Exam in Optimization I - Winter 2015/2016

Instructions: Answer the following 5 questions. Please start each question on a separate page and give **detailed explanations**. The grade of the exam is $\min\{X, 100\}$ with X being the sum of all accumulated points.

Good Luck!!!

Question 1 (22 pts.) Consider the function $f(x,y) = x^2 - x^2y^2 + y^4$. Find all the stationary points of f and classify them (strict/non-strict, local/global, minimum/maximum).

Question 2. (22 pts.) Prove that the following problem is convex in the sense that it consists of minimizing a convex function over a convex feasible set.

$$\begin{array}{ll} \min & \log\left(e^{x_1-x_2} + e^{x_2+x_3}\right) \\ \text{s.t.} & x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \leq 1, \\ & (x_1 + x_2 + 2x_3)(2x_1 + 4x_2 + x_3)(x_1 + x_2 + x_3) \geq 1, \\ & -e^{e^{x_1}} + [x_2]_+^3 \leq 7, \\ & x_1, x_2, x_3 \geq \frac{1}{10}. \end{array}$$

Question 3. (22 pts.)

(a) Prove that for any $\mathbf{a} \in \mathbb{R}^n$, the following holds:

$$\min \mathbf{a}^T\mathbf{x} + \|\mathbf{x}\|_\infty = \begin{cases} 0, & \|\mathbf{a}\|_1 \leq 1, \\ -\infty, & \text{else.} \end{cases}$$

(b) Consider the following minimization problem:

$$(\mathrm{P}) \quad \begin{array}{ll} \min & \sqrt{\|\mathbf{A}\mathbf{x}\|_2^2 + 1} + \|\mathbf{x}\|_\infty, \\ \text{s.t.} & \mathbf{B}\mathbf{x} \leq \mathbf{c}, \end{array}$$

where $\mathbf{A} \in \mathbb{R}^{d\times n}, \mathbf{B} \in \mathbb{R}^{m\times n}, \mathbf{c} \in \mathbb{R}^m$. Assume that the problem is feasible. Find a dual problem to (P). Do not perform any transformations that might ruin the convexity of the problem.

Question 4. (22 pts.) Consider the problem

$$(\mathrm{Q}) \quad \begin{array}{ll} \min & ax_1^2 + bx_2^2 + cx_1x_2 \\ \text{s.t.} & 1 \leq x_1 \leq 2, \\ & 0 \leq x_2 \leq x_1. \end{array}$$

where $a, b, c \in \mathbb{R}$.

(a) Prove that there exists a minimizer for problem (Q).

(b) Prove that if $a < 0, b < 0$ and $c^2 - 4ab \leq 0$, then the optimal value of problem (Q) is

$$4\min\{a, a+b+c\}.$$

(c) Prove that if $a > 0, b > 0$ and $c^2 < 4ab$, then problem (Q) has a unique solution.

Question 5. (22 pts.) Consider the problem

$$(\text{G}) \quad \begin{array}{ll} \min & \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} \\ \text{s.t.} & \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{c}^T\mathbf{x} + d \leq 0, \end{array}$$

where $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is positive definite, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $d \in \mathbb{R}$.

(a) Under which (explicit) condition on the data $(\mathbf{Q}, \mathbf{b}, \mathbf{c}, d)$ is the problem feasible?

(b) Under which (explicit) condition on the data $(\mathbf{Q}, \mathbf{b}, \mathbf{c}, d)$ does strong duality hold?

(c) Find a dual problem to (G) in one variable.

(d) Assume that $\mathbf{Q} = \mathbf{I}$. Find the optimal solution of the primal problem (G) assuming that the condition of part (b) holds.