Optimization 1 - 098311

Winter 2021 - HW 6

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Problem 1:

a)

We will show that $g(x) \triangleq f(x) - \sigma\frac{||x||^2}{2}$ is convex if and only if:

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y) - \frac{\sigma}{2}\lambda(1-\lambda)||x-y||^2$$

by the convex function definition it means that for every $x, y \in C$ and $\lambda \in [0,1]$:

$$g(\lambda x + (1-\lambda) y) \le \lambda g(x) + (1-\lambda) g(y) \iff f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y) - \frac{\sigma}{2}\lambda(1-\lambda)||x-y||^2$$

$$\begin{aligned}
& g(\lambda x + (1-\lambda) y) \le \lambda g(x) + (1-\lambda) g(y) \\
\iff & f((\lambda x + (1-\lambda) y)) - \sigma\frac{||(\lambda x + (1-\lambda) y)||^2}{2} \le \lambda\left(f(x) - \sigma\frac{||x||^2}{2}\right) + (1-\lambda)\left(f(y) - \sigma\frac{||y||^2}{2}\right) \\
\iff & f((\lambda x + (1-\lambda) y)) \le \sigma\frac{||(\lambda x + (1-\lambda) y)||^2}{2} + \lambda f(x) - \frac{\lambda\sigma}{2}||x||^2 + (1-\lambda) f(y) - \frac{(1-\lambda)\sigma}{2}||y||^2 \\
(\lambda \in [0,1]) \iff & f((\lambda x + (1-\lambda) y)) \le \lambda f(x) + (1-\lambda) f(y) + \frac{\sigma}{2}\left(||(\lambda x + (1-\lambda) y)||^2 - \lambda||x||^2 - (1-\lambda)||y||^2\right) \\
\iff & f((\lambda x + (1-\lambda) y)) \le \lambda f(x) + (1-\lambda) f(y) + \frac{\sigma}{2}\left(\lambda^2||x||^2 + 2\lambda(1-\lambda)\langle x, y\rangle + (1-\lambda)^2||y||^2 - \lambda||x||^2 - (1-\lambda)||y||^2\right) \\
\iff & f((\lambda x + (1-\lambda) y)) \le \lambda f(x) + (1-\lambda) f(y) + \frac{\sigma}{2}\left((\lambda^2 - \lambda)||x||^2 + 2\lambda(1-\lambda)\langle x, y\rangle + \left(1 - 2\lambda + \lambda^2 - 1 + \lambda\right)||y||^2\right) \\
\iff & f((\lambda x + (1-\lambda) y)) \le \lambda f(x) + (1-\lambda) f(y) - \frac{\sigma}{2}\left(\lambda(1-\lambda)||x||^2 - 2\lambda(1-\lambda)\langle x, y\rangle + \lambda(1-\lambda)||y||^2\right) \\
\iff & f((\lambda x + (1-\lambda) y)) \le \lambda f(x) + (1-\lambda) f(y) - \frac{\sigma}{2}\lambda(1-\lambda)\left(||x||^2 - 2\langle x, y\rangle + ||y||^2\right) \\
\iff & f((\lambda x + (1-\lambda) y)) \le \lambda f(x) + (1-\lambda) f(y) - \frac{\sigma}{2}\lambda(1-\lambda)\left(||x||^2 - 2\langle x, y\rangle + ||y||^2\right) \\
\iff & f((\lambda x + (1-\lambda) y)) \le \lambda f(x) + (1-\lambda) f(y) - \frac{\sigma}{2}\lambda(1-\lambda)||x-y||^2
\end{aligned}$$

$$\forall x, y \in C, \lambda \in [0,1]:$$

$$g(\lambda x + (1-\lambda) y) \le \lambda g(x) + (1-\lambda) g(y) \iff f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y) - \frac{\sigma}{2}\lambda(1-\lambda)||x-y||^2$$

which means $f(x)$ is strongly convex if and only if:

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y) - \frac{\sigma}{2}\lambda(1-\lambda)||x-y||^2$$

b)

let $x, y \in \mathbb{R}^n$

$f(x)$ is continuously differentiable over C and $-\sigma\frac{||x||^2}{2}$ is continuously differentiable over all $\mathbb{R}^n$, hence $g(x)$ is continuously differentiable over C.

therefore:

$$\begin{aligned}
& f(x) \text{ is strongly convex} \\
\text{by definition: } \iff & g(x) \text{ is convex} \\
\overset{(*)}{\iff} & g(y) \geq g(x) + \nabla g(x)^T (y-x) \\
\iff & f(y) - \sigma\frac{\|y\|^2}{2} \geq f(x) - \sigma\frac{\|x\|^2}{2} + \nabla\left(f(x) - \sigma\frac{\|x\|^2}{2}\right)^T (y-x) \\
\iff & f(y) - \sigma\frac{\|y\|^2}{2} \geq f(x) - \sigma\frac{\|x\|^2}{2} + \nabla f(x)^T (y-x) - \sigma x^T (y-x) \\
\iff & f(y) \geq f(x) + \nabla f(x)^T (y-x) - \sigma\frac{\|x\|^2}{2} + \sigma\frac{\|y\|^2}{2} - \sigma x^T y + \sigma\|x\|^2 \\
\iff & f(y) \geq f(x) + \nabla f(x)^T (y-x) + \frac{\sigma}{2}\left(\|x\|^2 - 2x^T y + \|y\|^2\right) \\
\iff & f(y) \geq f(x) + \nabla f(x)^T (y-x) + \frac{\sigma}{2}\|x-y\|^2
\end{aligned}$$

$(*)$ $g(x)$ is continuously differentiable over C

c)

let $x, y \in \mathbb{R}^n$

$$f(x) \text{ is strongly convex}$$

$$\text{from section b: } \iff \begin{cases} f(y) \geq f(x) + \nabla f(x)^T (y-x) + \frac{\sigma}{2}\|x-y\|^2 \\ f(x) \geq f(y) + \nabla f(y)^T (x-y) + \frac{\sigma}{2}\|x-y\|^2 \end{cases}$$

summing both inequalities:

$$\begin{aligned}
\iff & f(x) + f(y) \geq f(x) + f(y) + \nabla f(x)^T (y-x) - \nabla f(y)^T (y-x) + \sigma\|x-y\|^2 \\
\iff & 0 \geq (\nabla f(x) - \nabla f(y))^T (y-x) + \sigma\|x-y\|^2 \\
\iff & -(\nabla f(x) - \nabla f(y))^T (y-x) \geq \sigma\|x-y\|^2 \\
\iff & (\nabla f(y) - \nabla f(x))^T (y-x) \geq \sigma\|x-y\|^2
\end{aligned}$$

Problem 2:

a)

denote:

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{aligned} f(v) = f(x,y,z) &= \sqrt{2x^2 + 2y^2 + 5z^2 + 2xy + 2xz + 4yz - 4y + 64} \\ &= \sqrt{v^T \underbrace{\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}}_{Q} v + 2 \cdot \underbrace{(0,-2,0)}_{c^T} v + 64} \end{aligned}$$

let's first find the domain of $f(v)$.

first let's show that Q is a positive definite matrix:

$$M_1(Q) = 2 > 0$$

$$M_2(Q) = 4 - 1 = 3 > 0$$

$$\begin{aligned} M_3(Q) &= 2 \cdot (10-4) - 1 \cdot (10-1) + 1 \cdot (2-2) \\ &= 12 - 9 = 3 > 0 \end{aligned}$$

Since all the moments of Q are positive:

$$Q \succ 0$$

since Q is positive definite, the function under the square root is a quadratic function that has a strict global minimum at $v = -A^{-1}b$, let's find the minimum value attained:

$$\begin{aligned} b^T A^{-1}b - 2b^T A^{-1}b + c = c - b^T A^{-1}b \\ &= 64 - (0,2,0)\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}(0,2,0)^T \\ &= 64 - 2\cdot 1 \cdot 2 = 60 > 0 \end{aligned}$$

hence the function under the square root is always positive meaning that the domain of $f(v)$ is $\mathbb{R}^3$.

now for some $A \in \mathbb{R}^{3x3}$ and $b \in \mathbb{R}^3$ we can write:

$$\|Av + b\|^2 = (Av + b)^T (Av + b) = v^T A^T Av + 2b^T Av + b^T b$$

let's find a matrix A and a vector b such that:

$$A^T A = Q$$

$$b^T A = c^T$$

since Q is positive definite we can wright it using the Cholescky Decomposition:

$$Q = LL^T$$

since Q is positive definite then its invertible, hence L is invertible.

define:

$$A = L^T$$

$$b = L^{-1}c$$

then:

$$A^T A = LL^T = Q$$

$$b^T A = c^T \left(L^{-1}\right)^T L^T = c^T \left(LL^{-1}\right)^T = c^T$$

$$b^T b = \left(L^{-1}c\right)^T \left(L^{-1}c\right) = c^T \left(L^{-1}\right)^T L^{-1}c = c^T \left(LL^T\right)^{-1} c = c^T Q^{-1} c =$$

$$= \begin{pmatrix} 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = 4$$

now we can write:

$$\begin{aligned} f\left(v\right) &= \sqrt{v^TQv + 2\cdot c^Tv + 64} = \sqrt{v^TA^TAv + 2\cdot b^TAv + b^Tb + 60} = \\ &= \sqrt{\left\|Av+b\right\|^2 + 60} \end{aligned}$$

in the same way we showed in the tutorial that $f\left(x\right) = \sqrt{\left\|x\right\|^2+1}$ is convex we can show that $f\left(x\right) = \sqrt{\left\|x\right\|^2+60}$ is convex. thus also $f\left(v\right) = \sqrt{\left\|Av+b\right\|^2+60}$ is convex as a linear change in the coordinates of a convex function.

notice that:

$$f\left(v\right) = \left\| \begin{pmatrix} I \\ 0 \end{pmatrix} \left(Av+b\right) + \underbrace{\begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ \sqrt{60} \end{pmatrix}}_{d} \right\|$$

let's look at $Lev\left(-f\left(v\right), -2\right)$:

$$\left\{ v : -\left\| \begin{pmatrix} I \\ 0 \end{pmatrix} \left(Av+b\right) + d \right\| \le -2 \right\}$$

$$\left\{ v : \left\| \begin{pmatrix} I \\ 0 \end{pmatrix} \left(Av+b\right) + d \right\| \ge 2 \right\}$$

this is a complement of an ellipsoid, hence it is not convex, thus $f\left(v\right)$ is not quasi concave.

b)

The given function over $\mathbb{R}^2_{++}$ is given by:

$$\begin{aligned} f\left(x\right) &= \frac{x_1^4}{x_2^2} + \frac{x_2^4}{x_1^2} + 2x_1x_2 - \min\left\{\ln\left(x_1+x_2,\right), \ln\left(2x_1 + \frac{1}{2}x_2\right)\right\} \\ &= \underbrace{\left(\frac{x_1^2}{x_2} + \frac{x_2^2}{x_1}\right)^2}_{f_1(x)} + \underbrace{\left(-\min\left\{\ln\left(x_1+x_2,\right), \ln\left(2x_1+\frac{1}{2}x_2\right)\right\}\right)}_{f_2(x)} \end{aligned}$$

We will show that $f_1(x)$ and $f_2(x)$ are convex to conclude that f is convex as a summation of convex function

f_1:

$$f_1(x) = \left(\frac{x_1^2}{x_2} + \frac{x_2^2}{x_1}\right)^2 = \left(\frac{\left\|\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0_2\right\|^2}{(0,1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0} + \frac{\left\|\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0_2\right\|^2}{(1,0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0}\right)^2$$

we got a summation of two generalized quadratic-over-linear function which both are defined at the appropriate domain $(x_1 > 0, x_2 > 0)$.

Hence, each one of them is convex and the summation is convex.

Since $x_1, x_2 > 0$:

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_1} > 0$$

the image of the function inside the Parenthesis is contained in $\mathbb{R}_+$.

Moreover, as we saw in the lecture, $(\cdot)^2$ is a one-dimensional non-decreasing convex function when defined over $\mathbb{R}_{++}$.

Thus, $f_1(x)$ is convex.

f_2:

$$\begin{aligned} f_2(x) &= -\min\left\{\ln(x_1 + x_2,), \ln\left(2x_1 + \frac{1}{2}x_2\right)\right\} \\ (*) &= \max\left\{-\ln(x_1 + x_2), -\ln\left(2x_1 + \frac{1}{2}x_2\right)\right\} \\ &= \max\left\{\underbrace{-\ln\left(\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)}_{f_3(x)}, \underbrace{-\ln\left(\begin{pmatrix} 2 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)}_{f_4(x)}\right\} \end{aligned}$$

$(*)$ let $a, b \in \mathbb{R}$ such that $a \geq b$ (without the loss of generality):

$$-\min\{a, b\} \overset{a \geq b}{=} -a$$

$$\max\{-a,-b\} \overset{-a\geq -b}{=} -a = -\min\{a,b\}$$

Here as well, the image of of the argument of both ln's is contained in $\mathbb{R}_+(x_1,x_2>0)$.

first, let's show that $-\ln(\cdot)$ is a convex function over $\mathbb{R}_+$:

$$\frac{d(-\ln(s))}{ds} = -\frac{1}{s}$$
$$\frac{d^2(-\ln(s))}{ds^2} = \frac{1}{s^2} > 0, \forall s \in \mathbb{R}_+$$

hence, $-\ln(\cdot)$ is convex and since convexity is preserved under affine change of variables f_3, f_4 are convex as well.

finally, discrete max over convex functions is a convex function hence, f_2 is convex.

$\boxed{\Rightarrow f(x) \text{ is convex as a summation over two convex function}}$

Let's check if the function is quasi-concave:

for:

$$x = \begin{pmatrix} t \\ t \end{pmatrix}$$

$$\begin{aligned} f(x) &= \left(\frac{t^2}{t}+\frac{t^2}{t}\right)^2 - \min\{\ln(2t), \ln(2.5t)\} \\ &= 4t^2 - \ln(2t) \\ \Rightarrow -f(x) &= \ln(2t) - 4t^2 \end{aligned}$$

lets choose:

$$x = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, y = \begin{pmatrix} 0.225 \\ 0.225 \end{pmatrix}$$

we know:

$$-f(x) = \ln(1) - 1 = -1$$

and:

$$-f(y) = \ln(0.45) - 4 \cdot 0.225^2 = -1.001$$

but for:

$$z = \frac{1}{2}x + \frac{1}{2}y = \begin{pmatrix} 0.3625 \\ 0.3625 \end{pmatrix}$$

$$-f(z) = \ln(0.725) - 4 \cdot 0.3625^2 = -0.847$$

so for $a = -0.9$:

$$x, y \in \text{Lev}(-f, a)$$

but the convex combination:

$$z = \frac{1}{2}x + \frac{1}{2}y \notin \text{Lev}(-f, a)$$

hence $\text{Lev}(-f, a)$ is not a convex set

$\boxed{\text{There is an } a \in \mathbb{R} \text{ such that } \text{Lev}(-f, a) \text{ is not a convex set thus, } f \text{ is not quasi concave}}$

c)

We will show that the following function is convex over $\mathbb{R}^n_{++}$:

$$f(x) = \sum_{i=1}^{n} x_i \ln(x_i) - \left(\sum_{i=1}^{n} x_i\right) \ln\left(\sum_{i=1}^{n} x_i\right)$$

the partial derivatives are given by:

$$\begin{aligned}
\frac{\partial f(x)}{\partial x_j} &= x_j \cdot \frac{1}{x_j} + \ln(x_j) - 1 \cdot \ln\left(\sum_{i=1}^{n} x_i\right) - \left(\sum_{i=1}^{n} x_i\right) \cdot \frac{1}{\left(\sum_{i=1}^{n} x_i\right)} \cdot 1 \\
&= 1 + \ln(x_j) - \ln\left(\sum_{i=1}^{n} x_i\right) - 1 = \ln(x_j) - \ln\left(\sum_{i=1}^{n} x_i\right) \\
&= \ln\left(\frac{x_j}{\sum_{i=1}^{n} x_i}\right)
\end{aligned}$$

$$\ln\left(\sum_{j=1}^{n} \frac{x_j}{\sum_{i=1}^{n} x_i}\right) \geq \sum_{j=1}^{n} \ln\left(\frac{x_j}{\sum_{i=1}^{n} x_i}\right)$$

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_j} = \frac{1}{x_j} \cdot \mathbb{I}(x_j = x_k) - \frac{1}{\sum_{i=1}^{n} x_i}$$

$$\nabla^2 f(x) = \operatorname{diag}\left(\frac{1}{x}\right) - 1_n 1_n^T \frac{1}{\sum_{i=1}^n x_i}$$

let $v \in \mathbb{R}^n \backslash \{0_n\}$:

$$\begin{aligned}
v^T \nabla^2 f(x)\, v &= v^T \left(\operatorname{diag}\left(\frac{1}{x}\right) - 1_n 1_n^T \frac{1}{\sum_{i=1}^n x_i}\right) v \\
&= v^T \operatorname{diag}\left(\frac{1}{x}\right) v - v^T \left(1_n 1_n^T \frac{1}{\sum_{i=1}^n x_i}\right) v \\
&= \sum_{i=1}^n v_i^2 \cdot \frac{1}{x_i} - \frac{1}{\sum_{i=1}^n x_i} v^T \left(1_n 1_n^T\right) v \\
&= \sum_{i=1}^n v_i^2 \cdot \frac{1}{x_i} - \frac{1}{\sum_{i=1}^n x_i} v^T \begin{pmatrix} \sum_{i=1}^n v_i \\ \sum_{i=1}^n v_i \\ | \\ | \\ \sum_{i=1}^n v_i \end{pmatrix} \\
&= \sum_{i=1}^n v_i^2 \cdot \frac{1}{x_i} - \frac{1}{\sum_{i=1}^n x_i} \left(\sum_{i=1}^n v_i\right)^2 \\
(*) &\geq 0
\end{aligned}$$

define $f : \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ as follows:

$$f(z) = \frac{z_1^2}{z_2}$$

f is convex as a quadratic over linear function at the proper domain.

define:

$$z_i = \begin{pmatrix} v_i \\ x_i \end{pmatrix}$$

using Jensen inequality of convex function:

$$f\left(\frac{1}{n} z_1 + \frac{1}{n} z_2 + ... + \frac{1}{n} z_n\right) \leq \frac{1}{n} f(z_1) + \frac{1}{n} f(z_2) + ... + \frac{1}{n} f(z_n)$$

$$f\left(\frac{1}{n} \sum_{i=1}^n z_i\right) \leq \frac{1}{n} \sum_{i=1}^n f(z_i)$$

$$\frac{\left(\frac{1}{n}\sum_{i=1}^{n}v_i\right)^2}{\frac{1}{n}\sum_{i=1}^{n}x_i}\leq\frac{1}{n}\sum_{i=1}^{n}\frac{v_i^2}{x_i}$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{v_i^2}{x_i}-\frac{1}{n}\frac{\left(\sum_{i=1}^{n}v_i\right)^2}{\sum_{i=1}^{n}x_i}\geq 0$$

$$\sum_{i=1}^{n}\frac{v_i^2}{x_i}-\frac{\left(\sum_{i=1}^{n}v_i\right)^2}{\sum_{i=1}^{n}x_i}\geq 0$$

hence, the hessian of f is positive definite by definition.

Since f is twice continuously differentiable at the proper **open** domain we can conclude its convexity.

the function is not quasi concave, let's show a counter example:

we will show that $Lev\left(-f\left(x\right),0\right)$ is not convex.

first, if $n=1$ we will get:

$$f\left(x\right)=x\ln\left(x\right)-x\ln\left(x\right)=0$$

$f\left(x\right)$ is a constant function which is both concave and convex, and in particular quasi concave.

if $n\geq 2$:

let's take for example the vectors $x,y\in\mathbb{R}_{++}^{n}$

$$x=\begin{pmatrix}3\\1\\1\\.\\.\\.\\1\end{pmatrix},y=\begin{pmatrix}1\\3\\1\\.\\.\\.\\1\end{pmatrix}$$

then:

$$\begin{aligned}-f\left(x\right)&=-3\cdot\ln\left(3\right)-\left(n-1\right)1\ln\left(1\right)+\left(3+\left(n-1\right)\right)\cdot\ln\left(3+\left(n-1\right)\right)=\\&=-3\cdot\ln\left(3\right)+\left(n-1\right)\cdot\ln\left(n-1\right)\leq-3\cdot\ln\left(3\right)+\left(n-1\right)\cdot\ln\left(n-1\right)\end{aligned}$$

$$-f\left(y\right)=-3\cdot\ln\left(3\right)+\left(n-1\right)\cdot\ln\left(n-1\right)\leq-3\cdot\ln\left(3\right)+\left(n-1\right)\cdot\ln\left(n-1\right)$$

thus $x,y\in Lev\left(-f\left(x\right),-3\cdot\ln\left(3\right)+\left(n-1\right)\cdot\ln\left(n-1\right)\right)$

however:

$$\frac{1}{2}x+\frac{1}{2}y=\begin{pmatrix}2\\2\\1\\.\\.\\.\\1\end{pmatrix}$$

$$\begin{aligned}-f\left(\frac{1}{2}x+\frac{1}{2}y\right)&=-4\ln(2)+(4+n-2)\ln(4+n-2)=\\&=-4\ln(2)+(n+2)\ln(n+2)>-3\cdot\ln(3)+(n+2)\ln(n+2)>\\&\overset{(*)}{>}-3\cdot\ln(3)+(n-1)\cdot\ln(n-1)\end{aligned}$$

$(*)$ $g(x)=x\ln(x)$ is monotonically increasing for $x\geq e^{-1}$

$$\begin{aligned}&g^{'}(x)=\ln(x)+1\geq 0\\&\Longleftrightarrow \ln(x)\geq -1\\&\Longleftrightarrow x\geq e^{-1}\end{aligned}$$

thus $\frac{1}{2}x+\frac{1}{2}y\notin Lev(-f(x),-3\cdot\ln(3)+(n-1)\cdot\ln(n-1))$

we found 2 vectors $x,y\in Lev(-f(x),-3\cdot\ln(3)+(n-1)\cdot\ln(n-1))$ and $\lambda\in[0,1]$ for which $\lambda x+(1-\lambda)y\notin Lev(-f(x),-3\cdot\ln(3)+(n-1)\cdot\ln(n-1))$, therefore $Lev(-f(x),-3\cdot\ln(3)+(n-1)\cdot\ln(n-1))$ is not convex.

hence by definition, $f(x)$ is not quasi concave.

d)

$$f(x,y,z)\,,\ C=\mathbb{R}^n_+$$

$$f(x)=-\sqrt[n]{\prod_{i=1}^{n}x_i}=-\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}}$$

First we will show that f is convex in $\mathbb{R}^n_{++}$

Let $x, y \in \mathbb{R}^n_{++}$,we will show that the gradient inequality holds (f is continuous differentiable over $\mathbb{R}_+$).

Let's find the gradient of f at point $x \in \mathbb{R}^n_{++}$

$$\begin{aligned}
\frac{\partial f}{\partial x_j}(x) &= -\frac{1}{n}\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}-1} \cdot \frac{\prod_{i=1}^{n} x_i}{x_j} \\
&= -\frac{1}{nx_j}\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} = \frac{1}{nx_j} f(x) \\
&= \nabla f(x)_j
\end{aligned}$$

$$\begin{aligned}
f(x) + \nabla f(x)^T (y-x) &= f(x) + \sum_{i=1}^{n} \nabla f(x)_i (y_i - x_i) \\
&= f(x) + \sum_{i=1}^{n} \frac{1}{nx_i} f(x)(y_i - x_i) \\
&= f(x) \cdot \frac{1}{n}\left(n + \sum_{i=1}^{n} \frac{y_i - x_i}{x_i}\right) \\
&= \frac{f(x)}{n}\left(\sum_{i=1}^{n} 1 + \sum_{i=1}^{n} \frac{y_i - x_i}{x_i}\right) \\
&= \frac{f(x)}{n}\left(\sum_{i=1}^{n} \frac{y_i - x_i}{x_i} + 1\right) \\
&= f(x)\left(\frac{1}{n}\sum_{i=1}^{n} \frac{y_i}{x_i}\right) \\
(*) &\leq f(x)\left(\prod_{i=1}^{n} \frac{y_i}{x_i}\right)^{\frac{1}{n}} \\
&= f(x) \cdot \frac{-\left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}}}{-\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}} \\
&= f(x) \cdot \frac{f(y)}{f(x)} = f(y)
\end{aligned}$$

$(*)$ holds for two reasons:

1) since $\forall x \in \mathbb{R}^n_{++} : f(x) < 0$ as a negative multiplication of positive numbers.

2) since $\forall x_i, y_i \in \mathbb{R}^n_{++} : \frac{y_i}{x_i} > 0 \Rightarrow \frac{1}{n}\sum_{i=1}^{n} \frac{y_i}{x_i} \geq f(x)\left(\prod_{i=1}^{n} \frac{y_i}{x_i}\right)^{\frac{1}{n}}$ (by the AM-GM inequality)

Hence, f is convex over $\mathbb{R}_{++}$

Denote $S = \mathbb{R}^n_+ \backslash \mathbb{R}^n_{++}$.

we need to show that f is convex over $\mathbb{R}^n_+ = S \cup \mathbb{R}^n_{++}$

Let $x, y \in \mathbb{R}^n_+$

There are three cases:

1) $x, y \in \mathbb{R}^n_{++}$

shown above

2) $x, y \in S$

let $\lambda \in [0,1]$:

$$\begin{aligned} f(\lambda x + (1-\lambda) y) &\leq 0 = \lambda \cdot 0 + (1-\lambda) \cdot 0 \\ &= \lambda \cdot f(x) + (1-\lambda) \cdot f(y) \end{aligned}$$

where the inequality is true since one of the element of each x and y zeroes the multiplication defined by f.

3) $x \in \mathbb{R}_{++}, y \in S$ (without the loss of generality)

let $\lambda \in (0,1)$ (for $\lambda = \{0,1\}$ the inequality is trivial)

for the same reason stated above:

$$f(y) = 0$$

hence:

$$\begin{aligned} f(\lambda x + (1-\lambda) y) &= -\left(\prod_{i=1}^{n} \left(\underbrace{\lambda x_i}_{>0} + \underbrace{(1-\lambda) y_i}_{\geq 0}\right)\right)^{\frac{1}{n}} \\ &\leq -\left(\prod_{i=1}^{n} \lambda x_i\right)^{\frac{1}{n}} = -\left(\lambda^n \prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \\ &= \lambda f(x) + (1-\lambda) \cdot 0 \\ &= \lambda f(x) + (1-\lambda) f(y) \end{aligned}$$

The inequality which defined convexity holds in any case, hence:

$$\boxed{f(x) \text{ is convex over } \mathbb{R}^n_{++}}$$

for $n = 1$ we get:

$$f(x) = -x$$

which is both convex and concave, and in particular quasi concave.

if $n \geq 2$:

f is not quasi-concave:

Let's look at the following level set:

$$\begin{aligned}
Lev(-f, 0) &= \left\{x \in \mathbb{R}_+^n : -f(x) \leq 0\right\} \\
&= \left\{x \in \mathbb{R}_+^n : \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq 0\right\}
\end{aligned}$$

denote:

$$x = e_1, y = e - e_1$$

since both x and y have at least one zero element:

$$-f(x) = -f(y) = 0 \leq 0$$

hence:

$$x, y \in Lev(-f, 0)$$

but:

$$z = \frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}e$$

we get:

$$\begin{aligned}
-f(z) &= \left(\prod_{i=1}^{n} \frac{1}{2}\right)^{\frac{1}{n}} \\
&= \frac{1}{2} > 0
\end{aligned}$$

hence:

$$z \notin Lev(-f, 0)$$

we found a convex combination of elements in $Lev(-f, 0)$ which is not in the set thus:

$\boxed{Lev(-f, 0) \text{ is not convex and } f \text{ is not quasi-concave}}$

Problem 3:

a)

$$\left\{x \in \left(-\frac{1}{2}, \infty\right)^3 : (x_2 + x_3 + 1)(2x_1 + 2x_3 + 2)(3x_1 + 3x_2 + 3) \geq 1\right\}$$

define the function:

$$\begin{aligned} f(x_1, x_2, x_3) &= -\ln((x_2 + x_3 + 1)(2x_1 + 2x_3 + 2)(3x_1 + 3x_2 + 3)) = \\ &= -\ln(x_2 + x_3 + 1) - \ln(2x_1 + 2x_3 + 2) - \ln(3x_1 + 3x_2 + 3) = \\ &= -\ln(x_2 + x_3 + 1) - \ln(x_1 + x_3 + 1) - \ln(x_1 + x_2 + 1) - \ln(2) - \ln(3) \end{aligned}$$

$g(x_1, x_2, x_3) = x_2 + x_3 + 1 = (0, 1, 1)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + 1$ is a convex function in $\left(-\frac{1}{2}, \infty\right)^3$ because it is an affine function.

since $x_2, x_3 > -\frac{1}{2}$ the image of $g(x_1, x_2, x_3)$ is $\mathbb{R}^3_{++}$. In $\mathbb{R}^3_{++}$, the function $-\ln(x)$ is a non decreasing convex function, hence $-\ln(g(x_1, x_2, x_3))$ is a convex function.

in a similar matter we can show that $-\ln(x_1 + x_3 + 1)$ and $-\ln(x_1 + x_2 + 1)$ are convex functions in $\left(-\frac{1}{2}, \infty\right)^3$

in addition $-\ln(2) - \ln(3)$ is a convex function in $\left(-\frac{1}{2}, \infty\right)^3$ (an affine function).

Therefore we can conclude that $f(x_1, x_2, x_3)$ is a convex function in $\left(-\frac{1}{2}, \infty\right)^3$ as a summation of convex function in this domain.

Because $f(x_1, x_2, x_3)$ is a convex function in $\left(-\frac{1}{2}, \infty\right)^3$ than $Lev(f(x_1, x_2, x_3), 0)$ is convex in $\left(-\frac{1}{2}, \infty\right)^3$ meaning:

$$\left\{x \in \left(-\frac{1}{2}, \infty\right)^3 : f(x_1, x_2, x_3) \leq 0\right\} \text{ is convex}$$

$$\left\{x \in \left(-\frac{1}{2}, \infty\right)^3 : -\ln((x_2 + x_3 + 1)(2x_1 + 2x_3 + 2)(3x_1 + 3x_2 + 3)) \leq 0\right\} \text{ is convex}$$

$$\left\{x \in \left(-\frac{1}{2}, \infty\right)^3 : \ln((x_2 + x_3 + 1)(2x_1 + 2x_3 + 2)(3x_1 + 3x_2 + 3)) \geq 0\right\} \text{ is convex}$$

$$\boxed{\left\{x \in \left(-\frac{1}{2}, \infty\right)^3 : (x_2 + x_3 + 1)(2x_1 + 2x_3 + 2)(3x_1 + 3x_2 + 3) \geq 1\right\} \text{ is convex}}$$

b)

The set $S = \{x \in \mathbb{R}^n : x_1^2 \le x_2 x_3, x_2, x_3 \ge 0\}$ is convex

lets write the set as a union of two sets:

$$S = \left\{x \in \mathbb{R}^n : x_1^2 \le x_2 x_3, x_2, x_3 \ge 0\right\} = \underbrace{\left\{x \in \mathbb{R}^n : x_1^2 \le x_2 x_3, x_2 >, x_3 \ge 0\right\}}_{A} \cup \underbrace{\left\{x \in \mathbb{R}^n : x_1^2 \le x_2 x_3, x_2 = 0, x_3 \ge 0\right\}}_{B}$$

lets show that A is a convex set:

$$\begin{aligned} & x_1^2 \le x_2 x_3 \\ (x_2 > 0) \iff & \frac{x_1^2}{x_2} \le x_3 \\ \iff & \frac{x_1^2}{x_2} - x_3 \le 0 \end{aligned}$$

we got a summation between quadratic over linear function at its proper domain and a linear function which are both convex.

hence the summation is convex and the set A is a level set of a convex function.

hence, A is a convex set.

lets show that B is a convex set:

if x_2 is zero the only valid x_1 such that:

$$x_1^2 \le x_2 x_3 = 0$$

is:

$$x_1 = 0$$

hence:

$$B = \{x \in \mathbb{R}^n, x_1 = 0, x_2 = 0, x_3 \ge 0\}$$

which is of course a convex set

Now lets show that S is a convex set

Let $x, y \in \mathbb{S}, \lambda \in [0,1]$

denote:

$$z = \lambda x + (1 - \lambda) y$$

There are three cases:

Case 1

$x \in A, y \in A$

since A is a convex set:

$$z \in A \Rightarrow z \in A \cup B = S$$

Case 2

$x \in B, y \in B$

since B is a convex set:

$$z \in B \Rightarrow z \in A \cup B = S$$

Case 3

$x \in A, y \in B$ (without the loss of generality)

$$x \in A \Rightarrow x_1^2 \leq x_2 x_3, x_2 > 0, x_3 \geq 0$$

$$y \in B \Rightarrow y_1 = y_2 = 0, y_3 \geq 0$$

$$z_1 = \lambda x_1 + (1 - \lambda)\, y_1 = \lambda x_1$$

$$z_2 = \lambda x_2 + (1 - \lambda)\, y_2 = \lambda x_2$$

$$z_3 = \lambda x_3 + (1 - \lambda)\, y_3$$

$$\begin{aligned} z_1^2 &= \lambda^2 x_1^2 \leq \lambda^2 x_2 x_3 \leq \lambda x_2 \left[\lambda x_3 + \underbrace{(1 - \lambda)\, y_3}_{\geq 0} \right] \\ &= z_2 z_3 \Rightarrow z \in S \end{aligned}$$

in each one of the cases we got that the convex combination of x and y is inside S

$\boxed{\text{Thus, we can conclude that } S \text{ is a convex set.}}$

Section c

The set $\{x \in \mathbb{R}^n : ||x-u|| \leq ||x-v||\}$ is convex

Proof

First, let's try to define the set in a different way:

$$\begin{aligned} & ||x-u|| \leq ||x-v|| \\ \text{all positive } \Longleftrightarrow\ & ||x-u||^2 \leq ||x-v||^2 \\ \Longleftrightarrow\ & x^T x - 2u^T x + a^T a \leq x^T x - 2v^T x - v^T v \\ \Longleftrightarrow\ & -2\left(u^T - v^T\right) x + u^T u - v^T v \leq 0 \end{aligned}$$

hence:

$$\{x \in \mathbb{R}^n : ||x-u|| \leq ||x-v||\} = \left\{x \in \mathbb{R}^n : -2\left(u^T - v^T\right) x + u^T u - v^T v \leq 0\right\}$$

hence the set is half space which is convex

note:

if $u = v$ the inequality holds $\forall x \in \mathbb{R}^n$ and we know it is a convex set.