LQG in Discrete Time (Cont'd and End)

Note Title 5/2/2021

Comments :

1. the sequence of matrices S_k is obtained from the same RICCATI backward equation as in the deterministic case with full-state feedback. Hence the optimal gains for the LQG problem with partial information and the LQR problem with full information are IDENTICAL : the Certainty Equivalence Principle holds.

2. the control law is a Linear mapping of the state estimate $\hat{x}_{k|k}$: similar to the LQR controller with full information. this is a direct result from the assumptions underlying the LQG problem formulation : 1) Linearity of the Plant
   2) Quadratic Cost

3. The Kalman gains are calculated by a Forward RICCATI equation, which is independent of the Control input : the SEPARATION property between Estimation and Control holds for the LQG control.

4. Similarities / Differences between the solutions of the LQR/LQG problems

a. $\hat{x}_{k|k}$ instead of x_k, in the same control law

b. Random process noise $\Rightarrow$ terms of the cost containing W covariance.

c. Estimation error $\Rightarrow$ terms of the cost containing P covariance

5. The LQG algorithm will remain identical in the case where the matrices A, B, Q, R, W, and V are time-varying.

B.2 Continuous-time LQG.

a few words introduction ...

why continuous-time ? ✓ simpler than discrete-time

✓ lower bound on the performances

✓ first principles usually yield continuous-time equations

✓ stochastic differential equations are novel to us

B.2.1. Deterministic Case

B.2.1.1. Dynamic Programming for Continuous-Time systems

- Consider the following non linear differential equation

$$\frac{dx}{dt} = f(t, x) \qquad t_0 \leq t \leq T \qquad x(t_0) = x_0$$

$f(\cdot, \cdot)$ is such

Growth Condition: i) $\exists K > 0 \quad |f(t, x)| \leq K(1 + |x|) \qquad t \in [t_0, T] \quad x \in \mathbb{R}^n$

Uniform Lipshitz Condition: ii) $\exists K > 0 \quad |f(t, x) - f(t, y)| \leq K|x - y| \quad t \in [t_0, T] \quad x \in \mathbb{R}^n$

such that there is a Unique Solution to the diff. equation : $x(t)$

Integration Formula

For any function $\phi(t, x)$, where $\phi, \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}$ continuous

$$\boxed{\begin{aligned} &\phi[s, x(s)] = \phi[t, x(t)] + \int_t^s d\phi[\tau, x(\tau)] \qquad s \geq t \\ &\text{where} \quad d\phi[\tau, x(\tau)] = \left\{ \frac{\partial \phi[\tau, x(\tau)]}{\partial \tau} + \frac{\partial \phi[\tau, x(\tau)]}{\partial x^T} \cdot f[\tau, x(\tau)] \right\} d\tau \end{aligned}}$$

- Problem formulation

Find a control law $u(x,t) \in \mathcal{U}$ that minimizes

$$J(x_0, t_0) \doteq \int_{t_0}^{t_f} L(x,u,t)\, dt + g(x_f)$$

subject to

$$\frac{dx}{dt} \doteq f(x,u,t) \; ; \quad x(t_0) \doteq x_0$$

assumptions :

- $\mathcal{U}$: a - Growth $\exists K_1 > 0 \ |u(x,t)| \leq K_1 \sqrt{1 + |x|^2}$

  b - Lipschitz $\exists K_2 > 0 \ |u(x,t) - u(y,t)| \leq K_2 |x - y|$
- bounded state and cost: $|x_f| < \infty$

  $|L(x,u,t)| < C(1 + |x| + |u|)^k$

Notation: $\quad \gamma(t, t_f) = \{ u(x, \sigma) ; \; x \in \mathbb{R}^n , \; \sigma \in [t, t_f] \}$

- Optimal Return Function

$$J^o(x, t) = \min_{\gamma(t, t_f) \in \mathcal{U}} \left[\int_t^{t_f} L(x_\tau, u_\tau, \tau)\, d\tau + g(x_{t_f}) \right]$$

$$= J\left[\gamma^o(t, t_f) ; \; x, t \right]$$

$$\Longrightarrow \quad J^o(x_{t_f}, t_f) = g(x_{t_f})$$

- Sufficient Condition of Optimality $\equiv$ the Hamilton-Jacobi Equation

Consider $\gamma^1(t, t_f) = \begin{cases} u(x, \tau) & t \leq \tau \leq s \\ \text{optimal } \gamma^0(s, t_f) & s \leq \tau \leq t_f \end{cases}$

$$\begin{cases} J^0(x, t) = \displaystyle\int_t^s - \frac{dJ^0}{d\tau} d\tau + \cancel{J^0}(x_s, s) \\ J[\gamma^1(t, t_f); x, t] = \displaystyle\int_t^{t_f} L + g(x_{tf}) = \int_t^s L(x_\tau, u_\tau, \tau) d\tau + \underbrace{\int_s^{t_f} L(x_\tau, u^0_\tau, \tau) d\tau + g(x_{tf})}_{\equiv \cancel{J^0}(x_s, s)} \end{cases}$$

The candidate $J^\circ(x,t)$ is optimal iff

$$J^\circ(x,t) \leq J[\gamma'(t,t_f); x,t] \quad \forall \gamma' \in \mathcal{U} \quad \forall t \in [t,t_f]$$

$$0 \leq J[\gamma'(t,t_f); x,t] - J^\circ(x,t)$$

$$0 \leq \underbrace{\int_t^s L(x_\tau, u_\tau, \tau) + \frac{dJ^\circ}{d\tau}(x_\tau, \tau) \quad d\tau}_{\geq 0}$$

$$0 \leq L(x_\tau, u_\tau, \tau) + \frac{dJ^\circ}{d\tau}(x_\tau, \tau) \qquad \forall u \in \mathcal{U} \quad \forall \tau \, [t,s]$$

$$-\frac{dJ^{o}}{d\tau}(x_\tau,\tau) \leq L(x_\tau,u_\tau,\tau)$$

rate of increase of the Cost along the Optimal Traj.

rate of increase of the Cost along a nonoptimal trajectory

Remembering the Integration Formula; where $\Phi \longleftrightarrow J^{o}$

$$-\frac{\partial J^{o}}{\partial \tau}(x_\tau,\tau) \leq \frac{\partial J^{o}}{\partial x}(x,\tau)\cdot f(x_\tau,u_\tau^{o},\tau) + L(x_\tau,u_\tau,\tau)$$

Equality takes place when $u_\tau = u_\tau^{o}$ $\Longrightarrow$ H-J Equation

$$\begin{cases} -\dfrac{\partial J^{o}}{\partial \tau}(x_\tau,\tau) = \min\limits_{u\in\mathcal{U}}\left[\dfrac{\partial J^{o}}{\partial x}(x_\tau,\tau)\,f(x_\tau,u_\tau,t) + L(x_\tau,u_\tau,\tau)\right] & ;\ t_o \leq \tau \leq t_f \\ J^{o}(x_{tf},t_f) = g(x_{tf}) \end{cases}$$

B. 2.1.2 The LQR problem / algorithm

- Pb statement

Given

$$\frac{dx}{d\tau} = F_\tau x_\tau + B_\tau u_\tau \qquad (1)$$

$$J[\gamma(0,t_f)] = \int_0^{t_f} \|x_\tau\|^2_{Q_\tau} + \|u_\tau\|^2_{R_\tau} \, d\tau + \|x_{t_f}\|^2_{S_f}$$

$$Q_\tau^T = Q_\tau \geq 0 \qquad R_\tau^T = R_\tau > 0 \qquad S_f^T = S_f \geq 0$$

Solve

$$\min_{\gamma(0,t_f) \in \mathcal{U}} J[\gamma(0,t_f)] \text{ subject to } (1)$$

- Solution $\boxed{J^o(x,\tau) = \|x\|^2_{S_\tau}}$ a Guess ... $S_\tau^T = S_\tau \geq 0$

from this guess: $J^o(x_{tf}, t_f) = \|x_{tf}\|^2_{S(t_f)} = \|x_{tf}\|^2_{S_f} \Rightarrow S(\tau = t_f) = S_f$

unknown given

by inserting $J^o(x,\tau) = \|x\|^2_{S_\tau}$ into the H-J equation:

$$0 = \frac{\partial J^o}{\partial \tau}(x,\tau) + \min_{u \in \mathcal{U}} \left[\frac{\partial J^o}{\partial x}(x_\tau, \tau) f(x_\tau, u_\tau, \tau) + L(x_\tau, u_\tau, \tau) \right]$$

$$= x^T \dot{S} x + \min_{u \in \mathcal{U}} \left[x^T (S_\tau + S_\tau^T) \cdot (F_\tau x_\tau + B_\tau u_\tau) + \|x_\tau\|^2_{Q_\tau} + \|u_\tau\|^2_{R_\tau} \right]$$

$$\min_u \left[2x^T S B u + \|u\|_R^2 \right] \longmapsto \boxed{u^o(x,\tau) = -R_\tau^{-1} B_\tau^T S_\tau x_\tau}$$

Optimal Control Law

- Back to the H-J equation:

$$0 = x^T \dot{S}_\tau x + x^T(SF + F^T S)x - x^T(SBR^{-1}B^T S)x + \|x\|_Q^2 \quad \forall (x,\tau)$$

$$= x^T \Big[\underbrace{\dot{S} + SF + F^T S - SBR^{-1}B^T S + Q}_{=0 \quad \forall \tau} \Big] x \quad \forall (x,\tau)$$

A sufficient and necessary condition:

$$\dot{S} + SF + F^T S - SBR^{-1}B^T S + Q ;$$

equivalently

LQR in cont. time.

$$\left\{\begin{aligned} &\boxed{-\dot{S}_{\tau} = F^T_{\tau} S_{\tau} + S_{\tau} F_{\tau} - S_{\tau} B_{\tau} R^{-1}_{\tau} B^T_{\tau} S_{\tau} + Q_{\tau} \; ; \quad S(t_f) = S_f} \\ &\text{BACWARD RICCATI DIFFERENTIAL EQ.} \\ &\boxed{u^o = -R^{-1}_{\tau} B^T_{\tau} S_{\tau}\, x} \\ &\text{OPTIMAL CONTROL LAW} \end{aligned}\right.$$