Optimization 1 — Homework 5

November 19, 2020

Problem 1
Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and let $C \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^m$ be convex sets. Prove that the sets $\mathbf{A}(C) = \{\mathbf{Ax} \in \mathbb{R}^n \colon \mathbf{x} \in C\}$ and $\mathbf{A}^{-1}(D) = \{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{Ax} \in D\}$ are convex.

Problem 2
Let $\mathbf{a} \neq \mathbf{b} \in \mathbb{R}^n$. Find the values of $\mu \in \mathbb{R}$ for which the set

$$S_\mu = \{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x} - \mathbf{a}\| \leq \mu \|\mathbf{x} - \mathbf{b}\|\}$$

is convex.

Problem 3
Show that the conic hull of the set

$$S = \left\{\mathbf{x} \in \mathbb{R}^2 \colon (\mathbf{x}_1 - 1)^2 + \mathbf{x}_2^2 = 1\right\}$$

is the set $\left\{\mathbf{x} \in \mathbb{R}^2 \colon \mathbf{x}_1 > 0\right\} \cup \{(0,0)\}$ (it shows that the conic hull of a closed set is not necessarily a closed set).

Problem 4
Let $\emptyset \neq S \subseteq \mathbb{R}^n$ and let $\bar{\mathbf{x}} \in S$. Consider the set

$$C_{\bar{\mathbf{x}}} = \{\mathbf{y} \in \mathbb{R}^n \colon \mathbf{y} = \lambda(\mathbf{x} - \bar{\mathbf{x}}), \lambda \geq 0, \mathbf{x} \in S\}.$$

(a) Show that $C_{\bar{\mathbf{x}}}$ is a cone and interpret it geometrically.

(b) Show that if S is convex then $C_{\bar{\mathbf{x}}}$ is convex.

(c) Suppose that S is closed. Is it necessarily true that $C_{\bar{\mathbf{x}}}$ is closed? If not, find at least two conditions under which the set is closed.

Problem 5
Consider the optimization problem

$$(P_{\mathbf{a}}) \qquad \min\left\{\mathbf{a}^T\mathbf{x} \colon \mathbf{x} \in S\right\},$$

where $S \subseteq \mathbb{R}^n$. Let $\mathbf{x}^* \in S$ and let $K \subseteq \mathbb{R}^n$ be the set of all vectors $\mathbf{a}$ for which $\mathbf{x}^*$ is an optimal solution of $(P_{\mathbf{a}})$. Show that K is a convex cone.

Problem 6

(a) Show that the extreme points of $B_\infty[\mathbf{0}_n, 1] = \{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x}\|_\infty \leq 1\}$ are $\{-1, 1\}^n$.

(b) Let $X_i \subseteq \mathbb{R}^n$, $i = 1, 2, \ldots, k$. Prove that

$$\text{ext}(X_1 \times X_2 \times \ldots \times X_k) = \text{ext}(X_1) \times \text{ext}(X_2) \times \ldots \times \text{ext}(X_k).$$