Optimization 1 — Homework 2

October 29, 2020

Problem 1

(a) Let $\mathbf{a} \in \mathbb{R}^n$ be a nonzero vector and $\alpha \geq 0$. Explain why the problem of finding the maximum and minimum values of $\mathbf{a}^T\mathbf{x}$ over $\mathcal{B}[0,1] = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq \alpha\}$ attains its solutions. Also, show that the solutions are $\mathbf{x}_{\max} = \alpha\frac{\mathbf{a}}{\|\mathbf{a}\|}$ and $\mathbf{x}_{\min} = -\alpha\frac{\mathbf{a}}{\|\mathbf{a}\|}$, and that the maximal and minimal values are $\alpha\|\mathbf{a}\|$ and $-\alpha\|\mathbf{a}\|$, respectively.

(b) Using the previous section, find the maximum and minimum points and values of the function $3x - 5y$ over the constraint $x^2 + y^2 - 2y - 3 \leq 0$.

Problem 2

(a) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{m\times m}$ be two symmetric matrices. Prove that both $\mathbf{A}$ and $\mathbf{B}$ are positive semidefinite if and only if

$$\begin{pmatrix} \mathbf{A} & \mathbf{0}_{n\times m} \\ \mathbf{0}_{m\times n} & \mathbf{B} \end{pmatrix} \succeq 0.$$

(b) For the following two matrices, determine (without computing their eigenvalues) whether they are positive/negative semidefinite/definite or indefinite:

$$\mathbf{C} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(c) Find (without computing the eigenvalues) the intervals of $\alpha \in \mathbb{R}$ for which the matrix

$$\mathbf{E} = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ \alpha & 2 & \alpha & 0 \\ 0 & \alpha & 2 & \alpha \\ 0 & 0 & \alpha & 1 \end{pmatrix}$$

is positive/negative semidefinite/definite and indefinite.

Problem 3

For the following two functions, find the stationary points and classify them according to whether they are saddle, strict/nonstrict local/global minimum/maximum points.

(a) $f(x_1, x_2) = 2x_1^3 - 3x_1^2 - 6x_1x_2(x_1 - x_2 - 1)$.

(b) $f(x_1, x_2) = \left(x_1^2 + x_2^2\right)e^{-x_1^2 - x_2^2}$.

Problem 4

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function. Suppose that $\nabla^2 f(\mathbf{x}) \succ 0$ for any $\mathbf{x} \in \mathbb{R}^n$. Prove that a stationary point of f is necessarily a strict global minimum point.

Problem 5

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function. Let $\mathbf{x}^0$ be an interior point of U such that $\nabla^2 f\left(\mathbf{x}^0\right) \succ 0$. Prove that there exists $r > 0$ such that $\nabla^2 f(\mathbf{x}) \succ 0$ for any $\mathbf{x} \in B\left(\mathbf{x}^0, r\right)$.