Exam in Optimization I - Winter 2014/2015

Instructions: Answer the following 5 questions. Please start each question on a separate page and give **detailed explanations**.

Good Luck!!!

Question 1 (22 pts.) Consider the following convex optimization problem:

$$\begin{array}{ll} \min & \sqrt{2x_1^2+4x_1x_2+3x_2^2+1}+7 \\ \text{s.t.} & ((x_1^2+x_2^2)^2+1)^2 \leq 10x_1 \\ & \frac{x_1^2+4x_2^2+4x_1x_2}{2x_1+x_2+x_3} \leq 10 \\ & 1 \leq x_1, x_2, x_3 \leq 10. \end{array}$$

(a) Prove that the problem is convex.

(b) Write a CVX problem for solving the problem.

Question 2. (22 pts.) Consider the following convex optimization problem:

$$\text{(P)} \quad \begin{array}{ll} \min & \|\mathbf{Ax}+\mathbf{b}\|_2+\|\mathbf{Lx}\|_1+\|\mathbf{Mx}\|_2^2-\sum_{i=1}^n x_i \log x_i \\ \text{s.t.} & \mathbf{x} \geq \mathbf{0}, \end{array}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{L} \in \mathbb{R}^{p\times n}, \mathbf{M} \in \mathbb{R}^{q\times n}$. Write a dual problem of (P). Do not make any transformations that ruin the convexity of the problem.

Question 3. (22 pts.) Consider the set

$$S = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n w_i x_i^2 \leq 1 \right\},$$

where $w_1, w_2, \ldots, w_n \in \mathbb{R}_{++}$.

(a) Write the problem of finding the orthogonal projection of a vector $\mathbf{y} \in \mathbb{R}^n$ onto S (that is, computing $P_S(\mathbf{y})$) as a convex optimization problem.

(b) Describe an algorithm for computing $P_S(\mathbf{y})$ that relies on a one-dimensional root search procedure.

Question 4. (22 pts.) Consider the problem

$$\begin{array}{ll} \max & x_1^3 + x_2^3 + x_3^3 \\ \text{s.t.} & x_1^2 + x_2^2 + x_3^2 = 1. \end{array}$$

(a) Is the problem convex?

(b) Prove that all the local maximum points of the problem are also KKT points.

(c) Find all the KKT points of the problem.

(d) Find the optimal solution of the problem.

Question 5. (22 pts.) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Prove that the following two claims are equivalent.

(A) the system

$$\mathbf{Ax} = \mathbf{0}, \mathbf{x} > \mathbf{0}$$

has no solution.

(B) There exists a vector $\mathbf{y} \in \mathbb{R}^n$ for which $\mathbf{A}^T\mathbf{y} \leq \mathbf{0}$ and $\mathbf{A}^T\mathbf{y}$ is not the zeros vector.