Optimization 1 — Tutorial 6

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Convex Function

A function $f\colon C \subseteq \mathbb{R}^n \to \mathbb{R}$ for a convex set C is called convex if

$$f\left(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}\right) \leq \lambda f\left(\mathbf{x}\right) + (1-\lambda) f\left(\mathbf{y}\right),$$

for any $\mathbf{x}, \mathbf{y} \in C$ and $\lambda \in [0,1]$. We say that f is strictly convex if the above is satisfied with a strict inequality and for every $\lambda \in (0,1)$. We say that f is concave if $-f$ is convex.

Level Set

The level set of $f\colon S \subseteq \mathbb{R}^n \to \mathbb{R}$ with level $\alpha \in \mathbb{R}$ is

$$\operatorname{Lev}\left(f, \alpha\right) = \left\{\mathbf{x} \in S\colon f\left(\mathbf{x}\right) \leq \alpha\right\} \subseteq \mathbb{R}^n.$$

Epigraph

The epigraph of a function $f\colon C \subseteq \mathbb{R}^n \to \mathbb{R}$ for a convex set C is

$$\operatorname{epi}\left(f\right) = \left\{(\mathbf{x}, t) \in C \times \mathbb{R}\colon f\left(\mathbf{x}\right) \leq t\right\} \subseteq \mathbb{R}^{n+1}.$$

Quasi-Convex

A function $f\colon C \subseteq \mathbb{R}^n \to \mathbb{R}$ for a convex set C is called quasi-convex if $\operatorname{Lev}\left(f, \alpha\right)$ is a convex set for every $\alpha \in \mathbb{R}$.

Remark: a convex function is quasi-convex, but a quasi-convex function is not necessarily convex.

Characterization of Convex Functions

Let $f\colon C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function defined over a convex set C.

1. Gradient Inequality: suppose that f is continuously differentiable over C. Then, f is convex if and only if
$$f\left(\mathbf{y}\right) \geq f\left(\mathbf{x}\right) + \nabla f\left(\mathbf{x}\right)^T\left(\mathbf{y} - \mathbf{x}\right), \qquad \forall \mathbf{x}, \mathbf{y} \in C.$$
2. Second-order Characterization: suppose that f is twice continuously differentiable over C and that C is open. Then f is convex if and only if $\nabla^2 f\left(\mathbf{x}\right) \succeq 0$ for any $\mathbf{x} \in \mathbb{R}$. Additionally, if $\nabla^2 f\left(\mathbf{x}\right) \succ 0$ for any $\mathbf{x} \in \mathbb{R}$ then f is strictly convex.

Jensen's Inequality

Let $f\colon C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a convex function defined over a convex set C. Then, for every $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in C$ and $\boldsymbol{\lambda} \in \triangle_k$ we have

$$f\left(\sum_{i=1}^{k} \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^{k} \lambda_i f\left(\mathbf{x}_i\right).$$

Problem 1

For each of the following functions, determine whether it is convex, concave, quasi-convex or quasi-concave.

(a) $f(x) = e^x - 1$ over $\mathbb{R}$.

(b) $f(x, y) = xy$ over $\mathbb{R}^2_{++}$.

(c) $f(x, y) = \frac{1}{xy}$ over $\mathbb{R}^2_{++}$.

(d) $f(x, y) = \frac{x}{y}$ over $\mathbb{R}^2_{++}$.

(e) $f(\mathbf{x}) = \sqrt{\|\mathbf{x}\|^2 + 1}$ over $\mathbb{R}^n$.

Problem 2

Arithmetic-Geometric Mean (AGM) Inequality. Prove that for any $x_1, x_2, \ldots, x_n \geq 0$, the following inequality holds:

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \sqrt[n]{\prod_{i=1}^{n} x_i}.$$

In general, for any $\boldsymbol{\lambda} \in \triangle_n$ we have

$$\sum_{i=1}^{n} \lambda_i x_i \geq \prod_{i=1}^{n} x_i^{\lambda_i}.$$

Problem 3

Young's Inequality. Prove that for any $s, t \geq 0$ and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$st \leq \frac{s^p}{p} + \frac{t^q}{q}.$$

Problem 4

Hölder's Inequality. Prove that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left|\mathbf{x}^T \mathbf{y}\right| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q.$$