Optimization 1 — Tutorial 2

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Definition (Matrix Classification)
Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then,

1. If $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$ for any $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{A} \succeq 0$ is called positive semidefinite (PSD) (if and only if all eigenvalues are non-negative).
2. If $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ for any $\mathbf{0}_n \neq \mathbf{x} \in \mathbb{R}^n$, then $\mathbf{A} \succ 0$ is called positive definite (PD) (if and only if all eigenvalues are positive).
3. If there exist $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ and $\mathbf{y}^T \mathbf{A} \mathbf{y} < 0$, then $\mathbf{A}$ is called indefinite (if and only if there exist at least one positive and one negative eigenvalue).

Definition (Point Classification)
Let $f \colon U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Then,

1. Let $\mathbf{x}^* \in \text{interior}\,(U)$ such that f is differentiable in some neighborhood of $\mathbf{x}^*$. If $\nabla f\left(\mathbf{x}^*\right) = \mathbf{0}_n$ then $\mathbf{x}^*$ is called a stationary point.
2. Suppose that U is open and that f is continuously differentiable. Then a stationary point is called a saddle point if it is neither a local minimum point nor a local maximum point.

Proposition (Optimality Conditions)
Let $f \colon U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function.

1. **0-order conditions:**
   (a) (Weierstrass Theorem). Suppose that f is continuous over a non-empty and compact set $C \subseteq U$. Then f attains a minimum and a maximum value over C.
   (b) (Coerciveness). Suppose that f is continuous over $U = \mathbb{R}^n$ and that $\lim\limits_{\|\mathbf{x}\| \to \infty} f\left(\mathbf{x}\right) = \infty$, then f is called coercive. If f is a coercive function and $S \subseteq \mathbb{R}^n$ is a non-empty closed set, then f attains a minimum value over S. In particular, f attains a global minimum value over $\mathbb{R}^n$ if $S = \mathbb{R}^n$.
2. **First-order condition:** If $\mathbf{x}^* \in \text{interior}\,(U)$ is a local optimum point and all partial derivatives of f exist at $\mathbf{x}^*$, then $\nabla f\left(\mathbf{x}^*\right) = \mathbf{0}_n$.
3. **Second-order conditions:**
   (a) (Hessian Matrix). Suppose that U is open, and that f is twice continuously differentiable over U. Suppose that $\mathbf{x}^*$ is a stationary point. Then, if $\mathbf{x}^*$ is a local minimum [maximum] point of f, then $\nabla^2 f\left(\mathbf{x}^*\right) \succeq 0$ [$\nabla^2 f\left(\mathbf{x}^*\right) \preceq 0$]. Additionally, if $\nabla^2 f\left(\mathbf{x}^*\right)$ is indefinite, then $\mathbf{x}^*$ is a saddle point.
   (b) (Convexity). Suppose that f is twice continuously differentiable over $U = \mathbb{R}^n$. Suppose that $\nabla^2 f\left(\mathbf{x}\right) \succeq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Then, if $\mathbf{x}^*$ is a stationary point, it is a global minimum point over $\mathbb{R}^n$.

Problem 1

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. Show that $\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}\left(\mathbf{A}^T\mathbf{A}\right)}$.

Problem 2

Classify the stationary points of the following functions:

(a) $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

(b) $f(x,y) = \left(x^2 + y^2 - 1\right)^2 + \left(y^2 - 1\right)^2$.

Problem 3

Find the minimum and maximum points of the function $f(x,y) = x^2 + y^2$ over the set $C = \left\{(x,y) \in \mathbb{R}^2 \colon x + y \leq -1\right\}$.