Note Title 4/25/2021

F. Summary

reminder: the 'standard' state-space model

$$x_{k+1} = A_k x_k + B_k u_k + w_k$$

$$y_{k+1} = H_{k+1} x_{k+1} + v_{k+1}$$

x_k $n\times 1$ state

w_k $n \times 1$ process noise

$E\{w_k\} = 0$, $\operatorname{cov}\{w_k\} = Q_k$ $n\times n$, $w_k \sim \mathcal{N}(0, Q_k)$

$(p < n)$

z_k $p\times 1$ measurement

v_k $p\times 1$ measurement noise

$x_0 \sim \mathcal{N}(\bar{x}_0, P_0)$

(w_k, x_0) uncorrelated

$v_k \sim \mathcal{N}(0, R_k)$; (v_k, w_k) uncorrelated

the Kalman Filter

Time Propagation

time k+1

"time" k

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k|k} \qquad P_{k+1/k} = A_k P_{k|k} A_k^T + Q_k$$

measurement

"measurement" k

Linear

$$K_{k+1} = P_{k+1/k} H_{k+1}^T \left(H_{k+1} P_{k+1/k} H_{k+1}^T + R_{k+1} \right)^{-1} \quad \text{KALMAN GAIN}$$

Measurement Update

$$\hat{x}_{k+1/k+1} = \hat{x}_{k+1/k} + K_{k+1}\left(\underline{y_{k+1}} - \underline{H_{k+1} \hat{x}_{k+1/k}} \right)$$

$$P_{k+1/k+1} = (I - KH) P_{k+1/k} \left[\cancel{(I-KH) + KRK^T} \right]$$

linear

function of y_{k+1}

JOSEPH Formula

<u>Comments</u>

1. the Kalman Filter (KF) is the optimal estimator among all possible estimators when the system is a Linear Gaussian system.

2. the Kalman Filter (KF) is a Linear estimator, although we never require that!

3. The KF produces

$$\hat{x}_{k/k} = E\{x_k \mid y^k\} \quad \longrightarrow \text{function of } y^k$$

$$P_{k/k} = \mathrm{Cov}\{\tilde{x}_{k/k} \mid y^k\} \quad \longrightarrow \text{Independent of } y^k$$

Cond. covariance ≡ Uncond. covariance

4. The Covariance computations can be performed OFF-LINE. They can be done prior to "flying" the algorithm.

5. JOSEPH's Formula : numerical stability of order 2 w.r.t. errors in the Gain.
the standard formula $P^+ = (I - KH)P^-$, order 1 w.r.t. errors in the Gain.

6. Feedback nature of the KF. y : measurement

7. Prediction | Measurement Update $H\hat{x}$: predicted measurement $\Big\}\ y - H\hat{x}$

8. Time-varying recursive discrete-time plant

prediction measurement error
measurement residual
effective measurement

input : $\{y_k\}_{k=0\dots}$ output : $\{\hat{x}_{k|k}\}_{k=0,1,\dots}$

9. Time-propagation for P LYAPUNOV

$$P_{k+1|k} = A_k P_{k|k} A_k^T \boxed{+ Q_k}$$

process noise

$$P_{k+1|k} \geq P_{k|k} \qquad P_{k+1|k} \leq P_{k|k} \quad ?$$

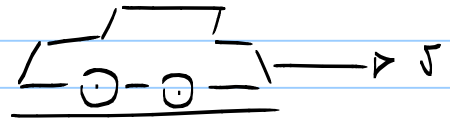
Measurement Update for P
RICCATI

$$P_{k+1|k+1} = (I-KH) P_{k+1|k} (I-KH)^T + KRK^T$$

$$= P_{k+1|k} - P_{k+1|k} H_{k+1}^T (H P_{k+1|k} H^T + R)^{-1} H P_{k+1|k}$$

$$\Rightarrow \boxed{P_{k+1|k+1} \leq P_{k+1|k}}$$

Example. estimating the distance traveled by a car



. the car travels during an hour at $v = 55 \, \frac{mi}{hr}$

after one hour, the odometer reads 55.3 mi

. model: $\dot{x} = v \quad \Rightarrow \quad x_{k+1} = x_k + v \Delta t$ ($x_k = 0$, $v = 55\ mi/hr$, $\Delta t = 1\ hr$) $\Rightarrow \quad z_1^{SPEED} = 55\ mi = x_1 + v_1^{SPEED}$ ($x_1 = 1$), $\quad v_1^{SPEED} \sim \mathcal{N}(0, \overbrace{0.03\ mi^2}^{q})$

$$z_1^{ODO} = 55.3 = x_1 + v_1^{ODO} \sim \mathcal{N}(0, \overbrace{0.001\ mi^2}^{r})$$

. estimator (KF)

Kalman Gain $\quad K = q(q+r)^{-1} = \frac{q}{q+r} = 0.996 \qquad q, r > 0$

$$P_1 \leq \min(q, r)$$

$$\hat{x}_1 = z_1 + K(\underbrace{z_2 - z_1}) \qquad P_1 = \left(1 - \frac{q}{q+r}\right) P_0$$

$$= 55 + (0.996)\underbrace{55.3 - 55}_{0.3} \qquad = \frac{q + r - q}{q+r} P_0$$

$$= 55.29\ mi \qquad = \frac{r}{q+r} P_0 \quad (P_0 = q) \Rightarrow \quad P_1 = \frac{qr}{q+r} = 0.00097\ mi^2$$

B.1.5. Stochastic Control with Partial Information

B.1.5.1 Problem formulation

Given 1) the following system

$$x_{k+1} = A_k x_k + \overset{\text{new}}{\boxed{B_k u_k}} + w_k \; ; \quad x_0 \sim N(\bar{x}_0, P_0)$$

$$y_{k+1} = H_{k+1} x_{k+1} + v_{k+1}$$

$\{v_k, w_k\}$ zero-mean, white, Gaussian, $V_k > 0$, $W_k \geq 0$

independent of x_0

2) the Cost Function

$$J = E\left\{ \sum_{k=0}^{N-1} \|x_{k+1}\|_Q^2 + \|u_k\|_R^2 \right\} \quad Q, R > 0$$

3) Find the Optimal sequence $\{u_k^*\}$ that minimizes J where $u_k^*\underbrace{(y^k, u^{k-1})}_{\text{Information Pattern}}$

- Kalman Filter with $\boxed{\text{known}}$ control input

Assumptions:

$$\begin{cases} x_{k+1} = A_k x_k + B_k u_k + w_k \\ y_{k+1} = H_{k+1} x_{k+1} + v_{k+1} \end{cases}$$

(v, w) uncorrelated, zero-mean, white, V, W

$\{u_k\}$ is a known sequence

How to modify the KF equations?

Time Propagation $\quad \hat{x}_{k+1/k} = A_k \hat{x}_{k/k} + B_k u_k$

error analysis:

$$\tilde{x}_{k+1/k} = x_{k+1} - \hat{x}_{k+1/k}$$

$$= A_k x_k + B_k u_k + w_k - (A_k \hat{x}_{k/k} + B_k u_k)$$

$$= A_k (x_k - \hat{x}_{k/k}) + \cancel{B_k u_k} - \cancel{B_k u_k} + w_k$$

$\Rightarrow$ the control term u_k does not alter the error dynamics.

Therefore the equations for P are independent of u!

estimation

Conclusion: the Estimation is said to be SEPARATED from the Control.

B.1.5.2 Solution by Dynamic Programming

■ LAST STAGE

$$\min_{u_{N-1}} J_{N-1} = \min_{u_{N-1}} E\left\{ |x_N|_Q^2 + |u_{N-1}|_R^2 \right\} \quad \text{given } y^{N-1}, u^{N-2}, \text{ assuming } u_{N-1} = f(y^{N-1}, u^{N-2})$$

$$= \min_{u_{N-1}} E\left\{ E\left\{ |x_N|_Q^2 + |u_{N-1}|_R^2 \mid y^{N-1}, u^{N-2} \right\} \right\}$$

$$= E\left\{ \underbrace{\min_{u_{N-1}} E\left\{ |x_N|_Q^2 + |u_{N-1}|_R^2 \mid y^{N-1}, u^{N-2} \right\}}_{\to\ u_{N-1}(y^{N-1}, u^{N-2})} \right\}$$

why? the Fundamental lemma of Stochastic Control (B.1.3)

Define $\boxed{S_N = Q}$

solve

$$\boxed{\min_{u_{N-1}} E\left\{ |x_N|_Q^2 + |u_{N-1}|_R^2 \mid y^{N-1}, u^{N-2} \right\} \qquad (*)}$$

- Solution via square completion

Insert the process equation $x_N = A x_{N-1} + B u_{N-1} + w_{N-1}$

$$|x_N|^2_{S_N} + |u_{N-1}|^2_R = |x_{N-1}|^2_{A^TS_NA} + |u_{N-1}|^2_{B^TS_NB} + \quad + 2 x_{N-1}^T A^T S_N B u_{N-1} + 2 x_{N-1}^T A^T S_N w_N + 2 w_{N-1}^T S_N B u_{N-1} + |u_{N-1}|^2_R$$

Apply $E\{\cdot \mid y^{N-1}, u^{N-2}\}$ and remember 1) u_{N-1} is deterministic at this stage

2) w_{N-1} is zero mean white noise

$$\Rightarrow |u_{N-1}|^2_{B^TS_NB+R} + 2u_{N-1}^T B^T S_N A \boxed{E\{x_{N-1} \mid y^{N-1}, u^{N-2}\}}\,! + \underbrace{E\{|x_{N-1}|^2_{A^TS_NA} \mid y^{N-1}, u^{N-2}\} + \operatorname{tr}(WS_N)}$$

$\hat{x}_{N-1}$: calculated by Kalman Filter.

$\Rightarrow$ by square completion

$$|u_{N-1} + \cancel{(B^TS_NB+R)^{-1}} B^TS_NA\hat{x}_{N-1}|^2_{B^TS_NB+R} - |\hat{x}_{N-1}|^2_{A^TS_NB(B^TS_NB+R)B^TS_NA}$$

By inspection, $\boxed{u^*_{N-1} = -(B^T S_N B + R)^{-1} B^T S_N A \underbrace{\hat{x}_{N-1}}_{KF}}$ the optimal control strategy

$\Rightarrow$ the Optimal Cost after substitution of u^*_{N-1}

$$- |\hat{x}_{N-1}|^2 + E\{ |x_{N-1}|^2_{A^T S_N A} \mid y^{N-1}, u^{N-2} \} + tr(W S_N)$$

$$= A^T S_N B (B^T S_N B + R)^{-1} B^T S_N A$$

objective : to blend the optimal cost at the last stage with the new cost at stage $N-2$.

<u>remember</u> : scalar $E\{x^2\} = E^2\{x\} + Var\{x\}$

vector $E\{x x^T\} = E\{x\} E^T\{x\} + Cov\{x\}$

remember $\hat{x}_{N-1} = E\{x_{N-1} \mid y^{N-1}, u^{N-2}\} \longleftarrow$ KF

$$P_{N-1} = \mathrm{Cov}\{ x_{N-1} \mid y^{N-1}, u^{N-2} \} \longleftarrow \text{KF}$$

So, we can rewrite the optimal cost :

$$E\{ |x_{N-1}|^2_{A^T S_N A - \square} \mid y^{N-1}, u^{N-2} \} + tr(P_{N-1} \square) + tr(W S_N)$$

In fact, the original problem is formulated on the unconditional expectation

then the Optimal Cost at stage N-1

$$\boxed{J_N^* = E\{ |x_{N-1}|^2_{A^T S_N A - \square} \} + tr(P_{N-1} \square) + tr(W S_N)}$$

Second-to-Last Stage

Principle of Optimality : $J_{N-1} = J^*_{N-1} + E\{ |x_{N-1}|^2_Q + |u_{N-2}|^2_R \}$

$$\min_{u_{N-2}} J_{N-1} = \min_{u_{N-2}} \left[E\left\{ |x_{N-1}|^2_{A^T S_N A - \square} + |x_{N-1}|^2_{Q} + |u_{N-2}|^2_R \right] + tr(P_{N-1} \square) + tr(W S_N) \right.$$

Given y^{N-2}, u^{N-3}, assuming $u_{N-2} = f(y^{N-2}, u^{N-3})$

equivalent problem

$$\min_{u_{N-2}} \left[E\left\{ |x_{N-1}|^2_{\underbrace{A^T S_N A - \square + Q}_{S_{N-1}}} + |u_{N-2}|^2_R \,\middle|\, y^{N-2}, u^{N-3} \right\} \right]$$

Define $S_{N-1} = A^T S_N A - \square + Q$

$$\min_{u_{N-2}} \left[E\left\{ |x_{N-1}|^2_{S_{N-1}} + |u_{N-2}|^2_R \,\middle|\, y^{N-2}, u^{N-3} \right\} \right] \qquad (**)$$

This problem is similar to the (*) problem solved in the last slide

Conclusion: the Algorithm Linear Quadratic Gaussian Control

For $k = N-1, N-2 \dots 0$ calculate

$$S_N = Q$$

$$S_k = A^T S_{k+1} A - A^T S_{k+1} B (B^T S_{k+1} B + R)^{-1} B^T S_{k+1} A + Q$$ ← Independent from the KF

SEPARATION PROPERTY OF CONTROL AND ESTIMATION

Then, for $k = 0, 1, 2, \dots, N-1$

Compute $\hat{x}_k$ using the Kalman Filter

Compute the optimal control

$$u_k^* = -(B^T S_{k+1} B + R)^{-1} B^T S_{k+1} A \hat{x}_k$$

"independent" of the Control Gain Calculation

$$J_k^* = |\hat{x}_k|^2_{A^T S_{k+1} A - A^T S_{k+1} B (B^T S_{k+1} B + R)^{-1} B^T S_{k+1} A} +$$

$$+ \operatorname{tr}\left(P_k A^T S_{k+1} A\right) + \sum_{i=k+1}^{N-1} \operatorname{tr}\left(P_i A^T S_{i+1} B\left(B^T S_{i+1} B + R\right)^{-1} B^T S_{i+1} A\right)$$

$$+ \sum_{i=k}^{N-1} \operatorname{tr}\left(W S_{i+1}\right)$$