Note Title 4/11/2021

B.1.4 Examples

Example 1

x_k state

Given $x_{k+1} = a x_k + b u_k + w_k$ a, b known scalars

random $\{w_k\}$ identically independently distributed, zero-mean, std σ

$$J = \sum_{k=0}^{N-1} x_{k+1}^2$$

Find the sequence of $\{u_k\}_0^N$ that minimizes the expected cost $E\{J\}$.

Solution by Dynamic Programming

■ Last Stage

u_{N-1} decision variable $\quad x_{N-1}$ known $\longrightarrow x_N$

$$\min_{u_{N-1}} E\{x_N^2\}$$

$$\min_{u_{N-1}} E\left\{\left[a x_{N-1} + b u_{N-1} + w_{N-1}\right]^2\right\} \qquad (1)$$

$$\min_{u_{N-1}} E\left\{a^2 x_{N-1}^2 + b^2 u_{N-1}^2 + w_{N-1}^2 + 2ab x_{N-1} u_{N-1} + 2a x_{N-1} w_{N-1} + 2b u_{N-1} w_{N-1}\right\}$$

assumptions :

1) x_{N-1} is known . it is deterministic

2) u_{N-1} is a deterministic variable

3) w_{N-1} is zero - mean $\quad E\{w_{N-1}\} = 0$

4) $E\{w_{N-1}^2\} = \sigma^2$

$$\min_{u_{N-1}} \left[a^2 x_{N-1}^2 + b^2 u_{N-1}^2 + \cancel{E\{w_{N-1}^2\}}^{\sigma^2} + 2ab\, x_{N-1} u_{N-1} + 2a x_{N-1} \cancel{E\{w_{N-1}\}}^{0} + 2b u_{N-1} \cancel{E\{w_{N-1}\}}^{0}\right]$$

$$\min_{u_{N-1}} \underbrace{\left[(a x_{N-1} + b u_{N-1})^2 \right]} + \sigma^2$$

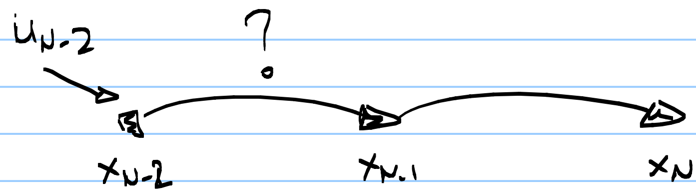
independent from U_{N-1}

this yields:

$$\boxed{u^*_{N-1} = -\frac{a}{b} x_{N-1}} \quad \text{and} \quad \boxed{J^*_{N-1} = \sigma^2}$$

- Second to last stage

x_{N-2} is known.



Applying Dynamic Programming

$$J_{N-2} = E\{ w_{N-2} + J^*_{N-1} \}$$

$$\min_{u_{N-2}} J_{N-2} = \min_{u_{N-2}} \left[E\{ w_{N-2} \} + E\{ J^*_{N-1} \} \right]$$

$$= \min_{u_{N-2}} \left[E\{ x_{N-1}^2 \} + \sigma^2 \right]$$

$$= \min_{u_{N-2}} E\{(x_{N-1})^2\} + \sigma^2 \qquad (2)$$

Clearly, (2) is similar to (1), with a different time index.

$$\boxed{u^*_{N-2} = -\frac{a}{b} x_{N-2}} \qquad \boxed{J^*_{N-2} = 2\sigma^2}$$

At the general stage

$$\boxed{u^*_k = -\frac{a}{b} x_k} \qquad \boxed{J^*_k = (N-k)\,\sigma^2}$$

Comments:

1. the optimal control law is a state-feedback linear control, identical to that of the same deterministic problem, $w_k \equiv 0$.
2. the randomness of the system, because of w_k, affects the cost.

For N stages the total cost optimized: $\underline{\underline{N\sigma^2}}$.

3. That is logical : the cost is $\sum_{k=0}^{N-1} E\{x_{k+1}^2\}$

at each stage

$$x_{k+1} = x_k + w_k$$

$$\Downarrow$$

$$E\{x_{k+1}^2\} \geq E\{w_k^2\} = \sigma^2 .$$

Example 2

Given $x_{k+1} = a_k x_k + b u_k$

$\{a_k\}$ random variables, i.i.d. $E\{a_k\} = \mu$; $Var\{a_k\} = \sigma^2$

$$J = \sum_{k=0}^{N-1} x_{k+1}^2$$

Find the optimal control law $\{u_k^*\}$

Solution by D.P.

- Last Stage.

$$J_{N-1} = E\{x_N^2\} \quad (1)$$

$$= E\{(a_{N-1} x_{N-1} + b u_{N-1})^2\}$$

$$= E\{\underbrace{a_{N-1}^2}_{\text{random}} \underbrace{x_{N-1}^2}_{\text{determ.}} + \underbrace{b^2 u_{N-1}^2}_{\text{deterministic}} + 2 a_{N-1} b \underbrace{x_{N-1} u_{N-1}}_{\text{determin.}}\}$$

!

$$= E\{\cancel{a_{N-1}^2}\}\cdot x_{N-1}^2 + b^2 u_{N-1}^2 + 2E\{\cancel{a_{N-1}}\}\, b\, x_{N-1} u_{N-1}$$

$\mu^2 + \sigma^2$ $\qquad$ μ

$$= (\mu^2+\sigma^2)\, x_{N-1}^2 + \underbrace{b^2 u_{N-1}^2} + 2\mu b\, x_{N-1} u_{N-1}$$

$$\min_{u_{N-1}} J_{N-1} \longmapsto \min_{u_{N-1}} \left[(\mu x_{N-1} + b u_{N-1})^2 \right] + \sigma^2 x_{N-1}^2$$

By inspection : $\boxed{u_{N-1}^* = -\frac{\mu}{b} x_{N-1}}$ $\quad$ $\boxed{J_{N-1}^* = \sigma^2 x_{N-1}^2}$

- <u>Second-to-last stage.</u>

x_{N-2} known

$$J_{N-2} = w_{N-2} + J_{N-1}^*$$

$$= x_{N-1}^2 + \sigma^2 x_{N-1}^2$$

$$= (1+\sigma^2)\, x_{N-1}^2$$

$$\min_{u_{N-2}} E\{(1+\sigma^2)\, x_{N-1}^2\}$$

$$(1+\sigma^2) \min_{u_{N-2}} E\{x_{N-1}^2\} \qquad (2)$$

(2) similar to (1) $\rightarrow$ solved

$$\boxed{u^*_{N-2} = -\frac{\mu}{b} x_{N-2}} \qquad \boxed{J^*_{N-2} = (1+\sigma^2)\left[\sigma^2 x^2_{N-2}\right]}$$

- The general stage

$$\boxed{u^*_k = -\frac{\mu}{b} x_k}$$

$$\boxed{\begin{array}{l} J^*_k = \alpha_k x_k^2 \\ \alpha_k = (1+\alpha_{k+1})\sigma^2 \qquad \alpha_{N-1} = \sigma^2 \end{array}}$$

<u>Comments</u> : 1. the linear structure of the control law is preserved

yet the control gain is calculated using the *expected* value of the random parameter a_k. This is consistent with the previous finding : the control law is a deterministic algorithm.

2. The optimal cost value results from the randomness of a_k & ξ.

$$J_0^* = (\sigma^2 + \sigma^4 + \sigma^8 + \dots) x_0^2 \underset{(\sigma \to 0)}{\sim} \sigma^2 x_0^2 .$$

Example 3

Given $x_{k+1} = a_k x_k + b_k u_k + w_k$

$\{a_k\}$ sequence i.i.d r.v. $E\{a_k\} = \alpha$; $Var\{a_k\} = \sigma_a^2$

$\{b_k\}$ " i.i.d " " β; " σ_b^2

$\{w_k\}$ " " " 0 σ_w^2

$$J = E\left\{\sum_{k=0}^{N-1} x_{k+1}^2\right\}$$

Find the optimal control law $\{u_k^*\}$

Solution via D.P.

- Last Step.

$$\min_{u_{N-1}} E\{x_N^2\} \qquad (1)$$

$$\min_{u_{N-1}} E\left\{(a_{N-1}x_{N-1} + b_{N-1}u_{N-1} + w_{N-1})^2\right\}$$

$$\min_{u_{N-1}} E\left\{ a_{N-1}^2 x_{N-1}^2 + b_{N-1}^2 u_{N-1}^2 + w_{N-1}^2 + 2 a_{N-1} b_{N-1} u_{N-1} x_{N-1} + 2 a_{N-1} x_{N-1} w_{N-1} + 2 b_{N-1} u_{N-1} w_{N-1} \right\}$$

$$\min_{u_{N-1}} \Big[E\{\underbrace{\cancel{a_{N-1}^2}}_{\mu_a^2+\sigma_a^2}\} x_{N-1}^2 + E\{\underbrace{\cancel{b_{N-1}^2}}_{\mu_b^2+\sigma_b^2}\} u_{N-1}^2 + E\{\underbrace{\cancel{w_{N-1}^2}}_{\sigma^2}\} + 2E\{a_{N-1} b_{N-1}\} u_{N-1} x_{N-1} + 2E\{a_{N-1} w_{N-1}\} x_{N-1} + 2E\{b w_{N-1}\} u_{N-1} \Big]$$

independent: $E\{a_{N-1}\} \cdot E\{b_{N-1}\} = \mu_a \mu_b$

independent: $E\{a_{N-1}\} E\{\cancel{w_{N-1}}\} = 0$

independent: $E\{b\} E\{\cancel{w}\} = 0$

$$\min_{u_{N-1}} \left[(\mu_a^2 + \sigma_a)^2 x_{N-1}^2 + (\mu_b^2 + \sigma_b^2) u_{N-1}^2 + 2\mu_a \mu_b x_{N-1} u_{N-1} \right]$$

basic property

If (a, b) are independent then they are uncorrelated.

$$\mathrm{Cov}\{a, b\} = E\{(a-\alpha)(b-\beta)\}$$
$$= E\{ab - \alpha a - \beta b + \alpha\beta\}$$
$$= E\{ab\} - \alpha E\{a\} - \beta E\{b\} + \alpha\beta$$

$$= E\{ab\} - \alpha\beta - \cancel{\alpha\beta} + \cancel{\alpha\beta}$$

$$= \iint ab \; p(a,b) \, d(a,b) \quad - \alpha\beta$$

! (independent)

$$= \qquad p(a) \cdot p(b)$$

$$= \left(\int a \, p(a) \, da \right) \left(\int b \, p(b) \, db \right) \quad - \alpha\beta$$

$$= E\{a\} \; E\{b\} - \alpha\beta$$

$$= \alpha\beta - \alpha\beta$$

$$= 0 \qquad \therefore$$

$$\min_{u_{N-1}} \left[(\alpha^2 + \sigma_a)^2 x_{N-1}^2 + (\beta^2 + \sigma_b^2) u_{N-1}^2 + 2\alpha\beta\, x_{N-1} u_{N-1} \right] + \sigma_w^2$$

$$\min_{u_{N-1}} \left[(\beta^2 + \sigma_b^2) u_{N-1}^2 + 2\alpha\beta\, x_{N-1} u_{N-1} \right] + (\alpha^2 + \sigma_a^2) x_{N-1}^2 + \sigma_w^2$$

$$\Rightarrow \quad \boxed{u_{N-1}^* = -\frac{\alpha\beta}{(\beta^2 + \sigma_b^2)} x_{N-1}} \qquad \boxed{J_{N-1}^* = C_{N-1}\, x_{N-1}^2 + \sigma_w^2}$$

By substitution

$$J_{N-1}^* = \underbrace{\left(\alpha^2 + \sigma_a^2 + \frac{\alpha^2\beta^2}{\beta^2 + \sigma_b^2} - \frac{2\alpha^2\beta^2}{\sigma_b^2 + \beta^2} \right)}_{C_{N-1}} x_{N-1}^2 + \sigma_w^2 =$$

$$= \alpha^2 + \sigma_a^2 - \frac{(\alpha\beta)^2}{\sigma_b^2 + \beta^2} = C_{N-1} \cdot x_{N-1}^2 + \sigma_w^2$$

- <u>2nd to last</u>

$$J_{N-2} = w_{N-2} + J^*_{N-1}$$
$$= E\{ x_{N-1}^2 + J^*_{N-1} \}$$
$$= E\{ x_{N-1}^2 + C_{N-1} x_{N-1}^2 + \sigma_w^2 \}$$
$$= (1 + C_{N-1}) E\{ x_{N-1}^2 \} + \sigma_w^2$$

$$\min_{u_{N-2}} J_{N-2} \Leftrightarrow \min_{u_{N-2}} E\{ x_{N-1}^2 \} \qquad (2)$$

(2) is similar to (1)

$$u^*_{N-2} = -\frac{\alpha\beta}{\beta^2 + \sigma_b^2} x_{N-2}$$

the general

- the general case:

$$\boxed{u_k^* = -\frac{\alpha\beta}{\beta^2+\sigma_b^2}\, x_k}$$

$$\boxed{\begin{aligned} J_k^* &= c_k x_k^2 + d_k \\ &\begin{cases} c_{N-1} = c \\ d_{N-1} = \sigma_w^2 \end{cases} \\ c_k &= (1 + c_{k+1})\, c \\ d_k &= (1 + c_{k+1})\, \sigma_w^2 + d_{k+1} \end{aligned}}$$

Comments:

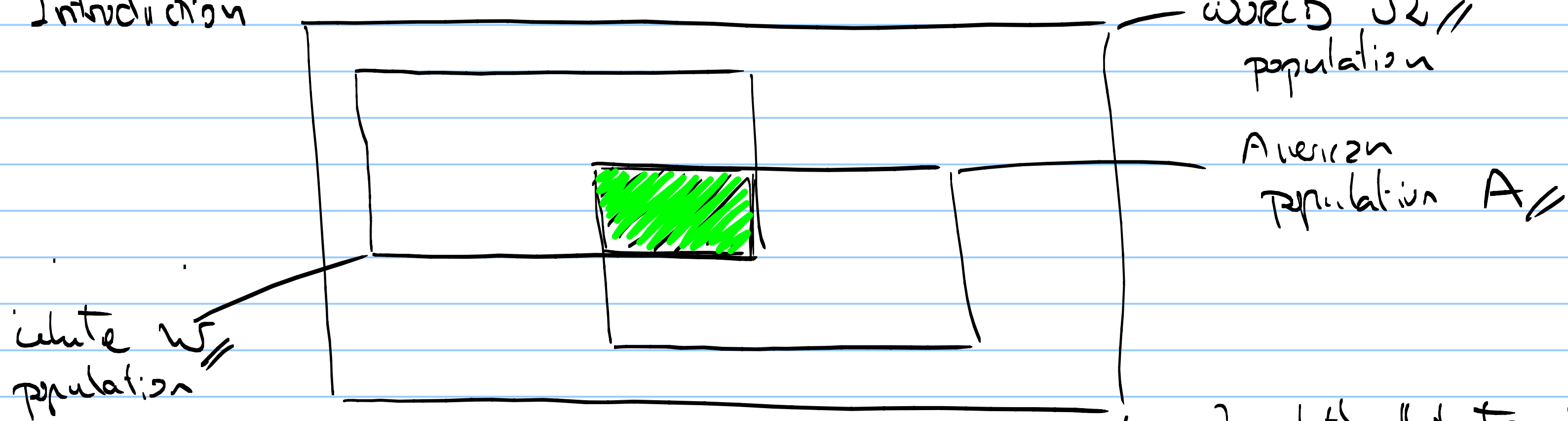
1. the same structure is preserved.

$$u_k^* = -\frac{E\{a_k\}}{\left[E\{b_k^2\} / E\{b_k\}\right]}\, x_k$$

2. $\sigma_b = \sigma_w = 0 \Rightarrow$ example 2.

B.1.4. Estimation

B.1.4.1 Introduction



$$P(W) = \frac{W}{\Omega} \quad (1)$$

$$P(A) = \frac{A}{\Omega} \quad (2)$$

$$P(\text{white}, \text{american}) = \frac{A \cap W}{\Omega} \quad (3)$$

$$P(\text{white} / \text{american}) = \frac{A \cap W}{A} \quad (4)$$

let's illustrate the definition of conditional probability

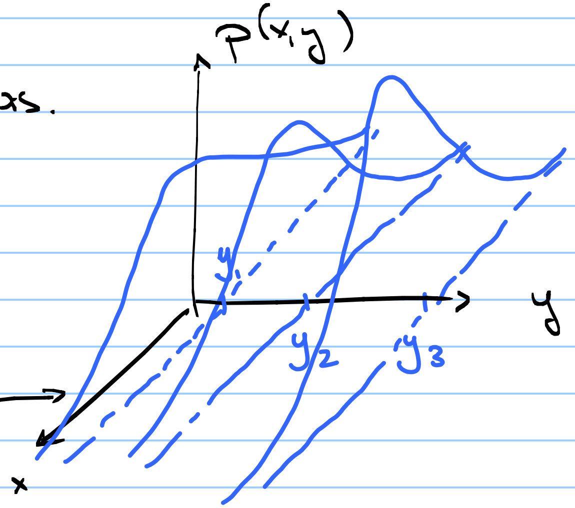
$$P(W/A) = \frac{P(A,W)}{P(A)} = \frac{(A \cap W)/\Omega}{A/\Omega} = \frac{A \cap W}{A}$$

As we see the probability $P(W/A) = \frac{A \cap W}{A}$ is necessarily greater $P(W,A) = \frac{A \cap W}{\Omega}$.
because $A \subseteq \Omega$.

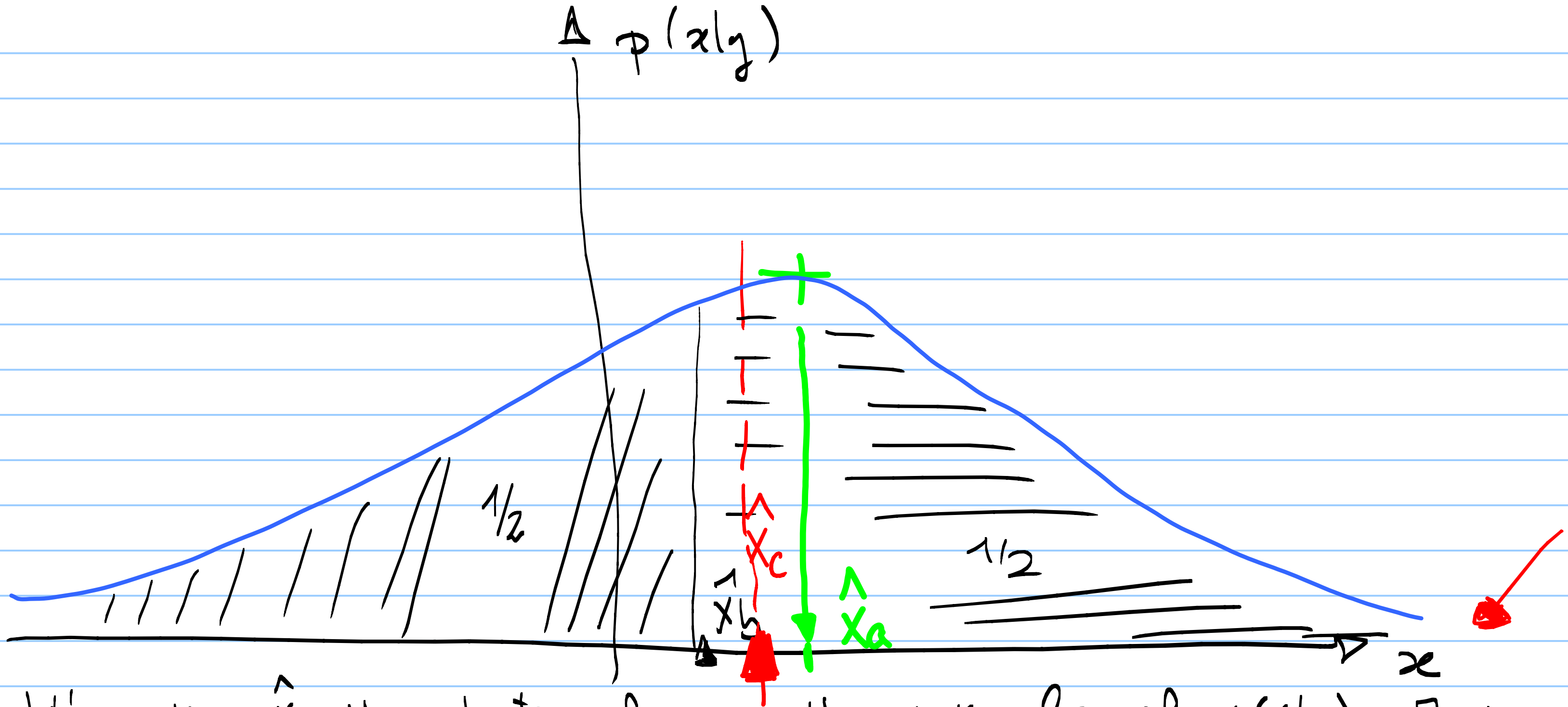
- The structure of conditional probability density functions.

example:

$$p(x,y) = \underbrace{p(y)}_{p(y_1)} \cdot p(x/y)$$ proportional to $p(x|y_1)$



- If $p(x/y)$ is known, while x is unknown, and we want to find a guess for x, then



Option a: let's pick $\hat{x}_a$ the estimate of x as the peak value of p(x|y): Maximum LIKELIHOOD estimate

Option b: $\hat{x}_b$ median estimate

Option c: let's pick $\hat{x}_c$ such that the Variance is minimized!

$$\min_{\hat{x}_c} J = \min_{\hat{x}_c}\left[E\{(x-\hat{x}_c)^2 \mid y\}\right]$$

<u>Solution:</u>

$$J = E\{(x-\hat{x})^2 \mid y\}$$

$$= \int_{-\infty}^{+\infty} (x-\hat{x})^2\, p(x|y)\, dx$$

$$= \int_{-\infty}^{+\infty} x^2 p(x|y)\,dx - 2\int_{-\infty}^{+\infty} x\hat{x}\, p(x|y)\,dx + \int_{-\infty}^{+\infty} \hat{x}^2 p(x|y)\,dx$$

$$= \underbrace{\int_{-\infty}^{+\infty} x^2 p(x|y)\,dx}_{\equiv E\{x^2|y\}} - 2\hat{x}\underbrace{\int_{-\infty}^{+\infty} x\,p(x|y)\,dx}_{\equiv E\{x|y\}} + \hat{x}^2 \underbrace{\int_{-\infty}^{+\infty} p(x|y)\,dx}_{=1}$$

$$=$$

pr. density $\int$!

$$= E\{x^2|y\} - 2\hat{x}\, E\{x|y\} + \hat{x}^2$$

$$\min_{\hat{x}} \left[\hat{x}^2 - 2E\{x|y\}\,\hat{x} \right] + E\{x^2|y\}$$

$$\min_{\hat{x}} \underbrace{\left[\hat{x} - E\{x|y\}\right]^2} \underbrace{- E^2\{x|y\} + E\{x^2|y\}}$$

by inspection

$$\hat{x}_c = E\{x|y\} \qquad J_c = E\{x^2|y\} - E^2\{x|y\}$$

Conditional expectation of the unknown r.v. x
Given the "measured" r.v. y

$$= \boxed{Var\{x|y\}}$$