Note Title 2/28/2021

OPTIMAL CONTROL 362-2-6221

A. INTRODUCTION

A.0 Preliminaries

A.0.1. Time-discretization of a Linear Time Invariant System (LTI)

Given the system
with initial conditions
$$\begin{cases} \dot{x}_t = A x_t + B u_t & t_0 \le t \le T \quad (1) \\ x(t_0) = x_0 \end{cases}$$

and given a partition $t_0 < t_1 < \ldots < t_N = T$

u_t is piece-wise constant $u_\tau = u(t_k) \quad t_k \le \tau \le t_{k+1}$

Then the solution to (1) is of the form

$$x_t = \Phi(t, t_0)\, x_0 + \int_{t_0}^{t} \Phi(t, \tau)\, B\, u(\tau)\, d\tau$$

where $\Rightarrow \dfrac{\partial \Phi(t,t_0)}{\partial t} = A\, \Phi(t, t_0)$ with I.C. $\Phi(t_0, t_0) = I$ (Identity Matrix)

For LTI systems, $\Phi(t,t_0) \triangleq e^{A(t-t_0)}$ exponential matrix

Consider the $[t_k, t_{k+1}]$ $k = 0, 1, \ldots, N-1$ / $t_{k+1} = t_k + \Delta t = (k+1)\Delta t$

$$x(t_{k+1}) = e^{A\Delta t} x(t_k) + \int_{t_k}^{t_{k+1}} e^{A\underbrace{[(k+1)\Delta t - \tau]}} B u(\tau)\, d\tau$$

Δt time Increment

change of variable $\xi = (k+1)\Delta t - \tau$

$d\xi = -d\tau$

$$x_{k+1} = e^{A\Delta t} x_k + \left(\int_0^{\Delta t} e^{A\xi} d\xi\right) B u_k$$

Difference Equation

$$x_{k+1} = F x_k + G u_k \quad , \quad IC \; x_0 \qquad (2)$$

where $\begin{cases} F = e^{A\Delta t} \\ G = \left[\int_0^{\Delta t} e^{A\xi} d\xi\right] B \end{cases}$

this is an <u>exact</u> representation

FIRST-ORDER approximation with respect to Δt

$$F = e^{A\Delta t}$$

$$\simeq I + A\Delta t \cancel{\left(+ \frac{A^2 \Delta t^2}{2!} + \frac{A^3 \Delta t^3}{3!} + \dots \right)}$$

$$G = \int_0^{\Delta t} e^{A\xi} d\xi \, B \simeq B\,\Delta t$$

we reject H.O.T. in Δt

As a Result

$$x_{k+1} = F x_k + G u_k \; ; \; x_0$$

will be our system representation.

a consequence :

$$x_k = F^k x_0 + \sum_{i=0}^{k-1} F^{k-1-i} G u_i$$

sequence of Input

A.0.2 Operators

Given X, Y two vector spaces. $x \in X$, $y \in Y$

F an operator on X with values in Y maps any element in X to a single element in Y

$$F : X \to Y$$
$$x \mapsto y = F(x)$$

example $X = \mathbb{R}^2$ $Y = \mathbb{R}^3$ F linear

$$\begin{bmatrix} y_1 \\ y_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad : y = F x$$

example X : { Continuous functions on $[0,T]$ }

$Y = X$

$$\Rightarrow y(t) = \int_0^t e^{A\xi} B\, x(\xi)\, d\xi \qquad y(t) = F\{x(t)\}$$

Linear Operator
$$\begin{cases} F(x_1 + x_2) = F(x_1) + F(x_2) \\ F(\alpha x) = \alpha F(x) \end{cases}$$

Functional : is an operator with the Real Line as the range.

example: $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

quadratic form

$$x \longmapsto f(x) = x^T Q x = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} \, x_i \, x_j$$

scalar

example: $f : \mathcal{L}_2^n[0,T] \longrightarrow \mathbb{R}$

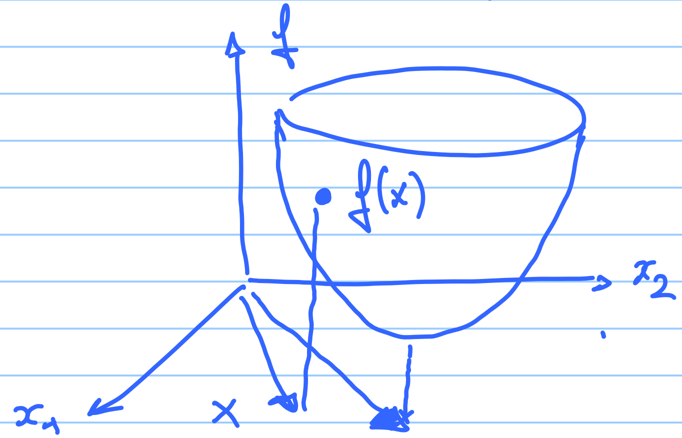
$$\int_0^T \|x_t\|^2 dt < \infty \qquad f(x(t)) = \int_0^T [x_t^T Q x_t] \, dt$$

example: $f : \ell_2 \longrightarrow \mathbb{R}$

$$\{x_k\}_0^\infty \,/\, \sum_{k=0}^{\infty} x_k^2 < \infty \qquad f(x_k) = \sum_{k=0}^{\infty} x_k^2$$

Question: How To Differentiate a Functional?

Directional Derivative of a Functional



Given a functional f $\mathcal{U} \subseteq X$ the directional derivative of f at x in direction h

$$\begin{cases} x \in \mathcal{U} \\ x + \varepsilon h \in \mathcal{U} \end{cases}$$

$$\boxed{\frac{d}{d\varepsilon} f(x + \varepsilon h)\Big|_{\varepsilon = 0}}$$

the functional f admits an extremum at x if

$$\boxed{\frac{d}{d\varepsilon} f(x + \varepsilon h)\Big|_{\varepsilon = 0} = 0 \quad \forall h \in X \mid x + \varepsilon h \in \mathcal{U}}$$

!! scalar differentiation

Example: Given $f(x) = x^T Q x \quad x \in \mathbb{R}^n \quad Q^T = Q$ (why?)

$\forall Q \quad Q = Q_S + Q_A \qquad Q_S = \frac{Q + Q^T}{2} \quad Q_A = \frac{Q - Q^T}{2}$

symmetric   anti-symmetric / skew-symmetric

check $x^T Q_A x \equiv 0$

$$x^T \left[\frac{1}{2} (Q - Q^T) \right] x$$

$$\frac{1}{2} x^T Q x - \frac{1}{2} \underbrace{x^T Q^T x}$$

$$[x^T Q^T x]^T$$

$$x^T \underbrace{(Q^T)^T}_{Q} x$$

$\forall h \in \mathbb{R}^n, \ \forall \varepsilon \in \mathbb{R}$

$F(x+\varepsilon h) = (x+\varepsilon h)^T Q \,(x+\varepsilon h)$

$= \underline{x^T Q x} + \underline{2x^T Q h \varepsilon} + h^T Q h \,\underline{\varepsilon^2}$

$\frac{d}{d\varepsilon} F(x+h\varepsilon) = 2x^T Q h + 2h^T Q h \,\varepsilon$

$\overset{(\varepsilon=0)}{=} 2x^T Q h$ the Directional Derivative of f

@ x along h

Example: Find the extremum of the functional

$f(x) = x^T Q x + 2y^T x \quad x \in \mathbb{R}^n \quad y \in \mathbb{R}^n \quad \begin{cases} Q^T = Q \\ Q > 0 \end{cases}$

$Q > 0$: Positive-Definite Matrix ?

by definition $\forall x \neq 0 \quad x^T Q x > 0$

$x = 0 \quad x^T Q x = 0$

by theorem, all eigenvalues of Q are positive numbers.

$$f_h'(x) \equiv \frac{d}{d\varepsilon} f(x + \varepsilon h) \Big|_{\varepsilon = 0}$$

$$= 2 x^T Q h + 2 y^T h$$

In order for x^* to be an extremum, the stationarity condition

$$f_h'(x^*) = 0 \qquad \forall h \in \mathbb{R}^n$$

$$2 x^{*T} Q h + 2 y^T h = 0 \qquad \forall h$$

$$2\left(x^{*T} Q + y^T\right) h = 0 \quad \forall h$$

$$2\, h^T \left(Q x^* + y\right) = 0 \quad \underline{\underline{\forall h}} \qquad (Q^T = Q)$$

Let's pick $\quad h = Q x^* + y$

then $\quad (Q x^* + y)^T (Q x^* + y) = 0$

$$\| Q x^* + y \|^2 = 0$$

$$\Rightarrow \quad Q x^* + y = 0$$

$$x^* = - Q^{-1} y \qquad (Q^{-1} \text{ exists } ! \,)$$

candidate for extremum. $\qquad$ because $\underline{Q > 0}$

Inserting x^* into $f(x)$ yields

$$f(x^*) = x^{*T} Q x^* + 2 y^T x^*$$

$$= (-Q^{-1}y)^T Q (-Q^{-1}y) + 2y^T(-Q^{-1}y)$$

$$= +y^T Q^{-1} \cancel{Q} \cancel{Q^{-1}} y - 2 y^T Q^{-1} y$$

$$= -y^T Q^{-1} y$$

By following the standard approach, we compute the gradient of $f(x)$

$$\nabla_x f(x) \equiv \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}_{n \times 1}$$

$$\nabla_x f(x) = \nabla_x \left[x^T Q x + 2y^T x \right]$$

$$= \nabla_x (x^T Q x) + 2 \nabla_x (y^T x)$$

$$= \underbrace{(Q + Q^T)}_{2Q} x + 2y = 2(Qx + y)$$

Example: $f(x_t) = \int_0^T (a_t \cdot x_t + b_t)^2 \, dt$ $\qquad a_t, b_t, x_t \in \mathcal{L}_2[0,T]$

$a_t \neq 0 \quad \forall t$

Find the minimum of $f(x_t)$ w.r.t. in $\mathcal{L}_2[0,T]$.

1. Directional Derivative

$\forall h_t \in \mathcal{L}_2[0,T]$

$$f(x_t + \varepsilon h_t) = \int_0^T (a_t(x_t + \varepsilon h_t) + b_t)^2 \, dt$$

$$= \int_0^T (a_t x_t + b_t)^2 \, dt + \int_0^T 2 a_t h_t (a_t x_t + b_t) \, dt \cdot \varepsilon + \int_0^T (a_t h_t)^2 \, dt \cdot \varepsilon^2$$

$$f_h'(x_t) = 2 \int_0^T a_t h_t (a_t x_t + b_t) \, dt$$

directional derivative of $f(x)$ at x_t along h_t.

Necessary condition for minimum:

$$\delta_h' (x_t) = 0 \quad \forall h_t \iff 2\int_0^T a_t h_t (a_t x_t + b_t)\, dt = 0 \quad \forall h_t$$

Pick up $\quad h_t = \dfrac{a_t x_t + b_t}{a_t} \quad \mapsto \quad \int_0^T \cancel{a_t} \dfrac{(a_t x_t + b_t)}{\cancel{a_t}} (a_t x_t + b_t)\, dt = 0$

$$\int_0^T (a_t x_t + b_t)^2\, dt = 0$$

$$\Rightarrow \quad a_t x_t + b_t = 0 \quad \forall t \in [0,T]$$

$$x_t^* = -\frac{b_t}{a_t} \quad \forall t \text{ "almost everywhere"}$$