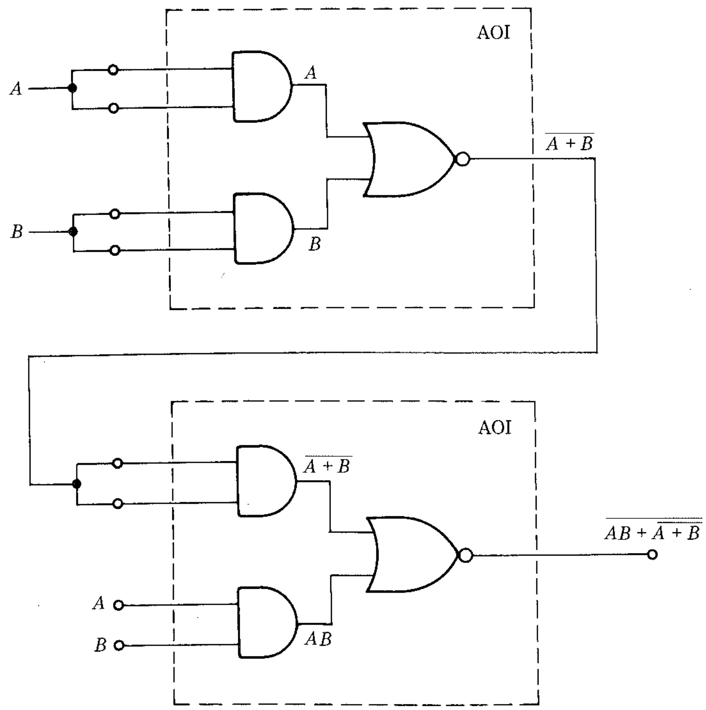


Figure 2.7-5 Two AND-OR-INVERT gates are used to generate the EXCLUSIVE-OR function.

Using AOI gates as inverters is hardly economical. With some ingenuity in algebraic manipulation we can devise a way to use the gates more effectively. As an example suppose that we have only the variables A and B and not their complements and want to generate the EXCLUSIVE-OR function $A \oplus B$. we can write

$$A \oplus B = \overline{AB + \bar{A}\bar{B}} \tag{2.7-8}$$

In this form we can generate $A \oplus B$ using three AOI gates, two of which would be used just to produce $\bar{A}$ and $\bar{B}$. Alternatively we can write

$$A \oplus B = \overline{AB + \bar{A}\bar{B}} = \overline{AB + \overline{A + B}} \tag{2.7-9}$$

In the last form the function can be realized using only two AOI gates, as shown in Fig. 2.7-5.

2.8 KARNAUGH MAPS

The Karnaugh map is an extremely useful device for the simplification and minimization of boolean algebraic expressions. In this and succeeding sections we shall discuss K maps and work examples with these maps to encourage the reader to develop some facility in their use.

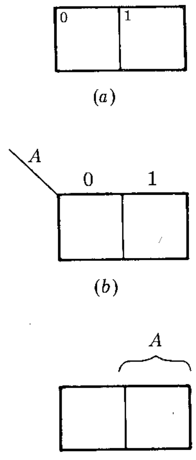
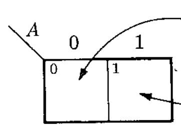


Figure 2.8-1 Three alternative schemes for identifying the boxes in a one-variable K map.

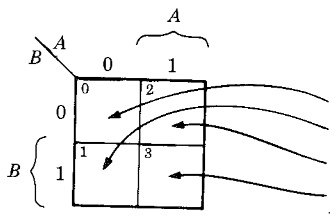
A K map is a geometrical figure which provides one region (box) for each row in a truth table. As we have noted, there is a one-to-one correspondence between truth-table rows and potential maxterms or minterms. We note as well that there is a one-to-one correspondence between K-map boxes and minterms and between such boxes and maxterms. Initially, we consider how the maps are constructed and the identification of boxes with truth-table rows, with minterms, and with maxterms, leaving for later the explanation of how the maps are used.

The K map for a single variable, say the variable A, is shown in Fig. 2.8-1. It consists simply of two adjacent boxes, corresponding to the fact that a truth table for a single variable has just two rows. Figure 2.8-1*a* to *c* shows three alternative ways in which the boxes are identified with the truth-table rows. In Fig. 2.8-1*a* the boxes have simply been numbered in the upper left-hand corner. The left-hand box corresponds to row 0, and the right-hand box corresponds to row 1. In Fig. 2.8-1*b* we have indicated the same information by noting that the left-hand box is the box corresponding to the truth-table row in which $A = 0$ and the right-hand box corresponds to the row in which $A = 1$. In Fig. 2.8-1*c* the understanding is that the box embraced by the bracket and labeled A is the box which corresponds to the truth-table row in which $A = 1$. In Fig. 2.8-2 we have drawn the one-variable K map and truth table side by side to show the correspondence again. In the truth table, the function $f(A)$ has been left blank to call attention to the fact that the form of the K map depends only on the



Row no.	A	$f(A)$
0	0	
1	1	

Figure 2.8-2 The correspondence between a one-variable truth table and the one-variable K map.



Row no.	A	B	$f(A, B)$
0	0	0	
1	0	1	
2	1	0	
3	1	1	

Figure 2.8-3 A two-variable truth table and its K map.

number of variables contemplated and not in any way on the boolean expression for which the map will be used.

A two-variable truth table and a two-variable K map are shown in Fig. 2.8-3. Note again the three alternative methods indicated to identify the boxes. Box 2 for, example, is not only labeled 2 but is also located at the intersection of $A = 1$ and $B = 0$, which are the entries in row 2 in the columns for A and B, respectively. Box 2 is also within the region of the A bracket and outside the region of the B bracket. An alternative two-variable K map as well as K maps for larger numbers of variables will be considered later. For the present, let us use the K map of Fig. 2.8-3 to replace a truth table.

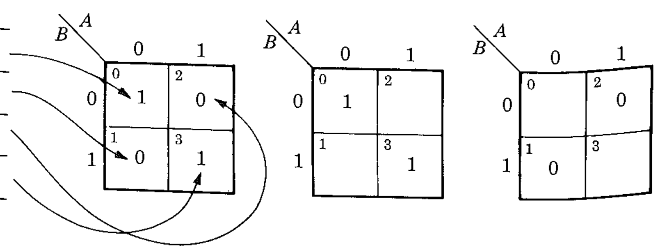
A particular function is defined in the truth table of Fig. 2.8-4a. This same function is represented in the K map in Fig. 2.8-4b. Here we have simply entered in the boxes the 1s or 0s, as the case may be, from the corresponding row of the truth table. Since we understand that where an entry is not a 1 it is a 0 and vice versa, we can simply enter the 1s as in Fig. 2.8-4c or the 0s as in Fig. 2.8-4d. (Rather generally, in a truth table *both* 1s and 0s are entered while in a K map *either* 1s or 0s are entered. As we shall see, in reading a K map we shall want to concentrate on either the 1s or the 0s.)

As we can readily verify, the function defined in Fig. 2.8-4 is

$$f(A,B) = \bar{A}\bar{B} + AB = \mathrm{m}_0 + \mathrm{m}_3 \qquad (2.8\text{-}1)$$

$$= (A + \bar{B})(\bar{A} + B) = \mathrm{M}_1 \cdot \mathrm{M}_2 \qquad (2.8\text{-}2)$$

Row no.	A	B	$f(A, B)$
0	0	0	1
1	0	1	0
2	1	0	0
3	1	1	1



(a) (b) (c) (d)

Figure 2.8-4 (a) A truth table defining a function. (b) Truth-table definition represented on a K map. (c) Only the 1s are entered on the map. (d) Only the 0s are entered on the map.

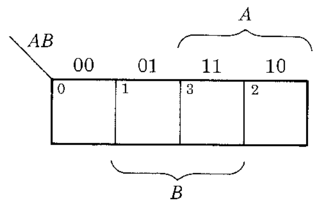


Figure 2.8-5 An alternative two-variable K map.

Thus we confirm that a function which is expressed as a sum of minterms m_0 and m_3 is represented by 1s in K-map boxes 0 and 3; i.e., box 0 is associated with m_0 and box 3 with m_3. Similarly, since the function is expressible also as the product of maxterms M_1 and M_2, we find 0s in boxes 1 and 2. Thus, if we were not given the function truth table explicitly but were given instead the minterms or maxterms, we could immediately represent the function on a K map.

A point especially to be noted in connection with entering minterms and maxterms on a K map is the following. A box in which some minterm, say m_i, is to be entered is the *same* box as the box in which maxterm M_i is to be entered. Thus if a function f has a maxterm $f = \bar{A} + B = M_2$ then a 0 is to be entered in box 2. If a (different) function g has a minterm $A\bar{B} = m_2$, then a 1 is to be entered in this *same* box.

An alternative two-variable K map is shown in Fig. 2.8-5. Note especially the ordering of the box numbers. The numbers proceed in the order 0, 1, 3, 2 rather than the natural order 0, 1, 2, 3. This pattern of ordering, as we shall see, appears as well in the K maps for larger number of variables. The purpose of this ordering will be discussed and explained below. The numbering across the top of the map, as is readily apparent, is consistent with the numbering in the boxes. In each case the left digit goes with the variable A and the right digit goes with the variable B. Observe that as we go across the top of the map from one box to the next, the two-digit numbers exhibit a change in just one or the other digit, never in both digits at the same time. This, as we shall see, is an essential feature of the ordering. Note, finally, that in the third scheme shown for identifying boxes the A bracket encompasses the boxes corresponding to $A = 1$ and the B bracket encompasses the boxes corresponding to $B = 1$.

The K map for three variables is shown in Fig. 2.8-6. Observe that, here

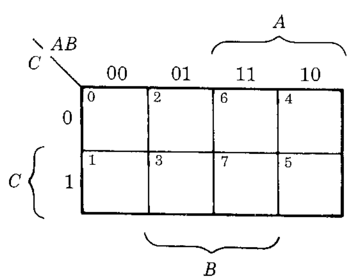


Figure 2.8-6 The K map for three variables.

-variable K map.

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.8-6. Observe that, here

three variables.

CD \ AB	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

Figure 2.8-7 The K map for four variables.

again, in numbering columns we have followed the pattern of the alternative two-variable map of Fig. 2.8-5.

The K map for four variables is shown in Fig. 2.8-7. Here the ordering pattern of the map of Fig. 2.8-5 has been applied to the rows as well as the columns. In moving vertically boxes in the third row are numbered last, while in moving horizontally the third column is numbered last.

Karnaugh maps for larger numbers of variables can also be drawn. A five-variable map has $2^5 = 32$ boxes while a six-variable map has $2^6 = 64$ boxes. We shall defer drawing such maps until Sec. 2.12, and we shall turn our attention in the next section to the matter of how such maps are used to simplify logical expressions.

Before going on, however, we note (again) that there is a measure of arbitrariness in assigning variables to rows and columns of a K map and also to assigning of numerical significance to the logical variables. Thus in Fig. 2.8-7 the digit assigned to the variable A is taken to be numerically most significant, the digit assigned to B is next most significant, etc. Thus, minterm $A\bar{B}\bar{C}\bar{D}$ is assigned the number $1000 = 8$, and the minterm is m_8. We might have decided to have arranged matters otherwise. For example, if we chose to reverse the order of numerical significance, the K map would appear as in Fig. 2.8-8a. Or we

BA \ DC	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	13

(a)

AB \ CD	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	9	11	10

(b)

Figure 2.8-8 Showing that alternative assignments of numerical significances and association of variables with rows and columns are allowed.

might have preserved the numerical significances of Fig. 2.8-7 but decided to associate the variables A and B with the rows and C and D with the columns. In this case the K map would appear and the boxes would be numbered as in Fig. 2.8-8*b*. We are at liberty to make any assignment of numerical significances and any association of variables with rows and columns. What is required, of course, is that having adopted an assignment and association, we observe them consistently throughout.

2.9 SIMPLIFICATION OF LOGICAL FUNCTIONS WITH KARNAUGH MAPS

The essential feature of the K map is that adjoining boxes horizontally and vertically (but not diagonally) correspond to minterms or maxterms which differ in only a single variable. This single variable will appear complemented in one term and uncomplemented in the other. It is precisely to achieve this end that the boxes have been ordered and numbered in the way we have described. To see the benefit of this feature, consider, for example, minterms m_8 and m_{12}, which adjoin horizontally on the K map of Fig. 2.9-1. We have

$$m_8\ (8 = 1000) = A\bar{B}\bar{C}\bar{D} \tag{2.9-1}$$

$$m_{12}\ (12 = 1100) = \mathrm{AB}\bar{\mathrm{C}}\bar{\mathrm{D}} \tag{2.9-2}$$

These two minterms differ only in that the variable B appears complemented in one and uncomplemented in the other. They can be combined to yield

$$A\bar{B}\bar{C}\bar{D} + AB\bar{C}\bar{D} = A\bar{C}\bar{D}(\bar{B} + B) = A\bar{C}\bar{D} \tag{2.9-3}$$

Thus two terms, each involving four variables, have been replaced by a single term involving three variables. The variable which appeared complemented in one term and uncomplemented in the other has been eliminated. Now if the terms of Eq. (2.9-3) had appeared together with other terms in a logical function, we would eventually, by dint of comparing each term with every other, have noted that they might be combined. On the other hand, suppose that we had noted the presence of these two minterms by placing 1s in the appropriate

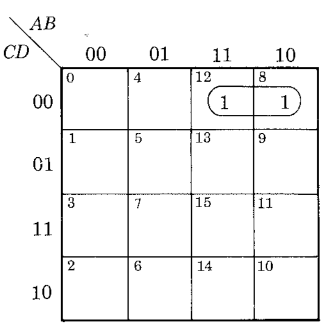

Figure 2.9-1 Minterms in adjacent boxes can be combined.

boxes of a K map, as shown in Fig. 2.9-1. Then we would have noted immediately that these minterms can be combined because the minterms correspond to *adjoining boxes*. The great merit of the K map is that it permits easy recognition through geometric visualization of combinations of minterms which can be combined into simpler expressions. The combining of minterms m_8 and m_{12} has been indicated on the K map of Fig. 2.9-1 by a line encircling the 1s in the adjacent boxes.

A general principle, then, which applies to a K map is that *any pair of adjoining minterms can be combined into a single term involving one variable fewer than do the minterms themselves*. This combined term is deduced by starting with either minterm and striking out the variable which appears complemented in one and uncomplemented in the other. Let us apply this rule to m_8 and m_{12}, which appear in Fig. 2.9-1. We note that in both these terms the variables A, C, and D are associated (by the numbering along the top and side of the map) with the *same* digits (A with 1 and C and D with 0). However, variable B is associated with 1 in minterm 12 and with 0 in minterm 8. Hence this variable is deleted. The two minterms combine into a single term in which A appears uncomplemented (since A is associated in *both* minterms with 1) and both C and D appear complemented (since these variables are associated in *both* minterms with 0). Thus

$$m_8 + m_{12} = A\bar{C}\bar{D} \tag{2.9-4}$$

Observe that in reading the K map of Fig 2.9-1 we used the numbering along the top and side of the map. We did not explicitly use the minterm numbering in the individual boxes. When a logical function is given in terms of its minterms or equivalently its truth table, however, the minterm numbering in the boxes is very useful since it makes it easy to place the specified minterms on the map.

The four-variable map with minterms m_8 and m_{12} marked as in Fig. 2.9-1 is reproduced in Fig. 2.9-2. Here we have used the alternative scheme of identifying boxes. The two columns in which $A = 1$ are bracketed are marked A. We

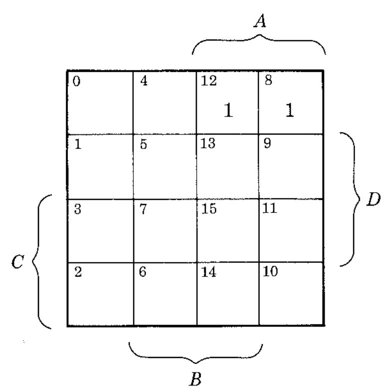

Figure 2.9-2 An alternative scheme for marking a K map.

have correspondingly bracketed and marked the two columns in which $B = 1$ and also the pairs of rows corresponding to $C = 1$ and $D = 1$.

Consider now how we would use the representation of Fig. 2.9-2 to read m_8 which appears in the map. We find that m_8 is *in* a column encompassed by the A bracket. Hence the A variable appears *uncomplemented.* We also find that m_8 is *outside* the columns encompassed by the B bracket. Hence the B variable appears *complemented.* Similarly m_8 is *outside* both the C and D bracketed rows; hence both C and D appear complemented. Altogether $m_8 = A\bar{B}\bar{C}\bar{D}$. In a similar way we would read $m_{12} = AB\bar{C}\bar{D}$.

When, now, it comes to reading the term corresponding to the *pair* m_8 and m_{12}, we proceed as follows. *Both* minterms are encompassed by A; hence A appears *uncomplemented. Neither* minterm is encompassed by C or by D; hence both C and D appear *complemented.* One minterm (m_{12}) is *in* the range encompassed by B while one minterm is *outside* the range of B. Hence variable B is deleted. Altogether $m_{12} + m_8 = A\bar{C}\bar{D}$.

2.10 ADDITIONAL LOGICAL ADJACENCIES

We have noted that minterms which are *geometrically* adjacent on a K map are also *logically* adjacent; i.e., the minterms differ in just a single variable. There are cases in which the boxes are not geometrically adjacent but are nonetheless logically adjacent. As can readily be verified, each box in the leftmost column is logically adjacent to the box in the rightmost column on the same row. Thus in Fig. 2.9-1 or 2.9-2 m_0 is adjacent to m_8, m_1 is adjacent to m_9, etc. Similarly, the boxes in the topmost row are adjacent to the boxes in the bottommost row, so that m_0 adjoins m_2, m_4 adjoins m_6, etc. We can visualize a geometrical as well as a logical adjacency between the left and right column by imagining the map wrapped around a vertical cylinder. We can visualize a geometrical as well as a logical adjacency between top and bottom rows by imagining the K map wrapped around a horizontal cylinder. Both left and right column adjacencies and top and bottom row adjacencies can be visualized simultaneously by imagining the K map wrapped around a doughnut.

Consider a K map with entries as in Fig. 2.10-1. As indicated by the circlings, we can combine the geometrically adjacent pairs with the result

$$m_8 + m_{12} = A\bar{C}\bar{D} \qquad (2.10\text{-}1)$$

and

$$m_2 + m_3 = \bar{A}\bar{B}C \qquad (2.10\text{-}2)$$

Next we can combine m_{10} with either m_8 or with m_2. Two alternative symbolisms to indicate a combining of m_{10} with m_2 are indicated in Fig. 2.10-1*a* and *b*. Using this combination of m_2 with m_{10}, we have

$$m_2 + m_{10} = \bar{B}C\bar{D} \qquad (2.10\text{-}3)$$

In this case, the logical function defined in the K map is, from Eqs. (2.10-1) to (2.10-3),

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$) = 1$.
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ited. We also find that m_8
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(2.10-1)

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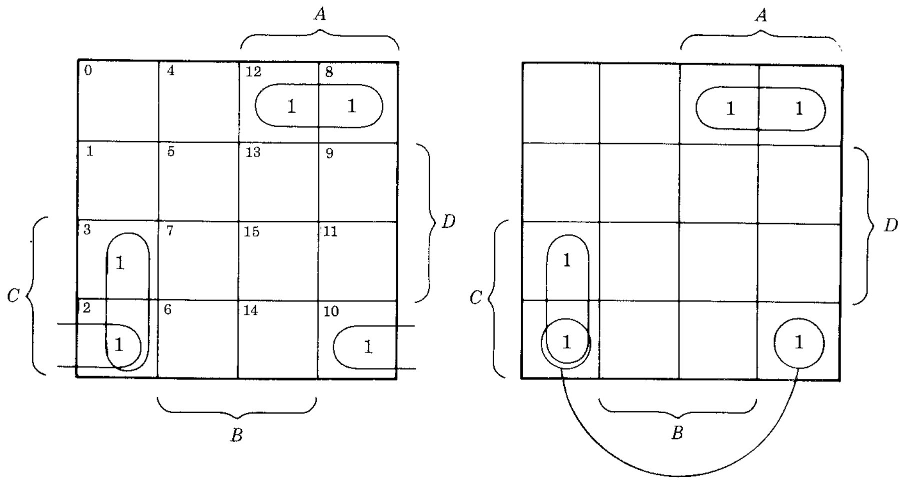


Figure 2.10-1 Alternative symbolism for indicating combinations of K-map boxes which are not geometrically adjacent.

$$f(A,B,C,D) = \Sigma m(2,3,8,10,12) = A\bar{C}D + \bar{A}\bar{B}C + \bar{B}C\bar{D} \qquad (2.10\text{-}4)$$

as shown in Fig. 2.10-2. If we choose, we can combine minterm m_{10} not with m_2 but with m_8. In this case we have

$$m_8 + m_{10} = A\bar{B}\bar{D} \qquad (2.10\text{-}5)$$

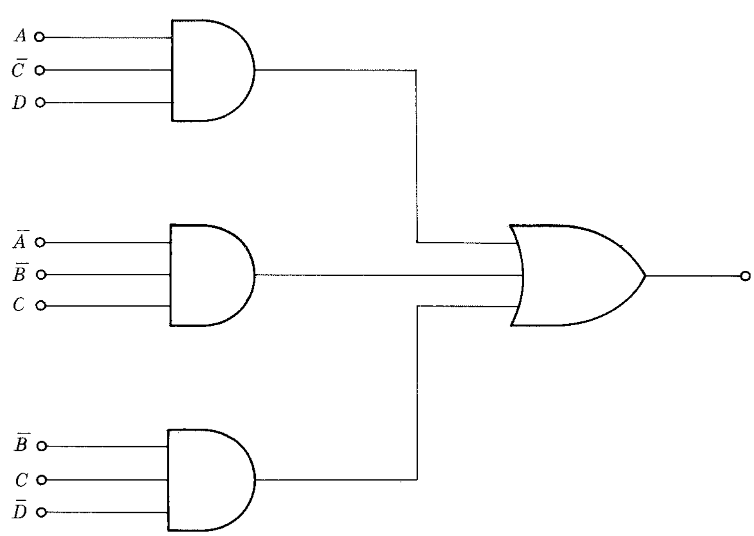


Figure 2.10-2 Implementation of the function of Eq. (2.10-4).

and from Eqs. (2.10-1), (2.10-2), and (2.10-5) we have

$$f(A,B,C,D) = \Sigma m(2,3,8,10,12) = A\bar{C}D + \bar{A}\bar{B}C + A\bar{B}\bar{D} \qquad (2.10\text{-}6)$$

Even though Eq. (2.10-6) appears to be different from Eq. (2.10-4), they are identical. If we derive truth tables for $f(A,B,C,D)$ using one equation and then the other, the tables will be identical. From the point of view of economy of hardware either result is equally acceptable. In both cases a single OR gate is required with a fan-in of 3, that is, with three inputs, and three AND gates are used, again each one with a fan-in of 3.

The following point needs a brief comment. In arriving at Eqs. (2.10-4) and (2.10-6) we have used a minterm twice in each case. In the first case, we combined m_2 with m_3 and then again combined m_2 with m_{10}. In the second case, we used m_8 twice. This repetitive use of a minterm is allowable since in using, say, m_2 twice we have simply taken advantage of the theorem of Eq. (1.15-5*a*), which in the present case yields

$$m_2 = \bar{A}\bar{B}C\bar{D} = \bar{A}\bar{B}C\bar{D} + \bar{A}\bar{B}C\bar{D} + \cdots \qquad (2.10\text{-}7)$$

2.11 LARGER GROUPINGS ON A K MAP

We have seen that two K-map boxes which adjoin can be combined, yielding a term from which one variable has been eliminated. In a similar way, whenever 2^n boxes adjoin, they can be combined to yield a single term from which n variables have been eliminated. Typical groups of four boxes are indicated in Fig. 2.11-1. In Fig. 2.11-1*a* the combinations $m_1 + m_5$ and $m_3 + m_7$ yield

$$m_1 + m_5 = \bar{A}\bar{C}D \qquad (2.11\text{-}1)$$

$$m_3 + m_7 = \bar{A}CD \qquad (2.11\text{-}2)$$

so that

$$(m_1 + m_5) + (m_3 + m_7) = \bar{A}\bar{C}D + \bar{A}CD = \bar{A}D(\bar{C} + C) = \bar{A}D \qquad (2.11\text{-}3)$$

The result would have been the same, of course, if the grouping had been made in the order $(m_1 + m_3) + (m_5 + m_7)$.

Reading the group $m_1 + m_3 + m_5 + m_7$ in a direct manner would proceed as follows. We note that all four are in boxes in columns in which $A = 0$. Hence this variable will remain in *complemented* form. In one column $B = 0$, and in one column $B = 1$. Hence the B variable is eliminated. Similarly we find that the C variable is eliminated and the D variable remains in uncomplemented form since in both rows $D = 1$.

Reading the remaining maps in Fig. 2.11-1, from Fig. 2.11-1*c* we have

$$f(A,B,C,D) = \Sigma m(1,5,9,13) = \bar{C}D \qquad (2.11\text{-}4)$$

since all four 1s fall in a row corresponding to $C = 0$ and $D = 1$. However the 1s are to be found in columns corresponding to $A = 0$ to $A = 1$ to $B = 0$ and to

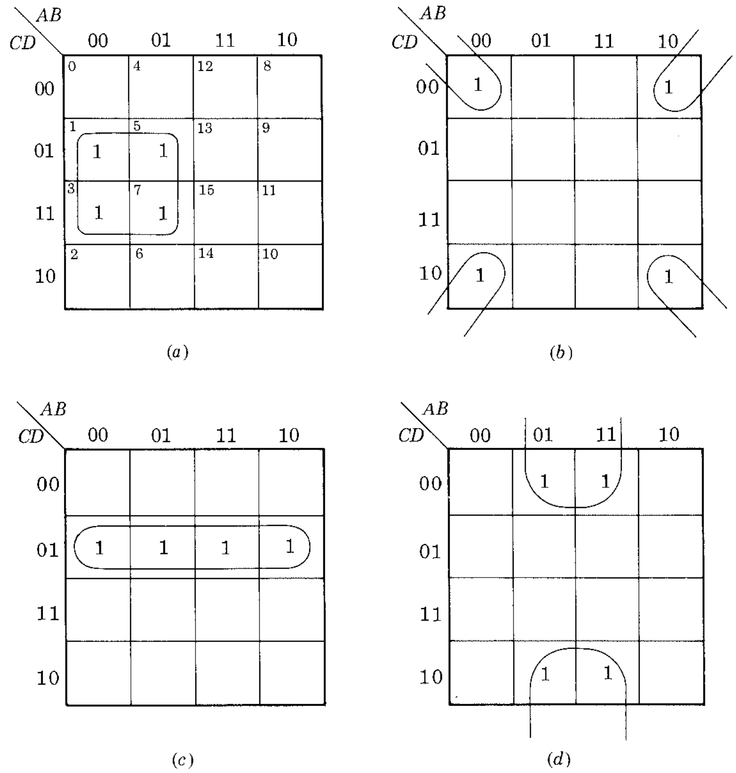


Figure 2.11-1 Representative adjacencies of four boxes.

$B = 1$. Reading Fig. 2.11-1*b*, we find, since the four corners are adjacent,

$$f(A,B,C,D) = \Sigma m(0,2,8,10) = \bar{B}\bar{D} \tag{2.11-5}$$

Reading Fig. 2.11-1*d*, we have

$$f(A,B,C,D) = \Sigma m(4,6,12,14) = B\bar{D} \tag{2.11-6}$$

Typical groups of eight boxes are shown in Fig. 2.11-2. In Fig. 2.11-2*a* we read $f = \bar{A}$ since the eight 1s lie all outside the range of the variable A but both inside and outside the ranges of all of the other variables. In Fig. 2.11-2*b* we read $f = \bar{D}$. In the four-variable case, sixteen 1s on a K map would mean that the function $f = 1$ independently of any variable. However, such groupings of sixteen are significant on a K map for five variables (Sec. 2.12) which has thirty-two boxes, etc.

For variety and completeness we now consider some cases in which the entries on the K map are 0s, representing maxterms, rather than 1s, representing minterms. The rule for grouping 0s is the same as for grouping 1s. The rule which determines whether or not a particular variable is eliminated remains the same in the two cases, but when a group of 0s is read, it leads to a *sum* rather

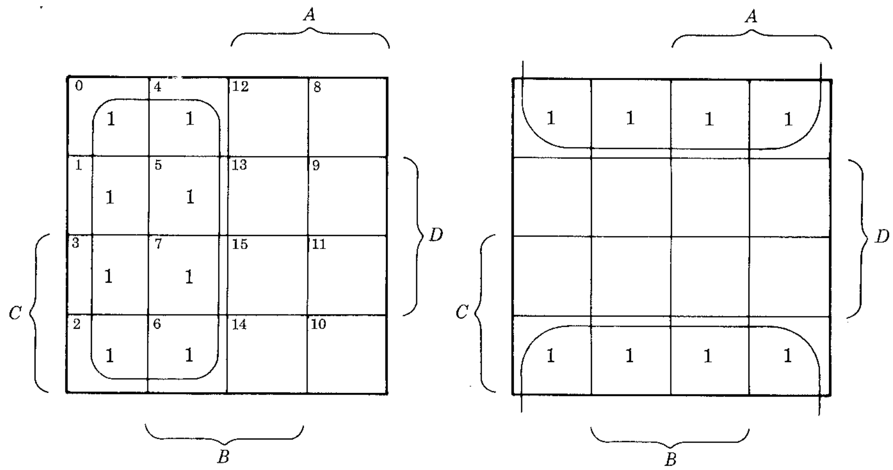


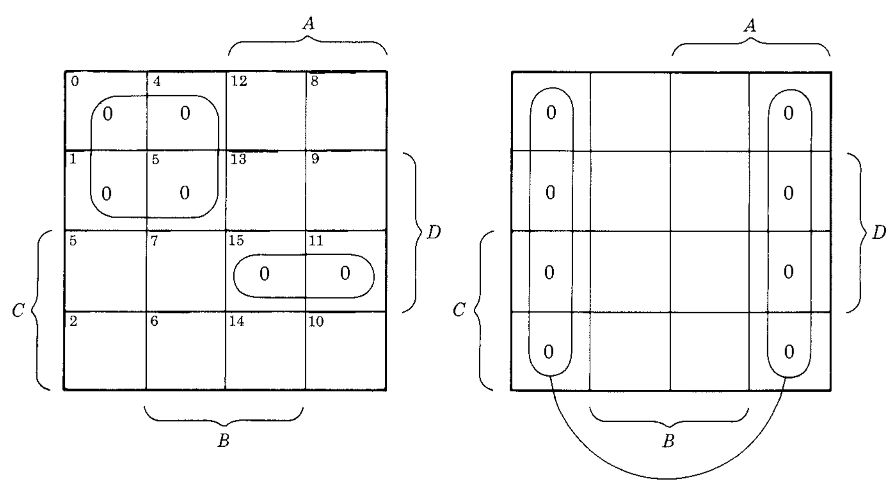
Figure 2.11-2 Representative adjacencies of eight boxes.

than a *product* of variables and the rule for determining whether or not a particular variable is to be complemented is reversed in the two cases. In Fig. 2.11-3*a* the group of two 0s is read

$$M_{11} \cdot M_{15} = \bar{A} + \bar{C} + \bar{D} \tag{2.11-7}$$

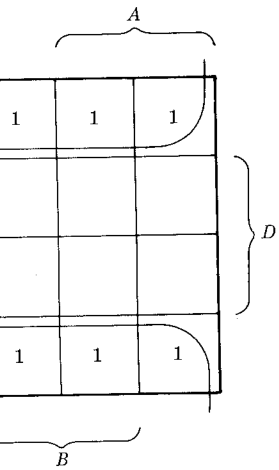
and the group of four 0s is read

$$M_0 \cdot M_1 \cdot M_4 \cdot M_5 = A + C \tag{2.11-8}$$



(*a*) (*b*)

Figure 2.11-3 Representative adjacencies using 0s.



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(2.11-7)

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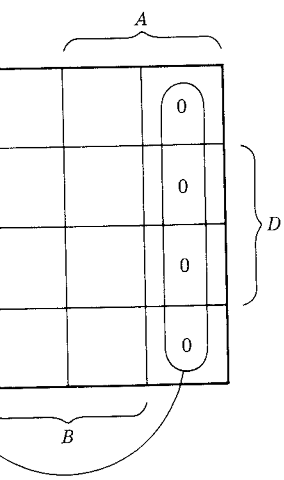

(b)

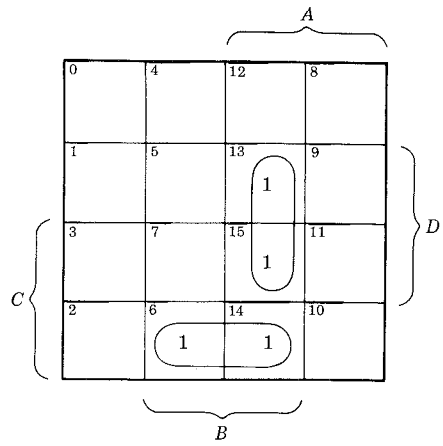

Figure 2.11-4 Minterm pairs which are physically but not logically adjacent and cannot be combined.

In Fig. 2.11-3*b* the group of eight 0s is read

$$M_0 \cdot M_1 \cdot M_2 \cdot M_3 \cdot M_8 \cdot M_9 \cdot M_{10} \cdot M_{11} = B \tag{2.11-9}$$

Finally, we note the following points.

1. The number of K-map boxes which are to be read as a group must be a power of 2. That is, we may read $2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8$, etc. We may not group three boxes, for example, even if they are all adjacent.
2. Suppose we have a situation as in Fig. 2.11-4. Here we have combined $m_6 + m_{14}$ and $m_{13} + m_{15}$. May we now also combine these groups of two into a single group of four? The answer is no in spite of the fact that these two groups appear to be adjacent; for one group was formed by a horizontal combination while the other group was formed by a vertical combination. Hence the variable (A) eliminated from $m_6 + m_{14}$ ($= BC\bar{D}$) is different from the variable (C) eliminated from $m_{13} + m_{15}$ ($= ABD$). Hence no further combining is possible.

2.12 KARNAUGH MAPS FOR FIVE AND SIX VARIABLES

Suppose that in establishing a K map for five variables we follow the pattern which led us from the one-variable map to the four-variable map. We would then have a five-variable map, as in Fig. 2.12-1. Here we have added a variable and plotted two four-variable maps side by side. In numbering the rows and columns we have followed the reflected binary code (Sec. 1.28). This map preserves the features of the previous maps. Geometrically neighboring boxes continue to be adjoining, and, as before, the leftmost and rightmost columns continue to adjoin, as do the top and bottom rows. But now, as can be verified, boxes symmetrically located with respect to the vertical centerline

DE \ ABC	000	001	011	010	110	111	101	100
00	0	4	12	8	24	28	20	16
01	1	5	13	9	25	29	21	17
11	3	7	15	11	27	31	23	19
10	2	6	14	10	26	30	22	18

Figure 2.12-1 A possible five-variable K map.

(the dashed line in Fig. 2.12-1) also adjoin. For example m_7 adjoins m_{23}, m_{13} adjoins m_{29}, etc. Similarly, the previous adjacencies of the four-variable K map persist. Thus, m_1 and m_9 adjoin, as do m_2 and m_{10}, etc.

Since we need to take maximum possible advantage of the adjoining of minterms, the merit of the K map is precisely that it makes such adjoining minterms immediately apparent by visual inspection. Keeping these new adjoining boxes in the map of Fig. 2.12-1, it is entirely feasible to use this map as the five-variable K map. There is, however, an alternative arrangement which makes the visualization much easier. This alternative five-variable K map, the one which is rather generally used, is shown in Fig. 2.12-2 with both notations for identifying boxes. Here, all the adjacent terms previously established for the four-variable K map continue to apply both for the left-hand four-variable section corresponding to $A = 0$ and for the right-hand four-variable section corresponding to $A = 1$. In addition, however, each box in the section $A = 0$ is adjacent to the corresponding box in the section $A = 1$. For example, m_5 is adjacent to m_{21}, m_{15} adjacent to m_{31}, etc. These adjacent terms between the two sections suggest that in our mind's eye we place one section on top of the other. It is then easy to keep in mind that boxes vertically above and below each other are adjacent.

Following the same considerations which led to the five-variable map, a six-variable K map is drawn in Fig. 2.12-3. The usual adjacent terms apply within each four-variable subsection of the map. In addition there are adjacent terms horizontally and vertically between corresponding boxes in the subsection. For example, m_5 is adjacent to m_{21}; m_{63} is adjacent to m_{31} and m_{47}; etc.

The Karnaugh map has the merit of allowing visualization of adjacent terms. When the number of variables becomes large, however, say seven or more, the K map becomes so expansive that its value as an aid to recognizing adjacent terms becomes questionable. Hence, for the many-variable case, tabular procedures are preferred.

Figure 2.12-1 A possible five-variable K map.

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cies of the four-variable K
d m_{10}, etc.
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kes such adjoining minterms
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se this map as the five-varia-
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to the five-variable map, a
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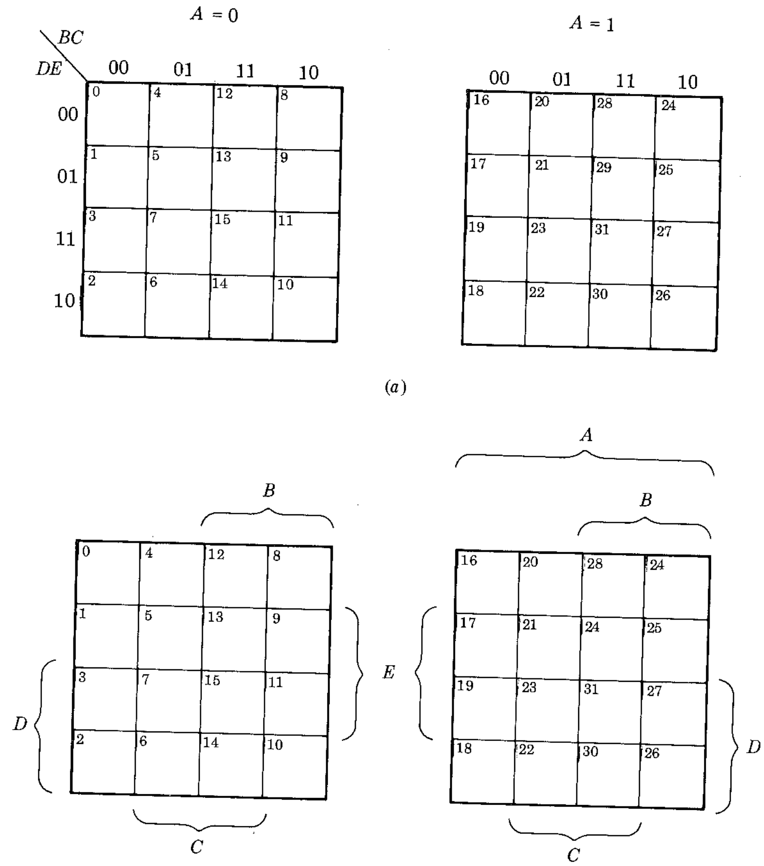


Figure 2.12-2 The more generally used five-variable K map. The two notations for identifying boxes are shown in (*a*) and (*b*).

2.13 USE OF KARNAUGH MAPS

When a logical function has been expressed in standard form in terms of its minterms, the K map can be used to simplify the function by applying the following principles:

1. The combination of boxes (minterms) which are selected must be such that each box is included at least once. As noted, however, a particular box may be involved in a number of different combinations.
2. The individual combinations should be selected to encompass as many boxes as possible so that all boxes will be included in as few different combinations as possible.

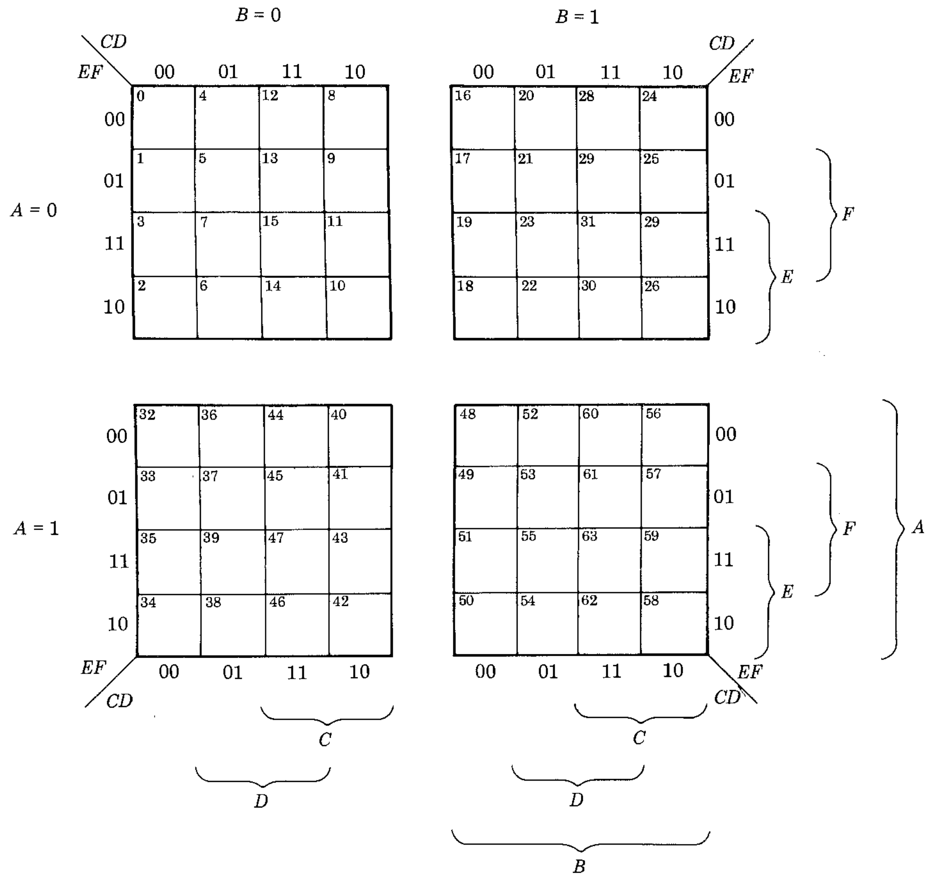
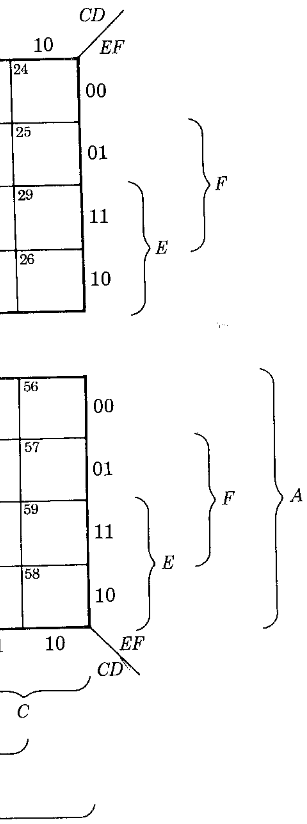


Figure 2.12-3 The K map for six variables.

The combinations are sometimes referred to as *products* and sometimes as *prime implicants.* (The reason for this terminology is presented in Prob. 2.13-1.) It may indeed turn out that it is not necessary to use all possible prime implicants to include every box at least once. In the example of Fig. 2.10-1 we saw just such a case. Here the prime implicants were $p_1 = m_2 + m_3$, $p_2 = m_8 + m_{12}$, $p_3 = m_2 + m_{10}$, and $p_4 = m_8 + m_{10}$, but the function in question could be expressed either as $f = p_1 + p_2 + p_3$ or as $f = p_1 + p_2 + p_4$. In either case we had no alternative but to use p_1 since otherwise m_3 would not be accounted for. For this reason p_1 is called an *essential* prime implicant. Similarly, p_2 is essential since, without p_2, m_{12} would not be accounted for. On the other hand, since we are at liberty to select or not to select p_3, this prime implicant is not essential. A similar comment applies to p_4.

When we have expressed a function as a sum of prime implicants, for each such implicant we shall require an AND gate. Further, the number of inputs to



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each such gate decreases as the number of boxes encompassed in the prime implicant increases. The economy of a gate structure is judged first of all by how few gates are involved. In different structures, with equal numbers of gates, economy is judged to be improved in the structure with the fewest total number of gate inputs.

Our preoccupation with finding prime implicants on a K map which encompass as many boxes as possible poses a potential hazard, illustrated in Fig. 2.13-1*a*, where we might be tempted to combine $m_5 + m_7 + m_{13} + m_{15}$ as noted by the dashed circle. If we were to do so, we would still find it necessary to add four additional prime implicants to take account of the four remaining 1s. Having done so, we would then find that all the boxes of the original combination of four have been accounted for and hence that this original combination is superfluous. The function is written out under the K map. A second, similar example is illustrated in Fig. 2.13-1*b*.

The following algorithm applied to a K map will lead to a minimal expression for a logical function and will avoid the hazard referred to above:

1. Encircle and accept as an essential prime implicant any box that cannot be combined with any other.
2. Identify the boxes that can be combined with a single other box in only one way. Encircle such two-box combinations. A box which can be combined into a grouping of two but can be so combined in more than one way is to be temporarily bypassed.
3. Identify the boxes that can be combined with three other boxes in only one way. If all four boxes so involved are not already covered in groupings of

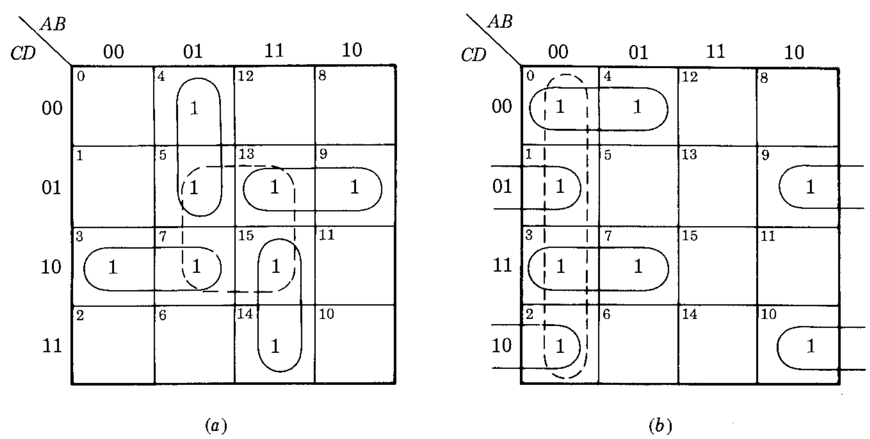

Figure 2.13-1 Two illustrations of a hazard associated with forming combinations on a K map:
(a) $f = (m_4 + m_5) + (m_3 + m_7) + (m_{14} + m_{15}) + (m_9 + m_{13})$
$= \bar{A}B\bar{C} + \bar{A}CD + ABC + A\bar{C}D$
(b) $f = (m_0 + m_4) + (m_1 + m_9) + (m_3 + m_7) + (m_2 + m_{10})$
$= \bar{A}\bar{C}\bar{D} + \bar{B}\bar{C}D + \bar{A}CD + \bar{B}C\bar{D}$

two, encircle these four boxes. Again, a box which can be encompassed in a group of four in more than one way is to be temporarily bypassed.

4. Repeat the preceding for groups of eight, etc.
5. After the above procedure, if there still remain some uncovered boxes, they can be combined with each other or with other already covered boxes in a rather arbitrary manner. Of course, we shall want to include these left-over boxes in as few groupings as possible.

This algorithm is illustrated in the following two examples. In the first example, the solution is completely determined by the algorithm. In the second example an easy exercise of judgment allows us to satisfy the requirements of step 5.

Example 2.13-1 A four-variable function is given as

$$f(A,B,C,D) = \Sigma\ \mathrm{m}(1,3,5,6,9,11,12,13,15) \qquad (2.13\text{-}1)$$

Use a K map to minimize the function.

SOLUTION The K map for the function of Eq. (2.13-1) is drawn in Fig. 2.13-2*a*. We note that m_6 can be combined with no other box. Hence, we encircle it and accept it as an essential prime implicant. Next we note that m_0 and m_{12} can be combined in groups of two in only one way. We therefore encircle each of these groups of two, as in Fig. 2.13-2*b*. Other boxes which can combine in a group of two in more than one way are passed over. We then observe that m_3, m_5, and m_{15} can be incorporated into groups of four in only one way, and we note also that the groups so formed involve other boxes not all of which are incorporated in groups of two. Hence, we encircle these three groups of four as indicated in Fig. 2.13-2*c*. Finally, in Fig. 2.13-2*d*, all encirclements have been combined, and we observe that all boxes have been accounted for. Reading from this map, we find

$$f(A,B,C,D) = \bar{A}BC\bar{D} + \bar{A}\bar{B}\bar{C} + AB\bar{C} + \bar{C}D + \bar{B}D + AD \qquad (2.13\text{-}2)$$

Example 2.13-2 A four-variable function is given as

$$f(A,B,C,D) = \Sigma\ \mathrm{m}(0,2,3,4,5,7,8,9,13,15) \qquad (2.13\text{-}3)$$

Use a K map to minimize the function.

SOLUTION The K map for the function of Eq. (2.13-3) is drawn in Fig. 2.13-3*a*. Applying steps 1 and 2 of the algorithm does not result in any selection of prime implicants. All four boxes m_5, m_7, m_{13}, and m_{15} satisfy the condition of step 3. Applying the procedure of step 3 to any one of them leads to the encirclement shown in Fig. 2.13-3*b*. Step 4 does not apply in the present case. We find that a number of boxes are not yet accounted for. As required by step 5, we combine them arbitrarily. It is rather obvious that the combinations indicated in Fig. 2.13-3*c* lead to the fewest additional

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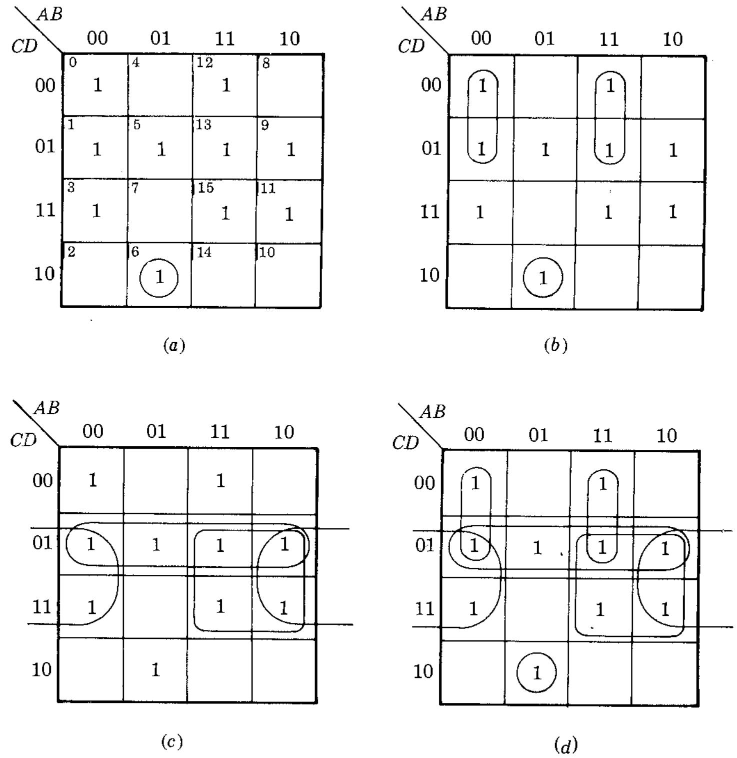


Figure 2.13-2 K map for Example 2.13-1.

prime implicants. The solution, read directly from Fig. 2.13-3*c*, is

$$f(A,B,C,D) = \bar{A}\bar{C}\bar{D} + \bar{A}\bar{B}C + A\bar{B}\bar{C} + BD \qquad (2.13\text{-}4)$$

For variety we now consider a problem in which a function is specified in terms of its 0s rather than its 1s, that is, in terms of its maxterms rather than its

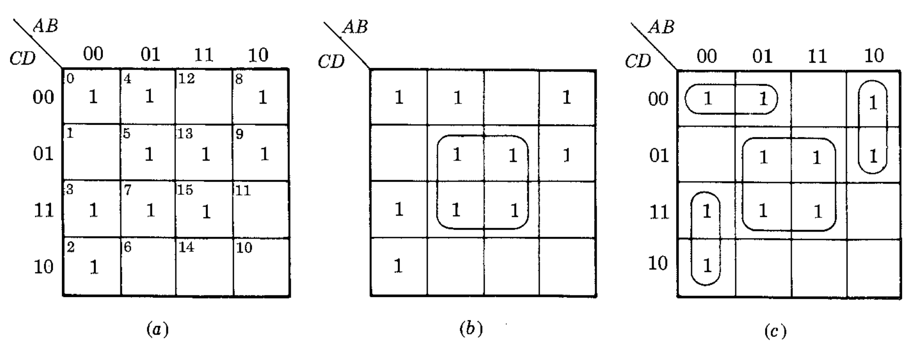


Figure 2.13-3 K map for Example 2.13-2.

minterms. In this case 0s rather than 1s are entered in the K map, and the solution appears in the form of a product of sums rather than a sum of products. The algorithm for combining minterms applies equally to maxterms, but with a change in terminology. Corresponding to the term prime implicant, defined as a product term in a sum representing a function, we have instead *prime implicate*, defined as a sum term in a product of sums.

Example 2.13-3 A four-variable function is given as

$$f(A,B,C,D) = \Pi M(0,3,4,5,6,7,11,13,14,15) \qquad (2.13\text{-}5)$$

Use a K map to minimize the function.

SOLUTION The K map is shown in Fig. 2.13-4. The algorithm leads uniquely to the groupings indicated. We then find directly from the map that

$$f(A,B,C,D) = (A + C + D)(\bar{C} + \bar{D})(\bar{B} + \bar{D})(\bar{B} + \bar{C}) \qquad (2.13\text{-}6)$$

The implementation of Eq. (2.13-6) using AND and OR gates is shown in Fig. 2.13-5. Observe that each term in Eq. (2.13-6) requires an OR gate and that all OR-gate outputs are combined in a single AND gate. Figure 2.13-5 is to be compared with Fig. 2.10-2, where to implement a sum-of-products function the order of the OR and AND gates is reversed.

Example 2.13-4 The minterms of a five-variable function have been entered on the K map of Fig. 2.13-6. Read the map.

SOLUTION Following the algorithm given above, the boxes with 1s have been combined as indicated. Readings of the combinations have been indicated on the map, and we find altogether

$$f(A,B,C,D,E) = A\bar{B}\bar{C}DE + ABCD + \bar{A}\bar{B}\bar{D} + B\bar{D}\bar{E} + C\bar{E} \qquad (2.13\text{-}7)$$

Example 2.13-5 Consider a K map as in Fig. 2.13-6, which was read in Ex-

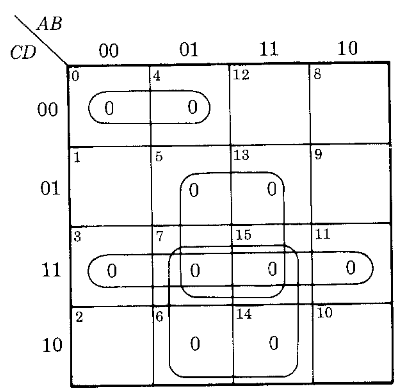


Figure 2.13-4 K map for Example 2.13-3.

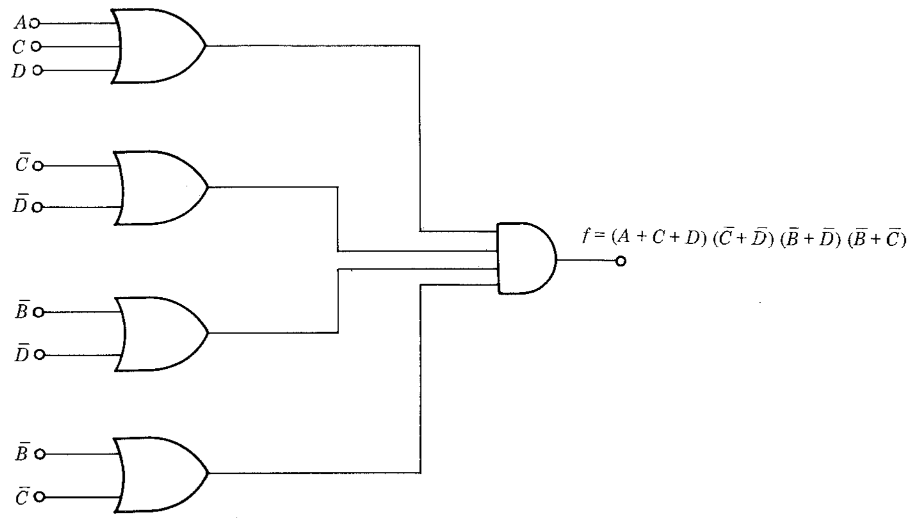


Figure 2.13-5 Implementation of gates in Eq. (2.13-6).

ample 2.13-4, but assume that the entries are not 1s but 0s, so that the map represents the function which is complementary to the function given in Eq. (2.13-7). Express this complementary function as a product of sums.

Solution The grouping of 0s is the same as the grouping of 1s in the previous problem, but now a box *within* the range of A is associated with $\bar{A}$, rather than with A, etc. We therefore find

$$f(A,B,C,D,E) = (\bar{A} + B + C + \bar{D} + \bar{E})(\bar{A} + \bar{B} + \bar{C} + \bar{D})$$
$$(A + B + D)(\bar{B} + D + E)(\bar{C} + E) \qquad (2.13\text{-}8)$$

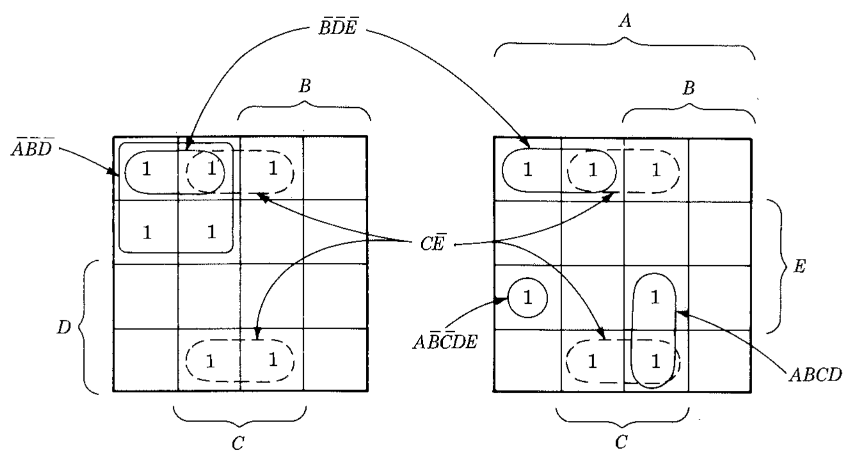


Figure 2.13-6 A five-variable K-map example.

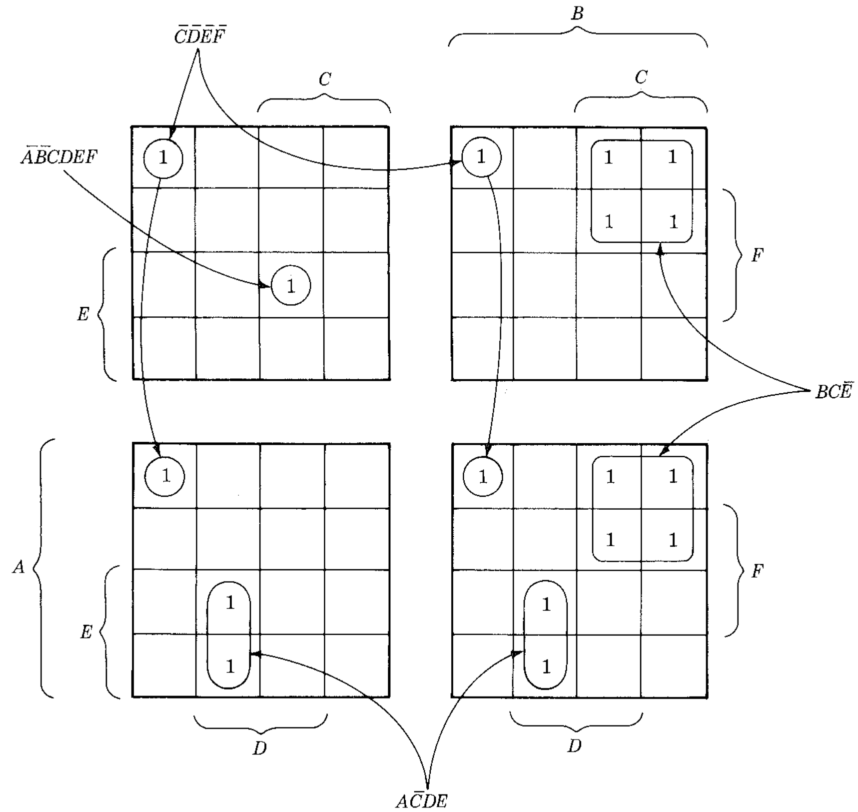


Figure 2.13-7 A six-variable K-map example.

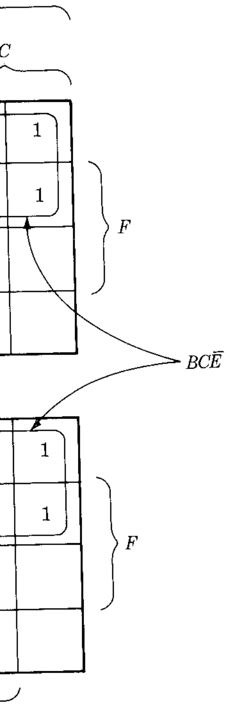
Example 2.13-6 Read the K map of Fig. 2.13-7.

SOLUTION Boxes have been combined as indicated. The readings of the combinations have been indicated on the map, and we find altogether

$$f(A,B,C,D,E,F) = \bar{A}\bar{B}CDEF + \bar{C}\bar{D}\bar{E}\bar{F} + A\bar{C}DE + BC\bar{E} \quad (2.13\text{-}9)$$

2.14 MAPPING WHEN THE FUNCTION IS NOT EXPRESSED IN MINTERMS

Our discussion of mapping suggests that if a function is to be entered on a K map, the function must first be expressed as a sum of minterms (or a product of maxterms). In principle, such is the case. As a matter of practice, however, if the function is not so expressed, it is not necessary to expand the function algebraically into its minterms. Instead, the expansion into minterms can be ac-



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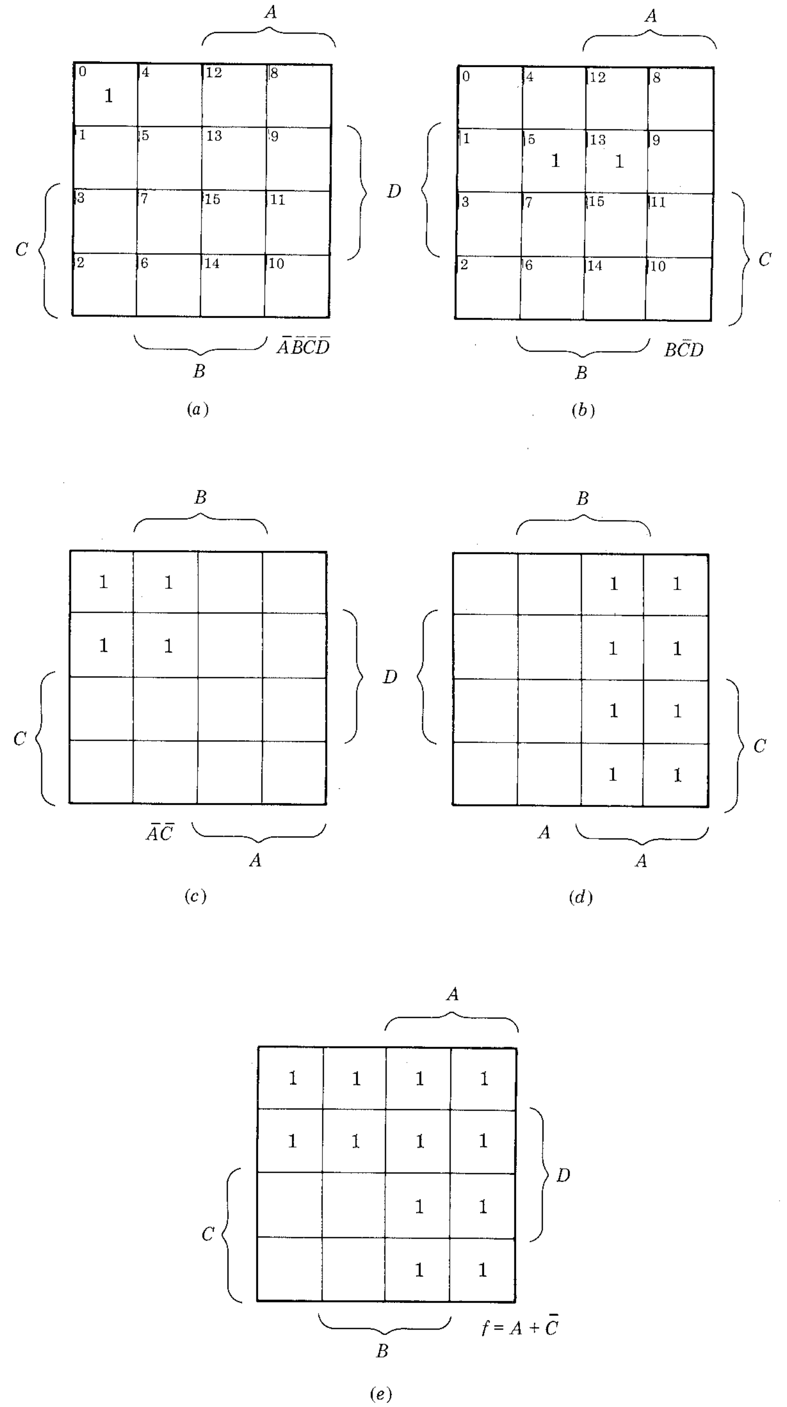

Figure 2.14-1 A logical function can be expanded into its minterms directly on a K map.

complished in the process of entering the terms of the function on the K map. To illustrate, consider that we propose to enter on a K map the function

$$f(A,B,C,D) = \bar{A}\bar{B}\bar{C}\bar{D} + B\bar{C}D + \bar{A}\bar{C} + A \qquad (2.14\text{-}1)$$

in which only the first term is a minterm. As in Fig. 2.14-1*a*, this first minterm can be entered directly on the K map. The second term $B\bar{C}D$ corresponds on the map to the boxes which lie inside the ranges of the variables B and D and outside the range of variable C *independently* of whether the box is inside or outside the range of variable A. Referring to Fig. 2.14-1*b*, we note that the range in which B and D overlap is the range encompassing boxes 5, 7, 13, and 15. Since we need also to be outside the range of C, we drop boxes 7 and 15 and are left with boxes 5 and 13. Since our term $B\bar{C}D$ is independent of A, we need make no further restrictions and the term is represented on the map in the manner shown. Similarly, as in Fig. 2.14-1*c*, the third term $\bar{A}\bar{C}$ is entered in all the boxes outside the range of A and C independently of B and D. Finally, as in Fig. 2.14-1*d*, the term A is entered every place in the range of A independently of all the other variables. The complete K map (Fig. 2.14-1*e*) results from combining the individual maps for the individual terms, and we find that

$$f(A,B,C,D) = A + \bar{C} \qquad (2.14\text{-}2)$$

We note in several instances, i.e., for m_0, m_5 and m_{13}, that 1s appear in the maps of more than just a single term. This situation causes no difficulty, for, as we have noted, a minterm added to itself still yields a single minterm.

2.15 INCOMPLETELY SPECIFIED FUNCTIONS

A logical function f is defined by specifying for each possible combination of variables whether the function has the value $f = 1$ or $f = 0$. Such a specification allows one to enter the minterms or maxterms on a K map immediately and thereafter to express the function in its simplest form.

Suppose that we undertake to write in simplest form a function f which is specified for some (but not all) possible combinations of the variables. In such a case a number of different functions are possible, all of which satisfy the specifications. They will differ from each other in the values assumed by the function for combinations of the variables which are not specified. The question then arises: How, from among the many allowable functions, can we arrive directly at the simplest function?

In practice, such incomplete specification arises in two ways. Sometimes we simply do not care what value is assumed by the function for certain combinations of variables. On other occasions, we may know that certain combinations of the variables will never occur. In this case, we may pretend that we do not care, since the net effect is the same.

To illustrate the procedure, using K maps, to simplify an incompletely specified function, consider that a function is defined by

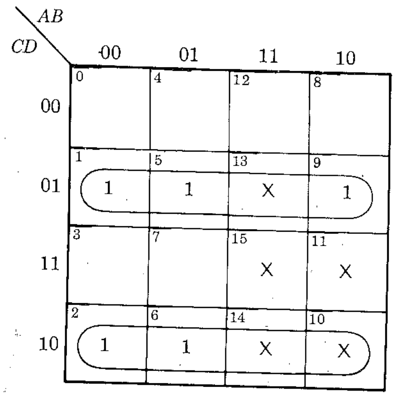

Figure 2.15-1 An incompletely specified function.

$$f(A,B,C,D) = \Sigma m(1,2,5,6,9) + d(10,11,12,13,14,15) \tag{2.15-1}$$

In this equation the d stands for "don't care," so that our function has the value $f = 1$, corresponding to minterms 1,2, . . . , and is unspecified for combinations of variables corresponding to minterms 10,11, ... On the K map, as in Fig. 2.15-1, we locate 1s where specified, and where a don't care is indicated we locate a cross. The procedure thereafter is to interpret a cross as a 1 if so doing effects a simplification and to ignore it otherwise. If the crosses were all ignored, the map of Fig. 2.15-1 would yield

$$f = (m_1 + m_5) + (m_1 + m_9) + (m_2 + m_6) \tag{2.15-2a}$$

$$= \bar{A}\bar{C}D + \bar{B}\bar{C}D + \bar{A}C\bar{D} \tag{2.15-2b}$$

If we interpret as 1s the crosses in m_{10}, m_{13}, and m_{14}, we find that the function simplifies to

$$f = (m_1 + m_5 + m_9 + m_{13}) + (m_2 + m_6 + m_{10} + m_{14}) \tag{2.15-3a}$$

$$= \bar{C}D + C\bar{D} \tag{2.15-3b}$$

The remaining crosses in m_{11}, m_{12}, and m_{15} cannot serve either to reduce the number of terms in the function or to reduce the number of variables in a term. Hence, these crosses are simply ignored, i.e., judged to be 0s.