Lecture # 5

ELEMENTARY ROW OPERATIONS

We can perform three types of operations on the rows (or column) of any matrix. These operations are known as elementary row (column) operations. These operations are:

(i) Multiplying a given row by a non-zero number. e. g 3 R1 = R1
(ii) Interchanging any two rows of a matrix. e.g. R1 $\leftrightarrow$ R2 or R12
(iii) Addition of any multiple of one row to another row. e.g. R2 – 3 R1 = R2

Explanation of point (iii) is that we may multiply any row of a matrix by a non-zero number and the result so obtained may be added to any other row.

Notations: We use the following notations for the three types of elementary row operations **(ERO)** as stated above.

(i) When a row R_i is multiplied by a non-zero number k, it is denoted by kR_i.

(ii) Interchanging the i^{th} row with j^{th} row is denoted by R_{ij}.

(iii) When the row R_j is multiplied by a non-zero number k and the result is added to row R_i ,it is denoted or shown by $R_i + kR_j$.

1.8.1 Row Equivalent Matrices

Two matrices A and B are called row equivalent matrices, written as $A \overset{R}{\sim} B$, if one can be obtained from the other by performing a finite sequence of elementary row operations. For example, let

$$A = \begin{bmatrix} 1 & 5 & 2 & 3 \\ 3 & 1 & 8 & -1 \\ 2 & 5 & 1 & 6 \end{bmatrix}$$

Then, if we multiply R_2 of matrix A by 3, (denoted by 3 R_2) we get a new matrix say B, given by:
New R2 = 3 old R2 = 3 (318-1) = (9324-3)

$$B = \begin{bmatrix} 1 & 5 & 2 & 3 \\ 9 & 3 & 24 & -3 \\ 2 & 5 & 1 & 6 \end{bmatrix} \sim \text{A}$$

We say that "matrix B is Row Equivalent to matrix A" and write $B \overset{R}{\sim} A$.
Similarly, if we interchange R_1 and R_3 of matrix A, (denoted by R_{13}) we get:
R1 $\leftrightarrow$ R3

$$\text{A} \sim C = \begin{bmatrix} 2 & 5 & 1 & 6 \\ 3 & 1 & 8 & -1 \\ 1 & 5 & 2 & 3 \end{bmatrix}$$

We say that "matrix C is Row Equivalent to matrix A" and write $C \overset{R}{\sim} A$.
Finally, if R_1 is multiplied by say 4 and the result is added into R_2, (denoted by $R_2 + 4R_1$), we get New R2 = old $R_2 + 4R_1$= (3 1 8 -1) + 4(1 5 2 3) = 3+4 1+20 8+8 -1+12 = 7 21 16 11

$$\text{A} \sim D = \begin{bmatrix} 1 & 5 & 2 & 3 \\ 7 & 21 & 16 & 11 \\ 2 & 5 & 1 & 6 \end{bmatrix}$$

It may be noted that row R_1 will remain same and only the row R_2 will be changed as the result obtained by multiplying the row R_1 is added into the row R_2. We say that "matrix D is Row Equivalent to matrix A" and write $D \overset{R}{\sim} A$.

1.8.2 Echelon and Reduced Echelon Matrices

An $m \times n$ matrix A is said to be an *echelon matrix* or *in echelon form* if it has the following structure:

- **(i)** The first element (if any) in each row is a non-zero element.
- **(ii)** Below this non-zero element all other elements in that column are zero.
- **(iii)** The number of zero's in each row is greater than the number of zero's in the preceding row.

For example,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} -2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $E = \begin{bmatrix} 0 & 5 & 2 & 3 \\ 0 & 0 & 8 & -1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ are echelon matrices, whereas the matrices,

$F = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, and $H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ are not echelon matrices.

The method of transforming a given matrix into a row equivalent echelon matrix by elementary row operations is also referred to as *reduction into echelon form.*

When a matrix is transformed into an echelon form, the first non-zero element in each row is called the ***leading or pivot element***. A column containing a pivot is called a ***pivot column***

If an echelon matrix has the additional property that each pivot is 1 and every other entry of the pivot column is zero, then the matrix is called ***REDUCED ECHELON MATRIX***. For example, the following matrices are Reduced Echelon Matrices.

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

EXAMPLE 01: Transform the matrix $A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & -7 \end{bmatrix}$

into an echelon matrix and then into reduced echelon matrix.

Solution: Given $A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & -7 \end{bmatrix} R_2 - 2R_1, R_3 - 2R_1$

$$A \approx \begin{bmatrix} 1 & 4 & -2 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} R_3 + R_2 \approx \begin{bmatrix} 1 & 4 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \text{ Rank} = 2$$

Rank of a matrix is the number of non zero rows (a row is called zero row if all elements of the row are zero) in echelon form

which is the required echelon matrix.

Now, $A \approx \begin{bmatrix} 1 & 4 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} (-1)R_2 \approx \begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} R_1 + (-4)R_2 \approx \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix},$

which is the desired reduced echelon matrix.

EXAMPLE 02: Transform the matrix $A = \begin{bmatrix} -1 & 3 & 0 & 1 & 4 \\ 1 & -3 & 0 & 0 & -1 \\ 2 & -6 & 2 & 4 & 0 \\ 0 & 0 & 1 & 3 & -4 \end{bmatrix}$ **into reduced echelon matrix.**

Solution: Given matrix is $A = \begin{bmatrix} -1 & 3 & 0 & 1 & 4 \\ 1 & -3 & 0 & 0 & -1 \\ 2 & -6 & 2 & 4 & 0 \\ 0 & 0 & 1 & 3 & -4 \end{bmatrix} \left(\frac{1}{2}\right) R_3$

$$\approx \begin{bmatrix} -1 & 3 & 0 & 1 & 4 \\ 1 & -3 & 0 & 0 & -1 \\ 1 & -3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -4 \end{bmatrix} R_{12} \qquad \approx \begin{bmatrix} 1 & -3 & 0 & 0 & -1 \\ -1 & 3 & 0 & 1 & 4 \\ 1 & -3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -4 \end{bmatrix} R_2 + R_1, R_3 - R_1$$

$$\approx\begin{bmatrix}1 & -3 & 0 & 0 & -1\\0 & 0 & 0 & 1 & 3\\0 & 0 & 1 & 2 & 1\\0 & 0 & 1 & 3 & -4\end{bmatrix}R_{24}\approx\begin{bmatrix}1 & -3 & 0 & 0 & -1\\0 & 0 & 1 & 3 & -4\\0 & 0 & 1 & 2 & 1\\0 & 0 & 0 & 1 & 3\end{bmatrix}R_3-R_2$$

$$\approx\begin{bmatrix}1 & -3 & 0 & 0 & -1\\0 & 0 & 1 & 3 & -4\\0 & 0 & 0 & -1 & 5\\0 & 0 & 0 & 1 & 3\end{bmatrix}R_4+R_3\approx\begin{bmatrix}1 & -3 & 0 & 0 & -1\\0 & 0 & 1 & 3 & -4\\0 & 0 & 0 & -1 & 5\\0 & 0 & 0 & 0 & 8\end{bmatrix}R_2+3R_3,\left(\frac{1}{8}\right)R_4$$

$$\approx\begin{bmatrix}1 & -3 & 0 & 0 & -1\\0 & 0 & 1 & 0 & 11\\0 & 0 & 0 & -1 & 5\\0 & 0 & 0 & 0 & 1\end{bmatrix}(-1)R_3$$

$$\approx\begin{bmatrix}1 & -3 & 0 & 0 & -1\\0 & 0 & 1 & 0 & 11\\0 & 0 & 0 & 1 & -5\\0 & 0 & 0 & 0 & 1\end{bmatrix}R_1+R_4, R_2+(-11)R_4, R_3+5R_4\approx\begin{bmatrix}1 & -3 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix},$$

which is the required reduced echelon matrix.

EXAMPLE 03: Transform the following matrices into (i) echelon form and (ii) reduced echelon form:

(a) $A=\begin{bmatrix}1 & 2 & 8\\1 & 0 & 2\\0 & 0 & 0\end{bmatrix}$ **(b)** $B=\begin{bmatrix}1 & -2 & 3 & 4\\3 & 2 & -1 & -3\\4 & 1 & 1 & 4\end{bmatrix}$ **(c)** $C=\begin{bmatrix}1 & 3 & 0 & 2 & 5\\0 & 0 & 0 & 0 & 0\\0 & 0 & 2 & 2 & 6\end{bmatrix}$

Solution: (a) Given matrix is $A=\begin{bmatrix}1 & 2 & 8\\1 & 0 & 2\\0 & 0 & 0\end{bmatrix}R_2-R_1\approx\begin{bmatrix}1 & 2 & 8\\0 & -2 & -6\\0 & 0 & 0\end{bmatrix}$. This is in Echelon form.

Furthermore,

$$A\approx\begin{bmatrix}1 & 2 & 8\\0 & -2 & -6\\0 & 0 & 0\end{bmatrix}\left(-\frac{1}{2}\right)R_2\qquad\approx\begin{bmatrix}1 & 2 & 8\\0 & 1 & 3\\0 & 0 & 0\end{bmatrix}R_1-2R_2\qquad\approx\begin{bmatrix}1 & 0 & 2\\0 & 1 & 3\\0 & 0 & 0\end{bmatrix}.$$

This is in Reduced Echelon form.

(b) $B=\begin{bmatrix}1 & -2 & 3 & 4\\3 & 2 & -1 & -3\\4 & 1 & 1 & 4\end{bmatrix}R_2+(-3)R_1, R_3+(-4)R_1$

$$\approx\begin{bmatrix}1 & -2 & 3 & 4\\0 & 8 & -10 & -15\\0 & 9 & -11 & -12\end{bmatrix}R_3+(-1)R_2\qquad\approx\begin{bmatrix}1 & -2 & 3 & 4\\0 & 8 & -10 & -15\\0 & 1 & -1 & 3\end{bmatrix}R_{23}$$

$$\approx \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 8 & -10 & -15 \end{bmatrix} R_3 + (-8)R_2 \qquad \approx \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -2 & -39 \end{bmatrix}.$$

This is in Echelon form.

Furthermore, $B \approx \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -2 & -39 \end{bmatrix} R_1 + 2R_2, \left(-\frac{1}{2}\right)R_3 \quad \approx \begin{bmatrix} 1 & 0 & 1 & 10 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 39/2 \end{bmatrix} R_1 + (-1)R_3, R_2 + R_3$

$$\approx \begin{bmatrix} 1 & 0 & 0 & -19/2 \\ 0 & 1 & 0 & 45/2 \\ 0 & 0 & 1 & 39/2 \end{bmatrix}.$$

This is in Reduced Echelon Form.

(c) $C = \begin{bmatrix} 1 & 3 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 6 \end{bmatrix} R_{23}$

$\approx \begin{bmatrix} 1 & 3 & 0 & 2 & 5 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. This is in Echelon form.

Furthermore,

$$C \approx \begin{bmatrix} 1 & 3 & 0 & 2 & 5 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \left(\frac{1}{2}\right)R_2 \qquad \approx \begin{bmatrix} 1 & 3 & 0 & 2 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is in Reduced Echelon form.

EXAMPLE 04: Show that $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \end{bmatrix} \approx \mathbf{I_3}$.

Solution: We have

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} R_2 + (-3)R_1 \qquad \approx \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 1 & 2 \end{bmatrix} R_{23}$$

$$\approx \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -5 & -1 \end{bmatrix} R_1 + (-2)R_2,\ R_3 + 5R_2 \qquad \approx \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 9 \end{bmatrix} \left(\frac{1}{9}\right)R_3$$

$$\approx \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} R_1 + 3R_3, R_2 + (-2)R_3 \qquad \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence the given matrix has been reduced to the identity matrix.