Optimization 1 — Homework 12

January 14, 2021

Problem 1

Consider the following geometric programming problem

$$
\begin{aligned}
(GP) \qquad \min_{t_1,t_2,t_2\in\mathbb{R}} \quad & \frac{1}{t_1t_2} + 10t_3 \\
\text{s.t.} \quad & \frac{t_1t_2}{t_3} \le 1, \\
& t_1, t_2, t_3 > 0.
\end{aligned}
$$

(a) Use the transformations $t_i = e^{x_i}$ for $i = 1, 2, 3$ and rewrite (GP) as a convex optimization problem denoted (P).

(b) Define appropriate linear transformations of the variables and find a dual of (P). Use the conventions $0\ln(0) = 0$, $0\ln\left(\frac{0}{0}\right) = 0$ and $c\ln\left(\frac{c}{0}\right) = -\infty$ for $c > 0$.

(c) Solve the dual problem and find all optimal solutions of (GP) and its optimal value.

Problem 2

Consider the problem

$$
\begin{aligned}
\min_{\mathbf{x},\mathbf{y}\in\mathbb{R}^n} \quad & \sqrt{\|\mathbf{x}\|^2 + 4} + \mathbf{a}^T\mathbf{y} + \|\mathbf{x}\| \\
\text{s.t.} \quad & \mathbf{B}\mathbf{x} + \mathbf{C}\mathbf{y} \le \mathbf{d}, \\
& \|\mathbf{y}\| \le 1,
\end{aligned}
$$

where $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{m\times n}$ and $\mathbf{d} \in \mathbb{R}^m$.

(a) Show that if $\mathbf{B}\mathbf{B}^T \succ 0$, then strong duality holds.

(b) Find a dual problem.

Problem 3

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ and consider the Fermat-Weber problem

$$
\min_{\mathbf{x}\in\mathbb{R}^n} \sum_{i=1}^{m} \|\mathbf{x} - \mathbf{a}_i\|.
$$

Find a dual problem.

Problem 4

Consider the problem

$$\begin{aligned} f^* = \min_{\mathbf{x},\mathbf{y}\in\mathbb{R}^n} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \ldots, m, \end{aligned}$$

where f, g_i are convex functions over $\mathbb{R}^n$. Suppose that there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $g_i(\hat{\mathbf{x}}) < 0$ for all i, and assume that $f^* > -\infty$. Consider the dual problem $\max\{q(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \in \mathrm{dom}(q)\}$ where $q(\boldsymbol{\lambda}) = \min_{\mathbf{x}\in\mathbb{R}^n} L(\mathbf{x}, \boldsymbol{\lambda})$ and $\mathrm{dom}(q) = \left\{\boldsymbol{\lambda} \in \mathbb{R}^m_+ : q(\boldsymbol{\lambda}) > -\infty\right\}$. Let $\boldsymbol{\lambda}^*$ be an optimal solution of the dual problem. Prove that

$$\sum_{i=1}^{n} \boldsymbol{\lambda}_i^* \leq \frac{f(\hat{\mathbf{x}}) - f^*}{\min\limits_{i=1,2,\ldots,m} \{-g_i(\hat{\mathbf{x}})\}}.$$

Problem 5

Consider the ℓ_1-ball with radius $\alpha \geq 0$ defined by

$$B_\alpha = \left\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} |\mathbf{x}_i| \leq \alpha\right\}.$$

(a) Let (P) be the orthogonal projection problem of vector $\mathbf{y} \in \mathbb{R}^n$ onto B_α. Find its dual.

(b) Describe an algorithm that computes the orthogonal projection onto the set B_α based on the dual problem found in the previous section.