Optimization 1 — Homework 9

December 24, 2020

Problem 1

Prove Motzkin's lemma: Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{k\times n}$. Prove that the system

$$(I) \quad \begin{cases} \mathbf{Ax} < \mathbf{0} \\ \mathbf{Bx} \leq \mathbf{0} \end{cases}$$

has a solution $\mathbf{x} \in \mathbb{R}^n$ if and only if the system

$$(II) \quad \begin{cases} \mathbf{A}^T\mathbf{u} + \mathbf{B}^T\mathbf{v} = \mathbf{0}_n \\ \mathbf{u} \neq \mathbf{0}, \quad \mathbf{u}, \mathbf{v} \geq \mathbf{0} \end{cases}$$

does not have a solution for any $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^k$.

Problem 2

For any set $C \subseteq \mathbb{R}^n$ define the set $C^* = \left\{\mathbf{y} \in \mathbb{R}^n \colon \mathbf{x}^T\mathbf{y} \geq 0 \text{ for all } \mathbf{x} \in C\right\}$ to be its dual cone.

(a) Prove that C^* is a closed and convex cone (even if C is a non-convex set).

(b) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. Show that the dual cone of $M = \{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{Ax} \geq \mathbf{0}\}$ is $M^* = \left\{\mathbf{A}^T\mathbf{v} \in \mathbb{R}^m \colon \mathbf{v} \geq \mathbf{0}\right\}$.
Hint: use the second formulation of Farkas' lemma.

Problem 3

Consider the problem

$$\begin{aligned} \min \quad & x^2 + y^2 + z^2 + xy + yz - 2x - 4y - 6z \\ \text{s.t.} \quad & x + y + z \leq 1. \end{aligned}$$

(a) Is the problem convex?

(b) Find all KKT points of the problem.

(c) Find the optimal solution of the problem.

Problem 4

Consider the problem

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{b}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{c}^T\mathbf{x} \le p_1, \\ & \mathbf{d}^T\mathbf{x} = p_2, \end{aligned}$$

where $n \ge 3$, $\mathbf{Q} \succ 0$, $\mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n$, $p_1, p_2 \in \mathbb{R}$. Assume that $\mathbf{c}^T\mathbf{Q}^{-1}\mathbf{c} = \mathbf{d}^T\mathbf{Q}^{-1}\mathbf{d}$, $\mathbf{d}^T\mathbf{Q}^{-1}\mathbf{c} = 0$ and $\mathbf{c}, \mathbf{d} \neq \mathbf{0}_n$.

(a) Are the KKT conditions necessary for optimality ({optimal points} $\subseteq$ {KKT points})? Are they sufficient ({KKT points} $\subseteq$ {optimal points})? Explain your answer.

(b) Prove that the problem is feasible.

Hint: start by showing that $\mathbf{c}$ and $\mathbf{d}$ are linearly independent.

(c) Write the KKT conditions explicitly.

(d) Find the optimal solution of the problem.

Problem 5

Consider the problem

$$\begin{aligned} (P_\alpha) \qquad \min \quad & \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 \\ \text{s.t.} \quad & \mathbf{e}^T\mathbf{x} = \alpha, \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{e} = (1, 1, ..., 1)^T \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ is a parameter.

(a) Prove that (P_α) has a unique solution if and only if $\text{Ker}(\mathbf{A}) \cap \text{Ker}(\mathbf{e}^T) = \{\mathbf{0}_n\}$.

(b) Assume that $\mathbf{A}$ has a full column rank. Employ the KKT conditions in order to solve (P_α).

(c) Let $f\colon \mathbb{R} \to \mathbb{R}_+$ be defined such that $f(\alpha)$ is the optimal value of (P_α). Without assuming any assumption on $\mathbf{A}$, prove that f is a convex function (you can assume that problem (P_α) attains a solution for every α).

(d) Assume that $\text{Ker}(\mathbf{A}) \cap \text{Ker}(\mathbf{e}^T) = \{\mathbf{0}\}$. Employ the KKT conditions in order to solve (P_α).