Lecture # 9

ROW EQUIVALENT AUGMENTED MATRICES

A general system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \qquad (1)$$

where a_{ij} and b_j for $i = 1,2,\cdots,m;\ j = 1,2,\cdots n$ are constants. These numbers a_{ij} are called the ***coefficients of the system.*** The system (1) is known as "*Non-homogeneous System of Linear Equations*".

If each b_i in (1) is zero, the system becomes:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned} \qquad (2)$$

The system of equations (2) is called the "H***omogeneous System of Linear Equations".*** A sequence of n numbers $s_1, s_2, s_3, \ldots s_n$, for which (1) is satisfied when we substitute $x_1 = s_1$, $x_2 = s_2, \ldots, x_n = s_n$, is called a ***solution*** of (1). The set of all such solutions is called ***solution set*** or the ***general solution*** of (1).

Matrix Notation of System of Linear Equations

Using the matrix notation, the linear system (1) may be written as:

$A\,\boldsymbol{x} = \boldsymbol{b}$ (3)

where $\boldsymbol{A}$, $\boldsymbol{x}$ and $\boldsymbol{b}$ are matrices given by:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},\ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The matrix $\boldsymbol{A}$ is called the system matrix, the right side constants b_i form a column vector $\boldsymbol{b}$ and the unknowns x_i form the column vector $\boldsymbol{x}$.

The matrix $A_b = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{mn} \end{bmatrix}$

is called the ***augmented matrix*** of the system (1). We see that A_b is obtained by augmenting the matrix A by column vector $\boldsymbol{b}$. The matrix A_b determines the system (1) completely, because it contains all the given numbers appearing in (1). The unknowns x_i are immaterial because instead of x_i if we use some other variable, the solution of the system will not change.

Let A_b be the augmented matrix for the system of equations (1) and C_d be its row equivalent matrix, obtained by the following three operations:

- **(i)** Any two equations of (1) are interchanged. This is shown by writing $R_{i\,j}$.
- **(ii)** Any equation of (1) is multiplied by a non-zero constant. This is shown by writing $c\,R_i$.
- **(iii)** A constant multiple of an equation (1) is added to another equation. This is shown by writing $R_i + c\,R_j$.

The three operations listed above are called ***elementary*** operations for the system (1). The resulting system of equations will be equivalent to the previous one, and therefore have the same solution. Thus by transforming the augmented matrix A_b into row equivalent augmented matrix C_d, we can easily solve the given system of equations. The two methods namely:

(i) GAUS'S Elimination Method and

(ii) GAUS'S-JORDAN Methods

are used to solve the system (1). These methods are known as ***"Direct Methods"*** or ***"Analytic Methods."***

Gauss Elimination Method

One of the several methods employed to solve a system of m linear equations in n variables, is known as Gaussian elimination method named after its inventor, the famous German mathematician Carl Friedrich Gauss (1777-1855). Let us consider the linear system (1)
In matrix form it can be expressed as $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A}$ is the matrix of coefficients of order $m \times n$, $\boldsymbol{x}$ and $\boldsymbol{b}$ are column matrices/vectors of order $n \times 1$ and $m \times 1$ respectively. The following four steps can be applied to solve a system using Gauss's elimination method.

Step1. Change the system of linear equations to the form $A\boldsymbol{x} = \boldsymbol{b}$.
Step2. Form the augmented coefficient matrix A_b by including the elements of $\boldsymbol{b}$ as anextra column in the matrix A.
Step3. Convert the augmented matrix into echelon form by using elementary row operations.
Step4. Find $\boldsymbol{x}$ by detaching the fourth column back to its original position on the right hand side of the matrix equation $\boldsymbol{Ax} = \boldsymbol{b}$.This procedure is known as "*Backward Substitution*".

NOTE:

(i) A system $\boldsymbol{Ax} = \boldsymbol{b}$ is called ***over-determined*** if it has more equations than unknowns i.e. $m > n$, ***determined*** if the number of equations is equal to the number of unknowns, i.e. $m = n$ and ***underdetermined*** if $\boldsymbol{Ax} = \boldsymbol{b}$ has fewer equations than unknowns i.e. $m < n$.

(ii) When an augmented matrix is written in the row echelon form, the variables corresponding to thefirst non-zero elements (or leading elements) in each row are called ***leading or pivotal variables***. All
other variables are called ***free or non-leading variables***.

(iii) If $m = n$ and $|A| \neq 0$, then system (1) can be solved by
(a) Cramer's rule discussed in Chapter 3.
(b) Using the inverse matrix method $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

EXAMPLE 01: Use Gauss's elimination method to solve the system of linear equations (*m* = *n*)

$$\begin{aligned} x_1 + 5x_2 + 2x_3 &= 9 \\ x_1 + x_2 + 7x_3 &= 6 \\ -3x_2 + 4x_3 &= -2 \end{aligned}$$

Solution:

Step1. We change the system of linear equations in matrix form $\boldsymbol{Ax} = \boldsymbol{b}$.

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 9 \\ 6 \\ -2 \end{bmatrix}.$$

Step2. We form the augmented coefficient matrix A_b by including the constants, an extra column in the matrix.

$$A_b = \begin{bmatrix} 1 & 5 & 2 & 9 \\ 1 & 1 & 7 & 6 \\ 0 & -3 & 4 & -2 \end{bmatrix}$$

Step3. We convert this augmented coefficient matrix into an *echelon form* using elementary row operations.

$$A_b = \begin{bmatrix} 1 & 5 & 2 & 9 \\ 1 & 1 & 7 & 6 \\ 0 & -3 & 4 & -2 \end{bmatrix} R_2 + (-1)R_1 \approx \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & -4 & 5 & -3 \\ 0 & -3 & 4 & -2 \end{bmatrix} R_3 + (-1)R_2$$

$$\approx \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & -4 & 5 & -3 \\ 0 & 1 & -1 & 1 \end{bmatrix} R_2 + 4R_3 \approx \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} R_{23}$$

$$\approx \begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot$$

Step4. Find ***x*** by detaching the fourth column back to its original position on the right hand side of the matrix equation $\boldsymbol{Ax} = \boldsymbol{b} \cdot$

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 1 \end{bmatrix} \cdot$$

$$\boldsymbol{x_1 + 5\,x_2 + 2\,x_3 = 9} \qquad (i)$$

$$\boldsymbol{x_2 - \quad x_3 = 1} \qquad (ii)$$

$$\boldsymbol{x_3 = -1} \qquad (iii)$$

From (*iii*), we get $x_3 = 1$. Substituting $x_3 = 1$ into (*ii*), we get $x_2 - 1 = 1 \Rightarrow x_2 = 2 \cdot$ Now put $x_3 = 1$ and $x_2 = 2$ in (*i*), we get $x_1 = 3 \cdot$

Thus, the required solution of the given system of equations is: $x_1 = 3$, $x_2 = 2$, $x_3 = 1$.