Lecture # 18

CRAMER'S RULE

Gabriel Cramer (1704 - 1752) was a Swiss mathematician who *published* this rule in 1750. This rule is useful for finding the solution of n linear equations in n unknowns. Although the method to solve the system of non-homogeneous linear equations $Ax = b$, presented by Cramer is straight forward, nevertheless it is not recommended to use it if the system contains more than four variables and equations.

In section 3.1, we have discussed that the system of two linear equations

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}$$

has a solution if $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$ and the solution is given by

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta} \qquad (I)$$

where

$$\Delta_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } \Delta_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Further, we studied that the system of three linear equations

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned}$$

has a unique solution if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0,$ and then the solution is given by

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta} \qquad (II)$$

where

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

Similarly, if we have n linear equations in n unknowns such as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ \ldots\ldots\ldots\ \ldots\ldots\ldots\ \ldots\ldots\ldots\ \ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

and

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0,$$ then the system has a solution which

is given by

$$x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta}, \cdots, x_n = \frac{\Delta_n}{\Delta} \cdots\cdots\cdots\cdots (III)$$

where

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix},\ \Delta_2 = \begin{vmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & b_n & \dots & a_{nn} \end{vmatrix}, \dots\dots\dots\ \dots\dots,\ \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & b_n \end{vmatrix};$$

that is, Δ is the determinant of the matrix of the coefficients and Δ_j is obtained by replacing the j^{th} column of Δ by the column on R.H.S. of the linear system.

To find the solution of a system of linear equations by (I), (II) or (III) is called ***Cramer's Rule***.

NOTE: Sometimes we also use D in place of $\Delta \cdot$ Thus, if $\det A = D = \Delta,$ then

$$x_1 = \frac{D_n}{D},\ x_2 = \frac{D_2}{D}, \cdots, x_n = \frac{D_n}{D};$$

where

$$D_1 = \Delta_1,\ D_2 = \Delta_2 \text{ and } D_n = \Delta_n \cdot$$

EXAMPLE 01: Solve the following system of linear equations by Cramer's Rule:

(i) $\begin{aligned} \boldsymbol{3x - 5y = 2} \\ \boldsymbol{2x - 4y = 3} \end{aligned}$ **(ii)** $\begin{aligned} \boldsymbol{x_1 + x_2 + x_3 = 1} \\ \boldsymbol{2x_1 + 3x_2 + 4x_3 = 3} \\ \boldsymbol{4x_1 + 9x_2 + 10x_3 = 11} \end{aligned}$ **(iii)** $\begin{aligned} \boldsymbol{x_1 + x_2 + x_3 + x_4 = 6} \\ \boldsymbol{2x_1 \quad - x_3 - x_4 = 4} \\ \boldsymbol{3x_3 + 6x_4 = 3} \\ \boldsymbol{x_1 \quad\quad - x_4 = 5} \end{aligned}$

Solution: (i) Given that

$$3x - 5y = 2$$
$$2x - 4y = 3$$

Let

$$A = \begin{bmatrix} 3 & -5 \\ 2 & -4 \end{bmatrix} \cdot \text{Then } \det A = D = \begin{vmatrix} 3 & -5 \\ 2 & -4 \end{vmatrix} = -12 + 10 = -2 \neq 0$$

$$D_1 = \begin{vmatrix} 2 & -5 \\ 3 & -4 \end{vmatrix} = -8 + 15 = 7 \text{ and } D_2 = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5;$$

Therefore,

$$x = \frac{D_1}{D} = \frac{7}{-2} = -\frac{7}{2} \text{ and } y = \frac{D_2}{D} = \frac{5}{-2} = -\frac{5}{2} \Rightarrow x = -7/2 \ \ and \ \ y = -5/2$$

(ii) Given system of linear equation is

$$x_1 + x_2 + x_3 = 1$$
$$2x_1 + 3x_2 + 4x_3 = 3$$
$$4x_1 + 9x_2 + 10x_3 = 11$$

Here

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 10 \end{vmatrix} \quad R_2 + (-2)R_1,\ \ R_3 + (-4)R_1 \Rightarrow D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 5 & 6 \end{vmatrix} \text{(expanding by } C_1\text{)}$$

$$\Rightarrow D = \begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} \Rightarrow D = 6 - 10 = -4 \neq 0$$

$$\Rightarrow D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 11 & 9 & 10 \end{vmatrix} = 2\ ,\ D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 11 & 10 \end{vmatrix} = -8\ , \qquad D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 4 & 9 & 11 \end{vmatrix}$$

Now, we have

$$x_1 = \frac{D_1}{D} = \frac{2}{-4} = -\frac{1}{2},\ x_2 = \frac{D_2}{D} = \frac{-8}{-4} = 2, \text{ and } x_3 = \frac{D_3}{D} = \frac{2}{-4} = -\frac{1}{2}.$$

Hence, the required solution of the above system of linear equations is:

$$x_1 = -1/2,\ x_2 = 2, \text{ and } x_3 = -1/2$$

(iii) Given system of linear equation is

$$x_1 + x_2 + x_3 + x_4 = 6$$
$$2x_1 \quad - x_3 - x_4 = 4$$
$$3x_3 + 6x_4 = 3$$
$$x_1 \quad - x_4 = 5$$

Here, $D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & -1 & -1 \\ 0 & 0 & 3 & 6 \\ 1 & 0 & 0 & -1 \end{vmatrix}$. Expanding by C_2, we have

$D = (-)\begin{vmatrix} 2 & -1 & -1 \\ 0 & 3 & 6 \\ 1 & 0 & -1 \end{vmatrix}$. Taking 3 common from R_2, we get, $D = (-)(3)\begin{vmatrix} 2 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{vmatrix} R_{13}$

$$D = -3(-)\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 2 & -1 & -1 \end{vmatrix} R_3 + (-2)R_1$$

$$= 3\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix} \text{ expanding by } C_1$$

$$= 3\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \Rightarrow D = 3(1+2) = 9.$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 & 1 \\ 4 & 0 & -1 & -1 \\ 3 & 0 & 3 & 6 \\ 5 & 0 & 0 & -1 \end{vmatrix} = 30,\ D_2 = \begin{vmatrix} 1 & 6 & 1 & 1 \\ 2 & 4 & -1 & -1 \\ 0 & 3 & 3 & 6 \\ 1 & 5 & 0 & -1 \end{vmatrix} = 0,\ D_3 = \begin{vmatrix} 1 & 1 & 6 & 1 \\ 2 & 0 & 4 & -1 \\ 0 & 0 & 3 & 6 \\ 1 & 0 & 5 & -1 \end{vmatrix} = 39\ \ D_4 = \begin{vmatrix} 1 & 1 & 1 & 6 \\ 2 & 0 & -1 & 4 \\ 0 & 0 & 3 & 3 \\ 1 & 0 & 0 & 5 \end{vmatrix} =$$

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Now, we have

$$x_1 = \frac{D_1}{D} = \frac{30}{9} = \frac{10}{3},\ x_2 = \frac{D_2}{D} = \frac{0}{9} = 0, x_3 = \frac{D_3}{D} = \frac{39}{9} = \frac{13}{3} \quad \text{and} \quad x_4 = \frac{D_4}{D} = \frac{-15}{9} = -\frac{5}{3}.$$

Hence, the required solution of the above system of equations is:

x_1=10/3, x_2=0, x_3=13/3, x_4= - 5/3.

EXAMPLE 02: The Sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times the first number we get 12. Use determinants to find the numbers?

Solution: Let the three numbers be *x, y* and *z*. Then,, from the given conditions, we have

$$\begin{array}{ccccccc} x & + & y & + & z & = & 6 \\ x & & & + & 2z & = & 7 \\ 3x & + & y & + & z & = & 12 \end{array}.$$

Here,

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = (0-2)-(1-6)+(1-0) = 4.$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 7 & 0 & 2 \\ 12 & 1 & 1 \end{vmatrix} = 6\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 7 & 2 \\ 12 & 1 \end{vmatrix} + 1\begin{vmatrix} 7 & 0 \\ 12 & 1 \end{vmatrix} = 6(0-2)-(7-24)+(7-0) = 12.$$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 7 & 2 \\ 3 & 12 & 1 \end{vmatrix} = 1\begin{vmatrix} 7 & 2 \\ 12 & 1 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 7 \\ 3 & 12 \end{vmatrix} = 1(7-24)-6(1-6)+1(12-21) = 4.$$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 0 & 7 \\ 3 & 1 & 12 \end{vmatrix} = 1\begin{vmatrix} 0 & 7 \\ 1 & 12 \end{vmatrix} - 1\begin{vmatrix} 1 & 7 \\ 3 & 12 \end{vmatrix} + 6\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1(0-7)-1(12-21)+6(1-0) = 8.$$

Therefore,

$$x = \frac{D_1}{D} = \frac{12}{4} = 3,\ y = \frac{D_2}{D} = \frac{4}{4} = 1,\ z = \frac{D_3}{D} = \frac{8}{4} = 2.$$

Hence the required three numbers are: 3, 1 and 2.