098311 Optimization 1 Spring 2018
HW 5

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Problem 1. Prove the following theorem:

Theorem.

1. Let $C_1, ..., C_k \subseteq \mathbb{R}^n$ be convex sets, and let $\mu_1, ..., \mu_k \in \mathbb{R}$. Then the set $\mu_1 C_1 + ... + \mu_k C_k$ is convex.

2. Let $C_i \in \mathbb{R}^{k_i}$, $i = 1, ..., m$ be convex sets. Then the Gaussian product

$$C_1 \times C_2 \times ... \times C_m = \{(x_1, ..., x_m) : x_i \in C_i, i = 1, ..., m\}$$

is convex.

3. Let $M \subseteq \mathbb{R}^n$ be a convex set, and let $A \in \mathbb{R}^{m \times n}$. Then the set $A(M) = \{Ax : x \in M\}$ is convex.

4. Let $D \subseteq \mathbb{R}^m$ be convex, and let $A \in \mathbb{R}^{m \times n}$. Then the set

$$A^{-1}(D) = \{x \in \mathbb{R}^n : Ax \in D\}$$

is convex.

Solution

1. We shall prove by induction. Let us assume $C_1, C_2 \subseteq \mathbb{R}^n$ are convex sets and $\mu_1, \mu_2 \in \mathbb{R}$. (*) Let x, y be two points contained in the set $\bar{C} = \mu_1 C_1 + \mu_2 C_2$:

$$x = \mu_1 x_1 + \mu_2 x_2 y = \mu_1 y_1 + \mu_2 y_2$$

where x_1, y_1 and x_2, y_2 are points contained in C_1, C_2 respectively.

$$\begin{aligned} z = \lambda x + (1-\lambda)y &= \mu_1 \lambda x_1 + \mu_2 \lambda x_2 + \mu_1 (1-\lambda) y_1 + \mu_2 (1-\lambda) y_2 \\ &= \mu_1(\lambda x_1 + (1-\lambda) y_1) + \mu_2(\lambda x_2 + (1-\lambda) y_2) \\ &= \mu_1 z_1 + \mu_2 z_2 \end{aligned}$$

where $\lambda \in [0,1]$. We have shown that $\forall x, y, \mu_1, \mu_2, \lambda$, the point z is a linear combination of two points z_1 and z_2 which are contained in C_1 and C_2 respectively. And as such $z \in \bar{C}$.

(**) We now assume that for some $n > 2$ the set $\hat{C} = \eta_1 C_1 + ... + \eta_n C_n$ is convex $\forall \eta_i \in \mathbb{R}$. Observe the set $\bar{C} = \gamma_1 \hat{C} + \gamma_2 C_{n+1}$. This set is convex as shown in (*). In addition, notice that we can write $\bar{C}$ in the following way:

$$\bar{C} = \gamma_1 \hat{C} + \gamma_2 C_{n+1} = \gamma_1(\eta_1 C_1 + ... + \eta_n C_n) + \gamma_2 C_{n+1} = \lambda_1 C_1 + ... + \lambda_n C_n + \lambda_{n+1} C_{n+1}$$

where $\lambda_i = \eta_i \gamma_1 \ \ \forall i \leq n$, $\lambda_{n+1} = \gamma_2$.

2. Consider two vectors in the Gaussian product of $C_i \in \mathbb{R}^{k_i}, i = 1, ..., m$: $x = (x_1, ..., x_m), x_i \in C_i$ and $y = (y_1, ..., y_m), y_i \in C_i$. Then, for any $\lambda \in [0,1]$ we have:

$$\begin{aligned}\lambda x + (1-\lambda)y &= (\lambda x_1, ..., \lambda x_m) + ((1-\lambda)y_1, ..., (1-\lambda)y_m) = \\ &= (\lambda x_1 + (1-\lambda)y_1, ..., \lambda x_m + (1-\lambda)y_m) \triangleq (z_1, ..., z_m)\end{aligned}$$

   Since each of the sets C_i is in itself convex, we have $z_i \in C_i \forall i$ and therefore the resulting vector z is also in the Gaussian product of all C_i, and the Gaussian product is convex.

3. Observe two vectors $x, y \in M$. We denote $z = \lambda x + (1 - \lambda y) \in M$ for $\lambda \in [0,1]$ by the definition of a convex set.

$$Az = A(\lambda x + (1 - \lambda y)) = \lambda Ax + (1-\lambda)Ay$$

   As for any point $z \in M$, Az is shown to be a linear combination of two points $Ax, Ay \in A(M)$ then by definition $A(M)$ is convex.

4. Let $x, y \in A^{-1}(D)$. Let $\lambda \in [0,1]$ and $z = \lambda x + (1-\lambda)y$.

$$Az = A(\lambda x + (1-\lambda)y) = \lambda Ax + (1-\lambda)Ay$$

   as $Ax, Ay \in D$ then $Az \in D$ as a linear combination of two points in D (for D is convex). Hence by definition, $z \in A^{-1}(D)$ and as such, by definition, $A^{-1}(D)$ is convex.

Problem 2. Let $a, b \in \mathbb{R}^n$, $(a \neq b)$. For what values of μ is the following set convex?

$$S_\mu = \{x \in \mathbb{R}^n : ||x - a|| \leq \mu ||x - b||\}$$

Solution

A set is convex if for any $s_1, s_2 \in S_\mu$ and $\lambda \in [0,1]$, $s_\lambda = \lambda s_1 + (1-\lambda)s_2 \in S_\mu$.

(I) Initially, we show that for $\mu > 1$ the set is non-convex:

For any $\mu > 1$, we define $y = \frac{b - \lambda a}{1-\lambda}$. We now have for $\lambda = \frac{1}{\mu} < 1$:

$$\begin{aligned}||y-a|| &= ||\frac{b-\lambda a}{1-\lambda} - a|| = \frac{1}{\lambda}||\frac{\lambda(b-\lambda a)}{1-\lambda} - \lambda a|| = \mu||\frac{\lambda b - \lambda a}{1-\lambda}|| \\ &= \mu||\frac{\lambda b - \lambda a + (1-\lambda)b - (1-\lambda)b}{1-\lambda}|| = \mu||\frac{b-\lambda a}{1-\lambda} - b|| = \mu||y-b||\end{aligned}$$

The consequence of the above equality is $y \in S_\mu$. Notice that the point a by definition is also in the set $S\mu$. However, $b \notin S_\mu \;\; \forall \mu$ as $||b-b|| = 0$ and as $a \neq b : ||a-b|| > 0$.
We have found two points in S_μ: a, y and a third point $b \notin S_\mu$ where $b = (1-\lambda)y + \lambda a$. Hence by definition S_μ is non-convex for any $\mu > 1$.

(II) Consider the scenario where $\mu = 1$. Due to symmetry, we assume that: $a = (1, 0, ..., 0)^T, b = (-1, 0, ..., 0)^T$. As for any two points a, b this can be reached through scaling, rotation and shift of the axis - operations which do not change the relative distance between points.

We denote the i-th index of the vector x by x_i. Notice that for any vector x such that $-1 < x_1 < 0$

$$\begin{aligned} ||x-a||^2 &= \sum_{i=1}^{n}(x_i - a_i)^2 = (x_1 - a_1)^2 + \sum_{i=2}^{n}(x_i - a_i)^2 \\ &> (x_1 - b_1)^2 + \sum_{i=2}^{n}(x_i - a_i)^2 = ||x-b||^2 \Rightarrow x \notin S_\mu \end{aligned}$$

additionally, for $x_1 < -1$

$$\begin{aligned} ||x-a||^2 = \sum_{i=1}^{n}(x_i - a_i)^2 &= (x_1 - a_1)^2 + \sum_{i=2}^{n}(x_i - a_i)^2 = (x_1 - b_1)^2 + (a_1 - b_1)^2 + \sum_{i=2}^{n}(x_i - a_i)^2 \\ &> (x_1 - b_1)^2 + \sum_{i=2}^{n}(x_i - a_i)^2 = (x_1 - b_1)^2 + \sum_{i=2}^{n}(x_i - b_i)^2 = ||x-b||^2 \Rightarrow x \notin S_\mu \end{aligned}$$

However, for any x such that $x > 1$

$$\begin{aligned} ||x-a||^2 = \sum_{i=1}^{n}(x_i - a_i)^2 &= (x_1 - a_1)^2 + \sum_{i=2}^{n}(x_i - a_i)^2 < (a_1 - b_1)^2 + (x_1 - a_1)^2 + \sum_{i=2}^{n}(x_i - a_i)^2 \\ &= (x_1 - b_1)^2 + \sum_{i=2}^{n}(x_i - a_i)^2 = \sum_{i=1}^{n}(x_i - a_i)^2 = ||x-b||^2 \Rightarrow x \in S_\mu \end{aligned}$$

We have shown that for $\mu = 1$, any point which resides on the right half-space defined by $\{x \in \mathbb{R}^n : x_1 > 0\}$ is contained in the set S_μ. As the half-space is by definition convex and any point outside of the half-space is not in S_μ we conclude that for $\mu = 1$ the set S_μ is convex.

(III) We now show that $\forall 0 \leq \mu < 1$ the set S_μ is indeed convex.

Let us look at the set S_μ. The set is defined by the inequality:

$$
\begin{aligned}
||x-a|| &\le \mu||x-b|| \\
||x-a||^2 &\le \mu^2||x-b||^2 \\
||x-a||^2-\mu^2||x-b||^2 &\le 0 \\
x^Tx-2x^Ta+a^Ta-\mu^2x^Tx+2\mu^2x^Tb-\mu^2b^Tb &\le 0 \\
x^T[(1-\mu^2)x-2a+2\mu^2b] &\le \mu^2b^Tb-a^Ta \\
(*)(1-\mu^2)x^T\left[x-2\frac{a-\mu^2b}{1-\mu^2}\right] &\le \mu^2b^Tb-a^Ta \\
x^T\left[x-2\underbrace{\frac{a-\mu^2b}{1-\mu^2}}_{x_0}\right] &\le \frac{\mu^2b^Tb-a^Ta}{1-\mu^2} \\
x^T(x-x_0)-x^Tx_0+x_0^Tx_0-x_0^Tx_0 &\le \frac{\mu^2b^Tb-a^Ta}{1-\mu^2} \\
(x-x_0)^T(x-x_0)=||x-x_0||^2 &\le \underbrace{\frac{\mu^2b^Tb-a^Ta}{1-\mu^2}+x_0^Tx_0}_{R^2}
\end{aligned}
$$

Where $(*)$ is true since $\mu<1$ and $x^Tx=||x||^2>0$. Notice the following regarding R^2:

$$
\begin{aligned}
R^2 &= \frac{\mu^2b^Tb-a^Ta}{1-\mu^2}+x_0^Tx_0 = \frac{(1-\mu^2)(\mu^2b^Tb-a^Ta)+a^Ta+\mu^4b^Tb-2\mu^2a^Tb}{(1-\mu^2)^2} \\
&= \frac{\mu^2b^Tb-a^Ta-\mu^4b^Tb+\mu^2a^Ta+a^Ta+\mu^4b^Tb-2\mu^2a^Tb}{(1-\mu^2)^2} \\
&= \frac{\mu^2b^Tb+\mu^2a^Ta-2\mu^2a^Tb}{(1-\mu^2)^2} = \frac{\mu^2}{(1-\mu^2)^2}||a-b||>0
\end{aligned}
$$

Hence, we have shown that all $S_\mu = B(x_0, R)$, a ball centered at x_0 with radius R. A ball is convex and as such by definition for $\mu<1$ the set S_μ is convex.
Note the special case where $\mu=0$: we receive a ball of radius 0, which is a point in $\mathbb{R}^n$ and therefore convex (by definition).
(IV) A final case we should consider is $\mu<0$. In this case, since norms are non-negative, we have $S_\mu=\emptyset$. The empty set is trivially convex.

Following the above (I), (II), (III) and (IV) we conclude that the set S_μ is convex for all $\mu\le 1$.

Problem 3. Show that the conic hull of the set

$$S=\{(x_1,x_2):(x_1-1)^2+x_2^2=1\}$$

is the set:

$$\{(x_1,x_2):x_1>0\}\bigcup\{(0,0)\}$$

Solution The conic hull of a set S is the set comprising of all conic combinations of vectors in S. Therefore, to show the required result, we show that for any vector $z = (z_1, z_2) : z_1 > 0$ we can represent it as a conic combination of vectors in S. Specifically, we shall show that for every such z, we can represent it as $(z_1, z_2) = (\eta x_1, \eta x_2)$ such that $(x_1, x_2) \in S$. We have:

$$(x_1 - 1)^2 + x_2^2 = 1 \iff \left(\frac{z_1}{\eta} - 1\right)^2 + \frac{z_2^2}{\eta^2} = 1 \iff \frac{z_1^2}{\eta^2} - 2\frac{z_1}{\eta} + 1 + \frac{z_2^2}{\eta^2} = 1$$
$$\iff z_1^2 - 2\eta z_1 + z_2^2 = 0 \iff \eta = \frac{z_1^2 + z_2^2}{2z_1}$$

Therefore, as long as $z_1 > 0$, there exists some $\eta \in \mathbb{R}_+$ such that z is a conic combination of some vector in x. Additionally, since for any $\eta \geq 0$ we have $z_1 \geq 0$, no conic combination of vectors in S would create a vector in which $z_1 \leq 0$.
Finally, the point $(0, 0)$ is also included in the cone(S) since it is in S (and also, $\eta = 0$ for some other point in S would give the same result).

Problem 4. Let S be a nonempty set in $\mathbb{R}^n$ and let $\bar{x} \in S$. Consider the set $C_{\bar{x}} = \{y : y = \lambda(x - \bar{x}), \lambda \geq 0, x \in S\}$.

a Show that $C_{\bar{x}}$ is a cone and interpret it geometrically.

b Show that $C_{\bar{x}}$ is convex if S is convex.

c Suppose that S is closed. Is it necessarily true that $C_{\bar{x}}$ is closed? If not, under what conditions would it be closed?

Solution

a The set $C_{\bar{x}}$ satisfies the cone property, and is therefore a cone:

$$\forall y \in C_{\bar{x}}, \forall \mu \geq 0 : \mu y = \mu\lambda(x - \bar{x}) \triangleq \eta(x - \bar{x}) \in C_{\bar{x}} \Leftarrow \eta \geq 0$$

Geometrically, $C_{\bar{x}}$ is the cone created by the set S shifted around $\bar{x}$, or to be exact, the conic hull of the set S shifted around $\bar{x}$.

b For $C_{\bar{x}}$ to be convex, we have seen in the lecture that it is enough to show $y + z \in C_{\bar{x}}$ for any $y, z \in C_{\bar{x}}$ (since we have already shown that $C_{\bar{x}}$ is a cone).

$$y + z = \lambda_y(x_y - \bar{x}) + \lambda_z(x_z - \bar{x}) = \lambda_y x_y + \lambda_z x_z - (\lambda_y + \lambda_z)\bar{x} =$$
$$= (\lambda_y + \lambda_z)\left[\frac{\lambda_y}{\lambda_y + \lambda_z}x_y + \frac{\lambda_z}{\lambda_y + \lambda_z}x_z - \bar{x}\right] (*)$$

We now define $\eta = \frac{\lambda_y}{\lambda_y + \lambda_z}$. Note $\eta \in [0, 1]$, and therefore, if S is convex, we have $x_* \triangleq \frac{\lambda_y}{\lambda_y+\lambda_z}x_y + \frac{\lambda_z}{\lambda_y+\lambda_z}x_z = \eta x_y + (1 - \eta)x_z \in S$. Now, we can continue from (*):

$$(*) = (\lambda_y + \lambda_z)(x_* - \bar{x}) \in C_{\bar{x}}$$

denoting $\mu = \lambda_y + \lambda_z \geq 0$, and since $x_* \in S$, we have by definition that $y + z \in C_{\bar{x}}$.

c $C_{\bar{x}}$ is not necessarily closed if S is closed. For instance, consider $S = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$ (which is a closed set), with $\bar{x} = (1, 0)$. The set $C_{\bar{x}}$ is then the conic hull of the set from question 3, therefore, as we've shown, it is $\{(x_1, x_2) : x_1 > 0\} \bigcup \{(0, 0)\}$, which is not closed (since it does not include the x_2 axis).

Requirements for $C_{\bar{x}}$ to be closed: Note that when $\bar{x} \notin \overline{S}$ (the closure of S) then $C_{\bar{x}}$ spans the entire space and as such is closed. However, when $\bar{x} \in \overline{S}$, the set is closed iff the tangent to the set S at $\bar{x}$ does not exist or it exists but has points that are also inside the set S.

Problem 5. Let $C \subseteq \mathbb{R}^n$ be a solution set of a quadratic inequality.

$$C = \{x \in \mathbb{R}^n : x^T A x + b^T x + c \leq 0\}$$

with $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbb{R}$.

Solution First, we present a lemma:

Lemma A. set C is convex iff its intersection with any arbitrary line $\hat{x} + tv, t \in \mathbb{R}$ is convex.

Proof. (*) Consider the set C and an arbitrary line $f(x, y) = x + \eta y$, where x is a constant point, y defines the direction and $\eta \in \mathbb{R}$. Following the definition of a convex set, for any two points a, b residing on the line $f(x, y)$ which are contained inside the set C, the point $p = \lambda a + (1 - \lambda) b$ is also contained in the set (for $\lambda \in [0, 1]$). This holds for any point x, and direction y.

(**) Note that the set of lines defined by $F = \{f(x, y), \forall x \in C, y \in \mathbb{R}^n, \eta\}$ spans all possible lines intersecting with the set C. If the intersection of f with C defines a convex sub-set $\forall f \in F$ then by definition we conclude that C is convex.

(***) Additionally, if C is convex, any two points $a, b \in C$ have $p = \lambda a + (1 - \lambda) b \in C$ for all $\lambda \in [0, 1]$. All three points (a, b, p) are also on some line $f \in F$, and therefore are also on the intersection of this line with C, making the intersection convex as well. □

Let us define the set $H = \{x : g^T x + h = 0\}$. Following our lemma, we note that if the intersection of all arbitrary lines $x + \eta y$ (where $x \in H \cap C$ and $x + \eta y \in H$) with C results in a convex sub-set then the set C is itself convex.

We begin by selecting some arbitrary line and then show for the private case in which $x \in H \cap C$:

$$\begin{aligned}
&(x + \eta y)^T A (x + \eta y) + b^T (x + \eta y) + c \leq 0 \\
\iff &x^T A x + \eta x^T A y + \eta y^T A x + \eta^2 y^T A y + b^T x + \eta b^T y + c \leq 0 \\
\iff &y^T A y \eta^2 + (x^T A y + y^T A x + b^T y)\eta + x^T A x + b^T x + c \leq 0 (*)
\end{aligned}$$

The resulting inequality $(*)$, together with the requirement $g^T(x + \eta y) + h = 0$ is an alternate definition of the set $C \cap H$. The set defined by $(*)$ is a one dimensional quadratic function

in η, and is therefore convex if $y^T A y \geq 0$.
WLOG, we can select x to be in the intersection $C \cap H$, meaning: $g^T x + h = 0$. Therefore, the requirement $g^T(x + \eta y) + h = 0$ gives us $g^T y = 0$.
Now, since we know $A + \lambda g g^T \succeq 0$, we have $y^T(A + \lambda g g^T)y \geq 0$ for any $y \in \mathbb{R}^n$. For our y, however, we have $g^T y = 0$, and therefore:

$$
\begin{aligned}
y^T A y + \lambda y^T g g^T y \geq 0 \\
y^T A y + \lambda (g^T y)^T g^T y \geq 0 \\
y^T A y \geq 0
\end{aligned}
$$

And therefore the set is convex.

Problem 6.

1. Show that the extreme points of $\Delta_n = \{x : \sum_{i=1}^n x_i = 1, x_i \geq 0\}$ are given by $\{e_1, ..., e_n\}$.

2. Prove that the extreme points of $\{x : ||x||_\infty \leq 1\}$ are given by $\{-1, 1\}^n$.

3. For each of the following sets specify the corresponding extreme points (no need to provide a formal proof):

   (a) $\left\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1, \max_{i=1,2} |x_i| \leq \frac{1}{\sqrt{2}+1}\right\}$

   (b) $\{(x_1, x_2) : 64x_1^2 + 36x_2^2 + 96x_1x_2 - 32x_1 - 24x_2 + 4 \leq 0, x_1 \geq 0, x_2 \geq 0\}$

   (c) $\left\{x \in \mathbb{R}^4 : \begin{pmatrix} 1 & 6 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 9 \\ 2 \end{pmatrix}, x \geq 0\right\}$

Solution

1. Let us define $A = (1, ..., 1) \in \mathbb{R}^{1 \times n}$, and $b = 1 \in \mathbb{R}^1$. Then, the set Δ_n can be defined as $\Delta_n = \{x : Ax = b, x_i \geq 1\}$. Now, since all the colums of A are linearly dependant, the only basic feasible solutions $\bar{x}$ are ones for which only one element of $\bar{x}$ is non-zero. The requirement that $\sum_{i=1}^n x_i = 1$ dictates that this single non-zero element is 1. Therefore, all the bfs's of the problem defined by the set are exactly $\{e_1, ..., e_n\}$. As we have seen in lecture 6 slide 31, for this type of set $\bar{x}$ is a bfs if and only if it is an extreme points, and therefore all of the extreme points of Δ_n are $\{e_1, ..., e_n\}$.

2. First, we shall prove that a point in the set but not in $\{-1, 1\}^n$ cannot be an extreme point. Let us assume such an extreme point x exists, then at least one element $|x_i| < 1$, meaning $-1 < x_i < 1$. Consider two vectors: $x' = (x_1, ..., x_{i-1}, 1, x_{i+1}, ..., x_n$ and $x'' = (x_1, ..., x_{i-1}, -1, x_{i+1}, ..., x_n$. We can represent our vector as $x = \frac{x_i+1}{2}x' + (1 - \frac{x_i+1}{2})x''$. Since $0 < \frac{x_i+1}{2} < 1$, our point x cannot be an extreme point.
   Now, let us verify that a point in $\{-1, 1\}^n$ is indeed an extreme point. Let us look at a point $y \in \{-1, 1\}^n$. Assuming y is not an extreme point, there exist some $y' \neq y''$

in the set and $\lambda \in (0,1)$ such that $y = \lambda y' + (1-\lambda)y''$. Then, for some element y_i we have $y'_i \neq y''_i$, and then:

$$\pm 1 = y_i = \lambda y'_i + (1-\lambda)y''_i$$

$$1 = |y_i| = |\lambda y'_i + (1-\lambda)y''_i| \leq \lambda|y'_i| + (1-\lambda)|y''_i| \overset{(*)}{<} \max\{|y'_i|, |y''_i|\} \leq 1$$

The inequality $(*)$ arises since $y'_i \neq y''_i$ and $\lambda \in (0,1)$, and therefore we get a contradiction $(1 < 1)$, and y must be an extreme point.

Finally, (i) we have shown that any point $x \in \{x : ||x||_\infty \leq 1\}$ in which one or more indices satisfy $|x_i| < 1$ is an interior point of the set $\{x : ||x||_\infty \leq 1\}$. Additionally, we have shown (ii) that any point x in which all indices $x_i = \pm 1$, is in-fact an extreme point. Combining (i) and (ii) we conclude that $x = \{-1, 1\}^n$ is the set of extreme points.

3. (a) Note that when $|x_1| = |x_2| = \frac{1}{\sqrt{2}+1}$ then:

$$x_1^2 + x_2^2 = \frac{2}{(\sqrt{2}+1)^2} = \frac{2}{2+2\sqrt{2}+1} = \frac{2}{3+2\sqrt{2}} < 1$$

Hence the extreme points of the set are:

$$\left(-\frac{1}{\sqrt{2}+1}, -\frac{1}{\sqrt{2}+1}\right), \left(-\frac{1}{\sqrt{2}+1}, \frac{1}{\sqrt{2}+1}\right), \left(\frac{1}{\sqrt{2}+1}, \frac{1}{\sqrt{2}+1}\right), \left(\frac{1}{\sqrt{2}+1}, -\frac{1}{\sqrt{2}+1}\right)$$

(b) Note that:

$$\begin{aligned} &64x_1^2 + 36x_2^2 + 96x_1x_2 - 32x_1 - 24x_2 + 4 \leq 0 \\ \Rightarrow &(8x_1 + 6x_2 - 2)^2 \leq 0 \end{aligned}$$

The above is non-negative and as such the set is defined by the points which satisfy $8x_1 + 6x_2 - 2 = 0, x_1, x_2 \geq 0$. For $x_1 = 0$ we have that $x_2 = \frac{2}{6} = \frac{1}{3}$ and for $x_2 = 0$ we have that $x_1 = \frac{2}{8} = \frac{1}{4}$ and these are the extreme points $(0, \frac{1}{4})$, $(\frac{1}{3}, 0)$.

(c) $\left\{x \in \mathbb{R}^4 : \begin{pmatrix} 1 & 6 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 9 \\ 2 \end{pmatrix}, x \geq 0\right\}$
we have the following equations:

$$\begin{aligned} &x_1 + 6x_2 - x_3 = 9 \\ &3x_1 - x_4 = 2 \\ &x_i \geq 0 \ \ \forall i \end{aligned}$$

Note that $x_1 \neq 0$. We know that BFSs are extreme points, hence the following solutions satisfy the requirements of a BFS:

$$\begin{aligned} &x_2 = x_3 = 0, \ \ x_1 = 9, \ \ x_4 = 25 \\ &x_3 = x_4 = 0, \ \ x_1 = \frac{2}{3}, \ \ x_2 = \frac{25}{18} \end{aligned}$$

any other solution is not a BFS (as any combination of more than 2 columns is linearly dependent), and as such the two solutions above are the extreme points of the set.