Optimization 1 - 098311

Winter 2021 - exam 2014-2015

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Question 1:

Consdier the following convex optimization problem:

$$\begin{aligned}
&\min\sqrt{2x_1^2+4x_1x_2+3x_2^2+1}+7\\
&s.t\left(\left(x_1^2+x_2^2\right)^2+1\right)^2\leq 10x_1\\
&\frac{x_1^2+4x_2^2+4x_1x_2}{2x_1+x_2+x_3}\leq 10\\
&1\leq x_1,x_2,x_3\leq 10
\end{aligned}$$

a)

prove that the problem is convex.

objective function:

$$\begin{aligned}
f\left(x_1,x_2,x_3\right) &= \sqrt{2x_1^2+4x_1x_2+3x_2^2+1}+7=\\
&=\sqrt{2x_1^2+4x_1x_2+2x_2^2+x_2^2+1}+7=\\
&=\sqrt{2\left(x_1^2+2x_1x_2+x_2^2\right)+x_2^2+1}+7=\\
&=\sqrt{2\left(x_1+x_2\right)^2+x_2^2+1}+7=\\
&=\left\|\begin{pmatrix}\sqrt{2}&\sqrt{2}&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|+7
\end{aligned}$$

the norm function is convex, hence $f\left(x_1,x_2,x_3\right)$ is a convex as a linear change in variables of a convex function (adding the constant of course preserve convexity).

first constraint:

$$\begin{aligned}
g_1\left(x_1,x_2,x_3\right) &= \left(\left(x_1^2+x_2^2\right)^2+1\right)^2-10x_1=\\
&=\left(\left\|\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right\|^4+1\right)^2-10x_1
\end{aligned}$$

the norm is a convex function hence $\left\|\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\right\|$ is convex as a linear change in the variables of a convex function. the norm is non negative and the function $f(x)=x^4$ is a non decreasing convex function over $\mathbb{R}_+$ thus $\left\|\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\right\|^4$ is convex. $\left\|\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\right\|^4+1$ is convex as a sum of two convex function, in addition the result is non negative and the function $f(x)=x^2$ is a non decreasing convex function over $\mathbb{R}_+$ thus $\left(\left\|\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\right\|^4+1\right)^2$ is a convex function. finally $g_1(x_1,x_2,x_3)$ is a convex function as a sum of two convex functions. thus the first constraint is convex as a level set of a convex function.

second constraint:

$$g_2(x_1,x_2,x_3)=\frac{x_1^2+4x_2^2+4x_1x_2}{2x_1+x_2+x_3}=\frac{(x_1+2x_2)^2}{2x_1+x_2+x_3}$$
$$=\frac{\left\|\begin{pmatrix}1 & 2 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\right\|^2}{\begin{pmatrix}2 & 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}}$$

since $1\leq x_1,x_2,x_3\leq 10$ from the third constraint, them $\begin{pmatrix}2 & 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}>0$ thus $g_2(x_1,x_2,x_3)$ is a convex function as a generalized quadratic over linear function in the correct domain. the second constraint is hence convex as a level set of a convex function.

third constraint

the third constraint is a bunch of linear constraints and hence defines a convex set.

the objective function is convex and all the constraints define a convex set, thus this is a convex optimization problem.

b)

Question 2:

Consider the following convex optimization problem

$$\min \|Ax+b\|_2 + \|Lx\|_1 + \|Mx\|_2^2 - \sum_{i=1}^{n} x_i \ln(x_i)$$

$$s.t x \geq 0$$

Write a dual problem.

define a new problem:

$$\min \|w\|_2 + \|y\|_1 + \|z\|_2^2 - \sum_{i=1}^{n} x_i \ln(x_i)$$

$$s.t w = Ax+b$$

$$y = Lx$$

$$z = Mx$$

$$x \geq 0$$

define the set:

$$X = \{x, w, y, z : x \geq 0\}$$

the Lagrangian:

$$L(x,y,w,z,\mu_1,\mu_2,\mu_3) = \|w\|_2 + \|y\|_1 + \|z\|_2^2 - \sum_{i=1}^{n} x_i \ln(x_i) + \mu_1^T(w - Ax - b) + \mu_2^T(y - Lx) + \mu_3^T(z - Mx)$$

where:

$$\mu_1 \in \mathbb{R}^m, \mu_2 \in \mathbb{R}^p, \mu_3 \in \mathbb{R}^q$$

now we can find the dual problem:

$$q(\mu_1,\mu_2,\mu_3) = \min_{(x,w,y,z)\in X} L(x,y,w,z,\mu_1,\mu_2,\mu_3) =$$

$$= \min_{(x,w,y,z)\in X} \|w\|_2 + \|y\|_1 + \|z\|_2^2 - \sum_{i=1}^{n} x_i \ln(x_i) + \mu_1^T(w - Ax - b) + \mu_2^T(y - Lx) + \mu_3^T(z - Mx) =$$

$$= \min_{x\in\mathbb{R}_+^n} \left(-\mu_1^T Ax - \mu_2^T Lx - \mu_3^T Mx - \sum_{i=1}^{n} x_i \ln(x_i)\right) + \min_{y\in\mathbb{R}^p} \left(\|y\|_1 + \mu_2^T y\right) + \min_{w\in\mathbb{R}^m} \left(\|w\|_2 + \mu_1^T w\right) + \min_{z\in\mathbb{R}^q} \left(\|z\|_2^2 + \mu_3^T\right.$$

let's solve the problem in y:

$$\min_{y\in\mathbb{R}^p} \left(\|y\|_1 + \mu_2^T y\right)$$

if $\|\mu_2\|_\infty > 1$ then the problem is unbounded from below because for $y = -\alpha \begin{pmatrix} 0 & . & . & . & 0 & sign\left(\mu_{2_k}\right) & 0 & . \end{pmatrix}$
for $k = \arg\max_i \left|\mu_{2_i}\right|$ we will get:

$$\|y\|_1 + \mu_2^T y = \alpha - \alpha \|\mu_2\|_\infty = \alpha \left(1 - \|\mu_2\|_\infty\right) \xrightarrow{\alpha \longrightarrow \infty} -\infty$$

if $\|\mu_2\|_\infty \leq 1$ then:

$$\|y\|_1 + \mu_2^T y \geq \|y\|_1 - \|\mu_1\|_\infty \|y\|_1 = \|y\|_1 \left(1 - \|\mu_1\|_\infty\right) \geq 0$$

and the lower bound is attained for $y = 0$, thus:

$$\min_{y \in \mathbb{R}^p} \left(\|y\|_1 + \mu_2^T y\right) = \begin{cases} 0 & \|\mu_2\|_\infty \leq 1 \\ -\infty & \|\mu_2\|_\infty > 1 \end{cases}$$

moving to w:

$$\min_{w \in \mathbb{R}^m} \left(\|w\|_2 + \mu_1^T w\right)$$

if $\|\mu_1\|_2 > 1$ then the problem is unbounded from below because for $w = -\alpha\mu_1$ we will get:

$$\|w\|_2 + \mu_1^T w = \|-\alpha\mu_1\|_2 - \alpha\mu_1^T \mu_1 = \alpha \|\mu_1\|_2 - \alpha \|\mu_1\|_2^2 = \alpha \|\mu_1\|_2 \left(1 - \|\mu_1\|_2\right) \xrightarrow{\alpha \longrightarrow \infty} -\infty$$

if $\|\mu_1\|_2 \leq 1$ then:

$$\|w\|_2 + \mu_1^T w \geq \|w\|_2 - \|\mu_1\|_2 \|w\|_2 = \|w\|_2 \left(1 - \|\mu_1\|_2\right) \geq 0$$

and the lower bound is attained for $w = 0$, thus:

$$\min_{w \in \mathbb{R}^m} \left(\|w\|_2 + \mu_1^T w\right) = \begin{cases} 0 & \|\mu_1\|_2 \leq 1 \\ -\infty & \|\mu_1\|_2 > 1 \end{cases}$$

moving to z:

$$\min_{z \in \mathbb{R}^q} \left(\|z\|_2^2 + \mu_3^T z\right) = \min_{z \in \mathbb{R}^q} \left(z^T I z + 2\left(\frac{1}{2}\mu_3\right)^T z\right)$$

this is an unconstrained optimization problem of a quadratic function with a P.D matrix, hence the optimal value is attained at:

$$z = -I^{-1}\frac{1}{2}\mu_3 = -\frac{1}{2}\mu_3$$

and the optimal value is:

$$-\left(\frac{1}{2}\mu_3\right)^T I^{-1} \left(\frac{1}{2}\mu_3\right) = -\frac{1}{4}\|\mu_3\|_2^2$$

and finally in x the problem is separable in the coordinates:

$$A = \min_{x\in\mathbb{R}_+^n}\left(-\mu_1^T Ax - \mu_2^T Lx - \mu_3^T Mx - \sum_{i=1}^n x_i \ln(x_i)\right) =$$
$$= -\min_{x\in\mathbb{R}_+^n}\left(\sum_{i=1}^n \left(\mu_1^T A\right)_i x_i + \left(\mu_2^T L\right)_i x_i + \left(\mu_3^T M\right)_i x_i + x_i \ln(x_i)\right) =$$
$$= -\min_{x\in\mathbb{R}_+^n}\left(\sum_{i=1}^n \left(\left(\mu_1^T A\right)_i + \left(\mu_2^T L\right)_i + \left(\mu_3^T M\right)_i + \ln(x_i)\right) x_i\right) =$$
$$= -\left(\sum_{i=1}^n \min_{x_i\in\mathbb{R}_+}\left(\left(\mu_1^T A\right)_i + \left(\mu_2^T L\right)_i + \left(\mu_3^T M\right)_i + \ln(x_i)\right) x_i\right)$$
$$\min_{x_i\in\mathbb{R}_+}\left(\left(\mu_1^T A\right)_i + \left(\mu_2^T L\right)_i + \left(\mu_3^T M\right)_i\right) x_i + x_i \ln(x_i)$$

denote $0\ln(0) = 0$.

the function is a convex function over $\mathbb{R}_+$ as a sum of convex functions, it is also coercive over $\mathbb{R}_+$ because as $x \longrightarrow \infty$ the function goes to ∞. thus a minimum is attained, and it must be attained at a stationery point:

$$\frac{\partial}{\partial x_i} = \left(\mu_1^T A\right)_i + \left(\mu_2^T L\right)_i + \left(\mu_3^T M\right)_i + x_i \cdot \frac{1}{x_i} + 1 \cdot \ln(x_i) = 0$$
$$\ln(x_i) = -1 - \left(\mu_1^T A\right)_i - \left(\mu_2^T L\right)_i - \left(\mu_3^T M\right)_i$$
$$x_i = e^{-\left(1+\left(\mu_1^T A\right)_i+\left(\mu_2^T L\right)_i+\left(\mu_3^T M\right)_i\right)}$$

and the function value is:

$$-e^{-\left(1+\left(\mu_1^T A\right)_i+\left(\mu_2^T L\right)_i+\left(\mu_3^T M\right)_i\right)}$$

hence:

$$-\left(\sum_{i=1}^n \min_{x_i\in\mathbb{R}_+}\left(\left(\mu_1^T A\right)_i + \left(\mu_2^T L\right)_i + \left(\mu_3^T M\right)_i + \ln(x_i)\right) x_i\right) =$$
$$= -\left(\sum_{i=1}^n -e^{-\left(1+\left(\mu_1^T A\right)_i+\left(\mu_2^T L\right)_i+\left(\mu_3^T M\right)_i\right)}\right) = \sum_{i=1}^n e^{-\left(1+\left(\mu_1^T A\right)_i+\left(\mu_2^T L\right)_i+\left(\mu_3^T M\right)_i\right)}$$

so sum, the dual problem is:

$$\max_{\mu_1\in\mathbb{R}^m,\mu_2\in\mathbb{R}^p,\mu_3\in\mathbb{R}^q} \sum_{i=1}^n \left(e^{-\left(1+\left(\mu_1^T A\right)_i+\left(\mu_2^T L\right)_i+\left(\mu_3^T M\right)_i\right)}\right) - \frac{1}{4}\|\mu_3\|_2^2 - \mu_1^T b$$
$$s.t \; \|\mu_2\|_\infty \le 1$$
$$\|\mu_1\|_2 \le 1$$

Question 3:

Consider the set:

$$S = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} w_i x_i^2 \leq 1 \right\}$$

where $w_1, w_2, ..., w_n \in \mathbb{R}_{++}$

a)

Write the problem of finding the orthogonal projection of a vector $y \in \mathbb{R}^m$ onto S (that is computing $P_S(y)$) as a convex optimization problem.

$$\min_{x \in \mathbb{R}^n} \|x - y\|^2$$
$$s.t \sum_{i=1}^{n} w_i x_i^2 \leq 1$$

let's see it is a convex problem.

the objective function:

the norm is a convex function, hence $\|x - y\|$ is a convex function as a linear change in variables of a convex function. the function $f(x) = x^2$ is a non decreasing convex function over $\mathbb{R}_+$ and $\|x - y\| \geq 0$, thus the objective function is convex.

the first constraint

$$\sum_{i=1}^{n} w_i x_i^2 = \left\| \begin{pmatrix} \sqrt{w_1} & & & & & \\ & \sqrt{w_2} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \sqrt{w_n} \end{pmatrix} x \right\|^2 = \|Wx\|^2$$

this is a convex function from the same reasons as the objective function, hence the constraint defines a convex set as a level set of a convex function.

the objective function is convex and the constraints defines a convex set, hence this is a convex optimization problem.

b)

$$\min_{x\in\mathbb{R}^n} \|x-y\|^2$$
$$s.t\ \|Wx\|^2 \leq 1$$

this is a convex optimization problem. the objective function is coercive over a closed set, thus a minimum is attained. genral slater holds, for example for $x = 0$

$$\|Wx\|^2 = 0 < 1$$

thus:

$$\{K.K.T\} = \{optimal\}$$

let's find the K.K.T points, the Lagrangian is:

$$\begin{aligned} L(x,\lambda) &= \|x-y\|^2 + \lambda\left(\|Wx\|^2 - 1\right) = \\ &= (x-y)^T(x-y) + \lambda\left((Wx)^T Wx - 1\right) = \\ &= x^Tx - 2x^Ty + y^Ty + \lambda\left(x^TW^TWx - 1\right) \end{aligned}$$

the K.K.T conditions are:

$$\begin{cases} \frac{\partial L(x,\lambda)}{\partial x} = 2x - 2y + 2\lambda W^TWx = 0 & (1) \\ \lambda\left(\|Wx\|^2 - 1\right) = 0 & (2) \\ \|Wx\|^2 \leq 1 & (3) \\ \lambda \geq 0 & (4) \end{cases}$$

if $\lambda = 0$ then (2) and (4) hold, and from (1):

$$2x - 2y = 0$$

$$x = y$$

from (3) we need to demand:

$$\|Wx\|^2 = \|Wy\|^2 \leq 1$$

which is no necessarily true, if it is true, then $x = y$ is a K.K.T point (basically y belongs to the set and we need no projection).

if $\lambda > 0$ then from (1):

$$x + \lambda W^T W x = y$$

$$\left(I + \lambda W^T W\right) x = y$$

since $w_1, w_2, ..., w_n \in \mathbb{R}_{++}$ then W is a P.D matrix, and hence $W^T W$ is invertiable, and also $I + \lambda W^T W$

$$x = \left(I + \lambda W^T W\right)^{-1} y$$

plug into (2):

$$\|Wx\|^2 = 1$$

$$\left\|W\left(I + \lambda W^T W\right)^{-1} y\right\|^2 = 1$$

$$\left\|W\left(I + \lambda W^T W\right)^{-1} y\right\|^2 - 1 = 0$$

denote:

$$\phi(\lambda) = \left\|W\left(I + \lambda W^T W\right)^{-1} y\right\|^2 - 1$$

since $\lambda > 0$, this is a strictly decreasing function in λ, thus has a unique root in $\mathbb{R}_{++}$. we can find the one dimensional root using the bisection algorithm

to sum:

$$P_S(y) = \begin{cases} y & \|Wy\|^2 \leq 1 \\ \left(I + \lambda W^T W\right)^{-1} y & \|Wy\|^2 > 1 \end{cases}$$

where λ is the sole root of $\phi(\lambda) = \left\|W\left(I + \lambda W^T W\right)^{-1} y\right\|^2 - 1$ over $\mathbb{R}_{++}$

Question 4:

Consider the problem:

$$\max x_1^3 + x_2^3 + x_3^3$$
$$s.t x_1^2 + x_2^2 + x_3^2 = 1$$

or

$$\min -x_1^3 - x_2^3 - x_3^3$$
$$s.t x_1^2 + x_2^2 + x_3^2 = 1$$

a)

is the problem convex?

No, the constraint is only the boundary of a 3D ball, which is of course not a convex set. Thus this is not a convex optimization problem.

b)

Prove that all the local maximum points of the problem are also K.K.T points.

first notice that the objective function is a continuously differentiable function, and the constraint defines a compact set(boundary of a ball) thus from Weierstrass theorem, the function attain a minimum value.

sine the constraint is is not convex, we only know that:

$$\emptyset \neq \{\text{optimal points}\} \subseteq \{\text{local optimal points}\} \subseteq \{K.K.T\} \cup \{\text{irregular points}\}$$

if we prove that there are no irregular points, then any local optimal point is also a K.K.T point. There is only one constaint whice is an equality constraint, thus an irregular points is a point that satisfies:

$$g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 1$$
$$\frac{\partial g(x_1, x_2, x_3)}{\partial x_1} = 2x_1 = 0 \Rightarrow x_1 = 0$$
$$\frac{\partial g(x_1, x_2, x_3)}{\partial x_2} = 2x_2 = 0 \Rightarrow x_2 = 0$$

$$\frac{\partial g(x_1,x_2,x_3)}{\partial x_3} = 2x_3 = 0 \Rightarrow x_3 = 0$$

but the point $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ doesn't satisfies the constraint:

$$x_1^2 + x_2^2 + x_3^2 = 0 \neq 1$$

$\boxed{\text{thus there are no irregular points, and every local optimal point is a K.K.T point.}}$

c)

Find all the K.K.T points of the problem.

the Lagrangian is:

$$L(x_1,x_2,x_3) = -x_1^3 - x_2^3 - x_3^3 + \mu\left(x_1^2 + x_2^2 + x_3^2 - 1\right)$$

where

$$\mu \in \mathbb{R}$$

the K.K.T conditions are:

$$\begin{cases} \begin{cases} \frac{\partial L(x_1,x_2,x_3)}{\partial x_1} = -3x_1^2 + 2\mu x_1 = 0 & (1) \\ \frac{\partial L(x_1,x_2,x_3)}{\partial x_2} = -3x_2^2 + 2\mu x_2 = 0 & (2) \\ \frac{\partial L(x_1,x_2,x_3)}{\partial x_3} = -3x_3^2 + 2\mu x_3 = 0 & (3) \end{cases} \\ x_1^2 + x_2^2 + x_3^2 = 1 & (4) \end{cases}$$

from (1):

$$x_1(2\mu - 3x_1) = 0$$

either $x_1 = 0$ or $2\mu = 3x_1$

in the same way to solve (2) we need either $x_2 = 0$ or $2\mu = 3x_2$ and to solve (3) we need either $x_3 = 0$ or $2\mu = 3x_3$

if $x_1 = x_2 = x_3 = 0$ then (4) is not satisfied.

if two of them are equal to 0 then from (4) the third one equals 1 and $\mu = \frac{3}{2}$.

if only one is equal to zero, let's say $x_1 = 0$, then:

$$\mu = \frac{3}{2}x_2 = \frac{3}{2}x_3$$

$$x_2 = x_3$$

thus from (4):

$$2x_2^2 = 2x_3^2 = 1$$

$$x_2 = x_3 = \pm\frac{1}{\sqrt{2}}$$

$$\mu = \pm\frac{3}{2\sqrt{2}}$$

if non of them equals 0 then:

$$\mu = \frac{3}{2}x_1 = \frac{3}{2}x_2 = \frac{3}{2}x_3$$

$$x_1 = x_2 = x_3$$

and from (4):

$$3x_1^2 = 3x_2^2 = 3x_3^2 = 1$$

$$x_1 = x_2 = x_3 = \pm\frac{1}{\sqrt{3}}$$

to summarize, the K.K.T points are:

(1) $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

(2) $\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \Big(- \frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}}$

(3) $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$

d)

Find the optimal solution of the problem.

We saw that an optimal solution is attained, and that it has to be a K.K.T point, so we just need to find which K.K.T points yeidls the optimal value.

for the set of points (1) the function value is

$$-x_1^3 - x_2^3 - x_3^3 = -1$$

for the set of points (2) with positive signs the function value is

$$-x_1^3 - x_2^3 - x_3^3 = -\frac{\sqrt{2}}{2}$$

for the set of points (2) with negative signs the function value is

$$-x_1^3 - x_2^3 - x_3^3 = \frac{\sqrt{2}}{2}$$

for the set of points (3) with positive signs the function value is

$$-x_1^3 - x_2^3 - x_3^3 = -\frac{\sqrt{3}}{3}$$

for the set of points (3) with negative signs the function value is

$$-x_1^3 - x_2^3 - x_3^3 = \frac{\sqrt{3}}{3}$$

thus the optimal value is attained at the non strict global minimum points:

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

and the value is:

$$-\frac{\sqrt{2}}{2}$$

Question 5:

Let $A \in \mathbb{R}^{mxn}$. Prove that the following two claims are equivalent:

(A) The system:

$$Ax = 0, x > 0$$

has no solution

(B) There exists a vector $y \in \mathbb{R}^n$for which $A^T y \leq 0$ and $A^T y$ is not the zeros vector

proof that $A \Rightarrow B$:

assume A is true, meaning the system:

$$Ax = 0, x > 0$$

has no solution.

then also the system :

$$A\left(x+e\right) = 0, x \geq 0$$

has no solution, since if it had a solution $y \geq 0$, then we can choose $z = \underbrace{y}_{\geq 0} + \underbrace{e}_{>0} > 0$, and:

$$Az = A\left(y+e\right) = 0$$

thus z was a feasible solution for the first problem.

The system

$$Ax = -Ae, x \geq 0$$

has no solution

then from Farkas lemma, there exists a vector $y \in \mathbb{R}^n$for which

$$A^T y \leq 0, -e^T A^T y > 0$$

in particular $A^T y \neq 0$, since if it was 0 then

$$-e^T A^T y = 0$$

proof that $B \Rightarrow A$:

assume that B is true, meaning that there exists a vector $y \in \mathbb{R}^n$for which $A^T y \leq 0$ and $A^T y$ is not the zeros vector. assume by contradiction that the system :

$$Ax = 0, x > 0$$

has a solution and z is the solution

then:

$$Az = 0, z > 0$$

$$0 = y^T \underbrace{Az}_{=0} = \underbrace{z^T}_{>0} \underbrace{A^T y}_{\substack{\leq 0 \\ \neq 0}} < 0$$

which is a contradiction.