Lecture # 12

CONSISTENCY CRITERIAN

A system of Linear algebraic equations AX = B orAX = 0 is called consistent if it has a solution. Otherwise called inconsistent. For example

1. Consistency Criteria for the Non-Homogeneous System of Linear Equations

(a) When the system of linear equations $A\,x = b$ is a square system and

i. The Rank of A = Rank of $A_b = n$ (where n is number of unknowns or equations) then the system has unique solution.
ii. The Rank of A = Rank $A_b < n$ then the system has infinite solutions.
iii. The Rank $A \neq$ Rank$A_b$the system is said to be inconsistent and hence has no solution.

(b) When the system of linear equations $A\,x = b$ is rectangular system and

i. The Rank of A = Rank of A_b, then the system has an infinite number of solutions.
ii. The Rank $A \neq$ Rank A_b, then the system is said to be inconsistent and hence has no solution.

2. Consistency Criteria for Homogenous System of Linear Equations

(a) When the homogeneous system $A\,x = O$ is a square system and

i. The Rank of matrix $A = n$, the system has only a trivial/zero solution.
ii. The Rank $A < n$ then the system has infinite many solutions.

(b) When the homogeneous system $A\,x = O$ is a rectangular system and

i. The Rank of $A = n$, where n is the number of unknowns or variables, then the system has an infinite solutions.

The following flowchart shows the systematic "*Consistency Criterion*" for the system of linear equations.

EXAMPLE 01: Show that the system of equations $2x+6y=-11,\ 6x+20y-6z=-3,\ 6y-18z=-1$ **is not consistent (*m* = *n*).**

Solution: The augmented matrix for the above system of equations is

$$A_b=\begin{bmatrix}2 & 6 & 0 & -11\\ 6 & 20 & -6 & -3\\ 0 & 6 & -18 & -1\end{bmatrix}.$$

Now we reduce the matrix into echelon form using elementary row operations:

$$A_b=\begin{bmatrix}2 & 6 & 0 & -11\\ 6 & 20 & -6 & -3\\ 0 & 6 & -18 & -1\end{bmatrix}R_2+(-3)R_1\approx\begin{bmatrix}2 & 6 & 0 & -11\\ 0 & 2 & -6 & 30\\ 0 & 6 & -18 & -1\end{bmatrix}\left(-\frac{1}{2}\right)R_2$$

$$\approx \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 1 & -3 & 15 \\ 0 & 6 & -18 & -1 \end{bmatrix} R_3 + (-6)R_2 \approx \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 1 & -3 & 15 \\ 0 & 0 & 0 & -91 \end{bmatrix}.$$

From the last matrix we can easily observe that $\text{Rank } A = 2$ and $\text{Rank } A_b = 3.$

$\because \text{Rank } A \neq \text{Rank } A_b$, (Since 0x + 0 y + 0 z = -91, that is, 0 = - 91 is not possible.)

Therefore the given equations are "inconsistent".

EXAMPLE 02: Discuss the consistency of the following system of equations

$$\boldsymbol{2x+3y+4z=11,\ \ x+5y+7z=15,\ \ 3x+11y+13z=25.}$$

If found consistent, solve it (*m* = *n*).

Solution: The augmented for the above system of equations is:

$$A_b = \begin{bmatrix} 2 & 3 & 4 & 11 \\ 1 & 5 & 7 & 15 \\ 3 & 11 & 13 & 25 \end{bmatrix}.$$

Now we reduce the matrix into echelon form using elementary row operations:

$$A_b = \begin{bmatrix} 2 & 3 & 4 & 11 \\ 1 & 5 & 7 & 15 \\ 3 & 11 & 13 & 25 \end{bmatrix} R_{12}$$

$$\approx \begin{bmatrix} 1 & 5 & 7 & 15 \\ 2 & 3 & 4 & 11 \\ 3 & 11 & 13 & 25 \end{bmatrix} R_2 + (-2)R_1,\ R_3 + (-3)R_1$$

$$\approx \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & -4 & -8 & -20 \end{bmatrix} (-1)R_2,\ \left(-\frac{1}{4}\right)R_3$$

$$\approx \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & 7 & 10 & 19 \\ 0 & 1 & 2 & 5 \end{bmatrix} R_{23} \approx \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & 1 & 2 & 5 \\ 0 & 7 & 10 & 19 \end{bmatrix} R_3 + (-7)R_2$$

$$\approx \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -4 & -16 \end{bmatrix} \left(-\frac{1}{4}\right) R_3 \approx \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

From the last matrix we can easily observe that $\text{Rank } A = \text{Rank } A_b = 3 \cdot$ Therefore the given system is consistent and has a unique solution. Also, we have

$$\begin{aligned} x + 5y + 7z &= 15 && (i) \\ y + 2z &= 5 && (ii) \\ z &= 4 && (iii) \end{aligned}$$

Solving, we have

$x = 2$, $y = -3$, $z = 4$

as a solution of the given system.

EXAMPLE 03: Test for consistency and solve the following system of linear equations

$\boldsymbol{5x + 3y + 7z = 4,\ 3x + 26y + 2z = 9,\ 7x + 2y + 10z = 5}$ **$(m = n)$**

Solution: The augmented for the above system of equations is:

$$A_b = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}.$$

Now we reduce the matrix into echelon form using elementary row operations:

$$A_b = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix} R_3 + (-2)R_2 \approx \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 1 & -50 & 6 & -13 \end{bmatrix} R_{13}$$

$$\approx \begin{bmatrix} 1 & -50 & 6 & -13 \\ 3 & 26 & 2 & 9 \\ 5 & 3 & 7 & 4 \end{bmatrix} R_2 + (-3)R_1, R_3 + (-5)R_1$$

$$\approx \begin{bmatrix} 1 & -50 & 6 & -13 \\ 0 & 176 & -16 & 48 \\ 0 & 253 & -23 & 69 \end{bmatrix} \left(\frac{1}{16}\right) R_2, \left(\frac{1}{23}\right) R_3$$

$$\approx \begin{bmatrix} 1 & -50 & 6 & -13 \\ 0 & 11 & -1 & 3 \\ 0 & 11 & -1 & 3 \end{bmatrix} R_3 + (-1)R_2 \approx \begin{bmatrix} 1 & -50 & 6 & -13 \\ 0 & 11 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the last matrix we can easily observe that

$$\text{Rank } A = \text{Rank } A_b = 2 < 3 \to \text{No. of unknowns.}$$

Thus the system is "consistent having infinite solutions". From the last matrix we have

$$x - 50y + 6z = -13 \qquad (i)$$
$$11y - z = 3 \qquad (ii)$$

From (ii), we have $y = (3+z)/11$. Substituting it into (i), we have

$$x - \frac{50}{11}(3+z) + 6z = -13 \Rightarrow x - \frac{50}{11}z - \frac{150}{11} + 6z = -13$$

$$x + \left(\frac{-50+66}{11}\right)z = \frac{150}{11} - 13 \Rightarrow x + \frac{16}{11}z = \frac{7}{11} \Rightarrow x = \frac{1}{11}(7 - 16z).$$

Let $z = k$ then, $x = (7 - 16k)/11$, $y = (3 + k)/11$.

EXAMPLE 04: Examine the following system for a non-trivial solution (m < n)

$$\boldsymbol{x_1 - x_2 + 2x_3 + x_4 = 0}$$
$$\boldsymbol{3x_1 + 2x_2 \quad + x_4 = 0}$$
$$\boldsymbol{4x_1 + x_2 + 2x_3 + 2x_4 = 0}$$

Solution: The matrix of the coefficients is: $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 4 & 1 & 2 & 2 \end{bmatrix}$.

Now we reduce the matrix into echelon form using elementary row operations:

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 4 & 1 & 2 & 2 \end{bmatrix} R_2 + (-3)R_1, R_3 + (-4)R_1$$

$$\approx \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -6 & -2 \\ 0 & 5 & -6 & -2 \end{bmatrix} R_3 + (-1)R_2 \approx \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -6 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\frac{1}{5}\right) R_2$$

$$\approx \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6/5 & -2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 + R_2 \approx \begin{bmatrix} 1 & 0 & 4/5 & 3/5 \\ 0 & 1 & -6/5 & -2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of A is 2 < 4 (the number of variables), therefore, the system of equations is consistent having infinite number of "non-trivial solutions". The first two rows of the last matrix give the following relations:

$$x_1 + \frac{4}{5}x_3 + \frac{3}{5}x_4 = 0 \qquad (i)$$

$$x_2 - \frac{6}{5}x_3 - \frac{2}{5}x_4 = 0 \qquad (ii)$$

From (i) and (ii), we have

$$x_1 = -\left(\frac{4}{5}x_3 + \frac{3}{5}x_4\right) \text{ and } x_2 = \frac{6}{5}x_3 + \frac{2}{5}x_4, \text{ where } x_3 \text{ and } x_4 \text{ arte arbitrary.}$$

Let $x_3 = k_1$ and $x_4 = k_2$, then

$$x_1 = -\left(\frac{4}{5}k_1 + \frac{3}{5}k_2\right) \text{ and } x_2 = \frac{6}{5}k_1 + \frac{2}{5}k_2 \cdot$$

Hence, we have

$$x_1 = -\left(\frac{4}{5}k_1 + \frac{3}{5}k_2\right), \; x_2 = \frac{6}{5}k_1 + \frac{2}{5}k_2, \; x_3 = k_1 \text{ and } x_4 = k_2 \cdot$$

We can get an infinite number of solutions by giving different values to k_1 and k_2 ·

EXAMPLE 05: Find the values of *k* such that the system of equations

$\boldsymbol{x + ky + 3z = 0,\;\; 4x + 3y + kz = 0,\;\; 2x + y + 2z = 0}$

has a non – trivial solution (m = n).

Solution: The matrix of coefficients is

$$A = \begin{bmatrix} 1 & k & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{bmatrix} \cdot$$

Now we reduce the matrix into echelon form using elementary row operations:

$$A = \begin{bmatrix} 1 & k & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{bmatrix} R_2 + (-4)R_1,\; R_3 + (-2)R_1 \approx \begin{bmatrix} 1 & k & 3 \\ 0 & 3-4k & k-12 \\ 0 & 1-2k & -4 \end{bmatrix} R_2 + (-2)R_3$$

$$\approx \begin{bmatrix} 1 & k & 3 \\ 0 & 1 & k-4 \\ 0 & 1-2k & -4 \end{bmatrix} R_3 + \{-(1-2k)\}R_2 \approx \begin{bmatrix} 1 & k & 3 \\ 0 & 1 & k-4 \\ 0 & 0 & 2k^2-9k \end{bmatrix}.$$

For non – trivial solution $2k^2 - 9k = 0 \Rightarrow k(2k-9) = 0 \Rightarrow k = 0 \text{ or } k = \frac{9}{2}\cdot$

The system has only trivial solution if 2 k2 – 9 k ≠ 0

EXAMPLE 06: Determine for what values of λ and μ the following system of equations

$$x + y + z = 6,\; x + 2y + 3z = 10,\; x + 2y + \lambda z = \mu$$

has (i) no solution (ii) a unique solution (iii) infinite number of solutions.
R(A)≠R(Ab) R(A)=R(Ab)=n R(A)=R(Ab)<n

Solution: The augmented matrix for the above system of equations is:

$$A_b = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}.$$

Now we reduce the matrix into echelon form using elementary row operations:

$$A_b = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix} R_2 + (-1)R_1,\; R_3 + (-1)R_1$$

$$\approx \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{bmatrix} R_3 + (-2)R_2 \approx \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix}\cdot$$

(i) There is no solution if $\text{Rank } A \neq \text{Rank } A_b$

i.e. $\lambda - 3 = 0 \Rightarrow \lambda = 3 \text{ and } \mu - 10 \neq 0 \Rightarrow \mu \neq 10\cdot$

(ii) There is a unique solution if $\text{Rank } A = \text{Rank } A_b = 3$

i.e. $\lambda - 3 \neq 0 \Rightarrow \lambda \neq 3 \text{ and } \mu \text{ may have any value.}$

(iii) There are infinite solutions if $\text{Rank } A = \text{Rank } A_b < 3$

i.e. $\lambda - 3 = 0 \Rightarrow \lambda = 3 \text{ and } \mu - 10 = 0 \Rightarrow \mu = 10\cdot$

EXAMPLE 7: For what value of λ the system of linear equations

$$\begin{aligned} (3-\lambda)x_1 - x_2 + x_3 &= 0 \\ x_1 - (1-\lambda)x_2 + x_3 &= 0 \\ x_1 - x_2 + (1-\lambda)x_3 &= 0 \end{aligned}$$

has a non-trivial solutions. Find these solutions.

Solution: The matrix of the coefficients is

$$A = \begin{bmatrix} (3-\lambda) & -1 & 1 \\ 1 & -(1-\lambda) & 1 \\ 1 & -1 & (1-\lambda) \end{bmatrix}.$$

We reduce this matrix to the echelon form by applying elementary row operations:

$$A = \begin{bmatrix} (3-\lambda) & -1 & 1 \\ 1 & -(1-\lambda) & 1 \\ 1 & -1 & (1-\lambda) \end{bmatrix} R_{12}$$

$$\approx \begin{bmatrix} 1 & -(1-\lambda) & 1 \\ (3-\lambda) & -1 & 1 \\ 1 & -1 & (1-\lambda) \end{bmatrix} R_2 + \{-(3-\lambda)\}R_1,\ R_3 + (-1)R_1$$

$$\approx \begin{bmatrix} 1 & -(1-\lambda) & 1 \\ 0 & 2-4\lambda+\lambda^2 & \lambda-2 \\ 0 & -\lambda & -\lambda \end{bmatrix} (-1)R_3$$

$$\approx \begin{bmatrix} 1 & \lambda-1 & 1 \\ 0 & 2-4\lambda+\lambda^2 & \lambda-2 \\ 0 & \lambda & \lambda \end{bmatrix} \left(\frac{1}{\lambda}\right) R_3;\ \lambda \neq 0 \approx \begin{bmatrix} 1 & \lambda-1 & 1 \\ 0 & 2-4\lambda+\lambda^2 & \lambda-2 \\ 0 & 1 & 1 \end{bmatrix} R_{23}$$

$$\approx \begin{bmatrix} 1 & \lambda-1 & 1 \\ 0 & 1 & 1 \\ 0 & 2-4\lambda+\lambda^2 & \lambda-2 \end{bmatrix} R_1 + \{-(\lambda-1)\}R_2,\ \ R_3 + \left\{-\left(2-4\lambda+\lambda^2\right)\right\}R_2$$

$$\approx \begin{bmatrix} 1 & 0 & 2-\lambda \\ 0 & 1 & 1 \\ 0 & 0 & -4+5\lambda-\lambda^2 \end{bmatrix}.$$

For non-trivial solutions rank of $A < 3$ (the number of unknowns), so we must have

$$-4+5\lambda-\lambda^2 = 0 \qquad or\ \lambda^2-5\lambda+4=0 \Rightarrow \lambda-4=0\ \ or\ \ \lambda-1=0 \Rightarrow \lambda = 1,\ 4 \cdot$$

Now when $\lambda = 1$, then from the last matrix, we have

$$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3 \text{ and } x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 \cdot$$

When $\lambda = 4$,

$$x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3 \quad \text{and} \quad x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 \cdot$$

$$Hence,\ \ for\ \ \lambda = 1,\ x_1 = x_2 = -x_3 \ \ : \ \ for\ \ \lambda = 4,\ x_1 = 2x_3,\ x_2 = -x_3 \cdot$$