HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

In the previous section we studied the system of linear equations ***A x = b***. Such a system of equations is known as "Non-Homogeneous System of Linear Equations". In this section we shall consider and study the "Homogeneous System of Linear Equations" where ***b*** = ***O***.

Consider a system of *m* homogeneous equations in *n* unknowns $x_1, x_2, \cdots, x_n$;

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}\right\}$$

In matrix form the above system of equations may be expressed as:

A x = O (1)

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Obviously $x_1 = 0,\ x_2 = 0,\ \cdots,\ x_n = 0,$ form a solution of (1). This is known as ***trivial solution***. All other solutions (if exist) are called ***non-trivial solutions*** of (1).

EXAMPLE 01: Solve the following system of homogeneous equations (*m* = *n*)

$$\begin{aligned} \boldsymbol{x + y + z} &= \boldsymbol{0} \\ \boldsymbol{4x + 5y + 2z} &= \boldsymbol{0} \\ \boldsymbol{2x + 3y} \quad\;\; &= \boldsymbol{0} \end{aligned}$$

Solution: We change the above system of equations to the matrix form $Ax = O$.

Or $$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A_b = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & 3 & 0 \end{bmatrix} R_2 + (-4)R_1, R_3 + (-2)R_1$$

$$\approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} R_3 + (-1)R_2 \approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

This last matrix implies that

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot$$

This gives
$$\begin{aligned} x + y + z &= 0 \qquad (i) \\ y - 2z &= 0 \qquad (ii) \end{aligned}$$

From equation (ii) we get $y = 2z \cdot$ Putting $y = 2z$ in (i) we get $x + 2z + z = 0 \Rightarrow x = -3z \cdot$

Let $z = k$, the solutions are $x = -3k,\ \ y = 2k$ and $z = k \cdot$ Here k is known as a "*free parameter*" or "*an arbitrary constant"*; hence it can hold any value. Thus for different values of k, there are *infinite solutions* of the above Homogeneous System of Equations. For example,

If k = 1, then $x = -3,\ \ y = 2$ and $z = 1 \cdot$

If k = 2, then $x = -6,\ \ y = 4$ and $z = 2 \cdot$ etc.

EXAMPLE 02: Solve the following linear homogeneous equations (m < n)

$$\begin{aligned} \boldsymbol{2w + 3x - \ y - z = 0} \\ \boldsymbol{4w - 6x - 2y + 2z = 0} \\ \boldsymbol{-6w + 12x + 3y - 4z = 0} \end{aligned}$$

Solution: Above equations can be written in the matrix form $Ax = O$**,** that is

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 4 & -6 & -2 & 2 \\ -6 & 12 & 3 & -4 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot$$

To solve the above system of equations, we use the Gauss-Jordan method and have

$$A_b = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 4 & -6 & -2 & 2 \\ -6 & 12 & 3 & -4 \end{bmatrix} R_2 + (-2)R_1,\ R_3 + 3R_1$$

$$\approx \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -12 & 0 & 4 \\ 0 & 21 & 0 & -7 \end{bmatrix} \left(\frac{1}{4}\right)R_2,\ \left(\frac{1}{7}\right)R_3 \qquad \approx \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -3 & 0 & 1 \\ 0 & 3 & 0 & -1 \end{bmatrix} R_3 + R_2$$

$$\approx \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \Rightarrow \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives

$$\begin{aligned} 2w + 3x - y - z &= 0 \qquad (i) \\ -3x \quad + z &= 0 \qquad (ii) \end{aligned}$$

From (ii), we get $x = z/3$. Putting this into (i), we get

$$2w + 3\left(\frac{z}{3}\right) - y - z = 0 \Rightarrow 2w + z - y - z = 0 \Rightarrow 2w - y = 0 \Rightarrow y = 2w \cdot$$

Thus, $x = z/3$ and $y = 2w$. Take $z = k_1$ and $w = k_2$, we get $x = k_1/3$ and $y = 2k_2$(It may be noted that there are two parameters k_1 and k_2).Hence, $x = k_1/3, y = 2k_2, z = k_1$ and $w = k_2$ constitute the general solution where k_1 and k_2 are arbitrary constants. We can obtain an infinite number of solutions by giving different values to k_1 and k_2.