Problem 4

(a) Calculate the directional derivative of the function $\|\mathbf{x}\|_2$.

(b) Calculate the directional derivative of the function $\|\mathbf{x}\|_1$.

Solution

(a) Let $\mathbf{d} \in \mathbb{R}^n$. First assume that $\mathbf{x} = \mathbf{0}_n$. We have

$$\lim_{t \to 0^+} \frac{\|\mathbf{0}_n + t\mathbf{d}\| - \|\mathbf{0}_n\|}{t} = \|\mathbf{d}\|.$$

Now assume that $\|\mathbf{x}\| > 0$. We have

$$\begin{aligned}
f'(\mathbf{x}; \mathbf{d}) &= \lim_{t \to 0^+} \frac{\|\mathbf{x} + t\mathbf{d}\| - \|\mathbf{x}\|}{t} = \lim_{t \to 0^+} \frac{(\|\mathbf{x} + t\mathbf{d}\| - \|\mathbf{x}\|)(\|\mathbf{x} + t\mathbf{d}\| + \|\mathbf{x}\|)}{t(\|\mathbf{x} + t\mathbf{d}\| + \|\mathbf{x}\|)} \\
&= \lim_{t \to 0^+} \frac{\|\mathbf{x} + t\mathbf{d}\|^2 - \|\mathbf{x}\|^2}{t(\|\mathbf{x} + t\mathbf{d}\| + \|\mathbf{x}\|)} = \lim_{t \to 0^+} \frac{\mathbf{x}^T\mathbf{x} + 2t\mathbf{d}^T\mathbf{x} + t^2\mathbf{d}^T\mathbf{d} - \mathbf{x}^T\mathbf{x}}{t(\|\mathbf{x} + t\mathbf{d}\| + \|\mathbf{x}\|)} \\
&= \lim_{t \to 0^+} \frac{2\mathbf{d}^T\mathbf{x} + t\mathbf{d}^T\mathbf{d}}{\|\mathbf{x} + t\mathbf{d}\| + \|\mathbf{x}\|} = \frac{2\mathbf{d}^T\mathbf{x}}{\|\mathbf{x}\| + \|\mathbf{x}\|} = \frac{\mathbf{d}^T\mathbf{x}}{\|\mathbf{x}\|}.
\end{aligned}$$

Note that the fourth equality is only true for the 2-norm. Overall we have

$$f'(\mathbf{x}; \mathbf{d}) = \begin{cases} \|\mathbf{d}\|, & \mathbf{x} = \mathbf{0}_n, \\ \frac{\mathbf{d}^T\mathbf{x}}{\|\mathbf{x}\|}, & \|\mathbf{x}\| > 0. \end{cases}$$

(b) Let $\mathbf{d} \in \mathbb{R}^n$. In the one-dimensional case we have

$$f'(x; d) = \lim_{t \to 0^+} \frac{|x + td| - |x|}{t} = \begin{cases} \lim_{t \to 0^+} \frac{x + td - x}{t} = d, & x > 0, \\ \lim_{t \to 0^+} \frac{-x - td + x}{t} = -d, & x < 0, \\ |d|, & x = 0, \end{cases}$$

where we used that fact that if $x > 0$ then there is a small enough $t > 0$ such that $|x + td| = x + td$, and if $x < 0$ then there is a small enough $t > 0$ such that $|x + td| = -x - td$. Now we have

$$\begin{aligned}
f'(\mathbf{x}; \mathbf{d}) &= \lim_{t \to 0^+} \frac{\sum_{i=1}^n |\mathbf{x}_i + t\mathbf{d}_i| - \sum_{i=1}^n |\mathbf{x}_i|}{t} = \lim_{t \to 0^+} \sum_{i=1}^n \left(\frac{|\mathbf{x}_i + t\mathbf{d}_i| - |\mathbf{x}_i|}{t} \right) \\
&= \sum_{i=1}^n \left(\lim_{t \to 0^+} \frac{|\mathbf{x}_i + t\mathbf{d}_i| - |\mathbf{x}_i|}{t} \right) = \sum_{\substack{i=1 \\ \mathbf{x}_i = 0}}^n |\mathbf{d}_i| + \sum_{\substack{i=1 \\ \mathbf{x}_i \neq 0}}^n |\mathbf{d}_i| \operatorname{sign}(\mathbf{x}_i).
\end{aligned}$$