A}{\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}} \underset{V}{\begin{pmatrix} X \\ Y \end{pmatrix}} \longrightarrow$$

This gives the transformed vector in normal coordinates (In this case rotated vector)

$$\underset{A^{-1}}{\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}^{-1}} \underset{T}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \underset{A}{\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}} \underset{V}{\begin{pmatrix} X \\ Y \end{pmatrix}} \longrightarrow$$

This gives the transformed vector in the new coordinate system

Therefore, the general formula for the transformation in new coordinates is:

$(A^{-1}TA)V$

T is the transformation and A and A^{-1} are the empathy or the change in perspective.

EIGEN VECTORS AND VALUE

Eigen vectors are some special vectors whose direction .i.e span, does not change after a particular linear transformation.

In 3D space Eigenvector represents the axis of rotation as no change is observed after transformation.

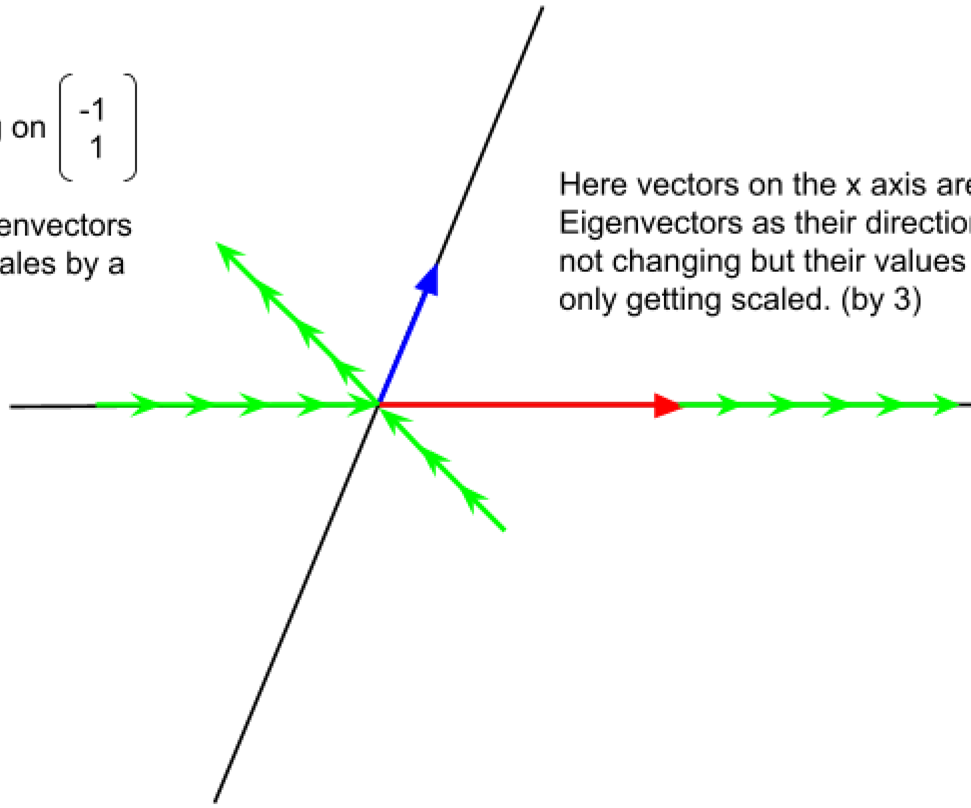
Eg.

Let there be a transformation of the form $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$

Vectors lying on $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Are also Eigenvectors which get scales by a factor of 2

Here vectors on the x axis are Eigenvectors as their direction is not changing but their values are only getting scaled. (by 3)



Therefore for Eigenvectors we can write

AV = λV

A is the transformation

V is the Eigenvector

And λV stands for scaling the vector as after transformation the vector is just scaled.

Eigenvalue is the factor by which an Eigenvector is scaled in a linear transformation. In this case λ.

So we can write AV = (λI)V

Here λI is $\lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

To make it in vector form on both sides so that we can do:
AV - (λI)V = 0
(A - λI)V = 0
(A - λI) = 0

This is true when V is a zero vector ut also when the determinant of (A - λI) is zero. This means that (A - λI) is a transformation that squishes a plane onto a line.

So by equating det(A - λI) to 0 we can get the value of λ, which is the Eigenvalue.

EIGEN BASIS

Choosing our basis vectors as Eigenvectors classifies them as Eigenbasis.

This is helpful in multiplying matrices multiple times.

If we are given a non-diagonal matrix that are actually the Eigenvectors then by transforming them back and reducing them to a diagonal form, makes it easier to multiply them since in a diagonal matrix the diagonals are simply raised to the power of how many times they want to be multiplied and then multiplying that matrix with the inverse of the transformation to get the desired output.