Problem 2

Classify the stationary points of the following functions:

(a) $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

(b) $f(x, y) = \left(x^2 + y^2 - 1\right)^2 + \left(y^2 - 1\right)^2$.

Solution

(b)

- Finding stationary points: by solving $\nabla f(x, y) = (0, 0)$ we derive that the stationary points are $(0, 0)$, $(\pm 1, 0)$ and $(0, \pm 1)$.
- Local and global classification: the Hessian matrix is

$$\nabla^2 f(x, y) = 4 \begin{pmatrix} 3x^2 + y^2 - 1 & 2xy \\ 2xy & x^2 + 6y^2 - 2 \end{pmatrix}$$

  - $(0, 0)$:

$$\nabla^2 f(0, 0) = \begin{pmatrix} -4 & 0 \\ 0 & -8 \end{pmatrix} \prec 0,$$

    and therefore is a strict local maximum. Since $f(x, 0) = \left(x^2 - 1\right)^2 + 1 \to \infty$ as $x \to \infty$, we derive that $(0, 0)$ is not a global maximum.
  - $(\pm 1, 0)$:

$$\nabla^2 f(\pm 1, 0) = \begin{pmatrix} 8 & 0 \\ 0 & -4 \end{pmatrix},$$

    which is indefinite, and therefore these are two saddle points.
  - $(0, \pm 1)$:

$$\nabla^2 f(0, \pm 1) = \begin{pmatrix} 0 & 0 \\ 0 & 16 \end{pmatrix} \succeq 0.$$

    Therefore, these points a local minima or saddles. We notice that $f(x, y) \geq 0$ for all $x, y \in \mathbb{R}$ and $f(0, 1) = f(0, -1) = 0$. Therefore, these are two non-strict global minima.