Exam in Optimization I - Winter 2012/2013

Instructions: Answer the following 5 questions. Please start each question on a separate page and give detailed explanations.

Question 1 (20 pts.) A cone K is called *pointed* if it contains no lines, meaning that if $\mathbf{x}, -\mathbf{x} \in K$, then $\mathbf{x} = \mathbf{0}$. Show that if C is a cone with nonempty interior, then the dual cone given by

$$C^* = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{y}^T\mathbf{x} \geq 0 \text{ for all } \mathbf{x} \in C\}$$

is pointed.

Question 2. (25 pts.) Let $f(x_1, x_2) = x_1^2 - 2x_1x_2^2 + \frac{1}{2}x_2^4$.

(a) Is the function f coercive? explain your answer.

(b) Find the stationary points of f and classify them (strict/nonstrict local/global minimum/maximum saddle point).

Question 3. (10 pts.) Let $\mathbf{Q}$ be an $n \times n$ positive semidefinite matrix and let $\mathbf{a} \in \mathbb{R}^n$. Prove that the set

$$C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{Q}\mathbf{x} \leq (\mathbf{a}^T\mathbf{x})^2, \mathbf{a}^T\mathbf{x} \geq 0\}.$$

is a closed and convex cone.

Question 4. (25 pts.) Let $\mathbf{E} \in \mathbb{R}^{k\times n}, \mathbf{f} \in \mathbb{R}^n, \mathbf{a} \in \mathbb{R}^m, \mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n \in \mathbb{R}^n$. Consider the problem

$$\text{(P)} \quad \begin{array}{ll} \min_{\mathbf{x}\in\mathbb{R}^n, \mathbf{z}\in\mathbb{R}^m} & \frac{1}{2}\|\mathbf{E}\mathbf{x}\|^2 + \frac{1}{2}\|\mathbf{z}\|^2 + \mathbf{f}^T\mathbf{x} + \mathbf{a}^T\mathbf{z} + \sum_{i=1}^n e^{\mathbf{c}_i^T\mathbf{x}} \\ \text{s.t.} & \mathbf{A}\mathbf{x} + \mathbf{z} = \mathbf{b}, \\ & \mathbf{z} \geq \mathbf{0}. \end{array}$$

Assume that $\mathbf{E}$ has full column rank.

(a) Show that the objective function is coercive.

(b) Show that if the set $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ is nonempty, then strong duality holds for problem (P).

(c) Write a dual problem for problem (P).

Question 5. (20 pts.) Consider the optimization problem

$$\begin{array}{ll} \max & x_1x_2x_3 \\ \text{s.t.} & x_1^2 + 2x_2^2 + 3x_3^2 \leq 1. \end{array}$$

(a) Find all the KKT points of the problem.

(b) Find all the optimal solutions of the problem.