Optimization 1 - 098311

Winter 2021 - HW 8

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Problem 1:

Denote the following optimization problem P:

$$(P) : \min\left\{f\left(x\right) : x \in \text{Box}\left[l, u\right]\right\}$$

where f is a continuously differentiable function over the box and $l \leq u \in \mathbb{R}^n$

We need to show that:

$$x^* \text{ is a stationary point of } P \iff \frac{\partial f}{\partial x_i}\left(x^*\right)\begin{cases} = 0 & l_i < x_i^* < u_i \\ \leq 0 & x_i^* = u_i \\ \geq 0 & x_i^* = l_i \end{cases}$$

proof of $\Rightarrow$

We know x^* is a stationary point.

Let's assume by contradiction that

$$\frac{\partial f}{\partial x_i}\left(x^*\right)\begin{cases} = 0 & l_i < x_i^* < u_i \\ \leq 0 & x_i^* = u_i \\ \geq 0 & x_i^* = l_i \end{cases}$$

is not satisfied.

There are three cases which it happens:

case 1

$$\exists i : l_i < x_i^* < u_i$$

but:

$$\frac{\partial f}{\partial x_i}\left(x^*\right) \neq 0$$

if $\frac{\partial f}{\partial x_i}\left(x^*\right) > 0$

lets choose $x \in \text{Box}\left[l, u\right]$ such that:

$$x_j = \begin{cases} x_j^* & j \neq i \\ l_i & j = i \end{cases}$$

$$
\begin{aligned}
\nabla f\left(x^*\right)^T\left(x-x^*\right) &= \sum_{k=1}^{n} \nabla f\left(x^*\right)_k\left(x_k-x_k^*\right) \\
&= \underbrace{\nabla f\left(x^*\right)_i}_{>0} \underbrace{\left(l_i-x_i^*\right)}_{<0} < 0
\end{aligned}
$$

We got a contradiction to the stationarity of x^*

if $\frac{\partial f}{\partial x_i}\left(x^*\right) < 0$

lets choose $x \in \text{Box}\left[l,u\right]$ such that:

$$
x_j = \begin{cases} x_j^* & j \neq i \\ u_i & j = i \end{cases}
$$

$$
\begin{aligned}
\nabla f\left(x^*\right)^T\left(x-x^*\right) &= \sum_{k=1}^{n} \nabla f\left(x^*\right)_k\left(x_k-x_k^*\right) \\
&= \underbrace{\nabla f\left(x^*\right)_i}_{<0} \underbrace{\left(u_i-x_i^*\right)}_{>0} < 0
\end{aligned}
$$

We got a contradiction to the stationarity of x^*

case 2

$$
\exists i : x_i^* = u_i
$$

but:

$$
\frac{\partial f}{\partial x_i}\left(x^*\right) > 0
$$

lets choose $x \in \text{Box}\left[l,u\right]$ such that:

$$
x_j = \begin{cases} x_j^* & j \neq i \\ \frac{l_j+u_j}{2} & j = i \end{cases}
$$

$$
\begin{aligned}
\nabla f\left(x^*\right)^T\left(x-x^*\right) &= \sum_{k=1}^{n} \nabla f\left(x^*\right)_k\left(x_k-x_k^*\right) \\
&= \nabla f\left(x^*\right)_i\left(\frac{l_i+u_i}{2}-u_i\right) \\
&= \underbrace{\nabla f\left(x^*\right)_i}_{>0} \underbrace{\left(\frac{l_i-u_i}{2}\right)}_{<0} < 0
\end{aligned}
$$

We got a contradiction to the stationarity of x^*

case 3

$$\exists i : x_i^* = l_i$$

but:

$$\frac{\partial f}{\partial x_i}(x^*) < 0$$

lets choose $x \in \text{Box}[l, u]$ such that:

$$x_j = \begin{cases} x_j^* & j \neq i \\ \frac{l_j + u_j}{2} & j = i \end{cases}$$

$$\begin{aligned} \nabla f(x^*)^T (x - x^*) &= \sum_{k=1}^{n} \nabla f(x^*)_k (x_k - x_k^*) \\ &= \nabla f(x^*)_i \left(\frac{l_i + u_i}{2} - l_i \right) \\ &= \underbrace{\nabla f(x^*)_i}_{<0} \underbrace{\left(\frac{u_i - l_i}{2} \right)}_{>0} < 0 \end{aligned}$$

We got a contradiction to the stationarity of x^*

each one of the possible cases led to a contradiction, thus:

$$x^* \text{ is a stationary point of } P \Rightarrow \frac{\partial f}{\partial x_i}(x^*) \begin{cases} = 0 & l_i < x_i^* < u_i \\ \leq 0 & x_i^* = u_i \\ \geq 0 & x_i^* = l_i \end{cases}$$

proof of $\Leftarrow$

we know that:

$$\frac{\partial f}{\partial x_i}(x^*) \begin{cases} = 0 & l_i < x_i^* < u_i \\ \leq 0 & x_i^* = u_i \\ \geq 0 & x_i^* = l_i \end{cases}$$

We will show that x^* is a stationary point of P by definition:

let $x \in \text{Box}\left[l, u\right]$, meaning:

$$\forall i \in \{1, ..., n\} : l_i \leq x_i \leq u_i$$

$$\begin{aligned}
\nabla f\left(x^*\right)^T\left(x - x^*\right) &= \sum_{i=1}^{n} \nabla f\left(x^*\right)_i\left(x_i - x_i^*\right) \\
&= \sum_{i=1}^{n} \nabla f\left(x^*\right)_i\left(x_i - x_i^*\right)\mathbb{I}\left\{l_i < x_i^* < u_i\right\} + \sum_{i=1}^{n} \nabla f\left(x^*\right)_i\left(x_i - x_i^*\right)\mathbb{I}\left\{x_i^* = l_i\right\} + \sum_{i=1}^{n} \nabla f\left(x^*\right)_i (\\
&= 0 + \sum_{i=1}^{n} \underbrace{\nabla f\left(x^*\right)_i}_{\geq 0}\underbrace{\left(x_i - l_i\right)}_{\geq 0}\underbrace{\mathbb{I}\left\{x_i^* = l_i\right\}}_{\geq 0} + \sum_{i=1}^{n} \underbrace{\nabla f\left(x^*\right)_i}_{\leq 0}\underbrace{\left(x_i - u_i\right)}_{\leq 0}\underbrace{\mathbb{I}\left\{x_i^* = u_i\right\}}_{\geq 0} \geq 0
\end{aligned}$$

Thus, x^* is a stationary point, meaning:

$$x^* \text{ is a stationary point of } P \Leftarrow \frac{\partial f}{\partial x_i}\left(x^*\right)\begin{cases} = 0 & l_i < x_i^* < u_i \\ \leq 0 & x_i^* = u_i \\ \geq 0 & x_i^* = l_i \end{cases}$$

To conclude:

$$\boxed{x^* \text{ is a stationary point of } P \iff \frac{\partial f}{\partial x_i}\left(x^*\right)\begin{cases} = 0 & l_i < x_i^* < u_i \\ \leq 0 & x_i^* = u_i \\ \geq 0 & x_i^* = l_i \end{cases}}$$

Problem 2:

Denote the following optimization problem P:

$$(P) : \min \{ f(x) : x \in \Delta_n \}$$

where f is a continuously differentiable function over Δ_n.

$$x^* \text{ is a stationary point of } P \iff \exists \mu \in \mathbb{R} : \frac{\partial f}{\partial x_i}(x^*) \begin{cases} = \mu & x_i^* > 0 \\ \geq \mu & x_i^* = 0 \end{cases}$$

proof of $\Rightarrow$

We know x^* is a stationary point.

Let's assume by contradiction that

$$\exists \mu \in \mathbb{R} : \frac{\partial f}{\partial x_i}(x^*) \begin{cases} = \mu & x_i^* > 0 \\ \geq \mu & x_i^* = 0 \end{cases}$$

is not satisfied.

meaning:

$$\forall \mu \in \mathbb{R}, \exists i : \frac{\partial f}{\partial x_i}(x^*) \neq \mu \cap x_i^* > 0 \text{ Or } \frac{\partial f}{\partial x_i}(x^*) < \mu \cap x_i^* = 0$$

especially for:

$$\mu = \min_{k \in \left\{ i : x_i^* > 0 \right\}} \nabla f(x^*)_k$$

The second condition will never hold since:

$$\mu = \min_{k \in \left\{ i : x_i^* > 0 \right\}} \nabla f(x^*)_k \leq \nabla f(x^*)_i < \mu \text{ (contradition)}$$

hence:

$$\exists i : \nabla f(x^*)_i \neq \mu \cap x_i^* > 0$$

since $\mu = \min_{k \in \left\{ i : x_i^* > 0 \right\}} \frac{\partial}{\partial x_k} f(x^*)$ we can conclude:

$$\exists i : \nabla f(x^*)_i > \mu$$

denote:

$$l = \arg\min_{k\in\{i:x_i>0\}} \nabla f\left(x^*\right)_k$$

notice that:

since $\nabla f\left(x^*\right)_i > \mu$:

$$l \neq i$$

Lets choose $x = e_l \in \Delta_n$:

$$\begin{aligned}
\nabla f\left(x^*\right)^T\left(x-x^*\right) &= \sum_{k=1}^{n} \nabla f\left(x^*\right)_k\left(x_k - x_k^*\right) \\
&= \sum_{k=1}^{n} \nabla f\left(x^*\right)_k x_k - \sum_{k=1}^{n} \nabla f\left(x^*\right)_k x_k^* \\
&= \nabla f\left(x^*\right)_l - \sum_{k=1}^{n} \nabla f\left(x^*\right)_k x_k^* \\
&= \nabla f\left(x^*\right)_l - \left(\underbrace{\nabla f\left(x^*\right)_i}_{>\mu}\underbrace{x_i^*}_{>0} + \sum_{\substack{k=1\\k\neq i}}^{n}\underbrace{\nabla f\left(x^*\right)_k}_{\geq\mu}\underbrace{x_k^*}_{\geq 0}\right) \\
&< \nabla f\left(x^*\right)_l - \left(\mu x_i^* + \mu\sum_{\substack{k=1\\k\neq i}}^{n} x_k^*\right) \\
&= \mu - \mu\sum_{k=1}^{n} x_k^* \\
&= \mu - \mu = 0
\end{aligned}$$

We got a contradiction to the stationarity of x^*, thus:

$$x^* \text{ is a stationary point of } P \Rightarrow \exists\mu\in\mathbb{R} : \frac{\partial f}{\partial x_i}\left(x^*\right)\begin{cases} = \mu & x_i^* > 0 \\ \geq \mu & x_i^* = 0\end{cases}$$

proof of $\Leftarrow$

we know that:

$$\exists\mu\in\mathbb{R} : \frac{\partial f}{\partial x_i}\left(x^*\right)\begin{cases} = \mu & x_i^* > 0 \\ \geq \mu & x_i^* = 0\end{cases}$$

We will show that x^* is a stationary point of P by definition:

let $x \in \Delta_n$, meaning:

$$\sum_{i=1}^{n} x_i = 1,\ \forall i : x_i \geq 0$$

$$\begin{aligned}
\nabla f\left(x^*\right)^T\left(x - x^*\right) &= \sum_{i=1}^{n} \nabla f\left(x^*\right)_i \left(x_i - x_i^*\right) \\
&= \sum_{i=1}^{n} \nabla f\left(x^*\right)_i \left(x_i - x_i^*\right) \mathbb{I}\left\{x_i^* > 0\right\} + \sum_{i=1}^{n} \nabla f\left(x^*\right)_i \left(x_i - x_i^*\right) \mathbb{I}\left\{x_i^* = 0\right\} \\
&= \mu \sum_{i=1}^{n} \left(x_i - x_i^*\right) \mathbb{I}\left\{x_i^* > 0\right\} + \sum_{i=1}^{n} \underbrace{\nabla f\left(x^*\right)_i}_{\geq \mu} \underbrace{x_i \mathbb{I}\left\{x_i^* = 0\right\}}_{\geq 0} \\
&\geq \mu \sum_{i=1}^{n} \left(x_i - x_i^*\right) \mathbb{I}\left\{x_i^* > 0\right\} + \mu \sum_{i=1}^{n} x_i \mathbb{I}\left\{x_i^* = 0\right\} \\
&= \mu \left(\sum_{i=1}^{n} x_i \mathbb{I}\left\{x_i^* > 0\right\} - \sum_{i=1}^{n} x_i^* \mathbb{I}\left\{x_i^* > 0\right\} + \sum_{i=1}^{n} x_i \mathbb{I}\left\{x_i^* = 0\right\} \right) \\
&= \mu \left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i^* \mathbb{I}\left\{x_i^* > 0\right\} \right) \\
&= \mu \left(1 - \sum_{i=1}^{n} x_i^* \mathbb{I}\left\{x_i^* > 0\right\} \right) \\
(*) &= \mu \left(1 - \sum_{i=1}^{n} x_i^* \mathbb{I}\left\{x_i^* \geq 0\right\} \right) \\
&= \mu \left(1 - 1\right) = 0
\end{aligned}$$

$(*)$ adding zeros to the summation

Thus, x^* is a stationary point, meaning:

$$x^* \text{ is a stationart point of } P \Leftarrow \exists \mu \in \mathbb{R} : \frac{\partial f}{\partial x_i}\left(x^*\right) \begin{cases} = \mu & x_i^* > 0 \\ \geq \mu & x_i^* = 0 \end{cases}$$

To conclude:

$$\boxed{x^* \text{ is a stationary point of } P \iff \exists \mu \in \mathbb{R} : \frac{\partial f}{\partial x_i}\left(x^*\right) \begin{cases} = \mu & x_i^* > 0 \\ \geq \mu & x_i^* = 0 \end{cases}}$$

Problem 3:

As was mentioned in class, stationarity in the unit ball is equivalence to the following condition:

$$\nabla f\left(x^{*}\right)=0 \text{ OR } \left(\|x^{*}\|=1 \text{ AND } \exists\lambda\leq 0:\nabla f\left(x^{*}\right)=\lambda x^{*}\right)$$

the first condition holds if and only if:

$$\begin{aligned}
&\nabla f\left(x^{*}\right)=0\\
\iff&2Ax^{*}=0\\
\iff&Ax^{*}=0\\
\iff&\boxed{x^{*}=0 \text{ OR } x^{*} \text{ is a singular eigen vector of } A}
\end{aligned}$$

the second condition holds if and only if

$$\begin{aligned}
&\exists\lambda\leq 0:\nabla f\left(x^{*}\right)=\lambda x^{*}\\
\iff&2Ax^{*}=\lambda x^{*}\\
\iff&Ax^{*}=\underbrace{\frac{\lambda}{2}}_{\leq 0}x^{*}\\
\iff&x^{*}=0 \text{ OR } x^{*} \text{ is a normalized eigen vector of } A \text{ with a non-positive eigen value}
\end{aligned}$$

To conclude, the stationary points are:

$$\begin{aligned}
&x^{*}=0\\
&\quad OR\\
&x^{*}=\text{singular eigen vector of } A \text{ such that } \|x^{*}\|\leq 1\\
&\quad OR\\
&x^{*}=\text{normalized eigen vector of } A \text{ with a negative eigen value}
\end{aligned}$$

Problem 4:

a)

Denote the following problems:

$$(ML)\ \min_{x\in\mathbb{R}^n}\sum_{i=1}^{m}\left(||x-a_i||-d_i\right)^2$$

$$(ML2)\ \min_{\substack{x\in\mathbb{R}^n\\ u_i\in B[0,1]}} f\left(x,u_1,u_2,...,u_m\right)=\sum_{i=1}^{m}\left(||x-a_i||^2-2d_iu_i^T\left(x-a_i\right)+d_i^2\right)$$

We need to show that x^* is an optimal solution of $(ML)\iff\exists u_1^*,...,u_m^*\in B\left[0,1\right]$ such that $x^*,u_1^*,...,u_m^*$ are an optimal solution of $(ML2)$.

$\Rightarrow$

we know that:

$$x^*\in\arg\min_{x\in\mathbb{R}^n}\sum_{i=1}^{m}\left(||x-a_i||-d_i\right)^2$$

notice that $\forall x\in\mathbb{R}^n,u_1,...,u_m\in B\left[0,1\right]$:

$$\begin{aligned}
f\left(x,u_1,u_2,...,u_m\right)&=\sum_{i=1}^{m}\left(||x-a_i||^2-2d_iu_i^T\left(x-a_i\right)+d_i^2\right)\\
(C.S,\ d_i\geq0)&\geq\sum_{i=1}^{m}\left(||x-a_i||^2-2d_i\,||u_i||\,||x-a_i||+d_i^2\right)\\
(||u_i||\leq1)&\geq\sum_{i=1}^{m}\left(||x-a_i||^2-2d_i\,||x-a_i||+d_i^2\right)\\
&=\sum_{i=1}^{m}\left(||x-a_i||-d_i\right)^2
\end{aligned}$$

which means that the objective function of (ML) is an lower bound of the objective of $(ML2)$.

Given x^* lets choose:

$$u_i^*=\begin{cases}\frac{x^*-a_i}{||x^*-a_i||} & x^*\neq a_i\\ e_1 & x^*=a_i\end{cases}$$

One can notice the $\forall i:u_i^*\in B\left[0,1\right]$.

$\forall x \in \mathbb{R}^n, u_1, ..., u_m \in B\left[0,1\right]$:

$$\begin{aligned}
f\left(x^*, u_1^*, ..., u_m^*\right) &= \sum_{i=1}^{m}\left(\left|\left|x^* - a_i\right|\right|^2 - 2d_i\left(u_i^*\right)^T\left(x^* - a_i\right) + d_i^2\right) \\
&= \sum_{i=1}^{m}\left(\left|\left|x^* - a_i\right|\right|^2 - 2d_i\left|\left|x^* - a_i\right|\right| + d_i^2\right) \\
&= \sum_{i=1}^{m}\left(\left|\left|x^* - a_i\right|\right| - d_i\right)^2 \\
(\text{optimality of } x^*) &\leq \sum_{i=1}^{m}\left(\left|\left|x - a_i\right|\right| - d_i\right)^2 \\
(\text{lower bound}) &\leq f\left(x, u_1, u_2, ..., u_m\right)
\end{aligned}$$

hence, we found $u_1^*, ..., u_m^* \in B\left[0,1\right]$ such that $\left(x^*, u_1^*, ..., u_m^*\right)$ are an optimal solution of $(ML2)$.

$\Leftarrow$

Let

$$\left(x^*, u_1^*, ..., u_m^*\right) \in \arg\min_{\substack{x \in \mathbb{R}^n \\ u_i \in B[0,1]}} f\left(x, u_1, u_2, ..., u_m\right)$$

we will show that:

$$f\left(x, u_1^*, ..., u_m^*\right) = \sum_{i=1}^{m}\left(\left|\left|x - a_i\right|\right| - d_i\right)^2$$

we know from C.S that:

$$\left(u_i^*\right)^T\left(x^* - a_i\right) \leq \left|\left|u_i^*\right|\right|\left|\left|x^* - a_i\right|\right|$$

we will prove that the equality holds.

assuming by contradiction that:

$$\left(u_i^*\right)^T\left(x^* - a_i\right) < \left|\left|u_i^*\right|\right|\left|\left|x^* - a_i\right|\right|$$

we get:

$$\begin{aligned}
f\left(x^*, u_1^*, ..., u_m^*\right) &= \sum_{i=1}^{m}\left(\left\|x^*-a_i\right\|^2 - 2d_i\left(u_i^*\right)^T\left(x^*-a_i\right) + d_i^2\right) \\
\text{(assumption)} &> \sum_{i=1}^{m}\left(\left\|x^*-a_i\right\|^2 - 2d_i\left\|u_i^*\right\|\left\|x^*-a_i\right\| + d_i^2\right) \\
\left(\left\|u_i\right\| \le 1\right) &\ge \sum_{i=1}^{m}\left(\left\|x^*-a_i\right\|^2 - 2d_i\left\|x^*-a_i\right\| + d_i^2\right) \\
&= \sum_{i=1}^{m}\left(\left\|x^*-a_i\right\| - d_i\right)^2
\end{aligned}$$

but if we choose $\hat{u}_i$ such that:

$$\hat{u}_i = \begin{cases} \frac{x^*-a_i}{\|x^*-a_i\|} & x^* \neq a_i \\ e_1 & x^* = a_i \end{cases}$$

we get:

$$f\left(x^*, u_1^*, ..., u_m^*\right) > \sum_{i=1}^{m}\left(\left\|x-a_i\right\| - d_i\right)^2 = f\left(x^*, \hat{u}_1, ..., \hat{u_m}\right)$$

The above equality comes from the first direction we have already proven.

and this is a contradiction to the optimality of $\left(x^*, u_1^*, ..., u_m^*\right)$

hence:

$$\left(u_i^*\right)^T\left(x^*-a_i\right) = \left\|u_i^*\right\|\left\|x^*-a_i\right\|$$

Similarly, we will show in the same way that $\forall i$:

$$x^* \neq a_i \Rightarrow \left\|u_i^*\right\| = 1$$

Assuming by contradiction that $x^* \neq a_i$ but $\left\|u_i^*\right\| < 1$.

by choosing the same $\hat{u}_i$ as above we get:

$$\begin{aligned}
f\left(x^*, u_1^*, ..., u_m^*\right) &= \sum_{i=1}^{m}\left(\left\|x^*-a_i\right\|^2 - 2d_i\left(u_i^*\right)^T\left(x^*-a_i\right) + d_i^2\right) \\
\text{(proven)} &= \sum_{i=1}^{m}\left(\left\|x^*-a_i\right\|^2 - 2d_i\left\|u_i^*\right\|\left\|x^*-a_i\right\| + d_i^2\right) \\
\left(\left\|u_i\right\| < 1\right) &> \sum_{i=1}^{m}\left(\left\|x^*-a_i\right\|^2 - 2d_i\left\|x^*-a_i\right\| + d_i^2\right) \\
&= \sum_{i=1}^{m}\left(\left\|x^*-a_i\right\| - d_i\right)^2 = f\left(x^*, \hat{u}_1, ..., \hat{u_m}\right)
\end{aligned}$$

and that is a contradiction to the optimality of $(x^*, u_1^*, ..., u_m^*)$. hence:

$$x^* \neq a_i \Rightarrow ||u_i^*|| = 1$$

by the two statement we have proven, we can conclude:

$$f\left(x^*, u_1^*, ..., u_m^*\right) = \sum_{i=1}^{m} \left(||x^* - a_i|| - d_i\right)^2$$

Finally, $\forall x \in \mathbb{R}^n$:

$$\begin{aligned}\sum_{i=1}^{m} \left(||x^* - a_i|| - d_i\right)^2 &= f\left(x^*, u_1^*, ..., u_m^*\right) \\ (\text{optimality of } x^*, u_i^*) &\leq f\left(x, \hat{u_1}, ..., \hat{u_m}\right) \\ &= \sum_{i=1}^{m} \left(||x - a_i|| - d_i\right)^2\end{aligned}$$

meaning x^* is the optimal solution of (ML)

b)

$$f\left(x, u_1, u_2, ..., u_m\right) = \sum_{i=1}^{m} \left(||x - a_i||^2 - 2d_i u_i^T\left(x - a_i\right) + d_i^2\right)$$

let's find the gradiant of f:

$$\begin{aligned}\frac{\partial f\left(x, u_1, u_2, ..., u_m\right)}{\partial x} &= \sum_{i=1}^{m} \left(2\left(x - a_i\right) - 2d_i u_i\right) = 2\sum_{i=1}^{m} \left(x - a_i - d_i u_i\right) = \\ &= 2\left(mx - \sum_{i=1}^{m} \left(a_i + d_i u_i\right)\right) \\ \frac{\partial f\left(x, u_1, u_2, ..., u_m\right)}{\partial u_j} &= -2d_j\left(x - a_j\right)\end{aligned}$$

denote

$$z = \begin{pmatrix} x \\ u_1 \\ u_2 \\ . \\ . \\ . \\ u_m \end{pmatrix}, \tilde{z} = \begin{pmatrix} \tilde{x} \\ \tilde{u_1} \\ \tilde{u_2} \\ . \\ . \\ . \\ \tilde{u_m} \end{pmatrix}$$

$$\|\nabla f(z) - \nabla f(\tilde{z})\| =$$

$$\left\| 2 \begin{pmatrix} mx - \sum_{i=1}^{m} a_i + d_i u_i \\ -d_1 (x - a_1) \\ . \\ . \\ . \\ -d_m (x - a_m) \end{pmatrix} - 2 \begin{pmatrix} m\tilde{x} - \sum_{i=1}^{m} a_i + d_i \tilde{u}_i \\ -d_1 (\tilde{x} - a_1) \\ . \\ . \\ . \\ -d_m (\tilde{x} - a_m) \end{pmatrix} \right\| =$$

$$= 2 \left\| \begin{pmatrix} m(x - \tilde{x}) - \sum_{i=1}^{m} d_i (u_i - \tilde{u}_i) \\ -d_1 (x - \tilde{x}) \\ . \\ . \\ . \\ -d_m (x - \tilde{x}) \end{pmatrix} \right\|$$

$$= 2 \left\| \underbrace{\begin{pmatrix} mI_n & -d_1 I_n & -d_2 I_n & . & . & . & d_m I_n \\ -d_1 I_n & & & & & & \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ -d_m I_n & & & & & & \end{pmatrix}}_{\triangleq A} \begin{pmatrix} x - \tilde{x} \\ u_1 - \tilde{u_1} \\ u_2 - \tilde{u_2} \\ . \\ . \\ . \\ u_m - \tilde{u_m} \end{pmatrix} \right\|$$

$$\leq 2 \|A\|_{2,2} \cdot \|z - \tilde{z}\|_2$$

where the last inequality comes from a lecture far away in the past.

Hence, a lipschitz constant of the gradient is given by:

$$L_{\nabla f} = 2 \|A\|_{2,2} = 2 \cdot \sqrt{|\lambda_{max} (A^T A)|}$$

Let's find the maximal eigenvalue of $A^T A$

denote:

$$A = \begin{pmatrix} B & C \\ C^T & D \end{pmatrix}$$

using the general property that (assuming D is invertible):

$$det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = det\left(D\right)det\left(A - BD^{-1}C\right)$$

let's find the eigenvalues of A:

$$\lambda I_{(m+1)\cdot n} - A = \begin{pmatrix} \lambda I_n - B & -C \\ -C^T & \lambda I_{m\cdot n} - D \end{pmatrix} = \begin{pmatrix} \lambda I_n - mI_n & -C \\ -C^T & \lambda I_{m\cdot n} \end{pmatrix} = \begin{pmatrix} \left(\lambda - m\right) I_n & -C \\ -C^T & \lambda I_{m\cdot n} \end{pmatrix}$$

$$\begin{aligned} det\left(\lambda I_{(m+1)\cdot n} - A\right) &= det\left(\lambda I_{m\cdot n}\right) det\left(\left(\lambda - m\right) I_n - C\left(\lambda I_{m\cdot n}\right)^{-1} C^T\right) = \\ &= \lambda^{m\cdot n} det\left(\left(\lambda - m\right) I_n - \frac{1}{\lambda} CC^T\right) = \\ &= \lambda^{m\cdot n} det\left(\left(\lambda - m\right) I_n - \frac{\sum_{i=1}^m d_i^2}{\lambda} I_n\right) = \\ &= \lambda^{m\cdot n} det\left(\left(\lambda - m - \frac{\sum_{i=1}^m d_i^2}{\lambda}\right) I_n\right) = \\ &= \lambda^{m\cdot n}\left(\lambda - m - \frac{\sum_{i=1}^m d_i^2}{\lambda}\right)^n \end{aligned}$$

$$\lambda - m - \frac{\sum_{i=1}^m d_i^2}{\lambda} = 0$$

$$\lambda^2 - m\lambda - \sum_{i=1}^m d_i^2 = 0$$

$$\lambda_{1,2} = \frac{m \pm \sqrt{m^2 + 4\sum_{i=1}^m d_i^2}}{2}$$

hence:

$\lambda = 0$ is an eigenvalue with $m \cdot n$ algebric multiplicity.

$\lambda = \frac{m \pm \sqrt{m^2 + 4\sum_{i=1}^m d_i^2}}{2}$ are eigenvalues with n algebric multiplicity.

since A is symmetric then:

$$\sqrt{\left|\lambda_{max}\left(A^T A\right)\right|} = max\left|\lambda\left(A\right)\right| = \frac{m + \sqrt{m^2 + 4\sum_{i=1}^m d_i^2}}{2}$$

and the Lipshictz constant is:

$$L = 2 \cdot \|A\|_{2,2} = 2 \cdot \sqrt{\left|\lambda_{max}\left(A^T A\right)\right|} = m + \sqrt{m^2 + 4\sum_{i=1}^m d_i^2}$$

c)

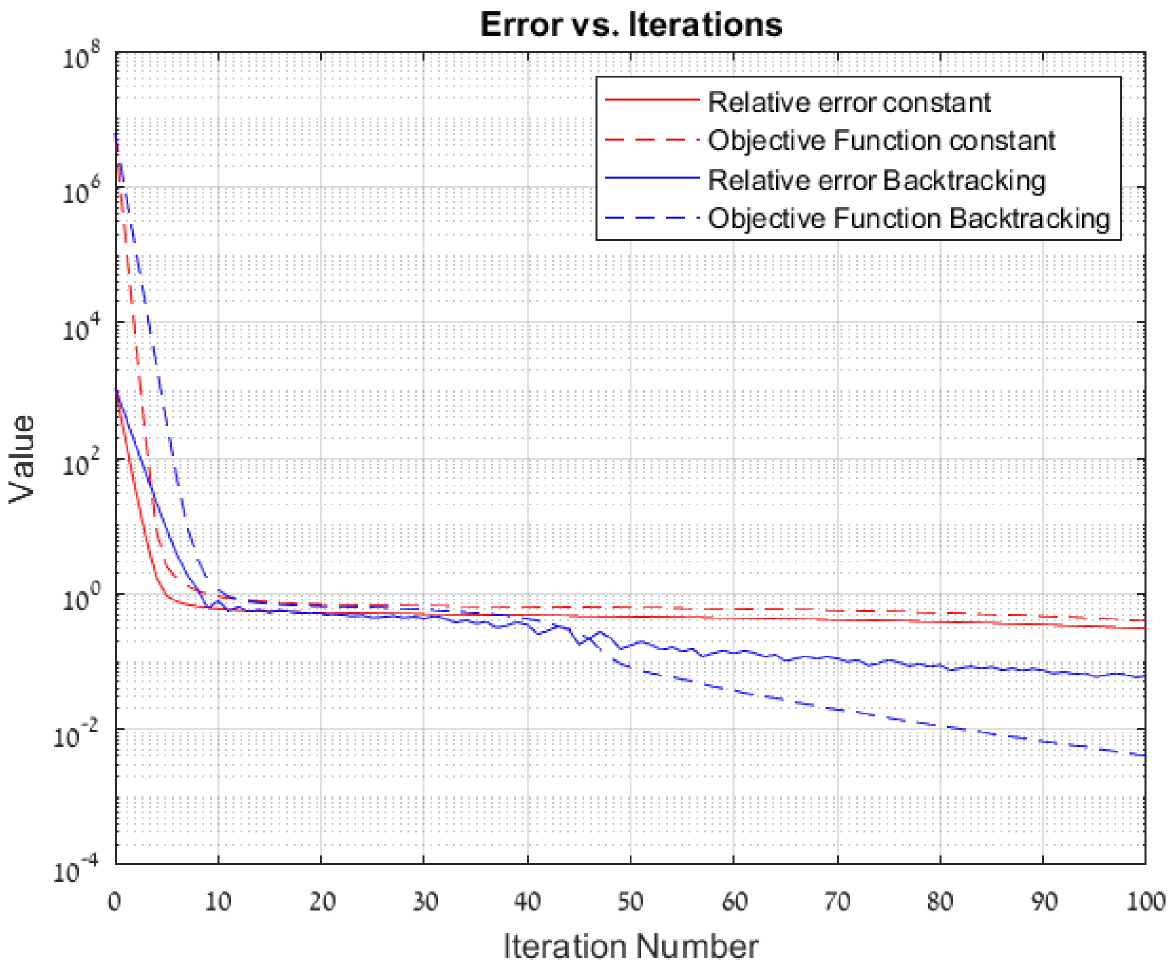

Figure 1: $L_{\nabla f} = 2 \cdot \sqrt{\lambda_{max}\left(A^T A\right)} = 12.117$

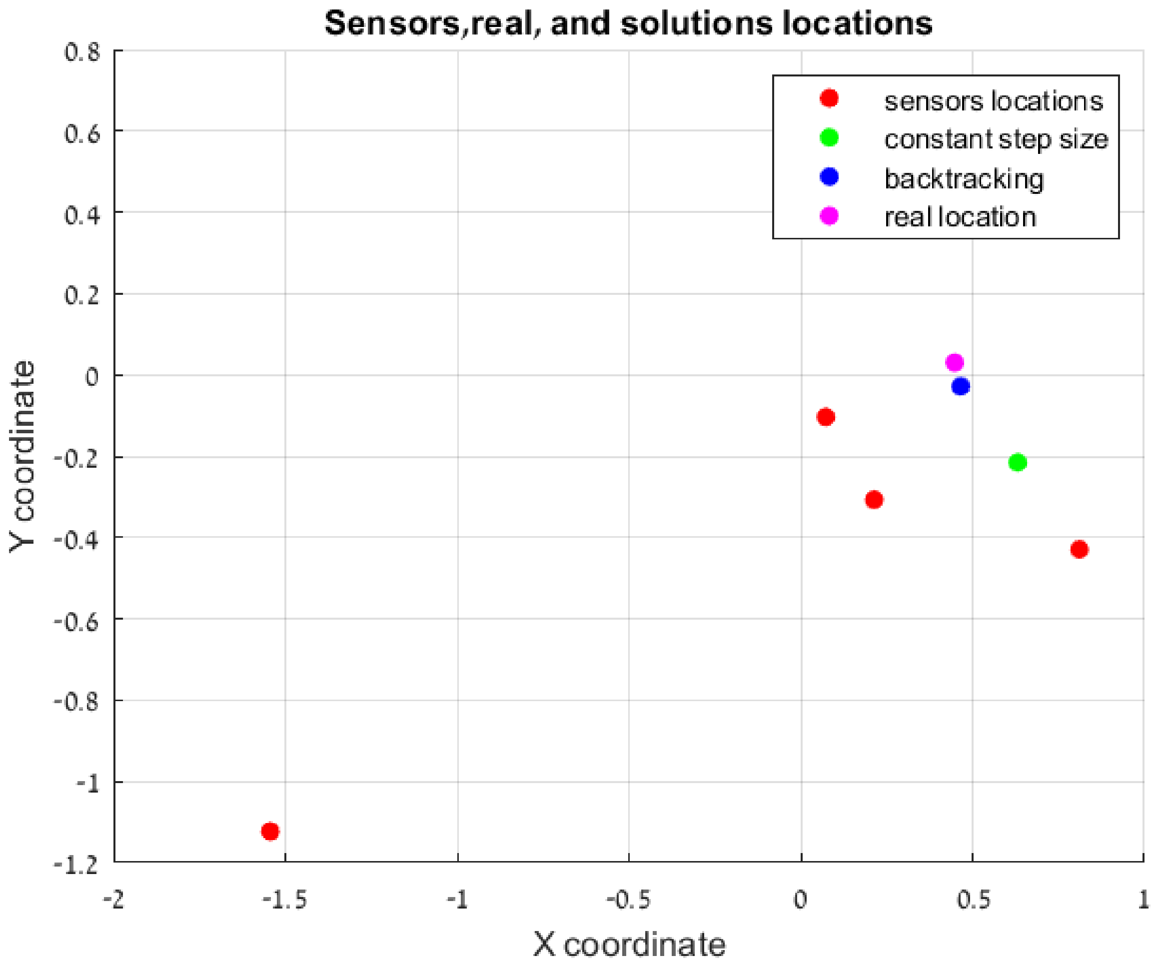
Sensors,real, and solutions locations
sensors locations
constant step size
backtracking
real location
Y coordinate
X coordinate
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-2
-1.5
-1
-0.5
0
0.5
1

Figure 2: