Note Title 6/13/2021

C.2 The JLQG problem under complete information

C.2.1 Problem Statement

* (1) $x_{k+1} = A(y_k)\, x_k + B(y_k)\, u_k + w_k \qquad k = 0, 1, 2, \ldots, N-1$

Jump-linear process model

x_k $n \times 1$ state vector

u_k $m \times 1$ control "

w_k $n \times 1$ process noise vector, white, zero-mean, W_k

$A(y_k)$ $n \times n$ $B(y_k)$ $n \times m$

* $\{ y_k, k = 0, 1, \ldots, N-1 \}$ scalar Markov chain, finite alphabet; $S_y = \{1, 2, \ldots, \nu\}$

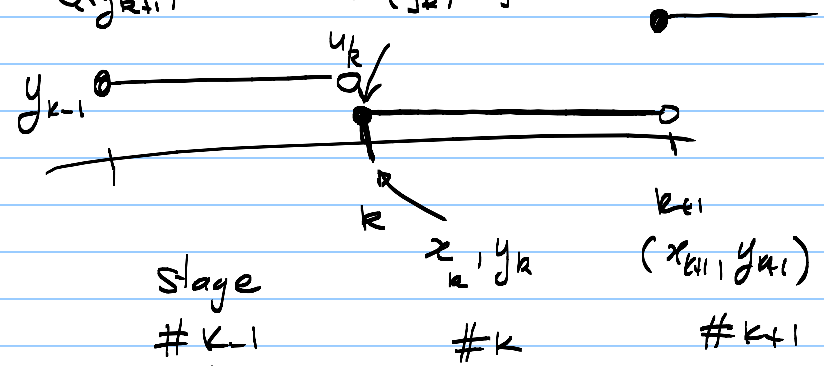
transition probability matrix P; $P(i,j) = Pr\{ y_{k+1} = j \mid y_k = i \} = p_{ij}$; $i, j \in S_y$

* x_0, y_k, w_k independent

Information pattern: x_k and y_k are known; $(X^k, Y^k) \equiv \{x_\ell, y_\ell\}_0^k$

• Cost Function $$J = E\left\{ \sum_{k=0}^{N-1} \|x_{k+1}\|^2_{Q(y_{k+1})} + \|u_k\|^2_{R(y_k)} \right\}$$

$$Q(y_k) \geq 0 \qquad R(y_k) > 0$$



• Goal: Let $\mathcal{U}$ denote the set of admissible controllers

$$\mathcal{U} = \left\{ \{u_k\} \mid u_k = u_k(\underbrace{x^k, y^k}_{\text{past history of know } (x,y)}) \right\}.$$

We want to minimize J with respect to $\{u_k\} \in \mathcal{U}$, subject (1).

Notation

$$\left\{\begin{array}{l} A(y_k) = A_k \\ B(y_k) = B_k \\ Q(y_k) = Q_k \\ R(y_k) = R_k \end{array}\right.$$ denote the random variables

$$X_k^\ell = \{ x_k \ldots x_\ell \}$$

$$Y_k^\ell = \{ y_k \text{---} y_\ell \}$$

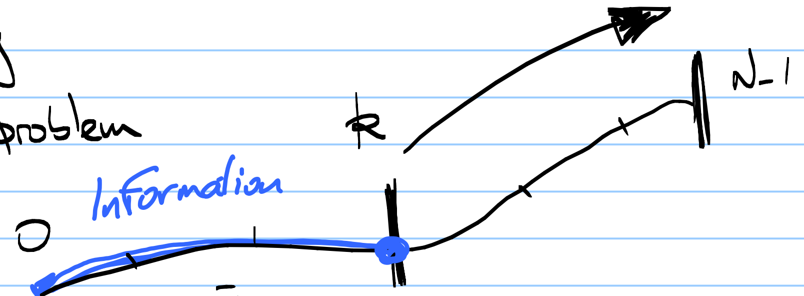
$$U_k^\ell = \{ u_k - u_\ell \}$$

$$\left\{\begin{array}{l} A(y_k = j) = A(j) \\ B(y_k = j) = B(j) \\ Q(y_k = j) = Q(j) \\ R(y_k = j) = R(j) \end{array}\right.$$ denote realizations

$j = 1, 2, \ldots, n$

C.2.2. Solution via Dynamic Programming

A. Recursive formulation of the control problem



Optimal Return Function J_k^*

$$J_k^*(X^k, Y^k) = \min_{U_k^{N-1}} \left[E\left\{ \sum_{i=k}^{N-1} \|x_{i+1}\|^2_{Q_{i+1}} + \|u_i\|^2_{R_i} \,\middle|\, X^k, Y^k \right\} \right]$$

goal : to formulate the above problem in a recursive manner

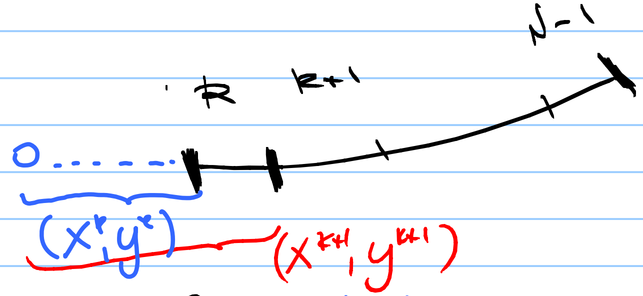
main tools : – Nested Property of the Conditionnal Expectation

$$A \subseteq B \quad \text{then} \quad E\{X \mid A\} = E\{E\{X \mid B\} \mid A\}$$

$$(x^0, y^0) \subseteq \dots \subseteq (X^k, Y^k) \subseteq (X^{k+1}, Y^{k+1}) \dots \subseteq (X^{N-1}, Y^{N-1})$$

– Fundamental Lemma of Optimal Control

the $\min_u [\ ]$ and $E\{\ \}$ operators may be interchanged provided that the control are in the admissible set.



Consider

$$J_k(X^k, y^k, u_k^{N-1}) = E\left\{ \sum_{i=k}^{N-1} |x_{i+1}|^2_{Q_{i+1}} + |u_i|^2_{R_i} \;\middle|\; X^k, y^k \right\}$$

$$= E\left\{ |x_{k+1}|^2_{Q_{k+1}} + |u_k|^2_{R_i} + \sum_{i=k+1}^{N-1} |x_{i+1}|^2_{Q_{i+1}} + |u_i|^2_{R_i} \;\middle|\; X^k, y^k \right\}$$

$$= E\left\{ E\left\{ |x_{k+1}|^2_{Q_{k+1}} + |u_k|^2_{R_k} + \sum_{i=k+1}^{N-1} |x_{i+1}|^2_{Q_{i+1}} + |u_i|^2_{R_i} \;\middle|\; X^{k+1}, y^{k+1} \right\} \;\middle|\; X^k, y^k \right\}$$

$$= E\left\{ |x_{k+1}|^2_{Q_{k+1}} + |u_k|^2_{R_k} + \underbrace{E\left\{ \sum_{i=k+1}^{N-1} |x_{i+1}|^2_{Q_{i+1}} + |u_i|^2_{R_i} \;\middle|\; X^{k+1}, y^{k+1} \right\}}_{\equiv J_{k+1}(X^{k+1}, y^{k+1}, u_{k+1}^{N-1})} \;\middle|\; X^k, y^k \right\}$$

to summarize : $J_k(X^k, Y^k, U_k^{N-1}) = E\left\{ |x_{k+1}|^2_{Q_{k+1}} + |u_k|^2_{R_k} + J_{k+1}(X^{k+1}, Y^{k+1}, U_{k+1}^{N-1}) \,\middle|\, X^k, Y^k \right\}$

then,

$$J_k^*(X^k, Y^k) = \min_{U_k^{N-1}} \left[J_k(X^k, Y^k, U_k^{N-1}) \right]$$

$$= \min_{u_k} \min_{U_{k+1}^{N-1}} \left[E\left\{ |x_{k+1}|^2_{Q_{k+1}} + |u_k|^2_{R_k} + J_{k+1}(X^{k+1}, Y^{k+1}, U_{k+1}^{N-1}) \,\middle|\, X^k, Y^k \right\} \right]$$

(Fundamental lemma!)

$$= \min_{u_k} \left[E\left\{ |x_{k+1}|^2_{Q_{k+1}} + |u_k|^2_{R_k} + \underbrace{\min_{U_{k+1}^{N-1}} \left[J_{k+1}(X^{k+1}, Y^{k+1}, U_{k+1}^{N-1}) \right]}_{\equiv\, J_{k+1}^*(X^{k+1}, Y^{k+1})} \,\middle|\, X^k, Y^k \right\} \right]$$

Thus we achieved a Recursion for the Optimal Cost To Go:

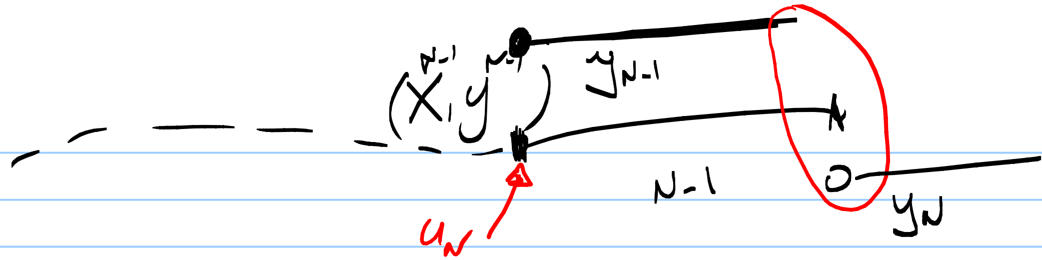
$$J_k^* = \min_{u_k} \left[E\left\{ |x_{k+1}|^2_{Q_{k+1}} + |u_k|^2_{R_k} + J_{k+1}^* \mid X^k, Y^k \right\} \right]$$

$\Rightarrow$ Dynamic Programming Rule.

Starting @ J_{N-1}^*

FINAL STATE

3. Recursive Algorithm



Last Stage $N-1$

Given x^{N-1}, y^{N-1}

$$J_{N-1} = E\left\{ |x_N|^2_{Q_N} + |u_{N-1}|^2_{R_{N-1}} \;\middle|\; X^{N-1}, y^{N-1} \right\} \qquad (*)$$

$Q(y_N)$ $\qquad$ $R(y_{N-1})$

$$J_{N-1} = E\left\{ | A_{N-1} x_{N-1} + B_{N-1} u_{N-1} + w_{N-1} |^2_{Q(y_N)} + |u_{N-1}|^2_{R_{N-1}} \;\middle|\; X^{N-1}, y^{N-1} \right\}$$

$$= E\left\{ (A_{N-1} x_{N-1} + B_{N-1} u_{N-1} + w_{N-1})^T Q(y_N) (A_{N-1} x_{N-1} + B_{N-1} u_{N-1} + w_{N-1}) + u_{N-1}^T R_{N-1} u_{N-1} \;\middle|\; X^{N-1}, y^{N-1} \right\}$$

$$= (A_{N-1} x_{N-1} + B_{N-1} u_{N-1})^T E\{Q_N | X^{N-1}, y^{N-1}\} (A_{N-1} x_{N-1} + B_{N-1} u_{N-1}) + u_{N-1}^T R_{N-1} u_{N-1}$$

$$+ \underbrace{E\{ w_{N-1}^T Q_N w_{N-1} \mid X^{N-1}, y^{N-1} \}}$$

remark: 1. all the terms that are linear in w_{N-1} cancel out. why?

w_{N-1}, zero-mean, independent from x_{N-1}.

2. $E\{Q_N \mid x^{N-1}, y^{N-1}\} = E\{Q(y_N) \mid x^{N-1}, y^{N-1}\} = E\{Q(y_N) \mid y_{N-1}\}$

why? $\{y_k\}$ is MARKOV!

Notation: $\overline{Q}_N \equiv E\{Q_N \mid y_{N-1}\}$

r.v. $\triangleleft$— $\overline{Q}_N(y_{N-1}) = \sum_{j=1}^{\nu} Q_N(j)\, \Pr\{y_N = j \mid y_{N-1}\}$; $y_{N-1} = 1, 2, \dots, \nu$

realiz^{e} $\triangleleft$— $\overline{Q}_N(i) = \sum_{j=1}^{\nu} Q_N(j)\, \underbrace{\Pr\{y_N = j \mid y_{N-1} = i\}}_{= p_{ij}}$ $\quad i = 1, 2, \dots, \nu$

$$\boxed{\overline{Q}_N(i) = \sum_{j=1}^{\nu} Q_N(j)\, p_{ij}} \quad i = 1, 2, \dots, \nu$$

this is an "averaging" over the weights Q_N according to the transition probability distribution.

To sum-up:

$$J_{N-1} = \left| \underset{\propto}{A_{N-1}} \underset{\propto}{x_{N-1}} + \underset{\propto}{B_{N-1}} u_{N-1} \right|^2_{\underset{\propto}{\bar{Q}_N}} + \left| u_{N-1} \right|^2_{\underset{\propto}{R_{N-1}}} + tr\left(W_{N-1} \underset{\propto}{\bar{Q}_N} \right)$$

$$\left\{ \begin{aligned} & E\left\{ w_{N-1}^T Q_N(y_N) w_{N-1} \mid X^{N-1}, Y^{N-1} \right\} \\ & tr\, E\left\{ w_{N-1} w_{N-1}^T Q_N(y_N) \mid X^{N-1}, Y^{N-1} \right\} \\ & tr\, E\left\{ w_{N-1} w_{N-1}^T \mid X^{N-1}, Y^{N-1} \right\} \cdot E\left\{ \underset{\sim}{Q}(y_N) \mid X^{N-1}, Y^{N-1} \right\} \\ & tr\left(W_{N-1} \cdot \bar{Q}_N \right) \end{aligned} \right\}$$

Now,

$$\min_{u_{N-1}} J_{N-1}(x_{N-1}, y_{N-1}, u_{N-1}) \longmapsto u^*_{N-1}(x_{N-1}, y_{N-1})$$

$$u^*_{N-1} = \underbrace{-\left(B_{N-1}^T \bar{Q}_N B_{N-1} + R_{N-1} \right)^{-1} B_{N-1}^T \bar{Q}_N A_{N-1}}_{\text{Control Gain } M_{N-1}} \cdot x_{N-1}$$

$\hookrightarrow$ Feedback Control, Linear in x_{N-1}.

Control Gain depends on y_{N-1}! $\rightarrow$ ν realization for M_{N-1}.

$$\bar{Q}_N = \sum_{j=1}^{n} Q_N(j)\, p_{ij} \qquad i = 1, 2, \dots, n \quad (= y_{N-1})$$

$$J^*_{N-1}(x_{N-1}, y_{N-1}) = |x_{N-1}|^2_{S_N(y_{N-1})} + \operatorname{tr}(W_{N-1} \bar{Q}_N)$$

where $S_N(y_{N-1}) = A^T_{N-1} \bar{Q}_N A_{N-1} - A^T_{N-1} \bar{Q}_{N-1} B_{N-1} (B^T_{N-1} \bar{Q}_N B_{N-1} + R_{N-1})^{-1} B^T_{N-1} \bar{Q}_N A_{N-1}$

! for each value of x_{N-1}, there n possible values for J^*.

Second to Last Stage:

Given x^{N-2}, y^{N-2}

$$J_{N-2} = E\left\{ |x_{N-1}|^2_{Q_{N-1}} + |u_{N-2}|^2_{R_{N-2}} + \underbrace{J^*_{N-1}(x_{N-1}, y_{N-1})}_{|x_{N-1}|^2_{S_N(y_{N-1})} + tr\left(W_{N-1}\overline{Q_N}(y_{N-1})\right)} \;\middle|\; x^{N-2}, y^{N-2} \right\}$$

$$= E\left\{ |x_{N-1}|^2_{\underbrace{Q_{N-1} + S_N}_{\equiv S_{N-1}(y_{N-1})}} \right.$$

$$= \underbrace{E\left\{ |x_{N-1}|^2_{S_{N-1}} + |u_{N-2}|^2_{R_{N-2}} \;\middle|\; x^{N-2}, y^{N-2} \right\}}_{\text{similar to } (*)} + \underbrace{E\left\{ tr\left(W_{N-1}\overline{Q_N}\right) \middle| x^{N-2}, y^{N-2} \right\}}_{\text{independent from } u_{N-2}}$$

$\min\limits_{u_{N-2}}$

Summary: The Jump-Linear Quadratic Gaussian Regulator

- Backward Propagation

$S_{N+1}(i) = 0 \qquad i = 1, 2, \dots, \nu$ a set of ν matrices S

For $k = N-1, N-2, \dots, 0$

ν matrices

$$\bar{S}_{k+1}(y_k = i) = \sum_{j=1}^{\nu} \left[S_{k+2}(y_{k+1} = j) + Q_{k+1}(y_{k+1} = j) \right] p_{ij} \, ; \quad y_k = i = 1, 2, \dots, \nu$$

$$S_k(y_k = i) = A_k^T \bar{S}_{k+1} A_k - A_k^T \bar{S}_{k+1} B_k \left(B_k^T \bar{S}_{k+1} B_k + R_k \right)^{-1} B_k^T \bar{S}_{k+1} A_k \; ; \quad i = 1, 2, \dots, \nu$$

$$M_k(y_k = i) = \left(B_k^T \bar{S}_{k+1} B_k + R_k \right)^{-1} B_k^T \bar{S}_{k+1} A_k \qquad i = 1, 2, \dots \nu$$

the Gains are calculated through Coupled Riccati equations.

Optimal Cost :

$$J_0^*(x_0, y_0) = \|x_0\|^2_{S_0(y_0)} + \Delta J_0(y_0) \qquad y_0 = 1, 2, \dots, \nu$$

- Gain | Computations
- Cost | OFF LINE.

$$\begin{cases} \Delta J_{N-1}(y_{N-1} = i) = tr\left(W_{N-1} \bar{Q}_N(y_{N-1}\right) & i = 1, \dots, \nu \\ \Delta J_k(y_k = i) = \sum_{j=1}^{\nu} \Delta J_{k+1}(y_{k+1} = j)\, p_{ij} + tr\left(W_k \bar{S}_{k+1}(y_k)\right) \end{cases}$$

- Forward Propagation :

$$\begin{cases} x_0 \text{ initial condition} \\ x_{k+1} = A(y_k) x_k + B(y_k) u_k^* \\ u_k^* = -M(y_k) x_k \end{cases}$$

State Feedback Control, Linear

mode dependent