Lecture # 4

SPECIAL SQUARE MATRICES

1. Periodic Matrix

A square matrix A is said to be *periodic* if $A^{k+1} = A \cdot$

If k is the least positive integer for which $A^{k+1} = A,$ then k is called the "period" of A.

If $A^2 = A$ it is periodic of period 1   If $A^5 = A$ it is periodic of period 4

EXAMPLE: Show that the matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

is a periodic matrix having period 2.

Solution: Consider

$$A^2 = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1+6-12 & -2-4-0 & -6-18+18 \\ -3-6+18 & 6+4+0 & 18+18-27 \\ 2-0-6 & -4+0-0 & -12+0+9 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix} \neq A.$$

Now consider,

$$A^3 = A^2 A = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+18-12 & 10-12-0 & 30-54+18 \\ 9-30+18 & -18+20+0 & -54+90-27 \\ -4+12-6 & 8-8-0 & 24-36+9 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} = A \cdot$$

We observe that $A^3 = A,$ hence A is a periodic matrix having period k =2.

2. Idempotent Matrix

A square matrix A is said to be *Idempotent* if $A^2 = A \cdot$

EXAMPLE: Show that

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$ **is an Idempotent matrix.**

Solution: Consider

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Since $A^2 = A$, the given matrix A is an Idempotent Matrix.

NOTE:Identity matrix is always a periodic of period one. It is also an Idempotent matrix for $I^2 = I$.

It may also be noted that an idempotent matrix is periodic matrix with period 1.

3. Nilpotent Matrix

A square matrix A is said to be *nilpotent* if $A^k = O, k \in N$.

If k is the least member of N such that $A^k = O$, then k is called the "index" of A.

EXAMPLE: Show that $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ **is a nilpotent matrix of index 2.**

Solution: Consider

$$A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$= \begin{bmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

Since, $A^2 = O$, the matrix A is a nilpotent matrix of index 2. You may observe that $|A| = 0$.

EXAMPLE: Show that $\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ **is nilpotent.**

Solution: Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$, then

$$A^2 = A \cdot A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3-4 & -3-9+12 & -4-12+16 \\ -1-3+4 & 3+9-12 & 4+12-16 \\ 1+3-4 & -3-9+12 & -4-12+16 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Thus, the given matrix is nilpotent of index 2. You may observe that $|A| = 0$.

4. Involutory Matrix

A square matrix A is said to be *Involutory* matrix if $A^2 = I$.

EXAMPLE: Show that $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ **is Involutory.**

Solution: Consider

$$A^2 = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ 0-12+12 & 3+9-12 & -3-12+16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Since, $A^2 = I$, hence A is an Involutory matrix.

NOTE: Since $A^2 = I \Rightarrow A\,A = I$. Pre-multiplying both sides by A^{-1}

$$A^{-1}(A.A) = A^{-1}.I \quad \Rightarrow \quad (A^{-1}A).A = A^{-1} \quad \Rightarrow I.\,A = A^{-1} \quad \Rightarrow A = A^{-1}.$$

This shows that if a matrix A is involutory matrix then it is equal to its inverse.

SPECIAL MATRICES IN GENERAL

1. Transpose of a Matrix

Let A be any matrix of order $m \times n$. The matrix obtained by interchanging rows and columns of A is called the *transpose* of a matrix. The transpose of matrix is denoted by A^t. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \end{bmatrix}_{2\times3}, \text{ then } A^t = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 7 \end{bmatrix}_{3\times2}.$$

Thus by definition, if matrix $A = [a_{ij}]$, then $A^t = [a_{ji}]$

Properties of Transpose of Matrices

If the matrices A and B are conformable for the sum $A+B$ and the product AB, then

(i) $(A+B)^t = A^t + B^t$ **(ii)** $(A^t)^t = A$ **(iii)** $(kA)^t = kA^t$, where $k \in R$ **(iv)** $(AB)^t = B^t A^t$.

2. Conjugate of a Matrix

if a + bi is a complex number than a – bi is called conjugate complex number

A complex matrix obtained by replacing its elements by their corresponding complex conjugates is called the *conjugate* of A and is denoted by $\overline{A}\cdot$ For example, if

$$A=\begin{bmatrix}2+3i & 4 & 5i\\ 0 & 4i & 8\\ 4-3i & 0 & 7+3i\end{bmatrix}, \text{ then } \overline{A}=\begin{bmatrix}2-3i & 4 & -5i\\ 0 & -4i & 8\\ 4+3i & 0 & 7-3i\end{bmatrix} \text{is the conjugate matrix of } A.$$

3. Tranjugate or Transpose of a Conjugate Matrix

The transpose of the conjugate of a matrix A is called the *tranjugate* or *transposed conjugate* of A. It is denoted by $\left(\overline{A}\right)^t\cdot$ For example, if

$$A=\begin{bmatrix}3+4i & 5-6i & 2+3i\\ 4-5i & 7 & 8i\\ 6 & 5+6i & 2-3i\end{bmatrix}, \text{ then } \overline{A}=\begin{bmatrix}3-4i & 5+6i & 2-3i\\ 4+5i & 7 & -8i\\ 6 & 5-6i & 2+3i\end{bmatrix}$$

and $$\left(\overline{A}\right)^t=\begin{bmatrix}\mathbf{3-4}i & \mathbf{4+5}i & \mathbf{6}\\ \mathbf{5+6}i & \mathbf{7} & \mathbf{5-6}i\\ \mathbf{2-3}i & \mathbf{-8}i & \mathbf{2+3}i\end{bmatrix}\cdot$$

Properties of Tranjugate Matrix: If A and B are two complex matrices confirmable for addition and multiplication, then

(i) $\overline{A\pm B}=\overline{A}\pm\overline{B}$ (ii) $\left(\overline{A}\right)^t=\left(\overline{A^t}\right)$ (iii) $\overline{A\cdot B}=\overline{A}\cdot\overline{B}$ (iv) $\overline{kA}=\overline{k}\,\overline{A}$, where k is a scalar, real or complex. (v) $\overline{((\overline{\mathrm{A}}))}=\mathrm{A}$ (vi) $A+\overline{A}=\mathrm{Re}\,al\ matrix$ (vii) $A-\overline{A}=Purely\ imaginary\ matrix$

4. Symmetric Matrix

Any square matrix A is said to be *symmetric* if $A^t=A\cdot$ For example, if

$$A=\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}, \text{ then } A^t=\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}=A\cdot$$

5. Skew-Symmetric Matrix

Any square matrix A is said to be *skew-symmetric* if $A^t=-A\cdot$ or $\mathbf{-A^t=A}$For example, if

$$A=\begin{bmatrix}0 & 1 & 2\\ -1 & 0 & -3\\ -2 & 3 & 0\end{bmatrix}, \text{ then } A^t=\begin{bmatrix}0 & -1 & -2\\ 1 & 0 & 3\\ 2 & -3 & 0\end{bmatrix}=-\begin{bmatrix}0 & 1 & 2\\ -1 & 0 & -3\\ -2 & 3 & 0\end{bmatrix}=-A\cdot$$

6. Hermitian Matrix

A square matrix A for which $(\overline{A})^t=A$ is called a *Hermitian matrix*. For example, if

$$A=\begin{bmatrix}a & b-ic\\ b+ic & d\end{bmatrix}, \text{ then } \overline{A}=\begin{bmatrix}a & b+ic\\ b-ic & d\end{bmatrix}$$

and $\left(\overline{A}\right)^t=\begin{bmatrix}a & b-ic\\ b+ic & d\end{bmatrix}=A\cdot$ So, A is a Hermitian matrix.

7. Skew-Hermitian Matrix

A square matrix A for which $(\overline{A})^t=-A$ or $-\mathbf{A_c^t}=\mathbf{A}$is called a *skew-Hermitian matrix.* For example, if

$$A=\begin{bmatrix}i & 1-i & 2\\ -1-i & 3i & i\\ -2 & i & 0\end{bmatrix}, \text{ then } \overline{A}=\begin{bmatrix}-i & 1+i & 2\\ -1+i & -3i & -i\\ -2 & -i & 0\end{bmatrix}$$

$$\left(\overline{A}\right)^t=\begin{bmatrix}-i & -1+i & -2\\ 1+i & -3i & -i\\ 2 & -i & 0\end{bmatrix}=-\begin{bmatrix}i & 1-i & 2\\ -1-i & 3i & i\\ -2 & i & 0\end{bmatrix}=-A\cdot$$

EXAMPLE 01: If $A=\begin{bmatrix}1 & 1+i & 2+3i\\ 1-i & 2 & -i\\ 2-3i & i & 0\end{bmatrix}$ **and** $B=\begin{bmatrix}i & 1+i & 2-3i\\ -1+i & 2i & 1\\ -2-3i & -1 & 0\end{bmatrix}$**, then show that**

(i) *iB* is Hermitian (ii) $\overline{A}$ is Hermitian (iii) $\overline{B}$ is Skew – Hermitian.

Solution:(i)

$$B=\begin{bmatrix}i & 1+i & 2-3i\\ -1+i & 2i & 1\\ -2-3i & -1 & 0\end{bmatrix}\Rightarrow iB=\begin{bmatrix}-1 & i-1 & 2i+3\\ -i-1 & -2 & i\\ -2i+3 & -i & 0\end{bmatrix}.$$

If $i\,B$ is Hermitian, then it should satisfy the condition: $\left(\overline{iB^t}\right)=iB\cdot$ Now,

$$\overline{iB}=\overline{\begin{bmatrix}-1 & -i-1 & -2i+3\\ i-1 & -2 & -i\\ 2i+3 & i & 0\end{bmatrix}}=\begin{bmatrix}-1 & i-1 & 2i+3\\ -i-1 & -2 & i\\ -2i+3 & -i & 0\end{bmatrix}$$

$$\therefore (\overline{iB})^t=\begin{bmatrix}-1 & i-1 & 2i+3\\ -i-1 & -2 & i\\ -2i+3 & -i & 0\end{bmatrix}=\begin{bmatrix}-1 & i-1 & 2i+3\\ -i-1 & -2 & i\\ -2i+3 & -i & 0\end{bmatrix}=iB\cdot$$

Hence iB is Hermitian.

(ii) We have

$$A = \begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix} \Rightarrow \overline{A} = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 0 \end{bmatrix}.$$

Again, if $\overline{A}$ is Hermitian then it should verify the following condition: $\overline{\left(\left(\overline{A}\right)^t\right)} = \overline{A}\cdot$ Now,

$$\overline{(\overline{A})^t} = \overline{\begin{bmatrix} \mathbf{1} & \mathbf{1}+i & \mathbf{2}+\mathbf{3}i \\ \mathbf{1}-i & \mathbf{2} & -i \\ \mathbf{2}-\mathbf{3}i & i & \mathbf{0} \end{bmatrix}} = \begin{bmatrix} \mathbf{1} & \mathbf{1}-i & \mathbf{2}-\mathbf{3}i \\ \mathbf{1}+i & \mathbf{2} & i \\ \mathbf{2}+\mathbf{3}i & -i & \mathbf{0} \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{1} & \mathbf{1}-i & \mathbf{2}-\mathbf{3}i \\ \mathbf{1}+i & \mathbf{2} & i \\ \mathbf{2}+\mathbf{3}i & -i & \mathbf{0} \end{bmatrix} = \overline{A}\cdot$$

Hence $\overline{A}$ is Hermitian.

(iii) We have

$$B = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & 1 \\ -2-3i & -1 & 0 \end{bmatrix} \Rightarrow \overline{B} = \begin{bmatrix} -i & 1-i & 2+3i \\ -1-i & -2i & 1 \\ -2+3i & -1 & 0 \end{bmatrix}.$$

If $\overline{B}$ is Skew – Hermitian then it should verify the following condition: $-\overline{\left(\left(\overline{B}\right)^t\right)} = \overline{B}\cdot$ Now,

$$-\ \overline{((\overline{B})^t)} = -\overline{\begin{bmatrix} -i & -\mathbf{1}-i & -\mathbf{2}+\mathbf{3}i \\ \mathbf{1}-i & -\mathbf{2}i & -\mathbf{1} \\ \mathbf{2}+\mathbf{3}i & \mathbf{1} & \mathbf{0} \end{bmatrix}} = \begin{bmatrix} -i & \mathbf{1}-i & \mathbf{2}+\mathbf{3}i \\ -\mathbf{1}-i & -\mathbf{2}i & \mathbf{1} \\ -\mathbf{2}+\mathbf{3}i & -\mathbf{1} & \mathbf{0} \end{bmatrix} = \overline{B}$$

Hence, $\overline{B}$ is skew-Hermitian.

8. Orthogonal Matrix

A square matrix A is said to be orthogonal if $A^t A = I = A A^t$

EXAMPLE 02: Show that the matrix

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

is orthogonal.

Solution: By definition consider,

$$AA^t = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & \cos\theta\sin\theta - \cos\theta\sin\theta \\ 0 & 1 & 0 \\ \cos\theta\sin\theta - \cos\theta\sin\theta & 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly we can show that $A^t A = I$. Hence given matrix is orthogonal.

NOTE: Since $A^t A = I \Rightarrow A^t = A^{-1}$

EXAMPLE 03: Prove that $A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ **is orthogonal.**

Solution:

$$\mathrm{A} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \Rightarrow \mathrm{A}^{t} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

Then

$$A^t A = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \frac{9}{9}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Since, $A^t A = I,$ therefore matrix A is orthogonal.

Theorem: If A and B are orthogonal matrices, then AB and BA are also orthogonal matrices.

9. Unitary Matrix

A square matrix A is said to be unitary if $\overline{\left(A^t\right)}A = I$.

EXAMPLE 04: Show that $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ **is a unitary matrix.**

Solution:

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{-1+i}{2} & \frac{1-i}{2} \end{bmatrix}. \text{ Also } \overline{\left(A^t\right)} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}.$$

Now

$$\overline{\left(A^t\right)}A = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\begin{bmatrix} 1-i & 1-i \\ -(1+i) & 1+i \end{bmatrix}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$\overline{\left(A^t\right)}A = \frac{1}{4}\begin{bmatrix} (1-i^2)+(1-i^2) & -(1-i)^2+(1-i)^2 \\ -(1+i)+(1+i)^2 & (1-i^2)+(1-i^2) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \frac{4}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Hence A is a unitary matrix.

SOME IMPORTANT RESULTS FOR SQUARE MATRICES

1. If A is a square matrix, prove that $A + A^t$ is a symmetric matrix and $A - A^t$ is a skew-symmetric matrix.

EXAMPLE 01: Let

$$A=\begin{bmatrix}2&8&5\\1&3&4\\7&9&5\end{bmatrix}\Rightarrow A^t=\begin{bmatrix}2&1&7\\8&3&9\\5&4&5\end{bmatrix}\therefore A+A^t=\begin{bmatrix}4&9&12\\9&6&13\\12&13&10\end{bmatrix}\ \text{and}\ A-A^t=\begin{bmatrix}0&7&-2\\-7&0&-5\\2&5&0\end{bmatrix}$$

2. If A be a Hermitian matrix, its diagonal elements are real. If A be a skew-Hermitian matrix, then its diagonal elements are either zero or purely imaginary.

EXAMPLE 02: Express $A=\begin{bmatrix}\boldsymbol{i}&\boldsymbol{2\text{-}3i}&\boldsymbol{4+5i}\\\boldsymbol{6+i}&\boldsymbol{0}&\boldsymbol{4\text{-}5i}\\\boldsymbol{\text{-}i}&\boldsymbol{2\text{-}i}&\boldsymbol{2+i}\end{bmatrix}$

as the sum of Hermitian and skew – Hermitian matrices.

Solution: We have

$$A=\begin{bmatrix}1+i&2&5-5i\\2i&2+i&4+2i\\-1+i&-4&7\end{bmatrix},\ \Rightarrow \overline{A}=\begin{bmatrix}1-i&2&5+5i\\-2i&2-i&4-2i\\-1-i&-4&7\end{bmatrix}$$

$$\left(\overline{A}\right)^t=\begin{bmatrix}1-i&-2i&-1-i\\2&2-i&-4\\5+5i&4-2i&7\end{bmatrix}$$

Thus,

$$A+\left(\overline{A}\right)^t=\begin{bmatrix}1+i&2&5-5i\\2i&2+i&4+2i\\-1+i&-4&7\end{bmatrix}+\begin{bmatrix}1-i&-2i&-1-i\\2&2-i&-4\\5+5i&4-2i&7\end{bmatrix}=\begin{bmatrix}2&2-2i&4-6i\\2+2i&4&2i\\4+6i&-2i&14\end{bmatrix}$$

$$\Rightarrow\qquad \frac{1}{2}\left\{A+\left(\overline{A}\right)^t\right\}=\begin{bmatrix}1&1-i&2-3i\\1+i&2&i\\2+3i&-i&7\end{bmatrix}\qquad\text{(i)}$$

Similarly,

$$A-\left(\overline{A}\right)^t=\begin{bmatrix}1+i&2&5-5i\\2i&2+i&4+2i\\-1+i&-4&7\end{bmatrix}-\begin{bmatrix}1-i&-2i&-1-i\\2&2-i&-4\\5+5i&4-2i&7\end{bmatrix}=\begin{bmatrix}2i&2+2i&6-4i\\-2+2i&2i&8+2i\\-6-4i&-8+2i&0\end{bmatrix}$$

$$\Rightarrow \frac{1}{2}\left\{A-\left(\overline{A}\right)^t\right\}=\begin{bmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix} \qquad \text{(ii)}$$

Adding (i) and (ii), we have

$$\frac{1}{2}\left\{A+\left(\overline{A}\right)^t\right\}+\frac{1}{2}\left\{A-\left(\overline{A}\right)^t\right\}=A=\begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix}+\begin{bmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}.$$

Hermitian Matrix Skew – Hermitian Matrix