Differentiability

Definition: A function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at point x if there exists a vector g such that

$$\lim_{\mathrm{d} \to 0} \frac{f(\mathrm{x}+\mathrm{d}) - f(\mathrm{x}) - \mathrm{g}^\top \mathrm{d}}{\|\mathrm{d}\|} = 0.$$

- If a function is differentiable at point x then all its directional derivative at point x exist and $f'(\mathrm{x}, \mathrm{d}) = \nabla f(\mathrm{x})^\top \mathrm{x}$.
- However, the existence of all partial derivative does not mean the function is differentiable.

Differentiability

Example: Consider the function

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0.0). \end{cases}$$

For every d it directional derivative at point $(0,0)$ exists, and is given by

$$f'(0,\mathrm{d}) = \lim_{t\to 0^+} \frac{t^3 d_1^2 d_2}{t^2(d_1^2+d_2^2)t} = \frac{d_1^2 d_2}{(d_1^2+d_2^2)},$$

and $\nabla f(0) = 0$. However, the function is not differentiable.

Differentiability

- If a function f is differentiable at point x it does not necessarily mean that its partial derivatives are continuous at x.

Example: The function

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,y) \neq (0,0), \\ 0 & (x,y) = (0.0). \end{cases}$$

is differentiable at point 0 but its partial derivatives are not continuous at 0.

- However, if the partial derivatives exist in a neighborhood of point x and are continuous at point x, then the function is differentiable at x.