Optimization 1 — Homework 8

December 10, 2020

Problem 1
Consider the set

$$\text{Box}\,[\mathbf{l},\mathbf{u}] \equiv \left\{\mathbf{x}\in\mathbb{R}^n\colon \mathbf{l}_i \le \mathbf{x}_i \le \mathbf{u}_i,\ i=1,2,\ldots,n\right\},$$

where $\mathbf{l},\mathbf{u}\in\mathbb{R}^n$ are given vectors that satisfy $\mathbf{l}\le\mathbf{u}$. Consider the minimization problem

$$(P)\qquad \min\left\{f\left(\mathbf{x}\right) : \mathbf{x}\in\text{Box}\,[\mathbf{l},\mathbf{u}]\right\},$$

where f is a continuously differentiable function over $\text{Box}\,[\mathbf{l},\mathbf{u}]$. Prove that $\mathbf{x}^*\in\text{Box}\,[\mathbf{l},\mathbf{u}]$ is a stationary point of (P) if and only if

$$\frac{\partial f}{\partial \mathbf{x}_i}\left(\mathbf{x}^*\right)\begin{cases}=0, & \mathbf{l}_i\le\mathbf{x}_i^*\le\mathbf{u}_i,\\ \le 0, & \mathbf{x}_i^*=\mathbf{u}_i,\\ \ge 0, & \mathbf{x}_i^*\le\mathbf{l}_i.\end{cases}$$

Problem 2
Consider the problem

$$\begin{aligned}\min\quad & f\left(\mathbf{x}\right)\\ \text{s.t.}\quad & \mathbf{x}\in\triangle_n,\end{aligned}$$

where $f\colon\mathbb{R}^n\to\mathbb{R}$ is a continuously differentiable function over $\triangle_n$. Show that $\mathbf{x}^*\in\triangle_n$ is a stationary point of the problem if and only if there exists $\mu\in\mathbb{R}$ such that

$$\frac{\partial f}{\partial \mathbf{x}_i}\left(\mathbf{x}^*\right)\begin{cases}=\mu, & \mathbf{x}_i^*>0,\\ \ge\mu, & \mathbf{x}_i^*=0.\end{cases}$$

Problem 3
Let $\mathbf{A}\in\mathbb{R}^{n\times n}$ be a symmetric matrix. Find all stationary points of the problem

$$\begin{aligned}\min\quad & \mathbf{x}^T\mathbf{A}\mathbf{x}\\ \text{s.t.}\quad & \|\mathbf{x}\|\le 1.\end{aligned}$$

Problem 4
In the source localization problem we are given m locations of sensors $\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_m\in\mathbb{R}^n$ and approximate distances $d_1,d_2,\ldots,d_m$ between the sensors and an unknown source located at $\mathbf{x}\in\mathbb{R}^n$, such that $d_i\approx\|\mathbf{x}-\mathbf{a}_i\|$ for all $i=1,2,\ldots,m$. The problem is to estimate the location of $\mathbf{x}$ by solving the following maximum-likelihood formulation

$$(ML)\qquad \min_{\mathbf{x}\in\mathbb{R}^n}\sum_{i=1}^{m}\left(\|\mathbf{x}-\mathbf{a}_i\|-d_i\right)^2.$$

(a) Show that (ML) is equivalent to the problem

$$(ML2) \qquad \min_{\substack{\mathbf{x}\in\mathbb{R}^n \\ \mathbf{u}_i\in\mathbb{R}^n}} f(\mathbf{x},\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_m) \equiv \sum_{i=1}^{m}\left(\|\mathbf{x}-\mathbf{a}_i\|^2 - 2d_i\mathbf{u}_i^T(\mathbf{x}-\mathbf{a}_i) + d_i^2\right).$$
$$\text{s.t. } \|\mathbf{u}_i\| \le 1 \ \forall i = 1,2,\ldots,m,$$

in the sense that $\mathbf{x}$ is an optimal solution of (ML) if and only if there exists $(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_m)$ such that $(\mathbf{x},\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_m)$ is an optimal solution of $(ML2)$.

(b) Find a Lipschitz constant $L_{\nabla f} \ge 0$ of the gradient of f.

(c) Consider the two-dimensional problem $(n = 2)$ with five anchors $(m = 5)$ and data generated by the MATLAB commands

```
randn('seed',317);
A = randn(2,5);
x_real = randn(2,1);
d = sqrt(sum((A - x * ones(1,5)).^2)) + 0.05 * randn(1,5);
d = d';
```

The columns of the 2×5 matrix $\mathbf{A}$ are the locations of the five sensors, $\mathbf{x}^{\text{real}}$ is the true location of the source and $\mathbf{d}$ is the vector of noisy measurements between the source and the sensors. Write a MATLAB function that implements the Projected Gradient algorithm employed on $(ML2)$ for the generated data. Use the following step-size selection strategies

(i) constant step size $\frac{1}{L_{\nabla f}}$.

(ii) backtracking with parameters $s = 1$ and $\alpha = \beta = \frac{1}{2}$.

Initialize both methods with the vectors $\mathbf{x}^0 = (1000, -500)^T$ and $\mathbf{u}_i = \mathbf{0}_2$ for all $i = 1,2,\ldots,m$. Run both methods for 100 iterations and compare their performance by plotting, in the same graph with a logarithmic scale in the y-axis, their original function values and relative errors along the iterations, where

$$(\text{relative error})_k = \left\|\mathbf{x}^{\text{real}} - \mathbf{x}^k\right\|, \ k = 0,1,2,\ldots,100.$$

Finally, plot in a second graph the location of the sensors and the output source $\mathbf{x}^{100}$ for each of the two methods.