Note Title 6/6/2021

B.2.5 Stationnary Optimal Control ($t_f \to +\infty$)

B.2.5.1

- Optimal Control under stationnary conditions
- LT Invariant systems OR Slowly-varying systems
- Advantage over the nonstationnary controller : gain unique, time-invariant

→ less computations, less memory.

B.2.5.2 General conditions

Ergodic random processes $\lim_{t \to \infty} \frac{1}{t} \int_0^t L[x_\tau, \Phi(x_\tau)] d\tau = E\{L\} = \int_{\mathbb{R}^n} L[x_t, \Phi(x_t)] \mu_\Phi(dx)$

here Φ – control law

μ_Φ – probability measure : $\mu_\Phi(B) = P\{x_0 \in B\} \iff \mu_\Phi(B) = P\{x(t) \in B\}$, $\forall t > 0 \quad \forall B \subseteq \mathbb{R}^n$

Loosely speaking: expectation is identical to time average

Theorem (Wonham) (1967)

If ϕ° an optimal control law, scalar λ, function $\bar{J}(x)$ such that

1. $E\left\{ \bar{J}(x) + |x| \left|\frac{\partial \bar{J}}{\partial x}\right| + |x|^2 \left|\frac{\partial^2 \bar{J}}{\partial x^2}\right| \right\} < \infty$

$\lambda = \mathcal{L}^{\phi^\circ}[\bar{J}(x)] + L(x, \phi^\circ(x)) \qquad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^n$

$\lambda \leq \mathcal{L}^{u}[\bar{J}(x)] + L(x, u)$

diff. generator of x cost

then ϕ° is optimal and $\lambda = E\{ L(x, \phi^\circ(x)) \} < \infty$.

B.2.5.3 the stationary Controller

Assume that $t_f \to \infty$ and that $L(x,u) = \|x\|_Q^2 + \|u\|_R^2$; Q, R are time-invariant.

Objective: to find $\gamma(0,\infty)$ that minimizes $J_\infty = E\left\{\lim_{t\to\infty} \frac{1}{t}\int_0^t L(x,u)\,dt\right\} = E\{L(x,u)\}$

where the dynamical constraint

$$dx = (Fx + Bu)\,dt + dw \; ; \quad F, B \text{ are time-invariant}$$

where w_t Brownian Motion, $E\{w_t w_\tau^T\} = W \min(t,\tau)$ $\{$i.e $E\{dw_t dw_\tau\} = W\delta(t-\tau)\}$

W - time invariant

and the measurement is

$$dz = Hx\,dt + d\vartheta$$

and ϑ_t Brownian motion, $E\{\vartheta_t \vartheta_\tau^T\} = V \min(t,\tau)$.

F, B, Q, R, H, V, W time invariant

$Q \geq 0$, $R > 0$, $V > 0$, $W \geq 0$

theorem: the stationary LQG

If $(F, B, \sqrt{Q})$ and $(F^T, H^T, \sqrt{W})$ are both "minimal realization"

i.e. simultaneously fully controllable and observable

then the controller $\phi^o(x) = -R^{-1}B^T S \hat{x}$

Algebraic Riccati Equation (A.R.E.) $0 = F^T S + SF + Q - SBR^{-1}B^T S$

$S^T = S \geq 0$

$d\hat{x} = (F - BR^{-1}B^T S)\hat{x}\,dt + K(dz - H\hat{x}\,dt)$

$K = PH^T V^{-1}$

Controller ARE

$0 = FP + PF^T + W - PH^T V^{-1} HP$

Filter ARE

C. The JUMP Linear Quadratic Gaussian Problem

C.1 Introduction to Markovian Jump systems

C.1.1 Motivation

A. TARGET TRACKING

- Air TRAFFIC CONTROL near crowded airports

Model a : equations of motions , physical variable : height, velocity, bank angle flight path angle ...

b : sudden changes in the acceleration due to switches between Modes i.e.

$$\begin{cases} \text{Mode 1 : Ascent} \\ \text{Mode 2 : Turn} \\ \text{Mode 3 : Accelerated Flight} \\ \text{Mode 4 : Cruise flight} \end{cases}$$

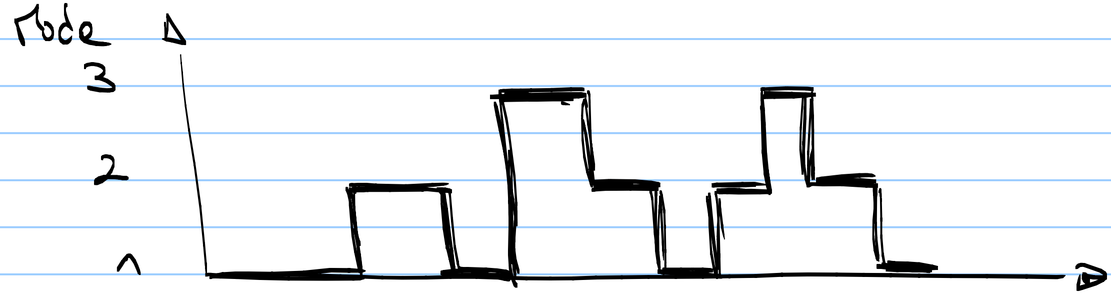
Transitions depend on the actions of the pilot, ATC, Weather

- Missile tracking

Pursuer - Evader : dynamics / kinematics of the problem

physical variables: height, relative velocity, line-of-sight angle rate.

Modes: the evader is trying to switch its acceleration among discrete levels in order to confuse the opponent.



- Manufacturing Processes

state $\quad \mathbb{N} \times \mathbb{R}^n \times S$

\# products $\quad$ Power $\quad$ Possible Failures $\quad$ Mode 1 : healthy
Mode 2 : failed

- Solar Thermal Receiver

Field of Movable Mirrors → Reflection Sunlight → Thermal Tank → turbine → electricity

Key challenge : changing insulation, sudden, incompletely predicted, creates hick-ups at the regulated Temperature Output.

{ Mode 1: Sunny $\quad$ MARKOV
Mode 2: Partial Cloudy → Random parameter; probabilities of transition.
Mode 3: Cloudy

- Fault-Tolerant Control
  - Large complex mechanical systems are prone to failure.
    Airplane \ Engine \ Space structures
  - Field: Fault-detection - isolation - reconfiguration (control)

C.1.2 Preliminaries

A. Discrete-time Markov Chain

$\{ y_k, k \in \mathbb{N} \}$ $\qquad S_y = \{1, 2, \dots, M\}$ finite alphabet
discrete state-space
sample space

"Chain" $\quad y_{k+1} = f(y_k, v_k)$

"Markov" $\quad Pr\{ y_{k+1} \mid \mathcal{Y}^k \} = Pr\{ y_{k+1} \mid y_k \}$; v_k independent random variables

"Transition Matrix" $\underset{\nu \times \nu}{P} = (p_{ij}) = \left(\Pr\{ y_{k+1} = j \mid y_k = i \} \right)$ $\quad i = 1 \ldots \nu$, $j = 1 \ldots \nu$

"Time homogenous" p_{ij} time-invariant

<u>example</u>

$$y_{k+1} = \begin{cases} +y_k & \text{w.p. } p \quad 0 \le p \le 1 \\ & \qquad p+q = 1 \\ -y_k & \text{w.p. } q \quad 0 \le q \le 1 \end{cases} \qquad y_0 \in \{-1, 1\}$$

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} \Pr\{y_{k+1} = -1 \mid y_k = -1\} & \Pr\{y_{k+1} = -1 \mid y_k = 1\} \\ \Pr\{y_{k+1} = 1 \mid y_k = -1\} & \Pr\{y_{k+1} = 1 \mid y_k = 1\} \end{bmatrix}$$

$$P = \begin{bmatrix} p + q \\ q + p \end{bmatrix} \begin{matrix} = 1 \\ = 1 \end{matrix}$$

- P is a "stochastic" matrix: $p_{ij} \geq 0$ $\sum_{j=1}^{\nu} p_{ij} = 1$

- $\boxed{\text{Chapman - Kolmogorov Equation}}$ for the Distribution.

Let

$$\underset{\nu \times 1}{p_k} \equiv \begin{bmatrix} \Pr\{y_k = 1\} \\ \vdots \\ \Pr\{y_k = i\} \\ \vdots \\ \Pr\{y_k = \nu\} \end{bmatrix}_{\nu \times 1}$$

vector of the unconditional probabilities of the mode at time t_k.

$$\boxed{p_{k+1} = P^T p_k}$$

proof: $\Pr\{y_{k+1} = j\} = \sum_{y_k=1}^{\nu} \Pr\{y_{k+1} = j, y_k = i\}$

$\forall j = 1, 2, \dots, \nu$

$$= \sum_{y_k=1}^{\nu} \Pr\{y_{k+1}=j \mid y_k=i\}\cdot \Pr\{y_k=i\}$$

$$\begin{array}{cc} & j \\ i & \begin{bmatrix} & & \\ & p_{ij} & \\ & & \end{bmatrix} \end{array} \qquad P_{k+1}(j) = \underbrace{\sum_{i=1}^{\nu} p_{ij}\cdot P_k(i)}$$

$$= \underbrace{\begin{bmatrix} p_{1j} & \cdots & p_{ij} & \cdots & p_{\nu j}\end{bmatrix}}_{j^{th}\text{ row of } P^T} \begin{bmatrix} P_k(1)\\ \vdots \\ P_k(i) \\ \vdots \\ P_k(\nu)\end{bmatrix}$$

$$\begin{bmatrix} P_{k+1}(1) \\ \vdots \\ P_{k+1}(\nu) \end{bmatrix} = \begin{bmatrix} & & \\ & P^T & \\ & & \end{bmatrix} \begin{bmatrix} P_k(1) \\ \vdots \\ P_k(r) \end{bmatrix} \iff P_{k+1} = P^T P_k \; ; \; P_0$$

B. Hybrid Markov Processes

$$\{ x_k, y_k : k \in \mathbb{N} \} \qquad x_k \in \mathbb{R}^n \qquad y_k \in S_y$$

$$\begin{cases} x_{k+1} = f_X(x_k, y_k, w_k) & w_k \text{ white noise} \\ y_{k+1} = f_Y(x_k, y_k, v_k) & v_k \text{ white noise} \end{cases}$$

x_k alone is NOT a MARKOV but (x_k, y_k) is MARKOV!

C.2. the JLQG control problem with complete information

C.2.1. Statement of the Problem

(1) $x_{k+1} = A(y_k)\, x_k + B(y_k)\, u_k + w_k$

continuous-value

control

y_k is a random jump parameter.