098311 Optimization 1 Spring 2018
HW 7

Chen Tessler 305052680
Orr Krupnik 302629027

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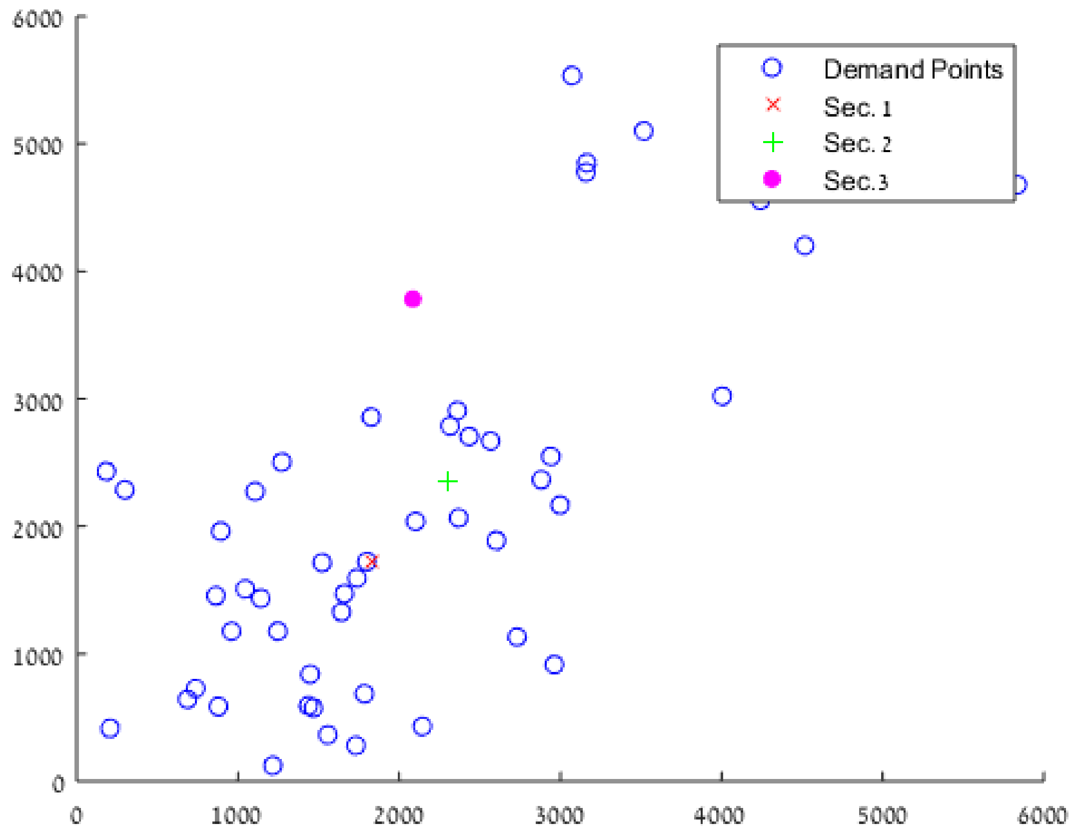
Solution 1.

1. The weekly cost of flight to a single demand point i is $\gamma g_i(x)p_i$. This cost is a convex function, since $g_i(x)$ is a norm of an affine function in x, and therefore a convex function in x. Therefore, a convex optimization problem solving the problem of placing the pizzeria is:
$$\min_x \sum_{i=1}^m \gamma p_i g_i(x)$$
2. let us define $y^+ = \max\{y, 0\}$. Then, we can add the compensation factor into the cost of delivering to a single location as: $p_i(\gamma g_i(x)+(g_i(x)-\eta_1)^+\mu_1+(g_i(x)-\eta_1)^+(\mu_2-\mu_1))$. The function $q(x) = \max\{f(x), 0\}$ is convex when f is convex, and therefore the complete cost is a sum of convex functions and hence convex. The optimization problem is now:
$$\min_x \sum_{i=1}^m p_i(\gamma g_i(x) + (g_i(x) - \eta_1)^+\mu_1 + (g_i(x) - \eta_2)^+(\mu_2 - \mu_1))$$
3. First, let us show the absolute variance of each element is a convex function.
$$\left| ||a_i - x||_2^2 - \frac{1}{m}\sum_{j=1}^m ||a_j - x||_2^2 \right| = \left| a_i^T a_i - 2a_i^T x + x^T x - \frac{1}{m}\sum_{j=1}^m \left(a_j^T a_j - 2a_j^T x + x^T x\right) \right| =$$
$$= \left| a_i^T a_i - 2a_i^T x - \frac{1}{m}\sum_{j=1}^m \left(a_j^T a_j - 2a_j^T x\right) \right| = \left| a_i^T a_i - \frac{1}{m}\sum_{j=1}^m a_j^T a_j - 2\left(a_i^T - \frac{1}{m}\sum_{j=1}^m a_j^T\right) x \right| \triangleq$$
$$\triangleq \left| v^T x - b \right|$$
since each absolute variance is an absolute value of an affine function of x, it is convex in x. Also, as we've seen in class, a maximum over convex functions is also a convex function. Therefore, we can write the convex optimization problem as:
$$\min_x \max_i \left| a_i^T a_i - \frac{1}{m}\sum_{j=1}^m a_j^T a_j - 2\left(a_i^T - \frac{1}{m}\sum_{j=1}^m a_j^T\right) x \right|$$

4. Below is the plot of the demand points and obtained locations. The results for each of the three scenarios are:

	Sec. 1	Sec. 2	Sec. 3
Position	(1826.1, 1728.1)	(2297.9, 2351.8)	(2085.7, 3783.0)
Optimal Loss value	15280.9	22193.5	8.256e6



Solution 2. Since f is strongly convex, we have for any x, y for which $Ax \neq Ay$, and for any $\lambda \in (0, 1)$:

$$f(\lambda Ax + (1-\lambda)Ay) < \lambda f(Ax) + (1-\lambda)f(Ay)$$

Now, assume $Ax^* \neq Ay^*$. We then have for some $\lambda \in (0, 1)$:

$$\begin{aligned} h(\lambda x^* + (1-\lambda)y^*) &= f(A(\lambda x^* + (1-\lambda)y^*) + g(\lambda x^* + (1-\lambda)y^*) \leq \\ &\leq f(A(\lambda x^* + (1-\lambda)y^*) + \lambda g(x^*) + (1-\lambda)g(y^*) < \\ &< \lambda f(Ax^*) + (1-\lambda)f(Ay^*) + \lambda g(x^*) + (1-\lambda)g(y^*) = \\ &= \lambda h(x^*) + (1-\lambda)h(y^*) = h(x^*) = h(y^*) \end{aligned}$$

We've found a better solution for the unconstrained problem of minimizing $h(x)$, namely $\lambda x^* + (1-\lambda)y^*$, in contradiction to the fact that x^* and y^* are optimal solutions to the problem. Therefore, we conclude $Ax^* = Ay^*$.

Solution 3. Denote the solutions to the constrained problem as

$$\bar{X} = \{x : x \in \text{argmin}_{x'} f(x') \text{ s.t. } g(x) \leq 0, x \in X\} \ .$$

Suppose that a solution x^{**} to the unconstrained problem $x^{**} \in \text{argmin}_{x \in X} f(x)$ satisfies $f(x^{**}) < f(x^*)$ and $g(x^{**}) > 0$, then $x^{**} \notin \bar{X}$ and particularly $x^{**} \neq x^*$. However, since X is a convex set and f is convex, for any $\lambda \in (0,1)$ we have $x' = \lambda x^* + (1-\lambda)x^{**} \in X$ and:

$$f(x') = f(\lambda x^* + (1-\lambda)x^{**}) \leq \lambda f(x^*) + (1-\lambda) f(x^{**}) < f(x^*)$$

Additionally, as g is convex, $g(x^*) < 0$ and $g(x^{**}) > 0$, there exists some $\lambda' = \frac{g(x^{**})}{g(x^{**}) - g(x^*)} \in (0,1)$ such that the following holds:

$$\begin{aligned} g(x') &= g(\lambda' x^* + (1-\lambda')x^{**}) \leq \lambda' g(x^*) + (1-\lambda') g(x^{**}) = \\ &= \frac{g(x^{**})}{g(x^{**}) - g(x^*)} g(x^*) + \left(1 - \frac{g(x^{**})}{g(x^{**}) - g(x^*)}\right) g(x^{**}) = \\ &= \frac{g(x^{**})}{g(x^{**}) - g(x^*)} g(x^*) - \frac{g(x^*)}{g(x^{**}) - g(x^*)} g(x^{**}) = 0 \end{aligned}$$

we have found some point $x' = \lambda' x^* + (1-\lambda')x^{**}$ for which $g(x') \leq 0$ and $f(x') < f(x^*)$ which contradicts the optimality of x^* for the constrained problem. Hence there cannot exist an optimal solution x^{**} to the unconstrained problem, such that $g(x^{**}) > 0$. As such, any optimal solution x^{**} which satisfies $g(x^{**}) \leq 0$ is also optimal for the constrained case, hence $f(x^{**}) = f(x^*)$ and x^* is also an optimal solution of the unconstrained case.

Solution 4. The problem is:

$$\begin{aligned} \max_{x \in \mathbb{R}^3} \ & 3x_1^2 + x_2^2 - 2x_3^2 + 2x_1 - 2x_2 + 5x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 1 \\ & x_i \geq 0 \ \ \forall i \in \{1,2,3\} \end{aligned}$$

The above can also be written as:

$$\begin{aligned} \max_{x \in \mathbb{R}^3} \ & a(x_1) + b(x_2) + c(x_3) \\ \text{s.t. } & x_1 + x_2 + x_3 = 1 \\ & x_i \geq 0 \ \ \forall i \in [1,3] \end{aligned}$$

where $a(\cdot), b(\cdot), c(\cdot)$ are functions equal to $3x^2 + 2x$, $x^2 - 2x$ and $-2x^2 + 5x$ respectively. The functions a, b, c have the following properties:

- $a(x) = 3x^2 + 2x$ is convex and positive for any $x \in (0,1]$.

- $b(x) = x^2 - 2x$ is convex and negative for any $x \in (0, 1]$.
- $c(x) = -2x^2 + 5x$ is concave and positive for any $x \in (0, 1]$.

Following the above, it is immediate that any point in which $x_2 \neq 0$ cannot be an optimal solution. As the optimal solution is $(x_1, 0, x_3)$, we can now solve the the following, equivalent, 1-dimensional constrained problem, denoting $x_1 = \lambda, x_2 = 0, x_3 = 1 - \lambda$:

$$\begin{aligned} &\max_{\lambda} f(\lambda) = \max_{\lambda} 3\lambda^2 + 2\lambda - 2(1-\lambda)^2 + 5(1-\lambda) \\ &\text{s.t. } \ \lambda \in [0, 1] \end{aligned}$$

expanding $f(\lambda)$ produces:

$$f(\lambda) = 3\lambda^2 + 2\lambda - 2 + 4\lambda - 2\lambda^2 + 5 - 5\lambda = \lambda^2 + \lambda + 3$$

$f(\lambda)$ is convex (a parabola) which attains the minimum at $-\frac{1}{2}$ and as such the maximal value over the interval $[0, 1]$ is attained at $f(1)$. Finally, we conclude that the optimal solution to the constrained problem is $(1, 0, 0)$.

Solution 5.

1. First, let us show the objective function is convex. The objective function can be written as:

$$\underbrace{\max\{\underbrace{|2x_1 - 3x_2|}_{(a)}, \underbrace{|x_2 - x_1 + x_3|}_{(b)}\}}_{(c)} + \underbrace{x_1^2 + x_2^2 + 2x_3^2 + (x_2 - x_3)^2}_{(d)}$$

   Where each part is convex because:

   (a) Absolute value (convex) of an affine function of x
   (b) Absolute value of an affine function of x
   (c) Maximum over convex functions is convex
   (d) Sum of convex functions (quadratic functions of affine transforms of x) is convex

   The whole objective function is a sum of convex functions and is therefore convex. We now show the convexity of the constraints. We can write the second constraint as:

$$x_1 + \frac{3}{2}x_2 \geq 1 \Rightarrow (2, 3, 0)x \geq 2$$

   this is a linear constraint and therefore convex. For the first constraint:

$$\underbrace{(\underbrace{2x_1^2 + 4x_2^2 + (2x_1 - 2x_2)^2 + 0.01}_{(a)})^8}_{(b)} + \underbrace{\frac{x_3^2}{2x_1 + 3x_2}}_{(c)} \leq 150$$

   where the different parts are convex because:

(a) Sum of convex functions. Note this sum is also always positive.

(b) The function $g(y) = y^8$ is increasing for $y > 0$ and also convex. Therefore, the composition of this function with (a) above is convex.

(c) Generalized quad over lin, with the denominator always positive due to the second constraint

Therefore, the problem can be written as:

$$\min \max\{|(2,-3,0)x|, |(-1,1,1)x|\} + x^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} x$$

$$\text{s.t. } \left(x^T \begin{pmatrix} 4 & -4 & 0 \\ -4 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + 0.01\right)^8 + \frac{x^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x}{(2,3,0)x} \leq 150$$

$$(2,3,0)x \geq 2$$

The optimal solution, after running CVX, is $x = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)^T$, and the optimal function value is $\frac{4}{9}$.

2. We can write the problem as:

$$\min \underbrace{\frac{1}{(2,0,4)x}}_{(a)} + \underbrace{x^T \begin{pmatrix} 5 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 8.6 \end{pmatrix} x}_{(b)} + \underbrace{\frac{||\left(\left(\frac{1}{2},0,0\right)x - 1\right)||^2}{(0,1,1)x}}_{(c)} + \underbrace{\frac{3}{4}\frac{||(1,0,0)x||^2}{(0,1,1)x}}_{(d)}$$

$$\text{s.t. } \underbrace{\left(x^T \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + 1\right)^2}_{(e)} \underbrace{- \min\{\underbrace{(1,1,0)x}_{(f)}, \underbrace{\sqrt{(1,0,0)x}}_{(g)}\}}_{(h)} \leq 10$$

$$x_i \geq 10^{-3} \; \forall i \in \{1,2,3\}$$

Where each part is convex because:

(a) Quad over lin

(b) Quadratic function with $A \succ 0$

(c) Quad over lin

(d) Quad over lin

(e) Similarly to the previous section, this is a non-decreasing function over the interval defined by the function and the last constraint. Therefore, this is a composition of a non-decreasing convex function over a convex function, and is therefore convex.

(f) An affine function is concave.

(g) Square root of an affine function of x is concave.

(h) Minimum of concave functions is concave, and the negative of a concave function is convex. Therefore, this whole part of the constraint is convex.

the objective function and the first constraints are sums of convex functions and therefore convex. The last constraint defines an epigraph and is therefore a convex constraint.
The result of optimizing with CVX is $x = (0.2034; 0.061; 0.3536)^T$ and the optimal function value is 4.001.

3. We can write the problem as:

$$\min \underbrace{\left\| \begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + (0,0,0,\sqrt{7})^T \right\|}_{(a)} + \underbrace{\left(x^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x + 1 \right)^2}_{(b)}$$

$$\text{s.t.} \quad \underbrace{\frac{||(1,1,0)x||^2}{(0,0,1)x+1}}_{(c)} + \underbrace{((1,0,0)x)^4}_{(d)} \leq 7$$

$$\underbrace{x^T \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 4 \end{pmatrix} x}_{(e)} \leq 10$$

$$x_1, x_2, x_3 \geq 0$$

where the various parts are convex since:

(a) Norm of an affine transform of x is convex since the norm is convex.

(b) The square of a non-negative convex function of x is convex, since the $g(x) = x^2$ is convex. The function in parentheses is convex due to the non-negativity constraint.

(c) Quad over lin

(d) $h(y) = y^4$ is convex, and therefore h over an affine transform of x is also covnex

(e) This is a quadratic function with $A \succeq 0$, and therefore convex. A is PSD based on the principal minors criterion - all principal minors are positive (starting from the bottom right corner), except the determinant which is equal 0.

Both the objective function and the first constraint are convex since they are sums of convex functions.
Solving with CVX, we obtain $x = (2.18371e-05, 2.09271e-05, 2.47071e-05)$ and an optimal value of 3.64575.