Lecture # 3

OPERATIONS ON MATRICES

In this section, we shall define some fundamental operations on matrices.

1. Multiplication of a Matrix by a Constant

If *A* is any matrix and *k* is a constant, then *kA* is a matrix whose elements are obtained by multiplying each element of matrix *A* by *k*. For example, if

$$A=\begin{bmatrix}6 & 5 & 2\\ 3 & 0 & -7\end{bmatrix} \text{ then } 3A=\begin{bmatrix}18 & 15 & 6\\ 9 & 0 & -21\end{bmatrix}.$$

2. Addition and Subtraction of Matrices

If *A* and *B* are two matrices of same order $m\times n$, then their sum *A+B* is an $m\times n$ matrix *C* such that each element of *C* is the sum of the corresponding elements of *A* and *B*.

Thus, if $A=\begin{bmatrix}6 & 5 & 4\\ -2 & 3 & 7\end{bmatrix}$ and $B=\begin{bmatrix}4 & 1 & -5\\ 3 & 9 & 2\end{bmatrix}$ then

$$C=A+B=\begin{bmatrix}6 & 5 & 4\\ -2 & 3 & 7\end{bmatrix}+\begin{bmatrix}4 & 1 & -5\\ 3 & 9 & 2\end{bmatrix}=\begin{bmatrix}6+4 & 5+1 & 4-5\\ -2+3 & 3+9 & 7+2\end{bmatrix}=\begin{bmatrix}10 & 6 & -1\\ 1 & 12 & 9\end{bmatrix}.$$

Similarly, "the difference *A* – *B*" of the matrices *A* and *B* is a matrix each element of which is obtained by subtracting the elements of *B* from the corresponding elements of *A*.

Thus if $A=\begin{bmatrix}6 & 2\\ 7 & -5\end{bmatrix}$ and $B=\begin{bmatrix}8 & 1\\ 3 & 4\end{bmatrix}$ then

$$D=A-B=\begin{bmatrix}6 & 2\\ 7 & -5\end{bmatrix}-\begin{bmatrix}8 & 1\\ 3 & 4\end{bmatrix}=\begin{bmatrix}6-8 & 2-1\\ 7-3 & -5-4\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 4 & -9\end{bmatrix}.$$

Using double subscript notations, we say that if the matrices *A* and *B* are confirmable for addition and subtraction and that *C* = *A* + *B* and *D* = *A* – *B* then,

$$c_{ij} = a_{ij} + b_{ij},\ \forall\ i, j$$

$$d_{ij} = a_{ij} - b_{ij},\ \forall\ i, j$$

NOTE:

i. Two matrices are, said to be confirmable for addition or subtraction if and only if they have the same order.

ii. If the matrices *A, B* and *C* are confirmable for addition/subtraction, then

a. *A* + *A* = 2 *A*	**b. *A* – *A* = *O***	**c. *A* + *A* + … + *A* = *n A***
d. *A* + *B* = *B* + *A*	**e. *A* – *B* ≠ *B* – *A***	**f. *A* + (*B* + *C*) = (*A* + *B*) + *C***

From (d) and (e) we observe that matrix addition is commutative but same is not true for matrix subtraction. In fact, matrix subtraction is anti-commutative. Property (f) is known as "Associative law of addition.

Multiplication of Matrices

Let *A* = (a_{ij}) be an $m\times n$ matrix and *B* = (b_{ij})be an $r\times p$ matrix, then the product AB (in this order) is defined only when $n=r,$ that is;

Number of columns of *A* = Number of rows of *B*

In this case, the product matrix AB is an $m\times p$ matrix. The elements of *C* = *AB* are determined as follows:

c_{ij} = SUM [(i^{th} row vector of *A*) · (j^{th} column vector of *B*)], that is;

$$c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}, i=1,2,\dots m;\ j=1,2,\dots p$$

In other words, c_{ij} is the sum of the products of the corresponding elements of the i^{th} row of A and j^{th} column of B.

IMPORTANT NOTES:

(i) A matrix can be multiplied by itself, if it is a square matrix. The product $A \cdot A$ in such case is written as A^2. Similarly A. $A^2 = A^3$, A^2. $A^2 = A^4$ etc.

(ii) If $AB = O$, it does not necessarily imply that A or B is a null matrix. For instance, if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

then the product

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is a null matrix, although neither A nor B is a null matrix.

Readers may note that if property (iii) holds then

- a. **$|A| = O$ or $|B| = O$.**
- b. **Matrices A and B are called "Divisors of zero".**
- c. **$AB = BA$.**

iv. If A B = A C it does not imply that B = C (This means in matrix theory, the cancellation law does not hold in general)

For instance, consider

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 3 & 3 & 3 \end{bmatrix}.$$

Then $$AB = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 2 \\ -3 & 2 & -7 \end{bmatrix} \text{ and } AC = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 2 \\ -3 & 2 & -7 \end{bmatrix}$$

Thus $A\ B = A\ C$ but $B \neq C$. This is contrary to the cancellation law w .r . t multiplication that is true for set of real numbers that if $a\,b = a\,c \Rightarrow b = c$ provided $a \neq 0$.

v. $(A^m)^n = (A^n)^m = A^{mn}$.

vi. $A^0 = I$

EXAMPLE 01: Find *AB* if $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 4 & 7 \\ 5 & 3 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} 6 & 1 \\ 0 & 4 \\ -2 & 3 \end{bmatrix}$.

3X3 **3X2**

Solution: The above matrices are conformable for multiplication because the number of columns of A is equal to the number of rows of B.

Now, $$AB = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 4 & 7 \\ 5 & 3 & 0 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 0 & 4 \\ -2 & 3 \end{bmatrix}.$$

$$= \begin{bmatrix} 1\times6+0\times0+(-1)\times(-2) & 1\times1+0\times4+(-1)\times3 \\ 2\times6+4\times0+7\times(-2) & 2\times1+4\times4+7\times3 \\ 5\times6+3\times0+0\times(-2) & 5\times1+3\times4+0\times3 \end{bmatrix} = \begin{bmatrix} 6+2 & 1-3 \\ 12-14 & 2+16+21 \\ 30+0 & 5+12+0 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ -2 & 39 \\ 30 & 17 \end{bmatrix}.$$

Here A is a 3×3 matrix and B is a 3×2 matrix and the resultant matrix AB is a 3×2 matrix. Note that matrix multiplication BA is not defined.

EXAMPLE 02: Find *AB* and *BA*, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

Solution: Here,

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1.1+1.0 & 1.1+2.2 \\ 3.1+4.0 & 3.1+4.2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 6 & 11 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1.1+1.3 & 1.2+1.4 \\ 0.1+2.3 & 0.2+2.4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 8 \end{bmatrix}$$

We may observe that $AB \neq BA$. Although, there are some special cases where $A\,B = B\,A$, for example $A\,I = I\,A$, where I is an identity matrix, but in general commutative law of matrix multiplication does not hold.

In general, the commutative law for matrix multiplication does not hold.

Using this property, we may prove that in general:

i. $\boldsymbol{A^2 - B^2 \neq (A - B)(A + B)}$ **(ii)** $\boldsymbol{(A \pm B)^2 \neq A^2 \pm 2AB + B^2}$

Proof: (i) Consider $(A - B)(A + B) = A(A + B) - B(A + B) = A^2 + AB - BA - B^2$.

Since $AB \neq BA$, hence $AB - BA \neq O$. This implies that in general, $A^2 - B^2 \neq (A - B)(A + B)$.

In case $AB = BA$ then, this property will be true.

(ii) Left as an exercise.

Properties of Matrix Multiplication

i. $A(BC) = (AB)C$ (Associative law).

ii. $A(B + C) = AB + AC$ (Left distributive law).

$(A + B)C = AC + BC$ (Right distributive law).

iii. $k(AB) = (kA)B = A(kB)$, k being some scalar.

EXAMPLE 03: Evaluate the matrix multiplication $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Solution: We have

First we have to check the conformability of multiplication

(1 x 3) x (3 x 3) x (3 x 1)

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x \times a + y \times h + z \times g & x \times h + y \times b + z \times f & x \times g + y \times f + z \times c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} ax + hy + gz & hx + by + fz & gx + fy + cz \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \left[(ax + hy + gz) \times x + (hx + by + fz) \times y + (gx + fy + cz) \times z \right]$$

$$= \left[ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gxz \right] \cdot \text{1X1}$$

EXAMPLE 04: Let $A=\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}$.

Find (i) $A+B$ (ii) $A-B$ (iii) $2A+3B$ (iv) $3A-5B$ (v) AB (vi) BA.

Solution:

(i) $$A+B=\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}+\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}=\begin{bmatrix}2-1 & -3+3 & -5+5\\ -1+1 & 4-3 & 5-5\\ 1-1 & -3+3 & -4+5\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

(ii) $$A-B=\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}-\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}=\begin{bmatrix}2+1 & -3-3 & -5-5\\ -1-1 & 4+3 & 5+5\\ 1+1 & -3-3 & -4-5\end{bmatrix}=\begin{bmatrix}3 & -6 & -10\\ -2 & 7 & 10\\ 2 & -6 & -9\end{bmatrix}.$$

(iii) $$2A+3B=2\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}+3\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}=\begin{bmatrix}4 & -6 & -10\\ -2 & 8 & 10\\ 2 & -6 & -8\end{bmatrix}+\begin{bmatrix}-3 & 9 & 15\\ 3 & -9 & -15\\ -3 & 9 & 15\end{bmatrix}$$

$$2A+3B=\begin{bmatrix}4-3 & -6+9 & -10+15\\ -2+3 & 8-9 & 10-15\\ 2-3 & -6+9 & -8+15\end{bmatrix}=\begin{bmatrix}1 & 3 & 5\\ 1 & -1 & -5\\ -1 & 3 & 7\end{bmatrix}.$$

(iv) $$3A-5B=3\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}-5\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}=\begin{bmatrix}6 & -9 & -15\\ -3 & 12 & 15\\ 3 & -9 & -12\end{bmatrix}-\begin{bmatrix}-5 & 15 & 25\\ 5 & -15 & -25\\ -5 & 15 & 25\end{bmatrix}$$

$$3A-5B=\begin{bmatrix}6+5 & -9-15 & -15-25\\ -3-5 & 12+15 & 15+25\\ 3+5 & -9-15 & -12-25\end{bmatrix}=\begin{bmatrix}11 & -24 & -40\\ -8 & 27 & 40\\ 8 & -24 & -37\end{bmatrix}.$$

(v) $$AB=\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}=\begin{bmatrix}-2-3+5 & 6+9-15 & 10+15-25\\ 1+4-5 & -3-12+15 & -5-20+25\\ -1-3+4 & 3+9-12 & 5+15-20\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

(vi) $$BA=\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}=\begin{bmatrix}-2-3+5 & 3+12-15 & 5+15-20\\ 2+3-5 & -3-12+15 & -5-15+20\\ -2-3+5 & 3+12-15 & 5+15-20\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

EXAMPLE 05: Evaluate (i) $\begin{bmatrix}1 & 2 & 1\\ 4 & 0 & 2\end{bmatrix}\begin{bmatrix}2 & -4\\ 1 & 5\\ -2 & 2\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 2\\ 4 & -3\end{bmatrix}^2$ (iii) $\begin{bmatrix}1 & 2\\ 4 & -3\end{bmatrix}^3$ (iv) $\begin{bmatrix}-1 & -1 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}^2$

EXAMPLE 06: Prove that if the product of matrices $A=\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$ and

$B=\begin{bmatrix}\cos^2\phi & \cos\phi\sin\phi\\ \cos\phi\sin\phi & \sin^2\phi\end{bmatrix}$ is the zero matrix then θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

AXB = O COS(θ + ϕ) = 0