**Lecture # 14**

# DETERMINANTS

## INTRODUCTION

Determinant is an important concept in matrix algebra. If a matrix is square, its elements are confined to compute a real number called the determinant.
Determinant is a function from vector to scalar => Dom(vector), Range(scalar)
Determinants have their origin in the solution of system of linear equations with two or more unknowns when $m = n$.

**A determinant of order three is represented and defined as:**

$$|\mathbf{A}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = \mathbf{d}$$

We conclude that to every square matrix $A$, there corresponds a real number, called the determinant of the matrix $A$ which is symbolically denoted by $\det(A)$ or $|A|\cdot$

**Definition:** The determinant of a square matrix $A$ of order $n$ (with real or complex elements) is a real or complex number associated with matrix $A$ of order $n$.

**Definition:** A determinant is a function whose domain is the set of all square matrices of order $n$ and whose range is a subset of real or complex numbers. That is

$$F : M \longrightarrow D$$

Here $M$ is the set of all square matrices and $D$ is the set of numbers (real or complex).

The converse is not true, that is, there are many matrices whose determinant may be same. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \text{etc,}$$

are different matrices but their determinant is zero. Thus determinant is many to one function.

Summarizing the above results, we see that if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

then $$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \qquad (1)$$

and if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{21} & a_{32} \end{vmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{21}) \qquad (2)$$

The expressions on the right of (1) and (2) are called "an expansion of the determinant with respect to the first row".

Every square matrix has a unique real value called Determinant. But every real does have a unique matrix

e.g: A = 2 3<br>
4 5 |A| = 2x5 – 3x4 = -2<br>
6 = 1 0 = 2 4 = 0 1<br>
5 6 5 13 -6 3

**3.1.2 Difference Between a Matrix and a Determinant**

- **(i)** A matrix cannot be reduced to a single number. A determinant can be reduced to a single number.
- **(ii)** In a matrix the number of rows may or may not be equal to the number of columns. In a determinant the number of rows must be equal to the number of columns.
- **(iii)** An interchange of rows (columns) gives a different matrix. An interchange of rows (columns) gives the same determinant with opposite sign.

**EXAMPLE 02: If**

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 3 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}, \text{ then find } |A|.$$

**Solution:**

Expanding by first row, we get

$$|A| = 4\begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 4(6-10) - (9-5) + 2(6-2) = -16 - 4 + 8 = -12 \cdot$$