Optimization 1 - 098311

Winter 2021 - HW 9

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Problem 1:

Prove Motzkin's lemma:

Let $A \in \mathbb{R}^{mxn}$ and $B \in \mathbb{R}^{kxn}$. Prove that the system.

$$(I) : \begin{cases} Ax < 0 \\ Bx \leq 0 \end{cases}$$

has a solution $x \in \mathbb{R}^n$ if and only if the system

$$(II) : \begin{cases} A^T u + B^T v = 0_n \\ u \neq 0, \qquad\qquad\qquad u, v \geq 0 \end{cases}$$

does not have a solution for any $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^k$.

direction 1:

we will prove that $(I) \Rightarrow (\neg II)$.

(I) has a solution, thus:

$$\exists x \in \mathbb{R}^n : \begin{cases} Ax < 0 \\ Bx \leq 0 \end{cases}$$

assume by contradiction that (II) also has a solution:

$$\exists u \in \mathbb{R}^m, v \in \mathbb{R}^k, u \neq 0, u, v \geq 0 : A^T u + B^T v = 0_n$$

$$A^T u + B^T v = 0_n$$

$$u^T A + v^T B = 0_n$$

$$u^T Ax + v^T Bx = 0_n$$

$$0_n = \underbrace{\underbrace{u^T}_{\substack{\geq 0 \\ \neq 0}} \underbrace{Ax}_{<0}}_{<0} + \underbrace{\underbrace{v^T}_{\geq 0} \underbrace{Bx}_{\leq 0}}_{\leq 0} < 0_n$$

$$\boxed{0_n < 0_n}$$

which is of course a contradiction.

thus (II) doesn't have a solution.

direction 2:

we will prove that $(\neg I) \Rightarrow (II)$.

(I) doesn't have a solution, specifically it means that:

$$\forall x \in \mathbb{R}^n : Bx \leq 0 \Rightarrow Ax \geq 0$$

looking at some row a_j of A as a column vector:

$$Bx \leq 0 \Rightarrow a_j^T x \geq 0$$

$$Bx \leq 0 \Rightarrow -a_j^T x \leq 0$$

using the second formulation of Farkas lemma:

$$\exists y \in \mathbb{R}_+^k : B^T y = -a_j$$

choosing $v = y \in \mathbb{R}_+^k$ and $u = e_j \in \mathbb{R}^m$ we get:

$$v, u \geq 0$$

$$u \neq 0$$

and:

$$A^T u + B^T v = A^T e_j + B^T y = a_j - a_j = 0_n$$

thus (II) has a solution.

Problem 2:

for any set $C \subseteq \mathbb{R}^n$ define the set:

$$C^* = \left\{ y \in \mathbb{R}^n : x^T y \geq 0 \text{ for all } x \in C \right\}$$

to be it's dual cone.

a)

let's prove that C^* is a cone.

let $z \in C^*$ and $\lambda \in \mathbb{R}_+$then:

$$\forall x \in C : x^T z \geq 0$$

which also means that:

$$\forall x \in C : x^T \lambda z \geq 0$$

$\boxed{\text{thus } \lambda z \in C^* \text{ which means that } C^* \text{ is a cone by definition.}}$

now let's prove it is a convex cone:

let $z_1, z_2 \in C^*$:

$$\forall x \in C : x^T z_1 \geq 0$$

$$\forall x \in C : x^T z_2 \geq 0$$

which also means that:

$$\forall x \in C : x^T (z_1 + z_2) = \underbrace{x^T z_1}_{\geq 0} + \underbrace{x^T z_2}_{\geq 0} \geq 0$$

$\boxed{\text{thus } z_1 + z_2 \in C^*, \text{ which means that } C^* \text{ is a convex cone.}}$

let's prove that C^* is a closed set.

let there be some converging series $\{z_n\}_{n=1}^{\infty} \subseteq C^*$, that converges to z.

assume by contradiction that $z \notin C^*$,it means that:

$$\exists x \in C : x^T z < 0$$

which also means that:

$$\exists r > 0 : \forall y \in B(z, r), x^T y < 0$$

since $z_n \xrightarrow{n\to\infty} z$ then:

$$\exists N : \forall n > N, \|z_n - z\| < r$$

thus $z_{N+1} \in B(z, r)$ hence $\exists x \in C : x^T z_{N+1} < 0$

this is a contradiction to the fact that $z_{N+1} \in C^*$.

$\boxed{\text{therefore } z \in C^* \text{ and the set is closed by definition.}}$

b)

Let $A \in \mathbb{R}^{mxn}$ and define:

$$M = \{x \in \mathbb{R}^n : Ax \geq 0\}$$

by definition of M^*:

$$M^* = \left\{y \in \mathbb{R}^n : x^T y \geq 0 \text{ for all } x \in M\right\}$$

denote:

$$\overline{M} = \left\{z \in \mathbb{R}^m : z = A^T v, v \geq 0\right\}$$

let $u \in M^*$.

we want to show that $u \in \overline{M}$, which means we need to show that there exists $y \geq 0$ such that

$$u = A^T y$$

from the definition of M^*:

$$u \in M^* \iff \forall x \in M : x^T u \geq 0$$

from M definition, we can also write it as:

$$\iff Ax \geq 0 \Rightarrow x^T u \geq 0$$

$$\iff -Ax \leq 0 \Rightarrow -u^T x \leq 0$$

now using the second formulation of Farkas lemma:

$$\iff \exists y \in \mathbb{R}^n_+ : -A^T y = -u$$

$$\iff u = A^T y$$

we showed that $\exists y \in \mathbb{R}^n_+ : u = A^T y$, meaning that $u \in \overline{M}$.

we showed that $u \in M^* \iff u \in \overline{M}$, thus:

$$\boxed{M^* = \overline{M}}$$

Problem 3:

$$\min\left\{f\left(x,y,z\right)=x^{2}+y^{2}+z^{2}+xy+yz-2x-4y-6z\right\}$$

$$s.t:x+y+z\leq1$$

a)

$$f\left(x,y,z\right)=\left(\begin{array}{ccc}x & y & z\end{array}\right)\underbrace{\left(\begin{array}{ccc}1 & \frac{1}{2} & 0\\ \frac{1}{2} & 1 & \frac{1}{2}\\ 0 & \frac{1}{2} & 1\end{array}\right)}_{Q}\left(\begin{array}{c}x\\ y\\ z\end{array}\right)+2\underbrace{\left(\begin{array}{ccc}-1 & -2 & -3\end{array}\right)}_{b^{T}}\left(\begin{array}{c}x\\ y\\ z\end{array}\right)$$

denote:

$$u=\left(\begin{array}{c}x\\ y\\ z\end{array}\right)$$

we got the problem:

$$\min\left\{f\left(u\right)=u^{T}Qu+2b^{T}u\right\}$$

$$s.t:e^{T}u\leq1$$

let's show that Q is positive definite.

$$M_{1}\left(Q\right)=1>0$$

$$M_{2}\left(Q\right)=1-\frac{1}{4}=\frac{3}{4}>0$$

$$M_{3}\left(Q\right)=1\left(1\cdot1-\frac{1}{2}\cdot\frac{1}{2}\right)-\frac{1}{2}\cdot\left(\frac{1}{2}\cdot1-0\right)=1-\frac{1}{4}-\frac{1}{4}=\frac{1}{2}>0$$

all the principal minors of Q are positive, thus Q is positive definite.

$f\left(u\right)$ is a quadratic function, with a P.D matrix, thus convex.

$$\boxed{\text{The problem is a minimization problem of a convex objective function, with a linear constraint which defines a convex set, thus this is a convex optimization problem.}}$$

b)

define the Lagrangian:

$$L\left(u,\lambda\right)=f\left(u\right)+\lambda\left(e^{T}u-1\right)$$

the KKT condition are:

$$\begin{cases}
(1) & \nabla_{u}L\left(u^{*},\lambda^{*}\right)=0\\
(2) & \lambda^{*}\left(e^{T}u^{*}-1\right)=0\\
(3) & e^{T}u^{*}\leq1\\
(4) & \lambda\geq0
\end{cases}$$

$$(1)\,\nabla_{u}L\left(u^{*},\lambda^{*}\right)=2Qu^{*}+2b+\lambda^{*}e=0$$

$$2Qu^{*}=-2b-\lambda^{*}e$$

$$Qu^{*}=-b-\frac{1}{2}\lambda^{*}e$$

$$u^{*}=-Q^{-1}\left(b+\frac{1}{2}\lambda^{*}e\right)$$

$$(2)\,\lambda^{*}\left(e^{T}u^{*}-1\right)=\lambda^{*}\left(e^{T}\left(-Q^{-1}\left(b+\frac{1}{2}\lambda^{*}e\right)\right)-1\right)=$$
$$=\lambda^{*}\left(-e^{T}Q^{-1}b-\frac{1}{2}\lambda^{*}e^{T}Q^{-1}e-1\right)=0$$

if $\lambda^{*}=0$,(2) and (4) hold and:

$$u^{*}=-Q^{-1}b=-\begin{pmatrix}1.5 & -1 & \frac{1}{2}\\ -1 & 2 & -1\\ \frac{1}{2} & -1 & 1.5\end{pmatrix}\begin{pmatrix}-1\\ -2\\ -3\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 3\end{pmatrix}$$

$$e^{T}u^{*}=4>1$$

thus (3) doesn't hold, meaning $\lambda^{*}>0$, therefore:

$$-e^{T}Q^{-1}b-\frac{1}{2}\lambda^{*}e^{T}Q^{-1}e-1=0$$

$$\frac{1}{2}\lambda^* e^T Q^{-1} e = -e^T Q^{-1} b - 1$$

$$\lambda^* e^T Q^{-1} e = -2e^T Q^{-1} b - 2$$

$$\lambda^* = \frac{-2e^T Q^{-1} b - 2}{e^T Q^{-1} e} = \frac{2\left(e^T \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - 1\right)}{2} = \frac{2(4-1)}{2} = 3 > 0$$

$$u^* = -Q^{-1}\left(b + \frac{1}{2}\lambda^* e\right) = u^* = -Q^{-1}\left(b + \frac{3}{2}e\right) = -\begin{pmatrix} 1.5 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 1.5 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1.5 \end{pmatrix}$$

now we need to check if (3) holds:

$$e^T u^* = 1 \leq 1$$

thus the point:

$$u^* = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1.5 \end{pmatrix}$$

is the only K.K.T point.

c)

$f(u)$ is continuously differentiable and the problem is convex thus:

$$\{u : u \text{ is a KKT point}\} = \{u : f(u) \text{ is the optimal solution}\}$$

which means the optimal solution is attained at the K.K.T point that we have found:

$$f(u^*) = u^T Q u + 2b^T u = \left(-\frac{1}{2}\right)^2 + (1.5)^2 - 2\left(-\frac{1}{2}\right) - 6 \cdot 1.5 = \frac{1}{4} + \frac{9}{4} + 1 = 3.5$$

Problem 4:

Consider the problem:

$$\min\left\{f\left(x\right)=\frac{1}{2}x^{T}Qx+b^{T}x\right\}$$

$$s.t:c^{T}x\leq p_{1}$$

$$d^{T}x=p_{2}$$

where $n\geq3$ $Q\succ0$ $b,c,d\in\mathbb{R}^{n}$ $p_{1},p_{2}\in\mathbb{R}$

$$c^{T}Q^{-1}c=d^{T}Q^{-1}d$$

$$d^{T}Q^{-1}c=0$$

$$c,d\neq0_{n}$$

a)

if $f\left(x\right)$ is continuously differentiable then the K.K.T conditions are necessary for optimality.

In addition, if the problem is a convex optimization problem, then the K.K.T conditions are sufficient for optimality

in our case, $f\left(x\right)$ is a quadratic function with positive definite matrix thus convex, and the constraints are linear hence defines a convex set, meaning the problem is a convex optimization problem. In addition $f\left(x\right)$ is continuously differentiable, thus the K.K.T conditions are sufficient for optimaility.

b)

let's assume by contradiction that c and d are linearly dependent then:

$$\exists\lambda\neq0:d=\lambda c$$

notice:

$$d^{T}Q^{-1}c=\underbrace{\lambda}_{\neq0}\underbrace{c^{T}Q^{-1}c}_{>0}\neq0$$

which is a contradiction to the fact that $d^T Q^{-1} c = 0$

thus d and c are linearly independent.

define:

$$\lambda_1 = \frac{p_1}{\underbrace{c^T Q^{-1} c}_{>0}}$$

$$\lambda_2 = \frac{p_2}{\underbrace{d^T Q^{-1} d}_{>0}}$$

$$\tilde{x} = \lambda_1 Q^{-1} c + \lambda_2 Q^{-1} d$$

then:

$$\begin{aligned} c^T \tilde{x} &= c^T \left(\lambda_1 Q^{-1} c + \lambda_2 Q^{-1} d\right) = \lambda_1 c^T Q^{-1} c + \lambda_2 c^T Q^{-1} d = \lambda_1 c^T Q^{-1} c = \\ &= \frac{p_1}{c^T Q^{-1} c} c^T Q^{-1} c = p_1 \leq p_1 \end{aligned}$$

$$\begin{aligned} d^T \tilde{x} &= d^T \left(\lambda_1 Q^{-1} c + \lambda_2 Q^{-1} d\right) = \lambda_1 d^T Q^{-1} c + \lambda_2 d^T Q^{-1} d = \lambda_2 d^T Q^{-1} d = \\ &= \frac{p_2}{d^T Q^{-1} d} d^T Q^{-1} d = p_2 \end{aligned}$$

thus $\tilde{x}$ is a feasible solution to the problem, hence the problem is feasible.

c)

define the Lagrangian:

$$L(x, \lambda, \mu) = f(x) + \lambda \left(c^T x - p_1\right) + \mu \left(d^T x - p_2\right)$$

the KKT condition are:

$$\begin{cases} (1) \quad \nabla_x L(x^*, \lambda^*, \mu^*) = Qx^* + b + \lambda^* c + \mu^* d = 0 \\ (2) \quad \lambda^* \left(c^T x^* - p_1\right) = 0 \\ (3) \quad \begin{cases} c^T x^* \leq p_1 \\ d^T x^* = p_2 \end{cases} \\ (4) \quad \lambda^* \geq 0 \end{cases}$$

d)

if $\lambda^* = 0$:

from (1):

$$Qx^* + b + \mu^* d = 0$$

$$x^* = -Q^{-1}\left(b + \mu^* d\right)$$

plug in (3):

$$d^T x^* = -d^T Q^{-1}\left(b + \mu^* d\right) = p_2$$

$$-d^T Q^{-1} b - \mu^* d^T Q^{-1} d = p_2$$

$$\mu^* d^T Q^{-1} d = -\left(p_2 + d^T Q^{-1} b\right)$$

$$\mu^* = \frac{-\left(p_2 + d^T Q^{-1} b\right)}{\underbrace{d^T Q^{-1} d}_{>0}}$$

plug back in (1):

$$x^* = -Q^{-1}\left(b + \frac{-\left(p_2 + d^T Q^{-1} b\right)}{d^T Q^{-1} d} d\right)$$

check if (3) holds:

$$\begin{aligned}
c^T x^* &= -c^T Q^{-1}\left(b + \frac{-\left(p_2 + d^T Q^{-1} b\right)}{d^T Q^{-1} d} d\right) = \\
&= -c^T Q^{-1} b + \underbrace{c^T Q^{-1} d}_{=0}\left(\frac{\left(p_2 + d^T Q^{-1} b\right)}{d^T Q^{-1} d}\right) = \\
&= -c^T Q^{-1} b
\end{aligned}$$

if $-c^T Q^{-1} b \leq p_1$ (3) holds and we have found a K.K.T point which attains the optimal solution as we saw in section a.

$$\begin{aligned} f\left(x^*\right) &= \frac{1}{2}\left(Q^{-1}\left(b+\frac{-\left(p_2+d^TQ^{-1}b\right)}{d^TQ^{-1}d}d\right)\right)^T Q\left(Q^{-1}\left(b+\frac{-\left(p_2+d^TQ^{-1}b\right)}{d^TQ^{-1}d}d\right)\right) - b^TQ^{-1}\left(b+\frac{-\left(p_2+d^T\right.}{d^TQ^{-1}}\right. \\ &= \frac{1}{2}\left(b^T+\frac{-\left(p_2+d^TQ^{-1}b\right)}{d^TQ^{-1}d}d^T\right)Q^{-1}\left(b+\frac{-\left(p_2+d^TQ^{-1}b\right)}{d^TQ^{-1}d}d\right) - b^TQ^{-1}\left(b+\frac{-\left(p_2+d^TQ^{-1}b\right)}{d^TQ^{-1}d}d\right) \\ &= \left[\frac{1}{2}\left(b^T+\frac{-\left(p_2+d^TQ^{-1}b\right)}{d^TQ^{-1}d}d^T\right) - b^T\right]Q^{-1}\left(b+\frac{-\left(p_2+d^TQ^{-1}b\right)}{d^TQ^{-1}d}d\right) \\ &= -\frac{1}{2}\left[b^T+\frac{p_2+d^TQ^{-1}b}{d^TQ^{-1}d}d^T\right]Q^{-1}\left(b-\frac{p_2+d^TQ^{-1}b}{d^TQ^{-1}d}d\right) \end{aligned}$$

otherwise, assuming $\lambda = 0$ did not yield any K.K.T points.

In this case, let's assume $\lambda > 0$,and $-c^TQ^{-1}b > p_1$ then:

from (3):

$$c^Tx^* = p_1$$

from (1):

$$Qx^* + b + \lambda^*c + \mu^*d = 0$$

$$x^* = -Q^{-1}\left(b + \lambda^*c + \mu^*d\right)$$

plugging in both equations of (3):

$$d^Tx^* = -d^TQ^{-1}\left(b + \lambda^*c + \mu^*d\right) = p_2$$

$$-d^TQ^{-1}b - \lambda^*\underbrace{d^TQ^{-1}c}_{=0} + \mu^*d^TQ^{-1}d = p_2$$

$$\mu^* = \frac{-\left(p_2 + d^TQ^{-1}b\right)}{\underbrace{d^TQ^{-1}d}_{>0}}$$

$$c^Tx^* = -c^TQ^{-1}\left(b + \lambda^*c + \mu^*d\right) = p_1$$

$$-c^TQ^{-1}b - \lambda^*c^TQ^{-1}c + \mu^*\underbrace{c^TQ^{-1}d}_{=0} = p_1$$

$$\lambda^* = \frac{-\left(p_1 + c^T Q^{-1} b\right)}{\underbrace{c^T Q^{-1} c}_{>0}} = \frac{\overbrace{-\left(p_1 + c^T Q^{-1} b\right)}^{>0}}{\underbrace{c^T Q^{-1} c}_{>0}} > 0$$

λ^* is indeed grater then zero.

plugging back to (1):

$$\begin{aligned} x^* &= -Q^{-1}\left(b + \lambda^* c + \mu^* d\right) = \\ &= -Q^{-1}\left(b + \frac{-\left(p_1 + c^T Q^{-1} b\right)}{c^T Q^{-1} c} c + \frac{-\left(p_2 + d^T Q^{-1} b\right)}{d^T Q^{-1} d} d\right) \end{aligned}$$

we have found a K.K.T point which attains the optimal solution as we saw in section a.

we will let you imagine what happens when we place x^* into $f(\cdot)$:)

to sum:

$$x^* = \begin{cases} -Q^{-1}\left(b - \frac{p_2 + d^T Q^{-1} b}{d^T Q^{-1} d} d\right) & p_1 + c^T Q^{-1} b \geq 0 \\ -Q^{-1}\left(b - \frac{p_1 + c^T Q^{-1} b}{c^T Q^{-1} c} c - \frac{p_2 + d^T Q^{-1} b}{d^T Q^{-1} d} d\right) & p_1 + c^T Q^{-1} b < 0 \end{cases}$$

Problem 5:

Consider the problem:

$$\min f(x) = \|Ax - b\|^2 = x^T A^T Ax - 2b^T Ax + b^T b$$

$$s.t : e^T x = \alpha$$

where $A \in \mathbb{R}^{mxn}$, $b \in \mathbb{R}^m$, $e = (1, 1, .., 1)^T \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ is a parameter.

a)

direction 1:

$Ker(A) \cap Ker(e^T) = \{0_n\}$

$f(x)$ is a quadratic function with positive semi definite matrix $A^T A$, thus convex.

the constraint is linear thus defines a convex set.

hence this is a convex optimization problem, and since $f(x)$ is continuously differentiable, any $K.K.T$ point is an optimizer of $f(x)$.

in section d we have found a $K.K.T$ point under the assumption that $Ker(A) \cap Ker(e^T) = \{0_n\}$. thus the set of optimal points is not empty, now we will show that it is also unique.

let x, y be points that attain the optimal solutions of the problem.

from the constraint we know:

$$e^T x = e^T y = \alpha$$

hence:

$$e^T (x - y) = 0 \longrightarrow (x - y) \in Ker(e^T)$$

The Lagrangian of the problem is given by:

$$L(x, \mu) = x^T A^T Ax - 2b^T Ax + b^T b + \mu(e^T x - \alpha)$$

since the $f(x)$ is continuously differentiable, and x, y are optimal solutions, they must satisfy the K.K.T conditions.

the K.K.T conditions are:

$$\begin{cases} (1) \quad 2A^TAx - 2A^Tb + \mu e = 0 \\ (2) \quad e^Tx = \alpha \end{cases}$$

plugging x, y into the first condition:

$$\begin{cases} 2A^TAx - 2A^Tb + \mu e = 0 \\ 2A^TAy - 2A^Tb + \mu e = 0 \end{cases}$$

subtract the equations:

$$2A^TA\left(x - y\right) = 0$$

$$\left(x - y\right)^T A^TA\left(x - y\right) = 0$$

$$\left\|A\left(x - y\right)\right\|^2 = 0$$

$$A\left(x - y\right) = 0 \longrightarrow \left(x - y\right) \in Ker\left(A\right)$$

we found that

$$\left(x - y\right) \in Ker\left(A\right) \cap Ker\left(e^T\right) = \{0_n\}$$

$$x - y = 0$$

$$x = y$$

thus, the optimizer is unique.

direction 2:

let x^* be the unique optimizer.

assume by contradiction that $Ker\left(A\right) \cap Ker\left(e^T\right) \neq \{0_n\}$

let :

$$0_n \neq u \in Ker\left(A\right) \cap Ker\left(e^T\right)$$

define:

$$y = x^* + u \neq x^*$$

notice:

$$e^T y = e^T (x^* + u) = e^T x^* + e^T u = \alpha + 0 = \alpha$$

in addition:

$$f(y) = \|Ay - b\|^2 = \|A(x^* + u) - b\|^2 = \|Ax^* + Au - b\|^2 = \|Ax^* - b\|^2 = f(x^*)$$

thus we found another optimizer of f which is different than x^*, which is a contradiction to the uniqueness of x^*.

Therefore,

$$Ker(A) \cap Ker\left(e^T\right) = \{0_n\}$$

b)

we already saw that the $K.K.T$ conditions are:

$$\begin{cases} (1) \quad 2A^T Ax^* - 2A^T b + \mu^* e = 0 \\ (2) \quad e^T x^* = \alpha \end{cases}$$

let's find the K.K.T points:

from (1):

$$2A^T Ax^* - 2A^T b + \mu^* e = 0$$

$$A^T Ax^* = A^T b - \frac{1}{2}\mu^* e$$

$$x^* = \left(A^T A\right)^{-1}\left(A^T b - \frac{1}{2}\mu^* e\right)$$

from (2):

$$e^T \left(A^T A\right)^{-1}\left(A^T b - \frac{1}{2}\mu^* e\right) = \alpha$$

$$e^T \left(A^T A\right)^{-1} A^T b - \frac{1}{2}\mu^* e^T \left(A^T A\right)^{-1} e = \alpha$$

$$\mu^* e^T \left(A^T A\right)^{-1} e = 2e^T \left(A^T A\right)^{-1} A^T b - 2\alpha$$

$$\mu^* = \frac{2e^T \left(A^T A\right)^{-1} A^T b - 2\alpha}{\underbrace{e^T \left(A^T A\right)^{-1} e}_{>0}}$$

plugging back to (1):

$$x^* = \left(A^T A\right)^{-1}\left(A^T b - \frac{e^T \left(A^T A\right)^{-1} A^T b - 2\alpha}{e^T \left(A^T A\right)^{-1} e} e\right)$$

we have found a K.K.T point to the convex optimization problem, thus the optimal value of the problem is $f(x^*)$.

c)

let: $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ such that $g(\alpha)$ is an optimal value of P_α.

let $\alpha_1, \alpha_2 \in \mathbb{R}$ and let x_1^*, x_2^* be the points at which P_{α_1} and P_{α_2} attain their optimal value.

$$g(\alpha_1) = f(x_1^*)$$

$$g(\alpha_2) = f(x_2^*)$$

notice that:

$$e^T(\lambda x_1^* + (1-\lambda) x_2^*) = \lambda e^T x_1^* + (1-\lambda) e^T x_2^* = \lambda\alpha_1 + (1-\lambda)\alpha_2$$

thus $\lambda x_1^* + (1-\lambda) x_2^*$ is a feasible solution to $P_{\lambda\alpha_1+(1-\lambda)\alpha_2}$therefore the solution $g(\lambda\alpha_1 + (1-\lambda)\alpha_2)$ holds :

$$\begin{aligned} g(\lambda\alpha_1 + (1-\lambda)\alpha_2) &\leq f(\lambda x_1^* + (1-\lambda) x_2^*) \\ \text{f is convex} &\leq \lambda f(x_1^*) + (1-\lambda) f(x_2^*) \\ &= \lambda g(\alpha_1) + (1-\lambda) g(\alpha_2) \end{aligned}$$

thus $g(\alpha)$ is convex by definition.

d)

since $Ker(A) \cap Ker\left(e^T\right) = \{0_n\}$, from section a we know that there exists a unique solution to P_α.

the K.K.T conditions as we saw are:

$$\begin{cases} (1) \quad 2A^T A x^* - 2A^T b + \mu^* e = 0 \\ (2) \quad e^T x^* = \alpha \end{cases}$$

subtracting and adding $2ee^T x^*$ from (1):

$$2A^TAx^* - 2A^Tb + \mu^* e + 2e^Tex^* - 2e^Tex^* = 2\left(A^TA + ee^T\right)x^* - 2A^Tb + \mu^* e - 2ee^Tx^* = 0$$

$$2\left(A^TA + ee^T\right)x^* = 2A^Tb - \mu^* e + 2\alpha e$$
$$x^* = \left(A^TA + ee^T\right)^{-1}\left(A^Tb + \left(\alpha - \frac{1}{2}\mu^*\right)e\right)$$

Let's justify the invertibility of $\left(A^TA + ee^T\right)$

let $x \in \mathbb{R}^n$:

$$x^TA^TAx + x^Tee^Tx = ||Ax||^2 + \left|\left|e^Tx\right|\right|^2 \geq 0$$

let's see when the equality holds:

$$||Ax||^2 + \left|\left|e^Tx\right|\right|^2 = 0$$
$$\Longleftrightarrow ||Ax||^2 = 0, \left|\left|e^Tx\right|\right|^2 = 0$$
$$\Longleftrightarrow Ax = 0, e^Tx = 0$$
$$\Longleftrightarrow x \in \text{Ker}\left(A\right) \cap \text{Ker}\left(e^T\right)$$
$$\Longleftrightarrow x = 0$$

hence $A^TA + ee^T$ is P.D and by that invertible. let's continue.

from the constraint we can conclude:

$$e^Tx = \alpha = e^T\left(\underbrace{A^TA + ee^T}_{\triangleq Q}\right)^{-1}\left(A^Tb + \left(\alpha - \frac{1}{2}\mu\right)e\right)$$

$$e^TQ^{-1}A^Tb + \alpha e^TQ^{-1}e - \frac{1}{2}\mu^* e^TQ^{-1}e = \alpha$$
$$\mu^* e^TQ^{-1}e = 2\left(e^TQ^{-1}A^Tb + \alpha e^TQ^{-1}e - \alpha\right)$$
$$\mu^* = \frac{2\left(e^TQ^{-1}A^Tb + \alpha e^TQ^{-1}e - \alpha\right)}{\underbrace{e^TQ^{-1}e}_{>0}} = \frac{2\left(e^TQ^{-1}A^Tb - \alpha\right)}{e^TQ^{-1}e} + 2\alpha$$

plugging back in (1):

$$
\begin{aligned}
x^* &= Q^{-1}\left(A^T b + \left(\alpha - \frac{1}{2}\mu^*\right) e\right) = \\
&= Q^{-1}\left(A^T b + \left(\alpha - \frac{\left(e^T Q^{-1} A^T b - \alpha\right)}{e^T Q^{-1} e} - \alpha\right) e\right) = \\
&= Q^{-1}\left(A^T b + \frac{\left(e^T Q^{-1} A^T b - \alpha\right)}{e^T Q^{-1} e} e\right)
\end{aligned}
$$

we have found a K.K.T point to the convex optimization problem, thus the optimal value of the problem is $f\left(x^*\right)$.