Optimization 1 — Homework 6

November 26, 2020

Problem 1
A function $f\colon C \subseteq \mathbb{R}^n$ for a convex set C is called strongly convex if there exists $\sigma > 0$ such that the function $f(\mathbf{x}) - \sigma\frac{\|\mathbf{x}\|^2}{2}$ is convex over C. The parameter σ is called the strong convexity parameter.

(a) Prove that f is strongly convex over C with parameter σ if and only if

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) - \frac{\sigma}{2}\lambda(1-\lambda)\|\mathbf{x}-\mathbf{y}\|^2,$$

for any $\mathbf{x}, \mathbf{y} \in C$ and $\lambda \in [0,1]$.

(b) Suppose that f is continuously differentiable over C. Prove that f is strongly convex over C with parameter σ if and only if

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \frac{\sigma}{2}\|\mathbf{x}-\mathbf{y}\|^2,$$

for any $\mathbf{x}, \mathbf{y} \in C$.

(b) Suppose that f is continuously differentiable over C. Prove that f is strongly convex over C with parameter σ if and only if

$$(\nabla f(\mathbf{x}) - \nabla f(\mathbf{x}))^T(\mathbf{y}-\mathbf{x}) \geq \sigma\|\mathbf{x}-\mathbf{y}\|^2,$$

for any $\mathbf{x}, \mathbf{y} \in C$.

Problem 2
Show that the following functions are convex. Determine whether they are quasi-concave or not.

(a) $f(x,y,z) = \sqrt{2x^2 + 2y^2 + 5z^2 + 2xy + 2xz + 4yz - 4y + 4}$ over $\operatorname{dom}(f)$ and find its domain.

(b) $f(\mathbf{x}) = \frac{\mathbf{x}_1^4}{\mathbf{x}_2^2} + \frac{\mathbf{x}_2^4}{\mathbf{x}_1^2} + 2\mathbf{x}_1\mathbf{x}_2 - \min\left\{\ln(\mathbf{x}_1+\mathbf{x}_2), \ln\left(2\mathbf{x}_1 + \frac{1}{2}\mathbf{x}_2\right)\right\}$ over $\mathbb{R}^2_{++}$. (Hint: use the quadratic-over-linear function).

(c) $f(\mathbf{x}) = \sum\limits_{i=1}^{n}\mathbf{x}_i\ln(\mathbf{x}_i) - \left(\sum\limits_{i=1}^{n}\mathbf{x}_i\right)\ln\left(\sum\limits_{i=1}^{n}\mathbf{x}_i\right)$ over $\mathbb{R}^n_{++}$.

(d) $f(\mathbf{x}) = -\sqrt[n]{\prod\limits_{i=1}^{n}\mathbf{x}_i}$ over $\mathbb{R}^n_+$ (Hint: prove it first over $\mathbb{R}^n_{++}$ using the gradient inequality).

Problem 3
Determine whether the following sets are convex or not.

(a) $\left\{\mathbf{x} \in \mathbb{R}^3_+ \colon (\mathbf{x}_2 + \mathbf{x}_3 + 1)(2\mathbf{x}_1 + 2\mathbf{x}_3 + 2)(3\mathbf{x}_1 + 3\mathbf{x}_2 + 3) \geq 1\right\}$ (Hint: show that this a level set of a convex function).

(b) $\left\{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{x}_1^2 \leq \mathbf{x}_2\mathbf{x}_3,\ \mathbf{x}_2, \mathbf{x}_3 \geq 0\right\}$.

(c) $\{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x}-\mathbf{u}\| \leq \|\mathbf{x}-\mathbf{v}\|\}$ for constants $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.