Optimization 1 — Tutorial 5

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Convex Set

A set $C \subseteq \mathbb{R}^n$ is convex if $\lambda \mathbf{x} + (1-\lambda)\,\mathbf{y} \in C$ for any $\mathbf{x}, \mathbf{y} \in C$ and $\lambda \in [0,1]$.

Algebraic Operation with Convex Sets

Let $C_i \subseteq \mathbb{R}^n$ be a convex set for all $i \in I$ and any I.

(a) $\bigcap\limits_{i \in I} C_i$ is convex.

(b) $\sum\limits_{i \in I} \mu_i C_i$ is convex for any $\mu_i \in \mathbb{R}$.

(c) $C_1 \times C_2 \times \ldots \times C_{|I|}$ is convex.

(d) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then $\mathbf{A}\,(C) = \{\mathbf{A}\mathbf{x} \in \mathbb{R}^n \colon \mathbf{x} \in C\}$ is convex.

(e) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then $\mathbf{A}^{-1}\,(C) = \{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{A}\mathbf{x} \in C\}$ is convex.

Basic Feasible Solution

Let $P = \left\{\mathbf{x} \in \mathbb{R}^n_+ \colon \mathbf{A}\mathbf{x} = \mathbf{b}\right\}$ for $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^n$. Suppose that the rows of $\mathbf{A}$ are linearly independent. Then $\bar{\mathbf{x}} \in P$ is a BFS if the columns of $\mathbf{A}$ corresponding to the indices of the non-zero elements of $\bar{\mathbf{x}}$ are linearly independent.

Extreme Point

Let $S \subseteq \mathbb{R}^n$ be a convex set. A point $\mathbf{x} \in S$ is called an extreme point of S if there do not exist $\mathbf{x}_1, \mathbf{x}_2 \in S$, $\mathbf{x}_1 \neq \mathbf{x}_2$, and $\lambda \in (0,1)$ such that $\mathbf{x} = \lambda \mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2$.

Problem 1

Given a convex set $P = \left\{\mathbf{x} \in \mathbb{R}^n_+ \colon \mathbf{A}\mathbf{x} = \mathbf{b}\right\}$, show that $\mathbf{x} \in P$ is a BFS if and only if $\mathbf{x}$ is an extreme point of P.

Solution

$\Longrightarrow$: Suppose that $\mathbf{x} \in P$ is a BFS and assume on the contrary that is not an extreme point.

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$\Longleftarrow$: Suppose that $\mathbf{x}$ is an extreme point of P. We have $\sum\limits_{i \in I} \mathbf{x}_i \mathbf{A}_i = \mathbf{A}\mathbf{x} = \mathbf{b}$. Need to show that $\{\mathbf{A}_i \colon i \in I\}$ is linearly independent.

- Assume otherwise:
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Problem 2

Prove/disprove convexity of the following sets:

(a) $\{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x}\|_2 \geq 1\}$.

(b) $\left\{\mathbf{x} \in \mathbb{R}^n \colon \max\limits_{1 \leq i \leq n} \mathbf{x}_i \leq 1\right\}$.

(c) $\left\{\mathbf{x} \in \mathbb{R}^n \colon \max\limits_{1 \leq i \leq n} \mathbf{x}_i \geq 1\right\}$.

Solution

(a)

(b)

(c)

Problem 3

Let $K \subseteq V$ where V is a vector space. The Minkowski functional $p \colon V \to \mathbb{R}_+$ is defined by

$$p(\mathbf{x}) = \inf \left\{\lambda > 0 \colon \frac{\mathbf{x}}{\lambda} \in K\right\}.$$

Suppose that K is a compact, convex and symmetric set ($\mathbf{x} \in K \Rightarrow -\mathbf{x} \in K$) and that $\mathbf{0}_V \in \operatorname{int}(K)$. Prove that p is a norm.

Solution

- Non-negativity: it is clear that $p \geq 0$. Suppose that $p(\mathbf{x}) = 0$. Since K is bounded, there exists $M \geq 0$ such that $\|\mathbf{y}\| \leq M$ for all $\mathbf{y} \in K$. Since $\operatorname{int}(K)$ contains a non-zero vector, we have that $M > 0$. Thus, for every $\lambda > 0$ such that $\frac{\mathbf{x}}{\lambda} \in K$ then $\frac{\|\mathbf{x}\|}{\lambda} \leq M$ and therefore $\frac{\|\mathbf{x}\|}{M} \leq \lambda$. Taking the infimum over $\lambda > 0$ we obtain that $\frac{\|\mathbf{x}\|}{M} \leq p(\mathbf{x}) = 0$ and therefore $\mathbf{x} = \mathbf{0}_V$. Finally, if $\mathbf{x} = \mathbf{0}_V$ then $p(\mathbf{x}) = \mathbf{0}_V$.

- Positive homogeneity:

- Triangle inequality: if $\mathbf{x} = \mathbf{y}$ then $p(\mathbf{x} + \mathbf{y}) = 2p(\mathbf{x}) = p(\mathbf{x}) + p(\mathbf{x})$. Suppose that $\mathbf{x} \neq \mathbf{y}$ are non-zero. We know that there exist $\lambda, \mu > 0$ such that $\frac{\mathbf{x}}{\lambda}, \frac{\mathbf{y}}{\mu} \in K$ (meaning, $p(\mathbf{x}) = \lambda$ and $p(\mathbf{y}) = \mu$). Since p is defined as an infimum, then for every $\epsilon > 0$ there exist $\lambda_\epsilon, \mu_\epsilon > 0$ such that $\lambda \leq \lambda_\epsilon < p(\mathbf{x}) + \epsilon$ and $\mu \leq \mu_\epsilon < p(\mathbf{y}) + \epsilon$. Notice that $0 < \frac{\lambda}{\lambda_\epsilon}, \frac{\mu}{\mu_\epsilon} \leq 1$ and therefore $\frac{\mathbf{x}}{\lambda_\epsilon} = \frac{\lambda}{\lambda_\epsilon}\frac{\mathbf{x}}{\lambda} + \left(1 - \frac{\lambda}{\lambda_\epsilon}\right)\mathbf{0}_V \in K$ since K is convex and contains the origin. Similarly, $\frac{\mathbf{y}}{\mu_\epsilon} \in K$. We have $\frac{\lambda_\epsilon}{\lambda_\epsilon + \mu_\epsilon}, \frac{\mu_\epsilon}{\lambda_\epsilon + \mu_\epsilon} \in (0,1)$ and $\frac{\lambda_\epsilon}{\lambda_\epsilon + \mu_\epsilon} + \frac{\mu_\epsilon}{\lambda_\epsilon + \mu_\epsilon} = 1$. Notice that

$$\frac{\mathbf{x} + \mathbf{y}}{\lambda_\epsilon + \mu_\epsilon} = \frac{\lambda_\epsilon}{\lambda_\epsilon + \mu_\epsilon}\frac{\mathbf{x}}{\lambda_\epsilon} + \frac{\mu_\epsilon}{\lambda_\epsilon + \mu_\epsilon}\frac{\mathbf{y}}{\mu_\epsilon} \in K,$$

  since K is convex. Therefore, by definition of $p(\mathbf{x} + \mathbf{y})$ we have $p(\mathbf{x} + \mathbf{y}) \leq \lambda_\epsilon + \mu_\epsilon < p(\mathbf{x}) + p(\mathbf{y}) + 2\epsilon$. Taking $\epsilon \to 0$ we obtain the required inequality.