Optimization 1 — Tutorial 1

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Definition (Matrix Norm)
A function $\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}$ is a norm if it satisfies the following three properties:

1. Non-negativity: $\|\mathbf{A}\| \geq 0$ and $\|\mathbf{A}\| = 0$ if and only if $\mathbf{A} = \mathbf{0}_{m\times n}$.
2. Positive homogeneity: $\|\lambda \mathbf{A}\| = |\lambda| \|\mathbf{A}\|$ for all $\lambda \in \mathbb{R}$.
3. Triangle inequality: $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$.

Definition (Induced Norm)
Given a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and two norms $(\mathbb{R}^n, \|\cdot\|_a)$ and $(\mathbb{R}^m, \|\cdot\|_b)$, the induced norm is defined by

$$\|\mathbf{A}\|_{a,b} = \max_{\mathbf{x}\in\mathbb{R}^n} \left\{\|\mathbf{A}\mathbf{x}\|_b : \|\mathbf{x}\|_a \leq 1\right\}.$$

Definition (Open Set)
A set $U \subseteq \mathbb{R}^n$ is said to be open if it contains only interior points. That is, for every $\mathbf{x} \in U$ there exists $r > 0$ such that $B(\mathbf{x}, r) \subseteq U$ (open ball with center $\mathbf{x}$ and radius r contained in U).

Definition (Closed Set)
A set $U \subseteq \mathbb{R}^n$ is said to be closed if it contains all the limits of convergent sequences of points in U. That is, for every sequence $\left\{\mathbf{x}^i\right\}_{i\geq 1} \subseteq U$ satisfying $\mathbf{x}^i \to \mathbf{x}^* \in \mathbb{R}^n$ as $i \to \infty$, it holds that $\mathbf{x}^* \in U$.

Theorem (Spectral Decomposition)
Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then there exists an orthogonal matrix $\mathbf{U} \in \mathbb{R}^{n\times n}$ ($\mathbf{U}^T\mathbf{U} = \mathbf{U}\mathbf{U}^T = \mathbf{I}_n$) and a diagonal matrix $\mathbf{D} \in \mathbb{R}^{n\times n}$ for which $\mathbf{U}^T\mathbf{A}\mathbf{U} = \mathbf{D}$.

Definition (Directional Derivative)
Let $f\colon S \subseteq \mathbb{R}^n \to \mathbb{R}$, let $\mathbf{x} \in \text{interior}(S)$ and let $\mathbf{0}_n \neq \mathbf{d} \in \mathbb{R}^n$. If the limit

$$\lim_{t\to 0^+} \frac{f(\mathbf{x} + t\mathbf{d}) - f(\mathbf{x})}{t}$$

exists, then it is called the directional derivative of $\mathbf{x}$ along the direction $\mathbf{d}$.

Problem 1

Prove that the induced ℓ_1 norm of $\mathbf{A} \in \mathbb{R}^{m\times n}$ is given by

$$\|\mathbf{A}\|_1 = \max_{j=1,2,\ldots,n} \sum_{i=1}^{m} |\mathbf{A}_{ij}| \,.$$

Problem 2

Prove that a set is closed if and only if its complement is open.

Problem 3

Solve the optimization problem $\max\limits_{\mathbf{x}\in\mathbb{R}^n} \left\{\mathbf{x}^T\mathbf{A}\mathbf{x}\colon\ \|\mathbf{x}\| = 1\right\}$ for some symmetric matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$.

Problem 4

(a) Calculate the directional derivative of the function $\|\mathbf{x}\|_2$.

(b) Calculate the directional derivative of the function $\|\mathbf{x}\|_1$.