098311 Optimization 1 Spring 2018
HW 11

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Solution 1. We know x^* is a feasible solution to $(QCQP)$ since it obeys the following:

$$(x^*)^T A_i(x^*) + 2b_i^T(x^*) + c_i \le 0$$

Denoting $f(x) = x^T A_0 x + 2b_0^T x$, we will now show $f(x^*) \le f(x)$ for any feasible x. The Lagrangian of the problem is:

$$\begin{aligned}\mathcal{L}(x,\lambda) &= x^T A_0 x + 2b_0^T x + \sum_{i=1}^m \lambda_i \left(x^T A_i x + 2b_i^T x + c_i\right) = \\ &= x^T\left(A_0 + \sum_{i=1}^m \lambda_i A_i\right)x + 2\left(b_0 + \sum_{i=1}^m \lambda_i b_i\right)^T x + \sum_{i=1}^m \lambda_i c_i\end{aligned}$$

Since $A_0 + \sum_{i=1}^m \lambda_i A_i \succeq 0$, the lagrangian is a quadratic function with a PSD quadratic coefficient matrix, and is therefore convex. As it is a convex, unconstrained function, we know that $\min_x \mathcal{L}(x,\lambda)$ is obtained where $\nabla_x \mathcal{L}(x,\lambda) = 0$.
We know

$$\nabla_x \mathcal{L}(x^*,\lambda) = \left(A_0 + \sum_{i=1}^m \lambda_i A_i\right)x^* + 2\left(b_0 + \sum_{i=1}^m \lambda_i b_i\right) = 0$$

with the given $\lambda_1, ..., \lambda_m$, and therefore $x^* \in \text{argmin}_x \mathcal{L}(x,\lambda)$.
Now, we have:

$$f(x^*) = (x^*)^T A_0(x^*) + 2b_0^T(x^*) = (x^*)^T A_0 x + 2b_0^T(x^*) + \underbrace{\sum_{i=1}^m \lambda_i \left((x^*)^T A_i(x^*) + 2b_i^T(x^*) + c_i\right)}_{(a)} =$$

$$= \mathcal{L}(x^*,\lambda) \le \mathcal{L}(x,\lambda) = x^T A_0 x + 2b_0^T x + \underbrace{\sum_{i=1}^m \lambda_i \left(x^T A_i x + 2b_i^T x + c_i\right)}_{(b)} \le x^T A_0 x + 2b_0^T x = f(x)$$

The above is true since

(a) $\lambda_i\left((x^*)^T A_i(x^*)+2b_i^T(x^*)+c_i\right)=0$ using x^* and the given $\lambda_1, ..., \lambda_m$.

(b) $\lambda_i \geq 0$ for each $i \in \{1, ..., m\}$ and $x^T A_i x + 2b_i^T x + c_i \leq 0$ for any feasible x and any $i \in \{1, ..., m\}$.

Solution 2.

1. Defining $x = (\alpha, q^T)^T$, the problem is equivalent to:

$$
\begin{aligned}
(P') \quad & \min_x \ (1, \mathbf{0})x \\
& \text{s.t. } (-f, A)x = 0 \\
& \quad\ \ ||(0, \mathbf{e})x||^2 \leq \epsilon
\end{aligned}
$$

   Therefore, x is limited to the convex set $||(0,\mathbf{e})x||^2 \leq \epsilon$, the equality constraint is affine and the objective function is affine and therefore convex. Additionally, there exists a point $\hat{x} \in int(||(0,\mathbf{e})x||^2 \leq \epsilon)$ which satisfies the equality: $\hat{x} = \mathbf{0}$, for instance. Under all of the above conditions, the general strong duality condition holds, and therefore strong duality holds for this problem.

2. The Lagrangian of the problem is:

$$\mathcal{L}(\alpha, q, \mu) = \alpha + \mu^T(Aq - \alpha f)$$

   where $\mu \in \mathbb{R}^m$. $\min \mathcal{L}(\alpha, q, \mu)$ is separable in α and q, and therefore we can minimize separately:

$$\min_\alpha \alpha(1 - \mu^T f)$$

   this does not have a finite value, unless $\mu^T f = 1$. Additionally,

$$\min_q \mu^T A q$$

   The minimum of the inner product $\langle A^T\mu, q\rangle$ is obtained when the direction of q is counter-parallel to the direction of $A^T\mu$. To obey the constraint on $||q||^2$, we can normalize the resulting direction vector and multiply it by $\sqrt{\epsilon}$. Therefore, the solution is obtained when:

$$
\begin{aligned}
q &= -\frac{A^T\mu}{||A^T\mu||}\sqrt{\epsilon} \\
\mu &\neq \mathbf{0}
\end{aligned}
$$

   and the resulting minimal value of the Lagrangian (and the objective function of the dual problem) is:

$$g(\mu) = -\sqrt{\epsilon}||A^T\mu||$$

and therefore the dual is:

$$(D) \quad \max_{\mu} \; -\sqrt{\epsilon}||A^T\mu|| \\ \text{s.t. } \mu^T f = 1$$

where the condition $\mu \neq 0$ can be ignored since it is implicitly included under $\mu^T f = 1$.

3. The dual is equivalent to:

$$(D') \quad -\sqrt{\epsilon}\min_{\mu} \; ||A^T\mu|| \\ \text{s.t. } \mu^T f - 1 = 0$$

Note that since $||A^T\mu|| > 0$ for any $\mu \neq 0$ we can solve an alternative minimization problem over $||A^T\mu||^2$ (since for any feasible μ_1, μ_2 such that $||A^T\mu_1|| < ||A^T\mu_2||$ then $||A^T\mu_1||^2 < ||A^T\mu_2||^2$).

$$\begin{aligned}
&L(\lambda,\mu) = ||A^T\mu||^2 - \lambda(\mu^T f - 1)\\
&\nabla_\mu L(\lambda,\mu) = 2AA^T\mu - \lambda f = 0\\
\Rightarrow&\mu = \frac{\lambda}{2}(AA^T)^{-1}f\\
\Rightarrow&\frac{\lambda}{2}(f^T(AA^T)^{-1}f) = 1\\
\Rightarrow&\lambda = \frac{2}{f^T(AA^T)^{-1}f}\\
\Rightarrow&\mu = \frac{(AA^T)^{-1}f}{f^T(AA^T)^{-1}f}
\end{aligned}$$

where AA^T is invertible as the rows of A are linearly independent (hence full rank) and $f^T(AA^T)^{-1}f = 0$ iff $\lambda = 0$ and $\lambda = 0$ iff the optimal solution to the unconstrained problem satisfies the constraints $\Rightarrow \mu = 0$ (which by definition does not hold, hence $\lambda \neq 0$).
Finally, the solution of the dual problem is:

$$-\sqrt{\epsilon}\left|\left|\frac{A^T(AA^T)^{-1}f}{f^T(AA^T)^{-1}f}\right|\right|$$

Since strong duality holds, this is also the optimal value of the primal problem, hence

$$\alpha^* = -\sqrt{\epsilon}\left|\left|\frac{A^T(AA^T)^{-1}f}{f^T(AA^T)^{-1}f}\right|\right|$$

the value of q for which the optimal solution is given by:

$$q^* = -\frac{A^T\mu}{||A^T\mu||}\sqrt{\epsilon} = -\frac{\frac{A^T(AA^T)^{-1}f}{f^T(AA^T)^{-1}f}}{\left|\left|\frac{A^T(AA^T)^{-1}f}{f^T(AA^T)^{-1}f}\right|\right|}\sqrt{\epsilon} = -\sqrt{\epsilon}\frac{A^T(AA^T)^{-1}f}{f^T(AA^T)^{-1}f\left|\left|\frac{A^T(AA^T)^{-1}f}{f^T(AA^T)^{-1}f}\right|\right|}$$

Solution 3.

1. Under the constraint $x_1 \geq 1$, $(x_1 - 1)^3$ is convex. Additionally, x_2^4 is convex since it is a composition of the convex function t^2 where it is a non-decreasing function on x_2^2, which is convex. Hence the objective is convex as a sum of convex $((x_1-1)^3, x_2^4)$ and affine functions $(6(x_1 - 1))$.
   Finally, the constraint is convex as a sum of two convex functions, in a similar fashion.

2. We start by assigning a multiplier to the first constraint and taking the minimum of the Lagrangian, noting that it is separable:
   $$\mathcal{L}(x,\lambda) = 2(x_1-1)^3 + x_2^4 + 6(x_1-1) + \lambda((x_1-1)^2 + x_2^2 - 4)$$
   $$\min_{x_1,x_2} \mathcal{L}(x,\lambda) = \min_{x_1 \geq 1}\{2(x_1-1)^3 + 6(x_1-1) + \lambda(x_1-1)^2\} + \min_{x_2}\{x_2^4 + \lambda x_2^2\} - 4\lambda = -4\lambda$$
   where the minimum is obtained for $x_1 = 1, x_2 = 0$.
   Hence, the dual is:
   $$\begin{aligned} &\max_{\lambda} -4\lambda \\ &\text{s.t. } \lambda \geq 0 \end{aligned}$$

3. Strong duality holds, since the problem and first constraint are convex over the set $x_1 \geq 1$, and the point $\hat{x} = (2,0)$ satisfies the constraint as a strict inequality: $(2 - 1)^2 + 0^2 = 1 < 4$.

4. Since strong duality holds, the optimal value of the dual is equal to the optimal value of the primal. Solving the dual gives us $\lambda = 0$, and this is also the optimal value of the primal. As the value of the primal is 0 when $x = (1,0)$, this is an optimal solution to the primal problem.

Solution 4.

1. To find a dual problem with one decision variable, we assign a multiplier to the constraint $a^T x \leq b$ and find the minimum of the Lagrangian:
   $$\mathcal{L}(x,\lambda) = \sum_{j=1}^{n} \frac{c_j}{x_j} + \lambda(a^T x - b) = \sum_{j=1}^{n} \left(\frac{c_j}{x_j} + \lambda a_j x_j\right) - \lambda b$$
   Minimizing the Lagrangian is a separable problem, therefore we can minimize on each of the coordinates of x separately.
   $$\begin{aligned} &\min_{x_j > 0} \frac{c_j}{x_j} + \lambda a_j x_j \\ 0 = \frac{\partial}{\partial x_j}\left(\frac{c_j}{x_j} + \lambda a_j x_j\right) &= -\frac{c_j}{x_j^2} + \lambda a_j \\ x_j &= \sqrt{\frac{c_j}{\lambda a_j}}, \lambda > 0 \end{aligned}$$

(Note if $\lambda = 0$ we get the impossible: $\frac{c_j}{x_j} = 0$ since $x, c \in \mathbb{R}^n_{++}$)
Therefore, the dual problem is:

$$\begin{aligned}
&\max_{\lambda} \sum_{j=1}^{n} \sqrt{\lambda c_j a_j} + \sum_{j=1}^{n} \sqrt{\lambda c_j a_j} - \lambda b = 2\sum_{j=1}^{n} \sqrt{\lambda c_j a_j} - \lambda b \\
&\text{s.t. } \lambda > 0
\end{aligned}$$

2. the function $f(t) = \frac{c}{t}$ is convex, and since x_j is an affine function of x, the objective function of the primal problem is a sum of convex functions (convex functions of affine functions of the variable). Additionally there exists some small enough $\hat{x} > 0$ such that $a^T x < b$ (for instance, select $\hat{x}$ such that $\hat{x}_j = \frac{b}{a_j n+\epsilon}$). Therefore, strong duality holds, and the optimal value of the dual problem is exactly the optimal value of the primal. Let us solve the dual problem by differentiation of the objective function (single variable maximization):

$$0 = \frac{\partial}{\partial \lambda}\left(2\sum_{j=1}^{n} \sqrt{\lambda c_j a_j} - \lambda b\right) = \frac{1}{\sqrt{\lambda}} \sum_{j=1}^{n} \sqrt{c_j a_j} - b$$

$$\Rightarrow \lambda = \left(\frac{1}{b}\sum_{j=1}^{n} \sqrt{c_j a_j}\right)^2$$

Plugging back in to the dual problem we have:

$$\max_{\lambda > 0} 2\sum_{j=1}^{n} \sqrt{\lambda c_j a_j} - \lambda b = 2\sum_{j=1}^{n} \sqrt{\left(\frac{1}{2}\sum_{i=1}^{n}\sqrt{c_i a_i}\right)^2 c_j a_j} - \left(\frac{1}{2}\sum_{j=1}^{n}\sqrt{c_j a_j}\right)^2 b =$$
$$\left(\frac{2}{b}\sum_{i=1}^{n}\sqrt{c_i a_i}\right)\sum_{j=1}^{n}\sqrt{c_j a_j} - \left(\frac{1}{b}\sum_{j=1}^{n}\sqrt{c_j a_j}\right)^2 b = \frac{1}{b}\left(\sum_{j=1}^{n}\sqrt{c_j a_j}\right)^2$$

This is also the optimal value of the primal due to duality, and as seen above,

$$x_i^* = \frac{1}{\sqrt{\lambda}}\sqrt{\frac{c_i}{a_i}} = \frac{b}{\sum_{j=1}^{n}\sqrt{c_j a_j}}\sqrt{\frac{c_i}{a_i}}$$

Solution 5.

1. The problem is convex as it can be represented in the following quadratic form $x^T A x + b^T x$, where $b^T = (-6, 2, 0)$ and $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. Since A is PSD, the above is convex. Similarly, the constraints can be represented as $c^T x \leq 0$ where $c^T = (2, 2, 1)$ and $x^T D x + d^T x = 0$ where $d^T = (-2, 4, 0)$ and $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Hence the constraints define convex sets (the first is an affine function and the second is a level set of a convex function).

2.

$$\mathcal{L}(\lambda, \mu, x) = -6x_1 + 2x_2 + 4x_3^2 + \lambda(2x_1 + 2x_2 + x_3) + \mu(-2x_1 + 4x_2 + x_3^2)$$

the objective function of the dual problem is:

$$\min_{x, x_2 \geq 0} -6x_1 + 2x_2 + 4x_3^2 + \lambda(2x_1 + 2x_2 + x_3) + \mu(-2x_1 + 4x_2 + x_3^2)$$

the problem is separable and can be written as:

$$\min_{x_1}(-6 + 2\lambda - 2\mu)x_1 + \min_{x_2 \geq 0}(2 + 2\lambda + 4\mu)x_2 + \min_{x_3}(4x_3^2 + \lambda x_3 + \mu x_3^2)$$

hence we require:

$$\begin{aligned}
&\lambda - \mu - 3 = 0 \Rightarrow \lambda = \mu + 3 \Rightarrow \mu \geq -3 \\
&\lambda + 2\mu + 1 \geq 0 \Rightarrow 3\mu + 4 \geq 0 \Rightarrow \mu \geq -\frac{4}{3} \\
&\mu \geq -4
\end{aligned}$$

The minimum over x_3 is attained at:

$$\begin{aligned}
&8x_3 + \lambda + 2\mu x_3 = 0 \\
&x_3(8 + 2\mu) = -\lambda \\
&\Rightarrow x_3 = \frac{-\lambda}{8 + 2\mu}
\end{aligned}$$

and $x_2 = 0$.

Hence the dual problem is:

$$\begin{aligned}
&\max_{\lambda, \mu} \quad -\frac{1}{2}\frac{\lambda^2}{8 + 2\mu} \\
&\text{s.t.} \quad \lambda \geq 0 \\
&\mu \geq -\frac{4}{3} \\
&\lambda - \mu - 3 = 0
\end{aligned}$$

3. The problem is equivalent to solving:

$$\begin{aligned}
-\min_{\lambda, \mu} \quad & \frac{1}{2}\frac{\lambda^2}{8 + 2\mu} \\
\text{s.t.} \quad & \lambda \geq 0 \\
& \mu \geq -\frac{4}{3} \\
& \lambda - \mu - 3 = 0
\end{aligned}$$

Since $8 + 2\mu \geq 0$ for any $\mu \geq -\frac{4}{3}$ the problem is convex (quad over lin) with affine equality/ineq constraints. Plugging in $\mu = \lambda - 3$ we have the following:

$$\begin{aligned} -\min_{\lambda,\mu} \ & \frac{1}{4}\frac{\lambda^2}{1+\lambda} \\ \text{s.t.} \quad & \lambda \geq \frac{5}{3} \end{aligned}$$

Assigning the multiplier $\eta \geq 0$ to the constraint, we have the following KKT conditions:

$$\begin{aligned} \nabla_\lambda \mathcal{L}(\lambda, \eta) &= \frac{1}{4}\frac{2\lambda(1+\lambda) - \lambda^2}{(1+\lambda)^2} - \eta = 0 \\ \eta\left(\frac{5}{3} - \lambda\right) &= 0 \end{aligned}$$

Either $\eta \neq 0$ and then $\lambda = \frac{5}{3}$, which gives us an optimal value of $-\frac{25}{96}$; or $\eta = 0$ and then:

$$\frac{1}{4}\frac{2\lambda(1+\lambda) - \lambda^2}{(1+\lambda)^2} = 0 \Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda = 0, 2$$

The solution $\lambda = 0$ is infeasible, and for $\lambda = 2$, we get a value of $-\frac{1}{3}$ which is smaller and therefore not optimal.

4. Since the equality constraint is a quadratic (and not an affine function) then general Slater's condition is not satisfied. As such, strong duality does not hold. This means the optimal value of the dual gives us a lower bound, but it does not allow us to deduce the optimal value of the primal.

5. Using the equality constraint, we can replace $x_1 = 2x_2 + \frac{1}{2}x_3^2$, and we get the following problem:

$$\begin{aligned} \min \ & x_3^2 - 10x_2 \\ \text{s.t.} \ & x_3^2 + 6x_2 + x_3 \leq 0 \\ & x_2 \geq 0 \end{aligned}$$

The objective function is convex since it is a quadratic function of x with a PSD coefficient matrix, and so is the constraint. Therefore, the problem is convex.
We can now assign a Lagrange multiplier to the first constraint and obtain the following Lagrangian:

$$\mathcal{L}(x, \lambda) = x_3^2 - 10x_2 + \lambda(x_3^2 + 6x_2 + x + 3)$$

and we can minimize separately, subject to $x_2 \geq 0$:

$$\begin{aligned} &\min_{x_2 \geq 0}(6\lambda - 10)x_2 = 0 \quad \Rightarrow \lambda \geq \frac{10}{6} \\ &\min_{x_3}(1+\lambda)x_3^2 + \lambda x_3 \quad \Rightarrow \left\{ x_3^* = -\frac{\lambda}{2(1+\lambda)} \right\} \Rightarrow \min_{x_3}(1+\lambda)x_3^2 + \lambda x_3 = -\frac{\lambda^2}{4(1+\lambda)} \end{aligned}$$

and the dual is:

$$\begin{aligned}
\max \quad & -\frac{\lambda^2}{4(1+\lambda)} \\
\text{s.t.} \quad & \frac{10}{6} - \lambda \leq 0
\end{aligned}$$

6. In this case strong duality holds, since the primal is convex and there exists some point $\hat{x}$ which satisfies Slater's condition, $\hat{x}_3^2 + 6\hat{x}_2 + \hat{x}_3 < 0$, for instance $\hat{x}_2 = 0.001, \hat{x}_3 = -\frac{1}{2}$. Therefore, the optimal value of the dual is equal to the optimal value of the primal. The dual problem can be solved with the KKT conditions (replacing $\max -q(\lambda)$ with $-\min q(\lambda)$):

$$\begin{aligned}
& 0 = \nabla_\lambda \mathcal{L}(\lambda, \eta) = \frac{2\lambda(1+\lambda) - \lambda^2}{4(1+\lambda)^2} - \eta \\
& \eta \left(\frac{10}{6} - \lambda \right) = 0 \\
& \Rightarrow 4\eta(1+\lambda)^2 = 2\lambda + \lambda^2 \Rightarrow (1-4\eta)\lambda^2 + 2(1-4\eta)\lambda - 4\eta = 0
\end{aligned}$$

if $\eta \neq \frac{1}{4}$:

$$\lambda_{1,2} = \frac{-2(1-4\eta) \pm \sqrt{4(1-4\eta)^2 + 16(1-4\eta)\eta}}{2(1-4\eta)} = -1 \pm \frac{1}{\sqrt{1-4\eta}}$$

$\Rightarrow 0 \leq \eta < \frac{1}{4}$. Since $\lambda \geq 0$ we have that $\lambda = -1 + \frac{1}{\sqrt{1-4\eta}}$. In addition, note that when $\eta = 0$ the optimal solution should satisfy the constraints, however, the optimal solution to the objective function ($\lambda \geq 0$) is $\lambda = 0 \Rightarrow \eta > 0 \Rightarrow 0 < \eta < \frac{1}{4}$.

Otherwise if $\eta = \frac{1}{4}$:

$$(1-4\eta)\lambda^2 + 2(1-4\eta)\lambda - 4\eta = 0 \Rightarrow -1 = 0$$

hence $\eta \neq \frac{1}{4}$.

Substituting $\lambda = -1 + \frac{1}{\sqrt{1-4\eta}}$ into the constraint:

$$\begin{aligned}
& \frac{5}{3} - \lambda = \frac{5}{3} + 1 - \frac{1}{\sqrt{1-4\eta}} = 0 \\
& \Rightarrow \sqrt{1-4\eta} = \frac{3}{8} \\
& \Rightarrow \eta = \frac{55}{256} \\
& \Rightarrow \lambda = \frac{10}{6}
\end{aligned}$$

plugging in, the optimal value of the dual (and since strong duality holds, the primal) function is $-\frac{25}{96}$.

Now, we can plug in λ^* and find that $x_3^* = -\frac{\lambda^*}{2+2\lambda^*} = -\frac{5}{16}$. Plugging in to the objective function of the primal, and using the optimal value, we obtain $x_2^* = \frac{55}{1536}$ (which obeys the original constraint). Finally, using the original equality constraint we have $x_1^* = 2x*_2 + \frac{1}{2}(x_3^*)^2 = \frac{77}{1536}$.