

The Probabilistic Method in Combinatorics 80721

Based on lectures by Dr. Yuval Peled, and the book by Alon and Spencer - *The probabilistic method*

Notes by Asaf Etgar
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For any corrections, requests for additions or notes - please email me at
asafetgar@gmail.com

These notes have not been revised by the course staff, and some things may appear differently than in the lectures/ recitations.

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Chapter 1

Introcuction

1.1 Ramsey Numbers

Claim 1.1.1. For any graph $G = (V, E)$ there exists a partitioning of $V = A \sqcup B$ such that at least half of the edges are $A - B$ edges.

Proof. Consider a random partition of V , A, B . That is, each vertex v is in A or in B w.p $\frac{1}{2}$ independently. Then:

$$\mathbb{E}[e(A, B)] \stackrel{\text{linearity}}{=} |E| \cdot \Pr[e \text{ is an } A, B \text{ edge}] = \frac{|E|}{2}$$

Which implies that there exists. a partition with said property. \square

Remark. One can prove this claim without the use of probability.

There are questions that we do not know yet how to solve without the use of probability:

Definition 1.1 (Ramsey Number). The number $R(k, l)$ is the minimal n such that every graph G over n vertices contains a k -clique or an l -anti-clique.

Theorem 1.1 (Ramsey). $R(k, l) \leq \binom{k-l-2}{k-1}$. In particular, $R(k, k) \leq \binom{2k-2}{k-1} \approx \frac{4^{k-1}}{\sqrt{\pi k}}$

Theorem 1.2. If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, then $R(k, k) > n$

Proof. Consider now a random graph $G \sim G(n, \frac{1}{2})$. For any $A \in \binom{[n]}{k}$, denote by M_A the event that A is a clique or anti-clique in G . Then:

$$\Pr[M_A] = \Pr[A \text{ is a clique}] + \Pr[A \text{ is an anti-clique}] = 2^{1-\binom{k}{2}}$$

And therefore

$$\Pr[\exists \text{ a clique or anti-clique of size } k] = \Pr\left[\bigcup_{A \in \binom{[n]}{k}} M_A\right] \leq \sum_{A \in \binom{[n]}{k}} \Pr[M_A] = \binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

Hence there exists a graph over n vertices without a clique or anti-clique of size k . \square

Remark. Note that

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1 \iff \binom{n}{k} < 2^{\binom{k}{2}-1}$$

And also, $\binom{n}{k} \leq \frac{n^k}{k!}$, and by Stirling's approximation - $k! \geq \left(\frac{k}{e}\right)^k$. Pluggin in the inequality:

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

Comparing this to the formula in 1.1, this is a very loose bound (at least for k, k).

Remark. If $n = 2^{k/2}$, then:

$$\Pr[\text{A random graph contains a clique or anti-clique}] \leq \binom{n}{k} 2^{1-\binom{k}{2}} \xrightarrow{n,k} 0$$

Which means "almost all graphs are Ramsey graphs", but we do not yet have any explicit construction.

This theorem implies that we know the existence of a graph of order n and without clique or anti-clique of size $k \approx 2 \log n$. The best construction known without the use of probability is for $k = \log^{C \cdot \log \log \log n} n$.

Chapter 2

Linearity of Expectation

2.1 Sum-Free Sets

Theorem 2.1. For any $B \in \binom{\mathbb{N}}{n}$ (with repetitions), there exists $A \in \binom{B}{n/3}$ such that there are no $a, b, c \in A$ with $a + b = c$

Proof (by Erdős). Denote $[x] = x - \lfloor x \rfloor$, and for any $t \in [0, 1]$ let $A_t = \{b \in B \mid [tb] \in (\frac{1}{3}, \frac{2}{3})\}$. For any t , A_t is sum-free: If $a, b \in A_t$ and $[ta], [tb] \in (\frac{1}{3}, \frac{2}{3})$, then $[a + b] \notin (\frac{1}{3}, \frac{2}{3})$. We consider the probability space of the coin tosses of t . Denote $X_i = \mathbf{1}_{b_i \in A_t}$, then $\Pr[X_i = 1] = \frac{1}{3}$. Hence consider the expectation of a size of a random A_t :

$$\mathbb{E}[|A_t|] = \sum_{i \in [n]} \mathbb{E}[X_i] = \frac{n}{3}$$

□

Remark. The general idea of probabilistic methods is to find an object in which the property always holds, and then average over these objects

2.2 Tournaments

Definition 2.1. A tournament is an orientation of K_n .

Definition 2.2. We say a vertex v *overcomes* some $A \subset V \setminus \{v\}$ if $v \rightarrow x$ for any $x \in A$ (that is, the orientation of $vx \in E(K_n)$ is $v \rightarrow x$).

Theorem 2.2. If $\binom{n}{k} (1 - 2^{-k})^{n-k}$, then there exists a tournament such that for any $A \in \binom{V}{k}$ there exists v that overcomes A .

Proof. Denote by S_k the event that for any A of size k there exists an overcoming v . Consider a random tournament, and let $A \in \binom{V}{k}$, what is the probability that no v overcomes A ?

$$\Pr[\text{No } v \text{ overcomes } A] = (1 - 2^{-k})^{n-k}$$

(some v overcomes A w.p 2^{-k} , and they are independent) Then:

$$\Pr[S_k^c] \leq \binom{n}{k} (1 - 2^{-k})^{n-k} < 1$$

□

Remark. The union bound is quite similar to linearity of expectation.

Theorem 2.3. *There exists a tournament with at least $n! \cdot 2^{-(n-1)}$ Hamiltonian cycles.*

Proof. Consider a random tournament. Then:

$$\mathbb{E}[\# \text{ of Hamiltonian cycles}] = \sum_{\pi \in S_n} \Pr[\pi(V) \text{ is a cycle}] = n! 2^{-(n-1)}$$

(the last equation is the probability of this permutation defining a cycle) Then there must exist a tournament with at least this number of cycles. \square

2.3 ??? If you have a suggestion for a name, let me know!

Theorem 2.4. *Let $v_1, \dots, v_n \in \mathbb{R}^d$ be unit vectors. Then there exists $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$ such that*

$$\left\| \sum_{i \in [n]} \varepsilon_i v_i \right\| \leq \sqrt{n}$$

and there exists such ε_i for the opposite inequality.

Proof. Consider a random choice of ε_i . Denote $X = \left\| \sum_{i \in [n]} \varepsilon_i v_i \right\|^2$. Then:

$$X = \left\| \sum_{i \in [n]} \varepsilon_i v_i \right\|^2 = \sum_{i \in [n]} \varepsilon_i^2 v_i \cdot v_i + 2 \cdot \sum_{i < j} \varepsilon_i \varepsilon_j \langle v_i, v_j \rangle$$

then

$$\mathbb{E}[X] = n + 2 \cdot \sum_{i < j} \langle v_i, v_j \rangle \mathbb{E}[\varepsilon_i \varepsilon_j] = n$$

Since $\mathbb{E}[\varepsilon_i \varepsilon_j] = \mathbb{E}[\varepsilon_j] \mathbb{E}[\varepsilon_i] = 0 \cdot 0 = 0$, and the claim follows as usual. \square

2.3.1 Derandomization

We would like to de-randomize the process and find an efficient algorithm of finding these ε_i . By the law of total expectation, $\mathbb{E}[X] = \frac{1}{2} \mathbb{E}[X \mid \varepsilon_1 = 1] + \frac{1}{2} \mathbb{E}[X \mid \varepsilon_1 = -1]$.

Claim 2.3.1. *If we've fixed $\varepsilon_1 \dots \varepsilon_{i-1}$ such that $\mathbb{E}[X \mid \varepsilon_1 \dots \varepsilon_{i-1}] \leq n$, then we can efficiently find ε_i such that $\mathbb{E}[X \mid \varepsilon_1 \dots \varepsilon_{i-1}, \varepsilon_i] \leq n$*

Proof.

$$\mathbb{E}[X \mid \varepsilon_1 \dots \varepsilon_{i-1}] \stackrel{\star}{=} \frac{1}{2} \mathbb{E}[X \mid \varepsilon_1 \dots \varepsilon_{i-1}, \varepsilon_i = 1] + \frac{1}{2} \mathbb{E}[X \mid \varepsilon_1 \dots \varepsilon_{i-1}, \varepsilon_i = -1]$$

with \star by law of total expectation (w.r.t the random variable ε_i). But

$$\mathbb{E}[X \mid \varepsilon_1 \dots \varepsilon_{i-1}, \varepsilon_i = 1] = n + 2 \sum_{j' < j \leq i} \varepsilon_j \varepsilon_{j'} \langle v_j, v_{j'} \rangle + 0$$

And we know the values of $\varepsilon_{j'}, \varepsilon_j$, so we can compute it efficiently. Then we choose the epsilon from the book \square

2.4 Turan's theorem

Theorem 2.5. *In any graph (V, E) , there exists an independent set of size at least $\sum_{v \in V} \frac{1}{\deg(v)+1}$*

Proof. Consider a random ordering of V . We choose a vertex to add to the set I ("independent") if he appears before all of his neighbors. Clearly I is independent. And:

$$\mathbb{E}[|I|] = \sum_{v \in V} \Pr[v \in I] = \sum_{v \in V} \Pr[v \text{ is the first of his neighbors in the ordering}] = \sum_{v \in V} \frac{1}{\deg(v)+1}$$

□

Corollary 2.6. *In G there exists a clique of size $\geq \sum_{v \in V} \frac{1}{n - \deg(v)}$*

Theorem 2.7 (Turán). *If the maximal clique is of size r , then*

$$r \geq \sum_{v \in V} \frac{1}{n - \deg(v)} \geq \frac{n^2}{n^2 - 2|E|}$$

Therefore $|E| \leq \left(1 - \frac{1}{n}\right) \cdot \frac{n^2}{2}$

2.5 Unbalancing Lights

Let A be an $n \times n$ matrix over $\{\pm 1\}$. There is a switch for every row and every column, which flips all bits corresponding to it.

Theorem 2.8. *There exists $x, y \in \{\pm 1\}^n$ such that $x^\top Ay \geq \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{\frac{3}{2}}$*

Proof. Choose a random y (that is, $y_i \sim U(\pm 1)$ iid. Let $R_i = \sum_{j=1}^n A_j^i y_j$. Since y_i are iid, $A_j^i y_j \sim$ a sum of n signs ± 1 iid. Then by CLT:

$$\frac{1}{\sqrt{n}} R_i \xrightarrow{\text{distribution}} \mathcal{N}(0, 1)$$

And therefore $\mathbb{E}\left[\frac{1}{\sqrt{n}} |R_i|\right] \xrightarrow{n \rightarrow \infty} \mathbb{E}[|z|] = \sqrt{\frac{2}{\pi}}$. Hence:

$$\mathbb{E}\left[\sum_{i=1}^n |R_i|\right] = \sum_{i=1}^n \mathbb{E}[|R_i|] = \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{\frac{3}{2}}$$

Then there exists y such that $\sum_{i=1}^n |R_i| \geq \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{\frac{3}{2}}$. As for x , note that

$$x^\top Ay = \sum_{i=1}^n \sum_{j=1}^n x_i A_j^i y_j = \sum_{i=1}^n x_i R_i \stackrel{*}{=} \sum_{i=1}^n |R_i| \geq \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{\frac{3}{2}}$$

with \star since we can take $x_i = \text{sign}(R_i)$.^{II}

□

2.6 2-colorings of hypergraphs

^Ithink of x as responsible of rows, and y of columns

^{II}In some sense, this is "the smartest move" in order to get as many light bulbs lit as possible.

Definition 2.3 (*k*-uniform Hypergraph). $H = (V, E)$ is a *k*-uniform Hypergraph with $V(H)$ its vertices and $E(H) \subset \binom{V(H)}{k}$. In particular, a 2-uniform hypergraph is just a graph.

Definition 2.4 (2-coloring of hypergraph). Let H be a *k*-graph. A 2-coloring of H is a function $f : V(H) \rightarrow \{0, 1\}$ such that there is no monochromatic edge, that is $\forall e \in E(H) \exists x, y \in e \ f(x) \neq f(y)$.

Definition 2.5. Denote $m(k)$ the minimal number of edges in a *k*-graph that is not 2-colorable.

Example 2.1. $m(2) = 3$, consider a triangle.

Example 2.2. $m(3) = 7$, consider the Fano Plane.

Theorem 2.9. $m(k) \geq 2^{k-1}$

Proof. Let H be a hypergraph with less than $m2^{k-1}$ edges. Let f be a uniformly random coloring of H Then

$$\Pr[e \text{ is monochromatic}] = 2^{1-k}$$

Therefore

$$\mathbb{E}[\#\text{monochromatic edges}] < m \cdot 2^{1-k} = 1$$

□

Theorem 2.10. $m(k) = O(k^2 2^k)$.

Proof. Let $n = k^2$, and choose $c \cdot k^2 2^k$ (we specify c later) edges uniformly IID. We show that $\mathbb{E}[\#\text{colorings without monochromatic edges}] < 1$. Fix a coloring $\varphi : [n] \rightarrow \{0, 1\}$. Let $a = |\varphi^{-1}(0)|$, then

$$\begin{aligned} \Pr[e \text{ is monochromatic under } \varphi] &= \\ \frac{\binom{a}{k} + \binom{n-a}{k}}{\binom{n}{k}} &\geq \frac{2 \cdot \binom{n/2}{k}}{\binom{n}{k}} \geq \frac{2 \cdot \frac{(\frac{n}{2}-k)^k}{k!}}{\frac{n^k}{k!}} = 2 \left(\frac{1}{2} - \frac{k}{n}\right)^k = 2 \cdot \left(\frac{1}{2}\right)^k \left(1 - \frac{2}{k}\right)^k \geq c \left(\frac{1}{2}\right)^k \end{aligned}$$

for some c . Now we have

$$\begin{aligned} \mathbb{E}[\#\text{colorings without monochromatic edges}] &= \sum_{\varphi} \Pr[\text{no edge is monochromatic under } \varphi] \leq \\ &\leq 2^n \left(1 - \frac{c}{2^k}\right)^m \leq e^{\log(2) \cdot n - c \cdot 2^{-k} \cdot m} \stackrel{*}{<} 1 \end{aligned}$$

With \star by choice of $m = \frac{2 \log(2)}{c} k^2 \cdot 2^k$. □

Theorem 2.11 (Improvement on the bound from 2.9). $m(k) \geq t 2^k \sqrt{\frac{k}{\log(k)}}$.

Proof. Or proof is algorithmic: Let H be with t edges, color all $V(H)$ in blue. Traverse $V(H)$ in a random order, let v the current vertex. If v is the last-visited vertex in a monochromatic blue edge - alter its color to blue. The algorithm fails only if there is a monochromatic red edge. This happens only if the vertex $v = e \cap f$, f is red, e is blue and v is the first in f and last in blue (this is a bad configuration). What is the probability of such configuration to occur? We consider the probability over the coin tosses of $\pi \sim U(S_n)$, and claim

$$\Pr[\text{there exists a bad configuration}] < 1$$

We note that:

$$\Pr[\text{there exists a bad configuration}] \leq \mathbb{E}[\#(e, f) \text{ are bad edges}] \stackrel{*}{\leq} m^2 \cdot \frac{((k-1)!)^2}{(2k-1)!} =$$

$$\frac{m^2}{(2k-1)\binom{2k-2}{k-1}} \stackrel{**}{=} \frac{m^2 \cdot (c + o(1))}{\sqrt{k} \cdot 4^k} \stackrel{?}{<} 1$$

with \star a bound on the number of edges that intersect in a unique vertex, times the probability of having a bad configuration, and $**$ since $\binom{2n}{n} = \frac{c+o(1)}{\sqrt{n}} 2^{2n}$. In order to have $?$, take $m < c' \cdot k^{\frac{1}{4}} \cdot 2^k$. This is not the bound we want - as the power of k is $\frac{1}{4}$. The problem is the expectation sometimes lies - that is, the expectation can be much larger than the probability we want to bound (see the remark below).

We traverse the vertices differently: For a vertex v , choose $r_v \sim U([0, 1])$ i.i.d, and traverse $V(H)$ according to r_v from the smallest to largest. Let p be some probability chosen later, and denote

$$L = \left[0, \frac{1-p}{2}\right] \quad M = \left[\frac{1-p}{2}, \frac{1+p}{2}\right] \quad R = \left[\frac{1+p}{2}, 1\right]$$

Now:

$$\Pr[\text{there exists a bad configuration}] \leq$$

$$\overbrace{\Pr[\exists e \in L \cup R]}^1 + \overbrace{\Pr[\exists \text{bad configuration whose intersection is in } M]}^2 \leq$$

$$\overbrace{m \cdot 2(|L|)^k}^1 + \overbrace{m^2 \int_{\frac{1-p}{2}}^{\frac{1+p}{2}} r_v^{k-1} (1-r_v)^{k-1} dr_v}^2 =$$

$$m \cdot 2 \frac{1-p^k}{2} + m^2 \int_{\frac{1-p}{2}}^{\frac{1+p}{2}} r_v^{k-1} (1-r_v)^{k-1} dr_v \leq$$

$$2m \frac{e^{-pk}}{2^k} + m^2 \cdot p \left(\frac{1}{4}\right)^{k-1} \stackrel{?}{<} 1$$

Choosing $p = \frac{\log k}{k}$ and $m < \frac{1}{4} 2^k \sqrt{\frac{k}{\log k}}$ yields the result. \square

Remark. Let $X_n = n^2$ with probability $1/n$ and 0 otherwise. Note that $\Pr[X_n > 0] = 1/n$, while $\mathbb{E}[X_n] = n$.

Chapter 3

Alterations Method

Up to this point, we made a random choice of object and use it. We now deal with the setting where a naïve random choice is not good enough - but we can alter it a little bit so it would be good. The idea here is to bound the expectations of alterations needed to the random object.

3.1 Dominating Sets

Definition 3.1. Let G be a graph. $A \subset V$ is *Dominating* if any $v \in V$ has a neighbor in A .

Theorem 3.1. Let G be of minimal degree δ , then there exists a dominating set of size $n \cdot \frac{\ln(1+\delta)}{1+\delta}$.

Proof. Let $B \subset V$ such that any $v \in B$ with probability p (will be chosen later) independently. Let C_B be the collection of vertices that all of their neighbors are not in B , that is $C_B = \{x \notin B \mid \forall vx \in E \quad v \notin B\}$. Clearly $A = B \cup C_B$ is dominating. Then

$$\mathbb{E}[|A|] = \mathbb{E}[|B|] + \mathbb{E}[|C_B|] = np + n\Pr[v \in C_B] \stackrel{\star}{\leq} npne^{-p(1+\delta)}$$

With \star since $\Pr[v \in C_B] = (1-p)^{1+\deg(x)} \leq (1-p)^{1+\delta} \leq e^{-p(1+\delta)}$. Find the optimal p by differentiating w.r.t p , and get $p = \frac{\ln(1+\delta)}{1+\delta}$, then $\mathbb{E}[|A|] \leq n \left(\frac{\ln(1+\delta)+1}{1+\delta} \right)$ \square

3.2 Ramsey Numbers - Revisited

Recall that 1.2 gives us a lower bound on Ramsey numbers. We will use alterations to improve this lower bound.

Theorem 3.2. For any n, k , $R(k, k) \geq n - \binom{n}{k} \cdot 2^{1-\binom{k}{2}}$

Proof. Consider $G \sim \mathcal{G}(n, \frac{1}{2})$. Note that $\mathbb{E}[\#\text{monochromatic sets of size } k] = \binom{n}{k} 2^{1-\binom{k}{2}}$ as we've seen, therefore there exists a graph with at most this amount of monochromatic sets of size k , denote it G . Let G' be the graph obtained from G by removing a single vertex of any monochromatic set of size k . Then $|V(G')|$ is at least $n - \binom{n}{k} \cdot 2^{1-\binom{k}{2}}$, and clearly in G' there is no monochromatic set of size k . \square

Corollary 3.3. $R(k, k) \geq n - \frac{e^n}{k} 2^{1-\binom{k}{2}}$ by the Stirling-esque estimation done in chapter 1. The optimal n is $\frac{2^{k/2} \cdot k}{e}$ which yields $R(k, k) \geq 2^{k/2} k \cdot \left(\frac{1+o(1)}{e} \right)$.

3.3 Girth and coloring

Let $G = (V, E)$ be a graph.

Definition 3.2 (Girth). The *girth* of G is the length of a minimal cycle in G .

Remark. In particular, if the girth is $\geq g$, then for any $v \in V$, its g -neighborhood looks like a tree.

Definition 3.3 (Chromatic Number). The *chromatic number* of G , denoted $\chi(G)$ is the minimal k such that there exists a proper coloring $c : V \rightarrow [k]$ of G .

Remark. It is difficult to know what $\chi(G)$ is - it is NP-hard

Definition 3.4 (Independence number). The *Independence Number* of a graph G , denoted $\alpha(G)$, is the size of a largest independent set in G .

Claim 3.3.1. If T is a tree, then $\chi(T) = 2$

Proof. It is bipartite - use BFS. □

Theorem 3.4 (Erdős). For any k, g there exists a graph G with $\chi(G) \geq k$ and $\text{girth} \geq g$.

Remark. This is surprising! Any neighborhood seems like χ should be small (as neighborhoods look like trees) - but it turns out it cannot be considered locally; χ is a *global* property of G .

For ease - we write $\alpha(G) = \alpha$, same for χ .

Lemma 3.4.1. $V(G) \leq \alpha \cdot \chi$.

Proof. If $c : V \rightarrow [\chi]$ is a proper coloring, any $c^{-1}(i)$ is independent. □

Lemma 3.4.2. There exists a graph G over n vertices (for a large enough $n = n(k, g)$) with the following properties:

1. The number of cycles of length $\leq g$ is smaller than $\frac{n}{2}$
2. $\alpha(G) \leq 3 \log n \cdot n^{1 - \frac{1}{2g}}$

Proof. Let $G \sim \mathcal{G}(n, p)$ with $p = n^{\frac{1}{2g} - 1}$. Let X be the number of cycles of length $\leq g$. Then:

$$\mathbb{E}[X] \stackrel{1}{=} \sum_{r=3}^g \binom{n}{r} \cdot \frac{(r-1)!}{2} \cdot p^r \stackrel{2}{\leq} \sum_{r=3}^g (n \cdot p)^r \stackrel{3}{\leq} g \cdot (n \cdot p)^g = g\sqrt{n}$$

Justifications:

1. Choose which vertices are in a cycle of length r ($\binom{n}{r}$) and order them in a cycle $((r-1)!/2$ options) and multiply by the probability of such cycle to exists.
2. Bound $\binom{n}{r} \cdot \frac{(r-1)!}{2}$ from above naturally.
3. Bound the sum with the largest element in the summation.

Hence by Markov:

$$\Pr[X > n/2] \leq \frac{g\sqrt{n}}{n/2} \xrightarrow{n \rightarrow \infty} 0$$

Which implies the first property. For the second property, let $t = 3 \log n \cdot n^{1 - \frac{1}{2g}}$. Now:

$$\Pr[\alpha(G) \geq t] \leq \binom{n}{t} (1-p)^{\binom{t}{2}} \leq n^t (e^{-p})^{\binom{t}{2}} \leq n^t e^{-p \binom{t}{2}} = e^{t(\log n - \frac{p \cdot t}{2} + 1)} = e^{t(-\frac{1}{2} \log n + 1)} \xrightarrow{n \rightarrow \infty} 0$$

□

Proof (of 3.4). Let G' be a graph obtained from G by removing a single vertex from any cycle of length smaller than g . Then G' 's girth is at least g . And $\alpha(G') \leq \alpha(G) \leq 3 \log n \cdot n^{1-\frac{1}{2g}}$, and note that

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{\frac{n}{2}}{3 \log n \cdot n^{1-\frac{1}{2g}}} \geq \frac{n^{\frac{1}{2g}}}{6 \cdot \log(n)} \xrightarrow{n \rightarrow \infty} \infty$$

□

3.4 Heilbronn triangle problem

Let $P \subset [0, 1]^2$, $|P| = n$, denote $T(P) = \min_{x,y,z \in P} \text{Area}(xyz)$, and let $T(n) = \max_{|P|=n} T(P)$. Heilbronn conjectured^I that $T(n) = \Theta(\frac{1}{n^2})$.

Theorem 3.5 (KPS, no proof). $T(n) = \Omega\left(\frac{\log(n)}{n^2}\right)$

Remark. We still do not know some f for which $T(n) = \Theta(f(n))$, the best upper bound is still not tight.

Theorem 3.6. $T(n) \geq \frac{1}{70n^2}$

Proof. Let $\varepsilon = \frac{1}{70n^2}$. Generate $2n$ points $\sim U([0, 1]^2)$ IID and remove a point from any triangle of area less than ε . Given a triangle xyz , Let t be the distance xy . then t has some density $f_{\text{dist}}(t)$. Then:

$$\Pr[\text{area}(xyz) \leq \varepsilon] \leq \int_0^{\sqrt{2}} \sqrt{2} \cdot 4 \frac{\varepsilon}{t} f_{\text{dist}}(t) dt = (\star)$$

Note that $f_{\text{dist}}(t) = \lim_{h \rightarrow 0} \frac{1}{h} \Pr[t \leq \text{dist}(x, y) \leq t + h]$ by the definition of density. Hence $f_{\text{dist}(x,y)}(t) \leq \lim_{h \rightarrow 0} \frac{1}{h} \pi((t+h)^2 - t^2) = 2\pi t$, then:

$$(\star) \leq \int_0^{\sqrt{2}} \sqrt{2} \frac{4\varepsilon}{t} 2\pi t dt = 16\pi\varepsilon$$

Which implies

$$\mathbb{E}[\text{number of triangles with area smaller than } \varepsilon] \leq \binom{2n}{3} \frac{16\pi}{70n^2} < n$$

□

Remark. Erdős has a non-combinatorial construction. Let n be some prime, and consider the grid $[n-1] \times [n-1]^{\text{II}}$ and take $\{(k, k^2 \bmod n)\}_{k \in [n-1]}$. Note that the smallest triangle of 3 points in \mathbb{Z}^2 is of area $1/2$, unless the three points are on the same diagonal. If they are on the diagonal $ax + b$, this means that there exists three values of k such that $(ak + b) = k^2 \bmod n$, but this is a quadratic polynomial in $\mathbb{F}_n[x]$, therefore it cannot have more than 2 solutions. Hence by scaling, $T(n) \geq \frac{1}{2(n-1)^2}$

^Ifalsely

^{II}Can rescale for the unit cube later...

Chapter 4

Second Moment Method

Up until now we discussed *first moment methods*. More formally, if $X = X_n \geq 0$ is an integer valued random variable, then the first moment method tells us that if $\mathbb{E}[X_n] \xrightarrow{n \rightarrow \infty} 0$, then $\Pr[X_n > 0] \xrightarrow{n \rightarrow \infty} 0$.

Example 4.1 (First Moment Method). When $G \sim \mathcal{G}(n, p)$ is triangle-free? Denote X the number of triangles in G . Then

$$\mathbb{E}[X] = \binom{n}{3} p^3 \leq (np^3)$$

Then taking $p = o\left(\frac{1}{n}\right)$ results in $\Pr[X > 0] \xrightarrow{n \rightarrow \infty} 0$. Is this bound *tight*? We saw that the expectation does not always give us a good bound - we need a way to reason about when is X concentrated about its expectation - that is the variance.

Definition 4.1 (Variance). The variance of X is $\text{Var} X = \mathbb{E}[(X - \mathbb{E}[X])^2]$

Definition 4.2 (Covariance). The covariance of X, Y is

$$\text{cov} X, Y = \mathbb{E}[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Theorem 4.1 (Chebyshev). $\Pr[|X - \mathbb{E}[X]| \geq t] \leq \frac{\text{Var} X}{t^2}$

Corollary 4.2. $\Pr[X = 0] \leq \frac{\text{Var}(X)}{\mathbb{E}[X]^2}$

Corollary 4.3. If $\text{Var} X = o(\mathbb{E}[X]^2)$ then $\Pr[X = 0] \rightarrow 0$

This results in the *Second moment method*: If $\text{Var} X = o(\mathbb{E}[X]^2)$ then $\Pr[X \geq 0] \xrightarrow{n \rightarrow \infty} 0$. An equivalent condition is $\mathbb{E}[X^2] = \mathbb{E}[X]^2 (1 + o(1))$ ^I An important case is when $X = \sum_{i=1}^m X_i$, in that case

$$\text{Var} X = \sum_{i=1}^m \text{Var} X_i + \sum_{i=1}^m \sum_{j \neq i}^m \text{cov} X_i, X_j$$

If we denote $i \sim j$ when X_i, X_j are dependent, then

$$\text{Var} X = \sum_{i=1}^m \text{Var} X_i + \sum_{i=1}^m \sum_{j \sim i}^m \text{cov} X_i, X_j$$

^ISince $\text{Var} X = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Assumptions:

1. $X_i = \mathbb{1}_{A_i}$, then $\text{Var} X_i = \Pr[A_i] \cdot (1 - \Pr[A_i]) \leq \Pr[A_i]$ and $\text{Cov} X_i, X_j = \Pr[A_i \cap A_j] - \Pr[A_i] \cdot \Pr[A_j] \leq \Pr[A_i \cap A_j] = \Pr[A_i] \cdot \Pr[A_j | A_i]$. Under this assumption, we get

$$\text{Var} X \leq \mathbb{E}[X] + \sum_i \Pr[A_i] \cdot \sum_{i \sim j} \Pr[A_j | A_i]$$

2. A symmetry assumption is $\sum_{i \sim j} \Pr[A_j | A_i]$ is independent of i . This is usually true in many cases. We denote $\sum_{i \sim j} \Pr[A_j | A_i] = \Delta^{*\text{II}}$. With this notation, $\text{Var} X \leq \mathbb{E}[X] (1 + \Delta^*)$.

Corollary 4.4. *If $\mathbb{E}[X] \rightarrow \infty$, $\Delta^* = o(\mathbb{E}[X])$, then $\Pr[X = 0] \rightarrow 0$.*

4.1 H -free graphs

The general question we deal with is

Question 4.1.1. *Given a small graph H , what is the threshold function of $G \sim \mathcal{G}(n, p)$ containing H ?*

More formally:

Definition 4.3 (Threshold Function). Let $G \sim \mathcal{G}(n, p)$ and H some fixed small graph. We say $f(n)$ is a *threshold function* for finding H in G if

$$\begin{aligned} p \ll f(n) &\Rightarrow \Pr[G \text{ contains a copy of } H] \xrightarrow{n \rightarrow \infty} 0 \\ \text{and} \\ p \gg f(n) &\Rightarrow \Pr[G \text{ contains a copy of } H] \xrightarrow{n \rightarrow \infty} 1 \end{aligned}$$

We now explore some threshold functions

4.1.1 Triangles in $\mathcal{G}(n, p)$

We've shown that if $G \sim \mathcal{G}(n, p)$ and $p \ll \frac{1}{n}$ then $\Pr[G \text{ contains a triangle}] \rightarrow 0$.

Claim 4.1.1. *If $p \gg \frac{1}{n}$ then $\Pr[G \text{ contains a triangle}] \rightarrow 1$*

Remark. In a case like this, we say that $\frac{1}{n}$ is the *threshold function* for triangle existence in $\mathcal{G}(n, p)$.

Proof. Denote by X the number of triangles in G , then:

$$\mathbb{E}[X] = \binom{n}{3} p^3 = (1_o(1)) \frac{n^3 p^3}{6} \xrightarrow{p \gg \frac{1}{n}} \infty$$

Denote by T a triangle in G .

$$\Delta^* = \sum_{T' \sim T} \Pr[A_{T'} | A_T] \stackrel{*}{=} 3 \cdot (n-3)p^2 \leq 3np^2$$

^{II}Usually $\sum_i \Pr[A_i] \cdot \sum_{i \sim j} \Pr[A_j | A_i] := \Delta$

With \star since any T' dependent on T is taken by choosing 2 vertices in T and a vertex not in T . Now we have

$$\frac{\Delta^*}{\mathbb{E}[X]} \leq \frac{18}{n^2 p} \rightarrow 0$$

And we are done. \square

Remark. In fact we've shown that $\frac{X}{\mathbb{E}[X]} \xrightarrow{\text{in probability}} 1$. This is some kind of law of large numbers. This can be seen by 4.1:

$$\Pr[|X - \mathbb{E}[X]| > \varepsilon \mathbb{E}[X]] \leq \frac{\text{Var} X}{\varepsilon^2 \mathbb{E}[X]^2} \rightarrow 0$$

4.1.2 K_4 in $\mathcal{G}(n, p)$

Denote X the number of K_4 copies in G . Then

$$\mathbb{E}[X] = \binom{n}{4} p^6 = \frac{1 + o(1)}{24} n^4 p^6$$

Then if $p \ll n^{-2/3}$, $\Pr[X = 0] \rightarrow 0$, and otherwise $\mathbb{E}[X] \rightarrow \infty$. Now denote by S a fixed copy of K_4 in G . Then:

$$\Delta^* = \sum_{S' \sim S} \Pr[A_{S'} \mid A_S] \leq \overbrace{6n^2 p^5}^{\text{Share 2 vertices}} + \overbrace{4np^3}^{\text{Share 3 vertices}}$$

Then

$$\frac{\Delta^*}{\mathbb{E}[X]} \leq O\left(\frac{1}{n^2 p} + \frac{1}{n^3 p^3}\right) \xrightarrow{p \gg n^{-2/3}} 0$$

4.1.3 $K_4 * e$ in $\mathcal{G}(n, p)$

In this case

$$\mathbb{E}[X] = 5 \binom{n}{5} p^7 = \frac{1 + o(1)}{24} n^5 p^7$$

Then $p \ll n^{-5/7}$ implies $\Pr[X > 0] \rightarrow 0$. Is it true that $p \gg n^{-5/7}$ implies $\Pr[X = 0] \rightarrow 1$? Note that $n^{-5/7} \ll n^{-2/3}$, and since existence of $K_4 * e$ implies existence of K_4 , had $n^{-5/7}$ been the threshold, it would contradict our previous proof.

Definition 4.4 (Maximal Subgraph Density). Given H , we define its maximal density by

$$m(H) := \max_{\emptyset \neq A \subset V(H)} \frac{e(A)}{|A|}$$

Remark. First moment argument shows that $p \ll n^{-\frac{1}{m(H)}}$.

Theorem 4.5 (Threshold characterization). *The threshold function of H is $n^{-\frac{1}{m(H)}}$*

4.2 Cliques in $\mathcal{G}(n, 1/2)$

Theorem 4.6. *Let $G \sim \mathcal{G}(n, 1/2)$. Denote X = size of maximal clique in G . There exists $k = k(n)$ and $k = \Theta(2 \log_2(n))$ such that*

$$\Pr[X \in \{k, k+1\}] \xrightarrow{n \rightarrow \infty} 1$$

Proof (Sketch). Define $f(k) = \mathbb{E}[\text{\#of } k\text{-cliques in } G] = \binom{n}{k} 2^{-\binom{k}{2}}$. We claim that if $f(k) \rightarrow \infty$ then there exists a clique of size k .

$$\Delta^* = \sum_{i=2}^{k-1} \binom{k}{i} \cdot \binom{n-k}{k-i} \left(\frac{1}{2}\right)^{\binom{k}{2} - \binom{i}{2}}$$

Now it can be shown that $k \sim 2 \log_2(n)$, if $f(k) \rightarrow \infty$ then $\frac{\Delta^*}{f(k)} \rightarrow 0$ by case analysis. We note that

$$\frac{f(k+1)}{f(k)} = \frac{\binom{n}{k+1} 2^{-\frac{k(k+1)}{2}}}{\binom{n}{k} 2^{-\frac{k(k-1)}{2}}} = \frac{n-k}{(k+1)} 2^{-k} \approx 2^{-2 \log_2(n)} < \frac{1}{n}$$

We now consider a k_0 such that $f(k_0) \geq 1$ and $f(k_0+1) < 1$. If $f(k_0(n))$ tends to ∞ with n and $f(k_0) = o(n)$, then $f((k_0+1)(n)) \xrightarrow{n \rightarrow \infty} 0$. In this case, there exists a maximal k_0 -clique, and no (k_0+1) -clique, so the maximal clique of k_0 . A similar argument may show the result, but it's quite annoying. The book has the complete proof \square

4.3 Distinct Sums Problems

The *Distinct Sums Problems* is a problem suggested by Erdős^{III}.

Problem. *What is the maximal size of $S \subset [n]$ with distinct partial sums?*

Solution (lower bound). Take $S = \{1, 2, 4, 8, \dots\}$, then $|S| = \log_2(n)$.

Question 4.3.1. *Is $|S| \leq \log_2(n) + o(1)$?*

Solution (Upper bound). $2^{|S|} \leq n \cdot |S|$

Corollary 4.7. $|S| \leq \log(n) + \log \log(n) + o(1)$

Claim 4.3.1. ^{IV} $|S| \leq \log(n) + 0.5 \log \log(n) + o(1)$

Proof. Let $S = \{s_1 \dots s_m\}$ and consider a random partial sum $X = \sum_{i=1}^m b_i s_i$ with $b_i \sim U(\{0, 1\})$.

$$\mu := \mathbb{E}[X] = \sum_{i=1}^m \frac{s_i}{2} \quad \text{and} \quad \text{Var} X = \sum_{i=1}^m \frac{s_i^2}{4}$$

Now try to bound the variation:

$$\text{Var} X = \frac{1}{4} \sum_{i=1}^m s_i^2 \stackrel{s_i \leq n}{\leq} \frac{mn^2}{4}$$

Then

$$\Pr[|x - \mu| \leq \lambda] \stackrel{\text{Chebyshev}}{\geq} 1 - \frac{mn^2}{4\lambda^2}$$

$$\Pr[|x - \mu| \leq \lambda] \stackrel{\text{distinct sums}}{\leq} (2\lambda + 2)2^{-m}$$

^{III}And it is still open with a \$300 prize awaiting to the solver!

^{IV}This is worth \$150!

So

$$(2\lambda + 2)2^{-m} \geq 1 - \frac{mn^2}{4\lambda^2} \quad \text{Take } \lambda = \sqrt{3}\sqrt{mn}$$

$$c\sqrt{mn}2^{-m} \geq \frac{4}{c'mn^2} \geq \frac{11}{12}$$

Therefore $2^m \leq \tilde{C}\sqrt{mn}$ And some computations result in the bound. \square

4.4 Hardy- Ramanujan Thoerem

The question we deal with is "how many prime numbers divide a random number $\sim U([n])$?"

Theorem 4.8 (Hardy- Ramanujan). *For $x \in \mathbb{N}$ let $\nu(x)$ be the number of prime divisors of x (without multiplicity). If $x \in U([n])$, then for a large enough n ,*

$$\forall \varepsilon > 0 \quad \exists A > 0 \quad \Pr \left[|\nu(x) - \log \log n| > A\sqrt{\log \log n} \right] < \varepsilon$$

Theorem 4.9 (Erdős-Kac). *With the same notations,*

$$\frac{\nu(x) - \log \log n}{\sqrt{\log \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$$

Theorem 4.10 (Merten). $\sum_{p \leq m} \frac{1}{p} = \log \log M = O(1)$

Proof(of 4.8). Denote by $X(x)$ the amount of prime divisors of x not greater then $n^{1/10}$. Clearly $X \leq \nu$. Since there cannot be more than 10 divisors larger than $n^{1/10}$, we have $\nu - 10 \leq X$. Denote $\chi_p \mathbf{1}_{p|x}$. Then:

$$X = \sum_{p \leq n^{1/10}} \chi_p \quad \text{and} \quad \mathbb{E}[\chi_p] = \frac{\lfloor \frac{n}{p} \rfloor}{n} = \frac{1}{p} + O(1/n) \Rightarrow$$

$$\mathbb{E}[X] = \sum_{p \leq n^{1/10}} \frac{1}{p} + O(1) \stackrel{4.10}{=} \log \log n + O(1)$$

Now we calculate the Δ^* part. For any p, q primes, we have $\Pr[\chi_p \cdot \chi_q] = \frac{\lfloor \frac{n}{pq} \rfloor}{n}$ (by the Chinese Reminder Theorem), therefore:

$$\text{cov}(\chi_p, \chi_q) = \mathbb{E}[\chi_p \chi_q] - \mathbb{E}[\chi_p] \mathbb{E}[\chi_q] = \frac{\lfloor \frac{n}{pq} \rfloor}{n} - \frac{\lfloor \frac{n}{p} \rfloor}{n} \frac{\lfloor \frac{n}{q} \rfloor}{n} = \frac{1}{pq} - \left(\frac{1}{p} - \frac{1}{n} \right) \left(\frac{1}{q} - \frac{1}{n} \right) \leq \frac{1}{n} \left(\frac{1}{p} + \frac{1}{q} \right)$$

So

$$\text{Var} X = \sum_p \text{Var} \chi_p + \sum_{p \neq q} \text{cov} \chi_p, \chi_q \leq \log \log n + O(1) + \sum_{p \neq q} \frac{1}{n} \left(\frac{1}{p} + \frac{1}{q} \right) \leq$$

$$\log \log n + O(1) + (n^{1/10})^2 \frac{1}{n} = \log \log n + o(1)$$

So the result follows from Chebyshev. \square

4.5 The Röde Nibble

The question deals with the existence of designs.

^vLet m be the number of divisors larger than $n^{1/10}$, then x is at least $n^{m/10} \leq x \leq n$.

Definition 4.5 $((n, k, r)$ - design). : An (n, k, r) - design is a k -graph over n vertices such that any r -set of vertices is contained exactly in one edge.

Example 4.2. $k = 2, r = 1$ means a perfect matching.

Question 4.5.1. *Is it true that for any $r < k \ll n$ there exists a corresponding design?*

Of course not! Take an odd n , one cannot find a perfect matching in a graph over an odd number of vertices. We get division requirements: Denote by e the number of edges in a design. Then

$$e \cdot \binom{k}{r} = \binom{n}{r} \Rightarrow \binom{k}{r} \mid \binom{n}{r}$$

Definition 4.6. The Complementary design with respect to $A \subset [n]$ with $|A| < r$ is with the edges

$$E_A = \{e \setminus A \mid A \subset e \in E\}$$

4.5.1 Approximations

Definition 4.7 (Covering). a covering is a relaxation of designs, when we demand that any r -tuple is contained in *at least* one edge

Definition 4.8 (Packing). a covering is a relaxation of designs, when we demand that any r -tuple is contained in *at most* one edge

Remark. In these cases, clearly $|E| \geq \frac{\binom{n}{r}}{\binom{k}{r}}$ and $|E| \leq \frac{\binom{n}{r}}{\binom{k}{r}}$ respectively.

The *Erdős Hananni conjecture* is that for any k, r , when $n \rightarrow \infty$ there exists a covering of size $VI (1 + o(1)) \frac{\binom{n}{r}}{\binom{k}{r}}$. This is equivalent of having a packing of size $(1 - o(1)) \frac{\binom{n}{r}}{\binom{k}{r}}$. This conjecture was proved by using the *Röde Nibble*.

Proof (for the case $r = 2, k = 3$). ^{VII} We look for a collection of $(1 + o(1)) \frac{\binom{n}{2}}{3}$ triplets that cover all edges. Had we tried to choose any triangle independently with probability $1/n$, we would have failed miserably:

$$\Pr[\text{A specific edge is not covered}] = (1 - 1/n)^{n-2} \approx \frac{1}{e}$$

Which means that this method "misses" a constant amount of edges!

We try to choose any triangle with probability $\frac{\varepsilon}{n}$, which results in approximately $\frac{\varepsilon n^2}{6}$ triangles. Then:

$$\Pr[\text{A specific edge is not covered}] = (1 - 1/n)^{n-2} \approx \frac{1}{e^\varepsilon}$$

So

$$\Pr[\text{a specific edge is covered}] \approx 1 - e^{-\varepsilon} \approx \varepsilon - \frac{\varepsilon^2}{2} \dots$$

□

^{VI}In the sense of $|E|$

^{VII}Steiner Triplet systems

Definition 4.9 (Typical Graph). A graph with m edges is called (D, δ, k) -Typical if:

1. Aside from $\delta \cdot m$ edges, all edge is contained in $D(1 \pm \delta)$ triangles.
2. Any edge is contained in at most kD triangles.

Lemma 4.10.1. For any $\varepsilon > 0$, large enough D , k and $\delta > 0$, there exists $\gamma > 0$ such that in any (D, δ, k) typical graph there is a collection of $\frac{\varepsilon}{3}(m \pm \gamma)$ triangles, denoted T such that $G \setminus T$ is a graph with $m \cdot e^{-\varepsilon}(1 \pm \gamma)$ edges, and is $(De^{-2\varepsilon}, \gamma, ke^{2\varepsilon})$ -typical

Proof. We sample each triangle i.i.d with probability $\frac{\varepsilon}{D}$. The number of triangles in the graph is at least $\frac{(1-\delta)mD(1-\delta)}{3} = \frac{mD}{3}(1-\delta_1)$, and at most $\frac{\delta mkD + (1-\delta)mD(1+\delta)}{3} = \frac{mD}{3}(1+\delta_1)$. Let T be the number of triangles. Then $T \sim \text{Bin}\left[\frac{mD}{3}(1 \pm \delta_1), \frac{\varepsilon}{D}\right]$ and the first item in the definition is gained by first moment argument.

Let X_e be the indicator of the event "e is not covered". Then if $d_e = D(1 \pm \delta)^{\text{VIII}}$, we get

$$\mathbb{E}[X_e] = \left(1 - \frac{\varepsilon}{D}\right)^{D(1 \pm \delta)} = e^{-\varepsilon}(1 + \delta_1)$$

Let $X = \sum_e X_e$, then $\mathbb{E}[X] = me^{-\varepsilon}(1 - \delta_1)$ and

$$\begin{aligned} \text{cov}X_e, X_{e'} &= \Pr[\text{both not covered}] - \Pr[e \text{ is not covered}] \Pr[e' \text{ is not covered}] \\ &= \left(1 - \frac{\varepsilon}{D}\right)^{d_e + d_{e'} - 1} - \left(1 - \frac{\varepsilon}{D}\right)^{d_e} \left(1 - \frac{\varepsilon}{D}\right)^{d_{e'}} \leq \frac{\varepsilon}{D} \end{aligned}$$

Then

$$\text{Var}X \leq me^{-\varepsilon}(1 \pm \delta_1) + mD(1 + \delta)2\frac{\varepsilon}{D} = O(m)$$

Then by Chebyshev, $\Pr[\text{The number of edges} \notin me^{-\varepsilon}(1 \pm \delta_2) < 0.01] \rightarrow 0$. It is left to show that $G \setminus T$ is typical.

Claim 4.5.1. Other than $\delta_1 m$ edges, all edges are both good and contained in $(1 \pm \delta_1)D$ triangles whose edges are good.

Proof.

$$\begin{aligned} \mathbb{E}[d_e(G \setminus T)] &= (1 \pm \delta_1)De^{-2\varepsilon}(1 \pm \delta_1)^2 \\ \text{Var}d_e(G \setminus T) &\leq \mathbb{E}[d_e(G \setminus T)] + D^2\frac{\varepsilon}{D} = O(D) \end{aligned}$$

Wo once again, Chebyshev we are done. □

□

Proof (Röde's Nibble - general case). Denote $p = e^{-\varepsilon}$. Let $G_0 = K_n$. Let G_{i+1} be obtained from G_i by removing each triangle with probability $\frac{\varepsilon}{p^{2i}D}$. Then $|E(G_i)| \approx p^i \binom{n}{2}$: With this step we've chosen

$$\approx \frac{\varepsilon}{p^{2i}D} \cdot \overbrace{p^i \binom{n}{2}}^{\# \triangle \text{ in typical edge}} \cdot \frac{\overbrace{p^{2i}n}}{3} = \varepsilon p^i \frac{n^2}{6}$$

Hence the number of triangles in the cover is

$$p^t \binom{n}{2} + \sum_{i=0}^{t-1} \varepsilon p^i \frac{n^2}{6} \leq \frac{n^2}{6} \left(3e^{-\varepsilon t} + \varepsilon \frac{1}{1 - e^{-\varepsilon}} \right) = \star$$

So when $\varepsilon \rightarrow 0$, we can choose a large enough t , we may have $\star \leq (1 + \delta) \frac{n^2}{6}$. The thing is - we've hidden all the error terms, but this can be dealt with. Super annoyingly. □

^{VIII}The number of triangles containing e

Chapter 5

Lovász's Local Lemma

Up to this point we used probability to find an object of interest with high probability. The *Local Lemma* is a tool to prove an object's existence even if the probability of finding them is small - even exponentially. In fact, this is an *algorithmic* approach.

Lemma 5.0.1 (The local Lemma, Symmetric). *Let $(A_i)_{i \in [n]}$ be events such that:*

1. *A_i is independent in all A_j , except for at most d of them^a*
2. *$\Pr[A_i] \leq p$*
3. *$ep(d+1) \leq 1$*

Then $\Pr\left[\bigcap_{i \in [n]} \overline{A_i}\right] > 0$

^aNot in pairs! A_i is dependent on the event $\bigcup_{j \in K} A_j$ for some $K \in \binom{[n] \setminus i}{d}$

5.0.1 Results from the lemma

Theorem 5.1 (Improvement on 1.1). *If $e \cdot 2^{1-\binom{k}{2}} \left(\binom{k}{2} \binom{n}{k-2} + 1 \right) < 1$, then $R(k, k) > n$*

Proof. Let $G \sim \mathcal{G}(n, 1/2)$. For any $S \in \binom{[n]}{k}$, denote A_S the event that S is a clique or an anti-clique. We know that $\Pr[A_S] = 2^{1-\binom{k}{2}}$. Note that A_S is independent in all A_M other than at most $\binom{k}{2} \binom{n}{k-2}$. Then by the 5.0.1 - there exists a graph in which no A_S occurs. \square

Exercise. Consider a k -SAT in which any variable appears in at most r clauses ($k > 3r$). Show a polynomial algorithm to decide satisfiability.