Optimization 1 - 098311

Winter 2021 - HW 7

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Problem 1:

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$

Direction 1:

$f(x)$ is a convex function.

by definition:

$$epi(f) = \{(x;t) : f(x) \leq t\}$$

let $(x;t_x), (y;t_y) \in epi(f)$,$\lambda \in [0,1]$ and denote:

$$z = \lambda x + (1-\lambda) y$$

$$t = \lambda t_x + (1-\lambda) t_y$$

we will show that $(z;t) \in epi(f)$:

because $(x;t_x), (y;t_y) \in epi(f)$:

$$f(x) \leq t_x$$
$$f(y) \leq t_y$$

and because $f(x)$ is convex

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

hence:

$$\begin{aligned} f(z) = f(\lambda x + (1-\lambda) y) &\leq \lambda f(x) + (1-\lambda) f(y) \\ &\leq \lambda t_x + (1-\lambda) t_y = t \end{aligned}$$

thus by definition $(z;t) \in epi(f)$

hence, $epi(f)$ is a convex set

Direction 2:

$epi\left(f\right)$ is a convex set.

let $x, y \in \mathbb{R}^n, \lambda \in (0,1)$

if one or both of $f\left(x\right)$ and $f\left(y\right)$ is ∞ (W.L.O.G $f\left(x\right) = \infty$):

$$f\left(\lambda x + \left(1-\lambda\right) y\right) \leq \infty = f\left(x\right) = \lambda f\left(x\right) = \lambda f\left(x\right) + \left(1-\lambda\right) f\left(y\right)$$

otherwise $f\left(x\right), f\left(y\right) < \infty$ hence we can choose:

$$t_x = f\left(x\right), t_y = f\left(y\right) \in \mathbb{R}$$

that satisfies:

$$f\left(x\right) \leq t_x$$

$$f\left(y\right) \leq t_y$$

hence:

$$\left(f\left(x\right); t_x\right), \left(f\left(y\right); t_y\right) \in epi\left(f\right)$$

from the convexity of $epi\left(f\right)$:

$$\lambda\left(x; t_x\right) + \left(1-\lambda\right)\left(x; t_y\right) = \left(\lambda x + \left(1-\lambda\right) y; \lambda t_x + \left(1-\lambda\right) t_y\right) \in epi\left(f\right)$$

by the definition of $epi\left(f\right)$:

$$\begin{aligned} f\left(\lambda x + \left(1-\lambda\right) y\right) &\leq \lambda t_x + \left(1-\lambda\right) t_y \\ &= \lambda f\left(x\right) + \left(1-\lambda\right) f\left(y\right) \end{aligned}$$

Thus f is convex by definition.

Problem 2:

$$f:\mathbb{R}^n\longrightarrow\mathbb{R}$$

$$h:\mathbb{R}^m\longrightarrow\mathbb{R}$$

$$A\in\mathbb{R}^{mxn}$$

$$h\left(y\right)=\inf_{x\in\mathbb{R}^n}\left\{f\left(x\right):Ax=y\right\}$$

prove that $h\left(y\right)$ is a convex function.

let $y_1,y_2\in\mathbb{R}^m,\lambda\in[0,1]$

$$\begin{aligned}\lambda h\left(y_1\right)+\left(1-\lambda\right)h\left(y_2\right)&=\lambda\inf_{x_1\in\mathbb{R}^n}\left\{f\left(x_1\right):Ax=y_1\right\}+\left(1-\lambda\right)\inf_{x_2\in\mathbb{R}^n}\left\{f\left(x_2\right):Ax=y_2\right\}\\&=\inf_{x_1\in\mathbb{R}^n}\left\{\lambda f\left(x_1\right):Ax=y_1\right\}+\inf_{x_2\in\mathbb{R}^n}\left\{\left(1-\lambda\right)f\left(x_2\right):Ax=y_2\right\}\\\text{(seperated variables)}&=\inf_{x_1,x_2\in\mathbb{R}^n}\left\{\lambda f\left(x_1\right)+\left(1-\lambda\right)f\left(x_2\right):Ax_1=y_1\cup Ax_2=y_2\right\}\\\text{(f is convex)}&\geq\inf_{x_1,x_2\in\mathbb{R}^n}\left\{f\left(\lambda x_1+\left(1-\lambda\right)x_2\right):Ax_1=y_1\cup Ax_2=y_2\right\}\\(*)&\geq\inf_{z\in\mathbb{R}^n}\left\{f\left(z\right):Az=\lambda y_1+\left(1-\lambda\right)y_2\right\}\\&=h\left(\lambda y_1+\left(1-\lambda\right)y_2\right)\end{aligned}$$

Thus, h is convex by definition.

$(*)$

We will show that $\left\{f\left(\lambda x_1+\left(1-\lambda\right)x_2\right):Ax_1=y_1\cup Ax_2=y_2\right\}\subseteq\left\{f\left(z\right):Az=\lambda y_1+\left(1-\lambda\right)y_2\right\}$

let

$$c\in\left\{f\left(\lambda x_1+\left(1-\lambda\right)x_2\right):Ax_1=y_1\cup Ax_2=y_2\right\}$$

and let $x_1,x_2\in\mathbb{R}_n$ be the corresponded points such that

$$c=f\left(\lambda x_1+\left(1-\lambda\right)x_2\right)$$

$$Ax_1=y_1$$

$$Ax_2=y_2$$

if we choose

$$z = \lambda x_1 + (1-\lambda)\, x_2$$

we get:

$$\begin{aligned} Az &= A\left(\lambda x_1 + (1-\lambda)\, x_2\right) \\ &= \lambda A x_1 + (1-\lambda)\, A x_2 \\ &= \lambda y_1 + (1-\lambda)\, y_2 \end{aligned}$$

hence:

$$c \in \{f(z) : Az = \lambda y_1 + (1-\lambda)\, y_2\}$$

meaning:

$$\{f(\lambda x_1 + (1-\lambda)\, x_2) : Ax_1 = y_1 \cup Ax_2 = y_2\} \subseteq \{f(z) : Az = \lambda y_1 + (1-\lambda)\, y_2\}$$

Since the the domain of the infimum on the right-hand side is containing the domain of the infimum on the left-hand side we get:

$$\inf_{x_1, x_2 \in \mathbb{R}^n} \{f(\lambda x_1 + (1-\lambda)\, x_2) : Ax_1 = y_1 \cup Ax_2 = y_2\} \geq \inf_{z \in \mathbb{R}^n} \{f(z) : Az = \lambda y_1 + (1-\lambda)\, y_2\}$$

Problem 3:

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$g : \mathbb{R}^n \longrightarrow \mathbb{R}$$

f and g are convex functions.

$X \subseteq \mathbb{R}^n$ is a convex set.

let x^* be an optimal solution of the first problem such that $g(x^*) < 0$.

assume by contradiction that x^* is not an optimal solution for the second problem, meaning that there exists $\hat{x} \neq x^* \in X$, which satisfies

$$f(\hat{x}) < f(x^*)$$

since $g(x^*) < 0$:

$$\exists r : \|\hat{x} - x^*\| > r > 0, B(x^*, r) \subseteq \{x : g(x) < 0\}$$

denote:

$$0 < \lambda = \frac{0.5r}{\|\hat{x} - x^*\|} < \frac{1}{2}$$

$$z = \lambda \hat{x} + (1 - \lambda) x^*$$

since $x^*, \hat{x} \in X$ as valid solutions for the first and second problem, and because X is a convex set:

$$z \in X$$

$$g(z) = g(\lambda \hat{x} + (1 - \lambda) x^*) = g(x^* + \lambda (\hat{x} - x^*)) = g\left(x^* + \frac{0.5r(\hat{x} - x^*)}{||\hat{x} - x^*||}\right)$$

$$\left\| x^* + \frac{0.5r(\hat{x} - x^*)}{||\hat{x} - x^*||} - x^* \right\| = \left\| \frac{0.5r(\hat{x} - x^*)}{||\hat{x} - x^*||} \right\| = 0.5r < r$$

hence:

$$x^* + \frac{0.5r(\hat{x} - x^*)}{||\hat{x} - x^*||} \in B(x^*, r)$$

and therefore:

$$g(z) = g\left(x^* + \frac{0.5r(\hat{x} - x^*)}{||\hat{x} - x^*||}\right) < 0$$

we can see that z is a valid solution for the first problem because $z \in X \cap \{x : g(x) \leq 0\}$, however:

$$\begin{aligned} f(z) &= f(\lambda \hat{x} + (1-\lambda) x^*) \\ &\overset{f \text{ is convex}}{\leq} \lambda f(\hat{x}) + (1-\lambda) f(x^*) \\ &\overset{f(\hat{x})<f(x^*)}{<} \lambda f(x^*) + (1-\lambda) f(x^*) = f(x^*) \end{aligned}$$

we have found a solution z for the first problem that attains a smaller value than x^*.

this is a contradiction to the fact that x^* is an optimal solution.

thus x^* is an optimal solution to the second problem as well.

Problem 4:

a)

Show that the extreme points of the unit simplex Δ_n are the unit vectors $e_1, e_2, ..., e_n$

$$\Delta_n = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, \forall i : x_i \geq 0 \right\}$$

first let's show that the unit simplex is a convex set for the problem to be well defined.

let $x, y \in \Delta_n$ and $\lambda \in [0,1]$ and define:

$$z = \lambda x + (1-\lambda) y$$

since $x, y \in \Delta_n$ then:

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1$$

$$\forall i \in \{1, 2, .., n\} : x_i, y_i \geq 0$$

notice that:

$$\sum_{i=1}^{n} z_i = \sum_{i=1}^{n} \lambda x_i + (1-\lambda) y_i = \lambda \sum_{i=1}^{n} x_i + (1-\lambda) \sum_{i=1}^{n} y_i = \lambda + (1-\lambda) = 1$$

$$\forall i \in \{1, 2, .., n\} : z_i = \lambda x_i + (1-\lambda) y_i \overset{\substack{y_i \geq 0 \\ x_i \geq 0 \\ 1-\lambda \geq 0 \\ \lambda \geq 0}}{\geq} 0$$

hence $z \in \Delta_n$

therefore the unit simplex is a convex set by definition.

first direction:

let $x = e_j \in \mathbb{R}^n$ for some $j \in 1, 2, ..., n$, lets prove it is an extreme point of Δ_n.

first:

$$\sum_{i=1}^{n} x_i = 1$$

$$\forall i : x_i \geq 0$$

hence $x \in \Delta_n$.

assume by contradiction, it is not an extreme point of Δ_n, then:

$$\exists y \neq z \in \Delta_n, \lambda \in (0,1) : x = \lambda y + (1-\lambda) z$$

since $y, z \in \Delta_n$:

$$\sum_{i=1}^{n} y_i = 1, \forall i \in \{1,2,..,n\} : y_i \geq 0$$

$$\sum_{i=1}^{n} z_i = 1, \forall i \in \{1,2,..,n\} : z_i \geq 0$$

in addition since $x = e_j$:

$$\forall i \neq j : \lambda y_i + (1-\lambda) z_i = 0$$

since $\forall i : y_i \geq 0, z_i \geq 0$ and $\lambda \in (0,1)$:

$$\forall i \neq j : \lambda y_i + (1-\lambda) z_i = 0$$

$$\iff \forall i \neq j : y_i = z_i = 0$$

but since $\sum_{i=1}^{n} y_i = 1$ and $\sum_{i=1}^{n} z_i = 1$ it means that:

$$\sum_{i=1}^{n} y_i = \sum_{\substack{i=1 \\ i \neq j}}^{n} y_i + y_j = 0 + y_j = y_j = 1$$

$$\sum_{i=1}^{n} z_i = \sum_{\substack{i=1 \\ i \neq j}}^{n} z_i + z_j = 0 + z_j = z_j = 1$$

hence:

$$y = z = e_j$$

this is a contradiction to the fact that $y \neq z$ Therefore $x = e_j$ is an extreme point of Δ_n

second direction:

let $x \in \Delta_n$ and $x \notin \{e_1, e_2, ..., e_n\}$. lets show x is not an extreme point of Δ_n.

since $x \in \Delta_n$ but $x \notin \{e_1, e_2, ..., e_n\}$ it means that:

$$\exists k \in \{1,2,...,n\} : 0 < x_k < 1$$

define:

$$\lambda_i = \frac{x_i}{1-x_k}, i \in \{1,2,...,n\} \setminus \{k\}$$

since $x \in \Delta_n$ then:

$$\sum_{i=1}^{n} x_i = 1 \Rightarrow \sum_{\substack{i=1 \\ i\neq k}}^{n} x_i = 1 - x_k$$

$$\sum_{\substack{i=1 \\ i\neq k}}^{n} \lambda_i = \sum_{\substack{i=1 \\ i\neq k}}^{n} \frac{x_i}{1-x_k} = \frac{\sum_{\substack{i=1 \\ i\neq k}}^{n} x_i}{1-x_k} = \frac{1-x_k}{1-x_k} = 1$$

now define $y \in \mathbb{R}^n$:

$$y = \sum_{\substack{i=1 \\ i\neq k}}^{n} \lambda_i e_i$$

since $e_1, e_2, ..., e_n \in \Delta_n$ (we showed that in the first direction) and $\sum_{\substack{i=1 \\ i\neq k}}^{n} \lambda_i = 1$ then y is a convex combination of points from the unit simplex, and since the unit simplex is a convex set it means that:

$$y \in \Delta_n$$

now notice that for $y, e_k \in \Delta_n$ and $\lambda = x_k \in (0,1)$

$$\begin{aligned}
\lambda e_k + (1-\lambda) y &= \lambda e_k + (1-\lambda) \sum_{\substack{i=1 \\ i\neq k}}^{n} \lambda_i e_i = \\
&= \lambda e_k + \sum_{\substack{i=1 \\ i\neq k}}^{n} (1-\lambda) \lambda_i e_i = \\
&= x_k e_k + \sum_{\substack{i=1 \\ i\neq k}}^{n} (1-x_k) \frac{x_i}{1-x_k} e_i = \\
&= x_k e_k + \sum_{\substack{i=1 \\ i\neq k}}^{n} x_i e_i \\
&= x
\end{aligned}$$

we found two vectors $y, e_k \in \Delta_n$ and a scalar $\lambda \in (0,1)$ such that $x \in \Delta_n$ can be written as:

$$x = \lambda e_k + (1-\lambda) y$$

hence x is not an extreme point of Δ_n.

b)

Find the optimal solution of the problem:

$$\max f(x) = 57x_1^2 + 65x_2^2 + 17x_3^2 + 96x_1x_2 - 32x_1x_3 + 8x_2x_3 + 27x_1 - 84x_2 + 20x_3$$
$$s.t x_1 + x_2 + x_3 = 1,$$
$$x_1, x_2, x_3 \geq 0$$

notice the restrictions simply tell us that the solution has to belong to the unit simplex, thus we can rewrite the problem as:

$$\max f(x)$$
$$s.t x \in \Delta_3$$

lets look at $f(x)$:

$$f(x) = x^T \underbrace{\begin{pmatrix} 57 & 48 & -16 \\ 48 & 65 & 4 \\ -16 & 4 & 17 \end{pmatrix}}_{A} x + 2 \underbrace{\begin{pmatrix} \frac{27}{2} & -42 & 10 \end{pmatrix}}_{b^T} x + \underbrace{0}_{c}$$

this is a quadratic function of x, let's show A is positive definite and hence $f(x)$ is a convex function.

$$M_1(A) = 57 > 0$$

$$M_2(A) = 57 \cdot 65 - 48^2 = 1401 > 0$$

$$M_3(A) = 57 \cdot \begin{vmatrix} 65 & 4 \\ 4 & 17 \end{vmatrix} - 48 \cdot \begin{vmatrix} 48 & 4 \\ -16 & 17 \end{vmatrix} - 16 \cdot \begin{vmatrix} 48 & 65 \\ -16 & 4 \end{vmatrix} =$$
$$= 57 \cdot 1089 - 48 \cdot 880 - 16 \cdot 1232 = 121 > 0$$

all the major minors of A are strictly positive, hence A is a positive definite matrix.

Therefore $f(x)$ is a quadratic strictly convex function over all $\mathbb{R}^n$ and also specifically over Δ_3.

notice that:

1) Δ_3 is a convex set as we proved in section a

2) Δ_3 is not empty, for example $e_1 \in \Delta_3$

3) Δ_3 is bounded because $\forall x \in \Delta_3 : \|x\|_1 = \sum_{i=1}^{3} |x_i| = \sum_{i=1}^{3} x_i = 1 \leq 1$

4) Δ_3 is closed as we saw in lecture 1

$f(x)$ is a convex function over the non empty convex and compact set Δ_3, therefore there exists at least one maximizer of $f(x)$ over Δ_3 which is an extreme point of Δ_3.

luckily in section a we saw that the extreme points of Δ_3 are $\{e_1, e_2, e_3\}$, we now know that one of them is a maximizer of $f(x)$ over Δ_3, we just need to find out who.

$$f(e_j) = e_j^T A e_j + 2b^T e_j = A_{jj} + 2b_j$$

$$f(e_1) = A_{11} + 2b_1 = 57 + 2 \cdot \frac{27}{2} = 57 + 27 = 84$$

$$f(e_2) = A_{22} + 2b_2 = 65 - 2 \cdot 42 = 65 - 84 = -19$$

$$f(e_3) = A_{33} + 2b_3 = 17 + 2 \cdot 10 = 17 + 20 = 37$$

thus we can see that the maximizer is e_1.

to conclude, the optimal solution of the problem can be attained at $x^* = e_1$ and it's value is $f(x^*) = 84$

Problem 5:

a)

Let's define the corresponded optimization problem:

$$\min_{x\in\mathbb{R}^n}\sum_{i=1}^{m}\alpha p_i\left|\left|x-a_i\right|\right|\gamma$$

We know that the norm $||x||$ is convex and thus $\forall i\ \alpha\gamma p_i\left|\left|x-a_i\right|\right|$ is convex as well as an affine transformation of the variable and a multiplication by a positive scalar. Finally, the whole target function is convex as a summation of convex functions.

b)

Let's define the following optimization problem:

denote:

$$\begin{aligned}f\left(x\right)&=\sum_{i=1}^{m}\alpha p_i\left|\left|x-a_i\right|\right|\gamma+\sum_{i=1}^{m}\mu_1 p_i\max\left\{0,\alpha\left|\left|x-a_i\right|\right|-\eta_1\right\}+\sum_{i=1}^{m}\left(\mu_2-\mu_1\right)p_i\max\left\{0,\alpha\left|\left|x-a_i\right|\right|-\eta_2\right\}\\&=\sum_{i=1}^{m}\left(\underbrace{\alpha p_i\left|\left|x-a_i\right|\right|\gamma+\mu_1 p_i\max\left\{0,\alpha\left|\left|x-a_i\right|\right|-\eta_1\right\}+\left(\mu_2-\mu_1\right)p_i\max\left\{0,\alpha\left|\left|x-a_i\right|\right|-\eta_2\right\}}_{h_i(x)}\right)\end{aligned}$$

$$\min_{x\in\mathbb{R}^n}f\left(x\right)$$

We can see that $\forall i$:

if $\alpha\left|\left|x-a_i\right|\right|\leq\eta_1$:

$$h_i\left(x\right)=\alpha p_i\left|\left|x-a_i\right|\right|\gamma$$

if $\eta_1\leq\alpha\left|\left|x-a_i\right|\right|\leq\eta_2$:

$$h_i\left(x\right)=\alpha p_i\left|\left|x-a_i\right|\right|\gamma+\mu_1 p_i\left(\alpha\left|\left|x-a_i\right|\right|-\eta_1\right)$$

and if $\alpha\left|\left|x-a_i\right|\right|>\eta_2$:

$$\begin{aligned}h_i\left(x\right)&=\alpha p_i\left|\left|x-a_i\right|\right|\gamma+\mu_1 p_i\left(\alpha\left|\left|x-a_i\right|\right|-\eta_1\right)+\left(\mu_2-\mu_1\right)p_i\left(\alpha\left|\left|x-a_i\right|\right|-\eta_2\right)\\&=\alpha p_i\left|\left|x-a_i\right|\right|\gamma+\mu_1 p_i\left(\alpha\left|\left|x-a_i\right|\right|-\eta_1-\alpha\left|\left|x-a_i\right|\right|+\eta_2\right)+\mu_2 p_i\left(\alpha\left|\left|x-a_i\right|\right|-\eta_2\right)\\&=\alpha p_i\left|\left|x-a_i\right|\right|\gamma+\mu_1 p_i\left(\eta_2-\eta_1\right)+\mu_2 p_i\left(\alpha\left|\left|x-a_i\right|\right|-\eta_2\right)\end{aligned}$$

which are all describe correctly the given situations.

$\forall i$ h_i is a convex function as a summation of maximum, scaling and affine transformation of convex functions.

Thus, f is convex a summation of convex functions.

c)

The convex optimization problem is given by:

$$\begin{aligned}
&\min_{x\in\mathbb{R}_n} \max_{i\in\{1,\dots,m\}} \left| ||x-a_i||^2 - \frac{1}{m}\sum_{j=1}^{m} ||x-a_j||^2 \right| \\
&\triangleq \min_{x\in\mathbb{R}^n} f(x)
\end{aligned}$$

Let's show that f is a convex function:

$$\begin{aligned}
f(x) &= \max_{i\in\{1,\dots,m\}} \left| ||x-a_i||^2 - \frac{1}{m}\sum_{j=1}^{m} ||x-a_j||^2 \right| \\
&= \max_{i\in\{1,\dots,m\}} \left| x^Tx - 2a_i^Tx + a_i^Ta_i - \frac{1}{m}\sum_{j=1}^{m} \left(x^Tx - 2a_j^Tx + a_j^Ta_j\right) \right| \\
&= \max_{i\in\{1,\dots,m\}} \left| 2\left(\frac{1}{m}\sum_{j=1}^{m}\left(a_j^T\right) - a_i^T\right)x + a_i^Ta_i - \frac{1}{m}\sum_{j=1}^{m}\left(a_j^Ta_j\right) \right|
\end{aligned}$$

Each element in the max operation is an linear transformation on the absolute value argument.

Since the absolute value function is convex, each element of the max operation is convex and by that the whole function is convex.

Figures

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%% section a

cvx_begin
variable x(n)
minimize alpha * gamma * sum(p .* norms(repmat(x,1,m)-A,2,1))

cvx_end
```

```
%% section b

cvx_begin
variable y(n)
minimize sum(gamma * alpha * p .* norms(repmat(y,1,m)-A,2,1) +...
            mu1 * p .* max(zeros(1,m),alpha*norms(repmat(y,1,m)-A,2,1)-eta1)+...
            (mu2-mu1) *p .* max(zeros(1,m),alpha*norms(repmat(y,1,m)-A,2,1)-eta2))
cvx_end
```

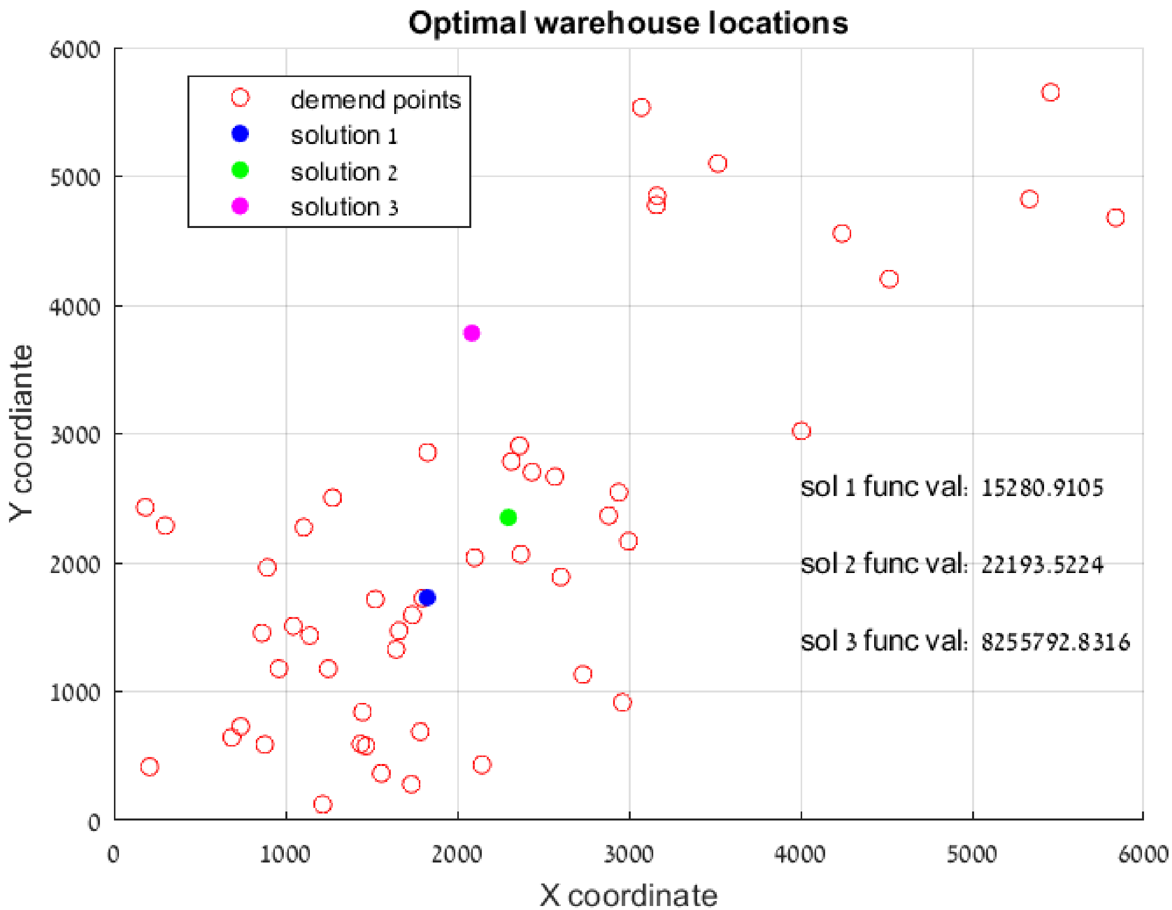
```
%% section c

c1 = (1/m) * sum(A,2)';
c2 = (1/m) * sum(vecnorm(A).^2);

cvx_begin
variable z(n)
minimize max(abs(2 * (repmat(c1,m,1) - transpose(A)) * z +transpose((vecnorm(A).^2)) - repmat(c2,m,1)))

cvx_end
```

Figure 1: The code



1	2	3
1826	2297	2085
1728	2351	3783

Figure 2: The problem solution

Problem 6:

a)

$$\begin{aligned}
&\min f(x) = \max\left\{\left|2x_1 - 3x_2\right|, \left|x_2 - x_1 + x_3\right|\right\} + x_1^2 + 2x_2^2 + 3x_3^2 - 2x_2x_3 \\
&s.t g_1(x) = \left(4x_1^2 + 6x_2^2 - 8x_1x_2 + 0.01\right)^8 + \frac{x_3^2}{2x_1 + 3x_2} \le 150 \\
&g_2(x) = x_1 + x_2 \ge 1 - \frac{x_2}{2}
\end{aligned}$$

objective function:

$$|2x_1 - 3x_2| = \left|\begin{pmatrix} 2 & -3 & 0 \end{pmatrix} x + 0\right| = |Ax + b| = \|Ax + b\|_1$$

$$|x_2 - x_1 + x_3| = \left|\begin{pmatrix} -1 & 1 & 1 \end{pmatrix} x + 0\right| = |Bx + c| = \|Bx + c\|_1$$

$\|x\|_1$ is a convex function, hence $\|Ax + b\|_1$ and $\|Bx + c\|_1$ are convex function as a linear change in the variables of a convex function.

a maximum between convex functions is a convex functions, hence $\max\left\{\left|2x_1 - 3x_2\right|, \left|x_2 - x_1 + x_3\right|\right\}$ is convex.

$$x_1^2 + 2x_2^2 + 3x_3^2 - 2x_2x_3 = x^T \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}}_{Q} x$$

Q is strictly diagonally dominant with non negative diagonal elements , hence positive definite, therefore $x^T Q x$ is a convex function as a quadratic function with a positive definite matrix.

we can conclude that $f(x)$ is a convex function as a summation of convex functions.

first constraint:

$$4x_1^2 + 6x_2^2 - 8x_1x_2 + 0.01 = x^T \underbrace{\begin{pmatrix} 4 & -4 & 0 \\ -4 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{P} x + 0.01$$

P is diagonally dominant with non negative diagonal elements , hence positive semi definite, therefore $x^T P x$ is a convex function as a quadratic function with a positive semi definite matrix.

in addition $x^T P x + 0.01 \geq 0.01 > 0$ hence the image of $x^T P x + 0.01$ is a subset of $\mathbb{R}_{++}$

the function $g(x) = x^8$ is a non decreasing convex function over $\mathbb{R}_{++}$, therefore we can conclude that $\left(4x_1^2 + 6x_2^2 - 8x_1x_2 + 0.01\right)^8$ is a convex function.

$$\frac{x_3^2}{2x_1 + 3x_2} = \frac{\left\| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x + 0 \right\|^2}{\begin{pmatrix} 2 & 3 & 0 \end{pmatrix} x + 0}$$

thus it is a generalized quadratic-over-linear function, however it is convex only if $\begin{pmatrix} 2 & 3 & 0 \end{pmatrix} x + 0 > 0$.

notice that from the second constraint we get:

$$x_1 + x_2 \geq 1 - \frac{x_2}{2}$$

$$x_1 + \frac{3}{2}x_2 \geq 1$$

$$2x_1 + 3x_2 \geq 1$$

hence $\begin{pmatrix} 2 & 3 & 0 \end{pmatrix} x + 0 > 0$ and the quadratic-over-linear function is indeed convex.

we can conclude that $g_1(x)$ is a convex function as a summation of convex function, thus the first constraint is a convex set as a level set of a convex function.

second constraint:

$$x_1 + x_2 \geq 1 - \frac{x_2}{2}$$

$$\iff -x_1 - \frac{3}{2}x_2 \leq -1$$

$$\iff \begin{pmatrix} -1 & -\frac{3}{2} & 0 \end{pmatrix} x \leq -1$$

$\begin{pmatrix} -1 & -\frac{3}{2} & 0 \end{pmatrix} x$ is a convex function because it is a linear function, hence the second constraint is a convex set as a level set of a convex function.

solution:

we showed the problem is a minimization problem of a convex function over a convex set(intersection of convex sets is convex), thus this is indeed a convex optimization problem.

attached is the CVX code for solving the problem:

Figure 3:

```
%% section a

Q = [1,0,0;0,2,-1;0,-1,3];
P = [4,-4,0;-4,6,0;0,0,0];
cvx_begin
variable x(3)
minimize max([norm([2,-3,0]*x,1),norm([-1,1,1]*x,1)]) + quad_form(x,Q)
subject to
    pow_pos(quad_form(x,P)+0.01,8) + quad_over_lin([0,0,1]*x,[2,3,0]*x) -150 <= 0;
    [-1,-1.5,0]*x + 1 <= 0;
cvx_end
```

Command Window

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```
------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +0.444444

The optimal solution is attained at x* =
    0.5000
    0.3333
    0.1667

and the optimal value is: f(x*) = 0.44444
```

b)

$$\begin{aligned}
&\min f(x) = 5x_1^2 + 4x_2^2 + 7x_3^2 + 4x_1x_2 + 2x_2x_3 + |x_1 - x_2| \\
&s.t. g_1(x) = \frac{x_1^2}{2x_1 + x_2} + \left(1 + e^{\sqrt{x_1^2 + x_2^2 + 1}}\right)^7 \leq 200 \\
&\quad g_2(x) = \max\left\{2, e^{(x_1+x_2)^3} + \frac{x_2^2 + x_2x_3 + x_3^2}{x_1} + x_2 - x_1\right\} - 2x_2 \leq 0 \\
&\quad x_1 \geq 1
\end{aligned}$$

objective function:

$$5x_1^2 + 4x_2^2 + 7x_3^2 + 4x_1x_2 + 2x_2x_3 = x^T \underbrace{\begin{pmatrix} 5 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 7 \end{pmatrix}}_{Q} x$$

Q is strictly diagonally dominant with non negative diagonal elements , hence positive definite, therefore x^TQx is a convex function as a quadratic function with a positive definite matrix.

$$|x_1 - x_2| = |(1,-1,0)\,x + 0| = |Ax+b| = \|Ax+b\|_1$$

$\|x\|_1$ is a convex function, hence $\|Ax+b\|_1$ is a convex function as a linear change in the variables of a convex function.

we can conclude that $f(x)$ is a convex function as a summation of convex functions.

first constraint:

$$\frac{x_1^2}{2x_1+x_2} = \frac{\left\|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x + 0\right\|^2}{\begin{pmatrix} 2 & 1 & 0 \end{pmatrix} x + 0}$$

thus it is a generalized quadratic-over-linear function, however it is convex only if $\begin{pmatrix} 2 & 1 & 0 \end{pmatrix} x + 0 > 0$.

notice that from the second constraint we get:

$$2x_2 \geq 2$$

$$x_2 \geq 1$$

and from the third constraint we get:

$$x_1 \geq 1$$

hence:

$$\begin{pmatrix} 2 & 1 & 0 \end{pmatrix} x + 0 = 2x_1 + x_2 \geq 2 + 1 = 3 > 0$$

and the quadratic-over-linear function is indeed convex.

$$\sqrt{x_1^2 + x_2^2 + 1} = \left\|\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{b}\right\|_2$$

$\|x\|_2$ is a convex function, thus $\|Ax+b\|_2$ is a convex function as well as a linear change in the variables of a convex function.

$g\left(x\right)=e^{x}$ is a non decreasing convex function over all $\mathbb{R}$, therefore $e^{\sqrt{x_1^2+x_2^2+1}}$ is a convex function. a constant function is of course convex, hence $1+e^{\sqrt{x_1^2+x_2^2+1}}$ is a convex function as a summation of convex function.

notice that:

$$\sqrt{x_1^2+x_2^2+1}\geq\sqrt{1}=1$$

$$e^{\sqrt{x_1^2+x_2^2+1}}\geq e$$

$$1+e^{\sqrt{x_1^2+x_2^2+1}}\geq 1+e>0$$

hence the image of $1+e^{\sqrt{x_1^2+x_2^2+1}}$ is a subset of $\mathbb{R}_{++}$

$u\left(x\right)=x^{7}$ is a non decreasing convex function over $\mathbb{R}_{++}$, therefore we can conclude that $\left(1+e^{\sqrt{x_1^2+x_2^2+1}}\right)^{7}$ is a convex function.

we can conclude that $g_1\left(x\right)$ is a convex function as a summation of convex function, thus the first constraint is a convex set as a level set of a convex function.

second constraint:

$x_1+x_2=\left(1,1,0\right)x$ is a convex function because it is a linear function.

in addition as we saw before:

$$x_1+x_2\geq 1+1=2>0$$

hence the image of $x_1+x_2=\left(1,1,0\right)x$ is a subset of $\mathbb{R}_{++}$

$r\left(x\right)=x^{3}$ is a non decreasing convex function over $\mathbb{R}_{++}$, therefore we can conclude that $\left(x_1+x_2\right)^{3}$ is a convex function.

$g\left(x\right)=e^{x}$ is a non decreasing convex function over all $\mathbb{R}$, therefore $e^{\left(x_1+x_2\right)^{3}}$ is a convex function.

$$\frac{x_2^2+x_2x_3+x_3^2}{x_1}=\frac{\begin{pmatrix} x_2 & x_3 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}}_{P}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}}{x_1}$$

P is a positive definite matrix because:

$$Tr\left(P\right)=1+1=2>0$$

$$det\left(P\right)=1-\frac{1}{4}=\frac{3}{4}>0$$

therefore we can find it's Cholesky decomposition:

$$P=LL^{T}$$

and we can write:

$$\frac{x_2^2+x_2x_3+x_3^2}{x_1}=\frac{\begin{pmatrix} x_2 & x_3 \end{pmatrix}P\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}}{x_1}=\frac{\begin{pmatrix} x_2 & x_3 \end{pmatrix}LL^{T}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}}{x_1}=\frac{\left(L^{T}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}\right)^{T}L^{T}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}}{x_1}=$$

$$=\frac{\left\|L^{T}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}\right\|^2}{x_1}=\frac{\left\|\underbrace{\begin{pmatrix} 0 & L_{1,1} & L_{2,1} \\ 0 & L_{1,2} & L_{2,2} \end{pmatrix}}_{D}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right\|^2}{x_1}=\frac{\left\|Dx+0\right\|^2}{\left(1,0,0\right)x+0}$$

thus it is a generalized quadratic-over-linear function, which it is convex only if $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}x+0>0$.

but we already saw that

$$x_1\geq 1$$

hence:

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}x+0=x_1>0$$

and the quadratic-over-linear function is indeed convex.

$x_2-x_1=\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}x$ is a convex function since it is a linear function.

thus $e^{(x_1+x_2)^3}+\frac{x_2^2+x_2x_3+x_3^2}{x_1}+x_2-x_1$ is a convex function as a summation of convex functions.

a constant function is of course convex, and a maximum between convex functions is also convex, thus $\max\left\{2,e^{(x_1+x_2)^3}+\frac{x_2^2+x_2x_3+x_3^2}{x_1}+x_2-x_1\right\}$is a convex function.

finally $-2x_2=\begin{pmatrix} 0 & -2 & 0 \end{pmatrix}x$ is a convex function since it is a linear function.

we can conclude that $g_2\left(x\right)$ is a convex function as a summation of convex functions, thus the second constraint is a convex set as a level set of a convex function.

third constraint:

$-x_1 = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix} x$ is a convex function since it is a linear function.

we can conclude that the third constraint is a convex set as a level set of a convex function.

solution:

we showed the problem is a minimization problem of a convex function over a convex set (intersection of convex sets is convex), thus this is indeed a convex optimization problem.

attached is the CVX code for solving the problem:

Figure 4:

```
%% section b

Q = [5,2,0;2,4,1;0,1,7];
A = [1,0,0;0,1,0;0,0,0];
b = [0;0;1];
P = [1,0.5;0.5,1];
L = chol(P,'lower');
D = [zeros(2,1),L'];
cvx_begin
variable x(3)
minimize quad_form(x,Q) + norm([1,-1,0]*x,1)
subject to
    quad_over_lin([1,0,0]*x,[2,1,0]*x) + pow_pos(1+exp(norm(A*x+b,2)),7) - 200 <= 0;
    max([2,exp(pow_pos([1,1,0]*x,3))+quad_over_lin(D*x,[1,0,0]*x)+[-1,1,0]*x])+ [0,-2,0]*x <= 0;
    [-1,0,0]*x + 1 <= 0;
cvx_end
```

```
Command Window
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Status: Infeasible
Optimal value (cvx_optval): +Inf

The optimal solution is attained at x* =
   NaN
   NaN
   NaN

and the optimal value is: f(x*) = Inf
```

as you can see CVX couldn't find a feasible solution.

this shouldn't surprise us, we already saw that the second and third constraint force:

$$x_1 \geq 1$$

$$x_2 \geq 1$$

thus if we look on the first constraint we will get:

$$\frac{x_1^2}{2x_1+x_2}+\left(1+e^{\sqrt{x_1^2+x_2^2+1}}\right)^7 \geq \left(1+e^{\sqrt{x_1^2+x_2^2+1}}\right)^7 \geq \left(1+e^{\sqrt{1+1+1}}\right)^7 = \left(1+e^{\sqrt{3}}\right)^7 = 576464 > 200$$

hence the set of constraints defines an empty set.