8

Boundary Value Problems

8.1 NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS BY FINITE DIFFERENCE METHOD

When the closed form solution is not possible in solving an ordinary differential equation, we transform it to an approximate difference equation and solve for the unknown for different values for the independent variable.

By Taylor Series:

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \dots \tag{8.1}$$

$$\frac{y(x+h) - y(x)}{h} = y'(x) + \frac{h}{2} y''(x) + \dots$$

Hence,
$$y'(x) = \frac{y(x+h) - y(x)}{h} - \frac{h}{2} y''(x) + \dots$$

i.e.,
$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h) \tag{8.2}$$

Equation (8.2) is forward difference approximation for $y'(x)$.

Also
$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2} y''(x) \tag{8.3}$$

$\therefore$
$$y'(x) = \frac{y(x) - y(x-h)}{h} + O(h) \tag{8.4}$$

Equation (8.4) is backward difference approximation for $y'(x)$.

A central difference approximation for $y'(x)$ can be got as follows. Subtracting (8.3) from (8.1), and dividing by $2h$, we have

$$\frac{y(x+h)-y(x-h)}{2h} = y'(x) + O(h^2) \tag{8.5}$$

Equation (8.5) gives a better approximation for $y'(x)$ than what is given in Eq. (8.2) or Eq. (8.4).

Adding Eq. (8.3) and (8.1), we get

$$y(x+h) + y(x-h) = 2y(x) + b^2 y''(x) + \frac{h^4}{24} y''''(x) + \ldots$$

$$\therefore \qquad y''(x) = \frac{y(x+h)+y(x-h)-2y(x)}{h^2} + O(h^2) \tag{8.6}$$

Equation (8.6) is taken as a difference approximate for $y''(x)$.

Therefore, at $x = x_i$, from Eqs. (8.5) and (8.6), we get

$$y_1' \simeq \frac{y_{i+1}-y_{i-1}}{2h} \tag{8.7}$$

and

$$y_i'' \simeq \frac{y_{i-1}+y_{i+1}-2y_i}{h^2} \tag{8.8}$$

neglecting $O(h^2)$, if h is small.

Suppose a boundary value problem.

$$y'' + a(x)\, y' + b(x)\, y(x) = c(x)$$

together with the boundary conditions $y(x_0) = \alpha$, $y(x_n) = \beta$ is given, when $x \in (x_0, x_n)$.

We replace $y'(x)$ and $y''(x)$ by the difference formulae (8.7) and (8.8), reduce to

$$\frac{y_{i+1}+y_{i-1}-2y_i}{h^2} + \frac{a(x_i)\,[y_{i+1}-y_{i-1}]}{2h} + b(x_i)\cdot y_i = c(x_i)$$

Simplifying, we get

$$y_{i+1}\left(1+\frac{h}{2}a_i\right) + y_i\left(h^2 b_i - 2\right) + y_{i-1}\left(1-\frac{h}{2}a_i\right) = a_i h^2 \tag{8.9}$$

where $i = 1, 2, \ldots n-1$

$$y_0 = \alpha$$
$$y_n = \beta$$
$$a_i = a(x_i)$$
$$b_i = b(x_i)$$
$$c_i = c(x_i)$$

Equation (8.9) will give $(n-1)$ equations for $i = 1, 2, 3 \ldots (n-1)$ which is a tridiagonal system and together with $y_0 = \alpha$, $y_n = \beta$, we get $(n+1)$ equations in the $(n+1)$ unknowns.

$$y_0, y_1, y_2, \ldots y_n$$

Solving from there $(n + 1)$ equations, we get
$y_0, y_1, y_2, \ldots y_n$ values, i.e., the values of y at $x = x_0, x_1, \ldots x_n$.

EXAMPLE 8.1 Using the finite difference method, solve $\dfrac{d^2y}{dx^2} = y$ in $(0, 2)$, given $y(0) = 0$, $y(2) = 3.63$, subdividing the rane of x into four equal parts.

Solution

$$nh = b - a \quad \Rightarrow \quad 4h = 2$$

$$\therefore \qquad h = \frac{1}{2}$$

Replacing the derivative y'' by differences, we get

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} = y_i$$

i.e.,

$$y_{i+1} - (2 + h^2)y_i + y_{i-1} = 0, \ i = 1, 2, 3$$

Putting $h = \dfrac{1}{2}$, we get

$$y_{i+1} - \frac{9}{4} y_i + y_{i-1} = 0, \quad i = 1, 2, 3 \tag{i}$$

Hence the equations to be solved are:

$$y_2 - \frac{9}{4} y_1 + y_0 = 0 \tag{ii}$$

$$y_3 - \frac{9}{4} y_2 + y_1 = 0 \tag{iii}$$

$$y_4 - \frac{9}{4} y_3 + y_2 = 0 \tag{iv}$$

Using $y_0 = 0$, $y(2) = y_4 = 3.63$, we have

$$y_2 - \frac{9}{4} y_1 = 0 \tag{v}$$

$$y_3 - \frac{9}{4} y_2 + y_1 = 0 \tag{vi}$$

$$3.63 - \frac{9}{4} y_3 + y_2 = 0 \tag{vii}$$

Eliminating y_1 from Eqs. (v) and (vi), we have

$$\frac{9}{4} y_3 - \frac{81}{16} y_2 + y_2 = 0$$

i.e.,
$$\frac{9}{4}y_3 - \frac{65}{16}y_2 = 0 \tag{viii}$$

From Eqs. (vii) and (viii), we get

$$\left(1 - \frac{65}{16}\right)y_2 = -3.63$$

$\therefore$
$$y_2 = 1.1853$$

Putting this value of y_2 in Eq. (viii), we get

$$y_3 = 2.1401$$

From Eq. (v), we get

$$y_1 = \frac{4}{9}y_2$$
$$= 0.5268$$

Taking the values, the solution is

x	0	0.5	1	1.5	2
y	0	0.5268	1.1853	2.1401	3.63

EXAMPLE 8.2 Using the finite difference method, solve $y'' - 64y + 10 = 0$, $x \in (0, 1)$, given $y(0) = y(1) = 0$, subdividing the interval into (i) four equal parts, and (ii) two equal parts.

Solution

Since $n = 4$ and $nh = 1$, $h = \frac{1}{4}$

Converting the differential equation into difference equation, we have

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} - 64y_i + 10 = 0$$

i.e.
$$y_{i+1} + y_{i-1} - (2 + 64h^2)\,y_i + 10h^2 = 0$$

Putting $h = \frac{1}{4}$, this becomes

$$y_{i+1} - 6y_i + y_{i-1} = \frac{-5}{8} \tag{i}$$

where $i = 1, 2, 3$, $y(0) = y(1) = 0$

Hence, using $y_0 = 0$, $y_4 = 0$, we get

$$y_2 - 6y_1 = \frac{-5}{8} \tag{ii}$$

$$y_3 - 6y_2 + y_1 = \frac{-5}{8} \tag{iii}$$

i.e.,
$$\frac{9}{4}y_3 - \frac{65}{16}y_2 = 0 \qquad \text{(viii)}$$

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Putting $h = \frac{1}{4}$, this becomes

$$y_{i+1} - 6y_i + y_{i-1} = \frac{-5}{8} \qquad \text{(i)}$$

where $i = 1, 2, 3$, $y(0) = y(1) = 0$

Hence, using $y_0 = 0$, $y_4 = 0$, we get

$$y_2 - 6y_1 = \frac{-5}{8} \qquad \text{(ii)}$$

$$y_3 - 6y_2 + y_1 = \frac{-5}{8} \qquad \text{(iii)}$$

$$-6y_3 + y_2 = \frac{-5}{8} \tag{iv}$$

Equation (ii) – (iv) gives

$$6(y_3 - y_1) = 0$$

$$y_1 = y_3$$

Equation (iii) becomes

$$2y_3 - 6y_2 = \frac{-5}{8}$$

$$-6y_3 + y_2 = \frac{-5}{8}$$

Eliminating y_3, we have

$$-17y_2 = \frac{-5}{2}$$

$$y(0.5) = y_2$$

$$= \frac{5}{34}$$

$$= 0.1471$$

Hence, Eq. (ii) reduces to

$$6y_1 = \frac{5}{34} + \frac{5}{8}$$

$$= \frac{105}{36}$$

$$\therefore \quad y_3 = y_1$$

$$= \frac{35}{272}$$

$$= 0.1287$$

Exact value of y_2 is 0.1505, when $n = 2$, y_2 is 0.1389.

EXAMPLE 8.3 Solve $y'' - xy = 0$, given $y(0) = -1$, $y(1) = 2$, by the finite difference method, taking $n = 2$.

Solution

If $n = 2$, then $h = \frac{1}{2}$, since range is (0, 1). The nodal points are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$.

The differential equation reduces to

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} - x_i y_i = 0$$

i.e.,
$$y_{i+1} - (2 + h^2x_i)\, y_i + y_{i-1} = 0$$

where $i = 1,\ h = \dfrac{1}{2},\ x_i = 0.5,\ y_0 = -1,\ y_2 = 2$

$$\therefore \quad y_2 - \left(2 + \frac{1}{8}\right) y_1 + y_0 = 0$$

$$2 - \frac{17}{8} y_1 - 1 = 0$$

$$y_1 = \frac{8}{17}$$

$$= 0.4706$$

Tabulating the values, we get

x	0	0.5	1
y	−1	0.4706	2

EXAMPLE 8.4 Using the finite difference method, solve for y, given the differential equation $\dfrac{d^2y}{dx^2} + y + 1 = 0, x \in (0, 1)$ and the boundary conditions $y(0) = y(1) = 0$, taking (i) $h = \dfrac{1}{2}$, and (ii) $h = \dfrac{1}{4}$.

Solution

Let the interval of x, that is, (0, 1) into n equal parts, each part being equal to h, so that $nh = 1$.

Case (1):

Suppose $n = 2$ (i.e., divide the range into 2 equal parts), hence $h = \dfrac{1}{2}$.

Using
$$y_i'' = \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2}$$

in the differential equation, we get
$$\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} + y_i + 1 = 0$$

i.e.,
$$y_{i+1} + y_i\,(h^2 - 2) + y_{i-1} + h^2 = 0 \qquad \text{(i)}$$

where
$$i = 1 \text{ and } h = \frac{1}{2}$$

Hence,
$$y_2 - \frac{7}{4} y_1 + y_0 = \frac{-1}{4}$$

where

$$y_0 = y(0)$$
$$= 0$$

and

$$y_2 = y(1)$$
$$= 0$$

$$y_1 = \frac{1}{7}$$

i.e.,

$$y_1 = y(0.5)$$
$$= \frac{1}{7}$$
$$= 0.1428$$

Tabulating the values of y, we get for $h = \frac{1}{2}$

x	0	0.5	1
y	0	$\frac{1}{7} = 0.1428$	0

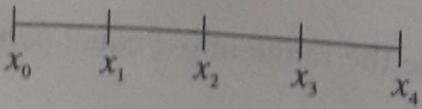
Solving analytically the given differential equation, we get

$$y(0.5) = 0.1395$$

Hence the error is only, 0.0133.

Case (2):

Taking $n = 4$ or $h = \frac{1}{4}$



Equation (i) reduces to

$$y_{i+1} - \frac{31}{16} y_i + y_{i-1} = \frac{-1}{16} \qquad \text{(ii)}$$

Setting i = 1, 2, 3, we get

$$y_2 - \frac{31}{16} y_1 + y_0 = \frac{-1}{16}$$
$$y_3 - \frac{31}{16} y_2 + y_1 = \frac{-1}{16}$$
$$y_4 - \frac{31}{16} y_3 + y_2 = \frac{-1}{16}$$

subject to

$$y_0 = y_4 = 0$$

$\therefore$

$$y_2 - \frac{31}{16} y_1 = \frac{-1}{16} \quad \text{(iii)}$$

$$y_3 - \frac{31}{16} y_2 + y_1 = \frac{-1}{16} \quad \text{(iv)}$$

$$\frac{-31}{16} y_3 + y_2 = \frac{-1}{16} \quad \text{(v)}$$

Equation (iii) – (v) gives

$$\frac{-31}{16}(y_1 - y_3) = 0$$

$\therefore$

$$y_1 = y_3$$

Equation (iv) becomes

$$2y_1 - \frac{31}{16} y_2 = \frac{-1}{16} \quad \text{(vi)}$$

$$\frac{-31}{16} y_1 + y_2 = \frac{-1}{16} \quad \text{(vii)}$$

Solving from Eqs. (vi) and (vii)

$$y_1 = \frac{47}{449}$$

$$y_2 = \frac{63}{449}$$

Tabulating the values of y, we get

x	0	0.25	0.5	0.75	1
y	0	$\frac{47}{449}$	$\frac{63}{449}$	$\frac{47}{449}$	0

$$y_2 = \frac{63}{449}$$

$$= 0.1403$$

The error is only 0.0008 when compared to the exact value.

Note

From the above result, we infer that the accuracy by finite difference method depends upon

(i) the number of subintervals (or width of the subinterval) chosen, and

(ii) also on the order of the approximation.

By increasing n, though the accuracy of the result increases, the number of equations to be solved also increases resulting more expenditure of time and energy.

EXERCISES

8.1 Using the finite differences, solve the following equations:

(i) $xy'' + y = 0$, given $y(1) = +1$, $y(2) = 2$, where $h = \frac{1}{4}$

[*Ans.* $y(1.25) = 1.3513$, $y(1.5) = 1.635$, $y(1.75) = 1.8505$]

(ii) $\frac{d^2y}{dx^2} - y = 0$, $x \in (0, 1)$, given $y(0) = 0$, $y(1) = 1$, $n = 2$

[*Ans.* Exact value of y (0.5) = 0.4434]

(iii) $\frac{d^2y}{dx^2} + y = 0, x \in (0, 1),$ given $y(0) = 0$, $y(1) = 1$

$\left[\textit{Ans.}\text{ Exact value of } y = \frac{\sin x}{\sin i}\right]$

(iv) $y'' + 2y' + y = 0, x \in (0, 1)$ given $y(0) = 0$, $y(1) = 1$

[*Ans.* Exact value of $y(0.5) = 0.824$]

(v) $y'' + y' - 2y = 0$, given $y(0) = 2$, $y(1) = 2.85$

[*Ans.* Exact value of $y(0.5) = 2.0165$]

(vi) $y'' - 3y' + 2y = 0$, given $y(0) = 2$, $y(1) = 10.1$

[*Ans.* Exact value of $y(0.5) = 4.367$]

(vii) $y'' + 6y' + 9y = 3$, given $y(0) = \frac{4}{3}$, $y(1) = 0.38$

[*Ans.* Exact value of $y(0.5) = 0.55646$]

8.2 NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations occur in many branches of applied mathematics, e.g. in hydrodynamics, elasticity, quantum mechanics and electromagnetic theory. Only a few of these equations can be solved by analytical methods which are also complicated requiring use of advanced mathematical methods. In most of the cases, it is easier to develop the approximate solutions of partial differential equations can be got by simple and efficient numerical methods.

The method of finite differences is most commonly used to solve partial differential equations. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference