Note Title 5/23/2021

B.2.2. Itô Stochastic Differential, Itô's Differentiation Rule

Theorem: Let x_t be the unique solution of the Itô SDE

$$dx_t = f(x_t, t)\,dt + G(x_t, t)\,d\beta_t$$

where β_t is a Brownian Motion (BM), $E\{d\beta_t \, d\beta_t^T\} = Q\,dt$

Let $\phi(x_t, t)$ be a scalar function of x_t, $\frac{\partial \phi}{\partial t}$, $\frac{\partial \phi}{\partial x}$, $\frac{\partial^2 \phi}{\partial x \partial x^T}$ continuous,

$$(2) \qquad d\phi = \frac{\partial \phi}{\partial t}\,dt + \frac{\partial \phi^T}{\partial x}\,dx_t + tr\left[GQG^T \frac{\partial^2 \phi}{\partial x \partial x^T}\right] dt$$

Discussion:

Taylor Series $\quad d\phi = \frac{\partial\phi}{\partial t}dt + \frac{\partial\phi}{\partial x^T}dx_t + \frac{1}{2}\underline{dx_t^T\frac{\partial^2\phi}{\partial x\partial x^T}dx_t} + HOT$

$$= tr\left(dx\, dx^T \frac{\partial^2\phi}{\partial x\partial x^T}\right)$$

(1) into $d\phi$

$$d\phi = \frac{\partial\phi}{\partial t}dt + \frac{\partial\phi^T}{\partial x}dx_t + \frac{1}{2}tr\left[G\, d\beta_t\, d\beta_t^T G^T \frac{\partial^2\phi}{\partial x\partial x^T}\right] + \dots$$

By integration between $t=a$ and $t=b$

$$\int_a^b d\phi = \int_a^b \frac{\partial\phi}{\partial t}dt + \int_a^b \frac{\partial\phi}{\partial x^T}dx_t + \frac{1}{2}tr\left[\int_a^b G\, d\beta_t\, d\beta_t^T\, G^T\frac{\partial^2\phi}{\partial x\partial x^T}\right]$$

<u>Reminder</u> : $\int_a^b g(x_t,t)\, d\beta_t^2$ is called a 2^{nd} order Itô integral

$$\overset{a.e.}{=} \int_a^b g(x_t,t)\, q\, dt \longleftarrow \text{if } \beta_t \text{ BM}, E\{d\beta_t^2\} = q\,dt$$

$$\phi(b) - \phi(a) = 0 + 0 + \frac{1}{2}\,\mathrm{tr}\int_a^b G\,(Q\,dt)\,G^T \frac{\partial^2\phi}{\partial x\,\partial x^T}$$

$$+ \frac{1}{2}\,\mathrm{tr}\left(\int_a^b GQG^T \frac{\partial^2\phi}{\partial x\,\partial x^T}\,dt\right)$$

$$d\phi = \frac{\partial\phi}{\partial t}\,dt + \frac{\partial\phi}{\partial x^T}\,dx_t + \frac{1}{2}\,\mathrm{tr}\left(GQG^T\frac{\partial^2\phi}{\partial x\,\partial x^T}\right).\,dt$$

Example 1

assume $\quad dx_t = \underbrace{\frac{\sigma^2}{2}x_t}\,dt + \underbrace{x_t}\,d\beta_t \quad ; \quad \beta_t$ BM, $E\{d\beta_t^2\} = \sigma^2\,dt$

Let $\quad \phi = x_t^2$, what is $d\phi$?

$$\frac{\partial \Phi}{\partial t} = 0, \qquad \frac{\partial \Phi}{\partial x} = 2x, \qquad \frac{\partial^2 \Phi}{\partial x^2} = 2 \quad \Rightarrow \quad d\Phi = (2x_t)\, dx_t + \sigma^2 x_t^2\, dt$$

$$= 2\sigma^2 x_t^2\, dt + 2 x_t^2\, d\beta_t$$

$$d\Phi = 2\sigma^2 \Phi_t\, dt + 2\Phi_t\, d\beta_t \quad \Leftarrow \quad d\Phi = f(\Phi)\, dt + g(\Phi)\, d\beta_t$$

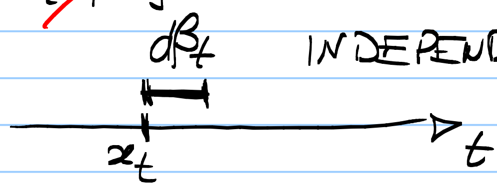
Conclusion: Itô SDE for Φ!

Let's develop $C_{xx} \equiv E\{x_t^2\}$

$$E\{d\Phi\} = 2\sigma^2 E\{\Phi_t\}\, dt + 2 E\{\Phi_t\, d\beta_t\}$$

$$d\, E\{\Phi_t\} = 2\sigma^2 E\{\Phi_t\}\, dt + 2 E\{\Phi_t\} \cdot \cancel{E\{d\beta_t\}} \rightarrow 0$$

! $d\beta_t$ and x_t are statistically INDEPENDENT



$$dC_{xx} = 2\sigma^2 C_{xx}\, dt$$

$$\dot{C}_{xx} = 2\sigma^2 C_{xx} \; ; \quad C_{xx}(0)$$

$$\Rightarrow \quad C_{xx}(t) = C_{xx}(0)\, e^{(2\sigma^2 t)}$$ grows unbounded.

Example.

Assume $dx_t = F x_t \, dt + G \, d\beta_t$; β_t BM, $E\{d\beta_t \, d\beta_t^T\} = Q \, dt$

what is $C_{xx} = E\{x_t x_t^T\}$?

since $C_{xx} = \begin{bmatrix} & \vdots & \\ \cdots & E\{x_i x_j\} & \\ & & \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & E\{\phi_{ij}\} & \\ & & \end{bmatrix}$ $\quad \phi_{ij} = x_i x_j$ scalar function of x

$\frac{\partial \phi_{ij}}{\partial t} = 0 \qquad \frac{\partial \phi_{ij}}{\partial x^T} = \begin{bmatrix} 0 \ldots \underset{i}{x_j} \ldots \underset{j}{x_i} \ldots 0 \end{bmatrix}$

$\Phi = XX^T \iff C_{xx} = E\{\Phi\}$

#1 / develop an SDE for $\phi_{ij} \longrightarrow \Phi$

#2 / apply the $E\{\cdot\}$ $\longrightarrow$ obtain an ODE for C_{xx}.

$$\boxed{\dot{C}_{xx} = F C_{xx} + C_{xx} F^T + GQG^T \qquad C_{xx}(0)}$$

LYAPUNOV equation

- The Hamilton-Jacobi-Bellman equation: Sufficient Condition for Optimality

  - Itô Integration FORMULA

Given $\phi(t, x)$ scalar function of x_t, one can write $t \leq s$

$$\phi(s, x_s) = \phi(t, x_t) + \int_t^s \frac{\partial \phi}{\partial \tau}(\tau, x_\tau) + \underbrace{\mathcal{L}[\phi(\tau, x_\tau)]}\, d\tau + \int_t^s \frac{\partial \phi}{\partial x^T} G(\tau, x_\tau)\, d\beta_\tau$$

$$\equiv \frac{\partial \phi}{\partial x^T} f(\tau, x_\tau) + \frac{1}{2} \operatorname{tr}\left(G(\tau, x_\tau) Q G^T(\tau, x_\tau) \frac{\partial^2 \phi}{\partial x_\tau \partial x_\tau^T} \right)$$

the Differential generator

- Dynamic Programming for Continuous-time MARKOV processes

  - Problem Statement

Find a control law $u(\cdot, \cdot) \in \mathcal{U}$ that minimizes the cost

time information

$$J(x_0, t) = E\left\{ \int_t^{t_f} h(x, u, \tau)\, d\tau + g(x(t_f)) \right\}$$

subject to

$$dx_\tau = f(x_\tau, u_\tau, \tau)\, d\tau + G(x_\tau, u_\tau, \tau)\, d\omega_\tau \quad ; \quad x(t) = x_0$$

where

ω_τ B.M. $\quad E\{ d\omega_\tau \, d\omega_\tau^T \} = W d\tau$

Assumptions :

1. Full State Information
2. The Class of admissible Controllers satisfy Growth and Lipschitz prop. (identical to the Deterministic Case)
3. State and Cost are bounded (see deterministic Case)
4. $\gamma(t, t_f) = \{ u(x, \sigma) \mid x \in \mathbb{R}^n, \ t \le \sigma \le t_f \}$

state : feedback control

Remark ! the problem of minimizing the Unconditional expectation is equivalent to the Minimization of :

$$E\left\{ \int_t^{t_f} L(x, u, \tau)\, d\tau + g[x(t_f)] \,\Big|\, x(t) = x_0 \right\}$$

<u>why</u> .

$$E\{A\} \equiv E\{E\{A \mid x\}\}$$

$$\min_{u \in \mathcal{U}} E\{A(u)\} = E\{\min_{u \in \mathcal{U}} \underbrace{E\{A(u) \mid x(t) = x_0\}}_{\text{because}}\}$$

▶ Optimal Return Function

<u>definition</u> it is the Value of the Cost Function when applying the optimizing Control

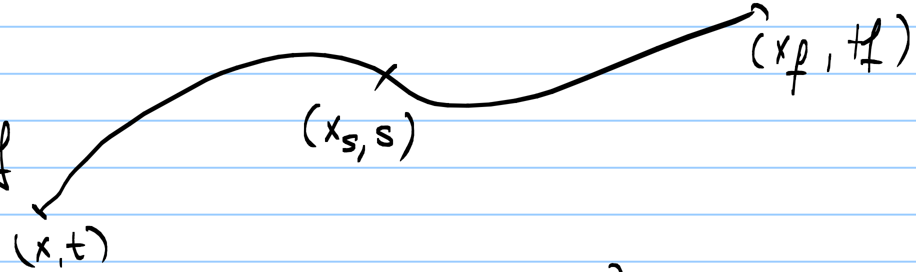
$$J^{o}(x,t) = \min_{\gamma(t,t_f) \in \mathcal{U}} J[\gamma(t,t_f); x, t]$$

$$= \min_{\gamma(t,t_f) \in \mathcal{U}} E\left\{ \int_t^{t_f} L(x,u,\tau)\, d\tau + g[x(t_f)] \,\middle|\, x(t) = x \right\}$$

$$= J[\gamma^{o}(t,t_f); x, t]$$

The H-J-B equation

Assume that $\gamma^1(t, t_f) = \begin{cases} u(x,\tau) & t \le \tau \le s \\ \gamma^0(s, t_f) & s \le \tau \le t_f \end{cases}$



$$J\left[\gamma^1(t,t_f); x, t\right] = E\left\{ \int_t^{t_f} L(x_\tau, \gamma^1_\tau, \tau)\, d\tau + g[x(t_f)] \;\Big|\; x(t) = x \right\}$$

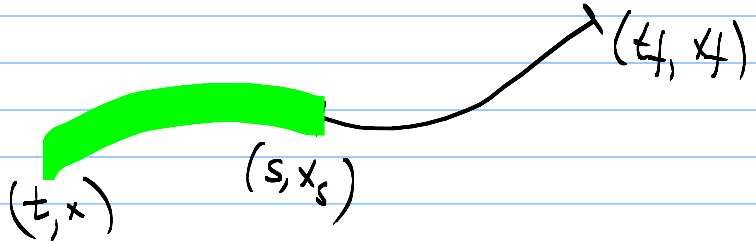
$$= E\left\{ \int_t^{s} L(x_\tau, u_\tau, \tau)\, d\tau + \int_s^{t_f} L(x_\tau, \gamma^0_\tau, \tau)\, d\tau + g[x(t_f)] \;\Big|\; x(t) = x \right\}$$

$$= E\left\{ \int_t^{s} \right\} + E\left\{ \int_s^{t_f} \quad + \quad \right\}$$

$$\boxed{J\left[\gamma^1(t,t_f); x, t\right] = E\left\{ \int_t^{s} L(x_\tau, u_\tau, \tau)\, d\tau \;\Big|\; x(t) = x \right\} + E\left\{ \int_s^{t_f} L(x_\tau, \gamma^0_\tau, \tau)\, d\tau + g[\;] \;\Big|\; x \right\}}$$

• Apply Itô Integration Formula to $\phi(t,x) \equiv J^o(t,x)$ the Optimal Return Function

$$\phi(s,x_s) = \phi(t,x_t) + \int_t^s \frac{\partial\phi}{\partial\tau}(\tau,x_\tau) + \underbrace{\mathcal{L}[\phi(\tau,x_\tau)]}_{} \, d\tau + \int_t^s \frac{\partial\phi}{\partial x^T} G(\tau,x_\tau)\, d\beta_\tau$$



$$\equiv \frac{\partial\phi}{\partial x^T} f(\tau,x_\tau) + \frac{1}{2} tr\left(G(\tau,x_\tau) Q G^T(\tau,x_\tau) \frac{\partial^2\phi}{\partial x_\tau \partial x_\tau^T} \right)$$

the Differential generator

$$\phi(t,x_t) = \phi(s,x_s) - \int_t^s \frac{\partial\phi}{\partial\tau}(\tau,x_\tau) + \underbrace{\mathcal{L}[\phi(\tau,x_\tau)]}_{} \, d\tau + \int_t^s \frac{\partial\phi}{\partial x^T} G(\tau,x_\tau)\, d\beta_\tau$$

$$\equiv \frac{\partial\phi}{\partial x^T} f(\tau,x_\tau) + \frac{1}{2} tr\left(G(\tau,x_\tau) Q G^T(\tau,x_\tau) \frac{\partial^2\phi}{\partial x_\tau \partial x_\tau^T} \right)$$

$$E\{ \cdot \mid x(t) = x \}$$

$$\phi(t,x) = J^o(x,t) = -E\left\{\int_t^s \frac{\partial J^o(x_\tau,\tau)}{\partial \tau} + \mathcal{L}^u[J^o(x_\tau,\tau)]\,d\tau \,\middle|\, x(t)=x\right\} + E\{\phi(s,x_s)\,|\,x(t)=x\}$$

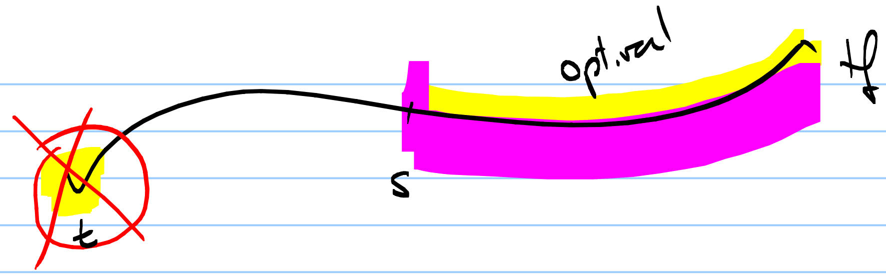
$$+ E\{J^o(s,x_s)\,|\,x(t)=x\}$$

$$\mathcal{L}^u[J^o(x_\tau,\tau)] = \frac{\partial\phi}{\partial x^T} f(x,u,\tau) + \frac{1}{2} tr\left(G(x,u,\tau)\,Q\,G^T(x,u,\tau)\frac{\partial^2\phi}{\partial x_\tau \partial x_\tau^T}\right)$$

$$J^o(x,t) = -E\left\{\int_t^s \frac{\partial J^o}{\partial \tau} + \mathcal{L}^u[J^o]\,d\tau \,\middle|\, x(t)=x\right\} + \underset{1}{E}\{J^o(s,x_s)\,|\,x(t)=x\}$$

$$J[\gamma(t,t_f),x,t] = E\left\{\int_t^s L(x_\tau,u_\tau,\tau)\,d\tau \,\middle|\, x(t)=x\right\} + \underset{2}{E}\left\{\int_s^{t_f} L(x_\tau,\gamma^o_\tau,\tau)\,d\tau + g[\,]\,\middle|\,x\right\}$$

? Is 1 = 2 ?



$$E\left\{ \int_s^{t_f} L(x_\tau, u^o, \tau)\, d\tau + g[x_{t_f}] \,\middle|\, x(t) = x \right\}$$

thanks to the MARKOV property of $x(t)$

$$= E\left\{ E\left\{ \int_s^{t_f} L(x_\tau, u^o, \tau)\, d\tau + g[x_{t_f}] \,\middle|\, \underbrace{\cancel{x(t) = x}, x(s) = x_s}_{B} \right\} \,\middle|\, \underbrace{x(t) = x}_{A} \right\}$$

$$= E\left\{ \underbrace{E\left\{ \int_s^{t_f} L(x_\tau, u^o_\tau, \tau)\, d\tau + g[x_{t_f}] \,\middle|\, \underline{x(s) = x_s} \right\}} \,\middle|\, x(t) = x \right\}$$

$\equiv$ the Optimal Return Function at time s, $x(s) = x_s$

$$J^o(s, x_s)$$

$$= E\left\{ J^{o}(s, x_s) \;\middle|\; x(t) = x \right\}$$

Iff $\qquad J^{o} - J^{\wedge} \leq 0$ (then γ^{o} Optimal)

Iff $\qquad 0 \leq J[\gamma'(t, I_f); x, t] - J^{o}(x, t) \qquad \forall \gamma' \in \mathcal{U}$

$$\text{Iff} \quad 0 \leq E\left\{ \int_t^s \frac{\partial J^{o}}{\partial \tau} + \mathcal{L}^{u}[J^{o}] + L(x_\tau, u_\tau, \tau) \;\middle|\; x(t) = x \right\} \qquad \forall\, t_0 \leq t \leq t_f$$

A sufficient Condition:

$$0 \leq \frac{\partial J^{o}}{\partial \tau} + \mathcal{L}^{o}[J^{o}] + L(x_\tau, u_\tau, \tau) \qquad \forall u \in \mathcal{U}, \; \forall t_0 \leq t \leq t_f$$

Equality takes places if (only if) $u = \gamma^{o}(\tau, I_f)$, hence:

$$
\begin{cases}
-\dfrac{\partial J^\circ(x_\tau,\tau)}{\partial \tau} = \min\limits_{u \in \mathcal{U}} \left[\mathcal{L}^u \left[J^\circ(x_\tau,\tau) \right] + L(x_\tau, u_\tau, \tau) \right] & t_0 \leq \tau \leq t_f \\
\text{with the Boundary Condition: } \quad J^\circ(x_{tf}, t_f) = g[x_{tf}]
\end{cases}
$$

this is the Hamilton - Jacobi - Bellman equation: the Sufficient Condition For Optimality

B.2.3 The LQG problem with Complete Information

We specialize the HJB equation to the case of Linear systems and Quadratic Costs.

- Problem Statement

Given $dx_\tau = [F(\tau)x(\tau) + B(\tau)u]d\tau + G(\tau)dw_\tau$ (1)

where w_τ is a B.M. with spectral density matrix W

Cost : $J[\gamma(0,t_f)] = E\left\{ \int_0^{t_f} \|x_\tau\|^2_{Q(\tau)} + \|u_\tau\|^2_{R(\tau)} d\tau + \|x_{t_f}\|^2_{S_f} \right\}$

where $R(\tau) = R^T(\tau) > 0$, $Q(\tau) = Q^T(\tau) \geq 0$; $S_f \geq 0$ $\forall\, 0 \leq \tau \leq t_f$

We seek to solve

$$\min_{\gamma(0,t_f)\in\mathcal{U}} J[\gamma(0,t_f)] \quad \text{s.t. } (1)$$

- Solution

In order to solve the HJB equation

Idea : Guess the structure of J^o

$$J^o = \|x\|^2_{S(t)} + \alpha(t)$$

$S(t) = S^T(t)$ with no loss of generality.

Development of sufficient conditions on $S(t)$ and $\alpha(t)$.

$$0 = \frac{\partial J^o(x_\tau, \tau)}{\partial \tau} + \min_{u \in \mathcal{U}} \Big[\underbrace{\mathcal{L}^u [J^o(x_\tau, \tau)]}_{\frac{\partial J^o}{\partial x^T}(x_\tau,\tau).(Fx + Bu) + \frac{1}{2}\mathrm{tr}\left[GWG^T \frac{\partial^2 J^o}{\partial x \partial x^T}\right]} + \underbrace{L(x_\tau, u, \tau)}_{\|x\|^2_{Q(\tau)} + \|u\|^2_{R(\tau)}} \Big]$$

$$\begin{cases} \dfrac{\partial J^0}{\partial x} = \dfrac{\partial}{\partial x}\left[x^T S(\tau)\, x + \alpha(\tau) \right] = \left[S(\tau) + S^T(\tau) \right] x = 2 S_\tau x \\[2ex] \dfrac{\partial^2 J^0}{\partial x \partial x^T} = 2 S_\tau \\[2ex] \dfrac{\partial J^0}{\partial \tau} = \dfrac{\partial}{\partial \tau}\left[x^T S(\tau)\, x + \alpha(\tau) \right] = x^T \dot{S}(\tau)\, x + \dot{\alpha}(\tau) \end{cases}$$

$$0 = \left(x^T \dot{S} x + \dot{\alpha} \right) + \min_u \left[2 x^T S \left(F x + B u \right) + \operatorname{tr}\left(G W G^T S \right) + \|x\|_Q^2 + \|u\|_R^2 \right]$$

We seek to minimize wrt u the following:

$$2 x^T S (B u) + \|u\|_R^2$$

$$\boxed{u^0(x,\tau) = -R_\tau^{-1} B_\tau^T S_\tau x}$$

Again we obtain a linear state-feedback control law.

Substituting the expression for u^o into the differential equation :

$$0 = x^T \dot{S} x + \dot{\alpha} + 2 x^T S F x$$

$$x^T (F^T S + S F) x + x^T Q x - x^T S B R^{-1} B^T S x + tr (G W G^T S)$$

$$0 = x^T \underbrace{(\dot{S} + F^T S + S F - S B R^{-1} B^T S + Q)}_{=0} x + \underbrace{\dot{\alpha} + tr (G W G^T S)}_{=0} \qquad \forall \tau \in [0, t_f] \quad \forall x \in \mathbb{R}^n$$

The Sufficient Condition is then :

$$\begin{cases} -\dot{S} = F^T S + S F - S B R^{-1} B^T S + Q \ ; \quad S(t_f) = S_f \\ -\dot{\alpha} = tr (G W G^T S) \ ; \quad \alpha(t_f) = 0 \end{cases}$$

Hence, the optimal Return Function : $J^o(x,t) = \|x\|^2_{S(t)} + \int_t^{t_f} tr (G_\tau W G_\tau^T S_\tau) \, d\tau$

Verification: $J^{o}(x_{tf}, tf) = \|x_{tf}\|^2_{S(tf)} = \|x_{tf}\|^2_{Sf} \equiv g(x_{tf})$ ✓

Concluding Remarks

1. The Optimal solution to the LQG in continuous-time with full state information is a LINEAR State Feedback Law.

2. the optimal gain $M(\tau) = -R_\tau^{-1} B_\tau^T S_\tau$ is obtained from the Backward RICCATI Differential equation for the S_τ with a boundary condition at tf. That equation is identical to the deterministic LQ problem!

decrease cost

$$\left[-\dot{S}_\tau\right] = F_\tau^T S_\tau + S_\tau F_\tau + \underbrace{Q_\tau}_{\text{increase cost}} - \underbrace{S_\tau B_\tau R_\tau^{-1} B_\tau^T S_\tau}_{\text{optimally}} \ ; \ S(tf) = Sf$$

Certainty Equivalence Principle holds : cancel the Noise and Solve for the Optimal Controller w/o impeding the stochastic case.

3. The Overall Cost, the Optimal Return Function at $t = 0$, is the Sum of the Cost of the LQ pb and the "Cost of the Randomness"

$$J^{0}(x_0, 0) = \| \underbrace{x_0}_{\text{given}} \|^{2}_{S(0)} + \int_0^{t_f} \mathrm{tr}\left(G \underbrace{W}_{\text{Process Noise Intensity}} G^{T} S_{\tau} \right) d\tau$$

B.2.4 The LQG problem with partial information

- **Problem Statement**

Given

$$dx_\tau = (F_\tau x_\tau + B_\tau u_\tau)\, d\tau + G_\tau \, dw_\tau$$

w_τ Brown Motion, W
process noise

$$dz_\tau = H_\tau x_\tau \, d\tau + d\vartheta_\tau$$

ϑ_τ Brownian Motion, V
measurement noise

The measurement history $Z_\tau = \{ z(s) : 0 \leq s \leq \tau \}$

The Control law

$$\gamma(0, t_f) = \{ u = u(\tau, Z_\tau), 0 \leq \tau \leq t_f \}$$

The objective

$$\min_{\gamma(0,t_f) \in \mathcal{U}} \left[J(\gamma(0,t_f)) = E \left\{ \int_0^{t_f} \|x_\tau\|^2_{Q_\tau} + \|u_\tau\|^2_{R_\tau} \, d\tau + \|x_{t_f}\|^2_{S_f} \right\} \right]$$

- Solution

It seems to be complicated ... because the expectation is now taken with also respect to the measurements.

Here is the Idea  to Recast the Problem as one with Full Information.

with respect to the state estimate as calculated by a Kalman Filter

- Let's see first how the Kalman Filter looks like.

$\hat{x}_\tau$ = state estimate of x_τ given Z_τ.

$$d\hat{x} = \underbrace{(F_\tau \hat{x} + B_\tau u)\, d\tau}_{\text{drift term}} + \underbrace{K_\tau}_{\text{Kalman Gain}} \underbrace{d\tilde{Z}_\tau}_{\text{White Noise}} \quad \text{where} \quad \underbrace{d\tilde{Z}_\tau}_{\text{error in measurement prediction}} = \overbrace{dZ_\tau}^{\text{measurement}} - \overbrace{H_\tau \hat{x}_\tau d\tau}^{\text{prediction of measurement}}$$

this is an ISDE !

It can be shown that $\tilde{Z}$ is a B.M. with Intensity Matrix V !!!!

- In order to recast the original partial information problem into a full information problem wrt the estimate $\hat{x}$:

$$J[r(0,t_f)] = E\left\{ \int_0^{t_f} \|x\|_Q^2 + \|u\|_R^2 \, d\tau + \|x_{t_f}\|_{S_f}^2 \right\}$$

$$= E\left\{ E\left\{ \int_0^{t_f} \|x\|_Q^2 + \|u\|_R^2 \, d\tau \,\Big|\, Z_0 \right\} + E\left\{ \|x_{t_f}\|_{S_f}^2 \,\Big|\, Z_{t_f} \right\} \right\}$$

$$= E\left\{ \int_0^{t_f} E\left\{ \|x\|_Q^2 + \|u\|_R^2 \,\Big|\, Z_\tau \right\} d\tau + E\left\{ \|x_{t_f}\|_{S_f}^2 \,\Big|\, Z_{t_f} \right\} \right\}$$

Remember

$$x = \hat{x} + \tilde{x} \qquad \tilde{x} : \text{ estimation error}$$

$$\bullet\; E\{\|x_\tau\|^2_{Q_\tau} \mid Z_\tau\} = E\{\|\hat{x}_\tau + \tilde{x}_\tau\|^2_{Q_\tau} \mid Z_\tau\}$$

$$= E\{\|\hat{x}_\tau\|^2_{Q_\tau} + \|\tilde{x}_\tau\|^2_{Q_\tau} + 2\hat{x}_\tau^T\tilde{x}_\tau \mid Z_\tau\}$$

function of Z_τ

$$= E\{\|\hat{x}_\tau\|^2_{Q_\tau} \mid Z_\tau\} + E\{\|\tilde{x}_\tau\|^2_{Q_\tau} \mid Z_\tau\} + 2E\{\hat{x}_\tau^T\tilde{x}_\tau \mid Z_\tau\}$$

$$= \|\hat{x}_\tau\|^2_{Q_\tau} + E\{\tilde{x}^TQ\tilde{x}/Z\}$$

$$E\{tr(\tilde{x}^TQ\tilde{x})/Z\}$$

$$E\{tr(\tilde{x}\tilde{x}^TQ)/Z\}$$

$$tr(E\{x\tilde{x}^TQ/Z\})$$

$$tr(\underline{E\{\tilde{x}\tilde{x}^T/Z\}}Q)$$

P_τ: Covariance Matrix of $\tilde{x}_\tau$ $\quad tr(P_\tau Q_\tau)$ as calculated by the KF.

$$\text{so}\quad E\{\|x\|^2_Q \mid Z\} = \|\hat{x}_\tau\|^2_{Q_\tau} + tr(P_\tau Q_\tau) + 2E\{\cancel{\hat{x}_\tau^T \tilde{x}_\tau} \mid Z_\tau\}\ 0$$

?

Fundamental Property of Optimal Est.

$$E\{f(Z)\tilde{x}^T \mid Z\} = f(Z).E\{\tilde{x}^T \mid Z\}$$

$$E\{(x-\hat{x})^T \mid Z\}$$

$$E\{x^T \mid Z\} - \hat{x}^T$$

$$\hat{x}^T - \hat{x}^T = 0 \;!$$

$$E\{f(Z)\tilde{x}^T / Z\} = 0$$

$$E\{\hat{x}\tilde{x}^T \mid Z\} = 0$$

$$E\{\hat{x}^T\tilde{x} \mid Z\} = 0$$

Hence $\quad J[\gamma(0,t_f)] = E\{\ \square\ \}$

$$\square = \underbrace{\int_0^{t_f} \|\hat{x}_\tau\|^2_{Q_\tau} + \|u_\tau\|^2_{R_\tau}\, d\tau + \|\hat{x}_{t_f}\|^2_{S_f}}_{\text{Quadratic Cost in } \hat{x}} + \underbrace{\int_0^{t_f} tr(P_\tau Q_\tau)\, d\tau + tr(P_{t_f} S_f)}_{\text{independent of the Control!}}$$

Re-statement of the Optimal Control Problem

Gain Matrix

"process noise" for $\hat{x}$; V intensity

Given $\quad d\hat{x} = (F\hat{x} + Bu)\,d\tau + K\,d\tilde{z} \quad$ where $d\tilde{z} = dz - H\hat{x}\,d\tau$

Cost $\quad \square$

Find $\quad u$ that minimizes the cost under full information on $\hat{x}$.

Conclusion: By inspection, applying the Full Information LQ algorithm, yields the Optimal Controller:

$$u^o = -R_\tau^{-1} B_\tau^T S_\tau \hat{x}_\tau$$

the Backward Riccati Diff. eq.

$$\begin{cases} -\dot{S} = F^T S + SF + Q - SBR^{-1}B^T S \, ; \quad S(t_f) = S_f \\ -\dot{\alpha} = tr(KVK^T S) \, ; \quad \alpha(t_f) = 0 \end{cases}$$

the Forward Riccati Diff. eq.

$$\begin{cases} \dot{P} = FP + PF^T + W - PH^T V^{-1} HP ; \quad P_0 \\ K = PH^T V^{-1} \\ \dot{\hat{x}} = F_\tau \hat{x}_\tau + B_\tau u_\tau^o + K_\tau (z_\tau - H_\tau \hat{x}) ; \quad \hat{x}_0 \end{cases}$$

ISDE $\Longleftrightarrow$

Cost-to-go From t to t_f.

$$J^o(\hat{x}, t) = \|\hat{x}\|^2_{S_t} + \int_t^{t_f} K_\tau V_\tau K_\tau^T S_\tau \, d\tau + \int_t^{t_f} tr(P_\tau Q_\tau) \, d\tau + tr(P_{t_f} S_f) .$$