The Probabilistic Method in Combinatorics 80721

Based on lectures by Dr. Yuval Peled, and the book by Alon and Spencer - *The probabilistic method*

Notes by Asaf Etgar

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For any corrections, requests for additions or notes - please email me at
asafetgar@gmail.com

These notes have not been revised by the course staff, and some things may appear differently than in the lectures/ recitations.

Contents

Chapter 1

Introcuction

1.1 Ramsey Numbers

Claim 1.1.1. *For any graph $G = (V, E)$ there exists a partitioning of $V = A \sqcup B$ such that at least half of the edges are $A - B$ edges.*

Proof. Consider a random partition of V, A, B. That is, each vertex v is in A or in B w.p $\frac{1}{2}$ independently. Then:

$$\mathbb{E}\left[e(A,B)\right] \overset{linearity}{=} |E| \cdot \Pr\left[e \text{ is an } A, B \text{ edge}\right] = \frac{|E|}{2}$$

Which implies that there exists. a partition with said property. $\square$

Remark. One can prove this claim without the use of probability.

There are questions that we do not know yet how to solve without the use of probability:

Definition 1.1 (Ramsey Number)**.** The number $R(k, l)$ is the minimal n such that every graph G over n vertices contains a k-clique or an l -anti-clique.

Theorem 1.1 (Ramsey)**.** $R(k, l) \le \binom{k-l-2}{k-1}$*. In particular,* $R(k, k) \le \binom{2k-2}{k-1} \approx \frac{4^{k-1}}{\sqrt{\pi k}}$

Theorem 1.2. *If* $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$*, then* $R(k, k) > n$

Proof. Consider now a random graph $G \sim G(n, \frac{1}{2})$. For any $A \in \binom{[n]}{k}$, denote by M_A the event that A is a clique or anti-clique in G. Then:

$$\Pr\left[M_A\right] = \Pr\left[A \text{ is a clique}\right] + \Pr\left[A \text{ is an anti-clique}\right] = 2^{1-\binom{k}{2}}$$

And therefore

$$\Pr\left[\exists \text{ a clique or anti-clique of size } k\right] = \Pr\left[\bigcup_{A \in \binom{[n]}{k}} M_A\right] \le \sum_{A \in \binom{[n]}{k}} \Pr\left[M_A\right] = \binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

Hence there exists a graph over n vertices without a clique or anti-clique of size k. $\square$

Remark. Note that

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1 \iff \binom{n}{k} < 2^{\binom{k}{2}-1}$$

And also, $\binom{n}{k} \leq \frac{n^k}{k!}$, and by Stirling's approximation - $k! \geq \left(\frac{k}{e}\right)^k$. Pluggin in the inequality:

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

Comparing this to the formula in 1.1, this is a very loose bound (at least for k, k).

Remark. If $n = 2^{k/2}$, then:

$$\Pr\left[\text{A random graph contains a clique or anti-clique}\right] \leq \binom{n}{k} 2^{1-\binom{k}{2}} \xrightarrow{n,k} 0$$

Which means "almost all graphs are Ramsey graphs", but we do not yet have any explicit construction.

This theorem implies that we know the existence of a graph of order n and without clique or anti-clique of size $k \approx 2 \log n$. The best construction known without the use of probability is for $k = \log^{C \cdot \log \log \log n} n$.

Chapter 2

Linearity of Expectation

2.1 Sum-Free Sets

Theorem 2.1. *For any $B \in \binom{\mathbb{N}}{n}$ (with repetitions), there exists $A \in \binom{B}{n/3}$ such that there are no $a, b, c \in A$ with $a + b = c$*

Proof (by Erdős). Denote $[x] = x - \lfloor x \rfloor$, and for any $t \in [0, 1]$ let $A_t = \left\{ b \in B \mid [tb] \in \left(\frac{1}{3}, \frac{2}{3}\right) \right\}$. For any t, A_t is sum-free: If $a, b \in A_t$ and $[ta], [tb] \in \left(\frac{1}{3}, \frac{2}{3}\right)$, then $[a + b] \notin \left(\frac{1}{3}, \frac{2}{3}\right)$. We consider the probability space of the coin tosses of t. Denote $X_i = \mathbb{1}_{b_i \in A_t}$, then $\Pr[X_i = 1] = \frac{1}{3}$. Hence consider the expectation of a size of a random A_t:

$$\mathbb{E}[|A_t|] = \sum_{i \in [n]} \mathbb{E}[X_i] = \frac{n}{3}$$

$\square$

Remark. The general idea of probabilistic methods is to find an object in which the property always holds, and then average over these objects

2.2 Tournaments

Definition 2.1. A tournament is an orientation of K_n.

Definition 2.2. We say a vertex v *overcomes* some $A \subset V \setminus x$ if $v \to x$ for any $x \in A$ (that is, the orientation of $vx \in E(K_n)$ is $v \to x$).

Theorem 2.2. *If $\binom{n}{k}\left(1 - 2^{-k}\right)^{n-k}$, then there exists a tournament such that for any $A \in \binom{V}{k}$ there exists v that overcomes A.*

Proof. Denote by S_k the event that for any A of size k there exists an overcoming v. Consider a random tournament, and let $A \in \binom{V}{k}$, what is the probability that no v overcomes A?

$$\Pr[\text{No } v \text{ overcomes } A] = \left(1 - 2^{-k}\right)^{n-k}$$

(some v overcomes A w.p 2^k, and they are independent) Then:

$$\Pr[S_k^c] \leq \binom{n}{k}\left(1 - 2^{-k}\right)^{n-k} < 1$$

$\square$

Remark. The union bound is quite similar to linearity of expectation.

Theorem 2.3. *There exists a tournament with at least* $n! \cdot 2^{-(n-1)}$ *Hamiltonian cycles.*

Proof. Consider a random tournament. Then:

$$\mathbb{E}\left[\text{\# of Hamiltonian cycles}\right] = \sum_{\pi \in S_n} \Pr\left[\pi(V)\text{is a cycle}\right] = n!2^{-(n-1)}$$

(the last equation is the probability of this permutation defininig a cycle) Then there must exist a tournament with at least this number of cycles. ☐

2.3 ??? If you have a suggestion for a name, let me know!

Theorem 2.4. *Let* $v_1, \dots v_n \in \mathbb{R}^d$ *be unit vectors. Then there exists* $\varepsilon_1, \dots \varepsilon_n \in \{\pm 1\}$ *such that*

$$\left\|\sum_{i\in[n]} \varepsilon_i v_i\right\| \leq \sqrt{n}$$

and there exists such ε_i *for the opposite inequality.*

Proof. Consider a random choice of ε_i. Denote $X = \left\|\sum_{i\in[n]} \varepsilon_i v_i\right\|^2$. Then:

$$X = \left\|\sum_{i\in[n]} \varepsilon_i v_i^2\right\| = \sum_{i\in[n]} \varepsilon_i^2 v_i + 2 \cdot \sum_{i<j} \varepsilon_i \varepsilon_j \left\langle v_i, v_j\right\rangle$$

then

$$\mathbb{E}\left[X\right] = n + 2 \cdot \sum_{i<j} \left\langle v_i, v_j\right\rangle \mathbb{E}\left[\varepsilon_i \varepsilon_j\right] = n$$

Since $\mathbb{E}\left[\varepsilon_i \varepsilon_j\right] = \mathbb{E}\left[\varepsilon_j\right]\mathbb{E}\left[\varepsilon_j\right] = 0 \cdot 0 = 0$, and the claim follows as usual. ☐

2.3.1 Derandomization

We would like to de-randomize the process and find an efficient algorithm of finding these ε_i. By the law of total expectation, $\mathbb{E}\left[X\right] = \frac{1}{2}\mathbb{E}\left[X \mid \varepsilon_1 = 1\right] + \frac{1}{2}\mathbb{E}\left[X \mid \varepsilon_1 = -1\right]$.

Claim 2.3.1. *If we've fixed* $\varepsilon_1 \dots \varepsilon_{i-1}$ *such that* $\mathbb{E}\left[X \mid \varepsilon_1 \dots \varepsilon_{i-1}\right] \leq n$*, then we can efficiently find* ε_i *such that* $\mathbb{E}\left[X \mid \varepsilon_1 \dots \varepsilon_{i-1}, \varepsilon_i\right] \leq n$

Proof.

$$\mathbb{E}\left[X \mid \varepsilon_1 \dots \varepsilon_{i-1}\right] \overset{\star}{=} \frac{1}{2}\mathbb{E}\left[X \mid \varepsilon_1 \dots \varepsilon_{i-1}, \varepsilon_i = 1\right] + \frac{1}{2}\mathbb{E}\left[X \mid \varepsilon_1 \dots \varepsilon_{i-1}, \varepsilon_i = -1\right]$$

with $\star$ by law of total expectation (w.r.t the random variable ε_i). But

$$\mathbb{E}\left[X \mid \varepsilon_1 \dots \varepsilon_{i-1}, \varepsilon_i = 1\right] = n + 2 \sum_{j'<j\leq i} \varepsilon_j \varepsilon_{j'} \left\langle v_j, v_i\right\rangle + 0$$

And we know the values of $\varepsilon_{j'}, \varepsilon_j$, so we can compute it efficiently. Then we choose the epsilon TodoCompete from the book ☐

2.4 Turan's theorem

Theorem 2.5. *In any graph (V,E), there exists an independent set of size at least* $\sum_{v\in V}\frac{1}{deg(v)+1}$

Proof. Consider a random ordering of V. We choose a vertex to add to the set I ("independent") if he appears before all of his neighbors. Clearly I is independent. And:

$$\mathbb{E}\left[|I|\right]=\sum_{v\in V}\Pr\left[v\in I\right]=\sum_{v\in V}\Pr\left[v\text{ is the first of hie neighbors in the ordering}\right]=\sum_{v\in V}\frac{1}{deg(v)+1}$$

□

Corollary 2.6. *In G there exists a clique of size* $\geq\sum_{v\in V}\frac{1}{n-deg(v)}$

Theorem 2.7 (Turán)**.** *If the maximal clique is of size r, then*

$$r\geq\sum_{v\in V}\frac{1}{n-\deg(v)}\geq\frac{n^2}{n^2-2|E|}$$

Therefore $|E|\leq\left(1-\frac{1}{n}\right)\cdot\frac{n^2}{2}$

2.5 Unbalancing Lights

Let A be an $n\times n$ matrix over $\{\pm1\}$. There is a switch for every row and every column, which flips all bits corresponding to it.

Theorem 2.8. *There exists $x,y\in\{\pm1\}^n$*[I] *such that* $x^\top Ay\geq\left(\sqrt{\frac{2}{\pi}}+o(1)\right)n^{\frac{3}{2}}$

Proof. Choose a random y (that is, $y_i\sim U(\pm1)$ iid. Let $R_i=\sum_{j=1}^n A_j^i y_j$. Since y_i are iid, $A_j^i y_j\sim$ a sum of n signs±1 iid. Then by CLT:

$$\frac{1}{\sqrt{n}}R_i\overset{distribution}{\longrightarrow}\mathcal{N}(0,1)$$

And therefore $\mathbb{E}\left[\frac{1}{\sqrt{n}}|R_i|\right]\overset{n\to\infty}{\longrightarrow}\mathbb{E}\left[|z|\right]=\sqrt{\frac{2}{\pi}}$. Hence:

$$\mathbb{E}\left[\sum_{i=1}^n|R_i|\right]=\sum_{i=1}^n\mathbb{E}\left[|R_i|\right]=\left(\sqrt{\frac{2}{\pi}}+o(1)\right)n^{\frac{3}{2}}$$

Then there exists y such that $\sum_{i=1}^n|R_i|\geq\left(\sqrt{\frac{2}{\pi}}+o(1)\right)n^{\frac{3}{2}}$. As for x, note that

$$x^\top Ay=\sum_{i=1}^n\sum_{j=1}^n x_iA_j^iy_j=\sum_{i=1}^n x_iR_i\overset{\star}{=}\sum_{i=1}^n|R_i|\geq\left(\sqrt{\frac{2}{\pi}}+o(1)\right)n^{\frac{3}{2}}$$

with $\star$ since we can take $x_i=sign(R_i)$.[II] □

2.6 2-colorings of hypergraphs

[I] think of x as responsible of rows, and y of columns
[II] In some sense, this is "the smartest move" in order to get as many light bulbs lit as possible.

Definition 2.3 (k-uniform Hypergraph)**.** $H = (V, E)$ is a k-uniform Hypergraph with $V(H)$ its vertices and $E(H) \subset \binom{V(H)}{k}$. In particular, a 2-uniform hypergraph is just a graph.

Definition 2.4 (2-coloring of hypergraph)**.** Let H be a k-graph. A *2 coloring* of H is a function $f : V(H) \to \{0, 1\}$ such that there is no monochromatic edge, that is $\forall e \in E(H) \quad \exists x, y \in e \quad f(x) \neq f(y)$.

Definition 2.5. Denote $m(k)$ the minimal number of edges in a k-graph that is not 2 colorable.

Example 2.1. $m(2) = 3$, consider a triangle.

Example 2.2. $m(3) = 7$, consider the Fano Plane.

Theorem 2.9. $m(k) \geq 2^{k-1}$

Proof. Let H be a hypergraph with less than $m2^{k-1}$ edges. Let f be a uniformly random coloring of H Then

$$\Pr[e \text{ is monochromatic}] = 2^{1-k}$$

Therefore

$$\mathbb{E}[\#\text{monochromatic edges}] < m \cdot 2^{1-k} = 1$$

□

Theorem 2.10. $m(k) = O(k^2 2^k)$.

Proof. Let $n = k^2$, and choose $c \cdot k^2 2^k$ (we specify c later) edges uniformly IID. We show that $\mathbb{E}[\#\text{colorings without monochromatic edges}] < 1$. Fix a coloring $\varphi : [n] \to \{0, 1\}$. Let $a = |\varphi^{-1}(0)|$, then

$$\Pr[e \text{ is monochromatic under } \varphi] =$$
$$\frac{\binom{a}{k} + \binom{n-a}{k}}{\binom{n}{k}} \geq \frac{2 \cdot \binom{n/2}{k}}{\binom{n}{k}} \geq \frac{2 \cdot \frac{(\frac{n}{2} - k)^k}{k!}}{\frac{n^k}{k!}} = 2\left(\frac{1}{2} - \frac{k}{n}\right)^k = 2 \cdot \left(\frac{1}{2}\right)^k \left(1 - \frac{2}{k}\right)^k \geq c\left(\frac{1}{2}\right)^k$$

for some c. Now we have

$$\mathbb{E}[\#\text{colorings without monochromatic edges}] = \sum_{\varphi} \Pr[\text{no edge is monochromatic under } \varphi] \leq$$
$$\leq 2^n \left(1 - \frac{c}{2^k}\right)^m \leq e^{\log(2) \cdot n - c \cdot 2^{-k} \cdot m} \overset{\star}{<} 1$$

With $\star$ by choice of $m = \frac{2\log(2)}{c} k^2 \cdot 2^k$. □

Theorem 2.11 (Improvement on the bound from 2.9)**.** $m(k) \geq t2^k \sqrt{\frac{k}{\log(k)}}$.

Proof. Or proof is algorithmic: Let H be with t edges, color all $V(H)$ in blue. Traverse $V(H)$ in a random order, let v the current vertex. If v is the last-visited vertex in a monochromatic blue edge - alter its color to blue. The algorithm fails only if there is a monochromatic red edge. This happens only if the vertex $v = e \cap f$, f is red, e is blue and v is the first in f and last in blue (this is a bad configuration). What is the probability of such configuration to occure? We consider the probability over the coin tosses of $\pi \sim U(S_n)$, and claim

$$\Pr[\text{there exists a bad configuration}] < 1$$

We note that:

$$\Pr[\text{there exists a bad configuration}] \leq \mathbb{E}[\#(e,f) \text{ are bad edges}] \overset{\star}{\leq} m^2 \cdot \frac{((k-1)!)^2}{(2k-1)!} =$$
$$\frac{m^2}{(2k-1)\binom{2k-2}{k-1}} \overset{\star\star}{=} \frac{m^2 \cdot (c+o(1))}{\sqrt{k \cdot 4^k}} \overset{?}{<} 1$$

with $\star$ a bound on the number of edges that intersect in a unique vertex, times the probability of having a bad configuration, and $\star\star$ since $\binom{2n}{n} = \frac{c+o(1)}{\sqrt{n}} 2^{2n}$. In order to have ?, take $m < c' \cdot k^{\frac{1}{4}} \cdot 2^k$. This is not the bound we want - as the power of k is $\frac{1}{4}$. The problem is the expectation sometimes lies - that is, the expectation can be much larger than the probability we want to bound (see the remark below).

We traverse the vertices differently: For a vertex v, choose $r_v \sim U([0,1])$ i.i.d, and traverse $V(H)$ according to r_v from the smallest to largest. Let p be some probability chosen later, and denote

$$L = \left[0, \frac{1-p}{2}\right] \quad M = \left[\frac{1-p}{2}, \frac{1+p}{2}\right] \quad R = \left[\frac{1+p}{2}, 0\right]$$

Now:

$$\begin{aligned}
&\Pr[\text{there exists a bad configuration}] \leq \\
&\overbrace{\Pr[\exists e \in L \cup R]}^{1} + \overbrace{\Pr[\exists \text{bad configuration whose intersection is in } M]}^{2} \leq \\
&\overbrace{m \cdot 2(|L|)^k}^{1} + \overbrace{m^2 \int_{\frac{1-p}{2}}^{\frac{1+p}{2}} r_v^{k-1}(1-r_v)^{k-1} dr_v}^{2} = \\
&m \cdot rbk\frac{1-p}{2}^k + m^2 \int_{\frac{1-p}{2}}^{\frac{1+p}{2}} r_v^{k-1}(1-r_v)^{k-1} dr_v \leq \\
&2m\frac{e^{-pk}}{2^k} + m^2 \cdot p \left(\frac{1}{4}\right)^{k-1} \overset{?}{<} 1
\end{aligned}$$

Choosing $p = \frac{\log k}{k}$ and $m < \frac{1}{4} 2^k \sqrt{\frac{k}{\log k}}$ yields the result. $\square$

Remark. Let $X_n = n^2$ with probability $1/n$ and 0 otherwise. Note that $\Pr[X_n > 0] = 1/n$, while $\mathbb{E}[X_n] = n$.

Chapter 3

Alterations Method

Up to this point, we made a random choice of object and use it. We now deal with the setting where a naïve random choice is not good enough - but we cat alter it a little bit so it would be good. The idea here is to bound the expectations of alterations needed to the random object.

3.1 Dominating Sets

Definition 3.1. Let G be a graph. $A \subset V$ is *Dominating* if any $v \in V$ has a neighbor in A.

Theorem 3.1. *Let G be of minimal degree δ, then there exists a dominating set of size $n \cdot \frac{\ln(1+\delta}{1+\delta}$.*

Proof. Let $B \subset V$ such that any $v \in B$ with probability p (will be chosen later) independently. Let C_B be the collection of vertices that all of their neighbors are not in B, that is $C_B = \{x \notin B \mid \forall vx \in E \quad v \notin B\}$. Clearly $A = B \cup C_B$ is dominating. Then

$$\mathbb{E}\left[|A|\right] = \mathbb{E}\left[|B|\right] + \mathbb{E}\left[|C_B|\right] = np + n\Pr\left[v \in C_B\right] \overset{\star}{\leq} npne^{-p(1+\delta)}$$

With $\star$ since $\Pr\left[v \in C_B\right] = (1-p)^{1+deg(x)} \leq (1-p)^{1+\delta)} \leq e^{-p(1+\delta)}$. Find the optimal p by differentiating w.r.t p, and get $p = \frac{\ln(1+\delta)}{1+\delta}$, then $\mathbb{E}\left[|A|\right] \leq n\left(\frac{\ln(1+\delta)+1}{1+\delta}\right)$ $\square$

3.2 Ramsey Numbers - Revisited

Recall that 1.2 gives us a lower bound on Ramsey numbers. We will use alterations to improve this lower bound.

Theorem 3.2. *For any n, k, $R(k,k) \geq n - \binom{n}{k} \cdot 2^{1-\binom{k}{2}}$*

Proof. Consider $G \sim \mathcal{G}(n, \frac{1}{2})$. Note that $\mathbb{E}\left[\#\text{monochromatic sets of size } k\right] = \binom{n}{k}2^{1-\binom{k}{2}}$ as we've seen, therefore there exists a graph with at most this amount of monochromatic sets of size k, denote it G. Let G' be the graph obtained from G by removing a single vertex of any monochromatic set of size k. Then $|V(G)|$ is at least $n - \binom{n}{k} \cdot 2^{1-\binom{k}{2}}$, and clearly in G' there is no monochromatic set of size k. $\square$

Corollary 3.3. *$R(k,k) \geq n - \frac{e^n}{k}2^{1-\binom{k}{2}}$ by the Stirling-esque estimation done in chapter 1. The optimal n is $\frac{2^{k/2} \cdot k}{e}$ which yields $R(k,k) \geq 2^{k/2}k \cdot \left(\frac{1+o(1)}{e}\right)$.*

3.3 Girth and coloring

Let $G = (V, E)$ be a graph.

Definition 3.2 (Girth)**.** The *girth* of G is the length of a minimal cycle in G.

Remark. In particular, if the girth is $\geq g$, then for any $v \in V$, its g-neighborhood looks like a tree.

Definition 3.3 (Chromatic Number)**.** The *chromatic number* of G, denoted $\chi(G)$ is the minimal k such that there exists a proper coloring $c : V \to [k]$ of G.

Remark. It is difficult to know what $\chi(G)$ is - it is NP-hard

Definition 3.4 (Independence number)**.** The *Independence Number* of a graph G, denoted $\alpha(G)$, is the size of a largest independent set in G.

Claim 3.3.1. *If T is a tree, then $\chi(T) = 2$*

Proof. It is bipartite - use BFS. □

Theorem 3.4 (Erdős)**.** *For any k, g there exists a graph G with $\chi(G) \geq k$ and girth $\geq g$.*

Remark. This is surprising! Any neighborhood seems like χ should be small (as neighborhoods look like trees) - but it turns out it cannot be considered locally; χ is a *global* property of G.

For ease - we write $\alpha(G) = \alpha$, same for χ.

Lemma 3.4.1. *$V(G) \leq \alpha \cdot \chi$.*

Proof. If $c : V \to [\chi]$ is a proper coloring, any $c^{-1}(i)$ is independent. □

Lemma 3.4.2. *There exists a graph G over n vertices (for a large enough $n = n(k, g)$) with the following properties:*

1. *The number of cycles of length $\leq g$ is smaller than $\frac{n}{2}$*
2. *$\alpha(G) \leq 3 \log n \cdot n^{1-\frac{1}{2g}}$*

Proof. Let $G \sim \mathcal{G}(n, p)$ with $p = n^{\frac{1}{2g}-1}$. Let X be the number of cycles of length $\leq g$. Then:

$$\mathbb{E}[X] \overset{1}{=} \sum_{r=3}^{g} \binom{n}{r} \cdot \frac{(r-1)!}{2} \cdot p^r \overset{2}{\leq} \sum_{r=3}^{g} (n \cdot p)^r \overset{3}{\leq} g \cdot (n \cdot p)^g = g\sqrt{n}$$

Justifications:

1. Choose which vertices are in a cycle of length r ($\binom{n}{r}$) and order them in a cycle ($(r-1)!/2$ options) and multiply by the probability of such cycle to exists.
2. Bound $\binom{n}{r} \cdot \frac{(r-1)!}{2}$ from above naturally.
3. Bound the sum with the largest element in the summation.

Hence by Markov:

$$\Pr[X > n/2] \leq \frac{g\sqrt{n}}{n/2} \overset{n\to\infty}{\longrightarrow} 0$$

Which implies the first property. For the second property, let $t = 3 \log n \cdot n^{1-\frac{1}{2g}}$. Now:

$$\Pr[\alpha(G) \geq t] \leq \binom{n}{t} (1-p)^{\binom{t}{2}} \leq n^t \left(e^{-p}\right)^{\binom{t}{2}} \leq n^t e^{-p \cdot \binom{t}{2}} = e^{t(\log n - \frac{p \cdot t}{2} + 1)} = e^{t(-\frac{1}{2}\log n + 1)} \overset{n\to\infty}{\longrightarrow} 0$$

□

Proof (of 3.4). Let G' be a graph obtained from G by removing a single vertex from any cycle of length smaller than g. Then G''s girth is at least g. And $\alpha(G') \leq \alpha(G) \leq 3 \log n \cdot n^{1-\frac{1}{2g}}$, and note that

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{\frac{n}{2}}{3 \log n \cdot n^{1-\frac{1}{2g}}} \geq \frac{n^{\frac{1}{2g}}}{6 \cdot \log(n)} \xrightarrow{n\to\infty} \infty$$

□

3.4 Heilbronn triangle problem

Let $P \subset [0,1]^2$, $|P| = n$, denote $T(P) = \min\limits_{x,y,z \in P} Area(xyz)$, and let $T(n) = \max\limits_{|P|=n} T(P)$. Heilbornn conjectured[I] that $T(n) = \Theta(\frac{1}{n^2})$.

Theorem 3.5 (KPS, no proof)**.** $T(n) = \Omega\left(\frac{\log(n)}{n^2}\right)$

Remark. We still do not know some f for which $T(n) = \Theta(f(n))$, the best upper bound is still not tight.

Theorem 3.6. $T(n) \geq \frac{1}{70n^2}$

Proof. Let $\varepsilon = \frac{1}{70n^2}$. Generate $2n$ points $\sim U([0,1]^2)$ IID and remove a point from any triangle of area less than ε. Given a tringle xyz, Let t be the distance xy. then t has some density $f_{dist}(t)$. Then:

$$\Pr\left[area(xyz) \leq \varepsilon\right] l \int_0^{\sqrt{2}} \sqrt{2} \cdot 4\frac{\varepsilon}{t} f_{dist}(t) dt = (\star)$$

Note that $f_{dist}(t) = \lim_{h\to 0} \frac{1}{h} \Pr\left[t \leq dist(x,y)) \leq t+h\right]$ by the definition of density. Hence $f_{dist(x,y)}(t) \leq \lim_{h\to 0} \frac{1}{h} \pi((t+h)^2 - t^2)) = 2\pi t$, then:

$$(\star) \leq \int_0^{\sqrt{2}} \sqrt{2} \frac{4\varepsilon}{t} 2\pi t dt = 16\pi\varepsilon$$

Which implies

$$\mathbb{E}\left[\text{number of triangles with area smaller than } \varepsilon\right] \leq \binom{2n}{3} \frac{16\pi}{70n^2} < n$$

□

Remark. Erdős has a non-combinatorial construction. Let n be some prime, and consider the grid $[n-1] \times [n-1]$[II] and take $\left\{(k, k^2 \mod n)\right\}_{k \in [n-1]}$. Note that the smallest triangle of 3 points in $\mathbb{Z}^2$ is of area 1/2, unless the three points are on the same diagonal. If they are on the diagonal $ax+b$, this means that there exists three values of k such that $(ak+b) = k^2 \mod n$, but this is a quadratic polynomial in $\mathbb{F}_n[x]$, therefore it cannot have more than 2 solutions. Hence by scaling, $T(n) \geq \frac{1}{2(n-1)^2}$

[I] falsely

[II] Can rescale for the unit cube later...

Chapter 4

Second Moment Method

Up until now we discussed *first moment methods*. More formally, if $X = X_n \geq 0$ is an integer valued random variable, then the first moment method tells us that if $\mathbb{E}[X_n] \overset{n\to\infty}{\longrightarrow} 0$, then $\Pr[X_n > 0] \overset{n\to\infty}{\longrightarrow} 0$.

Example 4.1 (First Moment Method)**.** When $G \sim \mathcal{G}(n, p)$ is triangle-free? Denote X the number of traiangels in G. Then

$$\mathbb{E}[X] = \binom{n}{3} p^3 \leq (np^3)$$

Then taking $p = o\left(\frac{1}{n}\right)$ results in $\Pr[X > 0] \overset{n\to 0}{\longrightarrow} 0$. Is this bound *tight*? We saw that the expectation does not always give us a good bound - we need a way to reason about when is X concentrated about its expectation - that is the variance.

Definition 4.1 (Variance)**.** The variance of X is $VarX = \mathbb{E}\left[(X - \mathbb{E}[x])^2\right]$

Definition 4.2 (Covariance)**.** The covariance of X, Y is

$$covX, Y = \mathbb{E}[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Theorem 4.1 (Chebyshev)**.** *$Pr[|X - \mathbb{E}[X]| \geq t] \leq \frac{VarX}{t^2}$*

Corollary 4.2. *$Pr[X = 0] \leq \frac{Var(X)}{\mathbb{E}[X]^2}$*

Corollary 4.3. *If $VarX = o\left(\mathbb{E}[X]^2\right)$ then $Pr[X = 0] \longrightarrow 0$*

This results in the *Second moment method*: If $VarX = o\left(\mathbb{E}[X]^2\right)$ then $\Pr[X \geq 0] \overset{n\to\infty}{\longrightarrow} 0$. An equivalent condition is $\mathbb{E}[X^2] = \mathbb{E}[X]^2 (1 + o(1))$[I] An important case is when $X = \sum_{i=1}^{m} X_i$, in that case

$$VarX = \sum_{i=1}^{m} VarX_i + \sum_{i=1}^{m} \sum_{j \neq i} covX_i, X_j$$

If we denote $i \sim j$ when X_i, X_j are dependent, then

$$VarX = \sum_{i=1}^{m} VarX_i + \sum_{i=1}^{m} \sum_{j \sim i} covX_i, X_j$$

[I]Since $VarX = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$

Assumptions:

1. $X_i = \mathbb{1}_{A_i}$, then $VarX_i = \Pr[A_i] \cdot (1 - \Pr[A_i]) \leq \Pr[A_i]$ and $CovX_i, X_j = \Pr[A_i \cap A_j] - \Pr[A_i] \cdot \Pr[A_j] \leq \Pr[A_i \cap A_j] = \Pr[A_i] \cdot \Pr[A_j \mid A_i]$. Under this assumption, we get

$$VarX \leq \mathbb{E}[X] + \sum_i \Pr[A_i] \cdot \sum_{i \sim j} \Pr[A_j \mid A_i]$$

2. A symmetry assumption is $\sum_{i \sim j} \Pr[A_j \mid A_i]$ is independent of i. This is usually true in many cases. We denote $\sum_{i \sim j} \Pr[A_j \mid A_i] = \Delta^*$[II]. With this notation, $VarX \leq \mathbb{E}[\cdot](1 + \Delta^*)$.

   Corollary 4.4. *If* $\mathbb{E}[X] \longrightarrow \infty$, $\Delta^* = o(\mathbb{E}[X])$, *then* $Pr[X = 0] \longrightarrow 0$.

4.1 H-free graphs

The general question we deal with is

Question 4.1.1. *Given a small graph H, what is the threshold function of $G \sim \mathcal{G}(n,p)$ containing H?*

More formally:

Definition 4.3 (Threshold Function)**.** Let $G \sim \mathcal{G}(n,p)$ and H some fixed small graph. We say $f(n)$ is a *threshold function* for finding H in G if

$$p \ll f(n) \Rightarrow \Pr[G \text{ contains a copy of } H] \overset{n\to\infty}{\longrightarrow} 0$$
and
$$p \gg f(n) \Rightarrow \Pr[G \text{ contains a copy of } H] \overset{n\to\infty}{\longrightarrow} 1$$

We now explore some threshold functions

4.1.1 Triangles in $\mathcal{G}(n,p)$

We've shown that if $G \sim \mathcal{G}(n,p)$ and $p \ll \frac{1}{n}$ then $\Pr[G \text{ contains a triangle}] \to 0$.

Claim 4.1.1. *If $p \gg \frac{1}{n}$ then $Pr[G$ contains a triangle$] \to 1$*

Remark. In a case like this, we say that $\frac{1}{n}$ is the *threshold function* for triangle existence in $\mathcal{G}(n,p)$.

Proof. Denote by X the number of triangles in G, then:

$$\mathbb{E}[X] = \binom{n}{3} p^3 = (1_o(1)) \frac{n^3 p^3}{6} \overset{p \gg \frac{1}{n}}{\longrightarrow} \infty$$

Denote by T a triangle in G.

$$\Delta^* = \sum_{T' \sim T} \Pr[A_{T'} \mid A_T] \overset{\star}{=} 3 \cdot (n-3)p^2 \leq 3np^2$$

[II]Usually $\sum_i \Pr[A_i] \cdot \sum_{i \sim j} \Pr[A_j \mid A_i] := \Delta$

With $\star$ since any T' dependent on T is taken by choosing 2 vertices in T and a vertex not in T. Now we have

$$\frac{\Delta^*}{\mathbb{E}[X]} \leq \frac{18}{n^2 p} \longrightarrow 0$$

And we are done. □

Remark. In fact we've shown that $\frac{X}{\mathbb{E}[X]} \overset{\text{in probability}}{\longrightarrow} 1$. This is some kind of law of large numbers. This can be seen by 4.1:

$$\Pr[|X - \mathbb{E}[X] > \varepsilon \mathbb{E}[X]] \leq \frac{VarX}{\varepsilon^2 \mathbb{E}[X]^2} \longrightarrow 0$$

4.1.2 K_4 in $\mathcal{G}(n,p)$

Denote X the number of K_4 copies in G. Then

$$\mathbb{E}[X] = \binom{n}{4} p^6 = \frac{1+o(1)}{24} n^4 p^6$$

Then if $p \ll n^{-2/3}$, $\Pr[X=0] \longrightarrow 0$, and otherwise $\mathbb{E}[X] \to \infty$. Now denote by S a fixed copy of K_4 in G. Then:

$$\Delta^* = \sum_{S' \sim S} \Pr[A_{S'} \mid A_S] \leq \overbrace{6n^2p^5}^{\text{Share 2 vertices}} + \overbrace{4np^3}^{\text{Share 3 vertices}}$$

Then

$$\frac{\Delta^*}{\mathbb{E}[X]} \leq O\left(\frac{1}{n^2 p} + \frac{1}{n^3 p^3}\right) \overset{p \gg n^{-2/3}}{\longrightarrow} 0$$

4.1.3 $K_4 * e$ in $\mathcal{G}(n,p)$

In this case

$$\mathbb{E}[X] = 5\binom{n}{5} p^7 = \frac{1+o(1)}{24} n^5 p^7$$

Then $p \ll n^{-5/7}$ implies $\Pr[X>0] \longrightarrow 0$. Is it true that $p \gg n^{-5/7}$ implies $\Pr[X=0] \longrightarrow 1$? Note that $n^{-5/7} \ll n^{-2/3}$, and since existence of $K_4 * e$ implies existence of K_4, had $n^{-5/7}$ been the threshold, it would contradict our previous proof.

Definition 4.4 (Maximal Subgraph Density)**.** Given H, we define its maximal density by

$$m(H) := \max_{\emptyset \neq A \subset V(H)} \frac{e(A)}{|A|}$$

Remark. First moment argument shows that $p \ll n^{-\frac{1}{m(H)}}$.

Theorem 4.5 (Threshold characterization)**.** *The threshold function of H is $n^{-\frac{1}{m(H)}}$*

4.2 Cliques in $\mathcal{G}(n, 1/2)$

Theorem 4.6. *Let $G \sim \mathcal{G}(n, 1/2)$. Denote X = size of maximal clique in G. There exists $k = k(n)$ and $k = \Theta(2\log_2(n))$ such that*

$$Pr[X \in \{k, k+1\}] \overset{n\to\infty}{\longrightarrow} 1$$

Proof (Sketch). Define $f(k) = \mathbb{E}[\#\text{of } k\text{-cliques in } G] = \binom{n}{k}2^{-\binom{k}{2}}$. We claim that if $f(k) \to \infty$ then there exists a clique of size k.

$$\Delta^* = \sum_{i=2}^{k-1} \binom{k}{i} \cdot \binom{n-k}{k-i} \left(\frac{1}{2}\right)^{\binom{k}{2}-\binom{i}{2}}$$

Now it can be shown that $k \sim 2\log_2(n$, if $f(k) \to \infty$ then $\frac{\Delta^*}{f(k)} \to 0$ by case analysis. We note that

$$\frac{f(k+1)}{f(k)} = \frac{\binom{n}{k+1}2^{-\frac{k(k+1)}{2}}}{\binom{n}{k}2^{-\frac{k(k-1)}{2}}} = \frac{n-k}{(k+1)}2^{-k} \approx 2^{-2\log_2(n)} < \frac{1}{n}$$

We now consider a k_0 such that $f(k_0) \geq 1$ and $f(k_0+1) < 1$. If $f(k_0(n))$ tends to ∞ with n and $f(k_0) = o(n)$, then $f((k_0+1)(n)) \overset{n\to\infty}{\longrightarrow} 0$. In this case, there exists a maximal k_0-clique, and no (k_0+1)-clique, so the maximal clique of k_0. A similar argument may show the result, but it's quite annoying. The book has the complete proof ☐

4.3 Distinct Sums Problems

The *Distinct Sums Problems* is a problem suggested by Erdős[III].

Problem. *What is the maximal size of $S \subset [n]$ with distinct partial sums?*

Solution (lower bound)**.** Take $S = \{1, 2, 4, 8, \ldots\}$, then $|S| = \log_2(n)$.

Question 4.3.1. *Is $|S| \leq \log_2(n) + o(1)$?*

Solution (Upper bound)**.** $2^{|S|} \leq n \cdot |S|$

Corollary 4.7. $|S| \leq \log(n) + \log\log(n) + o(1)$

Claim 4.3.1. [IV] $|S| \leq \log(n) + 0.5\log\log(n) + o(1)$

Proof. Let $S = \{s_1 \ldots s_m\}$ and consider a random partial sum $X = \sum_{i=1}^m b_i s_i$ with $b_i \sim U(\{0,1\}$.

$$\mu := \mathbb{E}[X] = \sum_{i=1}^m \frac{s_i}{2} \quad \text{and} \quad VarX \sum_{i=1}^m \frac{s_i^2}{4}$$

Now try to bound the variation:

$$VarX = \frac{1}{4}\sum_{i=1}^m s_i^2 \overset{s_i \leq n}{\leq} \frac{mn^2}{4}$$

Then

$$\Pr[|x-\mu| \leq \lambda] \overset{Chebyshev}{\geq} 1 - \frac{mn^2}{4\lambda^2}$$

$$\Pr[|x-\mu| \leq \lambda] \overset{\text{distinct sums}}{\leq} (2\lambda+2)2^{-m}$$

[III] And it is still open with a \$300 prize awaiting to the solver!
[IV] This is worth \$150!

So

$$(2\lambda+2)2^{-m} \geq 1-\frac{mn^2}{4\lambda^2} \qquad \overset{\text{Take } \lambda=\sqrt{3}\sqrt{m}n}{\Longleftrightarrow}$$
$$c\sqrt{m}n2^{-m} \geq \frac{4}{c'mn^2} \geq \frac{11}{12}$$

Therefore $2^m \leq \tilde{C}\sqrt{m}n$ And some computations result in the bound. $\square$

4.4 Hardy- Ramanujan Thoerem

The question we deal with is "how many prime numbers divide a random number $\sim U([n])$?

Theorem 4.8 (Hardy- Ramanujan)**.** *For $x \in \mathbb{N}$ let $\nu(x)$ be the number of prime divisors of x (without multiplicity). If $x \in U([n])$, then for a large enough n,*

$$\forall \varepsilon > 0 \quad \exists A > 0 \quad Pr\left[|\nu(x) - \log\log n| > A\sqrt{\log\log n}\right] < \varepsilon$$

Theorem 4.9 (Erdős-Kac)**.** *With the same notations,*

$$\frac{\nu(x)-\log\log n}{\sqrt{\log\log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

Theorem 4.10 (Merten)**.** $\sum_{p\leq m}\frac{1}{p} = \log\log M = O(1)$

Proof(of 4.8). Denote by $X(x)$ the amount of prime divisors of x not greater then $n^{1/10}$. Clearly $X \leq \nu$. Since there cannot be more than 10 divisors larger than $n^{1/10}$[V], we have $\nu - 10 \leq X$. Denote $\chi_p \mathbb{1}_{p|x}$. Then:

$$X = \sum_{p\leq n^{1/10}} \chi_p \quad \text{and} \quad \mathbb{E}\left[\chi_p\right] = \frac{\left\lfloor \frac{n}{p}\right\rfloor}{n} = \frac{1}{p} + O(1/n) \Rightarrow$$
$$\mathbb{E}\left[X\right] = \sum_{p\leq n^{1/10}} \frac{1}{p} + O(1) \overset{4.10}{=} \log\log n + O(1)$$

Now we calculate the Δ^* part. For any p, q primes, we have $\Pr\left[\chi_p\cdot\chi_q\right] = \frac{\left\lfloor \frac{n}{pq}\right\rfloor}{n}$ (by the Chinese Reminder Theorem), therefore:

$$cov(\chi_p,\chi_q) = \mathbb{E}\left[\chi_p\chi_q\right] - \mathbb{E}\left[\chi_p\right]\mathbb{E}\left[\chi_p\right] = \frac{\left\lfloor \frac{n}{pq}\right\rfloor}{n} - \frac{\left\lfloor \frac{n}{p}\right\rfloor}{n}\frac{\left\lfloor \frac{n}{q}\right\rfloor}{n} = \frac{1}{pq} - \left(\frac{1}{p}-\frac{1}{n}\right)\left(\frac{1}{q}-\frac{1}{n}\right) \leq \frac{1}{n}\left(\frac{1}{p}+\frac{1}{q}\right)$$

So

$$VarX = \sum_p Var\chi_p + \sum_{p\neq q} cov\chi_p,\chi_q \leq \log\log n + O(1) + \sum_{p\neq q}\frac{1}{n}\left(\frac{1}{p}+\frac{1}{q}\right) \leq$$
$$\log\log n + O(1) + (n^{1/10})^2\frac{1}{n} = \log\log n + o(1)$$

So the result follows from Chebyshev. $\square$

4.5 The Röde Nibble

The question deals with the existence of designs.

[V] Let m be the number of divisors larger than $n^{1/10}$, then x is at least $n^{m/10} \leq x \leq n$.

Definition 4.5 ((n,k,r) - design)**.** : An (n,k,r) - design is a k-graph over n vertices such that any r-set of vertices is contained exactly in one edge.

Example 4.2. $k=2, r=1$ means a perfect matching.

Question 4.5.1. *Is is true that for any $r<k\ll n$ there exists a corresponding design?*

Of course not! Take an odd n, one cannot find a perfect matching in a graph over an odd number of vertices. We get division requirements: Denote by e the number of edges in a design. Then

$$e\cdot\binom{k}{r}=\binom{n}{r}\Rightarrow\binom{k}{r}\mid\binom{n}{r}$$

Definition 4.6. The Complementary design with respect to $A\subset[n]$ with $|A|<r$ is with the edges

$$E_A=\{e\setminus A\mid A\subset e\in E\}$$

4.5.1 Approximations

Definition 4.7 (Covering)**.** a covering is a relaxation of designs, when we demand that any r-tuple is contained in *at least* one edge

Definition 4.8 (Packing)**.** a covering is a relaxation of designs, when we demand that any r-tuple is contained in *at most* one edge

Remark. In theres cases, clearly $|E|\geq\frac{\binom{n}{r}}{\binom{k}{r}}$ and $|E|\leq\frac{\binom{n}{r}}{\binom{k}{r}}$ respectively.

The *Erdős Hananni conjecture* is that for any k,r, when $n\to\infty$ there exists a covering of size[VI] $(1+o(1))\frac{\binom{n}{r}}{\binom{k}{r}}$. This is equivalent of having a packing of size $(1-o(1))\frac{\binom{n}{r}}{\binom{k}{r}}$. This conjecture was proved by using the *Röde Nibble.*

Proof (for the case $r=2,k=3$). [VII] We look for a collection of $(1+o(1))\frac{\binom{n}{2}}{3}$ triplets that cover all edges. Had we tried to choose any triangle independently with probability $1/n$, we would have failed miserably:

$$\Pr\left[\text{A specific edge is not covered}\right]=(1-1/n)^{n-2}\approx\frac{1}{e}$$

Which means that this method "misses" a constant amount of edges!
We try to choose any trianegl with probability $\frac{\varepsilon}{n}$, which results in approximately $\frac{\varepsilon n^2}{6}$ triangles. Then:

$$\Pr\left[\text{A specific edge is not covered}\right]=(1-1/n)^{n-2}\approx\frac{1}{e^{\varepsilon}}$$

So

$$\Pr\left[\text{a specific edge is covered}\right]\approx 1-e^{-\varepsilon}\approx\varepsilon-\frac{\varepsilon^2}{2}\dots$$

□

[VI] In the sense of $|E|$
[VII] Steiner Triplet systems

Definition 4.9 (Typical Graph)**.** A graph with m edges is called (D, δ, k)-*Typical* if:

1. Aside from $\delta \cdot m$ edges, all edge is contained in $D(1 \pm \delta)$ triangles.
2. Any edge is contained in at most kD triangles.

Lemma 4.10.1. *For any $\varepsilon > 0$, large enough D, k and $\delta > 0$, there exsists $\gamma > 0$ such that in any (D, δ, k) typical graph there is a collection of $\frac{\varepsilon}{3}(m \pm \gamma)$ triangles, denoted T such that $G \setminus T$ is a graph with $m \cdot e^{-\varepsilon}(1 \pm \gamma)$ edges, and is $(De^{-2\varepsilon}, \gamma, ke^{2\varepsilon})$-typical*

Proof. We sample each triangle i.i.d with probability $\frac{\varepsilon}{D}$. The number of triangles in the graph is at least $\frac{(1-\delta)mD(1-\delta)}{3} = \frac{mD}{3}(1 - \delta_1)$, and at most $\frac{\delta mkD+(1-\delta)mD(1+\delta)}{3} = \frac{mD}{3}(1 + \delta_1)$. Let T be the number of triangles. Then $T \sim \text{Bin}\left[\frac{mD}{3}(1 \pm \delta_1), \frac{\varepsilon}{D}\right]$ and the first item in the definition is gained by first moment argument.
Let X_e be the indicator of the event "e is not covered". Then if $d_e = D(1 \pm \delta)$[VIII], we get

$$\mathbb{E}\left[X_e\right] = \left(1 - \frac{\varepsilon}{D}\right)^{D(1\pm\delta)} = e^{-\varepsilon}(1 + \delta_1)$$

Let $X = \sum_e X_e$, then $\mathbb{E}\left[X\right] = me^{-\varepsilon}(1 - \delta_1)$ and

$$cov X_e, X_{e'} = \Pr\left[\text{both not covered}\right] - \Pr\left[e \text{ is not covered}\right]\Pr\left[e' \text{ is not covered}\right]$$
$$\left(1 - \frac{\varepsilon}{D}\right)^{d_e+d_{e'}-1} - \left(1 - \frac{\varepsilon}{D}\right)^{d_e}\left(1 - \frac{\varepsilon}{D}\right)^{d_{e'}} \le \frac{\varepsilon}{D}$$

Then

$$Var X \le me^{-\varepsilon}(1 \pm \delta_1) + mD(1 + \delta)2\frac{\varepsilon}{D} = O(m)$$

Then by Chebyshev, $\Pr\left[\text{The number of edges} \notin me^{-\varepsilon}(1 \pm \delta_2) < 0.01\right] \to 0$. It is left to show that $G \setminus T$ is typical.

Claim 4.5.1. *Other than $\delta_1 m$ edges, all edges are both good and contained in $(1\pm\delta_1)D$ triangles whose edges are good.*

Proof.

$$\mathbb{E}\left[d_e(G \setminus T)\right] = (1 \pm \delta_1)De^{-2\varepsilon}(1 \pm \delta_1)^2$$
$$Var d_e(G \setminus T) \le \mathbb{E}\left[d_e(G \setminus T)\right] + D^2\frac{\varepsilon}{D} = O(D)$$

Wo once again, Chebyshev we are done. ☐

☐

Proof (Röde's Nibble - general case). Denote $p = e^{-\varepsilon}$Let $G_0 = K_n$. Let G_{i+1} be obtained from G_i by removing each triangle with probability $\frac{\varepsilon}{p^{2i}D}$ .Then $|E(G_i)| \approx p^i\binom{n}{2}$: With this step we've chosen

$$\approx \frac{\varepsilon}{p^{2i}D} \cdot \overbrace{p^i\binom{n}{2}}^{|E(G_i)|} \cdot \qquad \frac{\overbrace{p^{2i}n}^{\#\,\triangle\text{ in typical edge}}}{3} \qquad = \varepsilon p^i\frac{n^2}{6}$$

Hence the number of triangles in the cover is

$$p^t\binom{n}{2} + \sum_{i=0}^{t-1}\varepsilon p^i\frac{n^2}{6} \le \frac{n^2}{6}\left(3e^{-\varepsilon t} + \varepsilon\frac{1}{1 - e^{-\varepsilon}}\right) = \star$$

So when $\varepsilon \to 0$, we can choose a large enough t, we may have $\star \le (1 + \delta)\frac{n^2}{6}$. The thing is - we've hidden all the error terms, but this can be dealt with. Super annoyingly. ☐

[VIII]The number of triangles containing e

Chapter 5

Lovász's Local Lemma

Up to this point we used probability to find an object of interest with high probability. The *Local Lemma* is a tool to prove an object's existence even if the probability of finding them is small - even exponentially. In fact, this is an *algorithmic* approach.

Theorem 5.1 (The local Lemma, Symmetric)**.** *Let $(A_i)_{i\in[n]}$ be events such that:*

1. *A_i is independent in all A_j, except for at most d of them*[a]
2. *$Pr[A_i] \leq p$*
3. *$e \cdot p \cdot (d+1) \leq 1$*

Then $Pr\left[\cap_{i\in[n]}\overline{A_i}\right] > 0$

[a]Not in pairs! A_i is dependent on the event $\cup_{j\in K} A_j$ for some $K \in \binom{[n]\setminus i}{d}$

5.1 Results from the lemma

Theorem 5.2 (Improvement on 1.1)**.** *If $e \cdot 2^{1-\binom{k}{2}} \left(\binom{k}{2}\binom{n}{k-2} + 1\right) < 1$, then $R(k,k) > n$*

Proof. Let $G \sim \mathcal{G}(n, 1/2)$. For any $S \in \binom{[n]}{k}$, denote A_S the even that S is a clique or an anti-clique. We know that $\Pr[A_S] = 2^{1-\binom{k}{2}}$. Note that A_S is independent in all A_M other than at most $\binom{k}{2}\binom{n}{k-2}$. Then by the 5.1 - there exists a graph in which no A_S occures. ☐

Exercise. Consider a k-SAT in which any variable appears in at most r clauses ($k > 3r$). Show a polynomial algorithm to decide satisfyability.

Theorem 5.3. *Any k-graph (a k-uniform hypergraph) in which any edge intersects at most $\frac{2^{k-1}}{e} - 1$ other edges is 2-colorable.*

Proof. Consider a random 2 coloring $c : X \to \{0, 1\}$. Assume A_i is the event "the i'th edge is monochromatic", then $\Pr[A_i] = 2^{1-k}$, and $d = \frac{2^{k-1}}{e} - 1$, and the result follows from LLL[I]. ☐

5.1.1 Colorings of $\mathbb{R}$

Consider a coloring $c : \mathbb{R} \to [k]$. We say T is *Colorful* if $c[T] = [k]$. The question Lovás and ??? asked is given a finite S, can we color $\mathbb{R}$ such that S *and all of its translations* are colorful.

[I]Lováz's Local Lemma

Theorem 5.4. *For any k and for any S of cardinality m such that*

$$e \cdot k \left(1 - \frac{1}{k}\right)^m (m(m-1)+1) \leq 1$$

there exists a coloring $c : \mathbb{R} \to [k]$ such that all of S's translations are colorful.[II]

Proof. Denote $c_x = \{c \mid x + S \text{ is colorful}\}$. We want to show that $\bigcap_{x\in\mathbb{R}} c_x \neq \emptyset$. By compactness arguments, it is sufficient to show that for any finite X, $\bigcap_{x\in X} c_x \neq \emptyset$. Consider a random coloring $c : \mathbb{R} \to [k]$. For any $x \in X$, the event $A_x =$ "$x + S$ is not colorful", hence

$$\Pr[A_x] \leq k\left(1 - \frac{1}{k}\right)^m$$

And we note that A_x, A_y are independent unless $(x + S) \cap (y + S) \neq \emptyset$, hence there are at most $m(m-1)$ such y's for which it happens. From LLL we are done. □

5.1.2 Coverings of $\mathbb{R}^3$

Definition 5.1 (k-covering)**.** A k covering of a metric space X is a covering in which any element is in at least k covering sets.

Definition 5.2 (Reducible)**.** We say a covering $\mathcal{U}$ is *reducible* if it can be partitioned into two coverings $\mathcal{U}_1, \mathcal{U}_2$ that are disjoint in their open sets.

Question 5.1.1. *Is there a k-covering in which any point is covered exactly k times? $O(k$ times?*

Theorem 5.5 (Nani-Levicko, Pach. No Proof)**.** *For any k there exists an irreducible k-covering of $\mathbb{R}^3$ by unit balls.*

Theorem 5.6. *Any k covering in which any point is covered at most $t := c \cdot 2^{\frac{k}{3}}$ times is reducible.*

Proof. Given a covering $\mathcal{U} = \{B_i\}_{i\in I}$, define a hypergraph H with vertex set $\mathcal{U}$ and edges indexed by $\mathbb{R}^3$: for any $x \in \mathbb{R}^3$, $e_x = \{B_i \mid x \in B_i\}$ and delete multiple edges. That is, edges correspond to "cells" in $\mathcal{U}$. for any x, $k \leq |e_x| \leq t$. We need to show that H is 2 colorable, which will correspond to two subcoverings. It suffices to show that any finite subgraph of H is 2-colorable (by compactness). Consider a random 2-coloring c, denote by A_x the event e_x is monochromatic, then $\Pr[A_x] \leq 2^{1-k}$. If A_x, A_y are dependent, then $d(x, y) \leq 4$: If $d(x, y) > 4$, any two balls containing x, y do not intersect. We now claim that any edge e_x intersects at most $c \cdot t^3$ other edges. By the previous claim - any ball intersecting some ball containing x is contained in $B_4(x)$, and any point is covered at most t times - thus the sum of volumes of balls intersecting some ball with x is $N \leq 4^3 \cdot B_1 \approx 4^3$. The number of sells is hence at most $N^3 \overset{\text{exercise}}{\leq} 4^9 t^3$, thus by LLL H is 2-colorable if $e \cdot 2^{1-k}(4^9 t^3 + 1) \leq 1$: choosing c appropriately guarantees this. □

Remark. If we start with a $2k$ cover and we want to partition into two k-coverings, this happens w.h.p polinomilally. From $3k$ to two k-coverings, we need to bound

$$\Pr[\text{Less than } k \text{ blue or less than } k \text{ red}] \geq 2 \cdot \lambda^{-k}$$

, and we get exponential h.p.

[II]Doing the calculations, we get $m \approx (3 + o(1))k \log k$

5.2 Proof of the Local Lemma

Definition 5.3 (Dependencies Graph)**.** Let $\mathcal{A} = \{A_1 \dots A_n\}$ be events in some probability space. The *Dependencies Graph* of $\mathcal{A}$ is the DiGraph with $V = \mathcal{A}$ and A_i is independent of all A_j such that $A_i A_j \notin E$. We identify A_i with i.

Theorem 5.7 (The True Local Lemma)**.** *Let $\mathcal{A}$ be some events with dependencies graph $\mathcal{D}$. If there exists $0 < x_i < 1$ with*

$$Pr[A_i] \le x_i \cdot Pr[i \to_{\mathcal{D}} j]\,(1 - x_j)$$

then

$$Pr\left[\bigcap_{i\in[n]} \overline{A_i}\right] \ge \prod_{i\in[n]} (1 - x_i)$$

In particular, if $\mathcal{A}$ are pairwise independent, then $Pr\left[\bigcap_i \overline{A_i}\right] = \prod_i (1 - x_i)$. In the symmetric case, taking $x_i = \frac{1}{d+1}$, by assumption $e \cdot p \cdot (d+1) \le 1$ we have $Pr[A_i] \le p \le \frac{1}{e\cdot(d+1)} \le x_i \left(1 - \frac{1}{d+1}\right)^d$.

Remark. This implies 5.1

Proof. For any i, for any $i \notin S \subset [n]$, $\Pr\left[A_i \mid \cap_{j\in S}\overline{A_j}\right] \le x_i$. This implies the lemma, since

$$\Pr\left[\bigcap \overline{A_i}\right] = \prod_{i=1}^{n} \Pr\left[\overline{A_i} \mid \bigcap_{j=1}^{i-1} \overline{A_i}\right] \ge \prod_i (1 - x_i)$$

So we prove the claim by induction on $|S|$:
$|S| = 0$ we get from the assumption.
Define $S_1 = \{j \in S \mid (i,j) \in \mathcal{D}\}$ and $S_2 = S \setminus S_1$, and let $B = \bigcap_{j\in S_1} \overline{A_j}$ and $C = \bigcap_{j\in S_2} \overline{A_j}$. Then:

$$\Pr[A_i \mid B \cap C] = \frac{\Pr[A_i \cap B \mid C]}{\Pr[B \mid C]} \le \frac{\Pr[A_i \mid C]}{\Pr[B \mid C]} = \frac{\Pr[A_i]}{\Pr[B \mid C]} \le \frac{x_i \prod_{i\to j}(1 - x_j)}{\Pr[B \mid C]}$$

So we need to show $\Pr[B \mid C] \ge \prod_{j\in S_1}(1 - x_j)$. Denote $S = \{j_k\}_{k\in[t]}$, then:

$$\Pr[B \mid C] = \prod_{k\in[t]} \Pr\left[\overline{A_{j_k}} \mid \prod_{k'=1}^{k-1} \overline{A_{j_{k'}}} \cap C\right] \overset{\text{induction}}{\ge} \prod_{k=1}^{t} (1 - x_{j_k}) = \prod_{j\in S_1} (1 - x_j)$$

□

5.2.1 The Algorithmic Version of the Lemma

Assume $A_1 \dots A_n$ are events in some product space Σ^N That satisfy the conditions of 5.7. Is it possible to efficiently find $(\sigma_1, \dots, \sigma_N) \in \Sigma^N$ such that no A_i holds? This is a generalization of 5.1.

The Moser-Tardös Algorithm for SAT: Consider a random σ. If some A_i holds - resample all of the σ_i on which he is dependent.

Claim 5.2.1. *The expected number of times A_i is resampled is at most $\frac{x_i}{1-x_i}$ - so the runtime of the algorithm is linear.*

Proof. No forml Proof. Consider the "log"[III] of the events we've taken care of: $A, B, A, C, B, D, C, A' \ldots$, and we ask "why is A' resampled?" - we build a tree rooted at A' - one of its predecessors in the log "broke it", say C (so C is a child of A' in the tree), and maybe D as well. Continue in this manner. We ask the probability for such a tree to occurre - and bound this. In fact, $\sum_{T \text{ Tree}} \Pr[T] \leq \frac{x_I}{1-x_i}$ □

[III] Computer-wise

Chapter 6

Concentration of Measure

Chapter 7

Extras

7.1 Crossing number

Turan worked on this in WWII.

Definition 7.1 (Crossing Number)**.** For a graph G, the *Crossing Number* $c(G)$ is the minimal amount of intersections between edges when drawn in $\mathbb{R}^2$

Claim 7.1.1. *For any G, $c(G) \geq e - (3v - 6) \geq e - 3v$*

Proof. From Euler's Formula $\square$

And now consider taking a random induced subgraph G' we have that $\mathbb{E}\left[c(G')\right] = p^4 c \geq p^2 e - 3pv$, thus

$$c \geq \frac{e}{p^2} - \frac{3v}{p^3}$$

optimizing p we get $p = \frac{9v}{2e}$, so when $e > \frac{9v}{2}$, then $c \geq \alpha \cdot \frac{e^3}{v^2}$ Plugging into LHS (when $p < 1$). Therefore when taking a dense subgraph $e \sim v^2$, then $c \gtrsim v^4$