Lecture # 16

MINORS and COFACTORS

Let A be a square matrix of order n. A determinant of order $(n - 1)$ obtained from A by deleting its i^{th} row and j^{th} column is called a ***Minor*** corresponding to a_{ij}. For instance, consider the matrix of order 3:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then}$$

$$M_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ M_{21}=\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix},\ M_{31}=\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, etc$$

are the minors corresponding the elements a_{11}, a_{21}, and a_{31} respectively.

Similarly, if A is a square matrix of order n, the ***cofactor*** of an element a_{ij}, denoted by A_{ij}, is defined as

$A_{ij}=(-1)^{i+j}M_{ij}$, where M_{ij} is the corresponding minor.

EXAMPLE 01: Find minors and cofactors of each element of the following matrix

$$\boldsymbol{A}=\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} \end{bmatrix}.$$

Solution: Let M_{ij} denote the minor of the element a_{ij}. Then

$$M_{11}=\begin{vmatrix} 3 & -1 \\ 0 & 5 \end{vmatrix}=15-0=15,\quad M_{12}=\begin{vmatrix} -2 & -1 \\ 0 & 5 \end{vmatrix}=-10-0=-10,$$

$$M_{13}=\begin{vmatrix} -2 & 3 \\ 0 & 0 \end{vmatrix}=0-0=0,\quad M_{21}=\begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix}=10-0=10,$$

$$M_{22}=\begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix}=5-0=5,\quad M_{23}=\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}=0-0=0,$$

$$M_{31}=\begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix}=-2-0=-2,\ M_{32}=\begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix}=-1-0=-1,$$

$$M_{33}=\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}=3+4=7\cdot$$

Now using the definition of cofactors, we have

$$A_{11}=(-1)^{1+1}M_{11}=15,\ A_{12}=(-1)^{1+2}M_{12}=10,$$

$$A_{13}=(-1)^{1+3}M_{13}=0,\ A_{21}=(-1)^{2+1}M_{21}=-10,$$

$$A_{22}=(-1)^{2+2}M_{22}=5,\ A_{23}=(-1)^{2+3}M_{23}=1,$$

$$A_{31}=(-1)^{3+1}M_{31}=-2,\ A_{32}=(-1)^{3+2}M_{32}=-1,\ C_{33}=(-1)^{3+3}M_{33}=7\cdot$$

Definition: For the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

of order n, we define det (A) by

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij} \qquad \forall\, i \qquad (1)$$

The expression on the right hand side of (1) is called an expansion of det (A) by cofactors. It is also known as an ***"Axiomatic definition of determinant"*** or "*Laplacian Expansion*".

EXAMPLE: Compute the determinant of

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{3} \\ \mathbf{5} & \mathbf{-7} & \mathbf{4} \\ \mathbf{1} & \mathbf{6} & \mathbf{-2} \end{bmatrix}.$$

Solution: We expand the matrix by the second row. Then

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = 5A_{21} + (-7)A_{22} + 4A_{23}.$$

Now,

$$A_{21} = (-1)^{2+1}M_{21} = -\begin{vmatrix} -1 & 3 \\ 6 & -2 \end{vmatrix} = -(2-18) = 16,\ A_{22} = (-1)^{2+2}M_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4-3 = -7, \text{ and}$$

$$A_{23} = (-1)^{2+3}M_{23} = -\begin{vmatrix} 2 & -1 \\ 1 & 6 \end{vmatrix} = -(12+1) = -13.$$

Thus, we have

$$|A| = 5(16) + (-7)(-7) + 4(-13) = 80 + 49 - 52 = 77.$$

Similarly, we can expand it by 3rd row or 1st row as well.

Adjugate or Adjoint of a Square Matrix

The adjoint of a square matrix A, denoted by $adj(A)$, is the transpose of the matrix formed by the cofactors of elements of A. Symbolically,

$$adj(A) = [A_{ij}]^t = [A_{ji}].$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

The matrix of the cofactors of the above matrix becomes

$$[A_{ij}]=\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}$$

or, $$adj(A)=[A_{ij}]^t=[A_{ji}]=\begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}.$$

EXAMPLE 12: Find the adjoint of the following square matrices

(i) $\boldsymbol{A}=\begin{bmatrix} \boldsymbol{1} & \boldsymbol{1} \\ \boldsymbol{3} & \boldsymbol{5} \end{bmatrix}$ **(ii)** $\boldsymbol{A}=\begin{bmatrix} \boldsymbol{1} & \boldsymbol{3} & \boldsymbol{5} \\ \boldsymbol{2} & \boldsymbol{4} & \boldsymbol{7} \\ \boldsymbol{9} & \boldsymbol{8} & \boldsymbol{6} \end{bmatrix}.$

Solution: (i) $$adjA=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^t=\begin{bmatrix} 5 & -3 \\ -1 & 1 \end{bmatrix}^t=\begin{bmatrix} 5 & -1 \\ -3 & 1 \end{bmatrix}.$$

Thus it is easy to find the adjoint of any square matrix of order 2. What we have to do is simply to keep the main diagonal elements unchanged and the remaining two elements are to be interchanged with each other. In addition to this, change the signs of these elements as well.

(ii) We have

$$\boldsymbol{A}=\begin{bmatrix} \boldsymbol{1} & \boldsymbol{3} & \boldsymbol{5} \\ \boldsymbol{2} & \boldsymbol{4} & \boldsymbol{7} \\ \boldsymbol{9} & \boldsymbol{8} & \boldsymbol{6} \end{bmatrix}$$

Now,

$$A_{11}=(-1)^{1+1}M_{11}=+\begin{vmatrix} 4 & 7 \\ 8 & 6 \end{vmatrix}=24-56=-32, \qquad A_{12}=(-1)^{1+2}M_{12}=-\begin{vmatrix} 2 & 7 \\ 9 & 6 \end{vmatrix}=(12-63)=51,$$

$$A_{13}=(-1)^{1+3}M_{13}=+\begin{vmatrix} 2 & 4 \\ 9 & 8 \end{vmatrix}=16-36=-20, \qquad A_{21}=(-1)^{2+1}M_{21}=-\begin{vmatrix} 3 & 5 \\ 8 & 6 \end{vmatrix}=-(18-40)=22,$$

$$A_{22}=(-1)^{2+2}M_{22}=+\begin{vmatrix} 1 & 5 \\ 9 & 6 \end{vmatrix}=6-45=-39, \qquad A_{23}=(-1)^{2+3}M_{23}=-\begin{vmatrix} 1 & 3 \\ 9 & 8 \end{vmatrix}=-(8-27)=19,$$

$$A_{31}=(-1)^{3+1}M_{31}=+\begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix}=21-20=1, \quad A_{32}=(-1)^{3+2}M_{32}=-\begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix}=-(7-10)=3,$$

$$A_{33}=(-1)^{3+3}M_{33}=+\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}=4-6=-2\cdot$$

Hence,

$$adjA = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t = \begin{bmatrix} -32 & 51 & -20 \\ 22 & -39 & 19 \\ 1 & 3 & -2 \end{bmatrix}^t = \begin{bmatrix} -32 & 22 & 1 \\ 51 & -39 & 3 \\ -20 & 19 & -2 \end{bmatrix}\cdot$$

Multiplicative Inverse of a Square Matrix by Adjoint Matrix Method

Let A be a square matrix of order n, if there exists a square matrix B of the same order n such that $AB = BA = I_n$, then B is called an inverse of A and is written as $B = A^{-1}\cdot$

Note:

(i) If A is a non-singular matrix, then $|A| \neq 0\cdot$

(ii) $\left(A^{-1}\right)^{-1} = A,$ that is, the inverse of the inverse of a matrix A is A itself.

(iii) $A^{-1}A = AA^{-1} = I_n\cdot$

Theorem: If A is non-singular matrix, then inverse of A is given by $\boldsymbol{A^{-1} = \dfrac{adjA}{|A|}}$**.**

EXAMPLE: Find by adjoint method the inverse of the matrix

$$\boldsymbol{A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{bmatrix}}\cdot$$

Solution: Here

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \\ 0 & 5 & -8 \end{vmatrix} R_2 - 2R_1,\ R_3 - 3R_1$$

Expanding by first column, we have

$$|A| = \begin{vmatrix} 3 & -5 \\ 5 & -8 \end{vmatrix} = -24 + 25 = 1 \neq 0\cdot$$

Hence A^{-1} exists.

Now,

$A_{11} = (-1)^{1+1}M_{11} = +\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} = -2 + 2 = 0,$ $\quad A_{12} = (-1)^{1+2}M_{12} = -\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} = -(-4+3) = 1,$

$A_{13} = (-1)^{1+3}M_{13} = +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1,$ $\quad A_{21} = (-1)^{2+1}M_{21} = -\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} = -(2-4) = 2,$

$A_{22} = (-1)^{2+2}M_{22} = +\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -2 - 6 = -8,$ $\quad A_{23} = (-1)^{2+3}M_{23} = -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = -(2+3) = -5,$

$A_{31} = (-1)^{3+1}M_{31} = +\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = 1 - 2 = -1,$ $\quad A_{32} = (-1)^{3+2}M_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -(-1-4) = 5,$

$A_{33} = (-1)^{3+3}M_{33} = +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1 + 2 = 3\cdot$

$$\therefore adjA = \begin{bmatrix} 0 & 1 & 1 \\ 2 & -8 & -5 \\ -1 & 5 & 3 \end{bmatrix}^t = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{bmatrix}\cdot \Rightarrow A^{-1} = \frac{adjA}{|A|} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{bmatrix}, \qquad [\text{since}, |A| = 1]\cdot$$