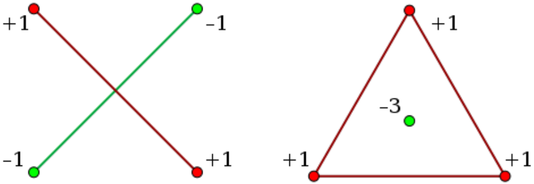
Radon's Theorem

Theorem (Radon's Theorem (1921))

Let S be a set of at least $n+2$ points in $\mathbb{R}^n$. Then, there exists a partition of S, i.e., sets S_1 and S_2 such that $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = S$, that satisfies $\mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \neq \emptyset$.

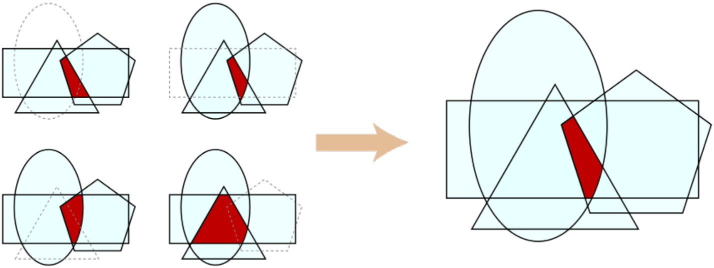


Proof. In class

Helly's Theorem

Theorem (Helly's Theorem (1923))

Let $S_1, \ldots, S_m$ be a finite collection of convex sets in $\mathbb{R}^n$, with $m \geq n+1$. If the intersection of every $n+1$ of these sets is nonempty, then the whole collection has a nonempty intersection; that is, $\bigcap_{j=1}^{m} S_j \neq \emptyset$.



Proof.

- Proof by induction.
- **Base case:** m=n+1, the claim is trivially true.
- **Induction step:** Assume that the statement is true for $m = n+1, \ldots, s$ we will prove that it is true for $s+1$.
  - By the induction hypothesis every subfamily of $S_1, \ldots, S_{s+1}$ with cardinality s has a nonempty intersection.
  - Let $\mathbf{x}_i \in \bigcap_{j \in \{1,\ldots,i-1,i+1,\ldots,m\}} S_j$, we have $s+1$ such points. Let $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_{s+1}\}$
  - Since $s+1 \geq n+2$, by Radon's Theorem there is a partition of X to X_1 and X_2 such that $\mathbf{y} \in \operatorname{conv}(X_1) \cap \operatorname{conv}(X_2) \neq \emptyset$.
  - W.l.o.g. let $x_j \in X_1$. Then $X_2 \subseteq S_j$. Since S_j is convex $\mathbf{y} \in \operatorname{conv}(X_2) \subseteq S_j$.
  - Therefore, $\mathbf{y} \in S_j$ for all j, and $\bigcap_{j=1}^{m} S_j \neq \emptyset$.

Application of Helly's Theorem

Assume that we have m equaly spaced vertical segments in a $\mathbb{R}^2$.
How can we verify that we can run a line through all of them?