Lecture # 08

## System of Linear Equations

### INTRODUCTION

The theory of matrices has been successfully applied in diverse fields and areas such as mathematics, engineering, social and biological sciences, economics etc. We shall apply this theory to a system of $m$ linear equations in $n$ unknowns or variables. A finite set of linear equations in $n$ unknowns $x_1, x_2, \ldots, x_n$ is called a ***system of linear equations*** or a ***linear system.***

A general system of $m$ linear equations in $n$ unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\cdots\cdots \\ &\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \qquad (1)$$

where $a_{i\,j}$ and $b_j$ for $i = 1,2,\cdots,m;\ j = 1,2,\cdots n$ are constants. These numbers $a_{i\,j}$ are called the ***coefficients of the system.*** The system (1) is known as "*Non-homogeneous System of Linear Equations*".

If each $b_i$ in (1) is zero, the system becomes:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ &\cdots\cdots \\ &\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned} \qquad (2)$$

The system of equations (2) is called the "***Homogeneous System of Linear Equations***". A sequence of $n$ numbers $s_1, s_2, s_3, \ldots s_n$, for which (1) is satisfied when we substitute $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$, is called a ***solution*** of (1). The set of all such solutions is called ***solution set*** or the ***general solution*** of (1).

A system of equations that has no solution is said to be ***inconsistent***. On the other hand, if there exists at least one solution of the system, the system is called ***consistent system.***

The system (2) always possesses one solution, namely the ***zero n-tupled*** (0, 0, 0, … , 0) solution called the ***zero*** or ***trivial solution.*** Any other solution, if it exists, is called a ***non-zero*** or ***non – trivial solution.***

**NOTE: The following examples will help the readers to understand the concept of consistent and inconsistent systems. The method is based on school algebra and does not need matrix concepts.**

### 2.1.1 Geometric Interpretation and Existence of Solution

From analytical geometry, we know that an equation $ax+by=c$ represents a straight line. Now consider two equations in two unknowns $x$ and $y$:

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

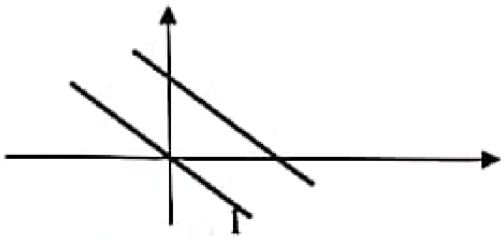
Each equation represents a straight line in the plane $R^2$. There are three possible cases:

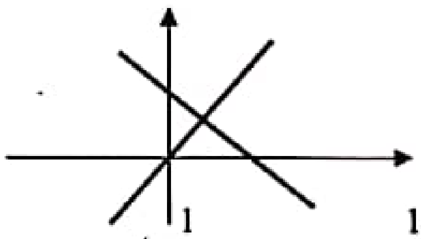
**(a)** No solution if the lines are parallel.

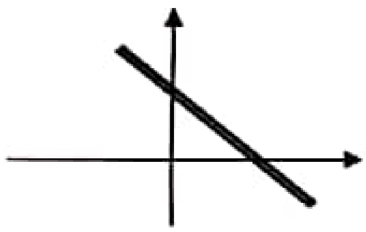
**(b)** Precisely one solution if lines intersect.

**(c)** Infinitely many solutions if lines coincide. For instance,

| | | |
|---|---|---|
| $x+y=1$ | $x+y=1$ | $x+y=1$ |
| $x+y=0$ | $x-y=0$ | $2x+2y=2$ |




**Case (a) No Solution exists** **Case (b) Precisely one solutions exist.** **Case (c) Infinitely many solution exists**

Mathematically speaking:

(a) No solution exists if $\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} \neq \frac{b_1}{b_2}$ (b) Precisely one solution if $\frac{a_{11}}{a_{21}} \neq \frac{a_{12}}{a_{22}}$

(c) Infinitely many solutions if $\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} = \frac{b_1}{b_2}$

**EXAMPLE 01: Solve the system of linear equations:**

**(a)** $3x+2y=12$      $(i)$

$7x-4y=2$      $(ii)$.

Multiply equation $(i)$ by 2 and add the two equations, we get

$$6x+4y=24$$
$$7x-4y=2$$

$13x=26 \Rightarrow x=2$.

Substituting $x=2$ into $(i)$, we get $6+2y=12 \Rightarrow y=3$. Thus, the system is ***"consistent"*** and has a unique solution $(x,y)=(2,3)$.

(b) $3x+2y=4$ (iii)

$6x+4y=9$ (iv)

Multiply equation (*iii*) by 2 and subtract it from equation (*iv*), we get

$$6x+4y=8$$
$$6x+4y=9$$
$$\underline{-\quad -\quad -}$$

$0=-1$ (This is not possible)

This shows that the given system is ***"inconsistent"*** and hence possesses no solution.

(c) $x+y=2$ (v)

$2x+2y=4$ (vi)

If we multiply equation (*v*) by 2, we shall get an equation that is identical to equation (*vi*). This means that equations (*v*) and (*vi*) are equivalent or identical. If we substitute $x=1$ and $y=1$, both equations are satisfied. Therefore, $(x, y)=(1, 1)$ forms a solution of the above system. You will notice that there are many other pairs of solutions, e.g.;

$$x=2 \text{ and } y=0, x=3 \text{ and } y=-1, x=4 \text{ and } y=-2 \text{ and so on}$$

This shows that this system of equations is ***"consistent and has infinitely many solutions"***.

System of linear equations helps us to solve practical problems. The following example shows one of the applications. We shall discuss many application problems in the coming sections.

**EXAMPLE 02: If 20 pounds of rice and 10 pounds of potatoes cost \$16.20 and 30 pounds of rice and 12 pounds of potatoes cost \$23.04, how much will 10 pounds of rice and 50 pounds of potatoes cost?**

**Solution:** Let the cost of 1 pound of rice be \$$x$ and the cost of 1 pound of potatoes be \$$y$. Then according to the question,

$$20x+10y=16.20 \qquad (i)$$
$$30x+12y=23.04 \qquad (ii)$$

Multiply (*i*) by 6 and (*ii*) by 4, we have

$$120x+60y=97.2 \qquad (iii)$$
$$120x+48y=92.16 \qquad (iv)$$

Subtracting (*iv*) from (*iii*), we have

$$12y=5.04 \Rightarrow y=0.42 \cdot$$

Substituting this value of $y$ into (*i*), we obtain

$$20x+10(0.42)=16.20 \Rightarrow x=0.6 \cdot$$

Thus, 10 pounds of rice cost $\$10\times 0.6=\$6$, and 50 pounds of potatoes cost $\$50\times 0.42=\$21\cdot$