Optimization 1 - 098311

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Question 1:

C is a cone with non empty interior $interior\left(C\right) \neq \emptyset$.

the dual cone is given by:

$$C^{*} = \left\{y \in \mathbb{R}^{n} : y^{T}x \geq 0, \forall x \in C\right\}$$

show that C^{*} is a pointed cone.

proof:

first let's show that C^{*} is a cone.

let $z \in C^{*}$ and $\lambda \in \mathbb{R}_{+}$, then since $z \in C^{*}$:

$$z^{T}x \geq 0, \forall x \in C$$

and:

$$\left(\lambda z\right)^{T}x = \underbrace{\lambda}_{\geq 0}\underbrace{z^{T}x}_{\geq 0} \geq 0, \forall x \in C$$

$\boxed{\text{hence } \lambda z \in C^{*}\text{, therefore } C^{*} \text{ is a cone by definition.}}$

now let's prove it is pointed.

let $z, -z \in C^{*}$.

since $z, -z \in C^{*}$:

$$z^{T}x \geq 0, \forall x \in C$$

$$-z^{T}x \geq 0 \Rightarrow z^{T}x \leq 0, \forall x \in C$$

meaning:

$$z^{T}x = 0, \forall x \in C$$

C has a non empty interior, let's take some $y \in interior\left(C\right)$.

since $y \in interior\left(C\right)$:

$$\exists \epsilon > 0 : \forall d \in \mathbb{R}^{n} : \|d\| = 1, \rightarrow y + \epsilon d \in C$$

since $y + \epsilon d \in C$:

$$z^{T}\left(y + \epsilon d\right) = 0$$

$$\underbrace{z^T y}_{=0} + \epsilon z^T d = 0$$

$$\epsilon z^T d = 0$$

$$z^T d = 0$$

in particular for

$$d = \frac{z}{\|z\|}$$

$$z^T \frac{z}{\|z\|} = 0$$

$$\frac{\|z\|^2}{\|z\|} = 0$$

$$\|z\| = 0 \Longrightarrow z = 0$$

hence C^* is a pointed cone.

Question 2:

a)

Let:

$$f\left(x_1,x_2\right)=x_1^2-2x_1x_2^2+\frac{1}{2}x_2^4$$

$$\boxed{\text{the function is not coercive for example:}\quad f\left(t^2,t\right)=t^4-2t^4+\frac{1}{2}t^4=-\frac{1}{2}t^4\xrightarrow{t\longrightarrow\infty}-\infty}$$

b)

let's find the stationary points:

$$\frac{\partial f\left(x_1,x_2\right)}{\partial x_1}=2x_1-2x_2^2$$

$$\frac{\partial f\left(x_1,x_2\right)}{\partial x_2}=-4x_1x_2+2x_2^3$$

$$\frac{\partial^2 f\left(x_1,x_2\right)}{\partial x_1^2}=2$$

$$\frac{\partial^2 f\left(x_1,x_2\right)}{\partial x_2^2}=-4x_1+6x_2^2$$

$$\frac{\partial^2 f\left(x_1,x_2\right)}{\partial x_1\partial x_2}=-4x_2$$

$$\begin{cases}\frac{\partial f(x_1,x_2)}{\partial x_1}=0\\ \frac{\partial f(x_1,x_2)}{\partial x_2}=0\end{cases}\Rightarrow\begin{cases}2x_1-2x_2^2=0\\ -4x_1x_2+2x_2^3=0\end{cases}\Rightarrow\begin{cases}x_1=x_2^2\\ -2x_1x_2+x_2^3=0\end{cases}$$

$$x_2^3-2x_2^3=0$$

$$x_2^3=0$$

$$x_2=0$$

$$x_1=0$$

so there is only one stationary point $(0,0)$.

$$\nabla^2 f\left(x_1,x_2\right)=\begin{pmatrix}2 & -4x_2\\ -4x_2 & -4x_1+6x_2^2\end{pmatrix}$$

$$\nabla^2 f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

this is a P.S.D matrix hence $(0,0)$ can be either a saddle or a local minimum.

we saw that:

$$f\left(t^2, t\right) = -\frac{1}{2}t^4 \leq 0$$

and

$$\lim_{t\to 0}\left(t^2, t\right) = (0,0)$$

meaning:

$$\forall\epsilon > 0 : f\left((t+\epsilon)^2, t+\epsilon\right) < 0$$

hence we have found a decent direction from the point $(0,0)$, so it can't be a local minimum point.

hence there is only one stationery point $(0,0)$ which is a saddle point.

the function is not bounded from below as we saw before hence doesn't have a global minimum.

Problem 3:

Let Q be an nxn P.S.D and let $a \in \mathbb{R}^n$.

Prove that the set:

$$C = \left\{ x \in \mathbb{R}^n : x^T Q x \leq \left(a^T x\right)^2, a^T x \geq 0 \right\}$$

is a closed convex cone.

first let's prove it is a cone.

let $z \in C$ and $\lambda \in \mathbb{R}_+$:

$$\begin{cases} z^T Q z \leq \left(a^T z\right)^2 \\ a^T z \geq 0 \end{cases}$$

$$(\lambda z)^T Q (\lambda z) = \lambda^2 z^T Q z \overset{\lambda \geq 0}{\leq} \lambda^2 \left(a^T z\right)^2 = \left(a^T (\lambda z)\right)^2$$

$$a^T (\lambda z) = \lambda a^T z \overset{\lambda \geq 0}{\geq} 0$$

$\boxed{\text{hence } \lambda z \in C \text{ and } C \text{ is a cone by definition.}}$

let's prove it is a convex cone.

let $z_1, z_2 \in C$:

$$\begin{cases} z_1^T Q z_1 \leq \left(a^T z_1\right)^2 \\ a^T z_1 \geq 0 \end{cases}, \begin{cases} z_2^T Q z_2 \leq \left(a^T z_2\right)^2 \\ a^T z_2 \geq 0 \end{cases}$$

and:

$$\begin{aligned}
(z_1 + z_2)^T Q (z_1 + z_2) &= z_1^T Q z_1 + z_1^T Q z_2 + z_2^T Q z_1 + z_2^T Q z_2 = \\
&= z_1^T Q z_1 + 2 z_1^T Q z_2 + z_2^T Q z_2 \leq z_1^T Q z_1 + 2 z_1^T Q^{\frac{1}{2}} Q^{\frac{1}{2}} z_2 + z_2^T Q z_2 = \\
&\overset{C.S}{\leq} z_1^T Q z_1 + 2 \left\| Q^{\frac{1}{2}} z_1 \right\| \left\| Q^{\frac{1}{2}} z_2 \right\| + z_2^T Q z_2 = z_1^T Q z_1 + 2 \sqrt{z_1^T Q z_1} \sqrt{z_2^T Q z_2} + z_2^T Q z_2 \\
&= \left(\sqrt{z_1^T Q z_1} + \sqrt{z_2^T Q z_2} \right)^2 \overset{\substack{a^T z_1 \geq 0 \\ a^T z_2 \geq 0}}{\leq} \left(a^T z_1 + a^T z_2\right)^2 \\
&= \left(a^T (z_1 + z_2)\right)^2
\end{aligned}$$

$$(z_1 + z_2)^T Q (z_1 + z_2) \leq \left(a^T (z_1 + z_2)\right)^2$$

in addition:

$$a^T (z_1 + z_2) = a^T z_1 + a^T z_2 \geq 0$$

hence $z_1 + z_2 \in C$ and thus C is a convex cone.

finally, C is an intersection of two closed sets, hence closed.

Problem 4:

Let $E \in \mathbb{R}^{kxn}$, $f \in \mathbb{R}^n$,$a \in \mathbb{R}^m$,$A \in \mathbb{R}^{mxn}$, $b \in \mathbb{R}^m$ and $c_1, c_2, ..., c_n \in \mathbb{R}^n$

consider the problem:

$$\min_{x \in \mathbb{R}^n, z \in \mathbb{R}^m} f(x,z) = \frac{1}{2}\|Ex\|^2 + \frac{1}{2}\|z\|^2 + f^T x + a^T z + \sum_{i=1}^{n} e^{c_i^T x}$$

$$s.t Ax + z = b$$

$$z \geq 0$$

assume that E has a full column rank.

a)

Show that the objective function is coercive.

$$\|Ex\|^2 = x^T E^T E x \geq \lambda_{min}\left(E^T E\right)\|x\|^2$$

$$f^T x + a^T z = \begin{pmatrix} f & a \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} \overset{C.S}{\geq} -\left\|\begin{pmatrix} f \\ a \end{pmatrix}\right\|\left\|\begin{pmatrix} x \\ z \end{pmatrix}\right\|$$

hence:

$$f(x,z) \geq \frac{1}{2}\lambda_{min}\left(E^T E\right)\|x\|^2 + \frac{1}{2}\|z\|^2 - \left\|\begin{pmatrix} f \\ a \end{pmatrix}\right\|\left\|\begin{pmatrix} x \\ z \end{pmatrix}\right\| \geq$$

$$\geq \frac{1}{2}\min\left\{\lambda_{min}\left(E^T E\right), 1\right\}\left(\|x\|^2 + \|z\|^2\right) - \left\|\begin{pmatrix} f \\ a \end{pmatrix}\right\|\left\|\begin{pmatrix} x \\ z \end{pmatrix}\right\|$$

$$= \frac{1}{2}\min\left\{\lambda_{min}\left(E^T E\right), 1\right\}\left\|\begin{pmatrix} x \\ z \end{pmatrix}\right\|^2 - \left\|\begin{pmatrix} f \\ a \end{pmatrix}\right\|\left\|\begin{pmatrix} x \\ z \end{pmatrix}\right\|$$

notice then since E has a full column rank, then $E^T E$ has a full column rank and $\lambda_{min}\left(E^T E\right) > 0$ which means

$$\frac{1}{2}\min\left\{\lambda_{min}\left(E^T E\right), 1\right\} > 0$$

so:

$$f(x,z) \geq \frac{1}{2}\min\left\{\lambda_{min}\left(E^T E\right), 1\right\}\left\|\begin{pmatrix} x \\ z \end{pmatrix}\right\|^2 - \left\|\begin{pmatrix} f \\ a \end{pmatrix}\right\|\left\|\begin{pmatrix} x \\ z \end{pmatrix}\right\| \xrightarrow{\left\|\begin{pmatrix} x & z \end{pmatrix}\right\| \to \infty} \infty$$

hence $f(x,z)$ is a coercive function.

b)

show that if the set:

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} \neq \emptyset$$

then strong duality holds.

let's first show this is a convex function.

$$f(x, z) = \frac{1}{2}\|Ex\|^2 + \frac{1}{2}\|z\|^2 + f^T x + a^T z + \sum_{i=1}^{n} e^{c_i^T x}$$

the objective function:

$\begin{pmatrix} f & a \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix}$ is a linear function hence convex.

$\|z\| = \left\| \begin{pmatrix} 0_{mxn} & I_{mxm} \end{pmatrix}_{mxn+m} \begin{pmatrix} x \\ z \end{pmatrix} \right\|$, the norm is convex hence this function is convex as a linear change in the variables of a convex function.

in addition $\|z\| \geq 0$ and convex, the function $g_1(x) = x^2$ is a non decreasing convex function over $\mathbb{R}^+$, hence $\|z\|^2$ is a convex function and of course also $\frac{1}{2}\|z\|^2$ is convex is multiplying by scalar preserve convexity.

$\|Ex\| = \left\| \begin{pmatrix} E_{kxn} & 0_{kxm} \end{pmatrix}_{kxn+m} \begin{pmatrix} x \\ z \end{pmatrix} \right\|$, so from the same reasons, $\frac{1}{2}\|Ex\|^2$ is convex.

$c_i^T x = \begin{pmatrix} c_i & 0 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix}$ is a convex function as a linear function, the function $g_2(x) = e^x$ is a non decreasing convex function over the entire $\mathbb{R}$, hence $e^{c_i^T x}$ is a convex function, and $\sum_{i=1}^{n} e^{c_i^T x}$ is a convex function as a sum of convex functions.

to conclude, the objective function is convex as a sum of convex functions.

the constraints:

both the constraints are linear constraints hence defines a convex set.

1) thus this is a convex optimization problem.

2) since all the constraints are linear then slater condition naturally holds, if the set of solutions is not empty. since we are given:

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} \neq \emptyset$$

let's take $y \in P$

$$Ay \leq b$$

and define:

$$z = b - Ay$$

then:

$$Ay + z = b$$

$$z \geq 0$$

hence this is a feasible solution for the problem and slater holds.

3) since we show the function is coercive, and is defined over a non empty closed set, that it must attain a global minimum value, in particular $f^* > -\infty$

$\boxed{\text{all three condition hold, thus strong duality holds.}}$

c)

Write a dual problem.

define the set:

$$X = \{x \in \mathbb{R}^n, z \in \mathbb{R}^m : z \geq 0\}$$

write the Lagrangian

$$L(x, z, \mu) = \frac{1}{2}\|Ex\|^2 + \frac{1}{2}\|z\|^2 + f^T x + a^T z + \sum_{i=1}^{n} e^{c_i^T x} + \mu^T (Ax + z - b)$$

where:

$$\mu \in \mathbb{R}^m$$

$$q(\mu) = \min_{(x,z)\in X} L(x, z, \mu) = \min_{(x,z)\in X} \frac{1}{2}\|Ex\|^2 + \frac{1}{2}\|z\|^2 + f^T x + a^T z + \sum_{i=1}^{n} e^{c_i^T x} + \mu^T (Ax + z - b) =$$
$$= \min_{x\in\mathbb{R}^n}\left(\frac{1}{2}\|Ex\|^2 + f^T x + \sum_{i=1}^{n} e^{c_i^T x} + \mu^T Ax\right) + \min_{z\in\mathbb{R}^m_+}\left(\frac{1}{2}\|z\|^2 + a^T z + \mu^T z\right) - \mu^T b$$

the problem in z is separable:

$$\min_{z\in\mathbb{R}^m_+}\left(\frac{1}{2}\|z\|^2 + a^T z + \mu^T z\right) = \min_{z\in\mathbb{R}^m_+}\left(\sum_{i=1}^{m}\frac{1}{2}z_i^2 + a_i z_i + \mu_i z_i\right) = \sum_{i=1}^{m}\min_{z_i\in\mathbb{R}_+}\left(\frac{1}{2}z_i^2 + a_i z_i + \mu_i z_i\right)$$

this is a simple minimum parable is z_i which is convex and coercive thus attains a minimum, either at a stationery point or on the boundary.

the stationery point is:

$$z_i = \frac{-a_i - \mu_i}{2 \cdot \frac{1}{2}} = -a_i - \mu_i$$

if $-a_i - \mu_i \geq 0$ then it is attained and it's value is:

$$\frac{1}{2}\left(a_i + \mu_i\right)^2 - \left(a_i + \mu_i\right)^2 = -\frac{1}{2}\left(a_i + \mu_i\right)^2$$

otherwise the minimal value is on the boundary $z_i = 0$ and it's value is 0

$$\min_{z_i \in \mathbb{R}_+}\left(\frac{1}{2}z_i^2 + a_i z_i + \mu_i z_i\right) = \begin{cases} -\frac{1}{2}\left(a_i + \mu_i\right)^2 & -a_i - \mu_i \geq 0 \\ 0 & -a_i - \mu_i < 0 \end{cases}$$

$$\sum_{i=1}^{m} \min_{z_i \in \mathbb{R}_+}\left(\frac{1}{2}z_i^2 + a_i z_i + \mu_i z_i\right) = \sum_{i: -a_i - \mu_i \geq 0} -\frac{1}{2}\left(a_i + \mu_i\right)^2$$

moving to x, the problem is not separable in the coordinate:

$$\min_{x \in \mathbb{R}^n}\left(\frac{1}{2}\|Ex\|^2 + f^T x + \sum_{i=1}^{n} e^{c_i^T x} + \mu^T A x\right)$$

this is an unconstrained convex problem and coercive (we proved it in section 1) thus, it must attains a global minimum at the stationery point.

$$\frac{\partial}{\partial x} = E^T E x + f + \sum_{i=1}^{n} e^{c_i^T x} c_i + A^T \mu = 0$$

we have no way to solve this equation, we will define new variable to help us, the new problem:

$$\min_{x, w \in \mathbb{R}^n, z \in \mathbb{R}^m} f\left(x, z\right) = \frac{1}{2}\|Ex\|^2 + \frac{1}{2}\|z\|^2 + f^T x + a^T z + \sum_{i=1}^{n} e^{w_i}$$

$$s.t Ax + z = b$$

$$\forall i \in \{1, ..., m\} : w_i = c_i^T x$$

$$z \geq 0$$

the new Lagrangian:

$$L\left(x, z, w, \mu, \gamma\right) = \frac{1}{2}\|Ex\|^2 + \frac{1}{2}\|z\|^2 + f^T x + a^T z + \sum_{i=1}^{n} e^{w_i} + \mu^T\left(Ax + z - b\right) + \sum_{i=1}^{n} \gamma_i\left(w_i - c_i^T x_i\right)$$

where:

$$\mu \in \mathbb{R}^m, \gamma \in \mathbb{R}^n$$

and now we need to solve:

$$q(\mu,\gamma) = \min_{(x,z)\in X} L(x,z,\mu,\gamma) = \min_{(x,z)\in X} \frac{1}{2}\|Ex\|^2 + \frac{1}{2}\|z\|^2 + f^T x + a^T z + \sum_{i=1}^{n} e^{w_i} + \mu^T(Ax+z-b) + \sum_{i=1}^{n}\gamma_i (w$$

$$= \min_{x\in\mathbb{R}^n}\left(\frac{1}{2}\|Ex\|^2 + f^T x + \mu^T Ax - \sum_{i=1}^{n}\gamma_i c_i^T x_i\right) + \min_{z\in\mathbb{R}_+^m}\left(\frac{1}{2}\|z\|^2 + a^T z + \mu^T z\right) + \min_{w\in\mathbb{R}^n}\left(\sum_{i=1}^{n} e^{w_i} + \gamma_i w_i$$

now the problem is x becomes:

$$\min_{x\in\mathbb{R}^n}\left(\frac{1}{2}\|Ex\|^2 + f^T x + \mu^T Ax - \gamma^T Cx\right) = \frac{1}{2}\min_{x\in\mathbb{R}^n}\left(x^T E^T Ex + 2\left(f + A^T\mu - C^T\gamma\right)^T x\right)$$

which is a quadratic function with a P.D matrix, hence the optimal solution is attained at:

$$x = -\left(E^T E\right)^{-1}\left(f + A^T\mu - C^T\gamma\right)$$

and the optimal value is:

$$-\frac{1}{2}\left(f + A^T\mu - C^T\gamma\right)^T\left(E^T E\right)^{-1}\left(f + A^T\mu - C^T\gamma\right)$$

$$\begin{aligned} x^T Ax + 2b^T x + c &= \left(-A^{-1}b\right)^T A\left(-A^{-1}b\right) + 2b^T\left(-A^{-1}b\right) + c = \\ &= b^T A^{-1} A A^{-1} b - 2b^T A^{-1} b + c = \\ &= b^T A^{-1} b - 2b^T A^{-1} b + c = \\ &= c - b^T A^{-1} b \end{aligned}$$

$$\min_{x\in\mathbb{R}^n}\left(\frac{1}{2}\|Ex\|^2 + f^T x + \mu^T Ax - \gamma^T Cx\right) = -\frac{1}{2}\left(f + A^T\mu - C^T\gamma\right)^T\left(E^T E\right)^{-1}\left(f + A^T\mu - C^T\gamma\right)$$

we still need to solve the problem in w:

$$\min_{w\in\mathbb{R}^n}\left(\sum_{i=1}^{n} e^{w_i} + \gamma_i w_i\right) = \sum_{i=1}^{n}\min_{w_i\in\mathbb{R}} e^{w_i} + \gamma_i w_i$$

this is seperable in the coordinates of w :

$$\min_{w_i\in\mathbb{R}} e^{w_i} + \gamma_i w_i$$

this is a convex problem, however notice that if $\gamma_i > 0$, the function is unbounded from below. for $\gamma_i < 0$ this is a coercive convex function, thus the globl minimum is attained at a stationary point:

$$\frac{\partial}{\partial w} = e^{w_i} + \gamma_i = 0$$

$$e^{w_i} = -\gamma_i$$

$$w_i = \ln\left(-\gamma_i\right)$$

the optimal value will be:

$$e^{\ln(-\gamma_i)} + \gamma_i \ln\left(-\gamma_i\right) = -\gamma_i + \gamma_i \ln\left(-\gamma_i\right)$$

for $\gamma_i = 0$ the minimal value of the exponent is 0, so if we denote $0 \ln(0) = 0$ then:

$$\min_{w_i \in \mathbb{R}} e^{w_i} + \gamma_i w_i = \begin{cases} -\gamma_i + \gamma_i \ln\left(-\gamma_i\right) & \gamma_i \leq 0 \\ -\infty & \gamma_i > 0 \end{cases}$$

and:

$$\min_{w \in \mathbb{R}^n} \left(\sum_{i=1}^{n} e^{w_i} + \gamma_i w_i \right) = \begin{cases} \sum_{i=1}^{n} -\gamma_i + \gamma_i \ln\left(-\gamma_i\right) & \gamma_i \leq 0, \forall i \in \{1, 2, ..., n\} \\ -\infty & else \end{cases}$$

to sum everything, the dual problem is:

$$\max_{\mu \in \mathbb{R}^m, \gamma \in \mathbb{R}^n} -\frac{1}{2}\left(f + A^T\mu - C^T\gamma\right)^T \left(E^T E\right)^{-1} \left(f + A^T\mu - C^T\gamma\right) + \sum_{i: -a_i - \mu_i \geq 0} -\frac{1}{2}\left(a_i + \mu_i\right)^2 + \sum_{i=1}^{n} -\gamma_i + \gamma_i \ln (-\gamma$$

$$s.t \gamma \leq 0$$

Problem 5:

Consider the optimization problem:

$$\max x_1 x_2 x_3$$
$$s.t x_1^2 + 2x_2^2 + 3x_3^2 \leq 1$$

a)

find all the K.K.T points of the problem.

write the Lagrangian:

$$L\left(x_1, x_2, x_3, \lambda\right) = x_1 x_2 x_3 + \lambda\left(x_1^2 + 2x_2^2 + 3x_3^2 - 1\right)$$

the K.K.T conditions:

$$\begin{cases} \begin{cases} \frac{\partial L(x_1,x_2,x_3,\lambda)}{\partial x_1} = x_2 x_3 + 2\lambda x_1 = 0 & (1) \\ \frac{\partial L(x_1,x_2,x_3,\lambda)}{\partial x_2} = x_1 x_3 + 4\lambda x_2 = 0 & (2) \\ \frac{\partial L(x_1,x_2,x_3,\lambda)}{\partial x_3} = x_1 x_2 + 6\lambda x_3 = 0 & (3) \end{cases} \\ \lambda\left(x_1^2 + 2x_2^2 + 3x_3^2 - 1\right) = 0 & (4) \\ x_1^2 + 2x_2^2 + 3x_3^2 \leq 1 & (5) \\ \lambda \geq 0 & (6) \end{cases}$$

if $\lambda = 0$, then (4) and (6) holds and:

$$x_2 x_3 = x_1 x_3 = x_1 x_2 = 0$$

meaning at least two of the three variables needs to be equal to 0.

if $x_1 = x_2 = 0$ then from (5):

$$3x_3^2 \leq 1$$
$$x_3^2 \leq \frac{1}{3}$$
$$-\frac{1}{\sqrt{3}} \leq x_3 \leq \frac{1}{\sqrt{3}}$$

if $x_1 = x_3 = 0$ then from (4):

$$2x_2^2 \leq 1$$

$$x_2^2 \leq \frac{1}{2}$$

$$-\frac{1}{\sqrt{2}} \leq x_2 \leq \frac{1}{\sqrt{2}}$$

if $x_2 = x_3 = 0$ then from (4):

$$x_1^2 \leq 1$$

$$-1 \leq x_1 \leq 1$$

hence we have found infinite number of K.K.T from this case :

$$\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{3}} \leq x_3 \leq \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \leq x_2 \leq \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \begin{pmatrix} -1 \leq x_1 \leq 1 & 0 & 0 \end{pmatrix}$$

all with $\lambda = 0$

if $\lambda > 0$, let's assume $x_1 = 0$, then from (1) either x_2 or x_3 has to be 0, let's assume $x_2 = 0$. from (3) it means that $x_3 = 0$, but $x_1 = x_2 = x_3$ contradicts (4), thus non of them can be zero.

from (1):

$$x_2 x_3 + 2\lambda x_1 = 0$$

$$2\lambda x_1 = -x_2 x_3$$

$$\lambda = -\frac{x_2 x_3}{2x_1}$$

plugging in (2):

$$x_1 x_3 + 4\lambda x_2 = 0$$

$$x_1 x_3 + 4\left(-\frac{x_2 x_3}{2x_1}\right) x_2 = 0$$

$$x_1 x_3 - 2\frac{x_2^2 x_3}{x_1} = 0$$

$$x_1^2 x_3 - 2x_2^2 x_3 = 0$$

$$x_3\left(x_1^2 - 2x_2^2\right) = 0$$

meaning:

$$x_1^2 = 2x_2^2$$

plugging in (3):

$$x_1 x_2 + 6\lambda x_3 = 0$$

$$x_1x_2 + 6\left(-\frac{x_2x_3}{2x_1}\right)x_3 = 0$$

$$x_1x_2 - 3\frac{x_2x_3^2}{x_1} = 0$$

$$x_1^2x_2 - 3x_2x_3^2 = 0$$

$$x_2\left(x_1^2 - 3x_3^2\right) = 0$$

meaning:

$$x_1^2 = 3x_3^2$$

plugging to (4):

$$x_1^2 + 2x_2^2 + 3x_3^2 - 1 = 0$$

$$3x_1^2 = 1$$

$$x_1^2 = \frac{1}{3}$$

$$x_1 = \pm\frac{1}{\sqrt{3}}$$

$$2x_2^2 = x_1^2 = \frac{1}{3}$$

$$x_2 = \pm\frac{1}{\sqrt{6}}$$

$$3x_3^2 = x_1^2 = \frac{1}{3}$$

$$x_3 = \pm\frac{1}{3}$$

$$\lambda = -\frac{x_2x_3}{2x_1} = \pm\frac{\frac{1}{\sqrt{6}}\frac{1}{3}}{\frac{1}{\sqrt{3}}} = \pm\frac{\sqrt{2}}{6}$$

in order for λ to be positive with either all the coordinates to be negative or exactly one of them, thus the K.K.T points from this case are:

$$\left(\begin{array}{ccc} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{3} \end{array}\right), \left(\begin{array}{ccc} -\frac{1}{\sqrt{3}} & +\frac{1}{\sqrt{6}} & +\frac{1}{3} \end{array}\right), \left(\begin{array}{ccc} +\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & +\frac{1}{3} \end{array}\right), \left(\begin{array}{ccc} +\frac{1}{\sqrt{3}} & +\frac{1}{\sqrt{6}} & -\frac{1}{3} \end{array}\right)$$

all with $\lambda = \frac{\sqrt{2}}{6}$

b)

Find all the optimal solutions of the problem.

The objective function is continuously differentiable over a compact set (a closed ellipsoid), thus an optimal solution is attained form Weierstrass theorem.

The objective function is not a convex function. The constraint, however, is a convex set as a level set of a quadratic function with a P.D matrix. Thus if Slater condition hold, then the K.K.T condition are necessary for optimality and we just need to search among them for the optimal solution.

Slater condition indeed holds, for example for $x_1 = x_2 = x_3 = 0$

$$x_1^2 + 2x_2^2 + 3x_3^2 = 0 < 1$$

let's calculate the value of the objective function at each K.K.T point:

at the four points with $\lambda = \frac{\sqrt{2}}{6}$ the function value is:

$$x_1 x_2 x_3 = -\frac{\sqrt{2}}{18}$$

at all the points with $\lambda = 0$ we will get a function value of

$$x_1 x_2 x_3 = 0$$

thus the optimal value of the problem is attained at all four points:

$$\begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{3} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{3}} & +\frac{1}{\sqrt{6}} & +\frac{1}{3} \end{pmatrix}, \begin{pmatrix} +\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & +\frac{1}{3} \end{pmatrix}, \begin{pmatrix} +\frac{1}{\sqrt{3}} & +\frac{1}{\sqrt{6}} & -\frac{1}{3} \end{pmatrix}$$

and the optimal value is $-\frac{\sqrt{2}}{18}$