Metric Repair - Hardness and Approximation(s)

Qualifying Exam Presentation

Based on *Metric Violation Distance: Hardness and Approximation* [FRB22] and *Fitting Metrics and Ultrametrics With Minimum Disagreements* [CAFLDM22]

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Talk outline

1. The Problem of Metric Repair
2. Hardness of Metric Repair
    - Decrease Only is Easy
    - Everything Else is Hard
3. Approximation Algorithms
    - First Approximation Algorithm
    - Second Approximation Algorithm
4. Future (read - current) Work

The Problem of Metric Repair

Motivation

- **Data Processing:** Many Data processing tasks (e.g, clustering) rely on the structural properties of data, such as satisfying the triangle inequality.
- **Graph Problems:** Hard graph problems become significantly easier when weights satisfy the triangle inequality (e.g, TSP).
- Perhaps a good idea would be to find a metric that is ”near” the given distance measures.

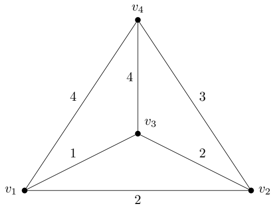
(K_n, w) is a complete, positively weighted graph.
M is its weighted adjacency matrix.

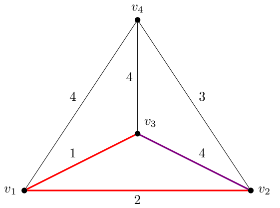
Definition (Broken Triangle)

We say that a triangle $\{i, j, k\} \subset V$ is ***broken*** if $M_{i,j} > M_{i,k} + M_{j,k}$. In this case, we say that $\{i, j\}$ is ***heavy*** in $\{i, j, k\}$, and the other two edges are ***light***. The ***deficit*** of $\{i, j, k\}$ is $\delta(\{i, j, k\}) := M_{i,j} - (M_{i,k} + M_{j,k})$.

Definition (Metric Graph)

We say M is ***metric*** if no triangle in M is broken.

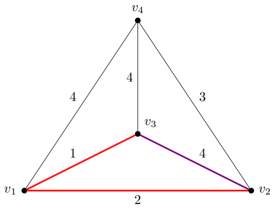




Problem (Metric Repair)

*Given (K_n, w) the problem of **Metric Repair** asks to find $w' : \binom{[n]}{2} \to \mathbb{R}_{>0}$ such that (K_n, w') is metric. A solution is **optimal** if $\|w' - w\|_0$ is minimal.*

- *$w' \geq w$: Increase Only Metric Repair (**IOMR**).*
- *$w' \leq w$: Decrease Only Metric Repair (**DOMR**).*
- *Otherwise: **General Metric Repair** (**MR**).*



Hardness of Metric Repair

Decrease Only

- Intuitively speaking, DOMR is equivalent to APSP [WW10]
- Observation: If uv is heavy in a triangle, then $w(uv) \neq \mathrm{dist}_w(u,v)$.
- Algorithm: [GJ17, FGR+19, FRB22]
  - Run APSP.
  - If $w(u,v) > \mathrm{dist}_w(u,v)$, set $w'(uv) = \mathrm{dist}_w(u,v)$
- **What about MR or IOMR?**

Theorem ([FRB22])

The decision version of MR (and IOMR) is NP *complete.*

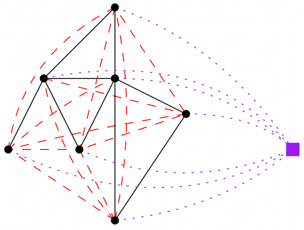
- k-***Vertex Cover***: Given $G = (V, E)$, is there a subset $V' \subset V$ of size $|V'| \leq k$ such that every edge in G is incident with a vertex in V'?
- k-***Metric Repair***: Is there a collection of at most k edges that changing them results in a metric graph?

We reduce $k - \mathrm{VTX} \leq_p k - \mathrm{MR}$.

Proof.

Given a graph $G = (V, E)$, we construct a complete graph $(V \cup \{v_0\}, \binom{V \cup \{v_0\}}{2})$ with weights:

$$w(u, v) = \begin{cases} 2 + \varepsilon & uv \in E \\ 2 & uv \notin E \text{ and } u, v \neq v_0 \\ 1 & u = v_0 \text{ or } v = v_0 \end{cases}$$



The reduction, from [FRB22]

All broken triangles in the constructed graph must be of the form $\{v_0, u, v\}$, and every edge $uv \in E$ defines such triangle. □

Approximation Algorithms

Theorem ([FRB22])

There exists an $O\left(OPT^{1/3}\right)$ *approximation algorithm for MR and IOMR that runs in* $O(n^6)$ *time.*

Theorem ([CAFLDM22])

There exists a randomized algorithm that gives an ***expected*** $O(\log(n))$ *approximation and runs in* $O(n^3)$ *time. This algorithm only holds for MR.*

Definition

A collection of edges S is a ***cover*** for $\mathcal{C}$ if it's a hitting set for $\mathcal{C}$. A cover is ***light*** if it contains a light edges from each broken cycle.

Theorem ([FGR+19])

Let $\mathcal{C}$ be the collection of all broken cycles in (K_n, w). Then $S \subset E$ is a valid solution to MR (IOMR) if and only if S is a (light) cover for $\mathcal{C}$.

Proof.

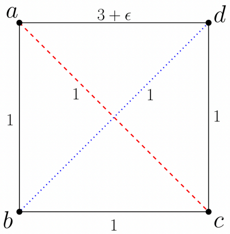
For hard direction:

- For $xy \in S$, set $w'(xy) = \min\left\{\mathrm{dist}_{K_n \setminus S, w}(x, y), \|w\|_\infty\right\}$
- When we remove S, no broken cycles remain - so edges are shortest paths between endpoints
- so no edge that was modified to $\mathrm{dist}_{K_n \setminus S}$ is heavy.
- For edges that were set $\|w\|_\infty$ - we must have disconnected the graph when removing S. We removed at least 2 edges, and a cycle with multiple maximal weights is not broken.

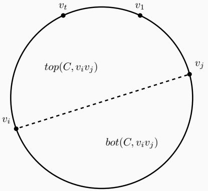
□

First Approximation Algorithm

- The theorem implies that MR is more like Set Cover - logarithmically hard to approximate.
- Problem - there could be exponentially many broken cycles.
- Do we need to cover large cycles? Can't we just cover triangles?



Small cycles are not enough. [CAFLDM22]



Definition

A ***unit cycle*** is a broken cycle C where for each chord e, either $bot(C, e)$ is not broken, or $bot(C, e)$ is broken with e not being the heavy edge.

$top(C, e)$ is broken for any unit cycle.

Claim

Let S be a light cover of all unit cycles in (K_n, w). Then S is a light cover of C.

Proof.

If C is non-unit, then there is a chord e such that $bot(C, e)$ is broken and e is heavy. "Propagate downwards" - smallest one must be unit □

First Approximation Algorithm: Strategy

1. Cover large ***enough*** cycles, S_k.
2. If we don't cover a large unit cycle - we cover ***important chords***.
3. Use cycles ***induced*** by chords to cover larger unit cycles S_c.
4. return $S_c \cup S_k$.

Claim

Let $\mathcal{C}_{\leq k}$ be the collection of all broken cycles of size at most k in (K_n, w). Then one can compute a light cover S_k for $\mathcal{C}_{\leq k}$ in $O(n^k)$ time, and $|S_k| \leq (k-1)|opt_k|$.

Proof.

We use a standard Hitting Set approximation for bounded size sets:

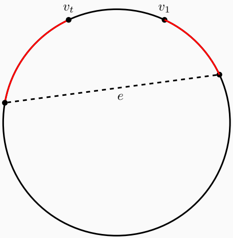
- Every $C \in \mathcal{C}_{\leq k}$ defines a collection of $k-1$ light edges $\ell(C)$.
- Set $S \leftarrow \emptyset$.
- Go over $(\ell(C))_{C \in \mathcal{C}}$ one by one. If $S \cap \ell(C) = \emptyset, S \leftarrow S \cup \ell(C)$.

$|S_k| \leq (k-1)|opt_k|$ since whenever we encounter an uncovered set, any solution would add at least one edge to opt_6, while we add at most $(k-1)$. □

Corollary

Let S_k be as before, and C be a unit cycle uncovered by S_k. Let e be a chord of C such that $|top(C, e)| \leq k$. Then $e \in S_k$.

Proof.

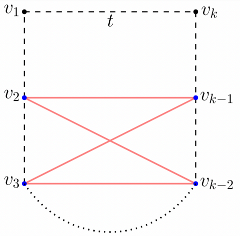


- *We need to cover uncovered unit cycles of size at least k.*
- *Each such cycle has many cords in S_k.*
- *These chords induce cycles that help us!*

$\square$

From now on $k = 6$

For a broken cycle C with $|C| > 6$ and heavy edge v_1v_k, and consider the edge-induced 4-cycle closest to (but not touching) v_1v_k



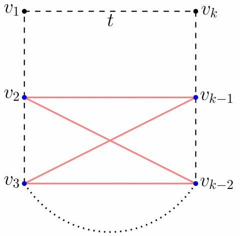
$embed4(C)$ [FRB22]

Observation: chords of this cycle are light edges of C. Moreover, these cycles appear in S_6 for each uncovered unit cycle C.

Lemma

Let S_6 be as before, and let S_c be a set of edges containing a chord of every 4-cycle induced by S_6. Then $S_6 \cup S_c$ is a solution to IOMR.

First Approximation Algorithm: Chording 4-cycles



$embed4(C)$ [FRB22]

We find a chord- cover for all 4-cycles in S_6:

1. $S_c \leftarrow \emptyset$
2. For each 4-cycle $v_2 v_{k-1} v_3 v_{k-2}$ in S_6, if $v_2 v_3 \notin S_c$ and $v_{k-1} v_{k-2} \notin S_c$[a], $S_c \leftarrow S_c \cup \{v_{k-1} v_{k-2}, v_2 v_3\}$

We know $|S_6| \leq 5|opt_5| \leq 5|OPT|$. If we can bound $|S_c|$ as a function of $|S_6|$, we will get an approximation.

[a] $v_{k-1} v_{k-2}, v_2 v_3$ need not be in S_6

First Approximation Algorithm: Chording 4-cycles: Not too many edges

Let $G = (V, S_6)$ be the graph induced by the edges in S_6, and let $\tilde{C}$ be the collection of cycle from which we added edges to S_c.

- No two cycles in $\tilde{C}$ can share a chord, so $|S_c|/2 = |\tilde{C}|$
- Clearly $|E(\tilde{C})| \leq |S_6|$.
- It sufficed to bound $|\tilde{C}|$ in terms of $|E(\tilde{C})|$.

Lemma

Let $G = (V, E)$ be a graph whose edge set comprised of a collection of 4-cycles $\tilde{C}$, no two of which share a chord. Then $|\tilde{C}| = O(|E|^{\frac{4}{3}})$.

Proof

Algorithm 1 Approximate IOMR

Input: K_n, w

1: Compute S_6 using standard HS approximation.
2: Compute S_c using S_6 as described before.
3: **return** $S_6 \cup S_c$.

Theorem

Algorithm 1 is an $O(OPT^{\frac{1}{3}})$ approximation to IOMR.

Proof.

OPT is the size of a minimum HS for C. Clearly $|opt_5| \le OPT$, and so $|S_6| = O(OPT)$. By the previous lemma, $|S_c| = O(|S_6|^{\frac{4}{3}}) = O(OPT^{\frac{4}{3}})$. □

Second Approximation Algorithm: Intro

- *Ransomized* algorithm.
- $\log(n)$ approximation ***in expectation*** (exponentially better!)
- $O(n^3)$ runtime.
- Cannot be (naively) modified to solve IOMR.

- If (K_n, w) is metric, then there are no broken triangle.
- Conversely - if there is a broken triangle, (K_n, w) is not metric.

Algorithm 2 Pivot

Input: K_n, w, i

for $j, k \in \binom{[n]\setminus i}{2}$ **do**
    if $w(jk) > w(ij) + w(ik)$ **then**
        $w(jk) = w(ij) + w(ik)$
    if $w(jk) < |w(ij) - w(ik)|$ **then**
        $w(jk) = |w(ij) - w(ik)|$

Claim

After pivoting at i, no broken triangles incident to i remain.

Algorithm 3 Randomized IOMR

Input: K_n, w

1: Pick $i \in [n]$ uniformly at random
2: Pivot at i
3: Call Randomized IOMR on $K_n \setminus \{i\}, w\mid_{[n]\setminus\{i\}}$

After pivoting at i, we don't change any edge incident to i anymore.

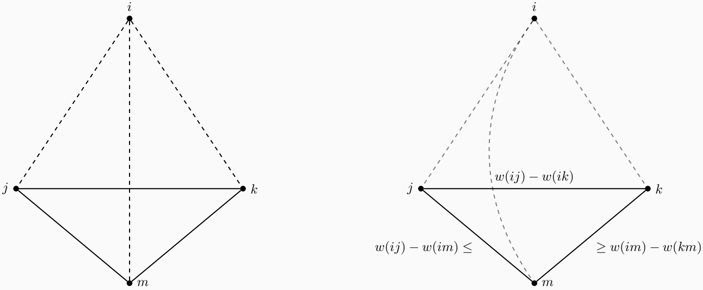
Lemma

After pivoting at i, no new broken triangles are created.

Lemma

After pivoting at i, no new broken triangles are created.

Proof.

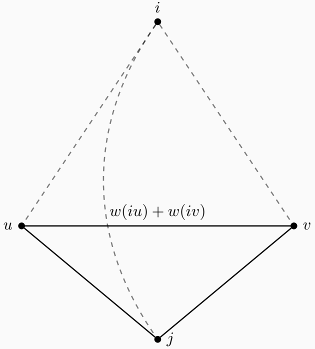


□

Second Approximation Algorithm: Approximation intuition

Lemma

Let uv be an edge that is adjacent to k broken triangles, in which uv is heavy. Order the third vertex of said triangles in order $1, \ldots k$ such that $\delta(uvi) \geq \delta(uvj)$ for $j \geq i$. Then pivoting at i fixes $\{u, v, j\}$ for all $j \geq i$.



w' are the weights after pivoting at i.

1. $w'(uv) \leq w'(ju) + w'(jv)$:
   - $\delta_i \leq \delta_j$, so $w(iu) + w(iv) \leq w(ju) + w(jv)$.
   - WLOG $w'(uj) = w(ui) + w(ij)$
   - But $w'(vj) \geq w(vi) - w(ij)$
2. $w'(uv) \geq |w'(ju) - w'(jv)|$.
   - $w(iu) + w(iv) < w'(ju) - w'(jv)$.
   - Since $w'(uj) \leq w(ij) + w(iu)$ and $w(ij) \leq w(iv) + w'(jv)$, we obtain contradiction.

Second Approximation Algorithm: Summary

- The algorithm makes progress
- The algorithm fixes ”about half” of the broken triangles incident to any edge at each step

How do we formalize this? Let $\mathcal{T}$ be the collection of triangles, and $\mathcal{T}'$ the broken triangles.

- Every solution to MR contains a ***Hitting Set*** for $\mathcal{T}'$
- So $|OPT_{MR}| \geq |HS_{\mathcal{T}'}|$
- By duality - $|HS_{\mathcal{T}'}| \geq |HS^f_{\mathcal{T}'}| \geq |PACK^f_{\mathcal{T}'}|$

Where the Fractional Packing problem is

$$\max_{p \in [0,1]^{|\mathcal{T}'|}} \sum_{t \in \mathcal{T}'} p_t \quad \text{s.t} \quad \forall e \in \binom{[n]}{2}, \sum_{t \ni e} p_t \leq 1$$

If we find some α and p_t such that $\sum_{t \ni e} \frac{p_t}{\alpha} \leq 1$, we have an α approximation

Second Approximation Algorithm: Proof

- $ALG :=$ the set of edges augmented by the algorithm.
- $A_{i,j,k} :=$ the event that one of $\{i, j, k\}$ was chosen as pivot, and $\{i, j, k\}$ was modified as a result. $p_t = \mathbb{E}[A_t]$.
- $\mathbb{E}[|ALG|] \leq \sum_{t \in \mathcal{T}} p_t = \sum_{t \in \mathcal{T}'} p_t$.

Fix e.

- When A_t happens, it changes e with probability 1/3, so $\alpha(e) := \mathbb{E}[\#\text{times } e \text{ was modified}] = \sum_{t \ni e} p_t/3$.
- We induct on n', the number of broken triangles e is incident to.
- Let $c_e(n')$ be an upper bound on the expected number of modifications of e when it is incident to n' broken triangles.
- Any partitioning $n_1 + n_2 = n'$ corresponds to possible increase/ decrease of e when pivoting.
- We use conditional probability on all such partitions, and later condition on "how big" the deficit was when we pivot.
- "Isoperimetric" qualities simplify our computation, as well as Stirling's approximation.

Details

A note on Ultrametrics

- An ***Ultrametric*** is a metric d with the property that

$$d(i,j) \leq \max d(i,k), d(j,k)$$

- Because of this additional structure - possible to get a better approximation ratio. Namely, $O(1)$ approximation!
- The idea is to use ***correlation clustering*** in a sophisticated way (because of ball structure of the metric).

Future (read - current) Work

- ***Generalized*** metric repair - $(G = (V, E), w)$ where triangle inequalities are ***cycle*** inequalities.
- Derandomization?
- ”Average case” graphs

Thank You! Questions?

Second Approximation Algorithm: Proof

For n_1, n_2 such that $n_1 + n_2 = n'$, let E_{n_1,n_2} be the event that pivoting at n_1 of the triangles induces an increase, and n_2 induces a decrease. Conditioned on E_{n_1,n_2}, for $k \in [n_1]$ let F_{k,n_2} be the event that the pivot chosen induced the kth largest increase (similarly define $G_{n_1,k}$). Back

Second Approximation Algorithm: Proof

$$
\begin{aligned}
&c_e(n') \leq \\
&\leq 1 + \sum_{i \in [n']} c_e(i) \Pr\left[i \text{ broken triangles remain after pivoting}\right] \leq \\
&\leq 1 + \sum_{n_1, n_2} \Pr\left[E_{n_1,n_2}\right] \left(\sum_{k=1}^{n_1} \Pr\left[F_{k,n_2}\right] c_e(n_2 + k - 1) + \sum_{k'=1}^{n_2} \Pr\left[G_{n_2,k'}\right] c_e(n_1 + k' - 1)\right) \\
&\leq 1 + \max_{n_1+n_2=n'} \left(\sum_{k=1}^{n_1} \frac{1}{n'} c_e(n_1 + k - 1) + \sum_{k'=1}^{n_2} \frac{1}{n'} c_e(n_2 + k' - 1)\right) \\
&\leq 1 + \max_{n_1+n_2=n'} \left(\sum_{k=n_1+1}^{n'} \frac{1}{n'} c_e(k - 1) + \sum_{k'=n_2+1}^{n'} \frac{1}{n'} c_e(k' - 1)\right) \\
&\leq 1 + \sum_{k=\lfloor n' \rfloor + 1}^{n'} \frac{1}{n'} c_e(k - 1) + \sum_{k=\lceil n' \rceil + 1}^{n'} \frac{1}{n'} c_e(k - 1) \\
&\leq 1 + \frac{1}{n'} c \ln\left(\frac{(n'!)^2}{\lfloor n' \rfloor! \lceil n' \rceil!}\right) \leq c \ln(n' + 1)
\end{aligned}
$$

Back

Lemma

Let $G = (V, E)$ be a graph whose edge set comprised of a collection of 4-cycles $\tilde{C}$, no two of which share a chord. Then $|\tilde{C}| = O(|E|^{\frac{4}{3}})$.

Proof.

- Let V_s be the vertices in G of degree $\leq |E|^{\frac{1}{3}}$ and $V_l := V \setminus V_s$.
- Partition V_s into sets $(g_i)_{i=1}^k$ such that $|E|^{\frac{1}{3}} \leq \sum_{v \in g_i} \deg(v) \leq 2|E|^{\frac{1}{3}}$.
- By degree-sum, $k|E|^{\frac{1}{3}} \leq 2|E|$ so $k \leq 2|E|^{\frac{2}{3}}$.
- Any two edges incident to v can appear in at most one 4 cycle, hence the number of 4 cycles v is in is bounded by $\binom{\deg(v)}{2} \leq \frac{\deg(v)^2}{2}$
- So total number of cycles is bounded by $\frac{1}{2} \sum_{i=1}^k \sum_{v \in g_i} \deg(v)^2 \leq \frac{1}{2} \sum_{i=1}^k \left(\sum_{v \in g_i} \deg(v)\right)^2 \leq k4|E|^{\frac{2}{3}} \leq 4|E|^{\frac{4}{3}}$
- By similar reasoning, $|V_l||E|^{\frac{1}{3}} \leq 2|E|$ and V_l can define at most $|E|^{\frac{4}{3}}$ cycles.

□

Back

References

- Vincent Cohen-Addad, Chenglin Fan, Euiwoong Lee, and Arnaud De Mesmay, ***Fitting metrics and ultrametrics with minimum disagreements***, 2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS), IEEE, 2022, pp. 301–311.
- Chenglin Fan, Anna C Gilbert, Benjamin Raichel, Rishi Sonthalia, and Gregory Van Buskirk, ***Generalized metric repair on graphs***, arXiv preprint arXiv:1908.08411 (2019).
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