Optimization 1 — Tutorial 3

November 5, 2020

Definition (Coercive Function)
A function $f : \mathbb{R}^n \to \mathbb{R}$ is called coercive if

$$\lim_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = \infty.$$

Definition (Diagonally Dominant Matrix)
Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then $\mathbf{A}$ is called [stirctly] diagonally dominant if for any $i = 1, 2, \ldots, n$ we have

$$|\mathbf{A}_{ii}| \underset{[>]}{\geq} \sum_{\{j \colon j \neq i\}} |\mathbf{A}_{ij}| .$$

Proposition (LS is quadratic)
Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^n$. Then the LS problem $\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|$ has a unique solution if $\mathbf{A}$ is of full column rank.

Problem 1

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $\mathbf{b} \in \mathbb{R}^n$. Prove that the function $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}+2\mathbf{b}^T\mathbf{x}+c$ is coercive if and only if $\mathbf{A} \succ 0$.

Problem 2

For each of the following functions, determine whether it is coercive or not.

(a) $f(x, y) = 4x^2 + 2xy + 2y^2$.

(b) $f(x, y) = 2x^2 - 8xy + y^2$.

(c) $f(x, y, z) = x^3 + y^3 + z^3$.

(d) $f(x, y) = x^4 + y^4$.

(e) $f(x, y) = (x - 2y)^4 + 2xy$.

Problem 3

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Suppose that $\mathbf{A}$ is [strictly] diagonally dominant with [positive] non-negative diagonal elements. Then $\mathbf{A}$ is [PD] PSD.