Lecture # 10

Gauss-Jordan Method

Gauss elimination process is based on:

(i) changing the augmented matrix A_b in triangular/echelon form and
(ii) making the backward substitution.

The Gauss's method is based upon transferring the augmented matrix into an "Echelon" form where backward substitution takes more operations (addition and multiplications). In Gauss-Jordan's method, one reduces the augmented matrix into "Reduced Echelon" form where one does not need the process of backward substitution. This saves a lot of manual or computing time. Let us see how this method works.

Consider the system (1) of section 2.1. Following four steps can be applied to solve a system of m linear equations in n variables using Gauss-Jordan method.

Step1. Change the system of linear equations to the form $\boldsymbol{Ax}=\boldsymbol{b}\cdot$

Step2. Form the augmented coefficient matrix A_b by including the constants, anextra column in the matrix.

Step3. Convert the augmented matrix into "Reduced Echelon" form by using elementary row operations.

Step4. Find **x** by detaching the fourth column back to its original position on the right hand side of the matrix equation $\boldsymbol{Ax}=\boldsymbol{b}\cdot$

EXAMPLE 03: Use Gauss-Jordan method to solve the system of linear equations (m = n)

$$\begin{aligned} x_1+5x_2+2x_3 &= 9\\ x_1+x_2+7x_3 &= 6\\ -3x_2+4x_3 &= -2 \end{aligned}$$

Solution:

Step1. We change the system of linear equations in matrix form $\boldsymbol{Ax}=\boldsymbol{b}\cdot$

$$\boldsymbol{A}=\begin{bmatrix}1 & 5 & 2\\ 1 & 1 & 7\\ 0 & -3 & 4\end{bmatrix},\ \boldsymbol{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \text{ and } \boldsymbol{b}=\begin{bmatrix}9\\ 6\\ -2\end{bmatrix}\cdot$$

Step2. We form the augmented coefficient matrix A_b by including the constants, an extra column in the matrix.

$$A_b=\begin{bmatrix}1 & 5 & 2 & 9\\ 1 & 1 & 7 & 6\\ 0 & -3 & 4 & -2\end{bmatrix}.$$

Step3. We convert this augmented coefficient matrix into reduced echelon form using elementary row operations.

$$A_b = \begin{bmatrix} 1 & 5 & 2 & 9 \\ 1 & 1 & 7 & 6 \\ 0 & -3 & 4 & -2 \end{bmatrix} R_2 + (-1)R_1 \approx \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & -4 & 5 & -3 \\ 0 & -3 & 4 & -2 \end{bmatrix} R_3 + (-1)R_2$$

$$\approx \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & -4 & 5 & -3 \\ 0 & 1 & -1 & 1 \end{bmatrix} R_2 + 4R_3 \approx \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} R_{23}$$

$$\approx \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_1 + (-5)R_2 \approx \begin{bmatrix} 1 & 0 & 7 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_1 + (-7)R_3, \quad R_2 + R_3$$

$$\approx \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot$$ x1 = -3, x2 = 2, x3 = 1

Step4. Find **x** by detaching the fourth column back to its original position on the right hand side of the matrix equation $\boldsymbol{Ax} = \boldsymbol{b} \cdot$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \cdot$$

Expanding from top, we have

$$x_1 + 0 \cdot x_2 + 0 \cdot x_3 = -3 \qquad (i)$$
$$0 \cdot x_1 + x_2 + 0 \cdot x_3 = 2 \qquad (ii)$$
$$0 \cdot x_1 + 0 \cdot x_2 + x_3 = 1 \qquad (iii)$$

From (i), (ii)and(iii)we have the required solution of the given system of linear equations

$x_1 = -3$, $x_2 = 2$, $x_3 = 1$.

EXAMPLE 04: Use Gauss-Jordan method to solve the system of linear equations

$$\boldsymbol{x_1 + x_2 + x_3 = a}$$
$$\boldsymbol{x_1 + (1 + a)x_2 + x_3 = 2a}$$
$$\boldsymbol{x_1 + x_2 + (1 + a)x_3 = 3a}$$

Solution: when a = your Roll No.

Step1. We change the system of linear equations in matrix form $\boldsymbol{Ax} = \boldsymbol{b} \cdot$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{and } \boldsymbol{b} = \begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix} \cdot$$

Step2. We form the augmented coefficient matrix A_b by including the constants, an extra column in the matrix.

$$A_b = \begin{bmatrix} 1 & 1 & 1 & a \\ 1 & 1+a & 1 & 2a \\ 1 & 1 & 1+a & 3a \end{bmatrix}.$$

Step3. We convert this augmented coefficient matrix into reduced echelon form using elementary row operations.

$$A_b = \begin{bmatrix} 1 & 1 & 1 & a \\ 1 & 1+a & 1 & 2a \\ 1 & 1 & 1+a & 3a \end{bmatrix} R_2 + (-1)R_1,\ \ R_3 + (-1)R_1$$

$$\approx \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & a & 0 & a \\ 0 & 0 & a & 2a \end{bmatrix} \left(\frac{1}{a}\right)R_1,\ \left(\frac{1}{a}\right)R_3 \approx \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} R_1 + (-1)R_2$$

$$\approx \begin{bmatrix} 1 & 0 & 1 & a-1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} R_1 + (-1)R_3 \approx \begin{bmatrix} 1 & 0 & 0 & a-3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Step4. Find **x** by detaching the fourth column back to its original position on the right hand side of the matrix equation $\boldsymbol{Ax} = \boldsymbol{b}\cdot$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a-3 \\ 1 \\ 2 \end{bmatrix}.$$

Expanding from top, we have

$$x_1 + 0\cdot x_2 + 0\cdot x_3 = a-3 \qquad (i)$$
$$0\cdot x_1 + x_2 + 0\cdot x_3 = 1 \qquad (ii)$$
$$0\cdot x_1 + 0\cdot x_2 + x_3 = 2 \qquad (iii)$$

From (i), (ii)and(iii)we get the required solution of the given system of linear equations:

$x_1 = a - 3$, $x_2 = 1$, $x_3 = 2$.

Solution of AX = B by Matrix Inverse Method

In this section we present the matrix inverse method to solve the system of linear equations $A\,\boldsymbol{x} = \boldsymbol{b}$, where A is square matrix and $|A| \neq 0$. $X = A^{-1} B$

As mentioned earlier, this method is based on the fact that if $A\,\boldsymbol{x} = \boldsymbol{b}$ and if A is invertible matrix, , then $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. For example, consider the system of equations:

$$2x + y - z = 1,\ 2y + z = 2,\ 5x + 2y - 3z = 1$$

In matrix form it may be written as:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$ ➔A ***x* = *b*** ➔***x*** =$\boldsymbol{A}^{-1}\boldsymbol{b}$

NOTE Inverse of matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ is $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ **(Find A $^{-1}$)**

➔ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ [See Example 01 part (v) Lec#7 LAAG]

Thus x = 3, y = -1 and z = 4 is the solution of the given system.

Verify: $2x + y - z = 1$, 2(3) -1 – 4 = 6-5 = 1

$2y + z = 2$, 2(-1)+4 =2

$5x + 2y - 3z = 1$ 5(3)+2(-1)-3(4)=1 verified.