Lecture # 15

Properties of Determinants

Finding the value of a determinant by expanding it or by any other method is called ***evaluation*** of the determinant. The evaluation of a determinant of order greater than 3 is a tedious task. However the use of certain ***properties*** of determinants makes calculation much easier. These properties are discussed below:

1. If A is any square matrix, then $\det(A) = \det(A^t)$. This means that the determinant of the square matrix has the same value as that of its transpose.

For, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \; then \; A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

➔ $$|A| = ad - bc = |A^t|$$

2. If B is the matrix obtained by interchanging any two rows of the square matrix A, then $\det(B) = -\det(A)$.

Similarly, if a matrix C is obtained by interchanging two columns of matrix A, then $\det(C) = -\det(A)$

For example, let,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; B = \begin{bmatrix} c & d \\ a & b \end{bmatrix} and \; C = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Here the matrix B is obtained by interchanging the 1st and 2nd rows of the matrix A. Similarly, matrix C is obtained by interchanging the 1st and 2nd columns of the matrix A.
[
Now,

$$\det(A) = ad - bc, \; \det(B) = bc - ad = -(ad - bc) = \det(C).$$

Thus,

$$\det(B) = -\det(A) \text{ and } \det(C) = -\det(A).$$

3. If two rows or two columns of a square matrix A are identical, then $\det(A) = 0$.

For, let

$A = \begin{bmatrix} a & b \\ a & b \end{bmatrix} \; then \; |A| = ab - ab = 0$. [Here two rows of the matrix A are equal]

Similarly, let

$A = \begin{bmatrix} a & a \\ b & b \end{bmatrix} \; then \; |A| = ab - ab = 0$. [Here two columns of the matrix A are equal]

4. If all entries of a row or a column of a square matrix are zero, then $\det(A) = 0$.

For, let

$$A=\begin{bmatrix}0 & 0\\ a & b\end{bmatrix} \text{ then } |A|=0.b-0.a=0$$

Similarly, all entries in a column of matrix *A* are zero, its determinant will be zero.

5. If *A* is a matrix of order *n* and *k* is any scalar, then $\det(kA)=k^n\det(A)\cdot$

For, let

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix} \text{ then } kA=\begin{bmatrix}ka & kc\\ kb & kd\end{bmatrix}$$ [Note: Matrix *A* is of order 2]

$$\Rightarrow \det(kA)=k^2ad-k^2bc=k^2(ad-bc)=k^2\det(A)$$

6. Let *B* be a square matrix obtained by multiplying any row or column of square matrix *A* by a non-zero number *k* and the result so obtained is added into another row or column respectively, then det(*B*) = *k* det(*A*).

For example, let

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix} \text{ and } B=\begin{bmatrix}ka & kc\\ b & d\end{bmatrix} \text{ then,}$$

det(*A*) = *ad* – *bc* and det(*B*) = *kad* – *kbc* = *k* (*ad* – *bc*) = *k* det(*A*)

7. If each element in a row or column of a determinant is the sum of two terms then the determinant can be expressed as the sum of two determinants.

For example, let

$$A=\begin{bmatrix}a+w & b\\ c+z & d\end{bmatrix}$$

Then,

$$\det\begin{bmatrix}a+w & b\\ c+z & d\end{bmatrix}=\det\begin{bmatrix}a & b\\ c & d\end{bmatrix}+\det\begin{bmatrix}w & b\\ z & d\end{bmatrix}$$

$$d(a+w)-b(c+z)=ad-bc+dw-bz$$
$$ad+dw-bc-bz=ad-bc+dw-bz$$

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8. If *A* be a diagonal or triangular matrix, then determinant of *A* is simply the product of diagonal elements.

For example, if

$$A=\begin{bmatrix}a & 0\\ c & d\end{bmatrix}, B=\begin{bmatrix}a & b\\ 0 & d\end{bmatrix}, C=\begin{bmatrix}a & 0\\ 0 & d\end{bmatrix}$$ then, det(*A*) = det(*B*) = det(*C*) = *a d*

9. The determinant of identity matrix is 1.

For example, if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then } |A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1-0) - 0 + 0 = 1 \cdot$$

NOTE: This is the reason that an Identity matrix is also called a unit matrix by some mathematicians. But we have said in chapter 2 that identity and unit matrices are two different concepts.

NOTE: Readers who are interested to see the proofs of the above properties may see "Mathematical Methods" By S. M. Yousuf.

EXAMPLE 03: If

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{1} & -\mathbf{4} \\ \mathbf{4} & \mathbf{5} & \mathbf{1} & \mathbf{2} \\ -\mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{5} \end{bmatrix}, \textbf{ find } |A|.$$

Solution: Expanding by first row, we get

$$|A| = 3\begin{vmatrix} 5 & 1 & 2 \\ 3 & 0 & 1 \\ 1 & 3 & 5 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 & 2 \\ -2 & 0 & 1 \\ 2 & 3 & 5 \end{vmatrix} + 1\begin{vmatrix} 4 & 5 & 2 \\ -2 & 3 & 1 \\ 2 & 1 & 5 \end{vmatrix} - (-4)\begin{vmatrix} 4 & 5 & 1 \\ -2 & 3 & 0 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= 3[5(0-3) - 1(15-1) + 2(9-0)] - 2[4(0-3) - 1(-10-2) + 2(-10-0)] +$$
$$[4(15-1) - 5(-10-2) + 2(-2-6)] + 4[4(9-0) - 5(-6-0) + 1(-2-6)] = 323$$

3.1.3 Singular and Non-singular Matrices

The matrix A is called ***singular matrix*** if $|A| = 0$ and ***non-singular*** if $|A| \neq 0 \cdot$

EXAMPLE 04: Find the value of *m* if the following matrix *A* is singular.

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & -\mathbf{3} & \mathbf{5} \\ \mathbf{1/2} & \mathbf{4} & \mathbf{6} \\ \boldsymbol{m} & \mathbf{0} & \mathbf{8} \end{bmatrix} \cdot$$

Solution: Expanding by the first row, we get

$$|A| = 2\begin{vmatrix} 4 & 6 \\ 0 & 8 \end{vmatrix} - (-3)\begin{vmatrix} 1/2 & 6 \\ m & 8 \end{vmatrix} + 5\begin{vmatrix} 1/2 & 4 \\ m & 0 \end{vmatrix}$$

$$= 2(32-0) + 3(4-6m) + 5(0-4m) = 64 + 12 - 18m - 20m = 76 - 38m$$

Since A is a singular matrix, therefore

$$|A| = 0 \quad \Rightarrow 76 - 38m = 0 \Rightarrow m = \frac{76}{38} = 2 \cdot$$