B. LINEAR QUADRATIC GAUSSIAN CONTROL (LQG)

B.1 Discrete-time

B.1.1 Deterministic Linear-Quadratic control

<u>Preliminary</u>. Factorization and Minimization of a quadratic form

- Scalar Case. $f(x) = ax^2 + 2bx + c$ $\quad a, b, c$ scalars $\quad a \neq 0$, $a > 0$

Solve $\min_{x} f(x)$

Way #1 : differentiation

$$2ax + 2b = 0$$

$$x^* = -\frac{b}{a}$$

Way #2 : Factorization
Square Completion

$$f(x) = a\left[x^2 + \frac{2b}{a}x + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)^2 + \frac{c}{a}\right]$$

$f^* = f(x^*) = a\left(-\frac{b}{a}\right)^2 + 2b\left(-\frac{b}{a}\right) + c$

$= c - \frac{b^2}{a}$

2nd order derivative : $2a > 0$

conclusion : $\left\{ \begin{array}{l} x^* \text{ is the optimizer} \\ f^* \text{ is the minimum} \end{array} \right.$

$= a\left(x + \frac{b}{a}\right)^2 + \underbrace{\left(c - \frac{b^2}{a}\right)}$.

Solve $\min_x f(x)$

$\min_x \left(x + \frac{b}{a}\right)^2$

By INSPECTION $\Rightarrow x^* = -b/a$

$\Rightarrow f^* = c - b^2/a$.

Vector case x $n \times 1$

$$\boxed{\begin{aligned} f(x) &= x^T A x + 2b^T x + c \\ &= \|x\|_A^2 + 2b^T x + c \end{aligned}}$$

A $n \times n$ $A^T = A > 0$

b $n \times 1$

c 1×1

Solve $\min_x f(x)$

Way #1 Differentiation

Way #2 Factorization

$$\underset{n\times 1}{\frac{df}{dx}} = \underbrace{\frac{d}{dx}\left(\|x\|_A^2\right)}\; + 2\underbrace{\frac{d}{dx}\left(b^T x\right)}\; + \frac{d}{\cancel{dx}}\cancel{c} \to 0$$

$$\longrightarrow \quad = \underbrace{(A+A^T)}_{2A}\,x + \underset{n\times 1}{2\,b}$$

$$\frac{df}{dx} = 2\,(Ax + b) = 0$$

$$\longrightarrow \quad x^* = -A^{-1}b$$

$$f^* = f(x^*) = \|-A^{-1}b\|_A^2 + 2\,b^T(-A^{-1}b) + c$$

$$= b^T \cancel{A^{-1}}\cancel{A}\,A^{-1}b - 2\,b^T A^{-1} b + c$$

$$= -\,b^T A^{-1} b + c$$

$$\underset{n\times n}{\frac{d^2 f}{dx\,dx^T}} = 2A > 0 \Rightarrow \text{minimum}$$

$$f(x) = \underbrace{x^T A x + 2b^T x} + c$$

$$= (x + A^{-1}b)^T A\,(x + A^{-1}b) + c - \|A^{-1}b\|_A^2$$

$$= \|x + A^{-1}b\|_A^2 + \underbrace{c - \|b\|_{A^{-1}}^2}$$

$$-\,b^T \underbrace{\cancel{A^{-1}}\cancel{A}}_{1}\,A^{-1}b$$

$$-\,b^T A^{-1} b = -\,\|b\|_{A^{-1}}^2$$

Solve $\min_x f(x)$

$$\min_x \|x + A^{-1}b\|_A^2$$

BY INSPECTION

$$\boxed{\begin{cases} x^* = -A^{-1}b \\ f^* = c - \|b\|_{A^{-1}}^2 \end{cases}}$$

Conclusion: $\begin{cases} x^* \text{ the minimizer } & -A^{-1}b \\ f^* \text{ the minimum } & c - \|b\|^2_{A^{-1}} \end{cases}$

- The LQR problem: a particular case – the "Cheap" Control case

Given $\blacksquare$ $x_{k+1} = A x_k + B u_k$ (1) linear, time-invariant, perfect state information

$\blacksquare$ $J = \sum_{k=0}^{N-1} \|x_{k+1}\|^2_{Q_k}$ $Q_k^T = Q_k \geq 0$ $x: n \times 1$ $u: m \times 1$

Solve $\min_{\{u_k\}_{k=0,\dots,N-1}} J$ subject to (1)

- SOLUTION via Dynamic Programming
  - Last Stage

Define $P_N = Q_{N-1}$

(1) $$\min_{u_{N-1}} J_{N-1} = \min_{u_{N-1}} \left\{ \|x_N\|^2_{P_N} \right\}$$

System equation $x_N = A x_{N-1} + B u_{N-1}$

$$\min_{u_{N-1}} J_{N-1} = \min_{u_{N-1}} \left\{ \|A x_{N-1} + B u_{N-1}\|^2_{P_N} \right\} \qquad (2)$$

scalars

$$\|A x_{N-1} + B u_{N-1}\|^2_{P_N} = (A x_{N-1} + B u_{N-1})^T P_N (A x_{N-1} + B u_{N-1})$$

$$= \underbrace{u^T_{N-1}}_{x^T} \underbrace{B^T P_N B}_{A} \underbrace{u_{N-1}}_{x} + \underbrace{u^T_{N-1} B^T P_N A x_{N-1} + (x^T_{N-1} A^T P_N B u_{N-1})^T}_{2\, x^T_{N-1} A^T P_N B\, u_{N-1}} + x^T_{N-1} A^T P_N A x_{N-1}$$

$$= x^T A x$$

b^T

$$= \|u_{N-1}\|^2_{B^T P_N B} + 2\,(B^T P_N A\, x_{N-1})^T u_{N-1} + \|x_{N-1}\|^2_{A^T P_N A}$$

must exists.

Solve $\min_{u_{N-1}} J_{N-1} \Longrightarrow$ Using the preliminary results

the optimal control $\quad u^*_{N-1} = -(B^T P_N B)^{-1} B^T P_N A\, x_{N-1}$

$$= -M_{N-1}\, x_{N-1}$$

and the optimal value of the cost

$$J^*_{N-1} = \|x_{N-1}\|^2_{A^T P_N A} - \|\underbrace{B^T P_N A}\, x_{N-1}\|^2_{\underbrace{(B^T P_N B)^{-1}}}$$

$$= \|x_{N-1}\|^2_{\underbrace{A^T P_N A - A^T P_N B (B^T P_N B)^{-1} B^T P_N A}}$$

$$= \|x_{N-1}\|^2_{\tilde{P}_{N-1}}$$

■ Second-to-last Stage

cost of the stage

$$J_{N-2} = w_{N-2} + J^*_{N-1}$$

COST-TO-GO optimized

From the Optimality Principle

$$J_{N-2} = \|x_{N-1}\|^2_{Q_{N-2}} + \|x_{N-1}\|^2_{\tilde{P}_{N-1}}$$

$$= \|x_{N-1}\|^2_{Q_{N-2}+\tilde{P}_{N-1}}$$

Define : $P_{N-1} = \tilde{P}_{N-1} + Q_{N-2}$

Solve $\displaystyle \min_{u_{N-2}} J_{N-2} = \min_{u_{N-2}} \|x_{N-1}\|^2_{P_{N-1}}$

System equation $x_{N-1} = A x_{N-2} + B u_{N-2}$

$$\min_{u_{N-2}} J_{N-2} = \min_{u_{N-2}} \left\{ \| A x_{N-2} + B u_{N-2} \|^2_{P_{N-1}} \right\} \qquad (3)$$

Problem (3) is similar to problem (2) solved at the last stage.

So we achieved a Recursion on the Problem formulation.

The solution of pb (3) is:

$$u^*_{N-2} = -M_{N-2}\, x_{N-2}$$

$$M_{N-2} = (B^T P_{N-1} B)^{-1} B^T P_{N-1} A$$

the optimal cost J^*_{N-2}

$$J^*_{N-2} = \|x_{N-2}\|^2_{\tilde{P}_{N-2}}$$

$$\tilde{P}_{N-2} = A^T P_{N-1} A - A^T P_{N-1} A (B^T P_{N-1} B)^{-1} B^T P_{N-1} A$$

General: Recursive algorithm for the computation of the Optimal Control and Optimal Cost-to-Go.

- BACKWARD $\quad k = N-1, \dots, 0 \; ; \; P_N = Q_{N-1}$ given

Time-varying, Optimal, OFF-LINE

1) Compute the Gain matrix M_k $\qquad M_k = (B^T P_{k+1} B)^{-1} B^T P_{k+1} A$

symmetric, numerically stable

2) Compute the P-matrix $\qquad P_k = A^T P_{k+1} A - A^T P_{k+1} B (B^T P_{k+1} B)^{-1} B^T P_{k+1} A$

Recursive BWD RICCATI Equation.

- FORWARD $\quad k = 0, \dots, N-1 \; ; \; x_0 \leftarrow$ given

3) Compute the Optimal control $\qquad u_k^* = -M_k x_k$

Linear Feedback Control

4) Propagate the state $\qquad x_{k+1} = A x_k + B u_k^*$

5) Compute the Optimal Cost-to-Go $\qquad J_k^* = \|x_k\|^2_{\tilde{P}_{k+1}} \qquad \tilde{P}_{k+1} = P_k - Q_{k-1}$

Quadratic in x

the cheap control problem is a particular case of the general LQR problem

$$J = \sum_{k=0}^{N-1} \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}^T \begin{bmatrix} Q_k & N_k \\ N_k^T & R_k \end{bmatrix} \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix}$$

$$Q_k \geq 0 \qquad R_k > 0$$

B.1.2 Random Sequences continuous-valued

definition: a collection of random variables (scalar or vector) that are indexed by integers.

$$\{\underline{x}_k\} = \{\underline{x}_0, \underline{x}_1, \underline{x}_2, \dots, \dots\}$$

A full characterization of a random sequence is provided through the Joint Probability Density function of any order k in the sequence.

example: 3 r.v. $\{x_0, x_1, x_2\}$

$$\begin{cases} p(x_0), p(x_1), p(x_2) \\ p(x_0, x_1) \quad p(x_1, x_2) \quad p(x_0, x_2) \\ p(x_0, x_1, x_3) \end{cases}$$

definition: MARKOV sequence

A random sequence such that $k = 1, 2, \dots$ $\boxed{p(x_{k+1} \mid x_k, \dots, x_1) = p(x_{k+1} \mid x_k)}$

Theorem: A MARKOV sequence is characterized by 1) the initial marginal p.d.f. of x_0 : $p(x_0)$

2) the transition p.d.f. of $x_{k+1} | x_k$.

$$p(x_{k+1} | x_k)$$

proof: By induction

$$\mathcal{P}(n) : \underbrace{p(x_0, \dots, x_n)}_{\text{joint p.d.f}} = p(x_n | x_{n-1})\, p(x_{n-1} | x_{n-2})\, p(x_{n-2} | x_{n-3}) \dots p(x_1 | x_0)\, \underbrace{p(x_0)}_{\text{initial p.d.f}}$$

transition p.d.f.

$$n = 1 \qquad p(x_0, x_1) = p(x_1 / x_0)\, p(x_0)$$

the most recent

$$n \Rightarrow n+1 \qquad p(x_0, \dots, x_n, x_{n+1}) = p(x_{n+1} | x_n, \cancel{x_{n-1}, \dots, x_0})\, \underbrace{p(x_n, x_{n-1}, \dots, x_0)}_{\mathcal{P}(n) \text{ true}}$$

MARKOV

$$= p(x_{n+1} | x_n) \cdot p(x_n | x_{n-1}) \cdot p(x_{n-1} | x_{n-2}) \dots p(x_1 | x_0)\, p(x_0)$$

Conclusion: $\mathcal{P}(n+1)$ true ∴

Example 1 A sequence of r.v. that statistically independent is a MARKOV sequence

Definition : Sequence of statistically independent variables

$$p(x_0, \dots, x_n) = p(x_0)\, p(x_1) \dots p(x_n)$$

a.k.a. "White" sequence ("White" Noise)

hint

$$p(x_0, x_1) = p(x_1 | x_0) \cdot \cancel{p(x_0)} = p(x_1) \cdot \cancel{p(x_0)}$$

statistical indep.

$\hookrightarrow \; p(x_1|x_0) = p(x_1)$ ✓

$$p(x_0, x_1, x_2) = p(x_2 | x_1, x_0) \cdot \cancel{p(x_1, x_0)} = p(x_2) \cdot p(x_1) \cdot p(x_0)$$

$$= p(x_2, x_1) \cdot p(x_0)$$

$$= p(x_2 | x_1) \cdot \underbrace{p(x_1) \cdot p(x_0)}$$

$$= p(x_2 | x_1) \quad p(\cancel{x_1}, x_0)$$

$$\Rightarrow \quad p(x_2 | x_1, \cancel{x_0}) = p(x_2 | x_1)$$

$\Rightarrow$ MARKOV property ...

Example 2

Consider the model $\quad x_{k+1} = x_k + w_k$

$p(x_0) \quad p(w_k) \quad \boxed{\{w_k\} \text{ ``white'' sequence}}$

question $\quad p(x_{k+1} | x_k, \ldots, x_0)$?

solution

Assume: $\quad x_k = \rho_k$

then $\quad x_{k+1} = x_k + \rho_k$ given that $x_k = \rho_k$

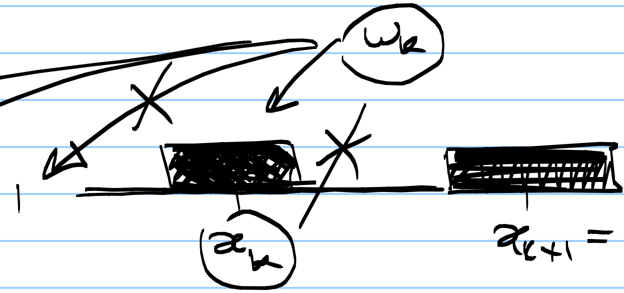
$$P\{\omega : a \le x_{k+1}(\omega) \le b \mid x_k(\omega)=\rho_k,\ x_{k-1}(\omega)=\rho_{k-1},\ \text{---},\ x_0(\omega)=\rho_0\}$$

$$= P\{\omega : a \le x_k(\omega)+w_k(\omega) \le b \mid x_k(\omega)=\rho_k,\ \ldots\ldots\ldots\}$$

$$= P\{\omega : a-\rho_k \le \underbrace{w_k(\omega)}_{\text{WHITE}} \le b-\rho_k \mid x_k(\omega)=\rho_k,\ x_{k-1}(\omega)=\rho_{k-1} - \cdots - \}$$

$$= P\{\omega : a-\rho_k \le w_k(\omega) \le b-\rho_k \mid x_k(\omega)=\rho_k\}$$

$= ?$



$x_{k+1} = x_k + w_k$

x_0

$$= P\{\omega : a-\rho_k \le w_k(\omega) \le b-\rho_k \mid x_k(\omega)=\rho_k\}$$

$$= P\{\omega : a \le \underbrace{w_k(\omega)+\rho_k}_{x_{k+1}(\omega)} \le b \mid x_k(\omega)=\rho_k\}$$

model

$$= P\{\omega : a \le x_{k+1}(\omega) \le b \mid x_k(\omega)=\rho_k\}$$

Conclusion $\Rightarrow$ $\{x_k\}$ is a 1st order MARKOV process.

$\{w_k\}$ white $\longrightarrow$ $\boxed{\text{Linear 1st order}}$ $\longrightarrow$ 1st order MARKOV.

- Generalized MARKOV Sequences

Given $$x_{k+1} = x_k + x_{k-1} + w_k \; ; \; \{w_k\}$$ statistically independent random vectors "white"

Define $$\underline{y}_k \equiv \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} \Rightarrow \underline{y}_{k+1} \equiv \begin{Bmatrix} x_{k+1} \\ x_k \end{Bmatrix}$$

$$\underline{y}_{k+1} = \begin{Bmatrix} x_{k+1} \\ x_k \end{Bmatrix} = \begin{Bmatrix} x_k + x_{k-1} + w_k \\ x_k \end{Bmatrix} = \begin{bmatrix} I & I \\ I & 0 \end{bmatrix} \underbrace{\begin{Bmatrix} x_k \\ x_{k-1} \end{Bmatrix}}_{y_k} + \begin{bmatrix} I \\ 0 \end{bmatrix} w_k$$

"white" noise

$$\underline{y}_{k+1} = A \underline{y}_k + B w_k$$

By State Augmentation we obtained a representation of a linear first-order system driven by White noise => this is called a generalized MARKOV Sequence.

- GAUSS-MARKOV Random Sequences (GMRS)

A GMRS $\{x_k\}$ is a MARKOV sequence where $\{x_0\}$ is Gaussian $[p(x_0) \text{ is } G]$ and $p(x_{k+1}|x_k)$ is G for all k.

In order to determine $p(x_{k+1}|x_k)$, one has to determine:

1) $E\{x_{k+1}|x_k\}$ conditional expectation of x_{k+1} given x_k

2) $E\{(x_{k+1} - E\{x_{k+1}|x_k\})(x_{k+1} - E\{x_{k+1}|x_k\})^T | x_k\}$

conditional covariance matrix of x_{k+1} given x_k

* A linear mapping on Gaussian vectors yields a Gaussian vector

- $x_{k+1} = A_k x_k + B_k w_k$; $\{w_k\}$ white sequence, x_0 and $\{w_k\}$ independent

then $\{x_k\}$ is MARKOV

$*$ $x_0, \{w_k\}$ are jointly Gaussian

$\Rightarrow$ $\boxed{\{x_k\} \text{ is a GMRS}}$.

Given

$$\begin{cases} x_{k+1} = A_k x_k + w_k \ ; \ w_k, x_0 \text{ are jointly Gaussian} \\ E\{w_k\} = \mu_{w_k} \ ; \ \mathrm{Cov}\{w_k\} = Q_k \ \ \underset{Q_k^T = Q_k \geq 0}{n \times n} \ ; \ \underset{k \neq \ell}{\mathrm{Cov}\{w_k, w_\ell\} = 0} \\ E\{x_0\} = \mu_0 \ ; \ \mathrm{Cov}\{x_0\} = \Pi_0 \ ; \ \underset{\Pi_0^T = \Pi_0 \geq 0}{n \times n} \ ; \ \mathrm{cov}\{w_k, x_0\} = 0 \end{cases}$$

$\mathrm{cov}\{w_k, x_k\} = 0$

Determine $p(x_{k+1} \mid x_k)$

1) Computation of $E\{x_{k+1} \mid x_k\}$

$$E\{x_{k+1} | x_k\} = E\{A_k x_k + w_k | x_k\}$$
$$= A_k \underbrace{E\{x_k | x_k\}}_{A_k x_k} + \underbrace{E\{w_k | x_k\}}_{E\{w_k\} = \mu_{w_k}}$$

independence by modeling

$$\boxed{E\{x_{k+1} / x_k\} = A_k x_k + \mu_{w_k}}$$

2) Computation of the Covariance Matrix

$$Cov\{x_{k+1} | x_k\} = E\{(x_{k+1} - E\{x_{k+1} | x_k\})(x_{k+1} - E\{x_{k+1} | x_k\})^T | x_k\}$$

System System

$$= E\{[A_k x_k + w_k - (A_k x_k + \mu_{w_k})] [A_k x_k + w_k - (A_k x_k + \mu_{w_k})]^T | x_k\}$$
$$= E\{(w_k - \mu_{w_k})(w_k - \mu_{w_k})^T | x_k\}$$

$$= E\{(w_k - \mu_{w_k})(w_k - \mu_{w_k})^T\} \quad \downarrow \text{independent}$$

$$= Q_k \quad \text{model}$$

$$\boxed{Cov(x_{k+1} \mid x_k) = Q_k}$$

<u>Comment</u> : • The conditioning upon x_k renders x_k deterministic

In the model equation $x_{k+1} = x_k + w_k$, only w_k is random

so the expected value is $x_{k+1} = x_k + \mu w_k$

the covariance is the covariance of the Noise w_k : Q_k.

Conclusion:

$$p(x_{k+1} \mid x_k) = \frac{1}{(2\pi)^{n/2} \det Q_k^{1/2}} e^{-\frac{1}{2} \| x_{k+1} - A_k x_k - \mu_{w_k} \|^2_{Q_k^{-1}}}$$

B.1.3 Stochastic Control for Systems with Full Information

B.1.3.1 Formulation of the Problem

Given:

$$\begin{cases} x_{k+1} = f_k(x_k, u_k, w_k) \\ y_k = x_k \end{cases}$$

x_k $n \times 1$ state

u_k $m \times 1$ control

w_k $p \times 1$ process noise

a. $\{w_k\}$ white sequence; $p(w_k)$ is known.

b. $\{u_k\}$ random sequence; $\underline{p(u_k \mid Y^k, U^{k-1})}$ where $Y_k^k = \{y_0 - y_k\}$

$U = \{u_0 - u_k\}$

c. Cost Function: $J = E\left\{ \sum_{k=0}^{N-1} \varpi_k(x_{k+1}, u_k) \right\}$; $\varpi_k \geq 0$; $\forall k$

We seek the random sequence of random vectors $\{u_k\}$ that the expected value of the cost.

Since $\{u_k\}$ is characterized by its p.d.f. then the control variable is the p.d.f.

Problem: Find the sequence of $p(u_k \mid Y^k, U^{k-1})$ that minimizes the cost.

B.1.3.2 Preliminary Result

a. Expression of the expectation

$$\boxed{\underset{z}{E}\{z\} = \underset{x,y,z}{E}\{z\}}$$

$$\underset{z}{E}\{z\} = \int_{Z} z\, p(z)\, dz = \int_{Z} z \left[\int_{x,y} p(x,y,z)\, dx\, dy \right] dz$$

$$= \int_{Z}\int_{x,y} z\, p(x,y,z)\, dx\, dy\, dz = \int_{x,y,z} z\, p(x,y,z)\, dx\, dy\, dz = \underset{x,y,z}{E}\{z\}$$

b. The Chain Rule

$$P(A,B,C,D) = P(A,B|C,D).P(C,D) = P(A|B,C,D).P(B|C,D).P(C,D)$$
$$= P(A|B,C,D).P(B,C,D)$$

c. System equat° $x_{k+1} = f_k(x_k, u_k, w_k)$

Full informat° on x_k then we can show $p(x_{k+1}|y^k, u^k) = p(x_{k+1}|x_k, u^k)$

d. Fundamental Lemma for Stochastic Control

Lemma: Suppose that the minimum to $\min_{u \in \mathcal{U}} g(x,u)$ exists and that $\mathcal{U}$ is a class of functions such that $E\{g(x,u)\}$ exists then

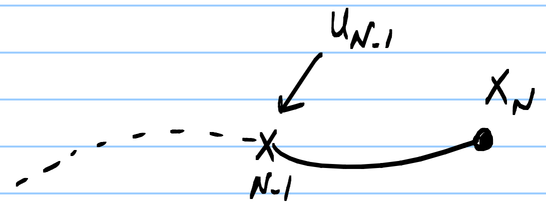
$$\min_{u \in \mathcal{U}} E\{g(x,u)\} = E\{\min_{u \in \mathcal{U}} g(x,u)\}$$

to be completed.

B.1.3.3. Solution Using DYNAMIC PROGRAMING

- LAST STAGE

we seek $p^*(u_{N-1} \mid y^{N-1}, u^{N-2}) \equiv p^*$



$$\min_p J_{N-1} = \min_p E\left\{ W_{N-1}(x_N, u_{N-1}) \right\}$$

$$= \min_p \int_{\Omega(x_N, u_{N-1})} W_{N-1} \, p(x_N, u_{N-1}) \, d(x_N, u_{N-1})$$

$$= \min_{P} \int_{\Omega(x_N, u_{N-1}, U^{N-2}, Y^{N-1})} W_{N-1}\, p(x_N, u_{N-1}, U^{N-2}, Y^{N-1})\, d(x_N, u_{N-1}, U^{N-2}, Y^{N-1})$$

$$= \min_{P} \int_{\Omega(x_N, u_{N-1}, U^{N-2}, Y^{N-1})} W_{N-1}\, P(x_N, u_{N-1} | U^{N-2}, Y^{N-1}) \cdot p(U^{N-2}, Y^{N-1})\, d(x_N, u_{N-1}) \cdot d(U^{N-2}, Y^{N-1})$$

$$= \min_{P} \int_{\Omega(U^{N-2}, Y^{N-1})} \left[\int_{\Omega(x_N, u_{N-1})} \underbrace{W_{N-1}\, p(x_N, u_{N-1} | U^{N-2}, Y^{N-1})\, d(x_N, u_{N-1})} \right] p(U^{N-2}, Y^{N-1})\, d(U^{N-2}, Y^{N-1})$$

$p(u_{N-1} | U^{N-2}, Y^{N-1})$

$$= \overbrace{p(x_N \mid u_{N-1}, U^{N-2}, Y^{N-1}) \cdot p(u_{N-1} \mid U^{N-2}, Y^{N-1})}$$

$$\min_p J_{N-1} = \min_p \int_{\Omega(U^{N-2}, Y^{N-1})} \int_{\Omega(x_{N}, u_{N-1})} w_{N-1} \, p(x_N \mid u_{N-1}, U^{N-2}, Y^{N-1}) \cdot p \; d(x_N, u_{N-1}) \; d(U^{N-2}, Y^{N-1})$$

p .. a func of (U^{N-2}, Y^{N-1})

$$= \int_{\Omega(U^{N-2}, Y^{N-1})} \left[\min_p \underbrace{\int_{\Omega} \cdots\cdots}\right.$$

B-1.3-2.C//

$$= \int_{\Omega(u_{N-1})} \underbrace{\left[\int_{\Omega(x_N)} \bar{W}_{N-1}\, \overbrace{p(x_N \mid U^{N-1}, y^{N-1})}^{p(x_N \mid x_{N-1},\, u_{N-1})}\, dx_N \right]} p \; du_{N-1}$$

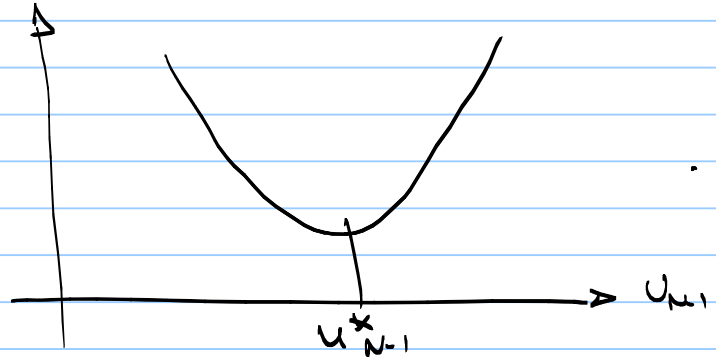
p is not f$^{\underline{o}}$ on x_{N-1}!

$$\min_p J_{N-1} = \int_{\Omega(U^{N-2}, y^{N-1})} \underbrace{\min_p \int_{\Omega(u_{N-1})} \lambda_N(x_{N-1}, u_{N-1})\, p \; du_{N-1}}_{I(p)}$$

$$\min_p I(p) = \min_p \int_{\Omega(u_{N-1})} \lambda_N(x_{N-1}, u_{N-1})\, p \; du_{N-1}$$

Considering that $\lambda_N(x_{N-1}, u_{N-1}) \geq 0$, has a minimum w.r.t. u_{N-1}

$$\min_{p} \int_{\delta(u_{N-1})}$$



$$\cdot\, p \;\; d(u_{N-1})$$

The desired value for p : is : delta function with its singularity @ u^*_{N-1} !

$$p = \delta(u_{N-1} - u^*_{N-1})$$

<u>Conclusion:</u> u_{N-1} should be the deterministic variable u^*_{N-1}.

the solution to $\min_{p} I(p) \Rightarrow I(p^*) = \lambda_N(x_{N-1}, u^*_{N-1})$

To conclude: 1. we transformed the problem from : finding $p(u_{N-1} \mid u^{N-2}, y^{N-1})$
to the problem of finding u_{N-1} as a vector in $\mathbb{R}^m$.

2. the (infinite dimensional) problem of minimizing w.r.t a funct$^=$ in a (functional) ∞-dimensional space is replaced by the (finite dimensional) problem of finding the minimizer of

$$\lambda_N(x_N, u_{N-1}) = E\left\{ \bar{W}_N(x_N, u_{N-1}) \right\}.$$