Optimization 1 — Homework 7

December 3, 2020

Note: this HW can only be submitted with MATLAB, using its CVX package (python with its packages are not allowed).

Problem 1
Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be an extended real-valued function. Show that f is convex if and only if $\operatorname{epi}(f)$ is a convex set.

Problem 2
Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be convex and let $\mathbf{A} \in \mathbb{R}^{m\times n}$. Consider $h\colon \mathbb{R}^m \to \mathbb{R}$ defined by

$$h(\mathbf{y}) = \inf_{\mathbf{x}\in\mathbb{R}^n} \{f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{y}\}.$$

Prove that h is convex.

Problem 3
Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be convex functions and let $X \subseteq \mathbb{R}^n$ be a convex set. Consider the problem

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g(\mathbf{x}) \le 0, \\ & \mathbf{x} \in X. \end{aligned}$$

Suppose that $\mathbf{x}^*$ is an optimal solution of the problem and that it satisfies $g(\mathbf{x}^*) < 0$. Prove that $\mathbf{x}^*$ is also an optimal solution of the problem

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in X. \end{aligned}$$

Problem 4

(a) Show that the extreme points of the unit simplex $\triangle_n$ are the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

(b) Find the optimal solution of the problem

$$\begin{aligned} \max\ & 57\mathbf{x}_1^2 + 65\mathbf{x}_2^2 + 17\mathbf{x}_3^2 + 96\mathbf{x}_1\mathbf{x}_2 - 32\mathbf{x}_1\mathbf{x}_3 + 8\mathbf{x}_2\mathbf{x}_3 + 27\mathbf{x}_1 - 84\mathbf{x}_2 + 20\mathbf{x}_3 \\ \text{s.t. } & \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = 1, \\ & \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \ge 0. \end{aligned}$$

Problem 5
In order to find an optimal location for a warehouse with a drone based delivery service, the city is divided into demand points $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$. The estimated mean number of orders per week is given by $p_1, p_2, \ldots, p_m \in \mathbb{R}$. Let $g_i(\mathbf{x}) = \alpha \|\mathbf{x} - \mathbf{a}_i\|$ be the estimated flight time (in minutes) from location $\mathbf{x}$ to the demand point i (we do not consider the trip back), and let γ be the travel cost per minute. Assume the following additional assumptions:

- Each drone can carry out only one order at a time.
- The takeoff, landing and production time are not considered in this model.
- The number of drones is unlimited.

(a) Write a convex optimization problem for finding a location for the warehouse that minimizes the weekly flight costs.

(b) Assume that we are interested to compensate customers for a late delivery according to the following policy: for each minute that exceeds a predefined flight time η_1, a discount of μ_1 per minute is given to the customer. If, in addition, the delivery time exceeds a predefined flight time $\eta_1 > \eta_1$, then for each minute that exceeds η_2 the discount is $\mu_2 > \mu_1$. Write a convex optimization problem that minimizes the total flight and compensation costs.

(c) As an alternative to the compensation policy, we can consider a warehouse location that is equally bad for all customers. Hence, we can consider a location with the smallest maximal absolute variance of the distances (if $\delta_i = \|\mathbf{x} - \mathbf{a}_i\|$ is the distance to demand point i, then its absolute variance with respect to the average distance is defined as $\left|\delta_i^2 - \frac{1}{m}\sum_{j=1}^{m}\delta_j^2\right|$. Write a convex optimization problem that corresponds to this alternative (note that in this case the flight costs are not considered).

(d) Use the following MATLAB code in order to generate data for the warehouse location problem:

```
m = 50; n=2; outliers_num=10;
rand('seed',314);
A = 3000*rand(n,m);
A(:,1:outliers_num) = A(:,1:outliers_num)+3000;
p = round(10*rand(m,1)+10);
alpha=0.01; gamma=1.2; eta1=20; eta2=30; mu1=2; mu2=5;
```

Solve each of the models considered in (a), (b) and (c) with CVX, and plot the demand points and the obtained locations on a single figure (your answer should also include the optimal solution and function value of each model).

Problem 6

For each of the following optimization problems:

- Prove that it is a convex programming problem.
- Write a CVX code that solves the problem.
- Write the optimal solution (by running CVX).

(a)

$$
\begin{aligned}
\min \quad & \max\{|2x_1 - 3x_2|, |x_2 - x_1 + x_3|\} + x_1^2 + 2x_2^2 + 3x_3^2 - 2x_2x_3 \\
\text{s.t.} \quad & \left(4x_1^2 + 6x_2^2 - 8x_1x_2 + 0.01\right)^8 + \frac{x_3^2}{2x_1 + 3x_2} \le 150, \\
& x_1 + x_2 \ge 1 - \frac{x_2}{2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\min \quad & 5x_1^2 + 4x_2^2 + 7x_3^2 + 4x_1x_2 + 2x_2x_3 + |x_1 - x_2| \\
\text{s.t.} \quad & \frac{x_1^2}{2x_1 + x_2} + \left(1 + e^{\sqrt{x_1^2 + x_2^2 + 1}}\right)^7 \le 200, \\
& \max\left\{2, e^{(x_1+x_2)^3} + \frac{x_2^2 + x_2x_3 + x_3^2}{x_1} + x_2 - x_1\right\} \le 2x_2, \\
& x_1 \ge 1.
\end{aligned}
$$