A.1 INTRODUCTION TO DYNAMIC PROGRAMING

A.1.1. Elements of a Deterministic Optimal Control Problem

1 MODEL

The mathematical model that describes the plant

exaple:

$$x_{k+1} = A_k x_k + B_k u_k$$

$$z_{k+1} = H_{k+1} x_{k+1}$$

x_k state
u_k control
z_k measurement

⋮

2 COST FUNCTIONAL

exaple: quadratic

k+1

$$J = \sum_{k=0}^{N-1} \| x_{k+1} - x_d \|^2$$

x_d : desired trajectory

$$J = \underbrace{\sum_{k=0}^{N-1} \| x_{k+1} - x_d \|^2}_{\text{Tracking}} + \underbrace{\| u_k \|^2}_{\text{Control}}$$

k

achieves a balance between

$$J = \sum_{k=0}^{N-1} \|x_{k+1} - x_d\|_Q^2 + \|u_k\|_R^2$$

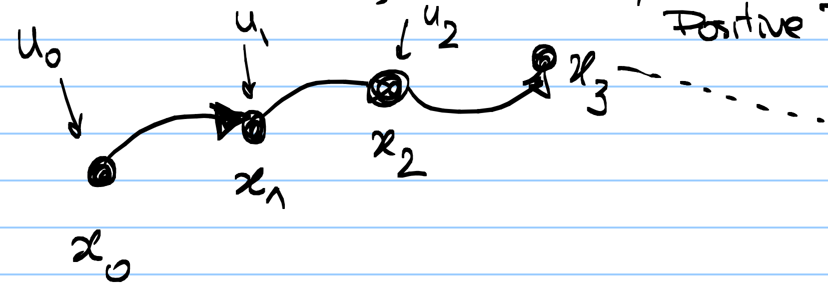
Positive Semi-Definite

Q weight matrix on the Tracking Error $\quad x \; n\times 1 \Rightarrow Q \; n\times n \; ; \; Q \geq 0$

R weight matrix on the Control effort $\quad u \; m\times 1 \Rightarrow R \; m\times m \; ; \; R > 0$

Positive Definite

Notation: $\begin{cases} \|x\|_Q^2 \equiv x^T Q x \\ \|u\|_R^2 \equiv u^T R u \end{cases}$



3. Constraints $\begin{cases} \text{on the state} \; ; \; \text{path constraints} \; ; \; \text{terminal constraints} \\ \text{on the control} \\ \text{hard constraints} \mid \text{soft constraints} \end{cases}$

A.1.2 DYNAMIC PROGRAMMING

RICHARD BELLMAN, 1960's RAND, optimization, Efficient computationally-speaking

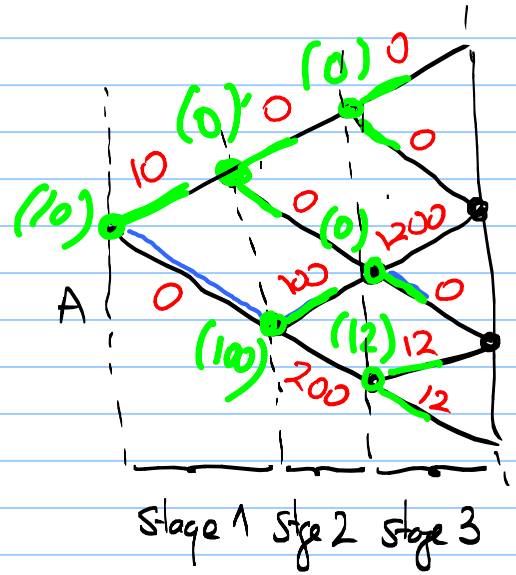
<u>example</u> Path planning

Open-loop strategy D U D

total cost = 0 ! the optimum.

For the new map of costs

the Open-loop strategy (D U D) yields a total cost of 100.



Plant: at each stage the traveler picks a direction U or D.

Objective: to minimize the total cost of travel from A to B after 3 stages.

Obviously, a better choice consists, for instance, of the strategy U_U_U : Cost = 10

A better strategy : to use Feedback at each stage to inform the traveler on the total cost as a function of his current position.

STEP 1 : backward building of the map of all values of the optimized cost and the related decisions

STEP 2 : forward propagation and construction of the Optimal Path

In this case :

U U U
D D D
U D D

3 possible optimal paths.

Opt. Cost = 10 .

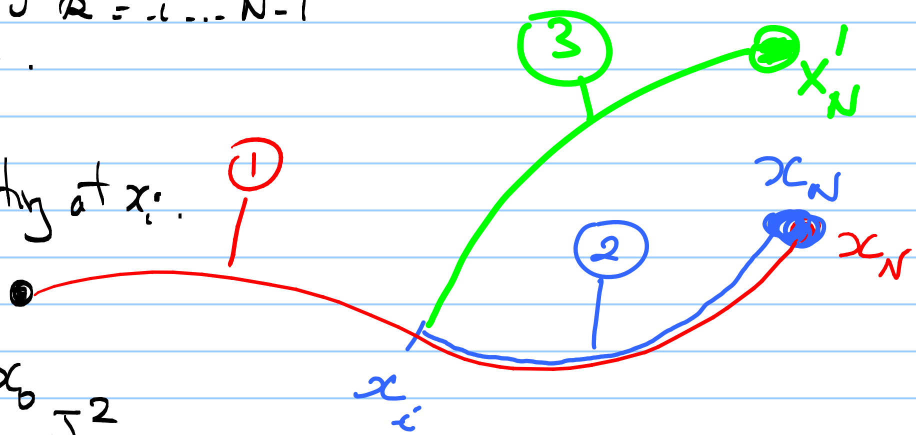
The PRINCIPLE OF OPTIMALITY

Given a certain optimal control problem, and the associated optimal policy $\{u_k^*\}_{k=0\ldots N-1}$

and the optimal trajectory $\{x_k^*\}_{k=0\ldots N-1}$

then any subsequence $\{u_k^*, x_k^*\}_{k=i\ldots N-1}$ represents an optimal solution

of the "same" optimal control problem.



- If 1 is optimal starting at x_0 then 2 is optimal starting at x_i.

JUSTIFICATION

If 3 is optimal starting at x_i: $J^3_{i\to N-1} < J^2_{i\to N-1}$

therefore

$$\underbrace{J^1_{0\to i-1} + J^3_{i\to N-1}}_{} < \underbrace{J^1_{0\to i-1} + J^2_{i\to N-1}}_{}$$

x_0

$$J^{13}_{0\to N-1} \leq J^{1}_{0\to N-1} \;?$$

thus (1) is Not the Optimal trajectory ∴

Example AIR TRAFIC CONTROL problem.

One airplane, three airports, three days

Everyday, the airplane either 1) stay at the same airport

Each step takes one day 2) flies to another airport

The costs of either staying for maintenance or travelling are given.

Notation: x_k : location (airport) every day (k) $x_k \in \{1, 2, 3\}$

Table of Costs

x_k \ x_{k+1}	1	2	3
1	0.25	0.5	0.75
2	0.5	0.25	0.1
3	0.75	0.1	0.25

Objective : To provide the optimal policy for a 3 days horizon as a function of the Initial airport.

Solution :

Intuition first. Suppose $x_0 = 1$: then stay @ the airport ...

" $x_0 = 2$ then fly to 3

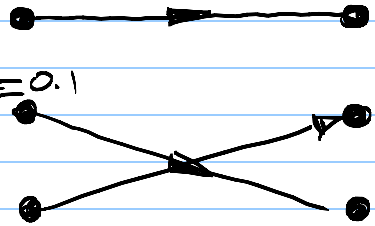
" $x_0 = 3$ then fly to 2

Let's apply D.P. START FROM THE END !

- Last Stage

x_2 is given

$J_2^*(1) = 0.25$



$J_2^*(2) = 0.1$

$J_2^*(3) = 0.1$

$$J_2^*(x_2) = \begin{bmatrix} 0.25 \\ 0.1 \\ 0.1 \end{bmatrix}$$

as a result, the optimal solution includes two things

1) the optimal policy
2) the optimal cost

as a function of x_2 !

Here is our Feedback.

• Second-to-Last Stage

cost of moving the A/C during Day 2

Given x_1 $$J_1(x_1) = W_1 + J_2^*$$ the optimized cost

Objective: $$\min J_1(x_1) = \min \left[W_1 + J_2^* \right]$$ D.P. idea

the optimal cost J_2^*, a.k.a. cost-to-go, is included in the Current Cost-to-Go

If $x_1 = 1$ $$\min J_1(1) = \min \begin{cases} 0.25 + 0.25 = 0.5 \longrightarrow 0.5 \\ 0.5 + 0.1 = 0.6 \\ 0.75 + 0.1 = 0.85 \end{cases}$$

So $J_1^*(1) = 0.5$ and the optimal decision Rule: $1 \longrightarrow 1$

If $x_1 = 2$ $$\min J_1(2) = \min \begin{cases} 0.5 + 0.25 = 0.75 \\ 0.25 + 0.1 = 0.35 \\ 0.1 + 0.1 = 0.2 \longrightarrow 0.2 \end{cases}$$

So $J_1^*(2) = 0,2$ $2 \longrightarrow 3$

If $x_1 = 3$ $\quad \min J_1(3) = \min \begin{cases} 0.75 + 0.25 = 1.0 \\ 0.1 + 0.1 = 0.2 \\ 0.25 + 0.1 = 0.35 \end{cases} \longrightarrow 0.2$

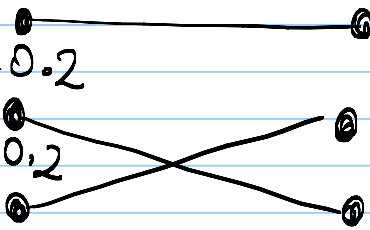
So $J_1^*(3) = 0.2$ " " " " $3 \rightarrow 2$

Sum-up : the Optimal policy and Optimal cost-to-go at Day 2

$J_1^*(1) = 0.5$

$J_1^*(2) = 0.2$

$J_1^*(3) = 0.2$



$$J_1^*(x_1) = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.2 \end{bmatrix}$$

- First Stage

Given x_0

$$J_0(x_0) = w_0 + J_1^*$$

$$\min J_0(x_0) = \min \left(w_0 + J_1^* \right)$$

If $x_0 = 1$ $\quad \min J_0(1) = \min \begin{cases} 0.25 + 0.5 = 0.75 \\ 0.5 + 0.2 = 0.7 \\ 0.75 + 0.2 = 0.95 \end{cases} \longrightarrow J_0^*(1) = 0.7$

the optimal first decision $1 \to 2$

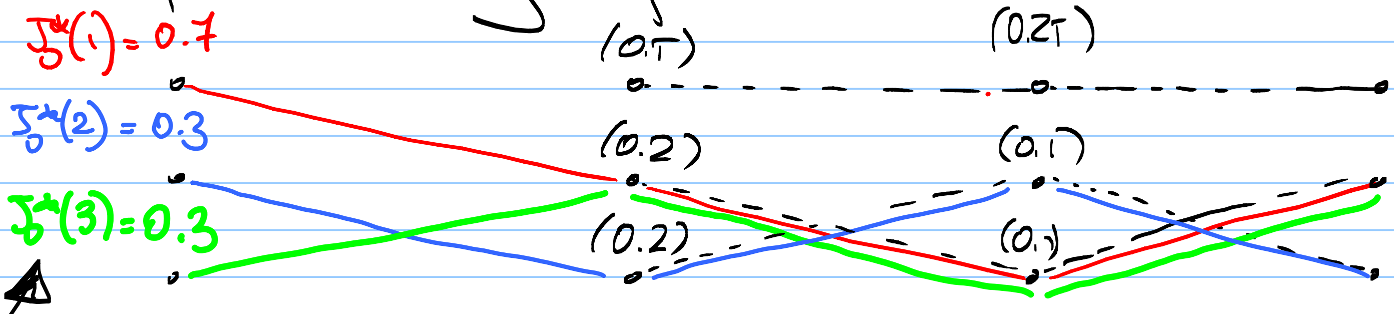
If $x_0 = 2$ $\quad \min J_0(2) = \min \begin{cases} 0.5 + 0.5 = 1.0 \\ 0.25 + 0.2 = 0.45 \\ 0.1 + 0.2 = 0.3 \end{cases} \longrightarrow J_0^*(2) = 0.3$

the optimal " " $2 \to 3$

If $x_0 = 3$ $\quad \min J_0(2) = \min \begin{cases} 0.75 + 0.5 = 1.25 \\ 0.1 + 0.2 = 0.3 \\ 0.25 + 0.2 = 0.45 \end{cases} \longrightarrow J_0^*(3) = 0.3$

the optimal " " $3 \to 2$

To Conclude: for a 3-Day trip



Given the initial conditions

In order to obtain the Optimal Policy and the Associated Optimal costs, using D.P., what we actually calculate, was a whole tree of optimal policies and their optimal costs-to-go.

- Dynamic Programing procedure

Given : 1) $x_{k+1} = f_k(x_k, u_k)$ system equation

$y_k = x_k$ measurement equation : perfect and full inform.

2) $J = \sum_{k=0}^{N-1} \omega_k(x_{k+1}, u_k)$; $\omega_k \geq 0$ cost functionnal

Find : the control sequence $\{u_k^*\}$ that minimizes J

Notation $J_k = \sum_{k=K}^{N-1} \omega_k$

General Solution

- **LAST STAGE**

Given x_{N-1}, assuming we followed the optimal path. We aim at minimizing the cost w.r.t. u_{N-1} in the last stage.

$$J_{N-1} \equiv \omega_{N-1}(x_N, u_{N-1})$$

$$\min_{u_{N-1}} J_{N-1} = \min_{u_{N-1}} \omega_{N-1}(x_N, u_{N-1}) \qquad (1)$$

! apply the system equation

$$x_N = f_{N-1}(x_{N-1}, u_{N-1})$$

$$\text{So } \min_{u_{N-1}} J_{N-1} = \min_{u_{N-1}} \underbrace{w_{N-1}\left(f_{N-1}(x_{N-1}, u_{N-1}), u_{N-1} \right)}$$

known ↙   the optimization variable ↙

$$= \min_{u_{N-1}} \tilde{W}_{N-1}(x_{N-1}, u_{N-1})$$

$$= \min_{u_{N-1}} J_{N-1}(x_{N-1}, u_{N-1}) \qquad (2)$$

(!) Perform minimization of $J_{N-1}(x_{N-1}, u_{N-1})$ over u_{N-1}

$\mapsto$ u^*_{N-1} as a function of x_{N-1} : $u^*_{N-1}(x_{N-1})$

$\mapsto$ J^*_{N-1} " " of x_{N-1} : $J^*_{N-1}(x_{N-1})$

to Conclude : the function $J^*_{N-1}(x_{N-1})$ is called the <u>Optimal Return Function</u>

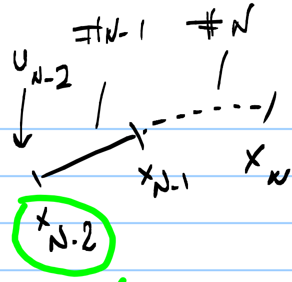
a function of the <u>Current</u> initial conditions.

• One before the Last Stage.

Opt. Ret. Funct°

$$J_{N-2} = W_{N-2}(x_{N-1}, u_{N-2}) + \bar{J}^*_{N-1}(x_{N-1})$$

final state   current control



Assume x_{N-2} is given

Here, we apply the principle of Optimality!

$$\min_{u_{N-2}} J_{N-2} = \min_{u_{N-2}} \left\{ w_{N-2}(x_{N-2}, u_{N-2}) + \bar{J}^*(x_{N-1}) \right\}$$

! Use the System equation in order to express the x_{N-1}, x_{N-2}, u_{N-2} relationship

$$x_{N-1} = f_{N-2}(x_{N-2}, u_{N-2}).$$

- **General Stage**

Given x_k

$$\min_{u_k} J_k = \min_{u_k} \{ g_k(x_{k+1}, u_k) + J_{k+1}^*(x_{k+1}) \}$$

Use the System equation $x_{k+1} = f_k(x_k, u_k)$

Perform the minimization w.r.t. u_k

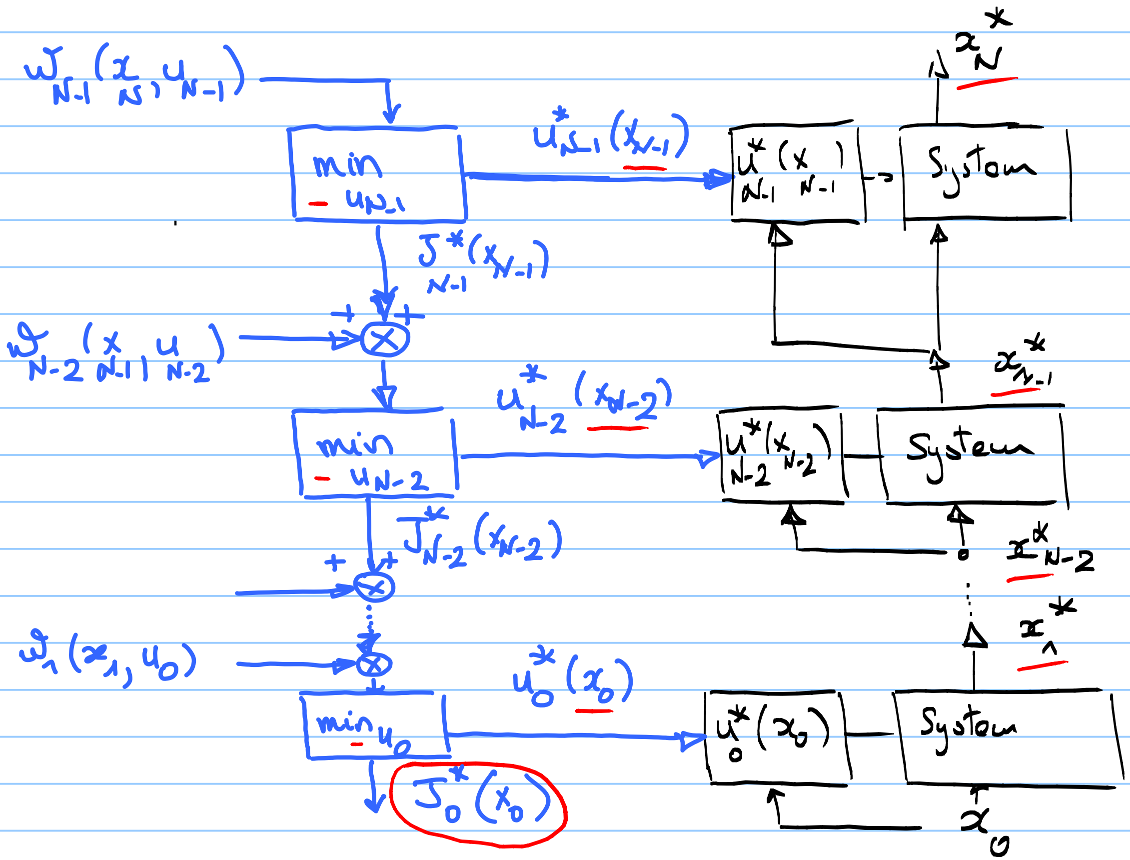
$$\min_{u_k} J_k = \min_{u_k} \left[g_k(f_k(x_k, u_k), u_k) + J_{k+1}^*(f_k(x_k, u_k)) \right\}$$

$$= \min_{u_k} J_k(x_k, u_k)$$

$\rightarrow u_k^*(x_k)$

$$J_k^* = \min_{u_k} J_k(x_k, u_k) = J_k(x_k, u_k^*)$$

Diagram of the D.P. procedure



BACKWARD PROPAGATION

Given $\{x_k\}$, the bwd propagation provides

$\{u_k^*(x_k)\}$ Feedback Control

$\{J_k^*(x_k)\}$

at $k=0$, $J_0^*(x_0)$ of the whole problem.

FORWARD PROPAGATION

Starts with x_0, build the optimal path and calculates the Optimal Controls $\{u_k^*\}$ $\{x_k^*\}$ $k=0 \dots N-1$

Comments:

1. each stage of the DP procedure solves a sub-problem of the multiple stages pb.
2. for each sub-problem the initial conditions x_k are the current state along the optimal path.
3. Typically, the DP procedure solves each subproblem w.r.t. a single control variable
4. The DP procedure builds the Control sequence as a State Feedback Law.
5. The Solution of the Optimal Control Problem $\{u_k^*\}$ $\{x_k^*\}$ is obtained

by a BACKWARD - FORWARD propagation scheme.

A.1.3 Example

Given $x_{k+1} = a x_k^2 + b u_k \qquad b \neq 0$

$\longrightarrow J = \sum_{k=0}^{N-1} x_{k+1}^{2}$

Find the sequence of $\{u_k^*\}$ such as to minimize J

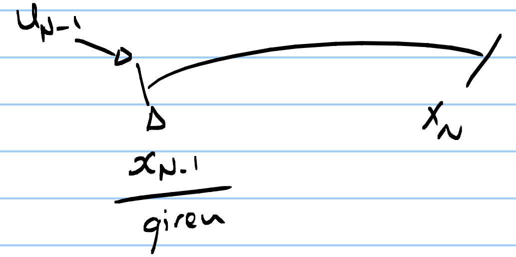
Solution via D.P.

① LAST STAGE. $\qquad J_{N-1} = \omega_{N-1}(x_N, u_{N-1}) = x_N^2$

Given x_{N-1}

$$\min_{u_{N-1}} x_N^2$$

$$\min_{u_{N-1}} \left[a x_{N-1}^2 + b u_{N-1} \right]^2 \qquad (*)$$



necessary conditions $\quad 2b\,(a x_{N-1}^2 + b u_{N-1}) = 0 \implies u_{N-1}^* = -\frac{a}{b} x_{N-1}^2$

$$\Rightarrow \quad J^*_{N-1} = 0$$

- Second to Last Stage

Given x_{N-2}

$$J_{N-2} = \omega_{N-2} + J^*_{N-1}$$

$$= \omega_{N-2}$$

$$\min_{u_{N-2}} J_{N-2} = \min_{u_{N-2}} \{ \omega_{N-2} \} = \min_{u_{N-2}} \{ x_{N-1}^2 \} = \min_{u_{N-2}} [a x_{N-2}^2 + b u_{N-2}]^2 \quad (**)$$

We can see that Problems (*) and (**) are SIMILAR.

therefore

$$\begin{cases} u^*_{N-2} = -\frac{a}{b} x_{N-2}^2 \\ J^*_{N-2} = 0 \end{cases}$$

- General stage

$$\begin{cases} u_k^* = -\frac{a}{b} x_k^2 \\ J_k^* = 0 \end{cases} \qquad k = N-1, N-2, \dots, 0$$

<u>Conclusion</u> :

1. the optimal control sequence is of the form : $u_k^* = -\frac{a}{b} x_k$
2. the optimal cost is 0 //.

A.1.4 Example

Given $x_{k+1} = a x_k + u_k \qquad \alpha > 0 \qquad (1)$

Solve $\min_{u_0 - u_{N-1}} J = \sum_{k=0}^{N-1} x_k^2$

subject to $u_k \geq 0$ and (1)

Solution D.P. . Enforce the constraint on the control $u_k = \hat{u}_k^2 \geq 0$

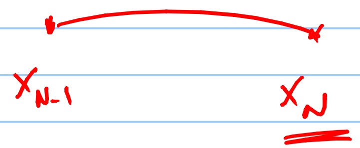
• Last Stage (1) becomes $x_{k+1} = \alpha x_k + \hat{u}_k^2$

x_{N-1} known. $J_{N-1} = \vartheta_{N-1} = x_{N-1}^2$

$$\min_{\hat{u}_{N-1}} J_{N-1} = \min_{\hat{u}_{N-1}} x_N^2$$

?

$$= \min_{\hat{u}_{N-1}} \left(\alpha x_{N-1} + \hat{u}_{N-1}^2 \right)^2$$



Necessary condition : $\frac{\partial J_{N-1}}{\partial \hat{u}_{N-1}} = 0 \implies 2(\alpha x_{N-1} + \hat{u}_{N-1}^2)\, 2\hat{u}_{N-1} = 0$

Two solutions

a) $\hat{u}_{N-1} = 0 \longrightarrow J_{N-1}(\hat{u}_{N-1} = 0) = (\alpha x_{N-1})^2$

b) $\hat{u}_{N-1}^2 = -\alpha x_{N-1}$

α) $x_{N-1} > 0$ not feasible

β) $x_{N-1} < 0$ then $\hat{u}_{N-1}^2 = -\alpha x_{N-1}$ feasible

In order to decide which solution is best, look at the cost!

$$\hat{u}_{N-1}^2 \quad (0,\ -\alpha x_{N-1})$$

control Cost $$J_{N-1} \quad ((\alpha x_{N-1})^2,\ 0)$$

So: if $x_{N-1} > 0 \implies u^*_{N-1} = 0$ $\quad J^*_{N-1} = \alpha^2 x_{N-1}^2$: "Hot $\implies$ don't Heat up"

if $x_{N-1} < 0 \implies u^*_{N-1} = -\alpha x_{N-1}$ $\quad J^*_{N-1} = 0$: "Cold $\implies$ Heat up"

the Optimal Solution

- Second-to-Last Stage.

DP $\quad J_{N-2} = \omega_{N-2} + J^*_{N-1}$

$$\min_{\hat{u}_{N-2}} J_{N-2} = \min_{\hat{u}_{N-2}} \left[\omega_{N-2} + J^*_{N-1} \right] = \begin{cases} \min\limits_{\hat{u}_{N-2}} \left(x^2_{N-1} + 0 \right) & x_{N-1} < 0 \checkmark \\ \min\limits_{\hat{u}_{N-2}} \left(x^2_{N-1} + \alpha^2 x^2_{N-1} \right) & x_{N-1} > 0 \checkmark \end{cases}$$

a) $x_{N-1} > 0$

$$\min_{\hat{u}_{N-2}} J_{N-2} = \min_{\hat{u}_{N-2}} \left[x^2_{N-1} + \alpha^2 x^2_{N-1} \right] = (1+\alpha^2) \min_{\hat{u}_{N-2}} \left[x^2_{N-1} \right]$$

this problem was solved at the Last Stage w.r.t $\hat{u}_{N-1}$

therefore: the Optimal solution

$$\begin{cases} \text{if} \quad x_{N-2} > 0 \Rightarrow u^*_{N-2} = 0 \;, \quad J^*_{N-2} = (1+\alpha^2)\alpha^2 x^2_{N-2} & \text{"Hot Case"} \\ \text{if} \quad x_{N-2} < 0 \Rightarrow u^*_{N-2} = -\alpha x_{N-2} \;; \quad J^*_{N-2} = 0 & \text{"Cold Case"} \end{cases}$$

b) $x_{N-1} < 0$

$$\min_{\hat{u}_{N-2}} J_{N-2} = \min_{\hat{u}_{N-2}} (x_{N-1}^2) \implies \text{solved @ the last Stage}$$

- <u>General Stage</u>

$$u_k^* \begin{cases} 0 & x_k > 0 \quad \text{Hot Case} \\ -\alpha x_k & x_k < 0 \quad \text{Cold Case} \end{cases}$$

A.1.5 A particular Vector Case

Given $x_{k+1} = Ax_k + Bu_k \qquad A = I_2 \qquad B = b = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad x_0$ given

Solve $J = \sum_{k=0}^{N-1} \|x_k\|_Q^2 + R u_k^2 \qquad Q = I_2 \qquad R = 1$

Solution $N = 3$. We are going to solve this optimal control via two approaches

Approach #1. the traditional approach

Approach #2. the D.P. approach

Approach #1. the cost J is a function of $\begin{cases} u_0, u_1, u_2 & \text{the optimization variables} \\ x_0 & \text{a parameter (known)} \end{cases}$

so $J(u_0, u_1, u_2)$

from the system equation $x_1 = x_0 + bu_0 \qquad (1)$

$$x_2 = x_1 + b u_1$$
$$= x_0 + b u_0 + b u_1$$
$$= x_0 + b(u_0 + u_1) \qquad (2)$$

$$x_3 = x_2 + b u_2$$
$$= x_0 + b(u_0 + u_1 + u_2) \qquad (3)$$

Substitute (1)(2)(3) into J :

$$J = \|x_1\|_Q^2 + R u_0^2 + \|x_2\|_Q^2 + R u_1^2 + \|x_3\|_Q^2 + R u_2^2$$
$$= x_1^T Q x_1 + \ldots$$
$$= \|x_0 + b u_0\|^2 + u_0^2 + \|x_0 + b(u_0+u_1)\|^2 + u_1^2 + \|x_0 + b(u_0+u_1+u_2)\|^2 + u_2^2$$
$$= 3\|x_0\|^2 + 2(-b^T x_0)(3u_0 + 2u_1 + u_2) + 3u_0^2 + 2(u_0+u_1)^2 + u_1^2 + 2(u_0+u_1+u_2)^2 + u_2^2$$

So minimizing J w.r.t. u_0, u_1, u_2 is done by differentiation:

$$\frac{\partial J}{\partial u_0} = 0 \qquad \frac{\partial J}{\partial u_1} = 0 \qquad \frac{\partial J}{\partial u_2} = 0$$

$$\begin{bmatrix} 7 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} (-b^T x_0) \quad x_0 = \begin{bmatrix} 20 \\ 21 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} u_0^* \\ u_1^* \\ u_2^* \end{bmatrix} = \begin{Bmatrix} -15 \\ -4 \\ -1 \end{Bmatrix}} \;\therefore$$

the Optimal Controls

Approach #2 : D.P.

- LAST STAGE

Assuming x_2 is Given

$$J_2 = \|x_3\|^2 + u_2^2$$

$$= \| x_2 + b u_2 \|^2 + u_2^2 \quad \text{by using the System Equation}$$

$$= \|x_2\|^2 + 2 x_2^T b \, u_2 + (1 + \|b\|^2) u_2^2$$

Perform a minimization of J_2 w.r.t. u_2

?

$$\frac{dJ_2}{du_2} = 0 \quad \Rightarrow \quad 2x_2^T b + 2(1+\|b\|^2)u_2 = 0$$

$$u_2^* = -\frac{1}{1+\|b\|^2} b^T x_2 \quad \Rightarrow \quad u_2^* = -\frac{1}{3} b^T x_2$$

Optimal Cost To Go : $J_2^* = \|x_2\|^2 + 2x_2^T b\left(-\frac{1}{3} b^T x_2\right) + 3\left(\frac{1}{9}\right)(b^T x_2)^2$

$$J_2^* = \|x_2\|^2 - \frac{1}{3}(b^T x_2)^2$$

- Second-to-Last Stage

$$J_1 = w_1 + J_2^*$$

x_1 is given

$$= \|x_2\|^2 + u_1^2 + \|x_2\|^2 - \frac{1}{3}(b^T x_2)^2$$

$$= 2\|x_2\|^2 - \frac{1}{3}(b^T x_2)^2 + u_1^2$$

Use the system equation $\quad x_2 = x_1 + bu_1$

$$J_1 = 2\|x_1 + bu_1\|^2 - \frac{1}{3}\left(b^T[x_1 + bu_1]\right)^2 + u_1^2$$

$$= 2\left(\|x_1\|^2 + 2x_1^T b u_1 + \|bu_1\|^2\right) - \frac{1}{3}\left(b^T x_1 + \|b\|^2 u_1\right)^2 + u_1^2$$

$$\frac{dJ}{du_1} = 0 \Rightarrow 8x_1^T b + 22u_1 = 0 \Rightarrow$$

Substituting u_1^* into J_1 yields J_1^*

$$\begin{cases} u_1^* = -\frac{4}{11} b^T x_1 \\ J_1^* = 2\|x_1\|^2 - \frac{9}{11}\left(b^T x_1\right)^2 \end{cases}$$

- First Stage

$$J_0 = \omega_0 + J_1^*$$

x_0 is given

$$= \|x_1\|^2 + u_0^2 + 2\|x_1\|^2 - \frac{9}{11}\left(b^T x_1\right)^2$$

Use the System equation $x_1 = x_0 + bu_0$

$$\Rightarrow J_0 = 3\left(\|x_0\|^2 + 2x_0^T b u_0 + \|bu_0\|^2\right) - \frac{9}{11}\left(b^T u_0 + \|b\|^2 u_0\right)^2 + u_0^2$$

Minimization w.r.t. u_0

$$\frac{dJ_0}{du_0} = 0 \Rightarrow$$

$$\begin{cases} u_0^* = -\frac{15}{41} b^T x_0 \\ J_0^* = 3\|x_0\|^2 - \frac{54}{41}(b^T x_0)^2 \end{cases}$$

Summary of BWD propagation:

$$\begin{cases} u_0^*(x_0) \\ u_1^*(x_1) \\ u_2^*(x_2) \end{cases} \qquad \begin{cases} J_0^*(x_0) \\ J_1^*(x_1) \\ J_2^*(x_2) \end{cases}$$

$$u_0^* = -\frac{15}{41}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 20 \\ 21 \end{bmatrix} = -15$$

For the I.C $x_0 = \begin{bmatrix} 20 \\ 21 \end{bmatrix}$: FWD propagation

$$x_0 = \begin{bmatrix} 20 \\ 21 \end{bmatrix} \xrightarrow[u_0^* = -15]{J_0^* = 309} x_1 = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \xrightarrow[u_1^* = -4]{J_1^* = 29} x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow[u_2^* = -1]{J_2^* = 2} x_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

General Comments : 1. each stage generates a Linear Feedback Control.

2. the optimal cost-to-go is a (Quadratic) value of the current state x_k.

3. the sequence of J_k^* is decreasing.

4. the DP solution is identical to that obtained through the traditional approach.

but ? * the traditional approach involves inverting a 3x3 matrix.

* " " " yields an OPEN-LOOP control algorithm u_0^* u_1^* u_2^* as a function of x_0 I.C.

as opposed the DP approach that delivers a Feedback Control.