Optimization 1 — Tutorial 7

December 3, 2020

Problem 1

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} + 2\mathbf{b}^T \mathbf{x} + c,$$

where $\mathbf{A} \succ 0$. Let $\mathbf{x}^0 \in \mathbb{R}^n$ be any starting point. Consider the Gradient Descent step $\mathbf{x}^{k+1} = \mathbf{x}^k - t\nabla f\left(\mathbf{x}^k\right)$ for $t > 0$. Find an explicit formula for $\mathbf{x}^0$ that will guarantee convergence to the global minimum point of f after one iteration.

Problem 2

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a convex function, and define the convex level set $Q = \{\mathbf{x} \in \mathbb{R}^n \colon f(\mathbf{x}) \leq d\}$ for some $d \in \mathbb{R}$. We say that a set $S \subseteq \mathbb{R}^n$ is contained in Q if

$$\max_{\mathbf{x}\in S} \quad f(\mathbf{x}) \leq d.$$

Formulate the problem which finds the ball with largest radius contained in Q (use $r \geq 0$ to denote its radius and $\mathbf{c} \in \mathbb{R}^n$ to denote its center). Is this problem convex in $\mathbf{c}$ and r?

Solution

- Plugging the ball $B(\mathbf{c}, r)$ in the definition gives us

$$\max_{\mathbf{x}\in B(\mathbf{c},r)} \quad f(\mathbf{x}) \leq d.$$

- Since we want the maximum r that still solves this problem, we will look at the problem

$$\begin{aligned} \max_{r\in\mathbb{R},\mathbf{c}\in\mathbb{R}^n} \quad & r \\ \text{s.t.} \quad & \max_{\{\mathbf{x}\in\mathbb{R}^n \colon \|\mathbf{x}-\mathbf{c}\|\leq r\}} f(\mathbf{x}) \leq d. \end{aligned}$$

- If the objective function is convex and the constraint defines a convex set in $\mathbf{c}$ and r, then this is a convex programming problem:

CVX

A MATLAB package that solves convex optimization problems. The package is available for download in http://cvxr.com/cvx/download/. The package uses external solvers.

A general CVX program is of the form:

```
cvx_begin
{variable declaration}
minimize ({convex function}) or maximize ({concave function})
subject to
{convex constraints}
cvx_end
```

MATLAB syntax is also available in a CVX program. Defining variables is performed as follows:

```
variable x;
variable y(4);
variable z(2,3);
or
variables x y(4) z(2,3);
```

The objective function in a minimization problem must be convex, and in a maximization problem it must be concave. A convex objective function is recognized as such only if it is composed using convex atomic functions with some allowed operations. The atomic functions are:

Function	Description	Properties
`norm(x,p)`	$\|\mathbf{x}\|_p$	Convex
`square(x)`	x^2	Convex
`sum_square(x)`	$\|\mathbf{x}\|_2^2$	Convex
`square_pos(x)`	$\max\left\{x^2, 0\right\}$	Convex, increasing
`sum_largest(x,k)`	$\mathbf{x}_{[1]} + \mathbf{x}_{[2]} + \ldots + \mathbf{x}_{[k]}$	Convex, increasing
`sqrt(x)`	$\sqrt{x}, \quad x \geq 0$	Concave, increasing
`inv_pos(x)`	$\frac{1}{x}, \quad x > 0$	Convex, decreasing
`max(x)`	$\max\left\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\right\}$	Convex, increasing
`quad_over_lin(x,y)`	$\frac{\|\mathbf{x}\|^2}{y}, \quad y > 0$	Convex, increasing in y
`quad_form(x,Q)`	$\mathbf{x}^T\mathbf{Q}\mathbf{x}, \ \mathbf{Q} \succeq 0$	Convex
`geo_mean(x)`	$\sqrt[n]{\prod_{i=1}^{n} \mathbf{x}_i}, \quad \mathbf{x} \geq 0$	Concave

The allowed operations between atomic functions to create a convex objective function are as follows:

1. Addition.
2. Multiplication with a non-negative scalar.
3. Composition of a convex function with a linear transformation.
4. Composition of a convex function with a convex increasing function.

In order to define the convex constraints, they must be of the form $f(\mathbf{x}) \leq 0$ for a convex f, or $h(\mathbf{x}) = 0$ for a linear h. Make sure you use $==$ and not $=$.

To read the results, type `x` (or any other variable) to read the optimal solution, type `cvx_optval` to read the optimal value, and type `cvx_status` to read the status of the solution (`Solved`, `Failed`, `Infeasible`, `Unbounded`).

Problem 3

Show that the following problem is a convex programming problem, and write a CVX code in MATLAB that solves the problem:

$$\begin{array}{ll}
\underset{\mathbf{x}\in\mathbb{R}^3}{\text{minimize}} & \dfrac{1}{\mathbf{x}_1} + \dfrac{1}{\mathbf{x}_1+\mathbf{x}_2} + \dfrac{1}{\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3} - (\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3)^{\frac{1}{3}} \\
\text{subject to} & \left(\mathbf{x}_1^2 + 2\mathbf{x}_1\mathbf{x}_2 + 3\mathbf{x}_2^2\right)^2 \le 10\mathbf{x}_1 - \mathbf{x}_2, \\
& (\mathbf{x}_1+\mathbf{x}_2)(\mathbf{x}_2 + 2\mathbf{x}_3 + 5) \ge \mathbf{x}_2^2, \\
& \displaystyle\sum_{i=1}^{3} \mathbf{x}_i \le 5, \\
& \mathbf{x} \ge 1.
\end{array}$$

Solution

- Objective function:
  - The function $\frac{1}{\mathbf{x}_1} + \frac{1}{\mathbf{x}_1+\mathbf{x}_2} + \frac{1}{\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3}$ is convex as a sum of convex functions under a linear change of variables, since the function $\frac{1}{t}$ is convex over the domain $\mathbb{R}_{++}$ and $\mathbf{x} \ge 1$.
  - The function $-(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3)^{\frac{1}{3}}$ is convex over $\mathbb{R}^3_+$ as the minus of the geometric mean.
  - Thus, the objective function is convex as the sum of convex functions.
- First constraint:
  - We have
  $$\left(\mathbf{x}_1^2 + 2\mathbf{x}_1\mathbf{x}_2 + 3\mathbf{x}_2\right)^2 = \left(\begin{pmatrix}\mathbf{x}_1 & \mathbf{x}_2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix}\right)^2,$$
  and since the matrix is PD, then the function inside the square is convex from the second-order condition. Since its range is in $\mathbb{R}_+$, we derive that this function is convex as a composition with the convex function t^2 over $\mathbb{R}_+$.
  - The function $10\mathbf{x}_1 - \mathbf{x}_2$ is linear and therefore convex.
  - Thus, the first constraint is convex as a level set of a convex function.
- Second constraint:
  - Since $\mathbf{x}_1 + \mathbf{x}_2 > 0$ (last constraint), we can divide by it the to obtain
  $$\frac{\mathbf{x}_2^2}{\mathbf{x}_1+\mathbf{x}_2} \le \mathbf{x}_2 + 2\mathbf{x}_3 + 5.$$
  The LHS is of the form $\frac{\|\mathbf{A}\mathbf{x}+\mathbf{b}\|^2}{\mathbf{c}^T\mathbf{x}+d}$, which is convex when $\mathbf{c}^T\mathbf{x}+d > 0$. The RHS is linear.
  - Thus, the second constraint is convex as a level set of a convex function.
- All remaining constraints are linear, thus this is indeed a convex programming problem.

A MATLAB implementation that solves this problem:

```
% Example 1
Q=[1 1; 1 3];
cvx_begin
variable x(3)
minimize inv_pos(x(1)) + inv_pos(sum(x(1:2))) + inv_pos(sum(x)) - geo_mean(x)
subject to
    square_pos(quad_form(x(1:2), Q)) <= 10*x(1) - x(2);
    quad_over_lin(x(2), x(1)+x(2)) <= x(2) + 2*x(3) + 5;
    sum(x) <= 5;
    x >= 1;
cvx_end

% Example 2
Q=[1 1; 1 3];
A=[1 0 0];
B=[1 1 0];
C=[1 1 1];
D=[0 1 0];
E=[0 1 2];
cvx_begin
variable x(3)
minimize inv_pos(A*x) + inv_pos(B*x) + inv_pos(C*x) - geo_mean(x)
subject to
    square_pos((B*x)'*Q*(B*x)) <= 10*x(1) - x(2);
    quad_over_lin(D*x, E*x) <= x(2) + 2*x(3) + 5;
    sum(x) <= 5;
    x >= 1;
cvx_end
```

Notice that we can also define the matrix $\mathbf{P} = \sqrt{\mathbf{Q}}$ and take the constraint $\texttt{pow\_pos}(\texttt{norm}(\texttt{P} * \texttt{B} * \texttt{x}), 4)$.