Note Title 4/18/2021

B.1.4.2 The KALMAN FILTER : a Baysian approach.

Given the following system model in state-space form

$$\begin{cases} x_{k+1} = A_k x_k + w_k \\ y_{k+1} = H_{k+1} x_{k+1} + \vartheta_{k+1} \end{cases}$$

x_k $n \times 1$ state
w_k $n \times 1$ noise
y_k $r \times 1$ measurement
ϑ_k $r \times 1$

$\{w_k\}$ $\{\vartheta_k\}$ sequences of independent r.v., Gaussian, zero-mean, covariance matrices $\underset{n \times n}{\overset{\geq 0}{Q_k}}$, $\underset{r \times r}{\overset{> 0}{R_k}}$

x_0 initial condition, Gaussian, mean $\hat{x}_0$, covariance P_0, independent of $\{w_k\}$ $\{\vartheta_k\}$.

Find the optimal state estimator $\hat{x}_k$ of x_k via the criteria of the Minimum Mean Square Error given the sequence of measurements $\mathcal{Y}^k = \{y_k, y_{k-1}, \dots, y_0\}$ $\forall k$

$$\boxed{\min_{\hat{x}_k} J = \min_{\hat{x}_k} E\left\{ \|x_k - \hat{x}_k\|^2_W \,\middle|\, y^k \right\}}$$

weighting matrix PSD

- Some Preliminary Results from Matrix Algebra.

MATRIX INVERSION LEMMA

Lemma: If $B^{-1} = A^{-1} + C^T D^{-1} C$ (1) then $B = A - AC^T\left(\underline{CAC^T + D}\right)^{-1} CA$ (2)

Proof:

$$AC^T = BC^T + \underline{BC^T D^{-1}}\,\underline{CAC^T}$$

$$= BC^T \underline{D^{-1} D} + BC^T D^{-1} CAC^T$$

$$= BC^T D^{-1}\left[\underline{D + CAC^T}\right]$$

motivation:

Using (1) in order to compute B requires a $(n \times n)$ inversion, while doing it using (2) requires $(r \times r)$ matrix inversion. $(r < n)$ GOOD!

$$\boxed{AC^T(CAC^T+D)^{-1} = BC^TD^{-1}}$$ remember it for later use.

$B(1)A=D$

$$A = B + BC^TD^{-1}CA$$

$$B = A - \boxed{BC^TD^{-1}CA}$$

$$= A - AC^T(CAC^T+D)^{-1}CA \quad \therefore$$

- <u>SCHUR Complement</u>

Given $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ A $n\times n$ B $n\times p$ C $p\times n$ D $p\times p$

A^{-1}, D^{-1} exist

Schur complement of A: $S_c(A) \equiv D - CA^{-1}B$

" " " D: $S_c(D) \equiv A - BD^{-1}C$

$$\boxed{|M| = |A|\,|S_c(A)| = |D|\,|S_c(D)|}$$

■ Now, we will prove that $\hat{x}_k$ is the Conditional Expectation of x_k given y^k.

The functional J to be minimized is applied to a "vector" $\{x_k\}$ within the linear vector space of infinite random sequences.

$$\left.\frac{\partial L(u^* + \epsilon h)}{\partial \epsilon}\right|_{\epsilon=0} = 0 \quad \forall h \ / \ u^* + \epsilon h \text{ stays inside the vector space.}$$

$$J = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \|x_k - \hat{x}_k\|_W^2 \, p(x_k | y^k) \, dx_k$$

$\hat{x}_k$ estimate

$\{\hat{x}_k\}$ sequence of estimates

$$\hat{x}_k = \hat{x}_k^* + \epsilon h_k$$

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \underbrace{\|x_k - \hat{x}_k^* - \epsilon h_k\|_W^2}_{} \, p(x_k | y^k) \, dx_k$$

$$\frac{d}{d\epsilon} \int_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \left\{ \| x_k - \hat{x}_k^* \|_W^2 - 2\left[(x_k - \hat{x}_k^*)^T W h_k \right] \epsilon + \| h_k \|_W^2 \epsilon^2 \right\} p(x_k | y^k) \, dx_k \Big|_{\epsilon = 0}$$

$$\left\{ -2\left[(x_k - \hat{x}_k^*)^T W h_k \right] + 2 \| h_k \|_W^2 \epsilon \right\} \Big|_{\epsilon = 0}$$

$$\int_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -2 (x_k - \hat{x}_k^*)^T W h_k \; p(x_k | y^k) \, dx_k = 0 \quad \forall h_k$$

$$\int_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} (x_k - \hat{x}_k^*) \, p(x_k | y^k) \, dx_k = 0$$

$$\hat{x}_k^* \cdot \underbrace{\int_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} p(x_k | y^k) \, dx_k}_{=1} = \underbrace{\int_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} x_k \, p(x_k | y^k) \, dx_k}_{= E\{x_k | y^k\}}$$

Conclusion: $$\hat{X}_k^* = E\{x_k | y^k\}$$

. If we are solving the Minimum Mean Square Error estimation problem the result is the Conditional Expectation of x_k given the y^k.

Computation of $E\{x_k \mid Y^k\}$

A. $\begin{cases} \hat{x}_k = E\{x_k \mid Y^k\} & \text{a posteriori estimate at time } k \\ \bar{x}_{k+1} = E\{x_{k+1} \mid Y^k\} & \text{prior estimate at time } k+1 \end{cases}$

$\begin{cases} P_k \equiv Cov\{x_k \mid Y^k\} \\ M_{k+1} \equiv Cov\{x_{k+1} \mid Y^k\} \end{cases}$

$$p(x_{k+1} \mid Y^{k+1}) = \frac{p(x_{k+1}, Y^{k+1})}{p(Y^{k+1})}$$

$$= \frac{p(y_{k+1}, x_{k+1}, Y^k)}{p(y_{k+1}, Y^k)}$$

$$= \frac{p(y_{k+1} \mid x_{k+1}, Y^k) \cdot p(x_{k+1} \mid Y^k) \cdot \cancel{p(Y^k)}}{p(y_{k+1} \mid Y^k) \cdot \cancel{p(Y^k)}}$$

$$p(x_{k+1} \mid y^{k+1}) = \frac{p(y_{k+1} \mid x_{k+1}, \cancel{y^k}) \cdot p(x_{k+1} \mid y^k)}{p(y_{k+1} \mid y^k)}$$

Since $y = Hx + v$, v independent of x

thus "$y \mid x$" is like noise, independent of y^k !

measurement Noise

$$\boxed{p(x_{k+1} \mid y^{k+1}) = \frac{p(y_{k+1} \mid x_{k+1}) \cdot p(x_{k+1} \mid y^k)}{p(y_{k+1} \mid y^k)}}$$

prediction of x_{k+1} given y^k

BAYES formula for the posterior p.d.f of x_{k+1} given y^{k+1}

prediction of y_{k+1} given y^k

B. Computation of $p(y_{k+1} | x_{k+1})$

Key idea : Gaussian $\longrightarrow$ Expectation , Covariance matrix

$$E\{y_{k+1} | x_{k+1}\} = E\{H_{k+1} x_{k+1} + v_{k+1} | x_{k+1}\}$$

$$= H_{k+1} E\{x_{k+1} | x_{k+1}\} + \cancel{E\{v_{k+1} | x_{k+1}\}}^{0}$$ (independent and zero mean)

$$= H_{k+1} x_{k+1}$$

$$E\{y_{k+1} | x_{k+1}\} = H_{k+1} x_{k+1}$$

$$\mathrm{Cov}\{y_{k+1} | x_{k+1}\} = E\{(y_{k+1} - E\{y_{k+1} | x_{k+1}\})(y_{k+1} - E\{y_{k+1} | x_{k+1}\})^T | x_{k+1}\}$$

$$= E\{(\cancel{Hx} + v - \cancel{Hx})(\cancel{Hx} + v - \cancel{Hx})^T | x_{k+1}\}$$

$$= E\{v v^T | x_{k+1}\}$$

$$= E\{v v^T\}$$ independent

$$\mathrm{Cov}\{y_{k+1} | x_{k+1}\} = R_{k+1}$$

C. Computation of $p(x_{k+1} | y^k)$

Gaussian $\rightarrow$ expectation, covariance matrix

$$E\{x_{k+1} | y^k\} = E\{A_k x_k + w_k | y^k\}$$

$$= A_k \underbrace{E\{x_k | y^k\}}_{\hat{x}_k} + \cancel{E\{w_k | y^k\}} \quad 0$$

independence

$$= A_k \hat{x}_k$$

$$E\{x_{k+1} | y^k\} = A_k \hat{x}_k \Rightarrow \bar{x}_{k+1} = A_k \hat{x}_k$$

$$\mathrm{Cov}\{x_{k+1}|y^k\} = E\left\{ (x_{k+1} - E\{x_{k+1}|y^k\})(x_{k+1} - E\{x_{k+1}|y^k\})^T \,|\, y^k \right\}$$

$$= E\left\{ [(A_k x_k + w_k) - (A_k \hat{x}_k)][\quad]^T | y^k \right\}$$

$$= E\left\{ [A_k(x_k - \hat{x}_k) + w_k][\quad] | y^k \right\}$$

$$= A_k \, E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T | y^k\} \, A_k^T \quad \equiv P_k$$

$$+ A_k E\{(x_k - \hat{x}_k) w_k^T | y^k\} + E\{\underbrace{w_k}\underbrace{(x_k - \hat{x}_k)^T} | y^k\} A_k^T$$

$$+ E\{w_k w_k^T | y^k\} \quad \equiv Q_k$$

(independent)

(x_k, w_k) independent

(y^k, w_k) independent

$$=$$

$$\underbrace{E\{\cancel{w_k}\}}_{0} \; \underbrace{E\{(x_k - \cancel{x_k})^T | y^k\}}_{0}$$

$$\mathrm{Cov}\{x_{k+1}|y^k\} = A_k P_k A_k^T + Q_k$$

$$\Rightarrow M_{k+1} = A_k P_k A_k^T + Q_k$$

D. Computation of $p(y_{k+1} | y^k)$

Gaussian $\rightarrow$ expectation, covariance matrix

$$E\{y_{k+1} | y^k\} = H_{k+1} \bar{x}_{k+1}$$

$$Cov\{y_{k+1} | y^k\} = H_{k+1} M_{k+1} H_{k+1}^T + R_{k+1}$$

F. Synthesis

$$p(x_{k+1} | y^{k+1}) = \frac{\dfrac{1}{(2\pi)^{r/2} |R_{k+1}|^{1/2}} e^{-\frac{1}{2}\| y_{k+1} - H_{k+1} x_{k+1} \|^2_{R^{-1}_{k+1}}} \dfrac{1}{(2\pi)^{n/2} |M_{k+1}|} e^{-\frac{1}{2} \| x_{k+1} - \bar{x}_{k+1} \|^2_{M^{-1}_{k+1}}}}{\dfrac{1}{(2\pi)^{r/2} |H_{k+1} M_{k+1} H^T_{k+1} + R_{k+1}|^{1/2}} e^{-\frac{1}{2} \| y_{k+1} - H_{k+1} \bar{x}_{k+1} \|^2_{(H_{k+1} M_{k+1} H^T_{k+1} + R_{k+1})^{-1}}}}$$

. .

• Collecting the exponentials' arguments

$$\|y - Hx\|^2_{R^{-1}} + \|x-\bar{x}\|^2_{M^{-1}} - \|y - H\bar{x}\|^2_{(HMH^T+R)^{-1}}$$

$$(y-Hx)^T R^{-1}(y-Hx) + (x-\bar{x})^T M^{-1}(x-\bar{x})$$

$$= x^T H^T R^{-1} H x + x^T M^{-1} x$$
$$- y^T R^{-1} H x - x^T H^T R^{-1} y - \bar{x}^T M^{-1} x - x^T M^{-1}\bar{x}$$
$$+ y^T R^{-1} y + \bar{x}^T M^{-1}\bar{x}$$

$$= x^T \left(M^{-1} + H^T R^{-1} H\right) x$$
$$- x^T\left(M^{-1}\bar{x} + H^T R^{-1} y\right) - \left(M^{-1}\bar{x} + H^T R^{-1} y\right)^T x$$
$$+ \dots$$

square completion

$$= \left[x - \left(M^{-1}\bar{x} + H^T R^{-1} y\right)\right]^T \left[M^{-1} + H^T R^{-1} H\right]\left[\ \cdot\ \right]$$ ← Square completed

$$- \left(M^{-1}\bar{x} + H^T R^{-1} y\right)^T \left(M^{-1} + H^T R^{-1} H\right)\left(M^{-1}\bar{x} + H^T R^{-1} y\right)$$
$$+ y^T R^{-1} y + \bar{x}^T M^{-1}\bar{x} + \dots$$
$$- (y - H\bar{x})^T (HMH^T + R)^{-1}(y - H\bar{x})$$

→ 0 !

check it ...

$$= \left\| x_{k+1} - \left(M^{-1}_{k+1}\bar{x}_{k+1} + H^T_{k+1} R^{-1}_{k+1} y_{k+1} \right) \right\|^2_{M^{-1} + H^T R^{-1} H}$$

$$P^{-1}_{k+1} = \left[\mathrm{Cov}\{ x_{k+1} | y^{k+1} \}_0 \right]^{-1}$$

Since the $p(x_{k+1} | y^{k+1})$ is a Gaussian then

$$E\{ x_{k+1} | y^{k+1} \} = \hat{X}_{k+1}$$

So KALMAN FILTER

$$\begin{cases} \hat{x}_{k+1} = M^{-1}_{k+1}\bar{x}_{k+1} + H^T_{k+1} R^{-1}_{k+1} y_{k+1} \; ; \; \hat{x}_0 & n \times 1 \\ P^{-1}_{k+1} = M^{-1}_{k+1} + H^T_{k+1} R^{-1}_{k+1} H_{k+1} \; ; \; P_0 & n \times n \end{cases}$$

INFORMATION form

Matrix Inversion Lemma

$$(M^{-1} + H^T R^{-1} H)^{-1} = M - MH^T (HMH^T + R)^{-1} HM$$

The Kalman filter equations are rewritten:

$$\hat{x}_{k+1} = \bar{x}_{k+1} + K_{k+1}\left(y_{k+1} - H_{k+1}\bar{x}_{k+1}\right)$$

$$P_{k+1} = M_{k+1} - M_{k+1}H_{k+1}^T\left(H_{k+1}M_{k+1}H_{k+1}^T + R_{k+1}\right)^{-1}H_{k+1}M_{k+1}$$ COVARIANCE FORM.

where

$$K_{k+1} = M_{k+1}H_{k+1}^T\left(H_{k+1}M_{k+1}H_{k+1}^T + R_{k+1}\right)^{-1}$$ inversion of $r \times r$ matrix

• the scalar coefficient in front of the exponential ?

Collection

It can be shown that :
$$\boxed{\frac{|HMH^T + R|}{|M| \; |R|} = \frac{1}{|P|}}$$

proof

augmented matrix $\mathcal{G} = \begin{bmatrix} M & MH^T \\ HM & HMH^T+R \end{bmatrix}$; M^{-1}, $(HMH^T+R)^{-1}$ exist

$\Downarrow$ SCHUR complement

$$|\mathcal{G}| = |M| \, |\underbrace{R}_{S_c(M)}| = |HMH^T+R| \, |\underbrace{M - MH^T(HMH^T+R)^{-1}HM}_{P}|$$

Finally

$$p(x_{k+1} \mid y^{k+1}) = \frac{1}{(2\pi)^{n/2} \, |P|^{1/2}} \, e^{-\frac{1}{2} \| x_{k+1} - \hat{x}_{k+1} \|^2_{P_{k+1}^{-1}}}$$

Calculated by the KF.