Optimization 1 — Homework 11

January 7, 2021

Problem 1

For $a \in \mathbb{R}$ consider the problem

$$\begin{aligned} \min_{x,y,z\in\mathbb{R}} \quad & x - 4y + az^4 \\ \text{s.t.} \quad & x + y + z^2 \leq 2, \\ & x \geq 0, \\ & y \geq 0. \end{aligned}$$

(a) Write a dual problem. For what values of a strong duality is guaranteed?

(b) Solve the dual problem for $a = 1$ and for $a = -1$.

Problem 2

Find a dual to the problem

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & \sum_{i=1}^{n} (\mathbf{x}_i \ln(\mathbf{x}_i) - \mathbf{x}_i) \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} > 0, \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$.

Problem 3

Consider the following optimization problem in the variables $\alpha \in \mathbb{R}$ and $\mathbf{q} \in \mathbb{R}^n$

$$\begin{aligned} (P) \quad \min_{\alpha,\mathbf{q}} \quad & \alpha \\ \text{s.t.} \quad & \mathbf{A}\mathbf{q} = \alpha\mathbf{f}, \\ & \|\mathbf{q}\|^2 \leq \epsilon, \end{aligned}$$

for $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{f} \in \mathbb{R}^m$ and $\epsilon > 0$. Assume in addition that the rows of $\mathbf{A}$ are linearly independent.

(a) Explain why strong duality holds for problem (P).

(b) Find a dual problem to problem (P) (do not assign a Lagrange multiplier to the quadratic constraint).

(c) Solve the dual problem obtained in section (b) and find the optimal solution of problem (P).

Problem 4

Consider the primal optimization problem

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & \sum_{i=1}^{n} \frac{\mathbf{c}_j}{\mathbf{x}_i} \\ \text{s.t.} \quad & \mathbf{a}^T\mathbf{x} \leq b, \\ & \mathbf{x} > 0, \end{aligned}$$

where $\mathbf{a} \in \mathbb{R}^n_{++}$, $\mathbf{c} \in \mathbb{R}^n_{++}$ and $b \in \mathbb{R}_{++}$.

(a) Find a dual problem with a single dual decision variable.

(b) Solve the dual and primal problems.

Problem 5

Consider the optimization problem

$$\begin{aligned} \min_{x,y,z\in\mathbb{R}} \quad & -6x + 2y + 4z^2 \\ \text{s.t.} \quad & 2x + 2y + z \leq 0, \\ & -2x + 4y + z^2 = 0, \\ & y \geq 0. \end{aligned}$$

(a) Is the problem convex?

(b) Assign Lagrange multipliers to the first two constraints and find a dual problem.

(c) Find the optimal solution of the dual problem.

(d) Can the solution to the dual problem help you deduce what is the optimal value of the primal problem? If so, give the optimal value and the optimal solution. If not, explain why.

(e) Reformulate the problem as a convex problem with two variables and two constraints, assign a Lagrange multiplier to only one constraint (associated with the first constraint of the original problem), and find its dual problem.

(f) Can the solution to this new dual problem help you deduce what is the optimal value of the primal problem? If so, give the optimal value and the optimal solution. If not, explain why.