Lecture # 17

EXAMPLE 01: Compute the determinant of A without expending

(i) $$A = \begin{bmatrix} \mathbf{5} & \mathbf{-3} & \mathbf{2} & \mathbf{3} \\ \mathbf{-4} & \mathbf{-6} & \mathbf{7} & \mathbf{-5} \\ \mathbf{3} & \mathbf{5} & \mathbf{6} & \mathbf{9} \\ \mathbf{2} & \mathbf{-3} & \mathbf{5} & \mathbf{4} \end{bmatrix}.$$

Solution: "1" does not appear as an entry of the matrix A, therefore first we reduce A to a matrix which has 1 as an entry. We must obtain 1 as an entry of the matrix A by adding a multiple of a row (column) to another row (column). Therefore,

$$|A| = \begin{vmatrix} 1 & -9 & 9 & -2 \\ -4 & -6 & 7 & -5 \\ 3 & 5 & 6 & 9 \\ 2 & -3 & 5 & 4 \end{vmatrix} R_1 + R_2$$

Now we reduce the elements -4, 3 and 2 in the 1st column to zero.

$$|A| = \begin{vmatrix} 1 & -9 & 9 & -2 \\ 0 & -42 & 43 & -13 \\ 0 & 32 & -21 & 15 \\ 0 & 15 & -13 & 8 \end{vmatrix} R_2 + 4R_1, \;\; R_3 + (-3)R_1, \;\; R_4 + (-2)R_1$$

$$= \begin{vmatrix} -42 & 43 & -13 \\ 32 & -21 & 15 \\ 15 & -13 & 8 \end{vmatrix} \text{ expanding by } C_1$$

$$= \begin{vmatrix} 1 & 43 & -13 \\ 11 & -21 & 15 \\ 2 & -13 & 8 \end{vmatrix} C_1 + C_2 \;\; \text{(This means add } C_1 \text{ in } C_2\text{)}$$

$$= \begin{vmatrix} 1 & 43 & -13 \\ 0 & -494 & 158 \\ 0 & -99 & 34 \end{vmatrix} R_2 + (-11)R_1, \;\; R_3 + (-2)R_1$$

$$= \begin{vmatrix} -494 & 158 \\ -99 & 34 \end{vmatrix} \text{ expanding by } C_1$$

$$|A| = (-494)(34) - (158)(-99) = -16796 + 15642 = -1154 \cdot$$

(ii) $$|A| = \begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{1} \\ \mathbf{3} & \mathbf{4} & \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{vmatrix}.$$

Solution:

$$|A| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & -2 & -8 & -10 \\ 0 & -7 & -10 & -13 \end{vmatrix} R_2 + (-2)R_1, \; R_3 + (-3)R_1, \; R_4 + (-4)R_1$$

Expanding by C_1 and taking $(-1),(-2)$ and (-1) common from R_2, R_3 and R_4 respectively.

$$|A| = (-1)(-2)(-1)\begin{vmatrix} 1 & 2 & 7 \\ 1 & 4 & 5 \\ 7 & 10 & 13 \end{vmatrix}$$

$$|A| = -2\begin{vmatrix} 1 & 2 & 7 \\ 0 & 2 & -2 \\ 0 & -4 & -36 \end{vmatrix} R_2 + (-1)R_1, \quad R_3 + (-7)R_1$$

Expanding by C_1 and taking (2) and (-4) common from R_2 and R_3 respectively.

$$|A| = (-2)(2)(-4)\begin{vmatrix} 1 & -1 \\ 1 & 9 \end{vmatrix} = 16(9+1) = 160 \cdot$$

EXAMPLE 02: Prove that

$$\begin{vmatrix} \boldsymbol{a-b} & \boldsymbol{b-c} & \boldsymbol{c-a} \\ \boldsymbol{b-c} & \boldsymbol{c-a} & \boldsymbol{a-b} \\ \boldsymbol{c-a} & \boldsymbol{a-b} & \boldsymbol{b-c} \end{vmatrix} = \boldsymbol{0} \cdot$$

Solution:

Adding second row into first row, we have

$$|A| = \begin{vmatrix} a-b+b-c & b-c+c-a & c-a+a-b \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = \begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Adding third row into first row, we have

$$|A| = \begin{vmatrix} a-c+c-a & b-a+a-b & c-b+b-c \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0 \qquad [\text{since } R_1 = 0] \cdot$$

EXAMPLE 03: Prove that

$$|A| = \begin{vmatrix} \boldsymbol{1} & \boldsymbol{\cos\alpha} & \boldsymbol{\cos\beta} \\ \boldsymbol{\cos\alpha} & \boldsymbol{1} & \boldsymbol{\cos(\alpha+\beta)} \\ \boldsymbol{\cos\beta} & \boldsymbol{\cos(\alpha+\beta)} & \boldsymbol{1} \end{vmatrix} = \boldsymbol{0} \cdot$$

Solution: Using the trigonometric formula, we have

$$|A| = \begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta & 1 \end{vmatrix}$$

we have

$$= \begin{vmatrix} 1 & \cos\alpha - \cos\alpha & \cos\beta \\ \cos\alpha & 1 - \cos^2\alpha & \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta - \cos\alpha\cos\beta & 1 \end{vmatrix} \quad (\text{By } C_2 - \cos\alpha C_1)$$

$$= \begin{vmatrix} 1 & 0 & \cos\beta \\ \cos\alpha & \sin^2\alpha & \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos\beta & -\sin\alpha\sin\beta & 1 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 & \cos\beta - \cos\beta \\ \cos\alpha & \sin^2\alpha & \cos\alpha\cos\beta - \sin\alpha\sin\beta - \cos\alpha\cos\beta \\ \cos\beta & -\sin\alpha\sin\beta & 1-\cos^2\beta \end{vmatrix} \qquad \text{(By } C_3 - \cos\beta C_1\text{)}$$

$$=\begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & \sin^2\alpha & -\sin\alpha\sin\beta \\ \cos\beta & -\sin\alpha\sin\beta & \sin^2\beta \end{vmatrix}$$

Taking $\sin\alpha$ and $-\sin\beta$ common from second and third column respectively, we have

$$=(\sin\alpha)(-\sin\beta)\begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & \sin\alpha & \sin\alpha \\ \cos\beta & -\sin\beta & -\sin\beta \end{vmatrix} = (\sin\alpha)(-\sin\beta)(0) = 0 \qquad \text{since } [C_2 = C_3]$$

EXAMPLE 04: Prove that

$$|A| = \begin{vmatrix} \boldsymbol{x} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{x} & \boldsymbol{a} & \boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{x} & \boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{x} \end{vmatrix} = (\boldsymbol{x}-\boldsymbol{a})^3(\boldsymbol{x}+\boldsymbol{3a}).$$

Solution: By adding second, third and fourth rows into the first row, we have

$$|A| = \begin{vmatrix} x+3a & x+3a & x+3a & x+3a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$$

Taking $(x+3a)$ common from first row, we have

$$|A| = (x+3a)\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$$

$$=(x+3a)\begin{vmatrix} 1 & 0 & 0 & 0 \\ a & x-a & 0 & 0 \\ a & 0 & x-a & 0 \\ a & 0 & 0 & x-a \end{vmatrix} \quad \text{by } \; C_2 - C_1,\, C_3 - C_1,\, C_4 - C_1$$

Expanding by first row, we get

$$=(x+3a)\begin{vmatrix} x-a & 0 & 0 \\ 0 & x-a & 0 \\ 0 & 0 & x-a \end{vmatrix}$$

Since the determinant is triangular, therefore its value will be equal to the product of its diagonal elements. It implies that

$$(x+3a)(x-a)(x-a)(x-a) = (x-a)^3(x+3a) = \text{L.H.S.}$$

EXAMPLE 05: Prove that

$$\begin{vmatrix} \boldsymbol{1+x} & \boldsymbol{1} & \boldsymbol{1} & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{1-x} & \boldsymbol{1} & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{1} & \boldsymbol{1+y} & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{1} & \boldsymbol{1} & \boldsymbol{1-y} \end{vmatrix} = \boldsymbol{x^2 y^2}.$$

Solution: Taking left hand side, we have

$$|A| = \begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ -x & -x & 0 & 1 \\ -x & 0 & y & 1 \\ 1-(1+x)(1-y) & y & y & 1-y \end{vmatrix} \text{ by } C_1-(1+x)C_4,\ C_2-C_4,\ C_3-C_4$$

Expanding by first row, we get

$$= \begin{vmatrix} -x & -x & 0 \\ -x & 0 & y \\ -x+y+xy & y & y \end{vmatrix} = -\begin{vmatrix} 0 & -x & 0 \\ -x & 0 & y \\ -x+xy & y & y \end{vmatrix} \text{ by } C_1-C_2$$

Taking $(-x)$ common from first row, we get

$$= -(-x)\begin{vmatrix} 0 & 1 & 0 \\ -x & 0 & y \\ -x(1-y) & y & y \end{vmatrix}.$$

Taking $(-x)$common from first column, we get

$$= x(-x)\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & y \\ 1-y & y & y \end{vmatrix}$$

Interchanging first and second column, we get

$$= -x^2(-)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ y & 1-y & y \end{vmatrix}$$

Taking y common from third column, we get

$$= x^2 y\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ y & 1-y & 1 \end{vmatrix}$$

Expanding by first row, we have

$$= x^2 y\begin{vmatrix} 1 & 1 \\ 1-y & 1 \end{vmatrix} = x^2 y(1-(1-y)) = x^2 y(1-1+y) = x^2 y^2 = \text{L.H.S.}$$