Lecture # 7

INVERSE OF A MATRIX

We know that real number 1 is an identity element under multiplication of a set real numbers because for any real number a, $a\cdot 1 = 1\cdot a = a\cdot$ In this section, an ***Identity matrix I*** is defined that has properties similar to those of the number 1in the set of real numbers. This identity matrix is then used to find the multiplicative inverse of any square matrix provided it exists.

Definition: If A and B be two square matrices of same order and $AB = I = BA$, then matrices A and B are the called multiplicative inverse of each other and are usually denoted by $A = B^{-1}$ and $B = A^{-1}$.

Similarly, if A and B are any two matrices of the same order and $A + B = O$, then B is called additive inverse of the matrix A and vice versa. This is denoted by $B = -A$ or $A = -B$.

Further note that if A and I are square matrices of order n, then $A \,.\, I = I \,.\, A = A$.
Similarly, if A and O are matrices of same order (not necessarily the square matrices) then, $A + O = O + A = A$. I and O are therefore called identity matrices under multiplication and addition respectively.
In this section we shall discuss the process of finding the multiplicative inverse of a square matrix, because finding the additive inverse of any matrix is straight forward.

NOTE:

i. **Only the square matrices may have multiplicative inverses.**
ii. **The symbol A^{-1} (A – inverse) does not mean $1/A$ or I/A. The symbol is just the notation for the inverse of matrix A because there is no such thing as matrix division.**
iii. **The order of multiplicative inverse of a square matrices (if it exists) will be same as that of the matrix A.**

Theorem: Prove that if the inverse of a square matrix exists then it will be unique.

Definition: The square matrix A, whose inverse A^{-1} exists, is called **non-singular or invertible matrix.** Square matrices which do not have inverses are called ***singular matrices.***

Theorem: If A and B are invertible matrices, so does the AB and that $(AB)^{-1} = B^{-1}A^{-1}$

1.9.1 Inverse of a Matrix Through Elementary Row Operations

A matrix obtained from the identity matrix by a single elementary row operation is called an *elementary matrix*. Thus elementary matrices are obtained

- **(i)** By interchanging any two rows of the identity matrix (R_{ij}).
- **(ii)** By multiplying a row of the identity matrix by a scalar ($k\,R_i$).
- **(iii)** From the identity matrix by adding a row after multiplying it by a scalar, into another row

($R_i + k\,R_k$).

Thus if $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are elementary matrices obtained respectively (i) by adding R_2 of I in R_1 (ii) by multiplying R_3 of I by -3 (iii) by multiplying R_1 of I by -7 and adding the result in the R_2.

Consider the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 5 & -1 & -2 \\ 3 & 2 & 4 \end{bmatrix}$. Let B be a matrix obtained from A by interchanging first and third rows of A, that is, $B = \begin{bmatrix} 3 & 2 & 4 \\ 5 & -1 & -2 \\ 2 & -3 & 1 \end{bmatrix}$ and let E be a matrix obtained from I_3 by interchanging first and third rows of I_3, that is, $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. We notice that

$$EA = \begin{bmatrix} 3 & 2 & 4 \\ 5 & -1 & -2 \\ 2 & -3 & 1 \end{bmatrix} = B.$$

Thus a matrix B obtained from a matrix A through an elementary row operation can also be obtained by applying the same row operations on I and then multiplying the resultant elementary matrix E with A. It can be verified that the square matrix A is row equivalent to B if and only if there exists elementary matrices $E_1, E_2, \cdots, E_r$ such that $E_r \cdot E_{r-1} \cdots E_2 \cdot E_1 \cdot A = B$. If $B = I$, then $E_r \cdot E_{r-1} \cdots E_2 \cdot E_1 \cdot I = A^{-1}$.

Thus, we deduce that:

If a square matrix A is reducible to an identity matrix I by a sequence of elementary row operations then A^{-1} may be obtained by applying the same sequence of elementary row operations on I.

EXAMPLE 01: Find the inverse of the following matrices by using elementary row operations:

(i) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 1 \\ 1 & 3 & 0 \end{bmatrix}$, (ii) $\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$, (iii) $\begin{bmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$, (iv) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 7 \end{bmatrix}$.

(v) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

Solution:

(i)

Given matrix Identity matrix(I_3)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 1 \\ 1 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2+(-2)R_1,\ R_3+(-1)R_1$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & -5 \\ 0 & 3 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} R_2+(-1)R_3,\ \left(-\frac{1}{3}\right)R_3$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 1/3 & 0 & -1/3 \end{bmatrix} (-1)R_3$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 2/3 & -1 & 4/3 \end{bmatrix} R_1+(-3)R_3,\ R_2+2R_3$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 3 & -4 \\ 1/3 & -1 & 5/3 \\ 2/3 & -1 & 4/3 \end{bmatrix}$$

Hence the inverse of the given matrix is $A^{-1} = \begin{bmatrix} -1 & 3 & -4 \\ 1/3 & -1 & 5/3 \\ 2/3 & -1 & 4/3 \end{bmatrix}$. Verify $A\,A^{-1} = I$

(ii)

Given matrix Identity matrix(I_3)

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2+2R_1,\ R_3+4R_1$$

$$=\begin{bmatrix}-1 & 2 & -3\\ 0 & 5 & -6\\ 0 & 6 & -7\end{bmatrix} \qquad \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 4 & 0 & 1\end{bmatrix}(-1)R_1,\ R_3+(-1)R_2$$

$$=\begin{bmatrix}1 & -2 & 3\\ 0 & 5 & -6\\ 0 & 1 & -1\end{bmatrix} \qquad \begin{bmatrix}-1 & 0 & 0\\ 2 & 1 & 0\\ 2 & -1 & 1\end{bmatrix}R_{23}$$

$$=\begin{bmatrix}1 & -2 & 3\\ 0 & 1 & -1\\ 0 & 5 & -6\end{bmatrix} \qquad \begin{bmatrix}-1 & 0 & 0\\ 2 & -1 & 1\\ 2 & 1 & 0\end{bmatrix}R_1+2R_2,\ R_3+(-5)R_2$$

$$=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & -1\end{bmatrix} \qquad \begin{bmatrix}3 & -2 & 2\\ 2 & -1 & 1\\ -8 & 6 & -5\end{bmatrix}R_1+R_3,\ R_2+(-1)R_3,\ (-1)R_3$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \begin{bmatrix}-5 & 4 & -3\\ 10 & -7 & 6\\ 8 & -6 & 5\end{bmatrix}$$

Hence the inverse of the given matrix is $A^{-1}=\begin{bmatrix}-5 & 4 & -3\\ 10 & -7 & 6\\ 8 & -6 & 5\end{bmatrix}$.

Students are advised to verify that $AA^{-1}=A^{-1}A=I$.

$$\begin{array}{ccc} & \text{Given matrix} & \text{Identity matrix}(I_3)\\ \text{(iii)} & \begin{bmatrix}i & -1 & 2i\\ 2 & 0 & 2\\ -1 & 0 & 1\end{bmatrix} & \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}R_{13}\end{array}$$

$$=\begin{bmatrix}-1 & 0 & 1\\ 2 & 0 & 2\\ i & -1 & 2i\end{bmatrix} \qquad \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}(-1)R_1,\ R_{23}$$

$$=\begin{bmatrix}1 & 0 & -1\\ i & -1 & 2i\\ 2 & 0 & 2\end{bmatrix} \qquad \begin{bmatrix}0 & 0 & -1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}R_2+(-i)R_1,\ R_3+(-2)R_1$$

$$=\begin{bmatrix}1 & 0 & -1\\ 0 & -1 & 3i\\ 0 & 0 & 4\end{bmatrix} \qquad \begin{bmatrix}0 & 0 & -1\\ 1 & 0 & i\\ 0 & 1 & 2\end{bmatrix}\left(\frac{1}{4}\right)R_3,\ (-1)R_2$$

$$=\begin{bmatrix}1 & 0 & -1\\ 0 & -1 & -3i\\ 0 & 0 & 1\end{bmatrix} \qquad \begin{bmatrix}0 & 0 & -1\\ -1 & 0 & -i\\ 0 & 1/4 & 1/2\end{bmatrix}R_1+R_3,\ R_2+3iR_3$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \begin{bmatrix}0 & 1/4 & -1/2\\ -1 & 3/4i & 1/2i\\ 0 & 1/4 & 1/2\end{bmatrix}$$

Hence the inverse of the given matrix is $A^{-1} = \begin{bmatrix} 0 & 1/4 & -1/2 \\ -1 & 3/4i & 1/2i \\ 0 & 1/4 & 1/2 \end{bmatrix}$.

(iv)

Given matrix $\qquad$ Identity matrix$\left(I_5\right)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} R_5 + (-3)R_4$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 \end{bmatrix} R_4 + (-2)R_5$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & -2 \\ 0 & 0 & 0 & -3 & 1 \end{bmatrix}$$

Hence the inverse of the given matrix is $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & -2 \\ 0 & 0 & 0 & -3 & 1 \end{bmatrix}$.

(v)

Given Matrix $\qquad$ Identity Matrix

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_3 + (-2)R_1$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} R_{13}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} R_3 + (-2)R_1$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 & 1 \\ -5 & 1 & 2 \\ 5 & 0 & -2 \end{bmatrix} R_3 + (-1)R_2$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 & 1 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} R_2 + R_3$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

Hence the inverse of the given matrix is $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$.

Students are advised to verify that $A A^{-1} = A^{-1} A = I$.