Optimization 1 — Homework 1

October 22, 2020

Problem 1
Prove that the induced ℓ_∞ norm of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is given by

$$\|\mathbf{A}\|_\infty = \max_{i=1,2,\ldots,m} \sum_{j=1}^{n} |\mathbf{A}_{ij}| .$$

Problem 2
Prove that for any $\mathbf{x} \in \mathbb{R}^n$ it holds that

$$\|\mathbf{x}\|_\infty = \lim_{p \to \infty} \|\mathbf{x}\|_p .$$

Problem 3

(a) Consider the normed spaces $(\mathbb{R}^m, \|\cdot\|_a)$ and $(\mathbb{R}^n, \|\cdot\|_b)$. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Prove the following two equalities:

$$\|\mathbf{A}\|_{b,a} = \max_{\mathbf{x} \in \mathbb{R}^n} \left\{ \|\mathbf{A}\mathbf{x}\|_a : \|\mathbf{x}\|_b = 1 \right\} = \max_{\mathbf{x} \in \mathbb{R}^n} \left\{ \frac{\|\mathbf{A}\mathbf{x}\|_a}{\|\mathbf{x}\|_b} : \mathbf{x} \neq \mathbf{0}_n \right\} .$$

(b) Consider the normed spaces $(\mathbb{R}^m, \|\cdot\|_a)$, $(\mathbb{R}^n, \|\cdot\|_b)$ and $\left(\mathbb{R}^k, \|\cdot\|_c\right)$. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times k}$. Prove the following inequality:

$$\|\mathbf{A}\mathbf{B}\|_{c,a} \leq \|\mathbf{A}\|_{b,a} \|\mathbf{B}\|_{c,b} .$$

Problem 4
Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Prove the following two properties:

(a) $\|\mathbf{A}\|_F^2 = \sum\limits_{i=1}^{n} \lambda_i \left(\mathbf{A}^T \mathbf{A}\right)$.

(b) $\frac{1}{\sqrt{n}} \|\mathbf{A}\|_\infty \leq \|\mathbf{A}\|_2 \leq \sqrt{m} \|\mathbf{A}\|_\infty$.

Problem 5
Let $\{\mathcal{A}_i\}_{i \in I} \subseteq \mathbb{R}^n$ be a collection of open sets, where I is a given index set.

(a) Show that $\bigcup\limits_{i \in I} \mathcal{A}_i$ is an open set.

(b) Show that if I is finite, then $\bigcap\limits_{i \in I} \mathcal{A}_i$ is open.

(c) Give an example of open sets $\{\mathcal{A}_i\}_{i \in I} \subseteq \mathbb{R}^n$ and an index set I, for which $\bigcap\limits_{i \in I} \mathcal{A}_i$ is not open.

Problem 6

(a) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be defined by $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ for some $\mathbf{A} \in \mathbb{R}^{n\times n}$. Explain why f is twice continuously differentiable and find its gradient vector and Hessian matrix.

(b) For any $\mathbf{x} \in \mathbb{R}^n$ and any non-zero vector $\mathbf{d} \in \mathbb{R}^n$, compute the directional derivative $f'(\mathbf{x};\mathbf{d})$ of

$$f(\mathbf{x}) = \left|\|\mathbf{x}-\mathbf{a}\|_2 - \delta\right|,$$

for $\mathbf{a} \in \mathbb{R}^n$ and $\delta \in \mathbb{R}_{++}$.