TECHNION - Israel Institute of Technology
The William Davidson Faculty of Industrial Engineering and Management

Lecturer: Shimrit Shtern — February 9, 2021

Optimization 1 - 098311

Final Exam Winter 2020-2021

Please read the instruction carefully before proceeding!

Instructions:

- The following exam is a home exam which starts on 9/2/2020 at 9:00 (9AM) and ends on 10/2/2020 at 9:00 (9AM).
- The exam is an individual exam. The solutions must be a result of your own work. You cannot consult with others (virtually or in person) regarding solutions to the exam. Any clarifications requests should be addressed only to the course's staff. Any violation of this clause will result in disciplinary action.
- You are allowed to use the course material available on the course's website as well as the course book.
- The exam contains 5 questions. Please start each question on a separate page and give detailed explanations. Missing explanations may results in a substantial reduction in your grade. The grade in the exam is $\min\{100, X\}$ where $X \leq 115$ is the number of accumulated points.
- The solutions of the exam need to be uploaded as a single PDF file and submitted on the course's site until 10/2/2020 at 9:00 (9AM) when the submission window will be closed. Late submissions will not be accepted.
- The solutions must be readable and organized. Unreadable solutions will not be graded.

Good Luck!

Question 1 (20 points)

(a) Let f be convex and continuously differentiable on a convex set C. Prove that $\mathbf{x}^*$ is an optimal point of f if and only if

$$\nabla f(\mathbf{x})^\top(\mathbf{x}^* - \mathbf{x}) \leq 0, \forall \mathbf{x} \in C.$$

(b) Consider the problem

$$\min\left\{\sum_{i=1}^{n} g(x_i) : \mathbf{x} \in \Delta_n\right\}$$

where g is a one-dimensional continuous function.

(i) Prove that if $\mathbf{x}^*$ is an optimal solution of the problem then any permutation of $\mathbf{x}^*$ is also an optimal solution. (A permutation of a vector $\mathbf{x}$ is a vector $\mathbf{y}$ such that $\mathbf{y}_i = \mathbf{x}_{\sigma(i)}$ where $\boldsymbol{\sigma}$ is a permutation of the index set $1, \ldots, n$).

(ii) Show that if g is convex then the vector $\mathbf{y} = (1/n, \ldots, 1/n)$ is an optimal solution of the problem.

Question 2 (15 points)

Consider the following optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 + \|\mathbf{L}\mathbf{x}\|_1$$

(a) Show that any vector in $\mathbf{x} \in \mathbb{R}^n$ can be written as $\mathbf{x} = \mathbf{y} + \mathbf{z}$ where $\mathbf{y} \in \text{Null}(A) \cap \text{Null}(L)$ and $\mathbf{z} \in (\text{Null}(A) \cap \text{Null}(L))_\perp$ where for any linear subspace L we denote $L_\perp$ as its orthogonal subspace (*i.e.*, $L_\perp = \{\mathbf{y} : \mathbf{y}^\top\mathbf{x} = 0,\ \forall \mathbf{x} \in L\}$).

(b) Show that this problem is coercive if and only if $\text{Null}(\mathbf{A}) \cap \text{Null}(\mathbf{L}) = \{0\}$.

Question 3 (20 points)

Let $\mathbf{A} \in \mathbb{R}^{n\times m}$. Prove that one and only one of the following is true

(I) $\exists \mathbf{x} > 0$ such that $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{B}\mathbf{x} < \mathbf{c}$

(II) $\exists \mathbf{u}, \mathbf{w} \geq \mathbf{0}$ such that $\mathbf{b}^\top\mathbf{u} + \mathbf{c}^\top\mathbf{w} \leq 0$, $\mathbf{A}^\top\mathbf{u} + \mathbf{B}^\top\mathbf{w} \geq \mathbf{0}$, and at least one of the following holds: (i) $\mathbf{w} \neq 0$ (ii) $\mathbf{A}^\top\mathbf{u} + \mathbf{B}^\top\mathbf{w} \neq \mathbf{0}$ (iii) $\mathbf{b}^\top\mathbf{u} + \mathbf{c}^\top\mathbf{w} < 0$.

Question 4 (25 points)

Let $\mathbf{Q} \in \mathbb{R}^{n\times n}$ be a PSD matrix, $\mathbf{b} \in \mathrm{Im}(\mathbf{Q})$ and let

$$f(\mathbf{x}) = \sqrt{\mathbf{x}^T\mathbf{Q}\mathbf{x} - 2\mathbf{b}^\top\mathbf{x} + c}.$$

(a) Prove that there exists a matrix $\mathbf{A} \in \mathbb{R}^{k\times n}$ which is full row rank such that $\mathbf{Q} = \mathbf{A}^\top\mathbf{A}$ where k is the rank of $\mathbf{Q}$.

(b) Let $\mathbf{z} = \mathbf{A}\tilde{\mathbf{x}}$, for some $\tilde{\mathbf{x}} \in \mathbb{R}^n$ where $\mathbf{A}$ is the matrix defined in (a). Show that $\mathbf{x}$ satisfies $\mathbf{A}\mathbf{x} = \mathbf{z}$ if and only if there exist a vector $\mathbf{y} \in \mathrm{Null}(\mathbf{Q})$ such that $\mathbf{x} = \tilde{\mathbf{x}} + \mathbf{y}$.

(c) Show that the domain of f is equivalent to a set of the form

$$\mathrm{dom}\, f = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{A}\mathbf{x} - g(\mathbf{A}, \mathbf{b})\|^2 \geq \|g(\mathbf{A}, \mathbf{b})\|^2 - c\},$$

where g is a function of $\mathbf{A}$ and $\mathbf{b}$, and find an explicit formulation of g.

(d) Find necessary and sufficient conditions on the problem parameters for $f(\mathbf{x})$ to be a convex function.

Question 5 (35 points)

Consider the set $C = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n p_i x_i^2 \leq \alpha\}$ where $\mathbf{p} \in \mathbb{R}^n_{++}$

(a) Consider the orthogonal projection problem over set C:

(i) Formulate the problem. Are KKT conditions necessary for optimality? Are they sufficient?

(ii) Use the KKT conditions in order do devise a simple algorithm that finds an optimal solution of the problem.

(b) Let $\mathbf{y}$ be a given vector. Consider problem (P) where we aim to find the furthest point from $\mathbf{y}$ which is contained in set C.

(i) Formulate problem (P) as an optimization problem. Is it convex?

(ii) Compute the dual of problem (P) as a single variable problem. Is it convex?

(iii) Show that optimal primal solution is attained and strong duality holds.

(c) Consider the problem of finding the minimum radius ball which contains set C.

(i) Formulate the problem as an optimization problem on the center of the ball $\mathbf{y}$ and the radius r.

(ii) Use the dual formulation in (b) to the reformulate the problem as

$$\begin{aligned} \min_{\mathbf{y},\rho,\lambda} \quad & \rho \\ \text{s.t.} \quad & \sum_{i:p_i\lambda-1>0} \mathbf{y}_i^2 \left(\frac{p_i\lambda}{p_i\lambda - 1}\right) + \alpha\lambda \leq \rho \\ & \lambda \geq 1/p^* \end{aligned} \tag{P'}$$

where $p^* = \min_i p_i$.

(iii) Show that (P') is jointly convex in $\mathbf{y}, \rho$ and λ

(iv) Solve problem (P').